

# ALGEBRAIC CYCLES AND FANO THREEFOLDS OF GENUS 8

ROBERT LATERVEER

ABSTRACT. We show that prime Fano threefolds  $Y$  of genus 8 have a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. As a consequence, a certain tautological subring of the Chow ring of powers of  $Y$  injects into cohomology.

## 1. INTRODUCTION

Given a smooth projective variety  $Y$  over  $\mathbb{C}$ , let  $A^i(Y) := CH^i(Y)_{\mathbb{Q}}$  denote the Chow groups of  $Y$  (i.e. the groups of codimension  $i$  algebraic cycles on  $Y$  with  $\mathbb{Q}$ -coefficients, modulo rational equivalence). The intersection product defines a ring structure on  $A^*(Y) = \bigoplus_i A^i(Y)$ , the Chow ring of  $Y$ . Motivated by the cases of K3 surfaces and abelian varieties, Beauville [1] has conjectured that for certain special varieties, the Chow ring should admit a multiplicative splitting. To make concrete sense of Beauville’s elusive “splitting property conjecture”, Shen–Vial [40] have introduced the concept of *multiplicative Chow–Künneth decomposition*. It seems both interesting and difficult to better understand the class of special varieties admitting such a decomposition (for some partial answers, cf. subsection 2.1 below).

This paper aims to contribute to this program. The main result is as follows:

**Theorem** (=Theorem 3.1). *Let  $Y$  be a Fano threefold that is prime (i.e., the Picard group is generated by the canonical divisor) of genus 8. Then  $Y$  has a multiplicative Chow–Künneth decomposition.*

The argument proving Theorem 3.1 relies on the classical fact that  $Y$  is birational to a cubic threefold (cf. subsection 2.3 below), plus the recently established fact that cubic threefolds have a multiplicative Chow–Künneth decomposition [7], [11].

We check that the multiplicative Chow–Künneth decomposition we construct behaves well with respect to the birationality with the cubic threefold (Proposition 4.1), and also with respect to birationalities among two genus 8 Fano threefolds (Proposition 4.2).

Using Theorem 3.1, we obtain a result concerning the *tautological ring*, which is a certain subring of the Chow ring of powers of  $Y$ :

**Corollary** (=Corollary 5.1). *Let  $Y$  be a prime Fano threefold of genus 8, and  $m \in \mathbb{N}$ . Let*

$$R^*(Y^m) := \langle (p_i)^*(h), (p_{ij})^*(\Delta_Y) \rangle \subset A^*(Y^m)$$

---

*Key words and phrases.* Algebraic cycles, Chow group, motive, Bloch–Beilinson filtration, Beauville’s “splitting property” conjecture, multiplicative Chow–Künneth decomposition, Fano threefolds, tautological ring.

2020 *Mathematics Subject Classification*: 14C15, 14C25, 14C30.

Supported by ANR grant ANR-20-CE40-0023.

be the  $\mathbb{Q}$ -subalgebra generated by pullbacks of the polarization  $h \in A^1(Y)$  and pullbacks of the diagonal  $\Delta_Y \in A^3(Y \times Y)$ . The cycle class map induces injections

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \text{ for all } m \in \mathbb{N}.$$

Corollary 5.1 can be loosely paraphrased by saying that prime genus 8 Fano threefolds behave like hyperelliptic curves from the point of view of intersection theory (cf. Remark 5.2 below).

To close this introduction, here is an open question:

**Question 1.1.** *Let  $Y$  be a Fano threefold with  $\text{Pic}(Y) = \mathbb{Z}$ . Does  $Y$  have a multiplicative Chow–Künneth decomposition? (And is Corollary 5.1 true for  $Y$ ?)*

(Without the hypothesis on  $\text{Pic}(Y)$ , the answer can be negative, as suggested by [1, Example 1.7].) In the final section 6, we record another question on a related note.

**Conventions.** *In this article, the word variety will refer to a reduced irreducible scheme of finite type over  $\mathbb{C}$ . A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.*

**All Chow groups will be with rational coefficients:** *we will denote by  $A_j(Y)$  the Chow group of  $j$ -dimensional cycles on  $Y$  with  $\mathbb{Q}$ -coefficients; for  $Y$  smooth of dimension  $n$  the notations  $A_j(Y)$  and  $A^{n-j}(Y)$  are used interchangeably. The notation  $A_{\text{hom}}^i(Y)$  will be used to indicate the subgroup of homologically trivial cycles. For a morphism  $f: X \rightarrow Y$ , we will write  $\Gamma_f \in A_*(X \times Y)$  for the graph of  $f$ .*

*The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [39], [34]) will be denoted  $\mathcal{M}_{\text{rat}}$ .*

## 2. PRELIMINARIES

### 2.1. MCK decomposition.

**Definition 2.1** (Murre [33]). *Let  $X$  be a smooth projective variety of dimension  $n$ . We say that  $X$  has a CK decomposition if there exists a decomposition of the diagonal*

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \text{ in } A^n(X \times X),$$

*such that the  $\pi_X^i$  are mutually orthogonal idempotents and  $(\pi_X^i)_* H^*(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$ .*

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

**Remark 2.2.** *The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [33], [19].*

**Definition 2.3** (Shen–Vial [40]). *Let  $X$  be a smooth projective variety of dimension  $n$ . Let  $\Delta_X^{sm} \in A^{2n}(X \times X \times X)$  be the class of the small diagonal*

$$\Delta_X^{sm} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$

*An MCK decomposition is a CK decomposition  $\{\pi_X^i\}$  of  $X$  that is multiplicative, i.e. it satisfies*

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) = 0 \text{ in } A^{2n}(X \times X \times X) \text{ for all } i + j \neq k.$$

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

**Remark 2.4.** *The small diagonal (seen as a correspondence from  $X \times X$  to  $X$ ) induces the multiplication morphism*

$$\Delta_X^{sm}: h(X) \otimes h(X) \rightarrow h(X) \text{ in } \mathcal{M}_{\text{rat}} .$$

*Suppose  $X$  has a CK decomposition*

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \text{ in } \mathcal{M}_{\text{rat}} .$$

*By definition, this decomposition is multiplicative if for any  $i, j$  the composition*

$$h^i(X) \otimes h^j(X) \rightarrow h(X) \otimes h(X) \xrightarrow{\Delta_X^{sm}} h(X) \text{ in } \mathcal{M}_{\text{rat}}$$

*factors through  $h^{i+j}(X)$ .*

*If  $X$  has an MCK decomposition, then setting*

$$A_{(j)}^i(X) := (\pi_X^{2i-j})_* A^i(X) ,$$

*one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends  $A_{(j)}^i(X) \otimes A_{(j')}^{i'}(X)$  to  $A_{(j+j')}^{i+i'}(X)$ .*

*It is expected that for any  $X$  with an MCK decomposition, one has*

$$A_{(j)}^i(X) \stackrel{??}{=} 0 \text{ for } j < 0 , \quad A_{(0)}^i(X) \cap A_{\text{hom}}^i(X) \stackrel{??}{=} 0 ;$$

*this is related to Murre's conjectures B and D, that have been formulated for any CK decomposition [33].*

*The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville's "splitting property" conjecture" [1]. To give an idea: hyperelliptic curves have an MCK decomposition [40, Example 8.16], but the very general curve of genus  $\geq 3$  does not have an MCK decomposition [11, Example 2.3]. For more detailed discussion, and examples of varieties with an MCK decomposition, we refer to [40, Section 8], as well as [48], [41], [12], [25], [26], [27], [28], [29], [30], [31], [11], [35].*

**2.2. Prime Fano threefolds of genus 8.** The classification of Fano threefolds is one of the glories of 20th century algebraic geometry [17]. Fano threefolds that are *prime* (i.e. with Picard group of rank 1 generated by the canonical divisor) come in 10 explicitly described families. In this paper we will be concerned with one of these families:

**Theorem 2.5** (Gushel [14], Mukai [32]). *Let  $Y$  be a smooth projective Fano threefold with  $\text{Pic}(Y) = \mathbb{Z}$ , of index 1 and genus 8. Then  $Y$  is a dimensionally transverse intersection of the Grassmannian  $\text{Gr}(2, 6) \subset \mathbb{P}^{14}$  with a codimension 5 linear subspace  $\mathbb{P}^9 \subset \mathbb{P}^{14}$ .*

*Conversely, any smooth dimensionally transverse intersection  $\text{Gr}(2, 6) \cap \mathbb{P}^9 \subset \mathbb{P}^{14}$  is a prime Fano threefold of genus 8.*

*Proof.* The last statement is just because  $\text{Gr}(2, 6)$  is a Fano variety of dimension 8 with Picard number 1, index 6 and degree 14; the codimension 5 intersection  $Y := \text{Gr}(2, 6) \cap \mathbb{P}^9$  thus has index 1 and degree  $d = 14$  (i.e. genus  $g = d/2 + 1 = 8$ ). The first statement is proven in [14] and independently in [32].  $\square$

**2.3. A birationality.** Any prime Fano threefold of genus 8 is birational to a cubic threefold; this was already known to Fano [8], and has been used in many works [15], [16], [38], [21], [23], [24]. The birationality is quite explicit:

**Theorem 2.6** (Puts [38], Iskovskikh [16]). *Let  $Y$  be a prime Fano threefold of genus 8. There exists a smooth cubic threefold  $X$  and a birational map*

$$\phi: Y \dashrightarrow X.$$

*More precisely, there exist a threefold  $Z$  and morphisms*

$$\begin{array}{ccc} & Z & \\ q \swarrow & & \searrow p \\ Y & & X, \end{array}$$

*with the following properties:*

- (i)  $p$  is the composition of the blow-up of a quintic elliptic curve  $E_X \subset X$ , and the blow-ups of (strict transforms of) lines  $L_X^1, \dots, L_X^{25} \subset X$ ; the lines  $L_X^j$  are pairwise disjoint and  $E_X \cap L_X^j$  consists of two points for each  $j$ ;
- (ii)  $q$  is the composition of the blow-up of a quintic elliptic curve  $E_Y \subset Y$ , and the blow-ups of (strict transforms of) lines  $L_Y^1, \dots, L_Y^{25} \subset Y$ ; the lines  $L_Y^j$  are pairwise disjoint and  $E_Y \cap L_Y^j$  consists of one point for each  $j$ ;
- (iii)  $p^{-1}(E_X) = q^{-1}(E_Y)$  and  $p^{-1}(L_X^j) = q^{-1}(L_Y^j)$ ,  $j = 1, \dots, 25$ .

