



HAL
open science

Percolation of repulsive particles on graphs

Nathalie Eisenbaum

► **To cite this version:**

Nathalie Eisenbaum. Percolation of repulsive particles on graphs. Catherine Donati-Martin, Antoine Lejay, Alain Rouault. Séminaire de Probabilités LI, 2301, Springer International Publishing, pp.381-396, 2021, Lecture Notes in Mathematics, 10.1007/978-3-030-96409-2_12 . hal-03401540

HAL Id: hal-03401540

<https://hal.science/hal-03401540v1>

Submitted on 25 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Percolation of repulsive particles on graphs

Nathalie Eisenbaum

Abstract What are the percolation properties of a repulsive point process on an infinite connected graph? To treat this question, one starts by clearly state which definition of repulsivity is adopted here. The first issue is then to identify some repulsive point processes. The most known examples of repulsive point process are determinantal point processes. Are there some others? We construct some examples and give a general answer on their percolation properties. In case of determinantal point processes, we establish stochastic domination relations to obtain other sufficient conditions for percolation.

Keywords : Percolation, repulsion, graph, determinantal process, Gaussian process, negative association, Markov chain, stochastic domination.

MSC2010 subject classifications : 60A10, 60G10, 60G15, 60G50, 60G55, 60J10, 60K35.

1 Introduction

Let \mathcal{G} be an infinite, connected, locally finite graph with vertex set \mathcal{V} and non-oriented edges set E . Given a point process χ on \mathcal{V} , we consider the random subgraph of \mathcal{G} with vertex set $\{x \in \mathcal{V} : \chi(x) > 0\}$ and edges set:

$$\{[x, y] \in E : \chi(x) > 0 \text{ and } \chi(y) > 0\}$$

and ask whether this random subgraph has an infinite connected component. In short, does this random subgraph percolate? One can also formulate the question as follows: does χ percolate on \mathcal{G} ?

This question has been first considered for χ Bernoulli point process (i.e. the variables $\chi(x), x \in \mathcal{V}$, are independent Bernoulli variables). In case all the Bernoulli's have the same parameter p ($0 \leq p \leq 1$), the probability for the Bernoulli point process to percolate, which is an increasing function of p with values in $\{0, 1\}$, has a critical value $p_c(\mathcal{G})$.

CNRS and Université de Paris, MAP5, France. E-mail: nathalie.eisenbaum@parisdescartes.fr

Here we will consider the percolation problem for repulsive point processes. A simple point process χ on \mathcal{V} is said to be **repulsive** if for every B_1, \dots, B_n measurable disjoint bounded subsets of \mathcal{V} , one has:

$$\mathbb{E}[\chi(B_1) \dots \chi(B_n)] \leq \mathbb{E}[\chi(B_1)] \dots \mathbb{E}[\chi(B_n)]. \quad (1.1)$$

In the literature (see e.g. [14]), one finds the following less restrictive definition of repulsivity:

$$\mathbb{E}[\chi(a)\chi(b)] \leq \mathbb{E}[\chi(a)]\mathbb{E}[\chi(b)], \text{ for any couple } (a, b) \text{ of distinct points of } \mathcal{V} \quad (1.2)$$

(sometimes written in term of the pair correlation function $\frac{\mathbb{E}[\chi(a)\chi(b)]}{\mathbb{E}[\chi(a)]\mathbb{E}[\chi(b)]}$).

Note that for χ Bernoulli point process on \mathcal{V} , (1.1) is always satisfied with equality. Also determinantal point processes (DPP in short) are well-known repulsive point processes. To give other examples we use negatively associated variables.

A sequence of real valued random variables $(X_n, n \geq 0)$ is **negatively associated** if for every disjoint pair of indexes subsets (I, J) and every increasing fonctionnals F and H from \mathbb{R}^I (resp. \mathbb{R}^J) into \mathbb{R} (i.e. they are increasing with respect to each component)

$$\mathbb{E}[F(X_n, n \in I)H(X_n, n \in J)] \leq \mathbb{E}[F(X_n, n \in I)]\mathbb{E}[H(X_n, n \in J)].$$

With a negatively associated sequence of Bernoulli variables $(X_a, a \in \mathcal{V})$, one can easily define a simple repulsive point process χ on \mathcal{V} , by setting:

$$\chi(f) = \sum_{a \in \mathcal{V}} f(a)X_a, \quad (1.3)$$

for any function f defined on \mathcal{V} .

To produce thanks to (1.3), repulsive simple point processes, the first issue is to find such a sequence of negatively associated Bernoulli variables. It can be easily obtained from a sequence of negatively associated variables. One can take advantage of known examples of negatively associated sequences (see e.g. [7]). In particular one can use negatively associated Gaussian sequences. In section 2, we give various ways to obtain negatively associated Gaussian sequences. Besides, we start section 2 by pointing that determinantal processes are illustrations of (1.3).

One might think that it should be harder for a repulsive simple point process χ on \mathbb{Z}^d to percolate than for a Bernoulli process with parameters $\mathbb{P}[\chi(x) = 1], x \in \mathcal{V}$. In section 3, we present a general result for the point processes obtained via (1.3) on \mathbb{Z}^d , that reinforces this intuition. We also present applications of this result to Gaussian free fields and to determinantal point processes.

In the case of determinantal point processes, we establish stochastic domination relations and use them to obtain other criterions of percolation. They are presented in section 4.

2 Sequences of negatively associated Bernoulli variables

2.1 Discrete determinantal processes

Definition 2.1 For \mathcal{S} locally compact metric space, for any A subset of \mathcal{S} , denote by $\mathcal{M}(A)$ the set of σ -finite measures on \mathcal{S} with support in A . Then a random measure M on \mathcal{S} is negatively associated if for every A and B disjoint subsets of \mathcal{S} , every non-decreasing real valued function f_1 (resp. f_2) on the set of σ -finite measures on \mathcal{S} , with support in $\mathcal{M}(A)$ (resp. $\mathcal{M}(B)$): $\mathbb{E}[f_1(M)f_2(M)] \leq \mathbb{E}[f_1(M)]\mathbb{E}[f_2(M)]$.

