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# POLARIMETRIC ANALYSIS USING THE ALGEBRAIC REAL REPRESENTATION OF THE SCATTERING MATRIX

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## ABSTRACT

Equivalent matrix representations in radar polarimetry have long been studied and used as tools for modeling and understanding the scattering mechanisms. We include here the Kennough, Graves, or covariance matrices which are today seen as alternative representations of the same physical quantity, the scattering matrix.

In this paper, we briefly explore some of the properties of the algebraic real representation of a complex matrix, a mathematical construction which has been introduced in the literature as an alternative way of performing consimilarity transformations (rather than by the usual Graves power decomposition, with applications limited only to those involving symmetric scattering matrices). Besides the theoretical presentation on the subject, the main goals of the paper are to study some of the advantages and limitations of using the  $4 \times 4$  real matrix form and to compare consimilarity transformation results obtained through the real representation to those given by the power representation.

**Index Terms**— Polarimetry, real matrix representation, consimilarity, similarity, SAR, coneigenvectors, coneigenvalues.

## 1. INTRODUCTION

The current paper proposes an investigation on the use of an alternative method to represent the scattering matrix of radar polarimetry and from there, to eventually retrieve new information associated to the backscattering process.

The rest of the document is structured as follows: Section 2 will provide a basic overview of concepts from linear algebra which are of primary importance for the current investigation, Section 3 will focus on implementation details and Section 4 will offer a brief conclusion.

Throughout the paper, the following notations will be used: bold characters with uppercase/lowercase format are utilized for matrices/vectors (e.g.,  $\mathbf{A}/\mathbf{b}$ ), while the notations for complex conjugate, transpose, and conjugate-transpose will be:  $(\cdot)^*$ ,  $(\cdot)^T$ ,  $(\cdot)^H$ .

## 2. THE REAL REPRESENTATION OF A COMPLEX MATRIX

### 2.1. General definition

Consider a generic square matrix having complex elements,  $\mathbf{A} \in \mathbb{C}^{n \times n}$ ,  $\mathbf{A} = \mathbf{A}_r + i \cdot \mathbf{A}_i$  and  $\mathbf{A}_r, \mathbf{A}_i \in \mathbb{R}^{n \times n}$ . We can find in the literature two definitions for its real representation [1]: a so-called, block-skew-circulant form ( $\mathbf{A}_{RR_1}$ ) and a block-Hankel-skew-circulant form ( $\mathbf{A}_{RR_2}$ ):

$$\mathbf{A}_{RR_1} = \begin{bmatrix} \mathbf{A}_r & -\mathbf{A}_i \\ \mathbf{A}_i & \mathbf{A}_r \end{bmatrix} \mathbf{A}_{RR_2} = \begin{bmatrix} \mathbf{A}_r & \mathbf{A}_i \\ \mathbf{A}_i & -\mathbf{A}_r \end{bmatrix} \quad (1)$$

In [2], Ling and Jiang have been the first to propose using the form  $\mathbf{A}_{RR_2}$  in the domain of radar polarimetry, while other resources have also supported the use of such a transformation for the case when a complex conjugation operator appears [1],[3]. As a result, this representation will also be used in the current article and hereafter referred generically as the real representation (RR) of a complex matrix. We will then investigate what information can be extracted when writing the scattering matrix -  $\mathbf{S}$  - in this equivalent form, where it becomes a  $4 \times 4$  matrix with only real elements,  $\mathbf{S}_{RR}$ .

### 2.2. Change of basis

The methodology of basis change has been linked in radar polarimetry, for a long time, to that of finding the optimal polarization for a given scattering matrix (or, finding the states for which the radar receives minim/maximum power on certain directions). Such an analysis is also used in the case of other polarization matrices, for example the Jones matrix. Moreover, the scattering and Jones matrices are connected themselves by a basis transformation [4], which appears as the result of using two different conventions for describing the scattering phenomenon on the receiving path.