*Proof.* In a nutshell, the argument goes as follows (this can be found in [38], cf. also [16, Chapter III Section 1] and [15, section 3]). Let  $V$  be a 6-dimensional vector space, and let us write  $G := \text{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$  for the Grassmannian and its Plücker embedding. The projective dual to  $G$  is the Pfaffian hypersurface  $X_{13} \subset \mathbb{P}(\wedge^2 V^*)$  (which is a cubic hypersurface of dimension 13, singular along  $\text{Gr}(2, V^*) \subset X_{13}$ ). Given a smooth genus 8 Fano threefold  $Y = G \cap h_1 \cap \dots \cap h_5$ , the intersection

$$X := X_{13} \cap \langle h_1^\vee, \dots, h_5^\vee \rangle \subset \mathbb{P}(\wedge^2 V^*)$$

is a smooth cubic threefold, called the *orthogonal cubic threefold*. A point  $x \in X$  represents a skew-symmetric form of rank 4 on  $V$ , whose kernel determines a line  $n_x \subset \mathbb{P}^5$ .

The *Palatini quartic* associated to  $Y$  is defined as

$$W_4 := \bigcup_{y \in Y} \ell_y \subset \mathbb{P}^5,$$

where  $\ell_y \subset \mathbb{P}(V) \cong \mathbb{P}^5$  is the line determined by  $y \in Y$ . This  $W_4$  is a quartic hypersurface (singular along a curve), and it turns out that this coincides with

$$W_4 = \bigcup_{x \in X} n_x \subset \mathbb{P}^5,$$

where  $X$  is the orthogonal cubic threefold. Choosing a general hyperplane  $H \subset \mathbb{P}^5$  and setting  $W := W_4 \cap H$ , one can define a map  $u_Y: Y \rightarrow W$  by sending  $y \in Y$  to the intersection point  $\ell_y \cap H$ . Likewise, one can define a map  $u_X: X \rightarrow W$  by sending  $x \in X$  to the intersection point

$n_x \cap H$ . One can check (cf. [38]) that the maps  $u_X$  and  $u_Y$  are birational, and so one obtains a birationality

$$Y \xrightarrow{u_Y} W \xleftarrow{u_X} X .$$

It remains to analyze the elimination of indeterminacy of this birational map. The indeterminacy locus of  $u_Y$  (resp.  $u_X$ ) is a quintic elliptic curve  $E_Y$  (resp.  $E_X$ ). For  $H$  sufficiently general, the singular locus of  $W$  consists of 25 singular points  $w_1, \dots, w_{25}$  (and these points correspond to the  $L_Y^j$  via  $u_Y$ , and to the  $L_X^j$  via  $u_X$ ). Let  $Z \rightarrow W$  be the blow-up of  $w_1, \dots, w_{25}$ . One can show [38, Section 6] that there are morphisms from  $Z$  to the blow-up of  $Y$  in  $E_Y$ , and to the blow-up of  $X$  in  $E_X$ ; this gives the statements (i), (ii), (iii).  $\square$

#### 2.4. An isomorphism of motives.

**Proposition 2.7.** *Let  $Y$  be a prime Fano threefold of genus 8, and let  $X$  be the orthogonal cubic threefold. There is an isomorphism*

$$h(Y) \xrightarrow{\cong} h(X) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

*Proof.* Let  $h \in A^1(Y)$  denote a hyperplane section (with respect to the embedding  $Y \subset \mathbb{P}^{14}$  given by Theorem 2.5). We define

$$\begin{aligned} \pi_Y^0 &:= \frac{1}{14} h^3 \times Y , \\ \pi_Y^2 &:= \frac{1}{14} h^2 \times h , \\ \pi_Y^4 &:= \frac{1}{14} h \times h^2 , \\ \pi_Y^6 &:= \frac{1}{14} Y \times h^3 , \\ \pi_Y^3 &:= \Delta_Y - \sum_{j \neq 3} \pi_Y^j \in A^3(Y \times Y) , \end{aligned}$$

and  $h^j(Y) := (Y, \pi_Y^j, 0) \in \mathcal{M}_{\text{rat}}$ . Using a hyperplane section  $h \in A^1(X)$ , we similarly define a decomposition  $h(X) = \bigoplus_j h^j(X)$ .

Let  $\iota: D \hookrightarrow Z$  be the inclusion morphism of the exceptional divisor  $D := q^{-1}(E_Y) = p^{-1}(E_X)$  in  $Z$  (where the curves  $E_Y, E_X$  are as in Theorem 2.6). The blow-up exact sequence gives equalities

$$(1) \quad \begin{aligned} \text{Im} \left( H^3(X, \mathbb{Q}) \xrightarrow{p^*} H^3(Z, \mathbb{Q}) \right) &= \ker \left( H^3(Z, \mathbb{Q}) \rightarrow H^3(D, \mathbb{Q}) \right) \\ &= \text{Im} \left( H^3(Y, \mathbb{Q}) \xrightarrow{q^*} H^3(Z, \mathbb{Q}) \right) , \end{aligned}$$

and so in particular there is an isomorphism

$$p_* q^*: H^3(Y, \mathbb{Q}) \xrightarrow{\cong} H^3(X, \mathbb{Q}) .$$

Using Manin's identity principle, this implies there is an isomorphism of homological motives

$$\Gamma_p \circ {}^t \Gamma_q: h^3(Y) \xrightarrow{\cong} h^3(X) \quad \text{in } \mathcal{M}_{\text{hom}} .$$

It is known (cf. [45, Theorem 4]) that Fano threefolds have finite-dimensional motives (in the sense of Kimura [20]), and so  $h^3(X)$  and  $h^3(Y)$  are finite-dimensional. It follows that there is also an isomorphism of Chow motives

$$\Gamma_p \circ {}^t\Gamma_q: h^3(Y) \xrightarrow{\cong} h^3(X) \text{ in } \mathcal{M}_{\text{rat}}.$$

For  $j = 0, \dots, 3$ , we have  $h^{2j}(Y) \cong h^{2j}(X) \cong \mathbb{1}(-j)$ . Combined with the above isomorphism  $h^3(Y) \cong h^3(X)$ , this proves the proposition.  $\square$

## 2.5. Franchetta property.

**Definition 2.8.** Let  $\mathcal{X} \rightarrow B$  be a smooth projective morphism, where  $\mathcal{X}, B$  are smooth quasi-projective varieties. We say that  $\mathcal{X} \rightarrow B$  has the Franchetta property in codimension  $j$  if the following holds: for every  $\Gamma \in A^j(\mathcal{X})$  such that the restriction  $\Gamma|_{X_b}$  is homologically trivial for the very general  $b \in B$ , the restriction  $\Gamma|_b$  is zero in  $A^j(X_b)$  for all  $b \in B$ .

We say that  $\mathcal{X} \rightarrow B$  has the Franchetta property if  $\mathcal{X} \rightarrow B$  has the Franchetta property in codimension  $j$  for all  $j$ .

This property is studied in [36], [3], [9], [10].

## 2.6. Franchetta property for $Y \times Y$ .

**Notation 2.9.** Let  $G$  be the Grassmannian  $G := \text{Gr}(2, 6)$ , and let  $\mathcal{O}_G(1)$  be the polarization corresponding to the Plücker embedding  $G \subset \mathbb{P}^{14}$ . Let

$$B \subset \bar{B} := \mathbb{P}H^0(G, \mathcal{O}_G(1)^{\oplus 5})$$

denote the Zariski open parametrizing smooth dimensionally transverse complete intersections, and let

$$\mathcal{Y} \rightarrow B$$

denote the universal family of smooth 3-dimensional complete intersections (in view of Theorem 2.5, this is the universal family of prime Fano threefolds of genus 8).

**Proposition 2.10.** Let  $\mathcal{Y} \rightarrow B$  be the universal family of prime Fano threefolds of genus 8. The families  $\mathcal{Y} \rightarrow B$  and  $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$  have the Franchetta property.

