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Graph-based Approximate Message Passing Iterations

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Abstract

Approximate-message passing (AMP) algorithms have become an important element of high-dimensional statistical inference, mostly due to their adaptability and concentration properties, the state evolution (SE) equations. This is demonstrated by the growing number of new iterations proposed for increasingly complex problems, ranging from multi-layer inference to low-rank matrix estimation with elaborate priors. In this paper, we address the following questions: is there a structure underlying all AMP iterations that unifies them in a common framework? Can we use such a structure to give a modular proof of state evolution equations, adaptable to new AMP iterations without reproducing each time the full argument? We propose an answer to both questions, showing that AMP instances can be generically indexed by an oriented graph. This enables to give a unified interpretation of these iterations, independent from the problem they solve, and a way of composing them arbitrarily. We then show that all AMP iterations indexed by such a graph admit rigorous SE equations, extending the reach of previous proofs, and proving a number of recent heuristic derivations of those equations. Our proof naturally includes non-separable functions and we show how existing refinements, such as spatial coupling or matrix-valued variables, can be combined with our framework.

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1 Introduction

Approximate Message Passing (AMP) algorithms are iterative equations solving inference problems involving high-dimensional random variables with random interactions [DMM09, ZK16]. For the typical case in which AMP iterations were initially studied, the interactions involve an i.i.d. Gaussian matrix. These algorithms are inspired from Bolthausen’s iterative solution of the celebrated Thouless-Anderson-Palmer (TAP) equations of spin glass theory [MPV87, Bol14, Bol19]. However, they are usually derived as heuristic relaxations of the belief propagation equations [Pea14] on dense factor graphs in a manner often encountered in the context of statistical physics of disordered systems. A central property of AMP iterations is that the distribution of their outputs can be tracked rigorously in the high-dimensional limit by low-dimensional equations called *state evolution* (SE). This property can be seen as similar to the concept of density evolution from coding theory [RU08], but in the case of dense factor graphs.

In recent years, the growing interest in high-dimensional inference and learning problems has motivated the introduction of approximate-message passing algorithms as solutions to many inference problems, and as analytical tools—thanks to the SE equations—to study the statistical properties of learned estimators, notably starting with the LASSO [BM11a, KMS⁺12, DMM09]. A number of extensions were then proposed for inference problems of growing complexity: generalized linear modelling and robust m-estimators [Ran11, DM16, ZK16], low-rank matrix reconstruction [RF12, LKZ17], principal component analysis (PCA) [DM14, LKZ15], inference in deep multilayer networks with random weights [MKMZ17], matrix-valued inference problems [AMB⁺19] or matrix recovery under generative priors [ALM⁺20], among others. Interestingly, AMP algorithms can be composed with one another to solve inference problems obtained by combining factor graphs, as demonstrated in [ALM⁺20], where each part of the factor graph represents an elaborate prior and inference process. This demonstrates the adaptability of such iterations, even more so as the state evolution equations are shown to hold, often heuristically, for these composite structures.

Contributions. As the diversity of inference problems and AMP iterations increases, it is important to identify a common structure underlying the known AMP algorithms. Such a partial unification was done in [JM13, BMN20]: symmetric and asymmetric AMP iterations are treated in a common framework. However, these results do not apply to the more recent AMP iterations designed for more complex problems presenting multilayered structures or ones obtained by combining factor graphs.

In this paper, our first contribution is to show how AMP algorithms are naturally indexed by a graph that determines its form. Seeing AMP algorithms as supported by this graph helps understanding the iterations, especially the multi-layer ones, in a unified way. In this regard, we hope that our framework will be used as a tool to generate new AMP iterations. Roughly speaking, the graph underlying the AMP iteration represents the interaction of the high-dimensional variables of the associated inference problem. However, this graph is not the factor graph representing the inference problem that sometimes appears in the derivation of AMP equations, see [KMS⁺12] for example. The factor graph is microscopic, in the sense that it disappears when taking the dense limit leading to the AMP equations. On the contrary, the graph that we consider here is macroscopic: it structures the AMP iteration itself. It is insensitive to the underlying inference problem that has generated the AMP equation; for instance, it can be used in both Bayes optimal or non-Bayes optimal scenarios.

The second contribution of this paper is to use the graph framework to show that all graph-based AMP iterations admit a rigorous SE description. This generalizes the previous works of [BM11a, JM13, BMN20] on SE to more complex iterations. Using our result, writing and proving the state evolution equations is reduced to the identification of a specific structure in the AMP iteration, instead of heuristically deriving or reproducing the rigorous proof entirely for problems of increasing complexity. In particular, it gives a theoretical grounding for the analysis of AMP on recent multi-layer structures [MKMZ17, AMB⁺19, ALM⁺20]. Related to [MKMZ17], this paper proves that AMP algorithms are a rigorously grounded approach to understanding multi-layer neural networks, albeit only when the weights are random and when we perform inference with an AMP algorithm. Still, in a context where theory struggles to explain the behavior of multi-layered neural networks, it is interesting to see that this particular case can be rigorously studied, even for deep architectures.

We illustrate the flexibility of our framework by applying it to diverse inference problems mentioned above, notably multilayer generalized linear estimation problems and low-rank matrix recovery with deep generative priors. We also show how our results can be extended to handle matrix-valued variables and combined with the spatial coupling framework introduced in [KMS⁺12, JM13].

Related work. There is a rich literature of proofs of state evolution equations, notably starting with Bolthausen’s iterative scheme [Bol14, Bol19] based on Gaussian conditioning. The technique was then adapted and extended to the case of a more generic AMP iteration related to the LASSO problem in [BM11a], where it is mentioned that Gaussian conditioning methods also appear in [Don06] to tackle fundamental random convex geometry problems. The analysis was then extended to matrix-valued variables with block-separable non-linearities in [JM13] and for vector-valued variables with non-separable non-linearities in [BMN20], which also show that symmetric AMP and asymmetric AMP can be treated in the same framework. Our proof is partly based on the same iterative Gaussian conditioning method but is additionally combined with an embedding specific to the graph framework. To the best of our knowledge, the latter part of the proof is novel.

Another line of work—called VAMP (vector approximate message passing) algorithms—handles rotationally invariant matrices [RSF19] with generic spectrum. This family of VAMP iterations is obtained using a Gaussian parametrization of *expectation propagation* [Min13, OWJ05], a variational inference algorithm based on iterative moment-matching between a chosen form of probability distribution (e.g., Gaussian nodes on a factor graph) and a target distribution observed through empirical data. These iterations also verify SE equations proven with a similar conditioning method [Tak17, RSF19], handling a different kind of randomness than i.i.d. Gaussian matrices. The SE

proof for VAMP iterations was then extended to multilayer inference problems and their matrix-valued counterparts in [FRS18, PSAR⁺20]. In these works, the conditioning method is applied in a sequential manner to each layer of the problem, making it specific to multilayer inference problems. On the contrary, our proof method is not restricted to sequential multilayer estimation as mentioned in the contributions, and does not rely on iterating through the graph. However, our proof does not apply to all rotationally invariant matrices. We handle mostly Gaussian or GOE matrices, with extensions to correlated Gaussian matrices, products of Gaussian matrices and spatially coupled Gaussian matrices. This is discussed in greater detail in Sections 3 and 4.

Outline of the paper. The paper is organised as follows: we start by presenting the indexation of AMP iterations by an oriented graph in Section 2. Several conceptual examples are provided. We present the state evolution equations on any graph-supported AMP iteration in Section 3, along with its proof, which constitutes the main technical contribution of this paper. We then move to applications to inference problems in Section 4 and conclude on related open problems in Section 5. All proofs of auxiliary results are deferred to the Appendix.

Notations. We adopt similar notations to those of [BMN20]. Differences are mainly due to the matrix variables framework.

We denote scalars with lowercase letters, vectors with bold lowercase letters and matrices with bold uppercase ones. Inner products are denoted by brackets $\langle \cdot, \cdot \rangle$, and the canonical inner products are chosen for vectors and matrices, i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y}$, $\langle \mathbf{X}, \mathbf{Y} \rangle = \text{Tr}(\mathbf{X}^\top \mathbf{Y})$. The associated norms are respectively denoted $\|\cdot\|_2$ and $\|\cdot\|_F$ for the Frobenius norm.

For two random variables X and Y , and a σ -algebra \mathfrak{G} , we use $X|_{\mathfrak{G}} \stackrel{d}{=} Y$ to mean that for any integrable function ϕ and any \mathfrak{G} -measurable bounded random variable Z , $\mathbb{E}[\phi(X)Z] = \mathbb{E}[\phi(Y)Z]$. For two sequences of random variables X_n, Y_n , we write $X_n \stackrel{P}{\simeq} Y_n$ when their difference converges in probability to 0, i.e., $X_n - Y_n \xrightarrow{P} 0$.

We use \mathbf{I}_N to denote the $N \times N$ identity matrix, and $0_{N \times N}$ the $N \times N$ matrix with zero entries. We use $\sigma_{\min}(\mathbf{Q})$ and $\sigma_{\max}(\mathbf{Q}) = \|\mathbf{Q}\|_{op}$ to denote the minimum and maximum singular values of a given matrix \mathbf{Q} . For two matrices \mathbf{Q} and \mathbf{P} with the same number of rows, we denote their horizontal concatenation with $[\mathbf{P}|\mathbf{Q}]$. The orthogonal projector onto the range of a given matrix \mathbf{M} is denoted \mathbf{P}_M , and let $\mathbf{P}_M^\perp = \mathbf{I} - \mathbf{P}_M$.

Let \mathcal{S}_q^+ denote the space of positive semi-definite matrices of size $q \times q$. For any matrix $\boldsymbol{\kappa} \in \mathcal{S}_q^+$ and a random matrix $\mathbf{Z} \in \mathbb{R}^{N \times q}$ we write $\mathbf{Z} \sim \mathbf{N}(0, \boldsymbol{\kappa} \otimes \mathbf{I}_N)$ if \mathbf{Z} is a matrix with jointly Gaussian entries such that for any $1 \leq i, j \leq q$, $\mathbb{E}[\mathbf{Z}^i (\mathbf{Z}^j)^\top] = \boldsymbol{\kappa}_{i,j} \mathbf{I}_N$, where $\mathbf{Z}^i, \mathbf{Z}^j$ denote the i -th and j -th columns of \mathbf{Z} . The i -th line of the matrix \mathbf{Z} is denoted \mathbf{Z}_i .

If $f : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^{N \times q}$ is a function and $i \in \{1, \dots, N\}$, we write $f_i : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^q$ the component of f generating the i -th line of its image, i.e., if $\mathbf{X} \in \mathbb{R}^{N \times q}$,

$$f(\mathbf{X}) = \begin{bmatrix} f_1(\mathbf{X}) \\ \vdots \\ f_N(\mathbf{X}) \end{bmatrix} \in \mathbb{R}^{N \times q}.$$

We write $\frac{\partial f_i}{\partial \mathbf{X}_i}$ the $q \times q$ Jacobian containing the derivatives of f_i with respect to (w.r.t.) the i -th

line $\mathbf{X}_i \in \mathbb{R}^q$:

$$\frac{\partial f_i}{\partial \mathbf{X}_i} = \begin{bmatrix} \frac{\partial(f_i(\mathbf{X}))_1}{\partial \mathbf{X}_{i1}} & \cdots & \frac{\partial(f_i(\mathbf{X}))_1}{\partial \mathbf{X}_{iq}} \\ \vdots & & \vdots \\ \frac{\partial(f_i(\mathbf{X}))_q}{\partial \mathbf{X}_{i1}} & \cdots & \frac{\partial(f_i(\mathbf{X}))_q}{\partial \mathbf{X}_{iq}} \end{bmatrix} \in \mathbb{R}^{q \times q}. \quad (1)$$

2 Graph-based AMP iterations

We start by defining the class of graphs indexing AMP iterations.

Definition 1 (graph notions). A finite directed graph—also simply called graph in the following—is a pair $G = (V, \vec{E})$ where V is a finite set, called the vertex set, and \vec{E} is a subset of $V \times V$, called the edge set. This definition of graphs uses directed edges and allows loops.

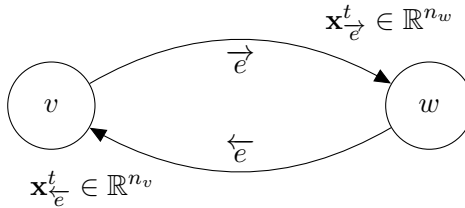
A graph $G = (V, \vec{E})$ is said to be symmetric if for all $v, w \in V$, $(v, w) \in \vec{E}$ if and only if $(w, v) \in \vec{E}$.

The degree $\deg v$ of a node $v \in V$ is the number of edges of which it is the end-node. In symmetric graphs, it is also the number of edges of which v is the starting-node.

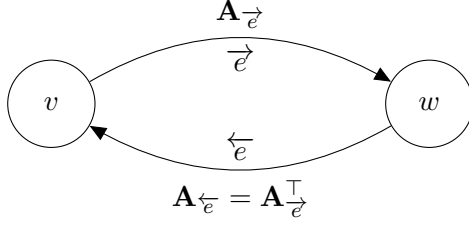
Graph notations. Given a symmetric graph $G = (V, \vec{E})$, the following notations are useful. We sometimes write $v \rightarrow w$ to mean that $\vec{e} = (v, w)$ is an edge of the graph. We say that v is the starting-node of \vec{e} and w the end-node of \vec{e} . We denote $\overleftarrow{e} = (w, v) \in \vec{E}$ the symmetric edge of \vec{e} . If \vec{e} is a loop, then $\overleftarrow{e} = \vec{e}$. We write $\vec{e} \rightarrow \vec{e}'$ as a shorthand to say that the end-node of $\vec{e} \in \vec{E}$ is the starting-node of $\vec{e}' \in \vec{E}$. Note that for any $\vec{e} \in \vec{E}$, $\overleftarrow{\overleftarrow{e}} \rightarrow \vec{e}$.

Iteration. We now fix a symmetric finite directed graph $G = (V, \vec{E})$. We associate an AMP iteration supported by the graph G as follows.

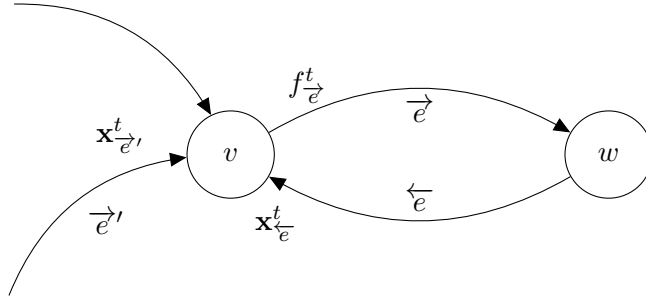
- The variables $\mathbf{x}_{\vec{e}}^t$ of the AMP iteration are indexed by the iteration number $t \in \mathbb{N}$ and the oriented edges of the graph $\vec{e} \in \vec{E}$.



- All variables associated to edges $\vec{e} = (v, w)$ with end-node $w \in V$ have a same dimension $n_w \in \mathbb{N}_{>0}$, i.e., $\mathbf{x}_{\vec{e}}^t \in \mathbb{R}^{n_w}$. We define $N = \sum_{(v,w) \in \vec{E}} n_w$ the sum of the dimensions of all variables.
- Matrices of the AMP iteration are also indexed by the edges of the graph. If $\vec{e} = (v, w) \in \vec{E}$, $\mathbf{A}_{\vec{e}} \in \mathbb{R}^{n_w \times n_v}$. These matrices must satisfy the symmetry condition $\mathbf{A}_{(v,w)} = \mathbf{A}_{(w,v)}^\top$. In particular, this implies that matrices $\mathbf{A}_{(v,v)} \in \mathbb{R}^{n_v \times n_v}$ associated to loops $(v, v) \in \vec{E}$ must be symmetric.



- Non-linearities of the AMP iteration are also indexed by the edges of the graph (and possibly by the iteration number t). If $t \geq 0$ and $\vec{e} = (v, w) \in \vec{E}$, $f_{(v,w)}^t \left((\mathbf{x}_{\vec{e}'}^t)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right)$ is a function of all the variables of the edges whose end-node is the starting-node v of \vec{e} , as denoted by the condition $\vec{e}' \rightarrow \vec{e}$. It is a function from $(\mathbb{R}^{n_v})^{\deg v}$ to \mathbb{R}^{n_w} .



Once these parameters $(\mathbf{A}_{\vec{e}})_{\vec{e} \in \vec{E}}$ and $(f_{\vec{e}}^t)_{t \geq 0, \vec{e} \in \vec{E}}$ are given, we can choose an arbitrary initial condition $\mathbf{x}_{\vec{e}}^0 \in \mathbb{R}^{n_w}$ for all oriented edges $\vec{e} \in \vec{E}$ of the graph. We define recursively the AMP iterates $(\mathbf{x}_{\vec{e}}^t)_{t \geq 0, \vec{e} \in \vec{E}}$, by the iteration: for all $t \geq 0$, $\vec{e} \in \vec{E}$,

$$\mathbf{x}_{\vec{e}}^{t+1} = \mathbf{A}_{\vec{e}} \mathbf{m}_{\vec{e}}^t - b_{\vec{e}}^t \mathbf{m}_{\overleftarrow{e}}^{t-1}, \quad (2)$$

$$\mathbf{m}_{\vec{e}}^t = f_{\vec{e}}^t \left((\mathbf{x}_{\vec{e}'}^t)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right), \quad (3)$$

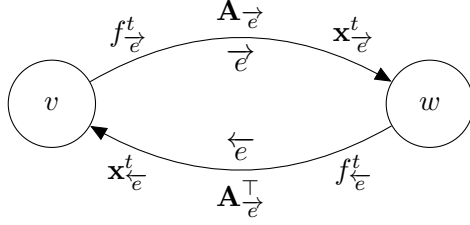
where $b_{\vec{e}}^t$ is the so-called *Onsager term*

$$b_{\vec{e}}^t = \frac{1}{N} \text{Tr} \frac{\partial f_{\vec{e}}^t}{\partial \mathbf{x}_{\overleftarrow{e}}} \left((\mathbf{x}_{\vec{e}'}^t)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right) \in \mathbb{R}. \quad (4)$$

The above partial derivative makes sense as $\overleftarrow{e} \rightarrow \vec{e}$, thus $\mathbf{x}_{\overleftarrow{e}}$ is a variable of $f_{\vec{e}}^t$. Note that in (2), the Onsager term multiplies the vector $\mathbf{m}_{\overleftarrow{e}}^{t-1}$ indexed by the symmetric edge \overleftarrow{e} of \vec{e} .

Let us derive some simple particular cases of this framework, first to recover the classical asymmetric and symmetric AMP iterations, and second to cover multi-layer AMP iterations.

Asymmetric AMP. The asymmetric AMP iteration appeared first in the literature to solve the compressed sensing problem [DMM09] and then more generally to tackle generalized linear estimation, see, e.g., [Ran11, SR14, DM16]. It corresponds to a simple underlying graph composed of two nodes and two symmetric directed edges between them.

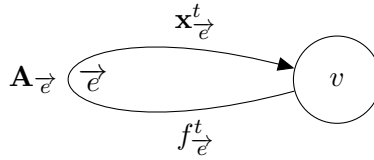


In this case, the graph AMP equations (2)-(3) give

$$\begin{aligned}
 \mathbf{x}_{\vec{e}}^{t+1} &= \mathbf{A}_{\vec{e}} \mathbf{m}_{\vec{e}}^t - b_{\vec{e}}^t \mathbf{m}_{\overleftarrow{e}}^{t-1}, \\
 \mathbf{m}_{\vec{e}}^t &= f_{\vec{e}}^t(\mathbf{x}_{\overleftarrow{e}}^t), \\
 \mathbf{x}_{\overleftarrow{e}}^{t+1} &= \mathbf{A}_{\overleftarrow{e}} \mathbf{m}_{\overleftarrow{e}}^t - b_{\overleftarrow{e}}^t \mathbf{m}_{\vec{e}}^{t-1}, \\
 \mathbf{m}_{\overleftarrow{e}}^t &= f_{\overleftarrow{e}}^t(\mathbf{x}_{\vec{e}}^t).
 \end{aligned} \tag{5}$$

The corresponding state evolution (SE) property was proved in [BM11a] for the separable case and in [BMN20] in the non-separable case. Note that the time indices proposed here are different from the ones appearing in these works. The time index convention adopted here generalizes better to more elaborate graphs. We show how to recover the usual time indices in Appendix A.

Symmetric AMP. The symmetric AMP iteration is central to our discussion as we show that all graph AMP iterations can be reduced to this case (with matrix-valued iterates, as detailed below). It is already known that the asymmetric case can be reduced to this case [JM13]. The symmetric AMP iteration appears, e.g., when solving the low-rank matrix recovery problem [RF12, DM14], or community detection in graphs [DAM17]. It corresponds to the degenerate graph with only one node and one loop.

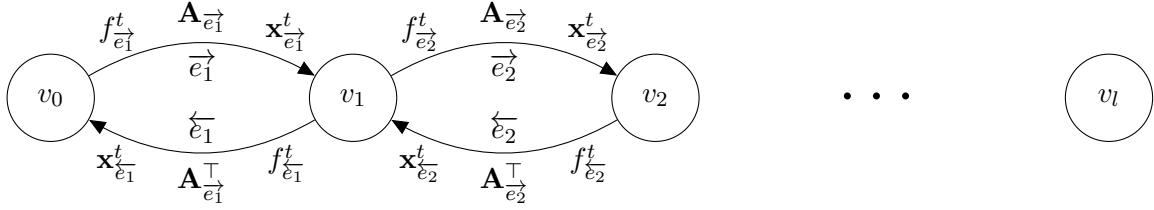


Recall that $\overleftarrow{e} = \vec{e}$ as \vec{e} is a loop. In this case, the graph AMP equations (2)-(3) give

$$\begin{aligned}
 \mathbf{x}_{\vec{e}}^{t+1} &= \mathbf{A}_{\vec{e}} \mathbf{m}_{\vec{e}}^t - b_{\vec{e}}^t \mathbf{m}_{\vec{e}}^{t-1}, \\
 \mathbf{m}_{\vec{e}}^t &= f_{\vec{e}}^t(\mathbf{x}_{\vec{e}}^t),
 \end{aligned} \tag{6}$$

Here, as there is a single edge \vec{e} , the indexes are superfluous and could be dropped. For these equations, the SE property was proved in [JM13] for the separable case and in [BMN20] in the non-separable case. Note that the results of [JM13] allow matrix-valued variables.