Transforming the Jones matrix to another basis requires the use of a similarity transformation (eq. 2), operation which can be quite straightforward for most cases, even for diagonalization (i.e., equivalent to an eigen-analysis). However, when performing a basis transformation of the scattering matrix, one must take into account, this time, a consimilarity (i.e., conjugate similarity) transformation (eq. 3):

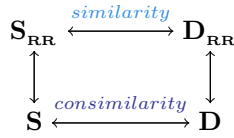
$$\mathbf{D}_J = \mathbf{Y}^{-1} \mathbf{J} \mathbf{Y} \quad (2) \quad \mathbf{D}_S = (\mathbf{X}^*)^{-1} \mathbf{S} \mathbf{X} \quad (3)$$

Using the similarity approach and the Jordan canonical form(s), [5] classifies the Jones matrices based on the relationship between their eigenvalues and eigenvectors, as: diagonalizable homogeneous (i.e., with orthogonal eigenvectors) and inhomogeneous (i.e., only independent eigenvectors), as well as defective matrices (the latter are not diagonalizable, but can be reduced to an upper triangular Jordan canonical form).

More earlier, Lüneburg has performed a similar investigation [6], [7] taking also into account the case of the scattering matrix and formulating some observations for its coneigenvectors and coneigenvalues. A coneigenvector is "a polarization vector that does not change its state of polarization in case of backscattering".

$$\mathbf{S} \mathbf{x} = \xi \mathbf{x}^* \quad (4)$$

In eq. 4, we identify  $\mathbf{x}$  as the coneigenvector and  $\xi$  as its associated coneigenvalue. In the literature, these quantities are computed by using some equivalent representations, which map the consimilarity transformation to that of simple similarity. Sometimes, these representations may also be coupled with simplifying assumptions (e.g.,  $\mathbf{S}$  is symmetric). As the conventional method for determining the coneigenvalues and coneigenvectors, we include the one based on the power (Graves) matrix [6]. It states that "every symmetric matrix is unitary congruent to a diagonal matrix with real nonnegative entries", making use of the equivalent power representation  $\mathbf{S} \mathbf{S}^H$ , but enforcing also the condition of symmetry.



**Fig. 1:** Transformation equivalences in/between the space of the scattering matrix (bottom) and the space of the its real representation (top).

While the earlier approaches in radar polarimetry have used the equivalent representation with the power matrix, we are going to explore the more recent method with the real representation (Fig. 1). [8] and [2] show that one link exists between the eigenvalues of the RR and the coneigenvalues of the matrix from the original space.

In the RR space, each real eigenvalue will be part of a pair  $(\lambda_k, -\lambda_k)$ , in which both its positive and negative values are present. According to [2], the value  $\lambda_k, k \in \{1, 2\}$  corresponds to the unique modulus of an otherwise phase ambiguous coneigenvalue  $\xi_k$ . This is however not the unique case. We recall that a square matrix with real elements may also presents pairs of complex conjugate eigenvalues. This general property adds to the one presented earlier and, as a result, when the eigenvalues of the RR matrix are complex, they will be part of a quad set:  $(\lambda_k, -\lambda_k, \lambda_k^*, -\lambda_k^*)$ .

Because  $\mathbf{S}_{RR} \in \mathbb{R}^{4 \times 4}$ , two main combinations are identified:  $\mathbf{S}_{RR}$  presents either only real eigenvalues (two distinct pairs or one double pair) or only complex ones (as a quad set). Each pair of real eigenvalues relates to a coneigenvalue, while each quad set of complex eigenvalues corresponds also to only one coneigenvalue.