In the special case when \mathcal{S} is a locally finite discrete set \mathcal{V} and M is a simple point process χ , one has:

*The random measure χ is negatively associated
iff the sequence $(\chi(a), a \in \mathcal{V})$ is negatively associated.*

In particular, consider any determinantal point process χ on a discrete set \mathcal{V} . It has been established by Lyons [10] that χ is negatively associated. Hence $(\chi(a), a \in \mathcal{V})$ is a sequence of negatively associated Bernoulli variables. Moreover χ admits the representation (1.3):

$$\chi(f) = \sum_{a \in \mathcal{V}} f(a)\chi(a).$$

2.2 Negatively correlated Gaussian variables

According to Joag-Dev and Proschan [5], a centered Gaussian variables sequence $(\eta_a, a \in \mathcal{V})$ is negatively associated iff it is negatively correlated (i.e. $\mathbb{E}[\eta_a\eta_b] \leq 0$, for $a \neq b$).

Once found a sequence of centered negatively correlated Gaussian variables $(\eta_a, a \in \mathcal{V})$, one easily constructs a sequence of negatively associated Bernoulli variables $(Y_a, a \in \mathcal{V})$, by, for example, choosing an arbitrary real constant h and setting:

$$Y_a = 1_{\eta_a > h}. \tag{2.1}$$

The problem becomes to find negatively correlated centered Gaussian sequences. The four propositions below present solutions.

Proposition 2.2 *For $(Y_a, a \in \mathcal{V})$ any sequence of pairwise negatively correlated variables, $(\mathbb{E}[Y_a Y_b] - \mathbb{E}[Y_a]\mathbb{E}[Y_b], (a, b) \in \mathcal{V} \times \mathcal{V})$ is the covariance of a negatively correlated centered Gaussian sequence indexed by \mathcal{V} .*

Proof One has for $a \neq b$: $\mathbb{E}[Y_a Y_b] - \mathbb{E}[Y_a] \mathbb{E}[Y_b] \leq 0$. Moreover, for every $x_i, 1 \leq i \leq n$ in \mathbb{R} and every $a_i, 1 \leq i \leq n$ in \mathcal{V}

$$\sum_{1 \leq i, j \leq n} x_i x_j (\mathbb{E}[Y_{a_i} Y_{a_j}] - \mathbb{E}[Y_{a_i}] \mathbb{E}[Y_{a_j}]) = \mathbb{E}[(\sum_{i=1}^n x_i Y_{a_i})^2] - \mathbb{E}[\sum_{i=1}^n x_i Y_{a_i}]^2 \geq 0. \quad (2.2)$$

□

Example 2.2.1 Use the Bernoulli variables corresponding to a DPP on \mathcal{V} with kernel $(K(a, b), (a, b) \in \mathcal{V}^2)$, to obtain a negatively correlated centered Gaussian sequence with covariance C given by:

$$C(a, b) = -(K(a, b))^2 \text{ for } a \neq b \text{ and } C(a, a) = K(a, a)(1 - K(a, a)).$$

In case K has only nonnegative entries ($K(x, y) \geq 0, \forall x, y \in \mathcal{V}$), there is another way to obtain a negatively correlated Gaussian sequence. Indeed $I - K$ is positive semidefinite and has nonpositive off diagonal entries.

More generally, one can use any simple point processes satisfying (1.2) to produce repulsive simple point processes (i.e. satisfying (1.1)). To do so, one notes the following immediate proposition.

Proposition 2.3 *Let χ be a simple point process on \mathcal{V} satisfying (1.2), then $(\mathbb{E}[\chi(a)\chi(b)] - \mathbb{E}[\chi(a)]\mathbb{E}[\chi(b)], (a, b) \in \mathcal{V} \times \mathcal{V})$ is the covariance of a negatively correlated centered Gaussian sequence indexed by \mathcal{V} .*

Example 2.3.1 Let $\tilde{\chi}$ be any DPP on \mathbb{R}^2 . For any (n, m) in \mathbb{Z}^2 , set: $A_{(n, m)} = (n, n + 1) \times (m, m + 1)$. Define the simple point process χ on \mathbb{Z}^2 by $\chi((n, m)) = 1_{\tilde{\chi}(A_{(n, m)}) > 0}$. Since $\tilde{\chi}$ is negatively associated in the sense of Definition 2.1 (see [11]), χ satisfies (1.2).

Remark 2.3.2 One could think that anti-ferromagnetic Ising spin systems should correspond to point processes satisfying (1.2). This intuition is false. Consider for example an anti-ferromagnetic Ising spin system with only pair interaction between nearest neighbors on \mathbb{Z}^2 . Denote by χ the corresponding point process of sites with positive spin on \mathbb{Z}^2 . Split \mathbb{Z}^2 into $A = \{(x, y) \in \mathbb{Z}^2 : |x + y| \text{ is odd}\}$ and A^c . Thanks to [8], one obtains that

$$\mathbb{E}[\chi(a)\chi(b)] \leq \mathbb{E}[\chi(a)]\mathbb{E}[\chi(b)] \text{ if } a \in A \text{ and } b \in A^c$$

and

$$\mathbb{E}[\chi(a)\chi(b)] \geq \mathbb{E}[\chi(a)]\mathbb{E}[\chi(b)] \text{ if } a, b \in A \text{ or } a, b \in A^c.$$

Another way to obtain negatively correlated Gaussian sequences, consists in using Markov processes.

Proposition 2.4 *Let P be the transition matrix of a symmetric Markov chain on a discrete space \mathcal{V} . Let I be the identity matrix on the discrete space \mathcal{V} . Then $I - P$ is the covariance matrix of a negatively correlated centered Gaussian sequence indexed by \mathcal{V} .*

Indeed $(I - P)$ is positive semi-definite (see for ex [1]) and the off diagonal entries of $(I - P)$ are all nonpositive.