Multi-layer AMP. The multi-layer AMP iteration appears when considering inference problems through a multi-layer random neural network, see [MKMZ17]. They correspond to a line graph whose length l is the number of layers.

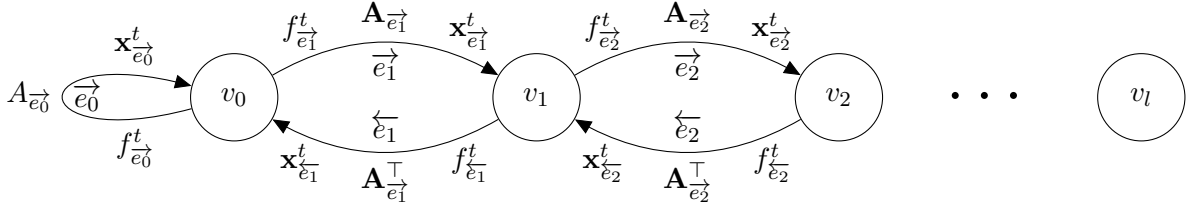


In this case, the graph AMP equations (2)-(3) give

$$\begin{aligned}
\mathbf{x}_{e_1}^{t+1} &= \mathbf{A}_{e_1}^{\vec{e}_1} \mathbf{m}_{e_1}^t - b_{e_1}^t \mathbf{m}_{e_1}^{t-1}, \\
\mathbf{m}_{e_1}^t &= f_{e_1}^t(\mathbf{x}_{e_1}^t), \\
\mathbf{x}_{e_1}^{t+1} &= \mathbf{A}_{e_1}^{\top} \mathbf{m}_{e_1}^t - b_{e_1}^t \mathbf{m}_{e_1}^{t-1}, \\
\mathbf{m}_{e_1}^t &= f_{e_1}^t(\mathbf{x}_{e_1}^t, \mathbf{x}_{e_2}^t), \\
\\
\mathbf{x}_{e_2}^{t+1} &= \mathbf{A}_{e_2}^{\vec{e}_2} \mathbf{m}_{e_2}^t - b_{e_2}^t \mathbf{m}_{e_2}^{t-1}, \\
\mathbf{m}_{e_2}^t &= f_{e_2}^t(\mathbf{x}_{e_1}^t, \mathbf{x}_{e_2}^t), \\
\mathbf{x}_{e_2}^{t+1} &= \mathbf{A}_{e_2}^{\top} \mathbf{m}_{e_2}^t - b_{e_2}^t \mathbf{m}_{e_2}^{t-1}, \\
\mathbf{m}_{e_2}^t &= f_{e_2}^t(\mathbf{x}_{e_2}^t, \mathbf{x}_{e_3}^t), \\
\\
&\vdots
\end{aligned} \tag{7}$$

Note that the non-linearities now take several variables as inputs when there are several incoming edges at a node.

Spiked matrix model under generative multi-layer priors. Of course, the structures described above can be combined to tackle new AMP iterations. For instance, the paper [AMB⁺19] studies the recovery of noisy symmetric rank-1 matrix when the spike comes from a known multi-layer generative prior. The associated AMP iteration corresponds to the following graph, where the loop corresponds to the spike recovery and the other edges correspond to multi-layer prior on the spike.



In this case, the graph AMP equations (2)-(3) give

$$\begin{aligned}
\mathbf{x}_{e_0}^{t+1} &= \mathbf{A}_{\vec{e}_0} \mathbf{m}_{e_0}^t - b_{e_0}^t \mathbf{m}_{e_0}^{t-1}, \\
\mathbf{m}_{e_0}^t &= f_{e_0}^t(\mathbf{x}_{e_0}^t, \mathbf{x}_{e_1}^t), \\
\mathbf{x}_{e_1}^{t+1} &= \mathbf{A}_{\vec{e}_1} \mathbf{m}_{e_1}^t - b_{e_1}^t \mathbf{m}_{e_1}^{t-1}, \\
\mathbf{m}_{e_1}^t &= f_{e_1}^t(\mathbf{x}_{e_0}^t, \mathbf{x}_{e_1}^t), \\
\mathbf{x}_{e_1}^{t+1} &= \mathbf{A}_{\vec{e}_1}^\top \mathbf{m}_{e_1}^t - b_{e_1}^t \mathbf{m}_{e_1}^{t-1}, \\
\mathbf{m}_{e_1}^t &= f_{e_1}^t(\mathbf{x}_{e_1}^t, \mathbf{x}_{e_2}^t), \\
&\vdots
\end{aligned} \tag{8}$$

3 State evolution for graph-based AMP iterations

In this section, we start by presenting the most straightforward form of our result, and show afterwards how several refinements can be added.

3.1 Main theorem

AMP algorithms admit a state evolution description under two major assumptions: that the interactions matrices $\mathbf{A}_{\vec{e}}$ are sufficiently random—in our case Gaussian or GOE—and that the dimensions $n = (n_v)_{v \in V}$ of all the variables converge to infinity with fixed ratios.

Assumptions. We make the following assumptions:

- (A1) The matrices $(\mathbf{A}_{\vec{e}})_{\vec{e} \in \vec{E}}$ are random and independent, up to the symmetry condition $\mathbf{A}_{\vec{e}} = \mathbf{A}_{\vec{e}}^\top$. Moreover, if $(v, w) \in \vec{E}$ is not a loop in G , i.e., $v \neq w$, then $\mathbf{A}_{(v,w)}$ has independent centered Gaussian entries with variance $1/N$. If $(v, v) \in \vec{E}$ is a loop in G , then $\mathbf{A}_{(v,v)}$ has independent entries (up to the symmetry $\mathbf{A}_{(v,v)} = \mathbf{A}_{(v,v)}^\top$), centered Gaussian with variance $2/N$ on the diagonal and variance $1/N$ off the diagonal.
- (A2) For all $v \in V$, $n_v \rightarrow \infty$ and n_v/N converges to a well-defined limit $\delta_v \in [0, 1]$. We denote by $n \rightarrow \infty$ the limit under this scaling.
- (A3) For all $t \in \mathbb{N}$ and $\vec{e} \in \vec{E}$, the non-linearity $f_{\vec{e}}^t$ is pseudo-Lipschitz of finite order, uniformly with respect to the problem dimensions $n = (n_v)_{v \in V}$ (see Definition 4 in Appendix D).
- (A4) For all $\vec{e} \in E$, $\|\mathbf{x}_{\vec{e}}^0\|_2/\sqrt{N}$ converges to a finite constant as $n \rightarrow \infty$.
- (A5) For all $\vec{e} \in E$, the following limit exists and is finite:

$$\lim_{n \rightarrow \infty} \frac{1}{N} \langle f_{\vec{e}}^0((\mathbf{x}_{\vec{e}'}^0)_{\vec{e}': \vec{e}' \rightarrow \vec{e}}), f_{\vec{e}}^0((\mathbf{x}_{\vec{e}'}^0)_{\vec{e}': \vec{e}' \rightarrow \vec{e}}) \rangle$$

(A6) Let $(\kappa_{\vec{e}})_{\vec{e} \in E}$ be an array of bounded non-negative reals and $\mathbf{Z}_{\vec{e}} \sim \mathbf{N}(0, \kappa_{\vec{e}} \mathbf{I}_{n_w})$ independent random variables for all \vec{e} . For all $\vec{e} \in E$, for any $t \in \mathbb{N}_{>0}$, the following limit exists and is finite:

$$\lim_{n \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\left\langle f_{\vec{e}}^0 \left((\mathbf{x}_{\vec{e}'}^0)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right), f_{\vec{e}}^t \left((\mathbf{Z}_{\vec{e}'}^t)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right) \right\rangle \right].$$

(A7) Consider any array of 2×2 positive definite matrices $(\mathbf{S}_{\vec{e}})_{\vec{e} \in E}$ and the collection of random variables $(\mathbf{Z}_{\vec{e}}, \mathbf{Z}'_{\vec{e}}) \sim \mathbf{N}(0, \mathbf{S}_{\vec{e}} \otimes \mathbf{I}_{n_w})$ defined independently for each edge \vec{e} . Then for any $\vec{e} \in E$ and $s, t > 0$, the following limit exists and is finite:

$$\lim_{n \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\left\langle f_{\vec{e}}^s \left((\mathbf{Z}_{\vec{e}'}^s)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right), f_{\vec{e}}^t \left((\tilde{\mathbf{Z}}_{\vec{e}'}^t)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right) \right\rangle \right].$$

Remark on the assumptions. In the literature, the random matrices $\mathbf{A}_{(v,w)}$ of AMP iterations are often scaled with variances $1/n_w$. To recover the desired scaling, it is sufficient to rescale the non-linearity on which a given matrix acts with the corresponding aspect ratio δ_w .

Definition 2 (State evolution iterates). *The state evolution iterates are composed of one infinite-dimensional array $(\kappa_{\vec{e}}^{s,r})_{r,s>0}$ of real values for each edge $\vec{e} \in \vec{E}$. These arrays are generated as follows. Define the first state evolution iterates*

$$\kappa_{\vec{e}}^{1,1} = \lim_{n \rightarrow \infty} \frac{1}{N} \left\| f_{\vec{e}}^0 \left((\mathbf{x}_{\vec{e}'}^0)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right) \right\|_2^2, \quad \vec{e} \in \vec{E}.$$

Recursively, once $(\kappa_{\vec{e}}^{s,r})_{s,r \leq t, \vec{e} \in \vec{E}}$ are defined for some $t \geq 1$, define independently for each $\vec{e} \in \vec{E}$, $\mathbf{Z}_{\vec{e}}^0 = \mathbf{x}_{\vec{e}}^0$ and $(\mathbf{Z}_{\vec{e}}^1, \dots, \mathbf{Z}_{\vec{e}}^t)$ a centered Gaussian random vector of covariance $(\kappa_{\vec{e}}^{r,s})_{r,s \leq t} \otimes \mathbf{I}_{n_w}$. We then define new state evolution iterates

$$\kappa_{\vec{e}}^{t+1,s+1} = \kappa_{\vec{e}}^{s+1,t+1} = \lim_{n \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\left\langle f_{\vec{e}}^s \left((\mathbf{Z}_{\vec{e}'}^s)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right), f_{\vec{e}}^t \left((\mathbf{Z}_{\vec{e}'}^t)_{\vec{e}': \vec{e}' \rightarrow \vec{e}} \right) \right\rangle \right]$$

for all $s \in \{1, \dots, t\}$, $\vec{e} \in \vec{E}$.

Theorem 1. *Assume (A1)-(A7). Define, as above, independently for each $\vec{e} = (v,w) \in \vec{E}$, $\mathbf{Z}_{\vec{e}}^0 = \mathbf{x}_{\vec{e}}^0$ and $(\mathbf{Z}_{\vec{e}}^1, \dots, \mathbf{Z}_{\vec{e}}^t)$ a centered Gaussian random vector of covariance $(\kappa_{\vec{e}}^{r,s})_{r,s \leq t} \otimes \mathbf{I}_{n_w}$. Then for any sequence of uniformly (in n) pseudo-Lipschitz function $\Phi : \mathbb{R}^{(t+1)N} \rightarrow \mathbb{R}$,*

$$\Phi \left((\mathbf{x}_{\vec{e}}^s)_{0 \leq s \leq t, \vec{e} \in \vec{E}} \right) \stackrel{\text{P}}{\simeq} \mathbb{E} \left[\Phi \left((\mathbf{Z}_{\vec{e}}^s)_{0 \leq s \leq t, \vec{e} \in \vec{E}} \right) \right]$$

3.2 Reduction of graph-based AMP iterations to the matrix-valued, non-separable symmetric case

The core strategy in the proof of Theorem 1 is to reduce the graph AMP iteration (2)-(4) into a symmetric AMP iteration with matrix-valued iteration, i.e., an iteration of the form

$$\mathbf{X}^{t+1} = \mathbf{A} \mathbf{M}^t - \mathbf{M}^{t-1} (\mathbf{b}^t)^\top \in \mathbb{R}^{N \times q}, \quad (9)$$

$$\mathbf{M}^t = f^t(\mathbf{X}^t) \in \mathbb{R}^{N \times q}, \quad (10)$$

$$\mathbf{b}_t = \frac{1}{N} \sum_{i=1}^N \frac{\partial f_i^t}{\partial \mathbf{X}_i}(\mathbf{X}^t) \in \mathbb{R}^{q \times q}. \quad (11)$$

- Let $q_{\vec{e}} \in \mathbb{N}_{>0}$ be the number of columns of $\mathbf{x}_{\vec{e}}^t$. We assume that, for all $\vec{e} \in E$, $q_{\vec{e}} = q_{\bar{\vec{e}}}$, and the $q_{\vec{e}}$ remain constant, independently of $n \rightarrow \infty$.
- The initial condition becomes $\mathbf{x}_{(v,w)}^0 \in \mathbb{R}^{n_w \times q_{(v,w)}}$, for all edges $\vec{e} = (v, w)$.
- Non-linearities f_t indexed by the edge $\vec{e} = (v, w) \in \vec{E}$, $f_{(v,w)}^t(x_{\vec{e}'}, \vec{e}' \rightarrow \vec{e})$ are now functions from $\times_{\vec{e}' \rightarrow \vec{e}} \mathbb{R}^{n_v \times q_{\vec{e}'}}$ to $\mathbb{R}^{n_w \times q_{(v,w)}}$.

The AMP iterates are then recursively defined with:

$$\mathbf{x}_{\vec{e}}^{t+1} = \mathbf{A}_{\vec{e}} \mathbf{m}_{\vec{e}}^t - \mathbf{m}_{\bar{\vec{e}}}^{t-1} (\mathbf{b}_{\vec{e}}^t)^\top \in \mathbb{R}^{n_w \times q_{\vec{e}}}, \quad (13)$$

$$\mathbf{m}_{\vec{e}}^t = f_{\vec{e}}^t \left((\mathbf{x}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e}) \right), \quad (14)$$

where each Onsager term is now a matrix given by:

$$\mathbf{b}_{\vec{e}}^t = \frac{1}{N} \sum_{i=1}^{n_w} \frac{\partial f_{\vec{e},i}^t}{\partial \mathbf{x}_{\bar{\vec{e}},i}} \left((\mathbf{x}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e}) \right) \in \mathbb{R}^{q_{\vec{e}} \times q_{\vec{e}}}.$$

where we used the notation from Eq.(1). The state evolution equations then read

$$\boldsymbol{\kappa}_{\vec{e}}^{1,1} = \lim_{n \rightarrow \infty} \frac{1}{N} f_{\vec{e}}^0(\mathbf{x}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e})^\top f_{\vec{e}}^0(\mathbf{x}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e}) \in \mathbb{R}^{q_{\vec{e}} \times q_{\vec{e}}}, \quad \vec{e} \in \vec{E}.$$

$$\boldsymbol{\kappa}_{\vec{e}}^{t+1,s+1} = \boldsymbol{\kappa}_{\vec{e}}^{s+1,t+1} = \lim_{n \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[f_{\vec{e}}^s(\mathbf{Z}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e})^\top f_{\vec{e}}^t(\mathbf{Z}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e}) \right] \in \mathbb{R}^{q_{\vec{e}} \times q_{\vec{e}}}$$

for all $1 \leq s \leq t$, $\vec{e} \in \vec{E}$.

where the Gaussian fields generalize straightforwardly to $\mathbf{Z}_{\vec{e}}^t \sim \mathbf{N}(0, \boldsymbol{\kappa}_{\vec{e}}^{t,t} \otimes \mathbf{I}_{n_w}) \in \mathbb{R}^{n_w \times q_{\vec{e}}}$ for each edge. Using these generalized definitions, the above statement of Theorem 1 and its proof can be adapted easily. We give examples throughout Section 4.

Additional random variables in the non-linearities. Many inference problems are formulated with a “planted” signal, i.e., a ground truth signal parametrizing the function the statistician tries to reconstruct, sometimes called *teacher* in statistical physics. This often leads to the dependence of certain non-linearities on additional random variables. As long as they appropriately concentrate and are independent on the rest of the problem, they can be treated in straightforward fashion with an additional average in the SE equations as done in [JM13], where the summability is reduced to second-order moments conditions due to the separability of the update functions. As we will see in Section 4, it is not always straightforward to isolate the independent contribution in the teacher which is often generated using the matrices found in the AMP algorithm, effectively introducing a correlation between the matrices and non-linearities. The state evolution iterates then become

$$\boldsymbol{\kappa}_{\vec{e}}^{t+1,s+1} = \boldsymbol{\kappa}_{\vec{e}}^{s+1,t+1} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\langle f_{\vec{e}}^s \left((\mathbf{Z}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e}), \mathbf{y}_{\vec{e}} \right), f_{\vec{e}}^t \left((\mathbf{Z}_{\vec{e}'}, \vec{e}' \rightarrow \vec{e}), \mathbf{y}_{\vec{e}} \right) \rangle \right]$$

for all $1 \leq s \leq t$, $\vec{e} \in \vec{E}$.

where the average is now over the Gaussian fields and additional $\mathbf{y}_{\vec{e}}$. Examples are given throughout Section 4 and Appendix E.

Structured and correlated matrices. Products of Gaussian matrices can be considered by choosing identities as non-linearities on given edges of the graph. This was done heuristically in [MKMZ17] to study structured inference problems. Gaussian matrices with generic covariances can also be considered, i.e., $\mathbf{A} = \mathbf{Z}\mathbf{\Sigma}^{1/2}$ where \mathbf{Z} is an i.i.d. $\mathbf{N}(0, \frac{1}{d})$ matrix and $\mathbf{\Sigma} \in \mathbb{R}^{d \times d}$ is a positive definite matrix. Indeed, the covariance matrix can be absorbed in the non-linearity as a non-separable component. Depending on the non-linearity, expressions may simplify as functions of the spectral distribution of $\mathbf{\Sigma}$. Examples are given in Section 4.5.

Spatial coupling. *Spatial coupling* was introduced and studied in [KMS⁺12, KMS⁺12, JM13, DJM13] as a mean to reach information theoretic limits in compressed sensing. The idea is to write the state evolution equations when the random matrices have a block structure of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1l} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \dots & \mathbf{A}_{kl} \end{bmatrix} \in \mathbb{R}^{N \times d},$$

each $\mathbf{A}_{ij} \in \mathbb{R}^{N_i \times d_j}$ has i.i.d. $\mathbf{N}(0, \frac{\sigma_{ij}}{d})$ entries and $N_i/N, d_j/d$ are constant aspect ratios, where $\sum_i N_i = N$ and $\sum_j d_j = d$. The proof of SE equations with this kind of matrices was proposed in [JM13] and relies on a matrix-valued symmetric AMP iteration similar to the one used in our proof, with a family of non-linearities acting on blocks of variables, with a separable effect on each block. Since our proof extends the matrix-valued, symmetric AMP iteration to the fully non-separable case, the same ideas can be applied to our framework to include spatially coupled matrices on each edge of the oriented graph presented in the previous section (with the added possibility of non-separable effects on each block). We give an example in Section 4.6.

4 Applications to inference problems

In this section we illustrate our main theorem by showing how several AMP iterations established heuristically in the literature are included in our framework, in particular [MKMZ17, AMB⁺19, ALM⁺20, LSG⁺21], and how straightforward generalizations can be considered. We adopt an optimization viewpoint for each problem, omitting the probabilistic inference formulation at the origin of these iterations for simplicity.

4.1 A building block: AMP for generalized linear models

We start with a known AMP iteration for which the state evolution equations were already proven, and build upon the intuition it gives to present more elaborate iterations. Consider the task of optimizing a penalized cost functions of the form

$$\hat{\mathbf{x}} \in \min_{\mathbf{x} \in \mathbb{R}^d} g(\mathbf{A}\mathbf{x}, \mathbf{y}) + f(\mathbf{x}) \quad (15)$$

where the vector of labels \mathbf{y} is typically assumed to be generated from another process as

$$\mathbf{y} = \phi(\mathbf{A}\mathbf{x}_0),$$

with $\mathbf{x}_0 \in \mathbb{R}^d$ generated from a given distribution $p_{\mathbf{x}_0}$ independent from the matrix \mathbf{A} , $\mathbf{A} \in \mathbb{R}^{N \times d}$ is a matrix with i.i.d. $\mathbf{N}(0, \frac{1}{d})$ elements, and ϕ a given function. The goal is then to reconstruct

the vector \mathbf{x}_0 . This formulation is at the basis of many of the fundamental estimation methods in machine learning: least-squares, LASSO, logistic regression, etc. Approximate-message passing algorithms were proposed for this task, notably in [DMM09, BM11a, Ran11, KMS⁺12, JM13], and take the generic form of the asymmetric AMP iteration (5) where $\mathbf{A}_{\mathcal{C}} = \mathbf{A}$. Intuitively, the functions $f_{\mathcal{C}}^t, f_{\mathcal{E}}^t$ each correspond to one of the functions g, f from (15) and respectively output an estimate of the quantities $\mathbf{A}\hat{\mathbf{x}}, \hat{\mathbf{x}}$. As prescribed by the form of the generative model, we expect the update function associated to the loss $g(\cdot, \mathbf{y})$ to be correlated with the matrix \mathbf{A} , thus preventing a direct application of the SE equations. We propose a self-contained derivation explaining how to recover (an instance of) the usual SE equations for GAMP as originally derived in [Ran11] in Appendix E.