*Proof.* We exploit the relation with cubic threefolds (Theorem 2.6 and Proposition 2.7). For a given  $Y_b$ , let  $X_b$  denote the orthogonal cubic threefold, let  $Z_b$  denote the resolution of indeterminacy with morphisms

$$(2) \quad \begin{array}{ccc} & Z_b & \\ p \swarrow & & \searrow q \\ Y_b & & X_b. \end{array}$$

We observe that the construction of Theorem 2.6 makes sense familywise. That is, let us write  $\mathcal{X} \rightarrow B$  for the universal family of smooth dimensionally transverse cubic threefolds  $X_b$  obtained as intersections  $X_b = X_{13} \cap \mathbb{P}_b^4$  (here a point  $b \in B$  determines a 4-dimensional linear subspace  $\mathbb{P}_b^4 \subset \mathbb{P}(\wedge^2 V^*)$  which is the orthogonal to  $\mathbb{P}_b^9 \subset \mathbb{P}(\wedge^2 V)$ ). Up to shrinking  $B$ , we may

assume the  $X_b$  avoid the singular locus of  $X_{13}$ , and thus  $\mathcal{X}$  is a smooth quasi-projective variety (it is an open in a projective bundle over  $X_{13} \setminus \text{Sing}(X_{13})$ ).

Fixing the choice of a hyperplane  $H \subset V$ , one has a birational map  $\mathcal{Y} \dashrightarrow \mathcal{X}$ , and doing the required blow-ups, one obtains a diagram of  $B$ -schemes

$$\begin{array}{ccc} & \mathcal{Z} & \\ p \swarrow & & \searrow q \\ \mathcal{Y} & & \mathcal{X}, \end{array}$$

such that the base change along  $b \hookrightarrow B$  gives back the diagram (2). In particular, the isomorphism  $h(Y) \cong h(X)$  of Proposition 2.7 is generically defined, and so the induced isomorphisms of Chow groups

$$A^*(Y_b) \xrightarrow{\cong} A^*(X_b), \quad A^*(Y_b \times Y_b) \xrightarrow{\cong} A^*(X_b \times X_b)$$

send generically defined cycles to generically defined cycles. Consequently, Proposition 2.10 reduces to proving the Franchetta property for the families  $\mathcal{X} \rightarrow B$  and  $\mathcal{X} \times_B \mathcal{X}$ . This is the content of the next proposition:

**Proposition 2.11.** *Let  $\mathcal{X} \rightarrow B$  be the universal family of orthogonal cubic threefolds, as above. (i) Let  $\Gamma \in A^*(\mathcal{X})$  be such that the restriction  $\Gamma|_{X_b}$  is homologically trivial, for the very general fiber  $X_b$ . Then*

$$\Gamma|_{X_b} = 0 \quad \text{in } A^*(X_b) \quad \text{for all } b \in B.$$

(ii) Let  $\Gamma \in A_*(\mathcal{X} \times_B \mathcal{X})$ <sup>1</sup> such that the restriction  $\Gamma|_{X_b \times X_b}$  is homologically trivial, for the very general fiber  $X_b$ . Then

$$\Gamma|_{X_b} = 0 \quad \text{in } A^*(X_b \times X_b) \quad \text{for all } b \in B.$$

*Proof.* (NB: note that this is not the standard presentation of cubic threefolds in  $\mathbb{P}^4$ , and so this does not follow directly from the Franchetta property for the universal cubic in  $\mathbb{P}^4$  (and its square) !)

(i) Let us assume  $j = 2$  (since  $A^1(X_b) = A^3(X_b) = \mathbb{Q}$ , the statement is non-trivial only for  $j = 2$ ).

Let  $\bar{\mathcal{X}} \rightarrow \bar{B}$  denote the Zariski closure. This is the universal family of all (possibly singular and degenerate) intersections of the Pfaffian  $X_{13} \subset \mathbb{P}^{14}$  with a  $\mathbb{P}^4$ , and so  $\bar{B} \cong \mathbb{P}^r$ . Since every point in  $X_{13}$  imposes one condition on hyperplanes, the projection  $\pi: \bar{\mathcal{X}} \rightarrow X_{13}$  is a  $\mathbb{P}^{r-10}$ -bundle.

Now given  $\Gamma \in A_{\dim \mathcal{X}-2}(\mathcal{X})$ , let  $\bar{\Gamma} \in A_{\dim \bar{\mathcal{X}}-2}(\bar{\mathcal{X}})$  be a cycle restricting to  $\Gamma$ . The projective bundle formula yields

$$\bar{\Gamma} = \pi^*(a_2) + \pi^*(a_1) \cdot \xi \quad \text{in } A_{\dim \bar{\mathcal{X}}-2}(\bar{\mathcal{X}}),$$

<sup>1</sup>Here we write  $A_*(\mathcal{X} \times_B \mathcal{X})$  rather than  $A^*(\mathcal{X} \times_B \mathcal{X})$  because  $\mathcal{X} \times_B \mathcal{X}$  may not be smooth. The Gysin homomorphism  $A_*(\mathcal{X} \times_B \mathcal{X}) \rightarrow A^*(X_b \times X_b)$  is still well-defined, because  $B$  is smooth.

where  $\xi \in A^1(\bar{\mathcal{X}})$  is relatively ample with respect to  $\pi$  and  $a_j \in A_{13-j}(X_{13})$ . Let  $h \in A^1(\bar{B})$  be a hyperplane section and let  $\nu: \mathcal{X} \rightarrow \bar{B}$  denote the projection. We have

$$\nu^*(h) = c\xi + \pi^*(z) \quad \text{in } \text{Pic}(\bar{\mathcal{X}}),$$

for some  $c \in \mathbb{Q}$  and  $z \in A_{12}(X_{13})$ . It is readily checked that the constant  $c$  is non-zero. (Indeed, assume for a moment  $c$  were zero. Then we would have  $\nu^*(h^r) = \pi^*(z^r)$  in  $H^{2r}(\bar{\mathcal{X}})$ . But the right-hand side is zero, since  $r := \dim \bar{B} > \dim X_{13} = 13$ , while the left-hand side is non-zero.)

The constant  $c$  being non-zero, we can write

$$\xi = \pi^*(z) + \nu^*(d) \quad \text{in } A_{\dim \mathcal{X}-1}(\bar{\mathcal{X}}),$$

where  $z \in A_{12}(X_{13})$  and  $d \in A^1(\bar{B})$  are non-zero elements. The restriction of  $\nu^*(d)$  to a fiber  $X_b$  is zero, and so we find that

$$\Gamma|_{X_b} = a'_2|_{X_b} \quad \text{in } A^2(X_b),$$

for some  $a'_2 \in A_{11}(X_{13})$ . (Note that the restriction map  $A_{11}(X_{13}) \rightarrow A^2(X_b)$  is well-defined, since  $X_b \rightarrow X_{13}$  is the base change of  $\mathbb{P}^4 \rightarrow \mathbb{P}^{14}$  which is lci.) The proposition now follows from Lemma 2.12 below.  $\square$

**Lemma 2.12.**  $A_{11}(X_{13}) \cong \mathbb{Q}$ .

To prove the lemma, we use the resolution of singularities constructed in [22, Section 3]. Let  $\tilde{X}_{13}$  be the variety parametrizing pairs  $(\omega, K)$ , where  $K \subset V$  is a 2-dimensional subspace and  $\omega$  is a skew-form containing  $K$  in the kernel. The variety  $\tilde{X}_{13}$  is smooth (it is a projective bundle over a Grassmannian). The forgetful morphism  $f: \tilde{X}_{13} \rightarrow X_{13}$  induces a diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & \tilde{X}_{13} \\ \downarrow f|_E & & \downarrow f \\ S := \text{Sing}(X_{13}) & \hookrightarrow & X_{13}, \end{array}$$

where  $f$  is an isomorphism over the non-singular locus  $X_{13}^\circ := X_{13} \setminus \text{Sing}(X_{13})$ , and  $f|_E: E \rightarrow S \cong \text{Gr}(2, 6)$  is a  $\text{Gr}(2, 4)$ -fibration.

The variety  $\tilde{X}_{13}$ , being a projective bundle over a Grassmannian, has trivial Chow groups and so in particular the cycle class map induces an isomorphism

$$(3) \quad A^2(\tilde{X}_{13}) \xrightarrow{\cong} H^4(\tilde{X}_{13}, \mathbb{Q}).$$

The exceptional divisor  $E$  has  $A_0(E) \cong \mathbb{Q}$  (this follows from the fibration result in [46, Theorem 3.2], in view of the fact that both  $S \cong \text{Gr}(2, 6)$  and the fibers  $\text{Gr}(2, 4)$  of  $f|_E$  have trivial Chow group of 0-cycles). The Bloch–Srinivas argument [5] (plus the fact that  $h^{pq}(E) = 0$  for  $p \neq q$ ) then implies that the cycle class maps induce isomorphisms

$$(4) \quad A^2(E) \xrightarrow{\cong} H^4(E, \mathbb{Q}).$$



The isomorphisms (3) and (4) fit into a commutative diagram with exact rows

$$(5) \quad \begin{array}{ccccccc} A_{11}(E) & \xrightarrow{i_*} & A_{11}(\tilde{X}_{13}) & \rightarrow & A_{11}(X_{13}) & \rightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \end{array}$$

$$H_{22}(E, \mathbb{Q}) \xrightarrow{i_*} H_{22}(\tilde{X}_{13}, \mathbb{Q}) \rightarrow W_{-22}H_{22}(X_{13}, \mathbb{Q}) \rightarrow 0$$

(Here vertical arrows are cycle class maps, and  $W_*$  refers to Deligne's weight filtration on homology [37].) It follows that the right vertical arrow is also an isomorphism, and so one is reduced to proving that  $W_{-22}H_{22}(X_{13}, \mathbb{Q})$  is one-dimensional.