Example 2.4.1 For the simple random walk on \mathbb{Z}^2 , the probability to jump to a neighbor is $1/4$: $P(a, b) = \frac{1}{4}1_{d(a,b)=1}$. The centered Gaussian process with covariance $(I - P)$ is negatively associated.

A Markov process with state space S , is said to be weakly symmetric with respect to a σ -finite measure m on S if its transition semigroup (P_t) satisfies:

$$\int_S f(x)P_t g(x)m(dx) = \int_S g(x)P_t f(x)m(dx)$$

for every f, g in $L^2(m)$.

Proposition 2.5 *Denote by \mathcal{L} the infinitesimal generator of a Markov process weakly symmetric with respect to m . Let $(f_n)_{n \geq 0}$ be a sequence of bounded nonnegative functions with disjoint compact supports, elements of the domain of \mathcal{L} . Then there exists a negatively correlated centered process $(\eta_n)_{n \geq 0}$ with a covariance given by:*

$$\mathbb{E}[\eta_n \eta_k] = - \int_E f_n(x) \mathcal{L} f_k(x) m(dx) = - \int_E f_k(x) \mathcal{L} f_n(x) m(dx).$$

Proof First we remind that $(-\mathcal{L})$ is a positive semi-definite operator on its domain $D(\mathcal{L})$. Hence there exists a centered Gaussian field $(\eta(u), u \in D(\mathcal{L}))$ with covariance

$$\mathbb{E}[\eta(u)\eta(v)] = - \langle u, \mathcal{L}v \rangle = - \int_E u(x)\mathcal{L}v(x)m(dx) = \langle (-\mathcal{L})^{1/2}u, (-\mathcal{L})^{1/2}v \rangle .$$

For every u and v in $D(\mathcal{L})$:

$$- \langle u, \mathcal{L}v \rangle = \lim_{t \rightarrow 0} \frac{1}{t} \langle u, v - P_t v \rangle$$

(see for ex. Lemma 1.3.4 in [2]).

Assume now that u and v are also nonnegative and have disjoint supports. We obtain:

$$\langle u, v - P_t v \rangle = - \langle u, P_t v \rangle \leq 0$$

and hence the centered Gaussian process $(\eta(f_n), n \geq 0)$ is negatively correlated. \square

Example 2.5.1 Consider the generator \mathcal{L} of any symmetric Lévy process on \mathbb{R} with no drift, no Gaussian component and Lévy measure ν . For any C^2 -function f in the domain of \mathcal{L} : $\mathcal{L}f(x) = \int_{\mathbb{R}} (f(x+y) - f(x) - \frac{y}{1+y^2}f'(x))\nu(dy)$.

Fix $\varepsilon > 0$. Denote by f_a a C^2 -function in the domain of \mathcal{L} with support equal to $(a-\varepsilon, a+\varepsilon)$. For any increasing sequence of reals $(a_n, n \geq 0)$ such that $d(a_i, a_{i+1}) = 2\varepsilon$, the centered Gaussian sequence $(\eta(f_{a_n}), n \geq 0)$ is negatively correlated with covariance:

$$\mathbb{E}[\eta(f_{a_i})\eta(f_{a_j})] = - \int_{\mathbb{R}} f_{a_i}(x)\mathcal{L}f_{a_j}(x)dx.$$

3 Percolation for repulsive particles on \mathbb{Z}^d

3.1 A criterion for percolation

In the proposition below, we present a criterion for the percolation of repulsive particles on \mathbb{Z}^d with a configuration satisfying (1.3).

Proposition 3.1 *For any positive integer d , there exist two real numbers in $(0, 1)$, p_{dis} and p_{agr} such that for every negatively associated sequence of Bernoulli variables $(X_k, k \in \mathbb{Z}^d)$, we have:*

- (i) *If $\forall k, \mathbb{P}[X_k = 1] \leq p_{dis}(d)$, then a.s. $\{k \in \mathbb{Z}^d : X_k = 1\}$ has no infinite connected component.*
- (ii) *If $\forall k, \mathbb{P}[X_k = 1] \geq p_{agr}(d)$, then a.s. $\{k \in \mathbb{Z}^d : X_k = 1\}$ has an infinite connected component.*

Proposition 3.1 is a consequence of a criterion of Molchanov and Stepanov [13] for the occurrence of a percolation transition. We remind it below. It uses the notion of *-adjacent. Two points x and y of \mathbb{Z}^d are *-adjacent if $\sup_{1 \leq i \leq d} |x_i - y_i| = 1$. A *-connected set is connected w.r.t. *-adjacency.

There exist two finite constants c_d^{dis} and c_d^{agr} depending of the dimension d such that for each $\{0, 1\}$ -valued random field $(X_k)_{k \in \mathbb{Z}^d}$, the following points hold:

- *If there exists $c > 0$ such that for every connected subset A*

$$\mathbb{P}[\forall k \in A : X_k = 1] \leq c \exp(-c_d^{dis}|A|),$$

then a.s. $\{k \in \mathbb{Z}^d : X_k = 1\}$ has no infinite connected component.

- If there exists $c > 0$ such that for every $*$ -connected subset A

$$\mathbb{P}[\forall k \in A : X_k = 0] \leq c \exp(-c_d^{agr} |A|),$$

then a.s. $\{k \in \mathbb{Z}^d : X_k = 1\}$ has an infinite connected component.

We set $p_{dis}(d) = \exp(-c_d^{dis})$ and $p_{agr}(d) = 1 - \exp(-c_d^{agr})$.