4.2 Multilayer generalized linear estimation

Consider now the problem of recovering a vector \mathbf{x}_0 from a more complex generative model involving a multilayer neural network with random weights:

$$\mathbf{y} = \phi_L(\mathbf{A}_L \phi_{L-1}(\mathbf{A}_{L-1}(\dots \phi_1(\mathbf{A}_1 \mathbf{x}_0))))$$

where one has access to the final output \mathbf{y} and would like to reconstruct the intermediate ones and input \mathbf{x}_0 . For each layer $1 \leq l \leq L$ the matrix $\mathbf{A}_l \in \mathbb{R}^{N_{l+1} \times N_l}$ has i.i.d. $\mathbf{N}(0, \frac{1}{N_l})$ with $N_{l+1}/N_l = \delta_l$. The idea is to solve this sequentially using asymmetric AMP iterations similar to the one presented in the previous section. This approach was originally proposed in [MKMZ17] under the name multilayer AMP (MLAMP). For any $1 \leq l \leq L + 1$, define

$$\begin{aligned} \mathbf{x}_l &= \phi_{l-1}(\mathbf{A}_{l-1} \phi_{l-2}(\dots \phi_1(\mathbf{A}_1 \mathbf{x}_0))), \\ \text{such that } \mathbf{x}_{l+1} &= \phi_l(\mathbf{A}_l \mathbf{x}_l) \quad \text{and} \quad \mathbf{x}_{L+1} = \mathbf{y} \end{aligned}$$

The intuition is the following : each \mathbf{x}_l is then estimated using the asymmetric AMP corresponding to the problem

$$\hat{\mathbf{x}}_l = \arg \min_{\mathbf{x} \in \mathbb{R}^{N_l}} g_l(\mathbf{A}_l \mathbf{x}, \mathbf{y}_l) + f_l(\mathbf{x})$$

the output of which is used to estimate the next, i.e., $\mathbf{y}_l = \hat{\mathbf{x}}_{l+1}$, whose statistical properties are given by the SE equations. The complete derivation of the iteration involves writing the belief-propagation equations on the factor graph corresponding to the multilayer inference problem, capturing all the interactions between the different iterates. These SE equations were derived heuristically in [MKMZ17] for Bayes-optimal inference, and this paper proves them in the generic case.

4.3 Spiked matrix with generative prior

In the same spirit as the composition of generalized linear models defining MLAMP, different tasks can be composed to obtain richer instances of inference problems. For instance in [AMB⁺19], the reconstruction of a low-rank matrix under a generative prior is considered using an AMP iteration. A rank-one matrix is observed, blurred by Gaussian noise:

$$\mathbf{Y} = \sqrt{\frac{\lambda}{d}} \mathbf{v}_0 \mathbf{v}_0^\top + \mathbf{W}$$

where $\mathbf{W} \in GOE(N)$, and the vector $\mathbf{v}_0 \in \mathbb{R}^N$ is assumed to be generated from a multilayer neural network with random weights

$$\mathbf{v}_0 = \phi_L(\mathbf{A}_L \phi_{L-1}(\mathbf{A}_{L-1}(\dots \phi_1(\mathbf{A}_1 \mathbf{x}_0))))$$

for a given ground truth vector $\mathbf{x}_0 \in \mathbb{R}^{N_1}$, matrices $\{\mathbf{A}_l \in \mathbb{R}^{N_{l+1} \times N_l}\}_{1 \leq l \leq L}$ and non-linearities $\{\phi_l\}_{1 \leq l \leq L}$. The AMP iteration to estimate \mathbf{v}_0 from \mathbf{Y} was first proposed in [RF12, DM14], and takes the form of a symmetric AMP (6). Similarly to MLAMP, the output of this iteration can then be used as input, leading to the AMP iteration proposed in [ALM⁺20], which corresponds to the AMP iteration (8). This paper proves the state evolution equations for this iteration.

4.4 An example with matrix-valued variables

Matrix valued variables are encountered in scenarios such as committee machines [AMB⁺19] or multiclass learning problems [LSG⁺21], or more generically when a finite ensemble of predictors is learned. Consider the matrix-valued extension of the generalized linear estimation problem Eq.(15).

$$\hat{\mathbf{X}} \in \arg \min_{\mathbf{X} \in \mathbb{R}^{N \times q}} g(\mathbf{A}\mathbf{X}, \mathbf{Y}) + f(\mathbf{X})$$

where $\mathbf{Y} = \phi(\mathbf{A}\mathbf{X}_0)$

where $\mathbf{X}_0 \in \mathbb{R}^{N \times q}$ and $q \in \mathbb{N}$ is kept finite. The SE equations for the asymmetric AMP with matrix valued-variables are included in the result of [JM13]. This can be directly generalized to a multilayer matrix inference problem by considering a generative model of the form

$$\mathbf{Y} = \phi_L(\mathbf{A}_L \phi_{L-1}(\mathbf{A}_{L-1}(\dots \phi_1(\mathbf{A}_1 \mathbf{X}_0))))$$

and successive application of the matrix-valued asymmetric AMP as proposed for MLAMP in Section 4.2. The state evolution equations for this problem is included in our framework using the results from Section 3.3.

4.5 An example with structured random matrices

Consider a generalized linear inference task where the data is now represented by a Gaussian matrix with a covariance $\Sigma \neq \mathbf{I}_d$. This can be dealt with using the non-separable framework. Assuming the covariance matrix is full-rank, we can equivalently work with the variable $\tilde{\mathbf{x}} = \Sigma^{1/2}\mathbf{x}$, and solve

$$\arg \min_{\tilde{\mathbf{x}}} g(\tilde{\mathbf{A}}\tilde{\mathbf{x}}, \mathbf{y}) + f(\Sigma^{-1/2}\tilde{\mathbf{x}}).$$

where $\tilde{\mathbf{A}}$ is now an i.i.d. Gaussian matrix. This will modify the update function associated to f , becoming $f(\Sigma^{-1/2}\cdot)$, which is non-separable, even if the function f is initially assumed to be separable. The validity of the SE equations for this case follows from the results of [BMN20]. This manipulation can also be done on any layer of MLAMP, for a given set of covariance matrices $\Sigma_1, \dots, \Sigma_L$ associated to each random matrix $\mathbf{A}_1, \dots, \mathbf{A}_L$, with vector or matrix-valued variables. The validity of the SE equations in this case follows from the results of this paper. In the convex GLM case (2-layer), the fixed point of the state evolution equations with a generic covariance gives the same result as (a particular case of) the exact asymptotics recently proposed in [LGC⁺21] to study different feature maps in generalized linear models.

4.6 An example of spatial coupling with non-separable non-linearities

Here we briefly describe an inference problem recently studied in [LSG⁺21] that can be solved using spatial coupling on a non-separable AMP iteration. Consider the problem of classifying a high-

dimensional Gaussian mixture with a finite number K of clusters, described by the joint density

$$P(\mathbf{x}|\mathbf{y}) = \sum_{k=1}^K y_k \pi_k \mathbf{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

where $\mathbf{x} \in \mathbb{R}^d$ is a sample, $\mathbf{y} \in \mathbb{R}^K$ is a binary label vector, $\{\pi_k\}_k$ are the cluster probabilities such that $\sum_{k=1}^K \pi_k = 1$, $\{\boldsymbol{\mu}_k\}_{1 \leq k \leq K}$ are the means and $\{\boldsymbol{\Sigma}_k\}_{1 \leq k \leq K}$ are positive definite covariances, using a convex generalized linear model, i.e.,

$$\mathbf{X} \in \arg \min_{\mathbf{X} \in \mathbb{R}^{d \times K}} g(\mathbf{A}\mathbf{X}, \mathbf{Y}) + f(\mathbf{X})$$

where $\mathbf{Y} \in \mathbb{R}^{N \times K}$ is the concatenated matrix of one-hot encoded labels. The matrix \mathbf{A} representing N samples of the Gaussian mixture can be written as a block diagonal matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{Z}_1 \boldsymbol{\Sigma}_1^{1/2} & & & \\ & \mathbf{Z}_2 \boldsymbol{\Sigma}_2^{1/2} & & \\ & & \dots & \\ & & & \mathbf{Z}_K \boldsymbol{\Sigma}_K^{1/2} \end{bmatrix} \in \mathbb{R}^{N \times Kd}$$

where the $\mathbf{Z}_k \in \mathbb{R}^{N_k \times d}$ are i.i.d. $\mathbf{N}(0, \frac{1}{d})$ independent matrices, with N_k the number of samples coming from each cluster. This type of matrix can be embedded into an AMP iteration using the spatial coupling technique to handle the block structure and the non-separable framework to deal with the covariances on each block. The validity of the SE equations for the combination of spatial coupling and non-separable effects is proven by this paper. This is also an example where the teacher distribution is independent of the Gaussian matrices that will appear in the AMP iteration, as the multinomial distribution prescribing cluster membership is independent of the Gaussian cloud of each cluster.

5 Perspectives

We have shown that AMP algorithms can be unified in an intuitive way by means of an oriented graph, and that this representation leads to a modular, effective and extended proof of state evolution equations. Several problems follow from the results presented here.

Connecting back to the factor graph. We do not relate our proposed graphical representation of the AMP iterations with the factor graphs of the probabilistic inference problems that generated them. Understanding this relation would clarify the statistical inference problems that can be solved using AMP iterations. The applications that motivated this paper use our framework with only very simple graphs—line graphs, sometimes with a loop. However, the framework accepts much more complicated graphs, potentially with more loops. In future work, we hope to explore the new statistical problems and AMP iterations that can be analyzed using these graphs.

Rotationally invariant matrices. As shown in [RSF19, FRS18, PSAR⁺20, Fan20], the Gaussian conditioning method at the core of AMP proofs can be reproduced with right rotationally invariant matrices with generic spectrum. Extending the results of the present paper to this family of matrices requires finding the appropriate form of the graph iteration and is an open problem.

Universality and finite size corrections. State evolution proofs are amenable to both finite size analysis [RV18, MRB17] and universality proofs [BLM15, CL21]. Although both problems were tackled in simpler settings in these papers, their techniques could be combined with the embedding proposed in the proof of Theorem 1 to prove finite size rates and universality properties for any graph supported AMP.

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A Changing time indices

Here we show how the time index convention usually encountered in earlier instances of the asymmetric AMP iteration can be recovered from the one used in this proof. Consider two successive iterations of the asymmetric AMP (5):

$$\begin{aligned}
 \mathbf{x}_{\varrho}^{t+1} &= \mathbf{A}_{\varrho} \mathbf{m}_{\varrho}^t - b_{\varrho}^t \mathbf{m}_{\varrho}^{t-1}, & \mathbf{x}_{\varrho}^t &= \mathbf{A}_{\varrho} \mathbf{m}_{\varrho}^{t-1} - b_{\varrho}^{t-1} \mathbf{m}_{\varrho}^{t-2}, \\
 \mathbf{m}_{\varrho}^t &= f_{\varrho}^t(\mathbf{x}_{\varrho}^t), & \mathbf{m}_{\varrho}^{t-1} &= f_{\varrho}^{t-1}(\mathbf{x}_{\varrho}^{t-1}), \\
 \mathbf{x}_{\varrho}^{t+1} &= \mathbf{A}_{\varrho}^{\top} \mathbf{m}_{\varrho}^t - b_{\varrho}^t \mathbf{m}_{\varrho}^{t-1}, & \mathbf{x}_{\varrho}^t &= \mathbf{A}_{\varrho}^{\top} \mathbf{m}_{\varrho}^{t-1} - b_{\varrho}^{t-1} \mathbf{m}_{\varrho}^{t-2}, \\
 \mathbf{m}_{\varrho}^t &= f_{\varrho}^t(\mathbf{x}_{\varrho}^t) & \mathbf{m}_{\varrho}^{t-1} &= f_{\varrho}^{t-1}(\mathbf{x}_{\varrho}^{t-1})
 \end{aligned} \tag{16}$$

which requires initializing both \mathbf{x}_{ϱ} and \mathbf{x}_{ϱ} , and updates them simultaneously at each iteration. We see that to evaluate $\mathbf{x}_{\varrho}^{t+1}$ (resp. $\mathbf{x}_{\varrho}^{t+1}$), we only need the previous value of \mathbf{x}_{ϱ}^t (resp. \mathbf{x}_{ϱ}^t) and $\mathbf{x}_{\varrho}^{t-1}$ (resp. $\mathbf{x}_{\varrho}^{t-1}$). Thus only half of the iterates can be computed, independently of the other half, using the following formulae (setting the other update functions to zero):

$$\begin{aligned}
 \mathbf{x}_{\varrho}^{2t+1} &= \mathbf{A}_{\varrho}^{\top} \mathbf{m}_{\varrho}^{2t} - b_{\varrho}^{2t} \mathbf{m}_{\varrho}^{2t-1}, \\
 \mathbf{m}_{\varrho}^{2t} &= f_{\varrho}^{2t}(\mathbf{x}_{\varrho}^{2t}), \\
 \mathbf{x}_{\varrho}^{2t} &= \mathbf{A}_{\varrho} \mathbf{m}_{\varrho}^{2t-1} - b_{\varrho}^{2t-1} \mathbf{m}_{\varrho}^{2t-2}, \\
 \mathbf{m}_{\varrho}^{2t-1} &= f_{\varrho}^{2t-1}(\mathbf{x}_{\varrho}^{2t-1})
 \end{aligned} \tag{17}$$

which only requires one value at initialization and at each iteration. The usual time indices found in , e.g., [BMN20] are then recovered with the following mapping:

$$\begin{aligned}
 \mathbf{x}_{\varrho}^{2t+1} &= \mathbf{u}^{t+1} \\
 \mathbf{x}_{\varrho}^{2t} &= \mathbf{v}^t \\
 f_{\varrho}^{2t}(\cdot) &= g_t(\cdot) \\
 f_{\varrho}^{2t-1}(\cdot) &= e_t(\cdot)
 \end{aligned}$$

Note that this simplification is specific to the graph structure underlying the asymmetric AMP iteration.

B Matrix-valued symmetric AMP iterations with non-separable non-linearities

B.1 State evolution description

In this section, we present the state evolution equations for a symmetric AMP iteration with non-separable non-linearities and matrix-valued variables. This is an extension of the results of [JM13, BMN20]. This result underlies the proof of state evolution equations for graph-based AMP iterations.

Consider an initial (deterministic) matrix $\mathbf{X}^0 \in \mathbb{R}^{N \times q}$ and a sequence of deterministic functions $\{f^t : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^{N \times q}\}_{t \in \mathbb{N}}$. For the reader's convenience, we recall here the symmetric AMP iteration (9)-(11).

Symmetric AMP iteration. Let $\mathbf{X}^0 \in \mathbb{R}^{N \times q}$ and define recursively,

$$\mathbf{X}^{t+1} = \mathbf{A}\mathbf{M}^t - \mathbf{M}^{t-1}(\mathbf{b}^t)^\top \in \mathbb{R}^{N \times q}, \quad (18)$$

$$\mathbf{M}^t = f^t(\mathbf{X}^t) \in \mathbb{R}^{N \times q}, \quad (19)$$

$$\mathbf{b}^t = \frac{1}{N} \sum_{i=1}^N \frac{\partial f_i^t}{\partial \mathbf{X}_i}(\mathbf{X}^t) \in \mathbb{R}^{q \times q}. \quad (20)$$

where \mathbf{b}^t is the Onsager correction term. We now list the necessary assumptions.

Assumptions.

(B1) $\mathbf{A} \in \mathbb{R}^{N \times N}$ is a GOE(N) matrix, i.e., $\mathbf{A} = \mathbf{G} + \mathbf{G}^\top$ for $\mathbf{G} \in \mathbb{R}^{N \times N}$ with i.i.d. entries $G_{ij} \sim \mathbf{N}(0, 1/(2N))$.

(B2) For each $t \in \mathbb{N}$, $f^t : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^{N \times q}$ is pseudo-Lipschitz of order k , uniformly in N .

(B3) $\|\mathbf{X}^0\|_F/\sqrt{N}$ converges to a finite constant as $N \rightarrow \infty$.

(B4) The following limit exists and is finite:

$$\lim_{N \rightarrow \infty} \frac{1}{N} f^0(\mathbf{X}^0)^\top f^0(\mathbf{X}^0) \in \mathbb{R}^{q \times q} \quad (21)$$

(B5) For any $t \in \mathbb{N}_{>0}$ and any $\boldsymbol{\kappa} \in \mathcal{S}_q^+$, the following limit exists and is finite:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[f^0(\mathbf{X}^0)^\top f^t(\mathbf{Z}) \right] \in \mathbb{R}^{q \times q} \quad (22)$$

where $\mathbf{Z} \in \mathbb{R}^{N \times q}$, $\mathbf{Z} \sim \mathbf{N}(0, \boldsymbol{\kappa} \otimes \mathbf{I}_N)$.

(B6) For any $s, t \in \mathbb{N}_{>0}$ and any $\boldsymbol{\kappa} \in \mathcal{S}_{2q}^+$, the following limit exists and is finite:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[f^s(\mathbf{Z}^s)^\top f^t(\mathbf{Z}^t) \right] \in \mathbb{R}^{q \times q} \quad (23)$$

where $(\mathbf{Z}^s, \mathbf{Z}^t) \in (\mathbb{R}^{N \times q})^2, (\mathbf{Z}^s, \mathbf{Z}^t) \sim \mathbf{N}(0, \boldsymbol{\kappa} \otimes \mathbf{I}_N)$.

Under these assumptions, we define the *state evolution* iteration related to the AMP iteration (18)-(20).

Definition 3 (state evolution iterates). *The state evolution iterates are composed of one infinite-dimensional array $(\boldsymbol{\kappa}^{s,r})_{r,s>0}$ of real matrices. This array is generated as follows. Define the first state evolution iterate*

$$\boldsymbol{\kappa}_{1,1} = \lim_{N \rightarrow \infty} \frac{1}{N} f^0(\mathbf{X}^0)^\top f^0(\mathbf{X}^0) \quad (24)$$

Recursively, once $\boldsymbol{\kappa}^{s,r}, 0 \leq s, r \leq t$ are defined for some $t \geq 1$, take $\mathbf{Z}^0 = \mathbf{X}^0$ and $(\mathbf{Z}^1, \dots, \mathbf{Z}^t) \in (\mathbb{R}^{n \times q})^t$ a centered Gaussian vector of covariance $(\boldsymbol{\kappa}^{s,r})_{s,r \leq t} \otimes \mathbf{I}_N$. We then define new state evolution iterates

$$\boldsymbol{\kappa}^{t+1,s+1} = \boldsymbol{\kappa}^{s+1,t+1} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[f^s(\mathbf{Z}^s)^\top f^t(\mathbf{Z}^t) \right], \quad s \in \{0, \dots, t\}.$$

The following property then holds for the AMP iteration (18)-(20).

Theorem 2. *Assume (B1)-(B6). Define, as above, $\mathbf{Z}^0 = \mathbf{X}^0$ and $(\mathbf{Z}^1, \dots, \mathbf{Z}^t) \in (\mathbb{R}^{N \times q})^t$ a centered Gaussian vector of covariance $(\boldsymbol{\kappa}^{s,r})_{s,r \leq t} \otimes \mathbf{I}_N$. Then for any sequence $\Phi_N : (\mathbb{R}^{N \times q})^{t+1} \rightarrow \mathbb{R}$ of pseudo-Lipschitz functions,*

$$\Phi_N(\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^t) \stackrel{P}{\simeq} \mathbb{E} [\Phi_N(\mathbf{Z}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t)].$$

Given the above result, we can expect the Onsager correction \mathbf{b}^t to verify

$$\mathbf{b}^t \stackrel{P}{\simeq} \frac{1}{N} \mathbb{E} \left[\sum_{i=1}^N \frac{\partial f_i^t}{\partial \mathbf{Z}_i}(\mathbf{Z}^t) \right] \in \mathbb{R}^{q \times q}. \quad (25)$$

where $\mathbf{Z}^t \sim \mathbf{N}(0, \boldsymbol{\kappa}_{t,t} \otimes \mathbf{I}_n)$. In fact, similarly to [BMN20], Theorem 2 can be shown to hold for the AMP iteration ((18)-(20)) with any estimator $\hat{\mathbf{b}}^t$ satisfying

$$\hat{\mathbf{b}}^t(\mathbf{X}^0, \hat{\mathbf{M}}^0, \dots, \hat{\mathbf{M}}^{t-1}, \hat{\mathbf{X}}^t) \stackrel{P}{\simeq} \frac{1}{N} \mathbb{E} \left[\sum_{i=1}^N \frac{\partial f_i^t}{\partial \mathbf{Z}_i}(\mathbf{Z}^t) \right] \in \mathbb{R}^{q \times q}. \quad (26)$$

B.2 Application: proof of Theorem 1

In Section 3.2, we have seen that the graph AMP iteration (2)-(4) can be rewritten as a symmetric AMP iteration of the form (9)-(11). Here, we check that applying Theorem 2 on the symmetric iteration after performing the reduction indeed gives Theorem 1.

Define the state evolution iterates as in Definition 3. Here, due to the expression (12) of the non-linearities, the state evolution iterates are diagonal:

$$\boldsymbol{\kappa}^{1,1} = \lim_{N \rightarrow \infty} \frac{1}{N} \begin{pmatrix} \left\| f_{\vec{e}_1}^0((\mathbf{x}_{\vec{e}}^0)_{\vec{e}: \vec{e} \rightarrow \vec{e}_1}) \right\|^2 & & 0 \\ & \ddots & \\ 0 & & \left\| f_{\vec{e}_m}^0((\mathbf{x}_{\vec{e}}^0)_{\vec{e}: \vec{e} \rightarrow \vec{e}_m}) \right\|^2 \end{pmatrix} \quad (27)$$

and

$$\boldsymbol{\kappa}^{t+1,s+1} = \boldsymbol{\kappa}^{s+1,t+1} = \lim_{N \rightarrow \infty} \frac{1}{N} \begin{pmatrix} \mathbb{E} f_{\vec{e}_1}^s(\dots)^\top f_{\vec{e}_1}^t(\dots) & & 0 \\ & \ddots & \\ 0 & & \mathbb{E} f_{\vec{e}_m}^s(\dots)^\top f_{\vec{e}_m}^t(\dots) \end{pmatrix}.$$

Let $\mathbf{Z}^t \in \mathbb{R}^{N \times q}$ be the variable from Definition 3. Decompose

$$\mathbf{Z}^t = \begin{pmatrix} \mathbf{Z}_{\vec{e}_1}^t & & * \\ & \ddots & \\ * & & \mathbf{Z}_{\vec{e}_m}^t \end{pmatrix}.$$

where $\mathbf{Z}_{(\vec{v},\vec{w})}^t \in \mathbb{R}^{n_w}$. The diagonal structure of the state evolution iterates means that $\mathbf{Z}_{\vec{e}}^t$ and $\mathbf{Z}_{\vec{e}'}$ are independent when $\vec{e} \neq \vec{e}'$. We thus find that

$$\boldsymbol{\kappa}^{s,t} = \lim_{N \rightarrow \infty} \frac{1}{N} \begin{pmatrix} \boldsymbol{\kappa}_{\vec{e}_1}^{s,t} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{\kappa}_{\vec{e}_m}^{s,t} \end{pmatrix}$$

where the $\boldsymbol{\kappa}_{\vec{e}}^{s,t}$ are those defined in Section 3 and the variables $\mathbf{Z}_{\vec{e}}^t$ are the same as those defined in Section 3.

