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On local times of Ornstein-Uhlenbeck processes

Nathalie Eisenbaum

Abstract : We establish expressions of the local time process of an Ornstein-Uhlenbeck process in terms of the local times on curves of a Brownian motion.

1 Introduction and main result

The purpose of this note is to provide expressions of the local time process of an Ornstein-Uhlenbeck process in terms of the local times of a Brownian motion. Indeed we have recently been confronted to the lack of such expression. More precisely, we have recently solved a stochastic differential equation modeling macrophage dynamics in atherosclerotic plaques [4]. The solution is a functional of the local time process of an Ornstein-Uhlenbeck process and surprisingly we could not find in the literature a tractable expression of this local time. Our contribution below (Theorem 1.1) is based on the notion of local time on curves for Brownian motion. Various definitions of this notion (see [1], [5], [3]) exist and all coincide. We remind in section 2, the ones that we need to establish Theorem 1.1. Theorem 1.1 is established in section 3. Finally in section 4, we make some remarks on the asymptotic behavior of the local time of the Ornstein-Uhlenbeck process.

Fix $\lambda > 0$. Let Y be an Ornstein-Uhlenbeck process solution of

$$Y_t = y_0 + W_t - \frac{\lambda}{2} \int_0^t Y_s ds, \quad (1.1)$$

where $(W_t, t \geq 0)$ is a real Brownian motion starting from 0 and y_0 a fixed real number. Doob [2] has established the following expression for Y

$$(Y_t, t \geq 0) = (e^{-\lambda t/2}(Z_{u(t)} + y_0), t \geq 0), \quad (1.2)$$

where $Z = (Z_t, t \geq 0)$ is a real Brownian motion starting from 0 and $u(t) = \frac{e^{\lambda t} - 1}{\lambda}$.

We will denote the local time process of Z by $(L_t^x(Z), x \in \mathbb{R}, t \geq 0)$.

As a continuous semimartingale Y admits a local time process $(L_t^y(Y), y \in \mathbb{R}, t \geq 0)$, satisfying the following occupation time formula ([8] Chap VI - Corollary 1.6)

$$\int_0^t f(Y_s) ds = \int_{\mathbb{R}} f(y) L_t^y(Y) dy. \quad (1.3)$$

Under the assumption that the Ornstein-Uhlenbeck process Y starts at 0, the expression of its local time at 0 has already been noticed in [6]

$$(L_t^0(Y), t \geq 0) = \left(\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \frac{1}{\sqrt{1 + \lambda s}} dL_s^0(Z), t \geq 0 \right).$$

In [6], the result is given without proof. In their introduction, the authors first mention that the local time at 0 of a continuous semimartingale $M = (\varphi(t)Z_t)_{t \geq 0}$ with Z real Brownian motion starting from 0, is given by $(L_t^0(M))_{t \geq 0} = (\int_0^t \varphi(s) d_s L_s^0(Z))_{t \geq 0}$. This can be actually obtained as a consequence of the extended occupation formula

$$\int_0^t h(s, Z_s) ds = \int_{\mathbb{R}} \int_0^t h(s, x) d_s L_s^x(Z) dx, \quad (1.4)$$

where $d_s L_s^x(Z)$ denotes integration with respect to the time variable.

Using (1.2), Y can be written under the form $(M_{u(t)})_{t \geq 0}$, with $u(t) = \frac{e^{\lambda t} - 1}{\lambda}$ and $\varphi(t) = \frac{1}{\sqrt{1 + \lambda t}}$. One then obtains immediately the expression of its local time at 0

$$L_t^0(Y) = L_{u(t)}^0(M) = \int_0^{u(t)} \varphi(s) d_s L_s^0(Z), \quad t \geq 0.$$

The problem is that this argument does not lead to a tractable expression of $L_t^a(Y)$ for a distinct from 0. The reason is that one has to deal with more general functional of Z than functionals of type $\int_0^t h(s, Z_s) ds$. More precisely, the key notion to handle $L_t^a(Y)$ is stochastic integration over the plane with respect to the local time process $(L_t^x(Z), x \in \mathbb{R}, t \geq 0)$ as a doubly-indexed process. We give a brief exposure of that notion in section 2.