To see that Proposition 3.1 is a consequence of the above criterion, note that thanks to the negative association property, one has, in case (i):

$$\mathbb{P}[\forall k \in A, X_k = 1] \leq \prod_{k \in A} \mathbb{P}[X_k = 1] \leq p_{dis}(d)^{|A|}$$

and in case (ii):

$$\mathbb{P}[\forall k \in A, X_k = 0] \leq \prod_{k \in A} \mathbb{P}[X_k = 0] \leq (1 - p_{agr}(d))^{|A|}.$$

Consider a family of repulsive particles on \mathbb{Z}^d with a spatial configuration χ given by a family of negatively associated Bernoulli variables. Assume that they all have the same parameter p . One might think that the occurrence of a percolation for this population of particles should be harder to happen than if they were all independent with the same marginals. Denote by $p_c^{site}(\mathbb{Z}^d)$ the critical probability for percolation of a Bernoulli point process with the same probability at each site of \mathbb{Z}^d . One should legitimately expect that in case χ percolates, one would have: $p \geq p_c^{site}(\mathbb{Z}^d)$. Thanks to Proposition 3.1, we know that in case χ percolates, we must have: $p \geq p_{dis}(d)$.

Since any Bernoulli point process is negatively associated, one obtains:

$$p_{dis}(d) \leq p_c^{site}(\mathbb{Z}^d) \leq p_{agr}(d). \quad (3.1)$$

According to Molchanov and Stepanov [13], one can take: $p_{dis}(d) = \frac{1}{2d-1}$. One also has the following asymptotic [6]: $\lim_{d \rightarrow \infty} 2d p_c^{site}(\mathbb{Z}^d) = 1$. Consequently one obtains:

$$\lim_{d \rightarrow \infty} 2d p_{dis}(d) = 1$$

and

$$p_{dis}(d) \sim_{d \rightarrow \infty} p_c^{site}(\mathbb{Z}^d), \quad (3.2)$$

which reinforces the intuition that negative association goes against percolation, at least in high dimension.

For other graphs, this intuition might be false. Here is an example suggested by the referee. Consider the graph with vertex set $\mathbb{Z} \times \{0, 1\}$ and the edges set corresponding to $*$ -adjacency. The point process χ is such that the probability to occupy a site is always $1/2$, the couples $(\chi(n, 0), \chi(n, 1))$ are all independent and: $\chi((n, 0)) + \chi((n, 1)) = 1$. This point process is repulsive. Obviously this point process a.s. percolates but the corresponding Bernoulli process a.s. does not percolate.

3.2 Determinantal point processes

One should note first that whatever the infinite connected graph \mathcal{G} , a DPP is living on, one has the following 0-1 law.

Proposition 3.2 *For \mathcal{G} any infinite connected, locally finite graph, the probability that a determinantal point process on \mathcal{G} contains an infinite connected component is 0 or 1.*

Indeed, Lyons [10] has shown that a determinantal point process on any infinite discrete set is always tail trivial.

As a direct application of Proposition 3.1, one obtains the following sufficient condition for the occurrence of a percolation for determinantal point processes. A DPP with kernel K is denoted by $\text{DPP}(K)$.

Proposition 3.3 *Let $(K(x, y), (x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d)$ be the kernel of a determinantal point process on \mathbb{Z}^d .*

- (i) *If for every $x \in \mathbb{Z}^d$, $K(x, x) < p_{\text{disc}}(d)$, then a.s. $\text{DPP}(K)$ does not percolate on \mathbb{Z}^d .*
- (ii) *If for every $x \in \mathbb{Z}^d$, $K(x, x) > p_{\text{agr}}(d)$, then a.s. $\text{DPP}(K)$ percolates on \mathbb{Z}^d .*

3.3 Comparing a Gaussian free field with its neighborhood

One can make use of Proposition 3.1 to obtain a result on the so-called Gaussian free fields associated to symmetric transient Markov chain on \mathbb{Z}^d . To do so, we will establish relations in law between the Gaussian free fields (which are positively correlated), and negatively correlated Gaussian processes (Proposition 3.6).

Theorem 3.4 *Fix $d \geq 1$. Let h_{agr} and h_{disc} be the two finite numbers such that: $\mathbb{P}[N > h_{\text{agr}}] = p_{\text{agr}}(d)$ and $\mathbb{P}[N > h_{\text{disc}}] = p_{\text{disc}}(d)$, where N is a real standard Gaussian variable.*

Let $(\eta_+(x))_{x \in \mathbb{Z}^d}$ be a centered Gaussian field with covariance the Green function of a transient symmetric Markov chain X on \mathbb{Z}^d with transition matrix

$P = (P(x, y))_{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d}$ such that $P(x, x) = 0, \forall x \in \mathbb{Z}^d$. Then we have:

- *for any $h < h_{\text{agr}}$, the set $\{x \in \mathbb{Z}^d : \eta_+(x) \geq h + \sum_{y \in \mathbb{Z}^d} P(x, y)\eta_+(y)\}$ has a.s. an infinite connected component.*
- *for any $h > h_{\text{disc}}$, the set $\{x \in \mathbb{Z}^d : \eta_+(x) \geq h + \sum_{y \in \mathbb{Z}^d} P(x, y)\eta_+(y)\}$ has a.s. only finite connected components.*

Note that Theorem 3.4 does not require from the symmetric transient Markov chain to have stationary nor independent increments, nor jumps limited to nearest neighbors. When the considered Markov chain is a random walk (i.e. $P(x, y) = p(y - x)$ for x, y in \mathbb{Z}^d), one has the following corollary.

Corollary 3.5 *Let $(\eta_+(x))_{x \in \mathbb{Z}^d}$ be a Gaussian free field associated to a stationary symmetric transient random walk on \mathbb{Z}^d . Then there exists a real h_c such that $|h_c| < \infty$ and*

- *for any $h < h_c$, the set $\{x \in \mathbb{Z}^d : \eta_+(x) \geq h + \sum_{y \in \mathbb{Z}^d} p(y-x)\eta_+(y)\}$ has a.s. an infinite connected component.*
- *for any $h > h_c$, the set $\{x \in \mathbb{Z}^d : \eta_+(x) \geq h + \sum_{y \in \mathbb{Z}^d} p(y-x)\eta_+(y)\}$ has a.s. only finite connected components.*

Moreover the critical level h_c satisfies: $p_{dis}(d) \leq \mathbb{P}[N > h_c] \leq p_{agr}(d)$, where N is a real standard Gaussian variable.