Each of the coneigenvalues,  $\xi_k$ , will have associated one coneigenvector,  $\mathbf{x}_k$ . In the space of the real representation, the eigenvector associated to the negative eigenvalue,  $-\lambda_k$ , can be obtained through a simple relationship from the one which corresponds to eigenvalue  $\lambda_k$ :

$$\mathbf{S}_{RR} \mathbf{y}_k = \lambda_k \mathbf{y}_k \quad \mathbf{S}_{RR} (\mathbf{Q}_2 \mathbf{y}_k) = -\lambda_k (\mathbf{Q}_2 \mathbf{y}_k) \quad (5)$$

where

$$\mathbf{Q}_2 = \begin{pmatrix} \mathbf{0} & -\mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

If  $\lambda_k$  would be a complex eigenvalue, we already know that the eigenvector associated to  $\lambda_k^*$  is itself a conjugated version of  $\mathbf{y}_k$ :

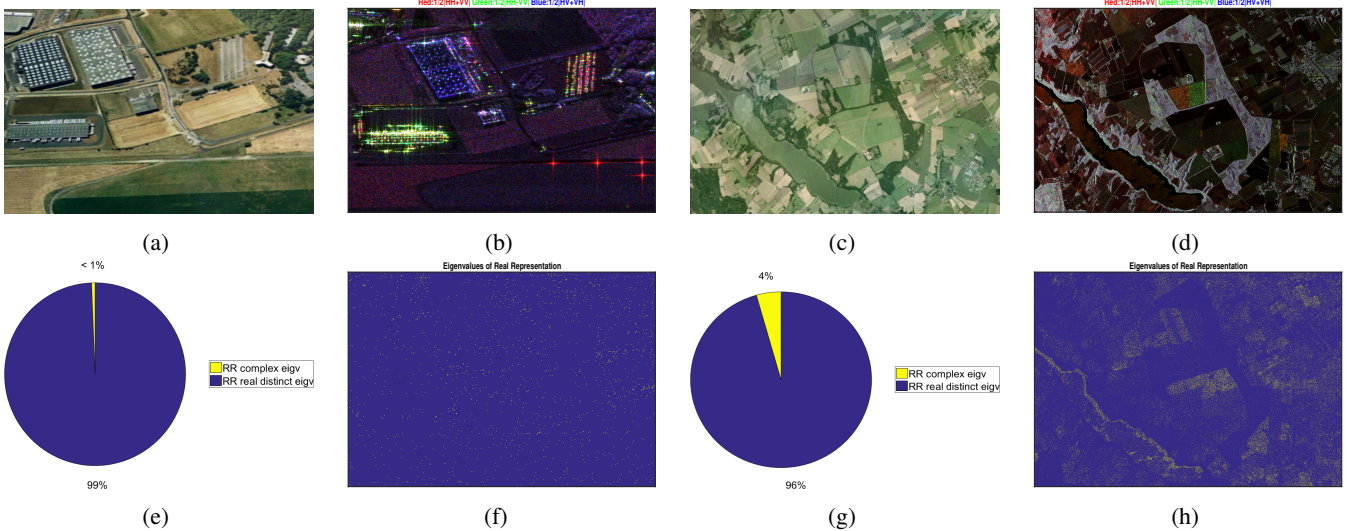
$$\mathbf{S}_{RR}^* \mathbf{y}_k^* = \lambda_k^* \mathbf{y}_k^* \rightarrow \mathbf{S}_{RR} \mathbf{y}_k^* = \lambda_k^* \mathbf{y}_k^* \quad (6)$$

Based on this observations, it is easier to understand why the case of a quad set in the RR space returns only one coneigenvalue/coneigenvector pair. Recently, it was also suggested [9] for the case of Jones matrices, that one complex eigenvalue may occur for defective  $2 \times 2$  complex Jones matrices.

By investigating its real space representation, we now can link a general interpretation for the scattering matrix,  $\mathbf{S}$ , which can may fall in one of the categories:

- *con-diagonalizable homogeneous*: when  $\mathbf{S}_{RR}$  has real and distinct eigenvalues with orthogonal eigenvectors.
- *con-diagonalizable inhomogeneous*: when  $\mathbf{S}_{RR}$  has real eigenvalues and only independent eigenvectors.
- *con-defective*: when  $\mathbf{S}_{RR}$  has complex eigenvalues, which will result in one unique complex coneigenvalue/coneigenvector pair and another generalized coneigenvector.