These elements show that Theorem 1 follows from the application of Theorem 2.

C Proof of Theorem 2

Once the concentration lemmas of Appendix D are established for matrix valued-variables, the proof follows closely that of [BMN20]. We include the main steps (with a few changes) for completeness nonetheless.

As an intermediate step, we introduce the following AMP iteration initialized with $\mathbf{X}^0 \in \mathbb{R}^{N \times q}$:

$$\mathbf{X}^{t+1} = \mathbf{A}\mathbf{M}^t - \mathbf{M}^{t-1}(\mathbf{b}^t)^\top \in \mathbb{R}^{N \times q} \quad (28)$$

$$\mathbf{M}^t = f^t(\mathbf{X}^t) \in \mathbb{R}^{N \times q}, \quad (29)$$

$$\mathbf{b}^t = \frac{1}{N} \mathbb{E} \left[\sum_{i=1}^N \frac{\partial f_i^t}{\partial \mathbf{Z}_i}(\mathbf{Z}^t) \right] \in \mathbb{R}^{q \times q}. \quad (30)$$

where the Onsager term has been replaced by the expectation in Eq.(25) using the state evolution recursion, i.e., $\mathbf{Z}^t \in \mathbb{R}^{N \times q} \sim \mathbf{N}(0, \boldsymbol{\kappa}_{t,t} \otimes \mathbf{I}_N)$. We denote this recursion with the shorthand $\{\mathbf{X}^t, \mathbf{M}^t | f^t, \mathbf{X}^0\}$. The following lemma is an analog of Theorem 2 for the iteration (28)-(30).

Lemma 2. *Define, as above, $\mathbf{Z}^0 = \mathbf{X}^0$ and $(\mathbf{Z}^1, \dots, \mathbf{Z}^t) \in (\mathbb{R}^{N \times q})^t$ a centered Gaussian vector of covariance $\begin{pmatrix} \boldsymbol{\kappa}^{1,1} & \dots & \boldsymbol{\kappa}^{1,t} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\kappa}^{t,1} & \dots & \boldsymbol{\kappa}^{t,t} \end{pmatrix} \otimes \mathbf{I}_N$. Then for any sequence $\Phi_N : (\mathbb{R}^{N \times q})^{t+1} \rightarrow \mathbb{R}$ of pseudo-Lipschitz functions, the iterates of (28)-(30) satisfy*

$$\Phi_N(\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^t) \stackrel{P}{\simeq} \mathbb{E} [\Phi_N(\mathbf{Z}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t)].$$

C.1 Proof outline and intermediate lemmas

The main idea is to analyze an iteration that behaves well under Gaussian conditioning and that asymptotically approximates (28)-(30).

Matrix LoAMP. We consider the following iteration, a matrix-valued version of the LoAMP iteration introduced in [BMN20]. The sequence of functions f^t and initialization \mathbf{X}^0 are the same as for the AMP iteration $\{\mathbf{X}^t, \mathbf{M}^t | f^t, \mathbf{X}^0\}$. Initialize $\mathbf{Q}^0 = f^0(\mathbf{X}^0)$, and recursively define

$$\mathbf{H}^{t+1} = \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{A} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t + \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \in \mathbb{R}^{N \times q}, \quad (31)$$

$$\mathbf{Q}^t = f^t(\mathbf{H}^t) \in \mathbb{R}^{N \times q}, \quad (32)$$

where at each step, the matrices $\mathcal{Q}_{t-1}, \boldsymbol{\alpha}^t, \mathcal{H}_{t-1}$ are defined as

$$\mathcal{Q}_{t-1} = [\mathbf{Q}^0 | \mathbf{Q}^1 | \dots | \mathbf{Q}^{t-1}] \in \mathbb{R}^{N \times tq}, \quad (33)$$

$$\boldsymbol{\alpha}^t = (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top \mathbf{Q}^t \in \mathbb{R}^{tq \times q}, \quad (34)$$

$$\mathcal{H}_{t-1} = [\mathbf{H}^1 | \mathbf{H}^2 | \dots | \mathbf{H}^t] \in \mathbb{R}^{N \times tq}, \quad (35)$$

$\mathbf{P}_{\mathcal{Q}_{t-1}} = \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top$ is the orthogonal projector on the subspace spanned by the columns of \mathcal{Q}_{t-1} , and $\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp = \mathbf{I}_N - \mathbf{P}_{\mathcal{Q}_{t-1}}$. We denote this recursion with the shorthand $\{\mathbf{H}^t, \mathbf{Q}^t | f^t, \mathbf{X}^0\}$. The inverse $(\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1}$ in the projector may not always be properly defined if \mathcal{Q}_{t-1} is either rank-deficient or has vanishing singular values. We thus introduce the following assumption as in [BMN20], which ensures the proper definition of the projector.

Assumption 1 (Non-degeneracy). *We say that the LoAMP iterates satisfy the non-degeneracy assumption if :*

- almost surely, for all t and all $N \geq t$, \mathcal{Q}_{t-1} has full column rank.
- for all t , there exists some constant $c_t > 0$ —independent of N —such that almost surely, there exists N_0 (random) such that, for $N \geq N_0$, $\sigma_{\min}(\mathcal{Q}_{t-1})/\sqrt{N} \geq c_t > 0$.

We now study the LoAMP iteration, starting with the non-degenerate case.

The non-degenerate case. The following lemma gives the distribution of the Long-AMP iterates when conditioned on the previous ones.

Lemma 3. *Consider the LoAMP iteration $\{\mathbf{H}^t, \mathbf{Q}^t | f^t, \mathbf{X}^0\}$ and assume it satisfies the non-degeneracy assumption. For any $t \in \mathbb{N}$, let \mathfrak{S}_t be the σ -algebra generated by the collection of random variables $\mathbf{H}^1, \mathbf{H}^2, \dots, \mathbf{H}^t$. Then*

$$\mathbf{H}^{t+1} |_{\mathfrak{S}_t} \stackrel{d}{=} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t + \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \quad (36)$$

where $\tilde{\mathbf{A}}$ is a copy of \mathbf{A} independent of \mathfrak{S}_t .

The next lemma characterizes the high-dimensional geometry and distribution of the LoAMP iterates, notably that they verify the state evolution equations.

Lemma 4. Consider the LoAMP recursion $\{\mathbf{H}^t, \mathbf{Q}^t | f_t, \mathbf{X}^0\}$ and suppose it satisfies the non-degeneracy assumption. Then

a) for all $0 \leq s, r \leq t$,

$$\frac{1}{N}(\mathbf{H}^{s+1})^\top \mathbf{H}^{r+1} \stackrel{P}{\simeq} \frac{1}{N}(\mathbf{Q}^s)^\top \mathbf{Q}^r \in \mathbb{R}^{q \times q}, \quad (37)$$

b) for any $t \in \mathbb{N}$, for any sequence of uniformly order- k pseudo-Lipschitz functions $\{\phi_N : (\mathbb{R}^{N \times q})^{t+2} \rightarrow \mathbb{R}\}$,

$$\Phi_N(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^{t+1}) \stackrel{P}{\simeq} \mathbb{E}[\Phi_N(\mathbf{X}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^{t+1})] \quad (38)$$

where

$$(\mathbf{Z}^1, \dots, \mathbf{Z}^{t+1}) \sim \mathbf{N}(0, (\boldsymbol{\kappa}^{s,r})_{s,r \leq t} \otimes \mathbf{I}_N) \quad (39)$$

The next two lemmas show that the iterates of the Long-AMP recursion are arbitrary close to those of the original symmetric AMP in the high-dimensional limit.

Lemma 5. For each iteration t of the LoAMP iteration $\{\mathbf{H}^t, \mathbf{Q}^t | f^t, \mathbf{X}^0\}$, consider the recursion

$$\hat{\mathbf{H}}^{t+1} = \mathbf{A}\mathbf{Q}^t - \mathbf{Q}^{t-1}(\mathbf{b}^t)^\top \quad \text{where} \quad \mathbf{b}^t = \frac{1}{N} \mathbb{E} \left[\sum_{i=1}^N \frac{\partial f_i^t}{\partial \mathbf{Z}_i}(\mathbf{Z}^t) \right] \in \mathbb{R}^{q \times q} \quad (40)$$

$$\mathbf{Q}^t = f^t(\mathbf{H}^t) \quad (41)$$

where we take $\hat{\mathbf{H}}^1 = \mathbf{A}\mathbf{Q}^0$ and $\mathbf{Z}^t \sim \mathbf{N}(0, \mathbf{K}_{t,t} \otimes \mathbf{I}_N)$ with $\mathbf{K}_{t,t}$ defined by the state evolution. Then for any $t \in \mathbb{N}$, $\frac{1}{\sqrt{N}} \|\mathbf{H}^{t+1} - \hat{\mathbf{H}}^{t+1}\|_F \xrightarrow[N \rightarrow \infty]{P} 0$.

Lemma 6. Consider the symmetric AMP iteration $\{\mathbf{X}^t, \mathbf{M}^t | f_t, \mathbf{X}^0\}$ and the LongAMP iteration $\{\mathbf{H}^t, \mathbf{Q}^t | f_t, \mathbf{X}^0\}$. Suppose that LongAMP satisfies the non-degeneracy assumption. Then for any $t \in \mathbb{N}$,

$$\frac{1}{\sqrt{N}} \|\mathbf{H}^{t+1} - \mathbf{X}^{t+1}\|_F \xrightarrow[N \rightarrow \infty]{P} 0 \quad \text{and} \quad \frac{1}{\sqrt{N}} \|\mathbf{Q}^t - \mathbf{M}^t\|_F \xrightarrow[N \rightarrow \infty]{P} 0 \quad (42)$$

Combining the previous results, and assuming the non-degeneracy is verified, Lemma 2 holds true.

Relaxing the non-degeneracy hypothesis This paragraph shows how the non-degeneracy assumption is relaxed using a perturbative argument as done in [BMN20]. Define the randomly perturbed functions

$$f_{\epsilon \mathbf{Y}^t}^t = f^t(\cdot) + \epsilon \mathbf{Y}^t \quad (43)$$

where $\mathbf{Y}^t \in \mathbb{R}^{N \times q}$ is a matrix with i.i.d. $\mathbf{N}(0, 1)$ entries independent of the original matrix \mathbf{A} . We denote \mathbf{Y} the set of random matrices $(\mathbf{Y}^0, \mathbf{Y}^1, \dots, \mathbf{Y}^t) \in (\mathbb{R}^{N \times q})^{t+1}$.

Lemma 7. The AMP iteration defined with the functions $f_{\epsilon \mathbf{Y}}^t$ and initialized with \mathbf{X}^0 verifies Assumptions (B4) – (B6). Furthermore, define the associated state evolution iteration $\{\boldsymbol{\kappa}_\epsilon^{s,t} | f_{\epsilon \mathbf{Y}}^t, \mathbf{X}^0\}$, initialized with

$$\boldsymbol{\kappa}_\epsilon^{1,1} = \lim_{N \rightarrow \infty} \frac{1}{N} (f_{\epsilon \mathbf{Y}}^0(\mathbf{X}^0))^\top (f_{\epsilon \mathbf{Y}}^0(\mathbf{X}^0)) \quad (44)$$

and

$$\boldsymbol{\kappa}_\epsilon^{s+1,t+1} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[(f_{\epsilon \mathbf{Y}}^s(\mathbf{Z}^{\epsilon,s}))^\top f_{\epsilon \mathbf{Y}}^t(\mathbf{Z}^{\epsilon,t}) \right] \quad (45)$$

where $(\mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t}) \sim \mathbf{N}(0, (\boldsymbol{\kappa}^{s,r})_{s,r \leq t}^\epsilon \otimes \mathbf{I}_N)$ and the expectations are taken w.r.t. $\mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t}$ but not on \mathbf{Y} . Then the state evolution $\{\boldsymbol{\kappa}_\epsilon^{s,t} | f_{\epsilon\mathbf{Y}}^t, \mathbf{X}^0\}$ is almost surely non-random.

Lemma 8. Denote $\mathcal{Q}_{t-1}^{\epsilon\mathbf{Y}}$ the $N \times tq$ matrix associated with the LoAMP iterates $\{\mathbf{H}^{\epsilon\mathbf{Y},t}, \mathbf{Q}^{\epsilon\mathbf{Y},t} | f_{\epsilon\mathbf{Y}}^t, \mathbf{X}^0\}$. Assume $\epsilon > 0$. Then for $N \geq t$, the matrix $\mathcal{Q}_{t-1}^{\epsilon\mathbf{Y}}$ almost surely has full column-rank. Furthermore, there exists a constant $c_{t,\epsilon}$, independent of n , such that, almost surely, there exists N_0 (random) such that, for $N \geq N_0$, $\sigma_{\min}(\mathcal{Q}_{t-1}^{\epsilon\mathbf{Y}})/\sqrt{N} \geq c_{t,\epsilon} > 0$.

The next two lemmas show uniform convergence of the perturbed state evolution averages to the original one when the perturbation vanishes.

Lemma 9. Let $\{\Phi_N : \mathbb{R}^{N \times tq} \rightarrow \mathbb{R}^{q \times q}\}_{N>0}$ be a sequence of uniformly pseudo-Lipschitz functions of order k . Let $\boldsymbol{\kappa}, \tilde{\boldsymbol{\kappa}}$ be two $tq \times tq$ covariance matrices and $\mathbf{Z} \sim \mathbf{N}(0, \boldsymbol{\kappa} \otimes \mathbf{I}_N)$, $\tilde{\mathbf{Z}} \sim \mathbf{N}(0, \tilde{\boldsymbol{\kappa}} \otimes \mathbf{I}_N)$. Then

$$\lim_{\tilde{\boldsymbol{\kappa}} \rightarrow \boldsymbol{\kappa}} \sup_{N \geq 1} \mathbb{E}[\Phi_N(\mathbf{Z})] - \mathbb{E}[\Phi_N(\tilde{\mathbf{Z}})] = 0. \quad (46)$$

Lemma 10. For any $s, t \geq 1$, $\boldsymbol{\kappa}_\epsilon^{s,t} \xrightarrow{\epsilon \rightarrow 0} \boldsymbol{\kappa}^{s,t}$.

This last lemma shows that the iterates of the AMP iteration defined with the randomly perturbed functions (43), denoted $\{\mathbf{X}^{\epsilon\mathbf{Y},t}, \mathbf{M}^{\epsilon\mathbf{Y},t} | f_{\epsilon\mathbf{Y}}^t, \mathbf{X}^0\}$, is arbitrarily close to the original AMP iteration $\{\mathbf{X}^t, \mathbf{M}^t | f^t, \mathbf{X}^0\}$ when the perturbation is taken to zero.

Lemma 11. Consider the symmetric AMP iteration defined by $\{\mathbf{X}^t, \mathbf{M}^t | f^t, \mathbf{X}^0\}$ and the corresponding perturbed iteration defined by $\{\mathbf{X}^{\epsilon\mathbf{Y},t}, \mathbf{M}^{\epsilon\mathbf{Y},t} | f_{\epsilon\mathbf{Y}}^t, \mathbf{X}^0\}$. Assume that, for some $t \in \mathbb{N}$. Then there exist functions $h_t(\epsilon)$, $h'_t(\epsilon)$, independent of N , such that

$$\lim_{\epsilon \rightarrow 0} h_t(\epsilon) = \lim_{\epsilon \rightarrow 0} h'_t(\epsilon) = 0 \quad (47)$$

and for all $\epsilon \leq 1$, with high probability,

$$\frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon\mathbf{Y},t} - \mathbf{M}^t\|_F \leq h'_t(\epsilon), \quad (48)$$

$$\frac{1}{\sqrt{N}} \|\mathbf{X}^{\epsilon\mathbf{Y},t+1} - \mathbf{X}^{t+1}\|_F \leq h_t(\epsilon). \quad (49)$$

Combining these lemmas, we now prove Lemma 2.

C.2 Proof of Lemma 2 and Theorem 2

Theorem 2 follows from Lemma 2 similarly to the proof of Corollary 2 from [BMN20].

Proof of Lemma 2. The lemmas presented in the previous section ensure the following:

- Lemma 8 and 2 ensure the AMP iteration defined with randomly perturbed functions verifies the non-degeneracy assumptions and the perturbed state evolution equations, i.e.,

$$\Phi_N(\mathbf{X}^0, \mathbf{X}^{\epsilon,1}, \dots, \mathbf{X}^{\epsilon\mathbf{Y},t}) \stackrel{\text{P}}{\simeq} \mathbb{E}[\Phi_N(\mathbf{Z}^{\epsilon,0}, \mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t})].$$

for any sequence of pseudo-Lipschitz functions Φ_N , where $(\mathbf{Z}^{\epsilon,0}, \mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t})$ are defined as in Eq.(44).

- We have shown that the perturbed state evolution converges to the original one for vanishing perturbations, i.e.,

$$\sup_{N \geq 1} |\mathbb{E} [\Phi_N(\mathbf{Z}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t)] - \mathbb{E} [\Phi_N(\mathbf{Z}^{\epsilon,0}, \mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t})]| \xrightarrow{\epsilon \rightarrow 0} 0$$

using Lemma 9 and 10.

- Lemma 11 ensures the AMP iteration $\{\mathbf{X}^{\epsilon\mathbf{Y},t}, \mathbf{M}^{\epsilon\mathbf{Y},t} | f_{\epsilon\mathbf{Y}}^t, \mathbf{X}^0\}$ uniformly approximates the $\{\mathbf{X}^t, \mathbf{M}^t | f^t, \mathbf{X}^0\}$ one.

In light of these results, consider the following decomposition: for any $\eta \geq 0$:

$$\begin{aligned} & \mathbb{P} (|\Phi_N(\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^t) - \mathbb{E} [\Phi_N(\mathbf{X}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t)]| \geq \eta) \\ & \leq \mathbb{P} \left(|\Phi_N(\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^t) - \Phi_N(\mathbf{X}^0, \mathbf{X}^{\epsilon\mathbf{Y},1}, \dots, \mathbf{X}^{\epsilon\mathbf{Y},t})| \geq \frac{\eta}{3} \right) \\ & \quad + \mathbb{P} \left(|\Phi_N(\mathbf{X}^0, \mathbf{X}^{\epsilon\mathbf{Y},1}, \dots, \mathbf{X}^{\epsilon\mathbf{Y},t}) - \mathbb{E} [\Phi_N(\mathbf{X}^0, \mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t})]| \geq \frac{\eta}{3} \right) \\ & \quad + \mathbb{P} \left(|\mathbb{E} [\Phi_N(\mathbf{X}^0, \mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t})] - \mathbb{E} [\Phi_N(\mathbf{X}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t)]| \geq \frac{\eta}{3} \right) \end{aligned}$$

Starting with the first term of the r.h.s., the pseudo-Lipschitz property and the triangle inequality give

$$\begin{aligned} & |\Phi_N(\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^t) - \Phi_N(\mathbf{X}^0, \mathbf{X}^{\epsilon\mathbf{Y},1}, \dots, \mathbf{X}^{\epsilon\mathbf{Y},t})| \leq \\ & L \left(1 + 2 \frac{\|\mathbf{X}^0\|_F^{k-1}}{n^{k-1}} + \sum_{i=1}^t \frac{\|\mathbf{X}^i\|_F^{k-1}}{n^{(k-1)/2}} + \sum_{i=1}^t \frac{\|\mathbf{X}^{\epsilon,i}\|_F^{k-1}}{N^{(k-1)/2}} \right) \sum_{i=1}^t \frac{\|\mathbf{X}^{\epsilon,i} - \mathbf{X}^i\|_F}{\sqrt{N}} \\ & \leq L \left(1 + 2 \frac{\|\mathbf{X}^0\|_F^{k-1}}{n^{(k-1)/2}} + \sum_{i=1}^t \frac{\|\mathbf{X}^i - \mathbf{X}^{\epsilon,i} + \mathbf{X}^{\epsilon,i}\|_F^{k-1}}{n^{(k-1)/2}} + \sum_{i=1}^t \frac{\|\mathbf{X}^{\epsilon,i}\|_F^{k-1}}{n^{(k-1)/2}} \right) \sum_{i=1}^t \frac{\|\mathbf{X}^{\epsilon,i} - \mathbf{X}^i\|_F}{\sqrt{N}} \\ & \leq L \left(1 + 2 \frac{\|\mathbf{X}^0\|_F^{k-1}}{n^{(k-1)/2}} + \sum_{i=1}^t \frac{\|\mathbf{X}^i - \mathbf{X}^{\epsilon,i}\|_F^{k-1}}{n^{(k-1)/2}} + 2 \sum_{i=1}^t \frac{\|\mathbf{X}^{\epsilon,i}\|_F^{k-1}}{n^{(k-1)/2}} \right) \sum_{i=1}^t \frac{\|\mathbf{X}^{\epsilon,i} - \mathbf{X}^i\|_F}{\sqrt{N}} \\ & \leq L \left(1 + 2C_0^{k-1} + \sum_{i=1}^t h_i(\epsilon)^{k-1} + 2 \sum_{i=1}^t C_{\epsilon\mathbf{Y},t}^{k-1} \right) \sum_{i=1}^t h_i(\epsilon) \quad \text{w.h.p.} \end{aligned}$$

where we used assumption (B3) for the convergence of $\|\mathbf{X}_0\|_F/\sqrt{N}$ to a finite constant, the well-defined state evolution of the perturbed AMP $\{\mathbf{X}^{\epsilon\mathbf{Y},t}, \mathbf{M}^{\epsilon\mathbf{Y},t} | f_{\epsilon\mathbf{Y}}^t, \mathbf{X}^0\}$ for convergence of $\|\mathbf{X}^{\epsilon,i}\|_F/\sqrt{N}$ to finite constants $C_{\epsilon\mathbf{Y},t}$ and Lemma 11 to replace the differences $\|\mathbf{X}^{\epsilon,i} - \mathbf{X}^i\|_F$ by the functions $h_i(\epsilon)$ with high probability. This gives, for any $\eta > 0$:

$$\lim_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P} \left(|\Phi_N(\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^t) - \Phi_N(\mathbf{X}^0, \mathbf{X}^{\epsilon\mathbf{Y},1}, \dots, \mathbf{X}^{\epsilon\mathbf{Y},t})| \geq \frac{\eta}{3} \right) = 0 \quad (50)$$

The state evolution for the perturbed AMP then gives

$$\lim_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P} \left(|\Phi_N(\mathbf{X}^0, \mathbf{X}^{\epsilon\mathbf{Y},1}, \dots, \mathbf{X}^{\epsilon\mathbf{Y},t}) - \mathbb{E} [\Phi_N(\mathbf{X}^0, \mathbf{Z}^{\epsilon,1}, \dots, \mathbf{Z}^{\epsilon,t})]| \geq \frac{\eta}{3} \right) = 0 \quad (51)$$

and Lemma 9 guarantees:

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left(|\mathbb{E} [\Phi_N(\mathbf{X}^0, \mathbf{Z}^{\epsilon\mathbf{Y},1}, \dots, \mathbf{Z}^{\epsilon\mathbf{Y},t})] - \mathbb{E} [\Phi_N(\mathbf{X}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t)]| \geq \frac{\eta}{3} \right) = 0 \quad (52)$$

for all N . From this we deduce

$$\mathbb{P}(|\Phi_N(\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^t) - \mathbb{E}[\Phi_N(\mathbf{X}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t)]| \geq \eta) \xrightarrow{N \rightarrow \infty} 0 \quad (53)$$

which is the desired result. \square

C.3 Proof of intermediate lemmas

Those proofs which are too close to the ones appearing in [BMN20] are not reminded.