From now on, given a symmetric transition matrix $P = (P(x, y))_{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d}$, we denote by $\mathcal{V}(x)$ the set $\{y \in \mathbb{Z}^d : P(x, y) > 0\}$.

The above corollary is satisfied in particular by the Gaussian free field η_+ associated to the simple symmetric random walk on \mathbb{Z}^d with $d \geq 3$, and takes in this case the following form:

There exists a critical real level h_c such that: $p_{dis}(d) \leq \mathbb{P}[N > h_c] \leq p_{agr}(d)$, and

(3.3)

- (i) *for any $h < h_c$, the set $\{x \in \mathbb{Z}^d : \eta_+(x) \geq h + \frac{1}{2d} \sum_{y \in \mathcal{V}(x)} \eta_+(y)\}$ has a.s. an infinite connected component.*
- (ii) *for any $h > h_c$, the set $\{x \in \mathbb{Z}^d : \eta_+(x) \geq h + \frac{1}{2d} \sum_{y \in \mathcal{V}(x)} \eta_+(y)\}$ has a.s. only finite connected components.*

Theorem 3.4 is a direct consequence of Proposition 3.1, Proposition 2.4 and of the following proposition.

Proposition 3.6 *For \mathcal{G} infinite connected graph, denote by \mathcal{V} its vertex set. Let $(X_n, n \geq 0)$ be a symmetric transient Markov chain on \mathcal{V} with transition matrix $P = (P(x, y), (x, y) \in \mathcal{V}^2)$ and Green operator $G = (G(x, y), (x, y) \in \mathcal{V}^2)$. One associates two centered Gaussian processes to X , $(\eta_+(x), x \in \mathcal{V})$ and $(\eta_-(x), x \in \mathcal{V})$, with respective covariances G and $(I - P)$. Then we have:*

$$\eta_- \stackrel{(\text{law})}{=} (I - P)\eta_+ \tag{3.4}$$

and

$$\eta_+ \stackrel{(\text{law})}{=} G\eta_- . \tag{3.5}$$

The two infinite matrices G and $I - P$ are positive semi-definite. This is the reason why Proposition 3.6 requires a proof. It is based on the following lemma.

Lemma 3.7 *For every nonnegative f on \mathcal{V} such that: $Gf(a) < \infty, \forall a \in \mathcal{V}$, one has:*

$$(I - P)Gf = f \quad (3.6)$$

and

$$G(I - P)f = f \quad (3.7)$$

Proof of Lemma 3.7 Since G is the Green operator associated to the Markov chain $(X_n)_{n \geq 0}$, for every nonnegative function f on \mathcal{V} : $Gf(a) = \mathbb{E}_a[\sum_{n=0}^{\infty} f(X_n)] = \sum_{b \in \mathcal{V}} G(a, b)f(b)$. Besides: $Pf(a) = \mathbb{E}_a[f(X_1)]$.

$$\begin{aligned} (I - P)Gf(a) &= Gf(a) - PGf(a) = \mathbb{E}_a[\sum_{n=0}^{\infty} f(X_n)] - \mathbb{E}_a[\mathbb{E}_{X_1}[\sum_{n=0}^{\infty} f(X_n)]] \\ &= \mathbb{E}_a[\sum_{n=0}^{\infty} f(X_n)] - \mathbb{E}_a[\sum_{n=1}^{\infty} f(X_n)] = \mathbb{E}_a[f(X_0)] = f(a), \end{aligned}$$

which gives (3.6).

For (3.7), one writes similarly:

$$\begin{aligned} G(I - P)f(a) &= Gf(a) - GPf(a) = \mathbb{E}_a[\sum_{n=0}^{\infty} f(X_n)] - \mathbb{E}_a[\sum_{n=0}^{\infty} Pf(X_n)] \\ &= \mathbb{E}_a[\sum_{n=0}^{\infty} f(X_n)] - \mathbb{E}_a[\sum_{n=0}^{\infty} \mathbb{E}_{X_n}[f(X_1)]] \\ &= \mathbb{E}_a[\sum_{n=0}^{\infty} f(X_n)] - \mathbb{E}_a[\sum_{n=1}^{\infty} f(X_n)] = \mathbb{E}_a[f(X_0)] = f(a). \end{aligned}$$

□

Proof of Proposition 3.6 By definition:

$$G(x, y) = \sum_{n=0}^{\infty} \mathbb{E}_x[X_n = y] = \sum_{n=0}^{\infty} P^n(x, y).$$

Set : $\varphi = (\varphi_x, x \in \mathcal{V}) = (I - P)\eta_+ = ((I - P)\eta_+(k), k \in \mathcal{V})$. The process φ is a centered Gaussian process. We compute the covariation of φ .

$$\begin{aligned} \mathbb{E}[\varphi(x)\varphi(y)] &= \mathbb{E}[(I - P)\eta_+(x)(I - P)\eta_+(y)] \\ &= \mathbb{E}[\sum_{k \in \mathcal{V}(x)} (I - P)(x, k)\eta_+(k) \sum_{q \in \mathcal{V}(y)} (I - P)(y, q)\eta_+(q)] \\ &= \sum_{k \in \mathcal{V}(x)} (I - P)(x, k) \sum_{q \in \mathcal{V}(y)} (I - P)(y, q)G(q, k) \end{aligned} \quad (3.8)$$

Using (3.6) for the function $f = 1_{\{k\}}$, one obtains: $(I - P)G1_{\{k\}}(y) = 1_{\{k\}}(y)$, for every y in \mathcal{V} , equivalently:

$$\sum_{q \in \mathcal{V}(y)} (I - P)(y, q)g(q, k) = 1_{\{k\}}(y). \quad (3.9)$$

Thanks to (3.9), (3.8) leads to

$$\mathbb{E}[\varphi(x)\varphi(y)] = \sum_{k \in \mathcal{V}(x)} (I - P)(x, k)1_{\{k\}}(y) = (I - P)(x, y)$$