The exact sequence

$$0 \rightarrow H^4(X_{13}, \mathbb{Q}) \rightarrow H^4(\tilde{X}_{13}, \mathbb{Q}) \oplus H^4(S, \mathbb{Q}) \rightarrow H^4(E, \mathbb{Q}) \rightarrow H^5(X_{13}, \mathbb{Q})$$

$$\begin{array}{ccc} \parallel & & \parallel \\ \mathbb{Q} & & 0 \end{array}$$

(where the vertical equalities follow from weak Lefschetz) yields that

$$H_{22}(\tilde{X}_{13}, \mathbb{Q}) \cong H^4(\tilde{X}_{13}, \mathbb{Q}) \cong H^4(X_{13}, \mathbb{Q}) \oplus H^2(S, \mathbb{Q}) \cong \mathbb{Q}^2.$$

Since  $f|_E: E \rightarrow S \cong \text{Gr}(2, 6)$  is a  $\text{Gr}(2, 4)$ -fibration, we find that

$$H_{22}(E, \mathbb{Q}) \cong H^2(E, \mathbb{Q}) \cong \mathbb{Q}^2.$$

The arrows labelled  $i_*$  in (5) cannot be identically zero (indeed, let  $h \in A^1(\tilde{X}_{13})$  be an ample divisor. Then  $i_*i^*(h) = h \cdot E$  is non-zero in  $H_{22}(\tilde{X}_{13}, \mathbb{Q})$ ), and so we find that  $A_{11}(X_{13})$  is one-dimensional, proving the lemma.

(ii) We consider again the Zariski closure  $B \subset \bar{B} \cong \mathbb{P}^r$ , and the induced family  $\bar{\mathcal{X}} \times_{\bar{B}} \bar{\mathcal{X}} \rightarrow \bar{B}$ . Since 2 distinct points in  $X_{13}$  impose 2 independent conditions on each hyperplane, the morphism

$$(\pi, \pi): \bar{\mathcal{X}} \times_{\bar{B}} \bar{\mathcal{X}} \rightarrow X_{13} \times X_{13}$$

has the structure of a  $\mathbb{P}^{r-20}$ -bundle over  $(X_{13} \times X_{13}) \setminus \Delta_{X_{13}}$ , and a  $\mathbb{P}^{r-10}$ -bundle over  $\Delta_{X_{13}}$ .

Using the projective bundle formula, this implies that for any  $b \in B$  one has equality

$$(6) \quad \begin{aligned} \text{Im}\left(A_*(\bar{\mathcal{X}} \times_{\bar{B}} \bar{\mathcal{X}} \rightarrow A_*(X_b \times X_b))\right) &= \text{Im}\left(A_*(X_{13} \times X_{13}) \rightarrow A_*(X_b \times X_b)\right) \\ &\quad + \Delta_* \text{Im}\left(A_*(X_{13}) \rightarrow A_*(X_b)\right), \end{aligned}$$

where  $\Delta: X_b \rightarrow X_b \times X_b$  denotes the diagonal embedding (the equality (6) is proven exactly as [9, Proposition 4.1]). As we have just seen (in the proof of (i) directly above), there is equality

$$\text{Im}\left(A_*(X_{13}) \rightarrow A_*(X_b)\right) = \langle h \rangle.$$

Moreover, the fact that  $X_{13}$  has a resolution of singularities  $\tilde{X}_{13}$  with trivial Chow groups implies that  $X_{13}$  has the Chow–Künneth property:

$$A_*(X_{13} \times X_{13}) = A_*(X_{13}) \otimes A_*(X_{13}).$$

(This is just because  $A_*(\tilde{X}_{13}) \rightarrow A_*(X_{13})$  is surjective, and  $\tilde{X}_{13}$ , being a smooth projective variety with trivial Chow groups, has the Chow–Künneth property.) In view of these two observations, equality (6) reduces to

$$\mathrm{Im}\left(A_*(\bar{\mathcal{X}} \times_{\bar{B}} \bar{\mathcal{X}}) \rightarrow A_*(X_b \times X_b)\right) = \langle h, \Delta_{X_b} \rangle .$$

It now remains to prove that for any smooth cubic threefold  $X_b$  the  $\mathbb{Q}$ -subalgebra

$$R^*(X_b \times X_b) := \langle h, \Delta_{X_b} \rangle \subset A^*(X_b \times X_b)$$

injects into cohomology via the cycle class map. In codimension 3 this is fine, because the diagonal  $\Delta_{X_b}$  is linearly independent from decomposable cycles  $A^*(X_b) \otimes A^*(X_b)$  in  $H^6(X_b \times X_b, \mathbb{Q})$  (indeed, if the diagonal were decomposable, all of  $H^*(X_b, \mathbb{Q})$  would be algebraic, which is absurd since  $h^{2,1}(X_b) \neq 0$ ). For  $j \neq 3$ , the equality

$$\Delta_{X_b} \cdot (p_j)^*(h) = \sum_{i=1}^4 \frac{1}{3} h^i \times h^{4-i} \quad \text{in } A^4(X_b \times X_b) \quad (j = 1, 2)$$

(which follows from the excess intersection formula, cf. [10, Equation (12)]) implies that

$$R^j(X_b \times X_b) \subset R^*(X_b) \otimes R^*(X_b) = \langle h \rangle \otimes \langle h \rangle \quad (j \neq 3),$$

and hence  $R^j(X_b \times X_b) \rightarrow H^{2j}(X_b \times X_b, \mathbb{Q})$  is injective also for  $j \neq 3$ . The proposition is proven.  $\square$

For later use, we also record the following, which is an easy consequence of the above.

**Proposition 2.13.** *Let  $\mathcal{Y} \rightarrow B$  and  $\mathcal{X} \rightarrow B$  be as above. The family  $\mathcal{Y} \times_B \mathcal{X} \rightarrow B$  has the Franchetta property.*

*Proof.* The (generically defined) isomorphism of Proposition 2.7 gives a commutative diagram

$$\begin{array}{ccc} A^*(\mathcal{Y} \times_B \mathcal{X}) & \rightarrow & A_*(\mathcal{X} \times_B \mathcal{X}) \\ \downarrow & & \downarrow \\ A^*(Y \times X) & \xrightarrow{\cong} & A^*(X \times X) . \end{array}$$

The statement thus reduces to the Franchetta property for  $\mathcal{X} \times_B \mathcal{X}$ , which is Proposition 2.11.  $\square$

## 2.7. Franchetta property for $Z \times Z$ .

**Notation 2.14.** *Let  $\mathcal{Y} \rightarrow B$  be as in Notation 2.9 the universal family of genus 8 prime Fano threefolds. Fixing a hyperplane  $H \subset V$ , one gets a family  $\mathcal{Z} \rightarrow B$  by blowing up  $\mathcal{Y}$  along certain closed  $B$ -subschemes (cf. proof of Proposition 2.10). (This family  $\mathcal{Z} \rightarrow B$  has the property that fiberwise one obtains the resolution of indeterminacy of Proposition 2.6.)*

**Proposition 2.15.** *Let  $\mathcal{Z} \rightarrow B$  be as in Notation 2.14. The family  $\mathcal{Z} \times_B \mathcal{Z} \rightarrow B$  has the Franchetta property in codimension  $\leq 3$ .*

*Proof.* The  $B$ -scheme  $\mathcal{Z}$  is obtained from the  $B$ -scheme  $\mathcal{Y}$  by blowing up the  $B$ -subscheme  $\mathcal{E} \rightarrow B$  (which is a family of elliptic curves), and then blowing up the strict transforms of the families of lines  $\mathcal{L}^i \rightarrow B$  ( $i = 1, \dots, 25$ ). Let us write  $Z := Z_B$  for a fiber of  $\mathcal{Z} \rightarrow B$  (and likewise  $E := E_b, L^i := L_b^i$  for a fiber of  $\mathcal{E} \rightarrow B$  resp.  $\mathcal{L} \rightarrow B$ ). The blow-up formula gives an isomorphism of motives (where arrows are the natural ones)

$$h(Z) \cong h(Y) \oplus h(E)(-1) \oplus \bigoplus_{i=1}^{25} \mathbb{1}(-1) \oplus \bigoplus_{i=1}^{25} \mathbb{1}(-2) \quad \text{in } \mathcal{M}_{\text{rat}}.$$