Proof of Lemma 3. Recall the σ -algebra $\mathfrak{S}_t = \sigma(\mathbf{H}^1, \mathbf{H}^2, \dots, \mathbf{H}^t)$. The LongAMP iteration verifies:

$$\mathbf{H}^{t+1} = (\text{Id} - \mathbf{P}_{\mathcal{Q}_{t-1}}) \mathbf{A} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t + \mathcal{H}_{t-1} \alpha^t \quad (54)$$

$$= \mathbf{A} \mathbf{Q}_\perp^t - \mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{A} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t \quad (55)$$

where $\mathbf{Q}_\perp^t = \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t$. We now show by an induction that conditioning on \mathfrak{S}_t is equivalent to conditioning on the linear observations $\mathbf{A} \mathbf{Q}^0, \mathbf{A} \mathbf{Q}^1, \dots, \mathbf{A} \mathbf{Q}^t$, and thus to conditioning on $\mathbf{A} \mathbf{Q}_{t-1}$. Consider the first iteration which initializes the induction:

$$\mathbf{H}^1 = \mathbf{A} \mathbf{Q}^0 \quad (56)$$

thus \mathbf{H}^1 is $\sigma(\mathbf{A} \mathbf{Q}^0)$ -measurable. Suppose now that \mathcal{H}_{t-1} is $\sigma(\mathbf{A} \mathbf{Q}_{t-1})$ -measurable. The LongAMP iteration then gives, remembering that $\mathbf{Q}_\parallel^t = \mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{Q}^t$:

$$\mathbf{H}^{t+1} = \mathbf{A} \mathbf{Q}^t - \underbrace{\mathbf{A} \mathbf{Q}_\parallel^t - \mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{A} \mathbf{Q}_\perp^t}_{\sigma(\mathbf{A} \mathbf{Q}_{t-1})\text{-measurable}} + \mathcal{H}_{t-1} \alpha^t \quad (57)$$

where the highlighted term is $\sigma(\mathbf{A} \mathbf{Q}_{t-1})$ -measurable by definition of \mathbf{Q}_\parallel^t and the induction hypothesis. This gives that \mathcal{H}_t is $\sigma(\mathbf{A} \mathbf{Q}_t)$ -measurable. We can now condition on the linear observation $\mathbf{A} \mathbf{Q}_{t-1}$ at each iteration. We thus have:

$$\mathbf{H}^{t+1} |_{\mathfrak{S}_t} \stackrel{d}{=} \mathbf{A} |_{\mathfrak{S}_t} \mathbf{Q}_\perp^t - \mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{A} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t \quad (58)$$

which amounts to condition the Gaussian space generated by the entries of \mathbf{A} on its subspace defined by the linear combinations $\mathbf{A} \mathbf{Q}_{t-1}$. Conditioning in Gaussian spaces amounts to doing orthogonal projections, which gives

$$\mathbf{A} |_{\mathfrak{S}_t} = \mathbb{E}[\mathbf{A} |_{\mathfrak{S}_t}] + \mathcal{P}_t(\tilde{\mathbf{A}}) \quad (59)$$

as shown in [BM11a],[JM13], where $\tilde{\mathbf{A}}$ is a copy of \mathbf{A} , independent of \mathfrak{S}_t and \mathcal{P}_t is the projector onto the subspace $\{\hat{\mathbf{A}} \in \mathbb{R}^{N \times N} | \hat{\mathbf{A}} \mathbf{Q}_{t-1} = 0, \hat{\mathbf{A}} = \hat{\mathbf{A}}^\top\}$:

$$\mathbb{E}[\mathbf{A} |_{\mathfrak{S}_t}] = \mathbf{A} - \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{A} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \quad (60)$$

$$\mathcal{P}_t(\tilde{\mathbf{A}}) = \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \quad (61)$$

where $\tilde{\mathbf{A}}$ is an independent copy of \mathbf{A} . Replacing in the original LongAMP iteration, we get :

$$\mathbf{H}^{t+1} |_{\mathfrak{S}_t} \stackrel{d}{=} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t + \mathcal{H}_{t-1} \alpha^t \quad (62)$$

where we used $\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbb{E}[\mathbf{A} |_{\mathfrak{S}_t}] \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp = 0$. \square

Proof of Lemma 4. We proceed by induction over t . Let S_t be the property at time t .

Initialization.

a) We have $\mathbf{H}^1 = \mathbf{A}\mathbf{Q}^0$. Then:

$$\begin{aligned} \frac{1}{N}(\mathbf{H}^1)^\top \mathbf{H}^1 &= \frac{1}{N}(\mathbf{A}\mathbf{Q}^0)^\top (\mathbf{A}\mathbf{Q}^0) \\ &\stackrel{P}{\simeq} \frac{1}{N}(\mathbf{Q}^0)^\top \mathbf{Q}^0 \end{aligned} \quad (63)$$

using Lemma 17. We then define $\boldsymbol{\kappa}^{1,1} = \frac{1}{N}(\mathbf{Q}^0)^\top \mathbf{Q}^0$.

b) We want to show that $\Phi_N(\mathbf{X}^0, \mathbf{H}^1) \stackrel{P}{\simeq} \mathbb{E}[\Phi_N(\mathbf{X}^0, \mathbf{Z}^1)]$ where $\mathbf{Z}^1 \sim \mathbf{N}(0, \boldsymbol{\kappa}^{1,1})$, where

$$\boldsymbol{\kappa}^{1,1} = \frac{1}{N}(\mathbf{Q}^0)^\top \mathbf{Q}^0 = \frac{1}{N} (f^0(\mathbf{X}^0))^\top f^0(\mathbf{X}^0) \quad (64)$$

For any sequence $\{\Phi_N\}_{N \in \mathbb{N}}$ of order k pseudo-Lipschitz function

$$\begin{aligned} \|\Phi_N(\mathbf{X}_0, \mathbf{A}\mathbf{Q}^0) - \mathbb{E}[\Phi_N(\mathbf{Z}^1)]\|_2 &\leq \|\Phi_N(\mathbf{A}\mathbf{Q}^0) - \Phi_N(\mathbf{Z}^1)\|_2 + \|\Phi_N(\mathbf{Z}^1) - \mathbb{E}[\Phi_N(\mathbf{Z}^1)]\|_2 \\ &\leq L_n \left(1 + \left(\frac{\|\mathbf{A}\mathbf{Q}^0\|_2}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\mathbf{Z}^1\|}{\sqrt{N}} \right)^{k-1} \right) \frac{\|\mathbf{A}\mathbf{Q}^0 - \mathbf{Z}^1\|_2}{\sqrt{N}} + \|\Phi_N(\mathbf{Z}^1) - \mathbb{E}[\Phi_N(\mathbf{Z}^1)]\|_2 \end{aligned} \quad (65)$$

where the large n limit of $\left(\frac{\|\mathbf{A}\mathbf{Q}^0\|_2}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\mathbf{Z}^1\|}{\sqrt{N}} \right)^{k-1}$ being bounded, $\frac{\|\mathbf{A}\mathbf{Q}^0 - \mathbf{Z}^1\|_2}{\sqrt{N}} \xrightarrow[n \rightarrow \infty]{a.s.} 0$ and $\|\Phi_N(\mathbf{Z}^1) - \mathbb{E}[\Phi_N(\mathbf{Z}^1)]\|_2 \xrightarrow[n \rightarrow \infty]{P} 0$ follow from Lemmas 16 and 17.

Induction. Here we assume that S_0, S_1, \dots, S_{t-1} are verified, and we prove S_t .

a) Consider the case $s < t$. Since \mathbf{H}^{s+1} and $\langle \mathbf{Q}^s, \mathbf{Q}^t \rangle$ are \mathfrak{S}_t measurable, using the conditioning lemma, we have :

$$\begin{aligned} \left((\mathbf{H}^{s+1})^\top \mathbf{H}^{t+1} - (\mathbf{Q}^s)^\top \mathbf{Q}^t \right) |_{\mathfrak{S}_t} &\stackrel{d}{=} \left((\mathbf{H}^{s+1})^\top \mathbf{H}^{t+1} |_{\mathfrak{S}_t} - (\mathbf{Q}^s)^\top \mathbf{Q}^t \right) \\ &= (\mathbf{H}^{s+1})^\top (\mathbf{P}_{\mathfrak{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{P}_{\mathfrak{Q}_{t-1}}^\perp \mathbf{Q}^t + \mathcal{H}_{t-1} \boldsymbol{\alpha}^t) - (\mathbf{Q}^s)^\top \mathbf{Q}^t \\ &= (\mathbf{H}^{s+1})^\top \mathbf{P}_{\mathfrak{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t + (\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t - (\mathbf{Q}^s)^\top \mathbf{Q}^t \end{aligned} \quad (66)$$

We thus have :

$$\begin{aligned} \frac{1}{N} \left\| \left((\mathbf{H}^{s+1})^\top \mathbf{H}^{t+1} - (\mathbf{Q}^s)^\top \mathbf{Q}^t \right) |_{\mathfrak{S}_t} \right\|_F &\leq \frac{1}{N} \left\| (\mathbf{H}^{s+1})^\top \mathbf{P}_{\mathfrak{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t \right\|_F \\ &\quad + \frac{1}{N} \left\| (\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t - (\mathbf{Q}^s)^\top \mathbf{Q}^t \right\|_F \end{aligned} \quad (67)$$

Starting with the term

$$\frac{1}{N} \left\| (\mathbf{H}^{s+1})^\top \mathbf{P}_{\mathfrak{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t \right\|_F = \frac{1}{N} \left\| (\mathbf{P}_{\mathfrak{Q}_{t-1}}^\perp \mathbf{H}^{s+1})^\top \tilde{\mathbf{A}} \mathbf{Q}_\perp^t \right\|_F \quad (68)$$

the induction ensures that $\frac{1}{\sqrt{N}} \|\mathbf{H}^{s+1}\|_F, \frac{1}{\sqrt{N}} \|\mathbf{Q}_\perp^t\|_F$ concentrate to finite values. Furthermore, $\|\mathbf{P}_{\mathfrak{Q}_{t-1}}^\perp \mathbf{H}^{s+1}\|_F \leq \|\mathbf{H}^{s+1}\|_F$, so according to Lemma 17, the first term on the right-hand-side

will concentrate to zero.

Moving to the second term, since $s < t$, $\mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{Q}^s = \mathbf{Q}^s$. Then:

$$\begin{aligned}
\frac{1}{N} \|(\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t - (\mathbf{Q}^s)^\top \mathbf{Q}^t\|_F &= \frac{1}{N} \|(\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t - (\mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{Q}^s)^\top \mathbf{Q}^t\|_F \\
&= \frac{1}{N} \|(\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t - (\mathbf{Q}^s)^\top \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top \mathbf{Q}^t\|_F \\
&= \frac{1}{N} \|(\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t - (\mathbf{Q}^s)^\top \mathcal{Q}_{t-1} \boldsymbol{\alpha}_t\|_F \\
&\leq \frac{1}{N} \|(\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} - (\mathbf{Q}^s)^\top \mathcal{Q}_{t-1}\|_F \|\boldsymbol{\alpha}_t\|_F \tag{69}
\end{aligned}$$

Here we consider $s < t$ thus $s+1 \leq t$. Hence the induction hypothesis includes the concentration properties of \mathbf{H}^{s+1} and $\boldsymbol{\alpha}_t$. We then have $\lim_{N \rightarrow \infty} \frac{1}{N} \|(\mathbf{H}^{s+1})^\top \mathcal{H}_{t-1} - (\mathbf{Q}^s)^\top \mathcal{Q}_{t-1}\|_F \rightarrow 0$ and $\|\boldsymbol{\alpha}_t\|_F$ has a finite and well-defined limit using the non-degeneracy assumption. Indeed:

$$\begin{aligned}
\|\boldsymbol{\alpha}_t\|_F &= \|(\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top \mathbf{Q}^t\|_F \\
&\leq \frac{1}{N c_t^2} \mathcal{Q}_{t-1}^\top \mathbf{Q}^t \tag{70}
\end{aligned}$$

using the induction hypothesis, $\lim_{n \rightarrow +\infty} \frac{1}{N} \mathcal{Q}_{t-1}^\top \mathbf{Q}^t$ is finite. This proves the property for $s < t$. Now consider the case $s = t$. We then have:

$$\begin{aligned}
(\|\mathbf{H}^{t+1}\|_F^2 - \|\mathbf{Q}^t\|_F^2) |_{\mathfrak{S}_t} &= (\|\mathbf{H}^{t+1}|_{\mathfrak{S}_t}\|_F^2 - \|\mathbf{Q}^t\|_F^2) \\
&= \|\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F^2 + 2\text{Tr} \left(\left(\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t \right)^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \right) + \|\mathcal{H}_{t-1} \boldsymbol{\alpha}^t\|_F^2 - \|\mathbf{Q}^t\|_F^2 \tag{71}
\end{aligned}$$

We then have

$$\frac{1}{N} \|\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F^2 = \frac{1}{N} \|\tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F^2 - \frac{1}{N} \|\mathbf{P}_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F^2 \stackrel{P}{\simeq} \frac{1}{N} \|\mathbf{Q}_\perp^t\|_F^2 \tag{72}$$

where we used

$$\frac{1}{N} \|\tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F^2 \stackrel{P}{\simeq} \frac{1}{N} \|\mathbf{Q}_\perp^t\|_F^2 \quad \text{and} \quad \frac{1}{N} \|\mathbf{P}_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F^2 \xrightarrow[n \rightarrow \infty]{P} 0 \tag{73}$$

which follows from Lemma 17 and the independence of $\tilde{\mathbf{A}}$. The second term then reads

$$\left(\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t \right)^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t = (\mathbf{Q}_\perp^t)^\top \tilde{\mathbf{A}} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \tag{74}$$

From the induction hypothesis, we know that $\boldsymbol{\alpha}^t$ has finite norm when $N \rightarrow \infty$. Moreover, $\|\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathcal{H}_{t-1} \boldsymbol{\alpha}^t\|_F \leq \|\mathcal{H}_{t-1} \boldsymbol{\alpha}^t\|_F$, and $\|\mathbf{Q}_\perp^t\|_F \leq \|\mathbf{Q}^t\|_F$. Also $\frac{1}{\sqrt{N}} \|\mathcal{H}_{t-1}\|_F$ and $\frac{1}{\sqrt{N}} \|\mathbf{Q}^t\|_F$ converge to finite constants, again according to the induction hypothesis. Using Lemma 17, we get

$$\frac{1}{N} \text{Tr} \left(\left(\mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{Q}_\perp^t \right)^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \right) \xrightarrow[n \rightarrow \infty]{P} 0 \tag{75}$$

Finally the third term can be decomposed

$$\begin{aligned}
\|\mathcal{H}_{t-1} \boldsymbol{\alpha}^t\|_F^2 &= \text{Tr}((\mathcal{H}_{t-1} \boldsymbol{\alpha}^t)^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t) \\
&= \text{Tr}((\boldsymbol{\alpha}^t)^\top \mathcal{H}_{t-1}^\top \mathcal{H}_{t-1} \boldsymbol{\alpha}^t) \\
&= \text{Tr}((\boldsymbol{\alpha}^t)^\top \left(\mathcal{H}_{t-1}^\top \mathcal{H}_{t-1} - \mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1} \right) \boldsymbol{\alpha}^t) + \text{Tr}((\boldsymbol{\alpha}^t)^\top \mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1} \boldsymbol{\alpha}^t) \\
&\leq \|\mathcal{H}_{t-1}^\top \mathcal{H}_{t-1} - \mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1}\|_F \|\boldsymbol{\alpha}_t\|_F + \|\mathcal{Q}_{t-1} \boldsymbol{\alpha}_t\|_F \tag{76}
\end{aligned}$$

Using the induction hypothesis and the non-degeneracy assumption, $\lim_{N \rightarrow \infty} \|\alpha^t\|_F$ is a finite constant, and $\frac{1}{N} \|\mathcal{H}_{t-1}^\top \mathcal{H}_{t-1} - \mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1}\|_F \xrightarrow[n \rightarrow \infty]{P} 0$. Furthermore, by definition of α_t , $\mathcal{Q}_{t-1} \alpha_t = \mathbf{Q}_t^\parallel$.

Grouping all the terms, we get

$$\begin{aligned} \frac{1}{N} (\|\mathbf{H}^{t+1}\|_F^2 - \|\mathbf{Q}^t\|_F^2) |_{\mathfrak{S}_t} &\simeq \frac{P}{N} \|\mathbf{Q}_\perp^t\|_F^2 + \frac{1}{N} \|\mathbf{Q}_\parallel^t\|_F^2 - \frac{1}{N} \|\mathbf{Q}^t\|_F^2 \\ &= 0 \end{aligned} \quad (77)$$

b) Using the conditioning lemma :

$$\begin{aligned} \Phi_N(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^t, \mathbf{H}^{t+1}) |_{\mathfrak{S}_t} &\stackrel{d}{=} \Phi_N(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^t, \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \tilde{\mathbf{A}} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t + \mathcal{H}_{t-1} \alpha^t) \\ &= \Phi_N(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^t, \tilde{\mathbf{A}} \mathbf{Q}_\perp^t - P_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t) \end{aligned} \quad (78)$$

Let $\Phi'_N(\tilde{\mathbf{A}} \mathbf{Q}_\perp^t - P_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t) = \Phi_N(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^t, \tilde{\mathbf{A}} \mathbf{Q}_\perp^t - P_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t)$ as a shorthand. Then, from the pseudo-Lipschitz property:

$$\begin{aligned} &|\Phi'_N(\tilde{\mathbf{A}} \mathbf{Q}_\perp^t - P_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t) - \Phi'_N(\tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t)| \\ &\leq L_N C(k, t) \left(1 + \left(\frac{\|\mathbf{X}^0\|_F}{\sqrt{N}} \right)^{k-1} + \sum_{s=1}^t \left(\frac{\|\mathbf{H}^s\|_F}{\sqrt{N}} \right)^{k-1} \right. \\ &\quad \left. + \left(\frac{\|\mathbf{H}^{t+1}\|_F}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\mathcal{H}_{t-1} \alpha^t\|_F}{\sqrt{N}} \right)^{k-1} \right) \frac{\|P_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F}{\sqrt{N}} \end{aligned} \quad (79)$$

where $C(k, t)$ is a constant depending only on k and t . The induction hypothesis ensures that $\left(\frac{\|\mathbf{X}^0\|_F}{\sqrt{N}} \right)^{k-1} + \sum_{s=1}^t \left(\frac{\|\mathbf{H}^s\|_F}{\sqrt{N}} \right)^{k-1}$ converges to a finite constant. Furthermore,

$$\frac{1}{\sqrt{N}} \|\tilde{\mathbf{A}}\|_F \leq \frac{1}{\sqrt{N}} \|\tilde{\mathbf{A}}\|_{op} \|\mathbf{Q}^t\|_F \quad (80)$$

which, using Proposition 2 and the induction hypothesis, converges to a finite constant. Also, using the fact that $\text{rank}(\mathbf{P}_{\mathcal{Q}_{t-1}}) \leq tq$ with t, q finite, and the independence of $\tilde{\mathbf{A}}$, Lemma 17 gives

$$\frac{1}{\sqrt{N}} \|P_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t\|_F \xrightarrow[n \rightarrow \infty]{P} 0. \quad (81)$$