Theorem 1.1 *Fix a real number y_0 . The Ornstein-Uhlenbeck process Y starting from y_0 , admits a local time process $(L_t^y(Y), y \in \mathbb{R}, t \geq 0)$, related to the local time process of Z , as follows*

$$(L_t^a(Y), a \in \mathbb{R}, t \geq 0) = \left(\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \frac{1}{\sqrt{1 + \lambda s}} d_s L_s^{F_a(\cdot)}(Z), a \in \mathbb{R}, t \geq 0 \right) \quad (1.5)$$

where $F : a \rightarrow F_a$ is a functional from \mathbb{R} into the continuous path from \mathbb{R}_+ into \mathbb{R} , defined as follows

$$F_a(t) = a\sqrt{1 + \lambda t} - y_0,$$

and $(L_t^{F_a(\cdot)}(Z), t \geq 0)$ denotes the local time process of Z along the curve F_a . In particular, one has

$$(L_t^0(Y), t \geq 0) = \left(\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \frac{1}{\sqrt{1 + \lambda s}} d_s L_s^{-y_0}(Z), t \geq 0 \right).$$

Remark 1.2 Note that (1.5) is equivalent to

$$L_t^a(Y) = a \int_0^{u(t)} \frac{1}{Z_s + y_0} d_s L_s^{F_a(\cdot)}(Z).$$

Besides, to avoid integration with respect to $(L_t^{F_a(\cdot)}(Z), t \geq 0)$, one can make an integration by parts, and obtain

$$L_t^a(Y) = \frac{L_{u(t)}^{F_a(\cdot)}(Z)}{(1 + \lambda u(t))^{1/2}} + \lambda/2 \int_0^{u(t)} L_s^{F_a(\cdot)}(Z) (1 + \lambda s)^{-3/2} ds, \quad a \in \mathbb{R}, t \geq 0. \quad (1.6)$$

Remark 1.3 As a diffusion, the Ornstein-Uhlenbeck process Y admits a bicontinuous local time $\ell = (\ell_t^x(Y), x \in \mathbb{R}, t \geq 0)$ with respect to its speed measure m ($m(dy) = 2e^{-\frac{\lambda}{2}y^2} dy$). One obviously has

$$(\ell_t^x(Y), x \in \mathbb{R}, t \geq 0) = \left(\frac{e^{\frac{\lambda}{2}x^2}}{2} L_t^x(Y), x \in \mathbb{R}, t \geq 0 \right).$$

But ℓ can also be related to the local time process of the Brownian motion W . Indeed denote by $(L_t^x(W), x \in \mathbb{R}, t \geq 0)$ the local time process of W . For simplicity assume that Y starts from 0. Following Orey [7], one obtains

$$(\ell_t^x(Y), x \in \mathbb{R}, t \geq 0) = (L_{\beta(t)}^{\rho(x)}(W), x \in \mathbb{R}, t \geq 0),$$

where ρ is the inverse of S the scale function of Y ($S(x) = \int_0^x e^{\frac{\lambda}{2}y^2} dy, x \in \mathbb{R}$) and β is the inverse of the function α defined by

$$\alpha(t) = \int_{\mathbb{R}} L_t^x(W) (m \circ \rho)(dx) = \int_{\mathbb{R}} L_t^{S(x)}(W) m(dx).$$

One finally obtains

$$(L_t^a(Y), a \in \mathbb{R}, t \geq 0) = (2e^{-\frac{\lambda}{2}a^2} L_{\beta(t)}^{\rho(a)}(W), a \in \mathbb{R}, t \geq 0). \quad (1.7)$$

Some questions (as questions of stochastic structure) on the local time of Y might be easier to handle using (1.7) than (1.5).