One concludes that: $(\varphi(x), x \in \mathcal{V}) \stackrel{(\text{law})}{=} (\eta_-(x), x \in \mathcal{V})$.
One uses similarly (3.7) to show (3.5). \square

Proof of Corollary 3.5 When $P(x, y) = p(y - x)$ for x, y in \mathbb{Z}^d , the two Gaussian centered fields η_- and η_+ associated to X , are stationary. Moreover one has: $\sum_{y \in \mathcal{V}} |\mathbb{E}[\eta_-(0)\eta_-(y)]| = \sum_{y \in \mathcal{V}(0)} P(0, y) \leq 1$. Hence η_- has the property of so-called finite susceptibility. This implies that all the translation invariant events involving η_- have probability 0 or 1. Together with Theorem 3.4, a simple monotony argument leads to the conclusion. \square

Remark 3.8 For the special case of the simple symmetric random walk on \mathbb{Z}^d , one could have taken advantage of a general result of Liggett, Schonmann and Stacey [9] (Theorem 0.0 (i)) on 3-dependent random fields to partially recover (3.3) (i). Indeed, making use of their result, for sufficiently small real h , there exists $\rho = \rho(h) > p_c^{\text{site}}(\mathbb{Z}^d)$, such that:

$$\text{Bernoulli}(\rho) \prec (1_{\eta_-(x) > h}, x \in \mathbb{Z}^d),$$

where $\text{Bernoulli}(\rho)$ denotes a Bernoulli process with parameter ρ .
Hence for sufficiently small h , the set $\{x \in \mathbb{Z}^d : \eta_-(x) \geq h\}$ has a.s. an infinite connected component. But this argument does not tell us that the percolation critical value of $\mathbb{P}[\eta_-(0) > h]$ should be greater than $p_c^{\text{site}}(\mathbb{Z}^d)$, nor than $p_{\text{dis}}(d)$.

We also mention that with the assumptions of Corollary 3.5, one can apply a result of Garet (Theorem 1, [3]) to η_- to obtain a result of percolation for $(|\eta_-(x)|)_{x \in \mathbb{Z}^d}$, which thanks to Lemma 3.7, translates into the same result for $(|\eta_+(x) - \sum_{y \in \mathcal{V}(x)} p(y - x)\eta_+(y)|)_{x \in \mathbb{Z}^d}$.

4 Stochastic domination for DPP

Stochastic domination relations are a usual tool for percolation results. We establish below some stochastic domination relations involving DPP on discrete sets. Georgii and Yoo [4] have shown similar relations for determinantal point processes on continuous

sets. In the later the DPP are dominated by Poisson point processes. Surprisingly the discrete case has not been treated already in full generality. The case of stationary DPP and $\mathcal{V} = \mathbb{Z}^d$ has been treated by Lyons and Steif [12]. We come back to this special case below.

For \mathcal{V} discrete set, let $(K(a, b), (a, b) \in \mathcal{V}^2)$ be the kernel of a determinantal point process on \mathcal{V} . If K has a spectrum included in $[0, 1)$, one sets:

$$J = K(I - K)^{-1} = (J(a, b), (a, b) \in \mathcal{V}^2).$$

Theorem 4.1 *Let K be the kernel of determinantal point process on a discrete set \mathcal{V} with a spectrum included in $[0, 1)$, then we have:*

$$\text{DPP}(K) \prec \text{Bernoulli}\left(\frac{J(a, a)}{1 + J(a, a)}, a \in \mathcal{V}\right)$$

and

$$\text{Bernoulli}\left(\frac{1}{1 + J(a, a)}, a \in \mathcal{V}\right) \prec \text{DPP}(I - K).$$

The proof of Theorem 4.1 is presented at the end of this section.

Remark 4.2 For K kernel of a DPP on a discrete set \mathcal{V} with a spectral radius ρ strictly smaller than 1, we have: $0 \leq K \leq \rho I$. Using the monotony property established by Lyons [10], this leads to: $\text{DPP}(K) \prec \text{Bernoulli}(\rho)$. But note that the domination relation of Theorem 4.1 is stronger. Indeed the eigenvalues of J are related to the ones of K as follows: λ is an eigenvalue of K iff $(\frac{1}{1-\lambda} - 1)$ is an eigenvalue of J . This implies that the spectral radius of J is equal to $(\frac{1}{1-\rho} - 1)$. Consequently for every a in \mathcal{V} , one has: $J(a, a) \leq \frac{1}{1-\rho} - 1$, and hence: $\text{Bernoulli}\left(\frac{J(a, a)}{1 + J(a, a)}, a \in \mathcal{V}\right) \prec \text{Bernoulli}(\rho)$.

Note that in case $K = pI$ with $p \in [0, 1)$, one has:

$$\text{DPP}(K) = \text{Bernoulli}\left(\frac{J(a, a)}{1 + J(a, a)}, a \in \mathcal{V}\right) = \text{Bernoulli}(\rho).$$

As an immediat application of Theorem 4.1, we give the following sufficient conditions of percolation and absence of percolation on an infinite connected graph \mathcal{G} with vertex set \mathcal{V} . The critical probability $p_c(\mathcal{G})$ is presented in the introduction. We assume that: $0 < p_c(\mathcal{G}) < 1$.

Proposition 4.3 *Let $\text{DPP}(K)$ be a determinantal process on \mathcal{V} such that the spectrum of its kernel K is in $[0, 1)$.*

- (i) *If for every x in \mathcal{V} , $J(x, x) < \frac{1-p_c(\mathcal{G})}{p_c(\mathcal{G})}$, then a.s. $\text{DPP}(K)$ does not percolate on \mathcal{G} .*
- (ii) *If for every x in \mathcal{V} , $J(x, x) < \frac{p_c(\mathcal{G})}{1-p_c(\mathcal{G})}$, then a.s. $\text{DPP}(I-K)$ percolates on \mathcal{G} .*

Remark 4.2 leads to the same kind of results with percolation sufficient conditions lying instead on the spectral radius.