Ultimately, we obtain

$$\begin{aligned} \Phi'_N(\tilde{\mathbf{A}} \mathbf{Q}_\perp^t - P_{\mathcal{Q}_{t-1}} \tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t) &\stackrel{P}{\simeq} \Phi'_N(\tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^t) \\ &\stackrel{P}{\simeq} \Phi'_N(\tilde{\mathbf{A}} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \alpha^{t,*}) \end{aligned} \quad (82)$$

where $\alpha_t^* = \lim_{N \rightarrow \infty} \alpha_t$ which are finite matrices, and $\alpha_t^* \in \mathbb{R}^{tq \times q}$. We write :

$$\begin{bmatrix} (\alpha_t^*)_1 \\ \vdots \\ (\alpha_t^*)_t \end{bmatrix} \quad (83)$$

where $\forall 1 \leq i \leq t$, $(\boldsymbol{\alpha}_i^*)_i \in \mathbb{R}^{q \times q}$. Then

$$\begin{aligned} \Phi'_N \left(\tilde{\mathbf{A}}\mathbf{Q}_\perp^t + \mathcal{H}_{t-1}\boldsymbol{\alpha}^{t,*} \right) &\stackrel{P}{\simeq} \Phi'_N \left(\tilde{\mathbf{A}}\mathbf{Q}_\perp^t + \mathcal{H}_{t-1}\boldsymbol{\alpha}^{t,*} \right) \\ &\stackrel{P}{\simeq} \Phi(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^t, \tilde{\mathbf{A}}\mathbf{Q}_\perp^t + \mathcal{H}_{t-1}\boldsymbol{\alpha}^{t,*}) \end{aligned} \quad (84)$$

Using Lemma 16, there exists $\mathbf{Z}_\perp^{t+1} \sim \mathbf{N}(0, \boldsymbol{\kappa}_\perp^{t+1} \otimes I_N)$ independent of \mathfrak{S}_t , where $\boldsymbol{\kappa}_\perp^{t+1} = \lim_{N \rightarrow \infty} \frac{1}{N} (\mathbf{Q}_\perp^t)^\top \mathbf{Q}_\perp^t$, such that:

$$\begin{aligned} \Phi(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^t, \tilde{\mathbf{A}}\mathbf{Q}_\perp^t + \mathcal{H}_{t-1}\boldsymbol{\alpha}^{t,*}) &\stackrel{P}{\simeq} \mathbb{E}_{\mathbf{Z}} \left[\Phi(\mathbf{X}^0, \mathbf{H}^1, \dots, \mathbf{H}^t, \mathbf{Z}_\perp^{t+1} + \mathcal{H}_{t-1}\boldsymbol{\alpha}^{t,*}) \right] \\ &\stackrel{P}{\simeq} \mathbb{E} \left[\Phi_N(\mathbf{X}^0, \mathbf{Z}^1, \dots, \mathbf{Z}^t, \mathbf{Z}_\perp^{t+1} + \sum_{i=1}^t \mathbf{Z}^i(\boldsymbol{\alpha}^{t,*})_i) \right] \end{aligned} \quad (85)$$

We now need to match the covariance matrices defined by the prescription of \mathbf{Z}^{t+1} we obtained with the ones from the state evolution. Let $\mathbf{Z}^{t+1} = \mathbf{Z}_\perp^{t+1} + \sum_{i=1}^t \mathbf{Z}^i(\boldsymbol{\alpha}^{t,*})_i \in \mathbb{R}^{q \times q}$. We then write $\mathbf{Z}^{t+1} \sim \mathbf{N}(0, \boldsymbol{\kappa}^{t+1, t+1} \otimes \mathbf{I}_N)$ where $\boldsymbol{\kappa}^{t+1, t+1} = \lim_{N \rightarrow \infty} \frac{1}{N} (\mathbf{Z}^{t+1})^\top \mathbf{Z}^{t+1}$. Then, using the isometry proved above and remembering that, for any $1 \leq i \leq t$, $\mathbf{Q}^t = f^t(\mathbf{H}^t)$:

$$\frac{1}{N} (\mathbf{Z}^{t+1})^\top \mathbf{Z}^{t+1} \stackrel{P}{\simeq} \frac{1}{N} (\mathbf{H}^{t+1})^\top \mathbf{H}^{t+1} \stackrel{P}{\simeq} \frac{1}{N} (\mathbf{Q}^t)^\top \mathbf{Q}^t \xrightarrow[n \rightarrow \infty]{P} \boldsymbol{\kappa}^{t+1, t+1} \quad (86)$$

similarly, for $s \geq 2$:

$$\boldsymbol{\kappa}^s = \frac{1}{N} (\mathbf{Z}^s)^\top \mathbf{Z}^{t+1} \stackrel{P}{\simeq} \frac{1}{N} (\mathbf{H}^s)^\top \mathbf{H}^{t+1} \stackrel{P}{\simeq} \frac{1}{N} (\mathbf{Q}^{s-1})^\top \mathbf{Q}^t \xrightarrow[n \rightarrow \infty]{P} \boldsymbol{\kappa}^{s, t+1} \quad (87)$$

and for $s = 1$:

$$\boldsymbol{\kappa}^s = \frac{1}{N} (\mathbf{Z}^1)^\top \mathbf{Z}^{t+1} \stackrel{P}{\simeq} \frac{1}{N} (\mathbf{H}^1)^\top \mathbf{H}^{t+1} \stackrel{P}{\simeq} \frac{1}{N} (\mathbf{Q}^0)^\top \mathbf{Q}^t \xrightarrow[n \rightarrow \infty]{P} \boldsymbol{\kappa}^{1, t+1} \quad (88)$$

□

Proof of Lemma 5. This lemma is proven by induction.

Initialization. The first iterates read $\mathbf{H}^1 = \mathbf{A}\mathbf{Q}^0$ and $\hat{\mathbf{H}}^1 = \mathbf{A}\mathbf{Q}^0$. This concludes the initialization.

Induction. Assume the proposition is true up to time t . Define the $(t+1)q \times (t+1)q$ block-diagonal matrix $\mathcal{B}_t = \text{diag}(0_{q \times q}, \mathbf{b}^1, \dots, \mathbf{b}^t)$ and $\hat{\mathcal{H}}_{t-1} = \begin{bmatrix} \hat{\mathbf{H}}^1 & & \\ & \hat{\mathbf{H}}^2 & \\ & & \dots \\ & & & \hat{\mathbf{H}}^t \end{bmatrix}$. We then have :

$$\begin{aligned} \mathbf{H}^{t+1} &= \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{A} \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t + \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \\ &= \mathbf{A} \mathbf{Q}_\perp^t - \mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{A} \mathbf{Q}_\perp^t + \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \end{aligned} \quad (89)$$

and

$$\begin{aligned} \hat{\mathbf{H}}^{t+1} &= \mathbf{A} \mathbf{Q}^t - \mathbf{Q}^{t-1} (\mathbf{b}^t)^\top \\ &= \mathbf{A} \mathbf{Q}_\perp^t + \mathbf{A} \mathbf{Q}_\parallel^t - \mathbf{Q}^{t-1} (\mathbf{b}^t)^\top \\ \text{where } \mathbf{A} \mathbf{Q}_\parallel^t &= \mathbf{A} \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top \mathbf{Q}^t \\ &= \mathbf{A} \mathcal{Q}_{t-1} \boldsymbol{\alpha}^t \end{aligned} \quad (90)$$

which gives

$$\hat{\mathbf{H}}^{t+1} - \mathbf{H}^{t+1} = \mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{A} \mathbf{Q}_{\perp}^t - \mathbf{Q}^{t-1} (\mathbf{b}^t)^\top + \mathbf{A} \mathcal{Q}_{t-1} \boldsymbol{\alpha}^t - \mathcal{H}_{t-1} \boldsymbol{\alpha}^t \quad (91)$$

using the definition of iteration (40), we have:

$$\mathbf{A} \mathcal{Q}_{t-1} = \hat{\mathcal{H}}_{t-1} + [0_{N \times q} | \mathbf{Q}^0 | \dots | \mathbf{Q}^{t-2}] \mathcal{B}_{t-1}^\top \quad (92)$$

$$\begin{aligned} \hat{\mathbf{H}}^{t+1} - \mathbf{H}^{t+1} &= \mathbf{P}_{\mathcal{Q}_{t-1}} \mathbf{A} \mathbf{Q}_{\perp}^t - \mathbf{Q}^{t-1} (\mathbf{b}^{t-1})^\top + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \boldsymbol{\alpha}^t + \left(\hat{\mathcal{H}}^{t-1} - \mathcal{H}^{t-1} \right) \boldsymbol{\alpha}^t \\ &= \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top \mathbf{A} \mathbf{Q}_{\perp}^t - \mathbf{Q}^{t-1} (\mathbf{b}^{t-1})^\top + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \boldsymbol{\alpha}^t \\ &\quad + \left(\hat{\mathcal{H}}^{t-1} - \mathcal{H}^{t-1} \right) \boldsymbol{\alpha}^t \end{aligned} \quad (93)$$

and

$$\begin{aligned} \mathcal{Q}_{t-1}^\top \mathbf{A} &= (\mathbf{A} \mathcal{Q}_{t-1})^\top \\ &= \left(\hat{\mathcal{H}}_{t-1} + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_t^\top \right)^\top \\ &= \hat{\mathcal{H}}_{t-1}^\top + \mathcal{B}_t [0_{N \times q} | \mathcal{Q}_{t-2}]^\top \end{aligned} \quad (94)$$

since $\mathbf{Q}_{\perp}^t = \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t$, it holds that:

$$\begin{aligned} \mathcal{Q}_{t-1}^\top \mathbf{A} \mathbf{Q}_{\perp}^t &= \left(\hat{\mathcal{H}}_{t-1}^\top + \mathcal{B}_t [0_{N \times q} | \mathcal{Q}_{t-2}]^\top \right) \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t \\ &= \hat{\mathcal{H}}_{t-1}^\top \mathbf{P}_{\mathcal{Q}_{t-1}}^\perp \mathbf{Q}^t \end{aligned} \quad (95)$$

which in turn gives:

$$\begin{aligned} \hat{\mathbf{H}}^{t+1} - \mathbf{H}^{t+1} &= \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \hat{\mathcal{H}}_{t-1}^\top \mathbf{Q}_{\perp}^t - \mathbf{Q}^{t-1} (\mathbf{b}^{t-1})^\top + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \boldsymbol{\alpha}^t \\ &\quad + \left(\hat{\mathcal{H}}^{t-1} - \mathcal{H}^{t-1} \right) \boldsymbol{\alpha}^t \\ &= \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{H}_{t-1}^\top \mathbf{Q}_{\perp}^t - \mathbf{Q}^{t-1} (\mathbf{b}^{t-1})^\top + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \boldsymbol{\alpha}^t \\ &\quad + \left(\hat{\mathcal{H}}^{t-1} - \mathcal{H}^{t-1} \right) \boldsymbol{\alpha}^t + \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \left(\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1} \right)^\top \mathbf{Q}_{\perp}^t \end{aligned} \quad (96)$$

We now study the limiting behaviour of this quantity, starting with:

$$\mathbf{C} = \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{H}_{t-1}^\top \mathbf{Q}_{\perp}^t - \mathbf{Q}^{t-1} (\mathbf{b}^{t-1})^\top + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \boldsymbol{\alpha}^t \quad (97)$$

We have :

$$\begin{aligned} \mathbf{Q}_{\perp}^t &= \mathbf{Q}^t - \mathbf{Q}_{\parallel}^t \\ &= \mathbf{Q}^t - \mathcal{Q}_{t-1} \boldsymbol{\alpha}_t \end{aligned} \quad (98)$$

and :

$$\mathbf{C} = \mathcal{Q}_{t-1} (\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{H}_{t-1}^\top (\mathbf{Q}^t - \mathcal{Q}_{t-1} \boldsymbol{\alpha}_t) - \mathbf{Q}^{t-1} (\mathbf{b}^{t-1})^\top + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \boldsymbol{\alpha}^t \quad (99)$$

Using Lemma 12, the state evolution, and the concentration properties of pseudo-Lipschitz functions Lemma 16, we get, for all $1 \leq j \leq t-1$ and $1 \leq i \leq t$:

$$\begin{aligned} \frac{1}{N}(\mathbf{H}^i)^\top f^j(\mathbf{H}^j) &\stackrel{P}{\simeq} \mathbb{E} \left[\frac{1}{N}(\mathbf{Z}^i)^\top f^j(\mathbf{Z}^j) \right] \\ &= \mathbf{K}_{i,j} \mathbb{E} \left[\frac{1}{N} \operatorname{div} f^j(\mathbf{Z}^j) \right] \\ &\stackrel{P}{\simeq} \frac{1}{N}(\mathbf{Q}^{i-1})^\top \mathbf{Q}^{j-1}(\mathbf{b}^j)^\top \end{aligned} \quad (100)$$

and for $j = 0$:

$$\begin{aligned} \frac{1}{N}(\mathbf{H}^i)^\top f(\mathbf{X}^0) &\stackrel{P}{\simeq} \mathbb{E} \left[\frac{1}{N}(\mathbf{Z}^i)^\top f_0(\mathbf{X}^0) \right] \\ &= 0 \end{aligned} \quad (101)$$

which in turn gives

$$\frac{1}{N}(\mathcal{H}_{t-1}^\top \mathbf{Q}^t) = \frac{1}{N}[\mathbf{H}^1 | \dots | \mathbf{H}^t]^\top f_t(\mathbf{H}^t) \stackrel{P}{\simeq} \frac{1}{N}(\mathcal{Q}_{t-1})^\top \mathbf{Q}^{t-1}(\mathbf{b}^{t-1})^\top \quad (102)$$

and

$$\frac{1}{N} \mathcal{H}_{t-1}^\top \mathcal{Q}_{t-1} = \frac{1}{N}[\mathbf{H}^1 | \dots | \mathbf{H}^t]^\top [\mathbf{Q}^0 | f_1(\mathbf{H}^1) | \dots | f_{t-1}(\mathbf{H}^{t-1})] \stackrel{P}{\simeq} \frac{1}{N} \mathcal{Q}_{t-1}^\top [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \quad (103)$$

which gives :

$$\begin{aligned} \frac{1}{N} \mathbf{C} &\stackrel{P}{\simeq} \frac{1}{N} \left(\mathcal{Q}_{t-1}(\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top (\mathbf{Q}^{t-1}(\mathbf{b}^{t-1})^\top - [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \alpha^t) \right. \\ &\quad \left. - \mathbf{Q}^{t-1}(\mathbf{b}^{t-1})^\top + [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \alpha^t \right) \end{aligned} \quad (104)$$

$$= \frac{1}{N} \left(\mathcal{Q}_{t-1}(\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \mathcal{Q}_{t-1}^\top \underbrace{(\mathbf{Q}^{t-1}(\mathbf{b}^{t-1})^\top - [0_{N \times q} | \mathcal{Q}_{t-2}] \mathcal{B}_{t-1}^\top \alpha^t)}_{\in \operatorname{span}(\mathcal{Q}_{t-1})} \right) \quad (105)$$

$$= 0 \quad (106)$$

At this point, we have :

$$\begin{aligned} \frac{1}{\sqrt{N}} \|\hat{\mathbf{H}}^{t+1} - \mathbf{H}^{t+1}\|_F &\leq \frac{1}{\sqrt{N}} \|\mathbf{C}\|_F + \frac{1}{\sqrt{N}} \left\| \left(\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1} \right) \alpha^t \right. \\ &\quad \left. + \mathcal{Q}_{t-1}(\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1} \left(\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1} \right)^\top \mathbf{Q}_\perp^t \right\|_F \end{aligned} \quad (107)$$

Where

$$\frac{1}{\sqrt{N}} \left\| \left(\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1} \right) \alpha^t \right\|_F \leq \frac{1}{\sqrt{N}} \|\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1}\|_F \|\alpha^t\|_F \quad (108)$$

As previously discussed, $\|\boldsymbol{\alpha}^t\|_F$ has a finite limit, and according to the induction hypothesis, $\frac{1}{\sqrt{N}}\|\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1}\|_F \xrightarrow[N \rightarrow \infty]{P} 0$. Then

$$\frac{1}{\sqrt{N}}\|\mathcal{Q}_{t-1}(\mathcal{Q}_{t-1}^\top \mathcal{Q}_{t-1})^{-1}(\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1})^\top \mathbf{Q}_\perp^t\|_F \leq \frac{1}{\sqrt{N}}\|\hat{\mathcal{H}}_{t-1} - \mathcal{H}_{t-1}\|_F \frac{1}{Nc_t^2}\|\mathcal{Q}_{t-1}\|_F\|\mathbf{Q}^t\|_F \quad (109)$$

where $\frac{1}{Nc_t^2}\|\mathcal{Q}_{t-1}\|_F\|\mathbf{Q}^t\|_F$ converges to a finite limit due to the state evolution proved above. This ultimately shows that

$$\frac{1}{\sqrt{N}}\|\hat{\mathbf{H}}^{t+1} - \mathbf{H}^{t+1}\|_F \xrightarrow[N \rightarrow \infty]{P} 0 \quad (110)$$

and concludes the induction. \square

Proof of Lemma 6. This one is another induction. Let S_t be the statement $\frac{1}{\sqrt{N}}\|\mathbf{Q}^t - \mathbf{M}^t\|_F \xrightarrow[N \rightarrow \infty]{P} 0$ and $\frac{1}{\sqrt{N}}\|\mathbf{H}^{t+1} - \mathbf{X}^{t+1}\|_F \xrightarrow[N \rightarrow \infty]{P} 0$.

Initialization. We have $\mathbf{Q}^0 = f^0(\mathbf{X}^0) = \mathbf{M}^0$ and $\mathbf{H}^1 = \mathbf{A}\mathbf{Q}^0, \mathbf{X}^1 = \mathbf{A}\mathbf{M}^0$.

Induction. We assume S_{t-1} is true, and we prove S_t . We have

$$\begin{aligned} \frac{1}{\sqrt{N}}\|\mathbf{Q}^t - \mathbf{M}^t\|_F &= \frac{1}{\sqrt{N}}\|f^t(\mathbf{H}^t) - f^t(\mathbf{X}^t)\|_F \\ &\leq L_t \left(1 + \left(\frac{\|\mathbf{H}^t\|_F}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\mathbf{X}^t\|_F}{\sqrt{N}} \right)^{k-1} \right) \frac{\|\mathbf{H}^t - \mathbf{X}^t\|_F}{\sqrt{N}} \end{aligned} \quad (111)$$

which goes to zero as n goes to infinity from the induction hypothesis. We then prove that $\frac{1}{\sqrt{N}}\|\hat{\mathbf{H}}^{t+1} - \mathbf{X}^{t+1}\|_F \xrightarrow[N \rightarrow \infty]{P} 0$.

$$\hat{\mathbf{H}}^{t+1} - \mathbf{X}^{t+1} = \mathbf{A}\mathbf{Q}^t - \mathbf{Q}^{t-1}(\mathbf{b}^t)^\top - \mathbf{A}\mathbf{M}^t + \mathbf{M}^{t-1}(\mathbf{b}^t)^\top \quad (112)$$

and

$$\frac{1}{\sqrt{N}}\|\hat{\mathbf{H}}^{t+1} - \mathbf{X}^{t+1}\|_F \leq \|\mathbf{A}\|_{op} \frac{1}{\sqrt{N}}\|\mathbf{Q}^t - \mathbf{M}^t\|_F + \frac{1}{\sqrt{N}}\|\mathbf{Q}^{t-1} - \mathbf{M}^{t-1}\|_F\|\mathbf{b}^t\|_F \quad (113)$$

using Proposition 2, $\|\mathbf{A}\|_{op} \xrightarrow[N \rightarrow \infty]{P} 2$. Using the induction hypothesis, $\frac{1}{\sqrt{N}}\|\mathbf{Q}^t - \mathbf{M}^t\|_F \xrightarrow[N \rightarrow \infty]{P} 0$, $\frac{1}{\sqrt{N}}\|\mathbf{Q}^{t-1} - \mathbf{M}^{t-1}\|_F \xrightarrow[N \rightarrow \infty]{P} 0$, and $\|\mathbf{b}^t\|_F$ is finite. This concludes the induction step. \square

Proof of Lemma 7. In this proof, we will consider the $2q \times 2q$ covariance matrix $\boldsymbol{\kappa} = \begin{bmatrix} \boldsymbol{\kappa}_{1,1} & \boldsymbol{\kappa}_{1,2} \\ \boldsymbol{\kappa}_{1,2} & \boldsymbol{\kappa}_{2,2} \end{bmatrix}$ and two matrices $\mathbf{Z}^1, \mathbf{Z}^2 \in (\mathbb{R}^{N \times q})^2$ following the distribution $\mathbf{N}(0, \boldsymbol{\kappa} \otimes \mathbf{I}_N)$, and we study the corresponding state evolution when the perturbed functions $f_{\epsilon\mathbf{Y}}^t$ are considered. We drop the ϵ

exponent on the covariance matrices since we are just studying the well-definiteness of the perturbed SE as an induction. The link with the original SE will be studied in subsequent lemmas.

$$\begin{aligned}
\mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f_{\epsilon \mathbf{Y}}^s(\mathbf{Z}^s))^\top f_{\epsilon \mathbf{Y}}^t(\mathbf{Z}^t) \right] &= \mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f^s(\mathbf{Z}^s))^\top f^t(\mathbf{Z}^t) \right] + \epsilon \mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f^s(\mathbf{Z}^s))^\top \mathbf{Y}^t \right] \\
&+ \epsilon \mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f^t(\mathbf{Z}^t))^\top \mathbf{Y}^s \right] + \epsilon^2 \frac{1}{N} (\mathbf{Y}^s)^\top \mathbf{Y}^t \\
&= \mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f^s(\mathbf{Z}^s))^\top f^t(\mathbf{Z}^t) \right] + \frac{\epsilon}{N} \mathbb{E}_{\mathbf{Z}} [f^s(\mathbf{Z}^s)]^\top \mathbf{Y}^t \\
&+ \frac{\epsilon}{N} \mathbb{E}_{\mathbf{Z}} [f^t(\mathbf{Z}^t)]^\top \mathbf{Y}^s + \frac{\epsilon^2}{N} (\mathbf{Y}^s)^\top \mathbf{Y}^t
\end{aligned}$$

- the first term does not depend on the perturbation and is deterministic. Using assumptions (A6), this quantity has a finite limit.
- second term is a $q \times q$ matrix where each element have zero mean and variance

$$\text{Var} \left[\frac{1}{N} \left(\mathbb{E} \left[f^s(\mathbf{Z}^s)^\top \mathbf{Y}^t \right]_j^i \right) \right] = \frac{1}{N^2} \|\mathbb{E} [f^s(\mathbf{Z}^s)]\|_2^2 \leq \frac{C}{N} \quad (114)$$

Using the Gaussian tail and the Borel-Cantelli lemma, this term converges almost surely to zero.