2 Local times on curves

Let Z be a real Brownian motion starting from 0. If one needs a local time of Z along a deterministic measurable curve $b = (b(t), t \geq 0)$, the first idea that comes to mind is to define it as follows

$$L_t^{b(\cdot)}(Z) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[b(s)-\varepsilon, b(s)+\varepsilon]}(Z_s) ds, \quad (2.1)$$

which is precisely the definition introduced by Bass and Burdzy in [1]. They show that the limit is uniform in t on compact sets, in L^2 .

When b is such that $(Z_t - b(t))_{t \geq 0}$ is a semi-martingale, then it admits a local time process and in particular a local time process at 0 which is given by Tanaka's formula

$$L_t^0(Z - b(\cdot)) = |Z_t - b(t)| - |b(0)| - \int_0^t \text{sgn}(Z_s - b(s)) d(Z_s - b(s)). \quad (2.2)$$

As noticed in [1], in that case the two definitions (2.1) and (2.2) coincide a.s.

$$L_t^{b(\cdot)}(Z) = L_t^0(Z - b(\cdot)), \quad t \geq 0.$$

In [3], we have defined a local time for Z along any measurable curve by using integration with respect to local time over the plane. We first remind this notion of integration of deterministic functions with respect to local time over the plane. Consider the space \mathcal{H}

$$\mathcal{H} = \{F : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} : \|F\| < \infty\},$$

where the norm $\|\cdot\|$ is defined by

$$\|F\| = 2 \left(\int_0^\infty \int_{\mathbb{R}} F^2(s, x) \exp\left(-\frac{x^2}{2s}\right) \frac{ds dx}{\sqrt{2\pi s}} \right)^{1/2} + \int_0^\infty \int_{\mathbb{R}} |xF(s, x)| \exp\left(-\frac{x^2}{2s}\right) \frac{ds dx}{s\sqrt{2\pi s}}.$$

For F elementary function i.e. such that there exist a finite sequence of \mathbb{R}_+ , $(s_i)_{1 \leq i \leq n}$, and a finite sequence of \mathbb{R} , $(x_j)_{1 \leq j \leq n}$, with

$$F(s, x) = \sum_{1 \leq i, j \leq n} F(s_i, x_j) 1_{[s_i, s_{i+1})}(s) 1_{[x_j, x_{j+1})}(x),$$

one sets

$$\int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x = \sum_{1 \leq i, j \leq n} F(s_i, x_j) (L_{s_{i+1}}^{x_{j+1}} - L_{s_i}^{x_{j+1}} - L_{s_{i+1}}^{x_j} + L_{s_i}^{x_j}).$$

We have shown in [3] that for every F in \mathcal{H} , and every sequence of elementary functions converging to F for the norm $\|\cdot\|$, $(F_n)_{n \geq 0}$, the sequence $(\int_0^t \int_{\mathbb{R}} F_n(s, x) dL_s^x)_{n \geq 0}$

converges in L^1 uniformly in t on compact sets. The limit does not depend on the choice of the sequence (F_n) and represents

$$\int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x.$$

Moreover

$$\mathbb{E}[\|\int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x\|] \leq \|F\|. \quad (2.3)$$

We remind that for every $t > 0$, the process $(Z_{t-s}, 0 \leq s \leq t)$ is a semimartingale. Extending a result of [5], we have established in [3], that

$$\int_0^v \int_{\mathbb{R}} F(s, x) dL_s^x = \int_0^v F(s, Z_s) dZ_s + \int_{t-v}^t F(t-s, Z_{t-s}) dZ_{t-s}, \quad 0 \leq v \leq t. \quad (2.4)$$