In particular, for Chow groups of the product we obtain isomorphisms (where arrows are the natural ones)

$$(7) \quad \begin{aligned} A^j(Z \times Z) \cong & A^j(Y \times Y) \oplus A^{j-1}(Y \times E) \oplus A^{j-1}(E \times Y) \oplus A^{j-2}(E \times E) \\ & \oplus \bigoplus A^*(Y) \oplus \bigoplus A^*(E) \oplus \bigoplus \mathbb{Q}. \end{aligned}$$

The isomorphism (7) fits into a commutative diagram

$$\begin{array}{ccccccc} A^j(\mathcal{Z} \times_B \mathcal{Z}) & \xrightarrow{\phi} & A^j(\mathcal{Y} \times_B \mathcal{Y}) & \oplus & A^{j-1}(\mathcal{Y} \times_B \mathcal{E}) & \oplus & A^{j-1}(\mathcal{E} \times_B \mathcal{Y}) & \oplus & A^{j-2}(\mathcal{E} \times_B \mathcal{E}) & \oplus & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ A^j(Z \times Z) & \xrightarrow{\cong} & A^j(Y \times Y) & \oplus & A^{j-1}(Y \times E) & \oplus & A^{j-1}(E \times Y) & \oplus & A^{j-2}(E \times E) & \oplus & \dots \end{array}$$

(Here vertical arrows are fiberwise restrictions, and the arrow  $\phi$  is defined because the isomorphism (7) is defined explicitly in terms of the various blow-ups linking  $Y$  to  $Z$ , and these blow-ups can be performed relatively.) The families  $\mathcal{Y} \rightarrow B$  and  $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$  have the Franchetta property (Proposition 2.10). Thus, in proving the proposition, we are reduced to proving the following two claims:

**Claim 2.16.** *The families  $\mathcal{E} \rightarrow B$  and  $\mathcal{E} \times_B \mathcal{E} \rightarrow B$  have the Franchetta property in codimension  $\leq 1$ .*

**Claim 2.17.** *The family  $\mathcal{Y} \times_B \mathcal{E} \rightarrow B$  has the Franchetta property in codimension  $\leq 2$ .*

It remains to prove these two claims. As for Claim 2.16, obviously it suffices to prove the Franchetta property for  $\mathcal{E} \times_B \mathcal{E} \rightarrow B$ . Recall that  $E := E_Y \subset Y$  is the indeterminacy locus of the map  $u_Y: Y \dashrightarrow W$  described in the proof of Theorem 2.6. We exploit the specific description of  $E$  given in [38, Section 6]: letting  $H \subset V \cong \mathbb{C}^6$  denote the fixed hyperplane defining  $W := W_4 \cap H$ , the elliptic curve  $E$  is given as

$$E = \text{Gr}(2, H) \cap H_1 \cap \dots \cap H_5 \subset \text{Gr}(2, V) \cap H_1 \cap \dots \cap H_5 =: Y.$$

This implies that  $\mathcal{E} \rightarrow B$  coincides with the universal complete intersection of  $\text{Gr}(2, H) \subset \mathbb{P}^{14}$  with a codimension 5 linear subspace. As in the proof of Proposition 2.10, the morphism

$$\bar{\mathcal{E}} \times_B \bar{\mathcal{E}} \rightarrow \text{Gr}(2, H) \times \text{Gr}(2, H)$$

has the structure of a  $\mathbb{P}^{r-10}$ -bundle over  $(\mathrm{Gr}(2, H) \times \mathrm{Gr}(2, H)) \setminus \Delta_{\mathrm{Gr}(2, H)}$ , and a  $\mathbb{P}^{r-5}$ -bundle over  $\Delta_{\mathrm{Gr}(2, H)}$ . Again as in the proof of Proposition 2.10, this implies

$$\mathrm{Im}\left(A^*(\bar{\mathcal{E}} \times_{\bar{B}} \bar{\mathcal{E}}) \rightarrow A^*(E \times E)\right) = \langle h, \Delta_E \rangle =: R^*(E \times E).$$

It is readily checked that  $R^j(E \times E)$  injects into cohomology for  $j \leq 1$  (it suffices to remark that  $\Delta_E \in H^2(E \times E, \mathbb{Q})$  is not decomposable). This proves Claim 2.16.

As for Claim 2.17, let  $g: \mathrm{Gr}(2, H) \hookrightarrow \mathrm{Gr}(2, V)$  denote the inclusion morphism, and let  $\Gamma_g$  denote its graph. The morphism

$$\bar{\mathcal{E}} \times_{\bar{B}} \bar{\mathcal{Y}} \rightarrow \mathrm{Gr}(2, H) \times \mathrm{Gr}(2, V)$$

has the structure of a  $\mathbb{P}^{r-10}$ -bundle over  $(\mathrm{Gr}(2, H) \times \mathrm{Gr}(2, V)) \setminus \Gamma_g$ , and a  $\mathbb{P}^{r-5}$ -bundle over  $\Gamma_g$ . This is what is called a *stratified projective bundle* in [9]. An application of [9, Proposition 5.2] then gives the inclusion

$$\begin{aligned} \mathrm{Im}\left(A^j(\bar{\mathcal{E}} \times_{\bar{B}} \bar{\mathcal{Y}}) \rightarrow A^j(E \times Y)\right) &\subset \mathrm{Im}\left(A^j(\mathrm{Gr}(2, H) \times \mathrm{Gr}(2, V)) \rightarrow A^j(E \times Y)\right) \\ &\quad + \mathrm{Im}\left(A^{j-3}(\Gamma_{g_b}) \rightarrow A^j(E \times Y)\right), \end{aligned}$$

where  $g_b: E \hookrightarrow Y$  is the inclusion morphism. For  $j \leq 2$ , the last summand obviously vanishes. Observing that

$$A^*(\mathrm{Gr}(2, H) \times \mathrm{Gr}(2, V)) = A^*(\mathrm{Gr}(2, H)) \otimes A^*(\mathrm{Gr}(2, V)),$$

and that the Franchetta property holds for  $\mathcal{E} \rightarrow B$  and for  $\mathcal{Y} \rightarrow B$ , this proves Claim 2.17.  $\square$

### 3. MAIN RESULT

**Theorem 3.1.** *Let  $Y$  be a prime Fano threefold of genus 8. Then  $Y$  has a multiplicative Chow–Künneth decomposition. The induced bigrading on the Chow ring is such that*

$$A_{(0)}^2(Y) = \mathbb{Q}[c_2(Y)].$$

*Proof.* Let us first construct a CK decomposition for  $Y$ . Letting  $h \in A^1(Y)$  denote a hyperplane section (with respect to the embedding  $Y \subset \mathbb{P}^{14}$  given by Theorem 2.5), we set

$$\begin{aligned} \pi_Y^0 &:= \frac{1}{14} h^3 \times Y, \\ \pi_Y^2 &:= \frac{1}{14} h^2 \times h, \\ \pi_Y^4 &:= \frac{1}{14} h \times h^2, \\ \pi_Y^6 &:= \frac{1}{14} Y \times h^3, \\ \pi_Y^3 &:= \Delta_Y - \sum_{j \neq 3} \pi_Y^j \in A^3(Y \times Y). \end{aligned}$$

We observe that this CK decomposition is *generically defined* with respect to the family  $\mathcal{Y} \rightarrow B$  (Notation 2.9), i.e. it is obtained by restriction from “universal projectors”  $\pi_{\mathcal{Y}}^j \in A^3(\mathcal{Y} \times_B \mathcal{Y})$ . (This is just because  $h$  and  $\Delta_Y$  are generically defined.)

Writing  $h^j(Y) := (Y, \pi_Y^j) \in \mathcal{M}_{\text{rat}}$ , we have

$$(8) \quad h^{2j}(Y) \cong \mathbf{1}(-j) \quad \text{in } \mathcal{M}_{\text{rat}} \quad (j = 0, \dots, 3).$$

What we need to prove is that this CK decomposition is MCK, i.e.

$$(9) \quad \pi_Y^k \circ \Delta_Y^{sm} \circ (\pi_Y^i \times \pi_Y^j) = 0 \quad \text{in } A^6(Y \times Y \times Y) \quad \text{for all } i + j \neq k,$$

or equivalently that

$$h^i(Y) \otimes h^j(Y) \xrightarrow{\Delta_Y^{sm}} h(Y)$$

coincides with

$$h^i(Y) \otimes h^j(Y) \xrightarrow{\Delta_Y^{sm}} h(Y) \rightarrow h^{i+j}(Y) \rightarrow h(Y),$$

for all  $i, j$ .

As a first step, let us assume that we have three integers  $(i, j, k)$  and at most one of them is equal to 3. The cycle in (9) is generically defined and homologically trivial. The isomorphisms (8) induce an injection

$$(\pi_Y^i \times \pi_Y^j \times \pi_Y^k)_* A^6(Y \times Y \times Y) \hookrightarrow A^*(Y),$$

and sends generically defined cycles to generically defined cycles (this is because the isomorphisms (8) are generically defined). As a consequence, the required vanishing (9) follows from the Franchetta property for  $Y$ , which is Proposition 2.10(i).

In the second step, let us assume that among the three integers  $(i, j, k)$ , exactly two are equal to 3. In this case, using the isomorphisms (8) we find an injection

$$(\pi_Y^i \times \pi_Y^j \times \pi_Y^k)_* A^6(Y \times Y \times Y) \hookrightarrow A^*(Y \times Y),$$

respecting the generically defined cycles. As such, the required vanishing (9) follows from the Franchetta property for  $Y \times Y$ , which is Proposition 2.10(ii).