- the third term is treated in the same way as the second one
- the last term follows from the strong law of large numbers:

$$\lim_{N \rightarrow \infty} \frac{1}{N} (\mathbf{Y}^s)^\top \mathbf{Y}^t \xrightarrow[n \rightarrow \infty]{a.s.} \mathbf{I}_{q \times q} \delta_{s=t} \quad (115)$$

Putting things together, we get, almost surely:

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f_{\epsilon \mathbf{Y}}^s(\mathbf{Z}^s))^\top f_{\epsilon \mathbf{Y}}^t(\mathbf{Z}^t) \right] = \lim_{N \rightarrow \infty} \mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f^s(\mathbf{Z}^s))^\top f^t(\mathbf{Z}^t) \right] + \epsilon^2 \mathbf{I}_{q \times q} \delta_{s=t} \quad (116)$$

Verifying the initialization assumptions (A4-A5) is very similar to the previous steps, thus we directly give the result. The initialization reads:

$$\lim_{N \rightarrow \infty} \frac{1}{N} (f_{\epsilon \mathbf{Y}}^0(\mathbf{X}^0))^\top f_{\epsilon \mathbf{Y}}^0(\mathbf{X}^0) = \lim_{N \rightarrow \infty} \frac{1}{N} (f^0(\mathbf{X}^0))^\top f^0(\mathbf{X}^0) + \epsilon^2 \mathbf{I}_{q \times q} \quad (117)$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[(f_{\epsilon \mathbf{Y}}^0(\mathbf{X}^0))^\top f_{\epsilon \mathbf{Y}}^t(\mathbf{Z}^t) \right] = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[(f^0(\mathbf{X}^0))^\top f^t(\mathbf{Z}^t) \right] \quad (118)$$

It follows straightforwardly from these equations and a short induction that the resulting state evolution is almost surely non-random. \square

Proof of Lemma 8. By definition, for any $t \in \mathbb{N}$:

$$\mathbf{Q}^{t, \epsilon \mathbf{Y}} = \mathbf{Q}^t + \epsilon \mathbf{Y}^t \quad (119)$$

Then

$$\mathbf{Q}_{\perp}^{\epsilon \mathbf{Y}, t} = \mathbf{P}_{\mathcal{Q}_{t-1}^{\epsilon \mathbf{Y}}}^{\perp} f^t(\mathbf{H}^{\epsilon \mathbf{Y}, t}) + \epsilon \mathbf{P}_{\mathcal{Q}_{t-1}^{\epsilon \mathbf{Y}}}^{\perp} \mathbf{Y}^t \quad (120)$$

with the parallel term a linear combination of the previous ones. Denote \mathcal{F}_t the σ -algebra generated by $\mathbf{H}^{\epsilon\mathbf{Y},1}, \dots, \mathbf{H}^{\epsilon\mathbf{Y},t}, \mathbf{Y}^1, \dots, \mathbf{Y}^{t-1}$. Since \mathbf{Y}^t is generated independently of \mathcal{F}_t , each column j of $\mathbf{Q}^{\epsilon\mathbf{Y},t}$ obeys the distribution:

$$(\mathbf{Q}_{\perp}^{\epsilon\mathbf{Y},t})_j |_{\mathcal{F}_t} \sim \mathbf{N}(\mathbf{P}_{\mathbf{Q}_{\perp}^{\epsilon\mathbf{Y}}}^{\perp}(f^t(\mathbf{H}^{\epsilon\mathbf{Y},t}))_j, \epsilon^2 \mathbf{P}_{\mathbf{Q}_{\perp}^{\epsilon\mathbf{Y}}}^{\perp}) \quad (121)$$

the variance of which is almost surely non-zero whenever $N \geq tq$. Thus, when $N \geq tq$, the matrix \mathbf{Q}_{t-1} has full column rank. We now need to control the minimal singular value of \mathbf{Q}_{t-1} . Following [BM11a], Lemma 9, we only need to check that, for any column j , almost surely, for N sufficiently large, there exists a constant $c_{\epsilon} > 0$ such that:

$$\frac{1}{N} \|(\mathbf{Q}_{\perp}^{\epsilon\mathbf{Y},t})_j\|^2 \geq c_{\epsilon} \quad (122)$$

which follows in almost identical fashion to [BMN20], Lemma 9 using the moments of a $N - tq$ chi-square variable, instead of $N - t$ in the original proof, which extends straightforwardly since q is kept finite. \square

Proof of Lemma 9. This result is proven for $q = 1$ in [BMN20] and the proof for the case of finite, integer q is identical. \square

Proof of Lemma 10. This lemma is proven by induction.

Initialization. From equation (117), it holds that

$$\mathbf{K}_{1,1}^{\epsilon} = \mathbf{K}_{1,1} + \epsilon^2 \xrightarrow{\epsilon \rightarrow 0} \mathbf{K}_{1,1} \quad (123)$$

Induction. Let t be a non-negative integer. Assume that, for any $r, s \leq t$, $\kappa_{\epsilon}^{r,s} \rightarrow \kappa^{r,s}$. Then:

$$\kappa_{\epsilon}^{s+1,t+1} = \lim_{N \rightarrow \infty} \mathbb{E} \left[\frac{1}{N} (f_{\epsilon\mathbf{Y}}^s(\mathbf{Z}_{\epsilon\mathbf{Y}}^s))^{\top} f_{\epsilon\mathbf{Y}}^t(\mathbf{Z}_{\epsilon\mathbf{Y}}^t) \right] \quad (124)$$

where $\mathbf{Z}_{\epsilon\mathbf{Y}}^s, \mathbf{Z}_{\epsilon\mathbf{Y}}^t$ are $n \times q$ Gaussian random matrices whose distributions are specified by $\kappa_{\epsilon}^{s,s}, \kappa_{\epsilon}^{t,t}$ and $\kappa_{\epsilon}^{s,t}$ which are $q \times q$ deterministic matrices. Then, from equation (116), we have

$$\kappa_{\epsilon}^{s+1,t+1} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}_{\mathbf{Z}} \left[\frac{1}{N} (f_s(\mathbf{Z}^{\epsilon,s}))^{\top} f_t(\mathbf{Z}^{\epsilon,t}) \right] + \epsilon^2 \mathbf{I}_{q \times q} \delta_{s=t} \quad (125)$$

From Lemma 13, the function $(\mathbf{Z}^s, \mathbf{Z}^t) \rightarrow \frac{1}{N} f_s(\mathbf{Z}^s)^{\top} f_t(\mathbf{Z}^t)$ is uniformly pseudo-Lipschitz. Moreover, from the induction hypothesis, we have :

$$\lim_{\epsilon \rightarrow 0} \kappa_{\epsilon}^{s,t} = \kappa^{s,t} \quad (126)$$

thus, using the uniform convergence Lemma 9, we get :

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[f_s(\mathbf{Z}^{\epsilon,s})^{\top} f_t(\mathbf{Z}^{\epsilon,t}) \right] = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[f_s(\mathbf{Z}^s)^{\top} f_t(\mathbf{Z}^t) \right] = \kappa^{s+1,t+1} \quad (127)$$

where $(\mathbf{Z}^s, \mathbf{Z}^t) \sim \mathbf{N}(0, \kappa \otimes \mathbf{I}_n)$ and $\kappa = \begin{bmatrix} \kappa^{s,s} & \kappa^{s,t} \\ \kappa^{t,s} & \kappa^{t,t} \end{bmatrix}$. This shows that

$$\kappa_{\epsilon}^{s+1,t+1} \xrightarrow{\epsilon \rightarrow 0} \kappa^{s+1,t+1} \quad (128)$$

which concludes the induction. Similar reasoning proves the convergence of correlations with the initial vector

$$\boldsymbol{\kappa}_\epsilon^{1,t+1} \xrightarrow{\epsilon \rightarrow 0} \boldsymbol{\kappa}^{1,t+1} \quad (129)$$

□

Proof of Lemma 11. This Lemma is proven by induction.

Initialization.

$$\frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, 0} - \mathbf{M}^0\|_F = f_{\epsilon \mathbf{Y}}^0(\mathbf{X}^0) - f^0(\mathbf{X}^0) = \frac{1}{\sqrt{N}} \epsilon \|\mathbf{Y}^0\|_F \quad (130)$$

Using the bound from Lemma 1, there exists an absolute constant $C_{\mathbf{Y}}$ independent of N such that, with high probability:

$$\frac{\epsilon}{\sqrt{N}} \|\mathbf{Y}^0\|_F \leq C_{\mathbf{Y}} \epsilon \quad (131)$$

Note that $C_{\mathbf{Y}}$ is the same for all \mathbf{Y}^t . We thus choose $h'_0(\epsilon) = C_{\mathbf{Y}} \epsilon$. Then

$$\frac{1}{\sqrt{N}} \|\mathbf{X}^{\epsilon \mathbf{Y}, 1} - \mathbf{X}^1\|_F \leq \|\mathbf{A}\|_{op} \frac{\epsilon}{\sqrt{N}} \|\mathbf{Y}^0\|_F \leq 2C_{\mathbf{Y}} \epsilon \quad (132)$$

using the bound on the operator norm of GOE matrices Proposition 2, and we can choose $h_0(\epsilon) = 2C_{\mathbf{Y}} \epsilon$.

Induction Assume the property is verified up to time t , i.e., the functions $h_0(\epsilon), h'_0(\epsilon), \dots, h_{t-1}(\epsilon), h'_{t-1}(\epsilon)$ exist and are known. We now need to show $h_t(\epsilon), h'_t(\epsilon)$ exist. By definition of the iteration:

$$\begin{aligned} \frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t} - \mathbf{M}^t\|_F &= \frac{1}{\sqrt{N}} \|f_{\epsilon \mathbf{Y}}^t(\mathbf{X}^{\epsilon \mathbf{Y}}) - f^t(\mathbf{X}^t)\|_F \\ &= \frac{1}{\sqrt{N}} \|f^t(\mathbf{X}^{\epsilon \mathbf{Y}}) - f^t(\mathbf{X}^t) + \epsilon \mathbf{Y}^t\|_F \\ &\leq L_t \left(1 + \left(\frac{\|\mathbf{X}^{\epsilon \mathbf{Y}, t}\|_F}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\mathbf{X}^t\|_F}{\sqrt{N}} \right)^{k-1} \right) \frac{\|\mathbf{X}^{\epsilon \mathbf{Y}, t} - \mathbf{X}^t\|_F}{\sqrt{N}} + \frac{1}{\sqrt{N}} \epsilon \|\mathbf{Y}^t\|_F \\ &\leq L_t \left(1 + \left(\frac{\|\mathbf{X}^{\epsilon \mathbf{Y}, t}\|_F}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\mathbf{X}^t\|_F}{\sqrt{N}} \right)^{k-1} \right) h_{t-1}(\epsilon) + C_{\mathbf{Y}} \epsilon \\ &\leq L_t \left(1 + C_{\epsilon \mathbf{Y}}(k) + \left(\frac{\|\mathbf{X}^{\epsilon \mathbf{Y}, t}\|_F}{\sqrt{N}} + \frac{\|\mathbf{X}^{\epsilon \mathbf{Y}, t} - \mathbf{X}^t\|_F}{\sqrt{N}} \right)^{k-1} \right) h_{t-1}(\epsilon) + C_{\mathbf{Y}} \epsilon \\ &\leq L_t \left(1 + C_{\epsilon \mathbf{Y}}(k) + 2^{k-2} C_{\epsilon \mathbf{Y}}(k)^{k-1} + 2^{k-2} h_{t-1}^{k-1}(\epsilon) \right) h_{t-1}(\epsilon) + C_{\mathbf{Y}} \epsilon \end{aligned} \quad (133)$$

where we used the state evolution of the perturbed AMP iteration to show that $\frac{\|\mathbf{X}^{\epsilon \mathbf{Y}, t}\|_F}{\sqrt{N}}$ has a finite limit and Hölder's inequality. We can thus choose

$$h'_t(\epsilon) = L_t \left(1 + C_{\epsilon \mathbf{Y}}(k) + 2^{k-2} C_{\epsilon \mathbf{Y}}(k)^{k-1} + 2^{k-2} h_{t-1}^{k-1}(\epsilon) \right) h_{t-1}(\epsilon) + C_{\mathbf{Y}} \epsilon \quad (134)$$

which goes to zero when ϵ goes to zero. Then

$$\begin{aligned}
\frac{1}{\sqrt{N}} \|\mathbf{X}^{\epsilon \mathbf{Y}, t+1} - \mathbf{X}^{t+1}\|_F &\leq \|\mathbf{A}\|_{op} \frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t} - \mathbf{M}^t\|_F + \frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t-1} (\mathbf{b}_{\epsilon \mathbf{Y}}^t)^\top - \mathbf{M}^{t-1} (\mathbf{b}^t)^\top\|_F \\
&\leq 2h'_t(\epsilon) + \frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t-1} (\mathbf{b}_{\epsilon \mathbf{Y}}^t)^\top - \mathbf{M}^{t-1} (\mathbf{b}^t)^\top\|_F \\
&\leq 2h'_t(\epsilon) + \frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t-1} - \mathbf{M}^{t-1}\|_F \|\mathbf{b}^t\|_F + \frac{1}{\sqrt{N}} \|\mathbf{b}_{\epsilon \mathbf{Y}}^t - \mathbf{b}^t\|_F \|\mathbf{M}^{t-1}\|_F
\end{aligned} \tag{135}$$

and

$$\begin{aligned}
\|\mathbf{b}^t\|_F &= \left\| \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \frac{\partial f_i^t}{\partial \mathbf{Z}_i}(\mathbf{Z}^t) \right] \right\|_F \\
&\leq \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \left\| \frac{\partial f_i^t}{\partial \mathbf{Z}_i}(\mathbf{Z}^t) \right\|_F \right]
\end{aligned} \tag{136}$$

where $\mathbf{Z}^t \sim \mathbf{N}(0, \boldsymbol{\kappa}_{t,t} \otimes \mathbf{I}_n)$. Since the function $f^t : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^{N \times q}$ is pseudo-Lipschitz of order k , the components $f_i^t : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^q$ are pseudo-Lipschitz of order k as well. So are the functions $f_{i,j}^t : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}$ for $1 \leq j \leq q$ generating each component of $f_i^t(\mathbf{Z}^t) \in \mathbb{R}^q$ and their $\mathbb{R}^q \rightarrow \mathbb{R}$ restrictions to the i -th line of \mathbf{Z}^t . Then

$$\|\mathbf{b}^t\|_F \leq \frac{1}{N} \sum_{i=1}^N q \max_j \left\{ \mathbb{E} \|\nabla_{\mathbf{Z}_i^t} f_{i,j}^t(\mathbf{Z}^t)\|_2 \right\} \tag{137}$$

where $\max_j \left\{ \mathbb{E} \|\nabla_{\mathbf{Z}_i^t} f_{i,j}^t(\mathbf{Z}^t)\|_2 \right\}$ is bounded using the pseudo-Lipschitz property and a similar argument to the proof of lemma 16. Let C_J be this upper bound, then

$$\frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t-1} (\mathbf{b}_{\epsilon \mathbf{Y}}^t)^\top - \mathbf{M}^{t-1} (\mathbf{b}^t)^\top\|_F \leq q C_J h'_{t-1}(\epsilon) + \frac{1}{\sqrt{N}} \|\mathbf{M}^{t-1}\|_F \|\mathbf{b}_{\epsilon \mathbf{Y}}^t - \mathbf{b}^t\|_F \tag{138}$$

Using the same decomposition as before

$$\begin{aligned}
\frac{1}{\sqrt{N}} \|\mathbf{M}^{t-1}\|_F \|\mathbf{b}_{\epsilon \mathbf{Y}}^t - \mathbf{b}^t\|_F &\leq \left(\frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t-1} - \mathbf{M}^{t-1}\| + \frac{1}{\sqrt{N}} \|\mathbf{M}^{\epsilon \mathbf{Y}, t-1}\| \right) \|\mathbf{b}_{\epsilon \mathbf{Y}}^t - \mathbf{b}^t\|_F \\
&\leq (h'_{t-1}(\epsilon) + C_{\epsilon \mathbf{Y}, t-1}) \|\mathbf{b}_{\epsilon \mathbf{Y}}^t - \mathbf{b}^t\|_F
\end{aligned} \tag{139}$$

The definition of the Onsager correction terms gives

$$\|\mathbf{b}_{\epsilon \mathbf{Y}}^t - \mathbf{b}^t\|_F = \left\| \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \frac{\partial f_i^t}{\partial \tilde{\mathbf{Z}}_i^{\epsilon \mathbf{Y}, t}}(\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y}, t}) \right] - \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \frac{\partial f_i^t}{\partial \tilde{\mathbf{Z}}_i^t}(\tilde{\mathbf{Z}}^t) \right] \right\|_F \tag{140}$$

where $\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y}, t} = \mathbf{Z}(\boldsymbol{\kappa}_{t,t}^{\epsilon \mathbf{Y}})^{1/2}$ where $\mathbf{Z} \in \mathbb{R}^{N \times q}$ is an i.i.d. standard normal matrix. Similarly $\tilde{\mathbf{Z}}^t = \mathbf{Z}(\boldsymbol{\kappa}_{t,t})^{1/2}$. Using the positive definiteness of $\boldsymbol{\kappa}_{t,t}$ along with Lemma 12, we can write, keeping in

mind that the perturbation $\epsilon \mathbf{Y}$ doesn't change the derivatives in the Onsager correction:

$$\begin{aligned} \|\mathbf{b}_{\epsilon \mathbf{Y}}^t - \mathbf{b}^t\|_F &= \|(\boldsymbol{\kappa}_{\epsilon \mathbf{Y}}^{t,t})^{-1} \mathbb{E} \left[\frac{1}{N} (\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t})^\top f^t(\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t}) \right] - (\boldsymbol{\kappa}^{t,t})^{-1} \mathbb{E} \left[\frac{1}{N} (\mathbf{Z}^t)^\top f^t(\mathbf{Z}^t) \right]\|_F \\ &\leq \|(\boldsymbol{\kappa}_{\epsilon \mathbf{Y}}^{t,t})^{-1} - (\boldsymbol{\kappa}^{t,t})^{-1}\|_F \mathbb{E} \left[\frac{1}{N} (\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t})^\top f^t(\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t}) \right] + \\ &\quad \|(\boldsymbol{\kappa}^{t,t})^{-1}\|_F \mathbb{E} \left[\frac{1}{N} (\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t})^\top f^t(\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t}) \right] - \mathbb{E} \left[\frac{1}{N} (\mathbf{Z}^t)^\top f^t(\mathbf{Z}^t) \right]\|_F \end{aligned}$$

The function $\mathbb{R}^{N \times q} \rightarrow \mathbb{R}^{q \times q}, \mathbf{Z} \rightarrow \mathbf{Z}^\top f^t(\mathbf{Z})$ is pseudo-Lipschitz of order $k+1$. Moreover, from Lemma 8, $\boldsymbol{\kappa}_{\epsilon \mathbf{Y}}^{t,t} \xrightarrow{\epsilon \rightarrow 0} \boldsymbol{\kappa}^{t,t}$. Thus using Lemma 9, we get

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\frac{1}{N} (\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t})^\top f^t(\tilde{\mathbf{Z}}^{\epsilon \mathbf{Y},t}) \right] - \mathbb{E} \left[\frac{1}{N} (\mathbf{Z}^t)^\top f^t(\mathbf{Z}^t) \right]\|_F = 0 \quad (141)$$

and Lemma 10 gives $\lim_{\epsilon \rightarrow 0} \|(\boldsymbol{\kappa}_{\epsilon \mathbf{Y}}^{t,t})^{-1} - (\boldsymbol{\kappa}^{t,t})^{-1}\|_F = 0$, which concludes the induction. \square

D Useful definitions and probability lemmas

In this section, we compile useful definitions and lemmas that appear throughout the proof. Most of those results are finite-width matrix generalizations of those appearing in [BMN20] and some are the same.

Proposition 1. (Norm of matrices with Gaussian entries [Ver18]) *Let \mathbf{Y} be an $M \times N$ random matrix with independent $\mathbf{N}(0, 1)$ entries. Then, for any $t > 0$, we have:*

$$\mathbb{P} \left(\|\mathbf{Y}\|_F \leq C \left(\sqrt{M} + \sqrt{N} + t \right) \right) \geq 1 - 2 \exp(-t^2) \quad (142)$$

where C is an absolute constant.

Proposition 2. (Operator norm of GOE(N) [BLM13])

Consider a sequence of matrices $\mathbf{A} \sim \text{GOE}(N)$. Then $\|\mathbf{A}\|_{op} \rightarrow 2$ almost surely as $N \rightarrow \infty$.

Proposition 3. (Gaussian Poincaré inequality [BLM13])

Let $\mathbf{Z} \in \mathbb{R}^N$ be a $\mathbf{N}(0, \mathbf{I}_N)$ random vector. Then for any continuous, weakly differentiable φ , there exists a constant $c \geq 0$ such that:

$$\text{Var}[\varphi(\mathbf{Z})] \leq c \mathbb{E} \left[\|\nabla \varphi(\mathbf{Z})\|_2^2 \right] \quad (143)$$

The next result is a matrix version of Gaussian integration by parts, or Stein's lemma.