We have also shown (Corollary 3.2 (ii) in [3]) that if $\frac{\partial F}{\partial x}(s, x)$ exists, then

$$\int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x = - \int_0^t \frac{\partial F}{\partial x}(s, Z_s) ds. \quad (2.5)$$

Finally, it has been established in [3] that for any measurable curve $(b(s), s \geq 0)$, one has a.s.

$$(L_t^{b(\cdot)}(Z), t \geq 0) = (\int_0^t \int_{\mathbb{R}} 1_{\{x < b(s)\}} dL_s^x(Z), t \geq 0). \quad (2.6)$$

Using (2.4), one obtains

$$(L_v^{b(\cdot)}(Z), 0 \leq v \leq t) = (\int_0^v 1_{\{Z_s < b(s)\}} dZ_s + \int_{t-v}^t 1_{\{Z_{t-s} < b(t-s)\}} dZ_{t-s}, 0 \leq v \leq t).$$

3 Proof of Theorem 1.1

As a continuous semimartingale, the process Y admits a local time process $(L_t^y(Y), y \in \mathbb{R}, t \geq 0)$ which satisfies the occupation time formula (1.3). This local time process admits a bicontinuous modification. Indeed, according to Theorem 1.7 in [8], the local time process of Y is a.s. continuous in t and cadlag in y with the following discontinuities

$$L_t^y(Y) - L_t^{y-}(Y) = -\frac{\lambda}{2} \int_0^t 1_{(Y_s=y)} Y_s ds = -\frac{\lambda y}{2} \int_0^t 1_{(Y_s=y)} ds = 0,$$

using (1.3). We work with a bicontinuous modification of the local time process of Y . Using (1.2), we have

$$\int_0^t f(Y_s) ds = \int_0^t f(e^{-\lambda s/2}(Z_{u(s)} + y_0)) ds = \int_0^{\frac{e^{\lambda t}-1}{\lambda}} f\left(\frac{1}{\sqrt{1+\lambda u}}(Z_u + y_0)\right) \frac{du}{1+\lambda u}. \quad (3.1)$$

We now choose the function f : $f(x) = f_{a,\varepsilon}(x) = \frac{1}{2\varepsilon}1_{[a-\varepsilon, a+\varepsilon]}(x)$.

On one hand, using the occupation time formula and the continuity of the local time process of Y , a.s. for every $t \geq 0$, every real a

$$\lim_{\varepsilon \rightarrow 0} \int_0^t f_{a,\varepsilon}(Y_s) ds = L_t^a(Y). \quad (3.2)$$

On the other hand, defining the function $F_{a,\varepsilon}$ on $\mathbb{R}_+ \times \mathbb{R}$ by

$$F_{a,\varepsilon}(s, x) = \int_x^\infty \frac{1}{2\varepsilon} 1_{[a-\varepsilon, a+\varepsilon]} \left(\frac{y + y_0}{\sqrt{1 + \lambda s}} \right) \frac{1}{1 + \lambda s} dy,$$

one notes that for every $T > 0$, the function $F_{a,\varepsilon}(s, x)1_{[0, T]}(s)$ belongs to \mathcal{H} . Using (2.5), one obtains

$$\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} f_{a,\varepsilon} \left(\frac{1}{\sqrt{1 + \lambda s}} (Z_s + y_0) \right) \frac{ds}{1 + \lambda s} = \int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \int_{\mathbb{R}} F_{a,\varepsilon}(s, x) dL_s^x(Z),$$

where $dL_s^x(Z)$ refers to integration over the plane $\mathbb{R}_+ \times \mathbb{R}$ wrt local times (see section 2).