In case $\mathcal{V} = \mathbb{Z}^d$, and there exists a measurable function f from $\mathbb{R}^d/\mathbb{Z}^d$ into $[0, 1]$ such that: $K(a, b) = \hat{f}(b - a)$, for $a, b \in \mathbb{Z}^d$ (where $\hat{f}(a) = \int_{\mathbb{R}^d/\mathbb{Z}^d} f(x)e^{-2\pi i \langle a, x \rangle} \lambda_d(dx)$ for $a \in \mathbb{Z}^d$ with λ_d unit Lebesgue measure on $\mathbb{R}^d/\mathbb{Z}^d$), one has: $J(a, a) = J(0, 0)$ for every a . Denote this common value by $J_f(0)$ and write K_f for K . The geometric mean of f is defined by

$$\text{GM}(f) = \exp\left(\int_{\mathbb{R}^d/\mathbb{Z}^d} \log f(x) \lambda_d(dx)\right).$$

According to Theorem 5.11 in [12]: $\text{DPP}(K_f) \prec \text{Bernoulli}(q)$ iff $q \geq 1 - \text{GM}(1 - f)$ and $\text{Bernoulli}(p) \prec \text{DPP}(K_f)$ iff $p \leq \text{GM}(f)$.

One immediately notes that Theorem 5.11 in [12] is stronger than Theorem 4.1 restricted to this framework. Indeed from Theorem 4.1 one has: $\frac{1}{1+J_f(0)} \leq 1 - K_f(0, 0)$. Since $K_f(0, 0) = \text{AM}(f)$ the arithmetic mean of f , one obtains: $1 - K_f(0, 0) = \text{AM}(1 - f) \leq \text{GM}(1 - f)$, by Jensen's inequality.

One can take advantage of the above result of [12] to obtain the following conditions of percolation:

- If $\text{GM}(f) > p_c^{\text{site}}(\mathbb{Z}^d)$ then a.s. $\text{DPP}(K_f)$ percolates on \mathbb{Z}^d .
- If $\text{GM}(1 - f) > 1 - p_c^{\text{site}}(\mathbb{Z}^d)$ then a.s. $\text{DPP}(K_f)$ does not percolate on \mathbb{Z}^d .

Proof of Theorem 4.1 Notation: $K_A = (K(a, b), (a, b) \in A^2)$ and $J[A] = K_A(I_A - K_A)^{-1}$.

Denote by $(Y_a, a \in \mathcal{V})$ the family of Bernoulli variables corresponding to $\text{DPP}(K)$. For any finite subset $A = \{a_1, \dots, a_n\}$ of \mathcal{V} , the joint law of $(Y_{a_1}, \dots, Y_{a_n})$ is given by

$$\mathbb{P}[Y_{a_1} = x_1, \dots, Y_{a_n} = x_n] = \det(I_A - K_A) \det(J[A]_{A_1}) \quad (4.1)$$

for any x_{a_1}, \dots, x_{a_n} in $\{0, 1\}$ and $A_1 = \{a \in A : x_a = 1\}$.

For any y_{a_1}, \dots, y_{a_n} in $\{0, 1\}$, set $B_1 = \{a \in A : x_a \wedge y_a = 1\}$. Since: $B_1 \subset A_1$, one has:

$$\det(J[A]_{A_1}) \leq \det(J[A]_{A_1 \setminus B_1}) \det(J[A]_{B_1}) \leq \det(J[A]_{B_1}) \prod_{a \in A_1 \setminus B_1} J[A](a, a). \quad (4.2)$$

Multiplying each member of (4.2) by $\det(I_A - K_A)$, one obtains:

$$\mathbb{P}[Y_{a_1} = x_{a_1}, \dots, Y_{a_n} = x_{a_n}] \leq \mathbb{P}[Y_{a_1} = x_{a_1} \wedge y_{a_1}, \dots, Y_{a_n} = x_{a_n} \wedge y_{a_n}] \prod_{a \in A_1 \setminus B_1} J[A](a, a) \quad (4.3)$$

For \tilde{K} diagonal matrix indexed by $A \times A$, such that for every a in A : $0 \leq \tilde{K}(a, a) < 1$, denote by $(\tilde{Y}_a, a \in A)$ the corresponding Bernoulli variables. One has similarly:

$$\begin{aligned} \mathbb{P} & [\tilde{Y}_{a_1} = x_{a_1} \vee y_{a_1}, \dots, \tilde{Y}_{a_n} = x_{a_n} \vee y_{a_n}] \\ & = \mathbb{P}[\tilde{Y}_{a_1} = y_{a_1}, \dots, \tilde{Y}_{a_n} = y_{a_n}] \prod_{a \in \tilde{B}_1 \setminus \tilde{A}_1} \tilde{K}(a, a)(1 - \tilde{K}(a, a))^{-1} \end{aligned}$$

where $\tilde{A}_1 = \{a \in A : y_a = 1\}$ and $\tilde{B}_1 = \{a \in A : x_a \vee y_a = 1\}$.