When performing the similarity transformation (eq. 2) in the  $4 \times 4$  real space, for a more covering analysis, the matrix  $\mathbf{J}$  will be represented in its Jordan canonical form (or its real Jordan canonical form, when dealing with the complex eigenvalue case). For two real, distinct, sets of eigenvalues of  $\mathbf{S}_{RR}$ :  $(\lambda_1, -\lambda_1), (\lambda_2, -\lambda_2)$ , matrix  $\mathbf{J}$  is diagonal. For one double set  $(\lambda, -\lambda), (\lambda, -\lambda)$ , matrix  $\mathbf{J}$  may be diagonal or upper/lower triangular, depending on the relation between eigenvectors. For a complex eigenvalues quad, it is more suggestive to use the real Jordan form instead of the Jordan canonical form, with complex diagonal entries.



**Fig. 2:** Polarimetric data and results. a) Google Maps optical image, Brétigny (France), May 2004. b) Pauli color composite (Brétigny data). c) Google Maps optical image, Foulum (Denmark), July 2005. d) Pauli color composite (Foulum data). e) Pie chart classification for eigenvalues of RR (Brétigny data). f) Colormap classification for eigenvalues of RR (Brétigny data). g) Pie chart classification for eigenvalues of RR (Foulum data). h) Colormap classification for eigenvalues of RR (Foulum data).

### 3. IMPLEMENTATION

In the followings, we will provide a practical investigation, in connection to the real representation theory discussed previously, in a coherent, pixel-by-pixel manner on two polarimetric datasets. We will inspect the classification of eigenvalues obtained from the RR matrices in each case, as well as compare coneigenvalues results computed through the RR and the Graves methods.

#### 3.1. Results and discussion

Two monostatic full-polarimetric sets of images are used for real data testing. The first set of images has been obtained by the RAMSES X-Band airborne radar instrument of French Aerospace Laboratory (ONERA), over Brétigny region, France [10]. The size of the images is  $501 \times 501$  (rows  $\times$  columns) and they are characterized by a range  $\times$  azimuth resolution of around  $1.5 \times 1.5$  meters. The second, EMISAR Foulum, is a well-known dataset in the polarimetric community, available for free as one of the data companions for the earlier versions of the PolSARPro software. The data has been acquired through an airborne campaign in Foulum, Denmark. The radar operated in C-Band and the data presents a  $2 \times 2$  m resolution. Here, the size of the images is  $1750 \times 1000$ .

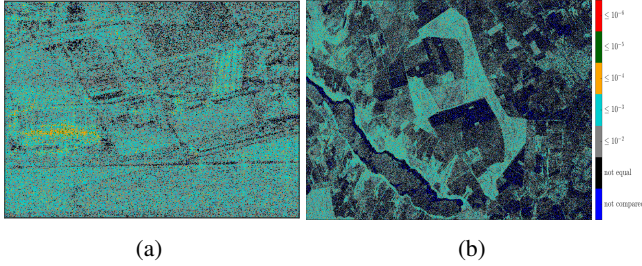
The RGB channels for the Pauli composites (Figs. 2b, 2d) are in the typical order:  $R = \frac{HH+VV}{2}$ ,  $G = \frac{HH-VV}{2}$ ,  $B = \frac{HV+VH}{2}$ . Moreover, it must be mentioned that the Google Maps images (Figs. 2a, 2c), displayed only for comparative reasons, are not quite from the exact same period as the radar acquisitions, therefore some differences may be present.