For every $T > 0$, as ε tends to 0, the function $F_{a,\varepsilon}(s, x)1_{[0, T]}(s)$ converges for the norm $\|\cdot\|$ to $\frac{1}{\sqrt{1 + \lambda s}} 1_{[x, +\infty)}(a\sqrt{1 + \lambda s} - y_0)1_{[0, T]}(s)$. Hence, thanks to (2.3), one obtains

$$\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \int_{\mathbb{R}} F_{a,\varepsilon}(s, x) dL_s^x(Z)$$

tends in L^1 as ε tends to 0, to

$$\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \int_{\mathbb{R}} \frac{1}{\sqrt{1 + \lambda s}} 1_{\{x < a\sqrt{1 + \lambda s} - y_0\}} dL_s^x(Z).$$

One easily shows that for any measurable bounded function h from \mathbb{R}_+ into \mathbb{R}

$$\int_0^T h(s) d_s L_s^{b(\cdot)}(Z) = \int_0^T \int_{\mathbb{R}} h(s) 1_{\{x < b(s)\}} dL_s^x(Z).$$

Consequently, $\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \int_{\mathbb{R}} F_{a,\varepsilon}(s, x) dL_s^x(Z)$ tends in L^1 to

$$\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \frac{1}{\sqrt{1 + \lambda s}} d_s L_s^{F_a(\cdot)}(Z).$$

Hence, thanks to (3.1), $\int_0^t f_{a,\varepsilon}(Y_s) ds$ tends to $\int_0^{\frac{e^{\lambda t} - 1}{\lambda}} \frac{1}{\sqrt{1 + \lambda s}} d_s L_s^{F_a(\cdot)}(Z)$ in L^1 uniformly in t on compact sets. This is still true finite-dimensionally in a . Using (3.2), one obtains then (1.5) by continuity arguments. \square

4 Some asymptotics of local times

Proposition 4.1 *For Y Ornstein-Uhlenbeck process starting from any fixed real number, the following properties hold for every real a .*

$$(i) \limsup_{t \rightarrow \infty} \frac{L_t^a(Y)}{e^{\frac{\lambda t}{2}} \sqrt{\log t}} \leq 2\sqrt{\frac{2}{\lambda}} \text{ a.s.}$$

$$\text{and in particular: } \limsup_{t \rightarrow \infty} \frac{L_t^0(Y)}{e^{\frac{\lambda t}{2}} \sqrt{\log(t)}} \leq \sqrt{\frac{2}{\lambda}} \text{ a.s.}$$

$$(ii) \limsup_{t \rightarrow \infty} \frac{L_t^a(Y)}{\sqrt{\log t}} \geq \sqrt{\frac{2}{\lambda}} \text{ a.s.}$$

Proof: Denote by y_0 the starting point of Y . Since $F_a(s)$ is an increasing continuous function of s , $M = (Z_t - F_a(t), t \geq 0)$ is a continuous semimartingale. Consequently (2.2) leads to

$$\begin{aligned} L_t^{F_a(\cdot)}(Z) &= |Z_t + y_0 - a\sqrt{1 + \lambda t}| - |a - y_0| - \int_0^t \operatorname{sgn}(Z_s + y_0 - a\sqrt{1 + \lambda s}) dZ_s \\ &\quad + \frac{a\lambda}{2} \int_0^t \operatorname{sgn}(Z_s + y_0 - a\sqrt{1 + \lambda s}) \frac{ds}{\sqrt{1 + \lambda s}}. \end{aligned} \quad (4.1)$$

Besides, Theorem 1.1 gives the following bounds

$$\frac{L_{u(t)}^{F_a(\cdot)}(Z)}{\sqrt{1 + \lambda u(t)}} \leq L_t^a(Y) \leq L_{u(t)}^{F_a(\cdot)}(Z). \quad (4.2)$$

Set: $H(t) = \sqrt{2t \log_2 t}$. One has: $H(u(t)) \sim_{t \rightarrow \infty} \sqrt{\frac{2}{\lambda}} e^{\frac{\lambda t}{2}} \sqrt{\log t}$.