Note that $\tilde{B}_1 \setminus \tilde{A}_1 = A_1 \setminus B_1$. We choose now : $\tilde{K}(a, a) = \frac{J[A](a, a)}{1 + J[A](a, a)}$, to obtain:

$$\mathbb{P}[\tilde{Y}_{a_1} = y_{a_1}, \dots, \tilde{Y}_{a_n} = y_{a_n}] \prod_{a \in A_1 \setminus B_1} J[A](a, a) = \mathbb{P}[\tilde{Y}_{a_1} = x_{a_1} \vee y_{a_1}, \dots, \tilde{Y}_{a_n} = x_{a_n} \vee y_{a_n}] \quad (4.4)$$

Multiplying member by member (4.3) and (4.4), one obtains:

$$\begin{aligned} & \mathbb{P}[Y_{a_1} = x_{a_1}, \dots, Y_{a_n} = x_{a_n}] \mathbb{P}[\tilde{Y}_{a_1} = y_{a_1}, \dots, \tilde{Y}_{a_n} = y_{a_n}] \prod_{a \in A_1 \setminus B_1} J[A](a, a) \\ & \leq \mathbb{P}[\tilde{Y}_{a_1} = x_{a_1} \vee y_{a_1}, \dots, \tilde{Y}_{a_n} = x_{a_n} \vee y_{a_n}] \\ & \times \mathbb{P}[Y_{a_1} = x_{a_1} \wedge y_{a_1}, \dots, Y_{a_n} = x_{a_n} \wedge y_{a_n}] \prod_{a \in A_1 \setminus B_1} J[A](a, a) \end{aligned}$$

In case $\prod_{a \in A_1 \setminus B_1} J[A](a, a) = 0$, then $\mathbb{P}[Y_{a_1} = x_{a_1}, \dots, Y_{a_n} = x_{a_n}] = 0$ and

$\mathbb{P}[\tilde{Y}_{a_1} = x_{a_1} \vee y_{a_1}, \dots, \tilde{Y}_{a_n} = x_{a_n} \vee y_{a_n}] = 0$, hence the following inequality is also true:

$$\begin{aligned} & \mathbb{P}[Y_{a_1} = x_{a_1}, \dots, Y_{a_n} = x_{a_n}] \mathbb{P}[\tilde{Y}_{a_1} = y_{a_1}, \dots, \tilde{Y}_{a_n} = y_{a_n}] \\ & \leq \mathbb{P}[\tilde{Y}_{a_1} = x_{a_1} \vee y_{a_1}, \dots, \tilde{Y}_{a_n} = x_{a_n} \vee y_{a_n}] \mathbb{P}[Y_{a_1} = x_{a_1} \wedge y_{a_1}, \dots, Y_{a_n} = x_{a_n} \wedge y_{a_n}] \end{aligned}$$

According to the Holey-Preston-Kemperman criterion, one obtains:

$$(Y_{a_1}, \dots, Y_{a_n}) \prec (\tilde{Y}_{a_1}, \dots, \tilde{Y}_{a_n})$$

which can be expressed as follows:

$$\text{DPP}(K_A) \prec \text{Bernoulli}\left(\frac{J[A](a, a)}{1 + J[A](a, a)}, a \in A\right) \quad (4.5)$$

Making use of the argument developed in [4] (Lemma 4.1 and Lemma A5) in the continuous framework, one shows that for every finite A : $J[A] \leq J_A$, where $J_A = (J(a, b), (a, b) \in A^2)$. Consequently: $J[A](a, a) \leq J(a, a)$, which leads with (4.5), to:

$$\text{DPP}(K) \cap A \prec \text{Bernoulli}\left(\frac{J(a, a)}{1 + J(a, a)}, a \in \mathcal{V}\right) \cap A$$

By letting A increase to \mathcal{V} , one obtains: $\text{DPP}(K) \prec \text{Bernoulli}\left(\frac{J(a, a)}{1 + J(a, a)}, a \in \mathcal{V}\right)$. \square

References

- [1] Diaconis F. and Evans S.N.: A different construction of Gaussian fields from Markov chains : Dirichlet covariances. *Ann. I. H. Poincaré - PR* 38, 6, 863-878 (2002).
- [2] Fukushima M, Oshima Y and Takeda M.: Dirichlet forms and symmetric Markov processes. *ed. De Gruyter, Second edition*(2011).
- [3] Garet O.: Percolation transition for some excursion sets. *Electron. Journal Probab.*, vol 9, 10 255-292 (2004).
- [4] Georgii H-O and Yoo H. J.: Conditional intensity and Gibbsianness of determinantal point processes. *Journal of Statistical Physics*, Vol. 118, Nos. 1/2, 55- 84 (2005).
- [5] Joag-Dev K. and Proschan F.: Negative association of random variables, with applications. *Annals of Stats.* - 11, 286-295 (1983).
- [6] Kesten H.: Asymptotics in high dimensions for percolation. *Disorder in physical systems (219-240)*, *Oxford Sci. Publ., Oxford Univ. Press, New York* (1990).
- [7] Last G., Szekli R. and Yogeshwaran D.: Some remarks on associated random fields, random measures and point processes. *arXiv:1903.06004* (2019).
- [8] Lebowitz J.L.: Griffiths inequalities for anti-ferromagnets. *Physics letters*, vol 38A, 2,99-100 (1971).
- [9] Liggett T.M., Schonmann R.H. and Stacey A.M.: Domination by product measures. *Annals of Probab.* 25, 1, 71-95 (1997).
- [10] Lyons R.: Determinantal probability measures. (English summary) *Publ. Math. Inst. Hautes tudes Sci.* No. 98, 167-212 (2003).
- [11] Lyons R.: Determinantal probability: basic properties and conjectures. *Proceedings of the International Congress of Mathematicians - Seoul 2014, Vol. IV*, 137-161, *Kyung Moon Sa, Seoul* (2014).
- [12] Lyons R. and Steif J.: Stationary determinantal processes: phase multiplicity, bernoullicity, entropy and domination. *Duke Math. J.* 120, no. 3, 515-575 (2003).
- [13] Molchanov S. A. and Stepanov A. K.: Percolation in random fields. I. *Teoret. Mat. Fiz.* 55 ,no. 2, 246-255 (1983).
- [14] Van Lieshout, M-C: Spatial point process theory. *Handbook of spatial statistics. Edited by Alan E. Gelfand, Peter J. Diggle, Montserrat Fuentes and Peter Guttorp. Chapman Hall/CRC Handbooks of Modern Statistical Methods. CRC Press, Boca Raton, FL*, 263-282 (2010).