When inspecting the classification of eigenvalues of the RR matrices,  $\mathbf{S}_{RR}$ , we have observed that in both datasets, the vast majority of the matrices are characterized by two real sets  $(\lambda_1, -\lambda_1)$ ,  $(\lambda_2, -\lambda_2)$  of distinct eigenvalues (indigo). Along it, a small fraction ( $< 1\%$ , yellow) is mapped to the case of complex eigenvalues, forming quad sets. As the two sets of polarimetric images are of different dimensions, for a proper comparison we must scale the percentage of complex eigenvalues obtained for the Foulum data as for an image having the same dimensions as the Brétigny set. In such case, the average percentage obtained is  $0.6\%$ , very similar to the one of the first set.

The results are not surprising. The complex eigenvalues of RR are expected generally for larger values (or comparable values to the co-pol) on the cross-pol components. As the data under study is of monostatic type, we expect dominant co-pol components and a symmetric (or, almost symmetric) form for  $\mathbf{S}$ . From a numerical point of view, for two particular cases it is important to offer the correct delimitation: RR complex eigenvalues with small imaginary part (orders of magnitude lower than the real part) and for the extraction of equal real eigenvalues of  $\mathbf{S}_{RR}$ , when the obtained solutions are still distinct, but of near values under a given tolerance.

For future studies, it is of particular interest to investigate the role of the RR in describing bistatic scattering phenomena.

We briefly recall the theory of Graves [11] for calculating the coneigenvalues and coneigenvectors in the case of symmetric (monostatic) matrices: for all real, nonnegative eigenvalues  $(\gamma_1, \gamma_2)$  of the  $2 \times 2$  Hermitian power matrix  $\mathbf{S}\mathbf{S}^H$ , the coneigenvalues will be the square roots  $(\sqrt{\gamma_1}, \sqrt{\gamma_2})$  and the



**Fig. 3:** Colormap of absolute differences  $\Delta d$  for distinct, real coneigenvalues (indigo, Fig. 2) obtained by RR and Graves methods.

coneigenvectors will equal the eigenvectors of the power matrix. As presented in Section 2, the real values ( $\lambda_1, \lambda_2$ ) obtained from RR distinct sets (case of indigo zones, Fig. 2) will be the coneigenvalues of the scattering matrix. We now study the equivalence between these conjugate eigenvalues results obtained by applying the two different methods.

An error parameter  $\Delta d = \text{abs}(\lambda - \sqrt{\gamma}) \leq \text{tolerance} \cdot \max\{\lambda, \sqrt{\gamma}\}$  is used to numerical quantify the similitude between the results. Figures 3a) b) present such a comparison, in a colormap representation, for the largest, real coneigenvalue. The results are very similar for the second coneigenvalue and have not been included for brevity. As colorbar conventions: blue pixels did not had real coneigenvalues through the RR method and their results have not been compared, and for the pixels marked as black the coneigenvalues returned by the two methods are marked as different. We have considered  $\text{tolerance} \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$ . If the error parameter,  $\Delta d$ , verified the above relation for at least one of the discrete tolerances (considered in the given order), the corresponding pixel in the image will get attributed a gray, cyan, orange, green or red color, respectively.

For the Brétigny dataset, the black and blue colors tend to match the zones from the image with shadows and small backscattering intensities. For the dominant part, the real coneigenvalues of the RR and Graves method are numerically comparable up to the third decimal (following comma). For the Foulum data, the real coneigenvalues obtained by the two methods remain, in general, equal up to the second/third decimal. While it is clear that for the case of real coneigenvalues, the RR representation and the power representation provide comparable results (the greater the backscattering intensity, the more alike), it must be emphasized that different coneigenvectors are obtained by applying the methods. In depth analysis regarding the coneigenvectors will be considered in future studies.

#### 4. CONCLUSIONS

In the present article we have examined some of the general properties of the RR of a complex matrix, with a particular focus on the RR of the radar scattering matrix and the consimilarity transformation. The theoretical discussion, quite new

to the domain, has been extended by a practical investigation which has confirmed, in the framework of the RR theory, the hypothesis of real coneigenvalues for monostatic (symmetric) scattering matrices and has numerically compared their values to those obtained through the Graves power representation.

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