(As an aside, we note that in this second step, for most combinations of  $(i, j, k)$  the vanishing (9) can also be proven by more elementary means; only for  $(i, j, k) = (3, 3, 4)$  do we need the full force of Proposition 2.10(ii).)

In the third and final step we treat the case  $i = j = k = 3$ . For this case, we exploit the birationality with the cubic threefold  $X$ , in the guise of the following lemma:

**Lemma 3.2.** *Let  $Y$  be a prime Fano threefold of genus 8, let  $X$  be its orthogonal cubic threefold, and let*

$$\begin{array}{ccc} & Z & \\ q \swarrow & & \searrow p \\ Y & & X \end{array}$$

be as in Theorem 2.6. There is equality

$${}^t\Gamma_p \circ \pi_X^3 \circ \Gamma_p = {}^t\Gamma_q \circ \pi_Y^3 \circ \Gamma_q \quad \text{in } A^3(Z \times Z)$$

(where  $\pi_X^3, \pi_Y^3$  are as above).

*Proof.* Both sides of the equality are generically defined (with respect to the family  $\mathcal{Z} \rightarrow B$  of Proposition 2.15). The equality holds true in cohomology, because we have seen in equality (1) that

$$\mathrm{Im}\left(H^3(X, \mathbb{Q}) \xrightarrow{p^*} H^3(Z, \mathbb{Q})\right) = \mathrm{Im}\left(H^3(Y, \mathbb{Q}) \xrightarrow{q^*} H^3(Z, \mathbb{Q})\right).$$

The Franchetta property in codimension 3 for  $Z \times Z$  (Proposition 2.15) then allows to conclude that the equality of the lemma holds true modulo rational equivalence.  $\square$

Armed with Lemma 3.2, one can do the final step of the proof. The generically defined CK decomposition  $\{\pi_X^j\}$  (cf. the proof of Proposition 2.7) is an MCK decomposition (cf. [11], or independently [7]). For brevity, let us write  $m := {}^t\Gamma_p \circ \pi_X^3 \circ \Gamma_p \in A^3(Z \times Z)$  for the correspondence of Lemma 3.2. We note that  ${}^t\pi_X^3 = \pi_X^3$  and so  ${}^tm = m$ . We have equalities

$$\begin{aligned} m \circ \Delta_Z^{sm} \circ (m \times m) &= (m \times m \times m)_*(\Delta_Z^{sm}) \\ &= \left( ({}^t\Gamma_p \circ \pi_X^3 \circ \Gamma_p)^{\times 3} \right)_*(\Delta_Z^{sm}) \\ &= (p, p, p)^*(\pi_X^3 \times \pi_X^3 \times \pi_X^3)_*(p, p, p)_*(\Delta_Z^{sm}) \\ &= (p, p, p)^*(\pi_X^3 \times \pi_X^3 \times \pi_X^3)_*(\Delta_X^{sm}) \\ &= (p, p, p)^*\left(\pi_X^3 \circ \Delta_X^{sm} \circ (\pi_X^3 \times \pi_X^3)\right) \\ &= 0 \quad \text{in } A^6(Z \times Z \times Z). \end{aligned}$$

(Here the first and fifth equality are Lieberman's lemma [47, Lemma 3.3]; the second equality is by definition of  $m$ ; the third follows from general properties of correspondences; the fourth is the fact that  $(p, p, p)_*(\Delta_Z^{sm}) = \Delta_X^{sm}$ , and the last equality expresses the fact that  $\{\pi_X^j\}$  is an MCK decomposition.)

On the other hand, we have equalities

$$\begin{aligned} \pi_Y^3 \circ \Delta_Y^{sm} \circ (\pi_Y^3 \times \pi_Y^3) &= (\pi_Y^3 \times \pi_Y^3 \times \pi_Y^3)_*(\Delta_Y^{sm}) \\ &= (q, q, q)_*(q, q, q)^*\left((\pi_Y^3 \times \pi_Y^3 \times \pi_Y^3)_*(\Delta_Y^{sm})\right) \\ &= (q, q, q)_*(q, q, q)^*(\pi_Y^3 \times \pi_Y^3 \times \pi_Y^3)_*(q, q, q)_*(\Delta_Z^{sm}) \\ &= (q, q, q)_*(m, m, m)_*(\Delta_Z^{sm}) \\ &= 0 \quad \text{in } A^6(Y \times Y \times Y). \end{aligned}$$

This proves the required vanishing (9) for  $i = j = k = 3$ , and shows that the CK decomposition is MCK.

Finally, let us prove that  $A_{(0)}^2(Y) = \mathbb{Q}[c_2(Y)]$ . We remark that  $c_2(Y) \in A_{(0)}^2(Y)$  because

$$(\pi_Y^i)_*c_2(Y) = (\pi_Y^i)_*c_2(T_{Y/B})|_Y = 0 \quad \text{in } A^2(Y) \quad \text{for all } i \neq 4,$$

as follows from the Franchetta property for  $\mathcal{Y} \rightarrow B$  (Proposition 2.10). One readily checks that  $c_2(Y)$  is non-zero (e.g. one can take a smooth anticanonical section  $S \subset Y$ ; if  $c_2(Y)$  were zero then by adjunction also  $c_2(S) = 0$ , which is absurd since  $S$  is a K3 surface). Since  $A_{(0)}^2(Y)$  injects into  $H^4(Y, \mathbb{Q}) \cong \mathbb{Q}$ , it follows that  $A_{(0)}^2(Y) = \mathbb{Q}[c_2(Y)]$ .  $\square$

## 4. COMPATIBILITIES

The MCK decomposition for  $Y$  behaves well with respect to the birationality of Theorem 2.6:

**Proposition 4.1.** *Let  $Y$  be a prime Fano threefold of genus 8, let  $X$  be the orthogonal cubic threefold and let  $\phi: Y \dashrightarrow X$  be the birational map of Theorem 2.6. One has*

$$(10) \quad \bar{\Gamma}_\phi \in A_{(0)}^3(Y \times X),$$

where  $Y \times X$  is given the product MCK decomposition.

In particular, one has

$$\phi_*c_2(Y) \in \mathbb{Q}[c_2(X)] \subset A^2(X).$$

*Proof.* The point is that the map  $\phi$ , and hence also the cycle  $\bar{\Gamma}_\phi$ , is generically defined (with respect to the base  $B$ ). Since the product MCK decomposition is also generically defined, the result (10) follows from the Franchetta property for  $Y \times X$  (Proposition 2.13).

For the second part, we have just seen that  $A_{(0)}^2(Y) = \mathbb{Q}[c_2(Y)]$ , and one also has  $A_{(0)}^2(X) = \mathbb{Q}[c_2(X)]$  (this is just because  $c_2(X)$  is a sum of intersections of divisors, by adjunction). The result (10) implies that

$$\phi_*A_{(0)}^2(Y) \subset A_{(0)}^2(X)$$

(cf. [41, Proposition 1.5]), and so  $\phi_*c_2(Y) \in \mathbb{Q}[c_2(X)]$ .  $\square$

The MCK decomposition for  $Y$  also behaves well with respect to birational maps between genus 8 Fano threefolds:

**Proposition 4.2.** *Let  $Y_1, Y_2$  be two prime Fano threefolds of genus 8 such that there is a birational map  $\psi: Y_1 \dashrightarrow Y_2$ . There is an isomorphism of motives*

$$(11) \quad \bar{\Gamma}_\psi: h^3(Y_1) \xrightarrow{\cong} h^3(Y_2) \text{ in } \mathcal{M}_{\text{rat}},$$

and one has

$$(12) \quad \bar{\Gamma}_\psi \in A_{(0)}^3(Y_1 \times Y_2)$$

(where  $Y_1 \times Y_2$  is given the product MCK decomposition).

In particular,

$$\psi_*c_2(Y_1) \in \mathbb{Q}[c_2(Y_2)] \subset A^2(Y_2).$$

*Proof.* Let  $X_1, X_2$  denote the cubic threefold orthogonal to  $Y_1$  resp.  $Y_2$ , and let  $\phi_i: Y_i \dashrightarrow X_i$  ( $i = 1, 2$ ) denote the birational maps of Theorem 2.6. Thanks to the Torelli theorem for cubic threefolds [6], [44] (cf. also [21, Remark 2.19]),  $X_1$  and  $X_2$  are isomorphic, and there is a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{\phi_1} & X_1 \\ \downarrow \psi & & \downarrow \psi_X \\ Y_2 & \xrightarrow{\phi_2} & X_2 \end{array}$$

where  $\psi_X$  is an isomorphism. Since the maps  $\phi_i$  induce isomorphisms  $h^3(Y_i) \cong h^3(X_i)$  (proof of Proposition 2.7), this proves (11).