(i) According to the well-known Strassen law of iterated logarithm [9], one has for B real Brownian motion

$$\limsup_{t \rightarrow \infty} \frac{B_t}{H(t)} = 1 \text{ a.s.} \quad (4.3)$$

Applying it to the two Brownian motions Z and $(\int_0^t \operatorname{sgn}(Z_s + y_0 - a\sqrt{1 + \lambda s}) dZ_s)_{t \geq 0}$, one obtains immediately from (4.1): $\limsup_{t \rightarrow \infty} \frac{L_{u(t)}^{F_a(\cdot)}}{e^{\frac{\lambda t}{2}} \sqrt{\log t}} \leq 2\sqrt{\frac{2}{\lambda}}$, which with the right hand inequality of (4.2) leads to (i).

From (4.3) one easily deduces the following law of iterated logarithm for local time:

$\limsup_{t \rightarrow \infty} \frac{L_t^{-y_0}(Z)}{H(t)} = 1$ a.s. It leads to: $\limsup_{t \rightarrow \infty} \frac{L_{u(t)}^{-y_0}(Z)}{e^{\frac{\lambda t}{2}} \sqrt{\log(t)}} = \sqrt{\frac{2}{\lambda}}$ a.s. Using the right

hand inequality of (4.2) for $a = 0$, one obtains: $\limsup_{t \rightarrow \infty} \frac{L_t^0(Y)}{e^{\frac{\lambda t}{2}} \sqrt{\log(t)}} \leq \sqrt{\frac{2}{\lambda}}$ a.s.

(ii) Using the left hand inequality of (4.2) one has: $\frac{L_t^a(Y)}{\sqrt{\frac{2}{\lambda}}e^{\frac{\lambda t}{2}}\sqrt{\log t}} \geq \frac{L_{u(t)}^{F_a(\cdot)}(Z)}{\sqrt{\frac{2}{\lambda}}e^{\frac{\lambda t}{2}}\sqrt{\log t}}$. Besides, denoting by B the Brownian motion $(-\int_0^t \text{sgn}(Z_s + y_0 - a\sqrt{1 + \lambda s})dZ_s, t \geq 0)$, one obtains with (4.1)

$$\limsup_{t \rightarrow \infty} \frac{L_{u(t)}^{F_a(\cdot)}(Z)}{\sqrt{\frac{2}{\lambda}}e^{\frac{\lambda t}{2}}\sqrt{\log t}} \geq \limsup_{t \rightarrow \infty} \frac{B_{u(t)}}{H(u(t))} + \frac{|Z_{u(t)}|}{H(u(t))} \geq \limsup_{t \rightarrow \infty} \frac{B_{u(t)}}{H(u(t))} = 1 \text{ a.s.},$$

which leads to the result. \square

References

- [1] Bass R. and Burdzy K.: The supremum of Brownian local times on hölder curves. *Ann. I.H. Poincaré*, 37, 6, 627-642 (2001).
- [2] J. L. Doob J.L.: The Brownian movement and stochastic equations. *Ann. of Math.* (2), 43 351-369 (1942).
- [3] Eisenbaum N.: Integration with respect to local time. *Potential Analysis* 13, 303-328 (2000).
- [4] Eisenbaum N. and Meunier N.: A lipid structured model for macrophage dynamics in atherosclerotic plaques. *Work in progress*.
- [5] Föllmer H., Protter P. and Shiriyayev A.N.: Quadratic covariation and an extension of Itô's formula. *Bernoulli* 1, 149-169 (1995).
- [6] Gradinaru M., Roynette B., Vallois P. and Yor M.: The laws of Brownian local time integrals. *Compt. Appl. Math*, 3, 259-331 (1999).
- [7] Orey S.: Conditions for the absolute continuity of two diffusions. *Trans. Amer. Math. Soc.*, V.193, 413-426 (1974).
- [8] Revuz D. and Yor M.: Continuous martingales and Brownian motion. *Springer - third edition* (1999).
- [9] Strassen V. : An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 3 211-226 (1964).

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