Using Manin's identity principle (applied to the diagram obtained from the above diagram by taking the cartesian product with a fixed variety), it follows that there is equality of cycle classes

$$\bar{\Gamma}_\psi = {}^t\bar{\Gamma}_{\phi_2} \circ \Gamma_{\psi_X} \circ \bar{\Gamma}_{\phi_1} \quad \text{in } A^3(Y_1 \times Y_2).$$

The  $\bar{\Gamma}_{\phi_i}$  are in  $A_{(0)}^3(Y_i \times X_i)$  (Proposition 4.1), and composition of correspondences preserves the subring  $A_{(0)}^*(\ )$  [41, Proposition 1.5]. To conclude the truth of equation (12), it only remains to establish the following:

**Lemma 4.3.**  $\Gamma_{\psi_X} \in A_{(0)}^3(X_1 \times X_2)$ .

To prove the lemma, we observe that the isomorphism  $\psi_X$  preserves (the canonical divisor and hence) the hyperplane class, which implies that

$$\pi_{X_2}^j = (\psi_X \times \psi_X)_* \pi_{X_1}^j = \Gamma_{\psi_X} \circ \pi_{X_1}^j \circ {}^t\Gamma_{\psi_X} \quad \text{in } A^3(X_2 \times X_2) \quad \forall j.$$

Consequently, we find that

$$\pi_{X_2}^k \circ \Gamma_{\psi_X} \circ \pi_{X_1}^j = 0 \quad \forall k \neq j.$$

Now writing

$$\begin{aligned} \Gamma_{\psi_X} &= \left( \sum_k \pi_{X_2}^k \right) \circ \Gamma_{\psi_X} \circ \left( \sum_j \pi_{X_1}^j \right) \\ &= \sum_j \pi_{X_2}^j \circ \Gamma_{\psi_X} \circ \pi_{X_1}^j \\ &= \sum_j (\pi_{X_1}^{6-j} \times \pi_{X_2}^j)_* (\Gamma_{\psi_X}) \quad \text{in } A^3(X_1 \times X_2), \end{aligned}$$

we find that  $\Gamma_{\psi_X} \in A_{(0)}^3(X_1 \times X_2)$  as requested. This closes the proof of the lemma and the proposition.  $\square$

**Remark 4.4.** Let  $Y_1, Y_2$  be 2 genus 8 Fano threefolds that are birational, as in Proposition 4.2. Using the relation with the orthogonal cubic threefold, Kuznetsov proves [21, Corollary 3.18] that there is an equivalence of categories

$$\text{Kuz}(Y_1) \cong \text{Kuz}(Y_2)$$

(where  $\text{Kuz}(Y_i)$  denotes the Kuznetsov component of the derived category of  $Y_i$ ). It follows from [4, Theorem 1.1] that the converse is also true: an equivalence of categories  $\text{Kuz}(Y_1) \cong \text{Kuz}(Y_2)$  implies that  $Y_1$  and  $Y_2$  are birational.

## 5. THE TAUTOLOGICAL RING

**Corollary 5.1.** Let  $Y$  be a prime Fano threefold of genus 8, and let  $m \in \mathbb{N}$ . Let

$$R^*(Y^m) := \langle (p_i)^*(h), (p_{ij})^*(\Delta_Y) \rangle \subset A^*(Y^m)$$



be the  $\mathbb{Q}$ -subalgebra generated by pullbacks of the polarization  $h \in A^1(Y)$  and pullbacks of the diagonal  $\Delta_Y \in A^3(Y \times Y)$ . (Here  $p_i$  and  $p_{ij}$  denote the various projections from  $Y^m$  to  $Y$  resp. to  $Y \times Y$ ). The cycle class map induces injections

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \text{ for all } m \in \mathbb{N}.$$

*Proof.* This is inspired by the analogous result for cubic hypersurfaces [10, Section 2.3], which in turn is inspired by analogous results for hyperelliptic curves [42], [43] (cf. Remark 5.2 below) and for K3 surfaces [49].

As in [10, Section 2.3], let us write  $o := \frac{1}{14}h^3 \in A^3(Y)$ , and

$$\tau := \Delta_Y - \frac{1}{14} \sum_{j=0}^3 h^j \times h^{3-j} \in A^3(Y \times Y)$$

(this cycle  $\tau$  is nothing but the projector on the motive  $h^3(Y)$  considered above). Moreover, let us write

$$\begin{aligned} o_i &:= (p_i)^*(o) \in A^3(Y^m), \\ h_i &:= (p_i)^*(h) \in A^1(Y^m), \\ \tau_{ij} &:= (p_{ij})^*(\tau) \in A^3(Y^m). \end{aligned}$$

We define the  $\mathbb{Q}$ -subalgebra

$$\bar{R}^*(Y^m) := \langle o_i, h_i, \tau_{ij} \rangle \subset H^*(Y^m, \mathbb{Q})$$

(where  $i$  ranges over  $1 \leq i \leq m$ , and  $1 \leq i < j \leq m$ ). One can prove (just as [10, Lemma 2.12] and [49, Lemma 2.3]) that the  $\mathbb{Q}$ -algebra  $\bar{R}^*(Y^m)$  is isomorphic to the free graded  $\mathbb{Q}$ -algebra generated by  $o_i, h_i, \tau_{ij}$ , modulo the following relations:

$$(13) \quad o_i \cdot o_i = 0, \quad h_i \cdot o_i = 0, \quad h_i^3 = 14 o_i;$$

$$(14) \quad \tau_{i,j} \cdot o_i = 0, \quad \tau_{i,j} \cdot h_i = 0, \quad \tau_{i,j} \cdot \tau_{i,j} = -10 o_i \cdot o_j;$$

$$(15) \quad \tau_{i,j} \cdot \tau_{i,k} = \tau_{j,k} \cdot o_i;$$

$$(16) \quad \sum_{\sigma \in \mathfrak{S}_{12}} \prod_{i=1}^6 \tau_{\sigma(2i-1), \sigma(2i)} = 0.$$

(NB: with reference to the notation of [10, Lemma 2.12], we have  $b_{pr} = \dim H^3(Y, \mathbb{Q}) = 10$ .)

To prove Corollary 5.1, it remains to check that all these relations are already verified modulo rational equivalence. The relations (13) take place in  $R^*(Y)$  and so they follow from the Franchetta property for  $Y$  (Proposition 2.10). The relations (14) take place in  $R^*(Y^2)$ . The first and the last relations are trivially verified, because ( $Y$  being Fano)  $A^6(Y^2) = \mathbb{Q}$ . As for the second relation of (14), this follows from the Franchetta property for  $Y \times Y$  (Proposition 2.10(ii)). (Alternatively, one can deduce the second relation from the MCK decomposition: the product  $\tau \cdot h_i$  lies in  $A_{(0)}^4(Y^2)$ , and it is readily checked that  $A_{(0)}^4(Y^2)$  injects into  $H^8(Y^2, \mathbb{Q})$ .)

Relation (15) takes place in  $R^*(Y^3)$  and follows from the MCK relation. Indeed, we have

$$\Delta_Y^{sm} \circ (\pi_Y^3 \times \pi_Y^3) = \pi_Y^6 \circ \Delta_Y^{sm} \circ (\pi_Y^3 \times \pi_Y^3) \quad \text{in } A^6(Y^3),$$

which (using Lieberman's lemma) translates into

$$(\pi_Y^3 \times \pi_Y^3 \times \Delta_Y)_* \Delta_Y^{sm} = (\pi_Y^3 \times \pi_Y^3 \times \pi_Y^6)_* \Delta_Y^{sm} \quad \text{in } A^6(Y^3),$$

which means that

$$\tau_{13} \cdot \tau_{23} = \tau_{12} \cdot o_3 \quad \text{in } A^6(Y^3).$$

Finally, relation (16), which takes place in  $R^*(Y^{12})$ , expresses the fact that

$$\text{Sym}^{12} \dim H^3(Y, \mathbb{Q}) = 0,$$

where  $H^3(Y, \mathbb{Q})$  is viewed as a super vector space. To check that this relation is also verified modulo rational equivalence (i.e. relation (16) holds true in  $A^{36}(Y^{12})$ ), we observe that relation (16) involves a cycle contained in

$$A^*(\text{Sym}^{12}(h^3(Y))).$$

But the Chow motive  $(\text{Sym}^{11} h^3(Y))$  and a fortiori  $\text{Sym}^{12} h^3(Y)$  is zero because  $\dim H^3(Y, \mathbb{Q}) = 10$  and  $h^3(Y)$  is oddly finite-dimensional (all Fano threefolds have finite-dimensional motive [45, Theorem 4]). This ends the proof.  $\square$

**Remark 5.2.** *Given a curve  $C$  and an integer  $m \in \mathbb{N}$ , one can define the tautological ring*

$$R^*(C^m) := \langle (p_i)^*(K_C), (p_{ij})^*(\Delta_C) \rangle \subset A^*(C^m)$$

(where  $p_i, p_{ij}$  denote the various projections from  $C^m$  to  $C$  resp.  $C \times C$ ). Tavakol has proven [43, Corollary 6.4] that if  $C$  is a hyperelliptic curve, the cycle class map induces injections

$$R^*(C^m) \hookrightarrow H^*(C^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

On the other hand, there are many (non hyperelliptic) curves for which the tautological ring  $R^*(C^3)$  does not inject into cohomology (this is related to the non-vanishing of the Ceresa cycle, cf. [43, Remark 4.2] and also [11, Example 2.3 and Remark 2.4]).

Corollary 5.1 shows that genus 8 Fano threefolds behave similarly to hyperelliptic curves. It would be interesting to understand what happens for other Fano threefolds: is Corollary 5.1 true for all of them or not? (This is related to Question 1.1.)

## 6. QUESTION

**Question 6.1.** *Let  $Y$  be a prime Fano threefold of genus 8, and let  $S \subset Y$  be a smooth anticanonical divisor. Then  $S$  is a K3 surface (and the general genus 8 K3 surface arises in this way). Is it true that*

$$\text{Im}(A^1(S) \rightarrow A^2(Y)) = \mathbb{Q}[h^2] \quad ??$$

(To prove this, it would suffice to prove that the graph of the inclusion morphism  $S \hookrightarrow Y$  is in  $A_{(0)}^3(S \times Y)$ , with respect to the product MCK decomposition. I have not been able to do this.)

This question also makes sense for cubic threefolds and their anticanonical divisors.

**Acknowledgements.** *Thanks to Charles Vial and the referee for helpful comments. This note is dedicated to Y.: you know why...☺*

## REFERENCES

- [1] A. Beauville, On the splitting of the Bloch–Beilinson filtration, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), London Math. Soc. Lecture Notes 344, Cambridge University Press 2007,
- [2] A. Beauville and C. Voisin, On the Chow ring of a K3 surface, *J. Alg. Geom.* 13 (2004), 417–426,
- [3] N. Bergeron and Z. Li, Tautological classes on moduli space of hyperkähler manifolds, *Duke Math. J.*, arXiv:1703.04733,
- [4] M. Bernardara, E. Macrì, S. Mehrotra and P. Stellari, A categorical invariant for cubic threefolds, *Adv. Math.* 229 no 2 (2012), 770–803,
- [5] S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles, *American Journal of Mathematics* Vol. 105, No 5 (1983), 1235–1253,
- [6] H. Clemens and P. Griffiths, The intermediate Jacobian of the cubic threefold, *Annals of Math.* 95 (1972), 281–356,
- [7] H. Diaz, The Chow ring of a cubic hypersurface, *International Math. Research Notices*,
- [8] G. Fano, Sulle sezione spaziale della varietà grassmanniana delle rette dello spazio a cinque dimensioni, *Rend. Acc. Naz. Lincei* 11 (1930), 329–335,
- [9] L. Fu, R. Laterveer and Ch. Vial, The generalized Franchetta conjecture for some hyper-Kähler varieties (with an appendix joint with M. Shen), *Journal Math. Pures et Appliquées* (9) 130 (2019), 1–35,
- [10] L. Fu, R. Laterveer and Ch. Vial, The generalized Franchetta conjecture for some hyper-Kähler varieties, II, *Journal de l’Ecole Polytechnique–Mathématiques* 8 (2021), 1065–1097,
- [11] L. Fu, R. Laterveer and Ch. Vial, Multiplicative Chow–Künneth decompositions and varieties of cohomological K3 type, *Annali Mat. Pura ed Applicata*, doi: 10.1007/s10231-021-01070-0,
- [12] L. Fu, Z. Tian and Ch. Vial, Motivic hyperkähler resolution conjecture for generalized Kummer varieties, arXiv:1608.04968,
- [13] W. Fulton, *Intersection theory*, Springer–Verlag Ergebnisse der Mathematik, Berlin Heidelberg New York Tokyo 1984,
- [14] N. Gushel, On Fano threefolds of genus 8, *St. Petersburg Math. J.* 4 no 1 (1993), 115–129,
- [15] A. Iliev and L. Manivel, Pfaffian lines and vector bundles on Fano threefolds of genus 8, *J. Alg. Geom.* 16 (2007), 499–530,
- [16] V. Iskovskikh, Birational automorphisms of three-dimensional algebraic varieties, *J. Sov. Math.* 13 (1980), 815–868,
- [17] V. Iskovskih and Yu. Prokhorov, *Algebraic Geometry V: Fano varieties*, Encyclopaedia of Math. Sciences 47, Springer-Verlag, Berlin 1999,
- [18] U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: *Motives* (U. Jannsen et alii, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
- [19] U. Jannsen, On finite-dimensional motives and Murre’s conjecture, in: *Algebraic cycles and motives* (J. Nagel and C. Peters, editors), Cambridge University Press, Cambridge 2007,
- [20] S.-I. Kimura, Chow groups are finite dimensional, in some sense, *Math. Ann.* 331 no 1 (2005), 173–201,
- [21] A. Kuznetsov, Derived categories of cubic and  $V_{14}$  threefolds, *Proc. V.A. Steklov Inst. Math* 246 (2004), 183–207,
- [22] A. Kuznetsov, Homological projective duality for Grassmannians of lines, arXiv:0610957,
- [23] A. Kuznetsov, Derived categories of Fano threefolds, *Proc. V. A. Steklov Inst. Math* 264 (2009), 110–122,
- [24] A. Kuznetsov, Y. Prokhorov and C. Shramov, Hilbert schemes of lines and conics and automorphism groups of Fano threefolds, *Japanese Journal of Mathematics* 13 no 1 (2018), 109–185,

- [25] R. Laterveer, A remark on the Chow ring of Küchle fourfolds of type  $d3$ , *Bulletin Australian Math. Soc.* 100 no. 3 (2019), 410—418,
- [26] R. Laterveer and Ch. Vial, On the Chow ring of Cynk–Hulek Calabi–Yau varieties and Schreieder varieties, *Canadian Journal of Math.* 72 no 2 (2020), 505—536,
- [27] R. Laterveer, Algebraic cycles and Verra fourfolds, *Tohoku Math. J.* 72 no. 3 (2020), 451—485,
- [28] R. Laterveer, On the Chow ring of certain Fano fourfolds, *Ann. Univ. Paedagog. Crac. Stud. Math.* 19 (2020), 39—52,
- [29] R. Laterveer, Algebraic cycles and Gushel–Mukai fivefolds, *Journal of Pure and Applied Algebra* 225 no. 5 (2021), doi: 10.1016/j.jpaa.2020.106582,
- [30] R. Laterveer, Algebraic cycles and intersections of 2 quadrics, *Mediterranean Journal of Mathematics* (2021), doi: 10.1007/s00009-021-01787-5,
- [31] R. Laterveer, Algebraic cycles and intersections of a quadric and a cubic, *Forum Mathematicum* 33 no. 3 (2021), 845—855,
- [32] S. Mukai, New developments in the theory of Fano threefolds: vector bundle method and moduli problems, *Sugaku Expositions* 15 no 2 (2002), 125—150,
- [33] J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II, *Indag. Math.* 4 (1993), 177—201,
- [34] J. Murre, J. Nagel and C. Peters, *Lectures on the theory of pure motives*, Amer. Math. Soc. University Lecture Series 61, Providence 2013,
- [35] A. Negut, G. Oberdieck and Q. Yin, Motivic decompositions for the Hilbert scheme of points of a K3 surface, arXiv:1912.09320v1,
- [36] N. Pavic, J. Shen and Q. Yin, On O’Grady’s generalized Franchetta conjecture, *Int. Math. Res. Notices* (2016), 1—13,
- [37] C. Peters and J. Steenbrink, *Mixed Hodge Theory*, *Ergebnisse der Math.* 52, Springer Verlag 2008,
- [38] P. Puts, On some Fano-threefolds that are sections of Grassmannians, *Indag. Math.* 44 (1982), 77—90,
- [39] T. Scholl, Classical motives, in: *Motives* (U. Jannsen et alii, eds.), *Proceedings of Symposia in Pure Mathematics* Vol. 55 (1994), Part 1,
- [40] M. Shen and Ch. Vial, The Fourier transform for certain hyperKähler fourfolds, *Memoirs of the AMS* 240 (2016), no.1139,
- [41] M. Shen and Ch. Vial, The motive of the Hilbert cube  $X^{[3]}$ , *Forum Math. Sigma* 4 (2016), 55 pp.,
- [42] M. Tavakol, The tautological ring of the moduli space  $M_2^{rt}$ , *International Math. Research Notices* 178 (2013),
- [43] M. Tavakol, Tautological classes on the moduli space of hyperelliptic curves with rational tails, *J. Pure Applied Algebra* 222 no 8 (2018), 2040—2062,
- [44] A. Tyurin, Geometry of the Fano surface of a nonsingular cubic  $F \subset \mathbb{P}^4$  and Torelli theorems for Fano surfaces and cubics, *Izv. USSR Math.* 35 (1971), 498—529,
- [45] Ch. Vial, Projectors on the intermediate algebraic Jacobians, *New York J. Math.* 19 (2013), 793—822,
- [46] Ch. Vial, Algebraic cycles and fibrations, *Documenta Math.* 18 (2013), 1521—1553,
- [47] Ch. Vial, Remarks on motives of abelian type, *Tohoku Math. J.* 69 (2017), 195—220,
- [48] Ch. Vial, On the motive of some hyperkähler varieties, *J. Reine Angew. Math.* 725 (2017), 235—247,
- [49] Q. Yin, Finite-dimensionality and cycles on powers of K3 surfaces, *Comment. Math. Helv.* 90 (2015), 503—511.

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, CNRS – UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE.

*Email address:* robert.laterveer@math.unistra.fr