Lemma 12. (Stein's lemma, matrix version) *Let $(\mathbf{Z}_1, \mathbf{Z}_2) \in (\mathbb{R}^{N \times q})^2$ be two $\mathbf{N}(0, \boldsymbol{\kappa} \otimes \mathbf{I}_N)$ random vectors, where $\boldsymbol{\kappa} \in \mathbb{R}^{(2q) \times (2q)}$.*

$$\boldsymbol{\kappa} = \begin{bmatrix} \boldsymbol{\kappa}_{11} & \boldsymbol{\kappa}_{12} \\ \boldsymbol{\kappa}_{12} & \boldsymbol{\kappa}_{22} \end{bmatrix} \quad (144)$$

Consider an almost everywhere differentiable function $f : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^{N \times q}$. For any $\mathbf{Z} \in \mathbb{R}^{N \times q}$ we can write:

$$f \left(\begin{bmatrix} \mathbf{Z}_{11}, \dots, \mathbf{Z}_{1q} \\ \dots \\ \mathbf{Z}_{n1}, \dots, \mathbf{Z}_{nq} \end{bmatrix} \right) = \begin{bmatrix} f_1(\mathbf{Z}) \\ \dots \\ f_n(\mathbf{Z}) \end{bmatrix} = \begin{bmatrix} f_1^1(\mathbf{Z}), \dots, f_1^q(\mathbf{Z}) \\ \dots \\ f_n^1(\mathbf{Z}), \dots, f_n^q(\mathbf{Z}) \end{bmatrix} \quad (145)$$

Then

$$\mathbb{E} \left[(\mathbf{Z}_1)^\top f(\mathbf{Z}_2) \right] = \boldsymbol{\kappa}_{1,2} \left(\sum_{k=1}^N \mathbb{E} \left[\frac{\partial f_k(\mathbf{Z}_2)}{\partial \mathbf{Z}_k} \right] \right)^\top \quad (146)$$

where $\frac{\partial f_k(\mathbf{Z}_2)}{\partial \mathbf{Z}_k} \in \mathbb{R}^{q \times q}$ is the Jacobian containing the partial derivatives of f_k w.r.t. the line $\mathbf{Z}_k \in \mathbb{R}^q$.

Proof.

$$\begin{aligned} \mathbb{E} \left[(\mathbf{Z}_1)^\top f(\mathbf{Z}_2) \right]_{ij} &= \sum_{k=1}^N \mathbb{E} [((\mathbf{Z}_1)_{ki} f_{kj}(\mathbf{Z}_2))] \\ &= \sum_{k=1}^N \sum_{l=1}^q \mathbb{E}[\mathbf{Z}_{ki}^1 \mathbf{Z}_{kl}^2] \mathbb{E} \left[\frac{\partial f_{kj}}{\partial (\mathbf{Z}_2)_{kl}}(\mathbf{Z}_2) \right] \quad \text{since } (\mathbf{Z}_1, \mathbf{Z}_2) \sim \mathbf{N}(0, \boldsymbol{\kappa} \otimes I_N) \\ &= \sum_{l=1}^q (\boldsymbol{\kappa}_{12})_{il} \sum_{k=1}^N \mathbb{E} \left[\frac{\partial f_{kj}}{\partial (\mathbf{Z}_2)_{kl}}(\mathbf{Z}_2) \right] \\ &= \sum_{l=1}^q (\boldsymbol{\kappa}_{12})_{il} \left(\sum_{k=1}^N \mathbb{E} \left[\frac{\partial f_k(\mathbf{Z}_2)}{\partial \mathbf{Z}_k} \right] \right)_{jl} \\ &= \left(\boldsymbol{\kappa}_{12} \left(\sum_{k=1}^N \mathbb{E} \left[\frac{\partial f_k(\mathbf{Z}_2)}{\partial \mathbf{Z}_k} \right] \right) \right)^\top_{ij} \end{aligned} \quad (147)$$

where the second step is obtained by iteratively conditioning on the entries of \mathbf{Z}_2 and applying one dimensional Gaussian integration by parts, see e.g. [Ver18] Lemma 7.2.5. \square

Definition 4 (pseudo-Lipschitz function). For $k \in \mathbb{N}^*$ and any $N, m \in \mathbb{N}^*$, a function $\Phi : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}^{m \times q}$ is said to be pseudo-Lipschitz of order k if there exists a constant L such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{N \times q}$,

$$\frac{\|\Phi(\mathbf{x}) - \Phi(\mathbf{y})\|_F}{\sqrt{m}} \leq L \left(1 + \left(\frac{\|\mathbf{x}\|_F}{\sqrt{N}} \right)^{k-1} + \left(\frac{\|\mathbf{y}\|_F}{\sqrt{N}} \right)^{k-1} \right) \frac{\|\mathbf{x} - \mathbf{y}\|_F}{\sqrt{N}} \quad (148)$$

A family of pseudo-Lipschitz functions is said to be uniformly pseudo-Lipschitz if all functions of the family are pseudo-Lipschitz with the same order k and the same constant L .

We now remind useful properties of pseudo-Lipschitz functions from [BMN20].

Lemma 13. Let k be any positive integer. Consider two sequences $f : \mathbb{R}^N \rightarrow \mathbb{R}^N, N \geq 1$ and $g : \mathbb{R}^N \rightarrow \mathbb{R}^N, N \geq 1$ of uniformly pseudo-Lipschitz functions of order k . The sequence of functions $\Phi_N : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}, N \geq 1$ such that $\Phi_N(\mathbf{x}, \mathbf{y}) = \langle f(\mathbf{x}), g(\mathbf{y}) \rangle$ is uniformly pseudo-Lipschitz of order $2k$.

Lemma 14. Let t, s and k be any three positive integers. Consider a sequence (in N) of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s \in \mathbb{R}^N$ such that $\frac{1}{\sqrt{N}} \|\mathbf{x}_j\| \leq c_j$ for some constant c_j independent of N , for $j = 1, \dots, s$ and a sequence of order- k uniformly pseudo-Lipschitz functions $\varphi_N : (\mathbb{R}^N)^{t+s} \rightarrow \mathbb{R}$. The sequence of functions $\phi_N(\cdot) = \varphi_N(\cdot, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s)$ is also uniformly pseudo-Lipschitz of order k .

Lemma 15. Let t be any positive integer. Consider a sequence of uniformly pseudo-Lipschitz functions $\varphi_N : (\mathbb{R}^N)^t \rightarrow \mathbb{R}$ of order k . The sequence of functions $\Phi_N : (\mathbb{R}^N)^t \rightarrow \mathbb{R}$ such that $\Phi_N(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t) = \mathbb{E} [\varphi_N(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{x}_t + \mathbf{Z})]$, in which $\mathbf{Z} \sim \mathbf{N}(0, a\mathbf{I}_N)$ and $a \leq 0$, is also uniformly pseudo-Lipschitz of order k .

We now state a result on Gaussian concentration of matrix-valued pseudo-Lipschitz functions. This is an extension to the matrix case (of finite width) of Lemma C.8 from [BMN20].

Lemma 16. *Let $\mathbf{Z} \sim \mathbf{N}(0, \boldsymbol{\kappa} \otimes \mathbf{I}_N)$ where $\boldsymbol{\kappa} \in \mathcal{S}_q^+$. Let $\Phi_N : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}$ be a sequence of random functions, independent of \mathbf{Z} , such that $\mathbb{P}(\mathcal{E}_N) \rightarrow 1$ as $N \rightarrow \infty$, where \mathcal{E}_N is the event that Φ_N is pseudo-Lipschitz of (deterministic) order k with (deterministic) pseudo-Lipschitz constant L . Then $\Phi_N(\mathbf{Z}) \stackrel{P}{\simeq} \mathbb{E}[\Phi_N(\mathbf{Z})]$.*

Proof. First, it is straightforward to see that

$$\Phi_N(\mathbf{Z}) = \Phi_N(\tilde{\mathbf{Z}}\boldsymbol{\kappa}^{1/2}) = \tilde{\Phi}_N(\tilde{\mathbf{Z}}) \quad (149)$$

where $\tilde{\mathbf{Z}} \in \mathbb{R}^{N \times q}$ is an i.i.d. standard normal matrix, and $\tilde{\Phi}_N = \Phi_N(\cdot\boldsymbol{\kappa}^{1/2})$. Since $\|\boldsymbol{\kappa}\|_{op}$ is bounded for all N , $\tilde{\Phi}_N$ is also pseudo-Lipschitz of order k , with constant $L \max(\|\boldsymbol{\kappa}\|_{op}^{1/2}, \|\boldsymbol{\kappa}\|_{op}^{k/2})$. $\tilde{\Phi}_N$ can then be considered as a function acting on a vector of size $N \times q$ with i.i.d. standard normal components. The proof is then identical to that of Lemma C.8 from [BMN20] with an additional finite factor q . We remind this proof for completeness. Under \mathcal{E}_N , using the definition of pseudo-Lipschitz functions and proposition 3:

$$\mathbb{E}_{\mathbf{Z}} [\|\nabla \Phi_N(\mathbf{Z})\|_2^2] \leq \frac{L^2}{Nq} \mathbb{E}_{\mathbf{Z}} \left[\left(1 + 2 \left(\frac{1}{\sqrt{Nq}} \|\mathbf{Z}\|_2 \right)^{k-1} \right)^2 \right] \leq \frac{L^2}{Nq} C(k) \quad (150)$$

for a constant $C(k)$ that only depends on k . Then for any $\epsilon > 0$, there exists a constant $c > 0$, independent of N , such that:

$$\begin{aligned} \mathbb{P}\{|\Phi_N(\mathbf{Z}) - \mathbb{E}_{\mathbf{Z}}[\Phi_N(\mathbf{Z})]| > \epsilon\} &\leq \mathbb{E}\{\mathbb{P}\{|\Phi_N(\mathbf{Z}) - \mathbb{E}_{\mathbf{Z}}[\Phi_N(\mathbf{Z})]| > \epsilon\} \mathbb{I}_{\mathcal{E}_N}\} + \mathbb{P}(\bar{\mathcal{E}}_N) \\ &\leq \frac{\text{Var}[\Phi_N(\mathbf{Z})]}{\epsilon^2} + \mathbb{P}(\bar{\mathcal{E}}_N) \\ &\leq \frac{L^2 C(k)}{Nq\epsilon^2} + \mathbb{P}(\bar{\mathcal{E}}_N) \end{aligned} \quad (151)$$

where the second and third line are obtained by applying Chebyshev's inequality and proposition 3 with the variance bound evaluated at Eq.(150). \square

The next lemmas are matrix generalizations of the ones used in [BMN20].

Lemma 17. *Consider a sequence of matrices $\mathbf{A} \sim \text{GOE}(N)$ and two sequences of non-random matrices, $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{N \times q}$ such that the columns of \mathbf{U} and \mathbf{V} verify $\|\mathbf{U}^i\|_2 = \|\mathbf{V}^i\|_2 = \sqrt{N}$. Under this hypothesis, define the finite quantity $\mathbf{G} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{U}^\top \mathbf{U}$, the limiting Gram matrix of the columns of \mathbf{U} . We then have:*

- a) $\frac{1}{N} \mathbf{V}^\top \mathbf{A} \mathbf{U} \xrightarrow[N \rightarrow \infty]{P} 0_{q \times q}$ and $\frac{1}{N} \|\mathbf{V}^\top \mathbf{A} \mathbf{U}\|_F \xrightarrow[N \rightarrow \infty]{P} 0$.
- b) Let $\mathbf{P} \in \mathbb{R}^{N \times N}$ be a sequence of non-random projection matrices such that there exists a constant t that satisfies, for all N , $k = \text{rank}(\mathbf{P}) \leq t$. Then $\frac{1}{N} \|\mathbf{P} \mathbf{A} \mathbf{U}\|_F^2 \xrightarrow[N \rightarrow \infty]{P} 0$.
- c) There exists a sequence of random matrices $\mathbf{Z} \in \mathbb{R}^{N \times q}$, such that $\frac{1}{N} \|\mathbf{A} \mathbf{U} - \mathbf{Z}\|_F^2 \xrightarrow[N \rightarrow \infty]{P} 0$ where $\mathbf{Z} \sim \mathbf{N}(0, \mathbf{G} \otimes \mathbf{I}_N)$.

$$d) \frac{1}{N}(\mathbf{A}\mathbf{U})^\top \mathbf{A}\mathbf{U} \xrightarrow[N \rightarrow \infty]{P} \mathbf{G}.$$

Proof. In this proof, the i -th line of a given matrix \mathbf{Z} is denoted \mathbf{Z}_i and its j -th column \mathbf{Z}^j .

a) For any $1 \leq i, j \leq q$, the i -th element of the j -th column verifies:

$$\begin{aligned} \frac{1}{N}(\mathbf{V}^\top \mathbf{A}\mathbf{U})_i^j &= \frac{1}{N}(\mathbf{V}^i)^\top \mathbf{A}\mathbf{U}^j \\ &= \frac{1}{N}(\mathbf{V}^i)^\top \mathbf{H}\mathbf{U}^j + \frac{1}{N}(\mathbf{V}^i)^\top \mathbf{H}^\top \mathbf{U}^j \end{aligned} \quad (152)$$

where \mathbf{H} is a matrix with i.i.d. $\mathbf{N}(0, \frac{1}{2N})$ elements. The random variable $\frac{1}{N}(\mathbf{V}^i)^\top \mathbf{H}\mathbf{U}^j$ is centered Gaussian with variance

$$\frac{1}{N^2} \sum_{k,l=1}^N (\mathbf{V}_k^i)^2 (\mathbf{U}_l^j)^2 \frac{1}{2N} = \frac{\|\mathbf{V}^i\|_2^2 \|\mathbf{U}^j\|_2^2}{2N^3} = \frac{1}{2N} \rightarrow 0 \quad (153)$$

which shows that $\frac{1}{N}(\mathbf{V}^i)^\top \mathbf{H}\mathbf{U}^j$ converges in probability to zero. A similar argument shows that $\frac{1}{N}(\mathbf{V}^i)^\top \mathbf{H}^\top \mathbf{U}^j$ also converges in probability to zero. The union bound then immediately gives that $\frac{1}{N}(\mathbf{V}^\top \mathbf{A}\mathbf{U})_i^j \xrightarrow[N \rightarrow \infty]{P} 0$. Thus each element of the finite size $q \times q$ matrix $\frac{1}{N}\mathbf{V}^\top \mathbf{A}\mathbf{U}$ goes to zero. Since q is finite, the union bound then gives the desired result on the Frobenius norm.

b) For any $1 \leq i \leq q$:

$$\frac{1}{N}(\mathbf{P}\mathbf{A}\mathbf{U})^i = \frac{1}{N}(\mathbf{P}\mathbf{A}\mathbf{U}^i) \quad (154)$$

Now let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be an orthogonal basis of the image of \mathbf{P} , such that $\|\mathbf{v}_1\| = \dots = \|\mathbf{v}_k\| = \sqrt{N}$, and $\mathbf{V} \in \mathbb{R}^{N \times t}$ the matrix of concatenated \mathbf{v} . Note that k can depend on N , but k is uniformly bounded by t . Then, using point (a) and the fact that q and k are finite for all N :

$$\frac{1}{N} \|\mathbf{P}\mathbf{A}\mathbf{U}\|_F^2 = \frac{1}{N} \|\mathbf{V}^\top \mathbf{A}\mathbf{U}\| \xrightarrow[N \rightarrow \infty]{P} 0 \quad (155)$$

This proves point (b).

c) The matrix $\mathbf{A}\mathbf{U}$ is a $\mathbb{R}^{N \times q}$ correlated Gaussian matrix. For any two columns $\mathbf{U}^l, \mathbf{U}^m$, the vector $(\mathbf{A}\mathbf{U}^l, \mathbf{A}\mathbf{U}^m)$ is a Gaussian vector with zero mean, whose covariance matrix has elements:

$$\begin{aligned} \mathbb{E} \left[(\mathbf{A}\mathbf{U}^l) (\mathbf{A}\mathbf{U}^m)^\top \right]_i^j &= \mathbb{E} \left[(\mathbf{A}\mathbf{U}^l)_i (\mathbf{A}\mathbf{U}^m)_j \right] \\ &= \mathbb{E} \left[\sum_{k=1}^N \mathbf{A}_i^k \mathbf{U}_k^l \sum_{k'=1}^N \mathbf{A}_j^{k'} \mathbf{U}_{k'}^m \right] \\ &= \mathbb{E} \left[\sum_{k,k'} \mathbf{H}_i^k \mathbf{H}_j^{k'} \mathbf{U}_k^l \mathbf{U}_{k'}^m + \mathbf{H}_i^k \mathbf{H}_{k'}^j \mathbf{U}_k^l \mathbf{U}_{k'}^m + \mathbf{H}_k^i \mathbf{H}_j^{k'} \mathbf{U}_k^l \mathbf{U}_{k'}^m + \mathbf{H}_k^i \mathbf{H}_{k'}^j \mathbf{U}_k^l \mathbf{U}_{k'}^m \right] \\ &= \frac{1}{N} \left(\delta_{ij} \sum_k \mathbf{U}_k^l \mathbf{U}_k^m + \mathbf{U}_i^l \mathbf{U}_j^m \right) \end{aligned} \quad (156)$$

which gives the block

$$\mathbb{E} \left[(\mathbf{A}\mathbf{U}^l) (\mathbf{A}\mathbf{U}^m)^\top \right] = \frac{1}{N} (\mathbf{U}^l)^\top \mathbf{U}^m \mathbf{I}_N + \frac{1}{N} \mathbf{U}^l (\mathbf{U}^m)^\top \quad (157)$$

and the covariance matrix

$$\Sigma = \begin{bmatrix} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^l (\mathbf{U}^l)^\top & \frac{(\mathbf{U}^l)^\top \mathbf{U}^m}{N} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^l (\mathbf{U}^m)^\top \\ \frac{(\mathbf{U}^l)^\top \mathbf{U}^m}{N} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^m (\mathbf{U}^l)^\top & \mathbf{I}_N + \frac{1}{N} \mathbf{U}^m (\mathbf{U}^m)^\top \end{bmatrix} \quad (158)$$

and in turn the following covariance matrix for the joint law of the q vectors $\mathbf{A}\mathbf{U}^1, \dots, \mathbf{A}\mathbf{U}^q$.

$$\begin{bmatrix} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^1 (\mathbf{U}^1)^\top & \dots & \dots & \dots & \dots & \frac{(\mathbf{U}^1)^\top \mathbf{U}^q}{N} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^1 (\mathbf{U}^q)^\top \\ \dots & \frac{(\mathbf{U}^{i-1})^\top \mathbf{U}^i}{N} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^i (\mathbf{U}^{i-1})^\top & \mathbf{I}_N + \frac{1}{N} \mathbf{U}^i (\mathbf{U}^i)^\top & \frac{(\mathbf{U}^i)^\top \mathbf{U}^{i+1}}{N} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^i (\mathbf{U}^{i+1})^\top & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{(\mathbf{U}^q)^\top \mathbf{U}^1}{N} \mathbf{I}_N + \frac{1}{N} \mathbf{U}^q (\mathbf{U}^1)^\top & \dots & \dots & \dots & \dots & \mathbf{I}_N + \frac{1}{N} \mathbf{U}^q (\mathbf{U}^q)^\top \end{bmatrix}$$

which can be rewritten

$$\begin{aligned} \Sigma &= \frac{1}{N} \mathbf{U}^\top \mathbf{U} \otimes \mathbf{I}_N + \frac{1}{N} \begin{bmatrix} \mathbf{U}^1 (\mathbf{U}^1)^\top & \dots & \dots & \dots & \mathbf{U}^1 (\mathbf{U}^q)^\top \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \mathbf{U}^i (\mathbf{U}^{i-1})^\top & \mathbf{U}^i (\mathbf{U}^i)^\top & \mathbf{U}^i (\mathbf{U}^{i+1})^\top & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{U}^q (\mathbf{U}^1)^\top & \dots & \dots & \dots & \mathbf{U}^q (\mathbf{U}^q)^\top \end{bmatrix} \\ &= \frac{1}{N} \mathbf{U}^\top \mathbf{U} \otimes \mathbf{I}_N + \frac{1}{N} \tilde{\mathbf{U}} \tilde{\mathbf{U}}^\top \end{aligned} \quad (159)$$

where $\tilde{\mathbf{U}} \in \mathbb{R}^{Nq}$ is the vector of vertically concatenated columns of \mathbf{U} . Now consider two independent $\mathbf{N}(0, \mathbf{I}_{Nq})$ vectors $\tilde{\mathbf{Z}}^1, \tilde{\mathbf{Z}}^2$ and $\tilde{\mathbf{V}} \in \mathbb{R}^{Nq}$ the vector of vertically concatenated columns of $\mathbf{A}\mathbf{U}$. We can write that the quantity:

$$\frac{\|\tilde{\mathbf{V}} - (\frac{1}{N} \mathbf{U}^\top \mathbf{U} \otimes \mathbf{I}_N)^{1/2} \tilde{\mathbf{Z}}^1\|_2}{\sqrt{N}} \quad (160)$$

is distributed as

$$\begin{aligned} \frac{\|(\frac{1}{N} \mathbf{U}^\top \mathbf{U} \otimes \mathbf{I}_N)^{1/2} \tilde{\mathbf{Z}}^1 + (\frac{1}{N} \tilde{\mathbf{U}} \tilde{\mathbf{U}}^\top)^{1/2} \tilde{\mathbf{Z}}^2 - (\frac{1}{N} \mathbf{U}^\top \mathbf{U} \otimes \mathbf{I}_N)^{1/2} \tilde{\mathbf{Z}}^1\|_2}{\sqrt{N}} &= \frac{1}{N \sqrt{N}} \|\tilde{\mathbf{U}} \tilde{\mathbf{U}}^\top \tilde{\mathbf{Z}}^2\|_2 \\ &= \frac{\sqrt{q}}{N} |\tilde{\mathbf{U}}^\top \tilde{\mathbf{Z}}^2| \xrightarrow[N \rightarrow \infty]{P} 0 \end{aligned} \quad (161)$$

where the last convergence follows from the fact that $\frac{1}{N} \tilde{\mathbf{U}} \tilde{\mathbf{U}}^\top \tilde{\mathbf{Z}}^2$ is a centered Gaussian random variable with variance $\|\tilde{\mathbf{U}}\|_2^2 / N^2 = q/N$, where q is kept finite. This concludes the proof of point (c).

- d) The function $\Phi : \mathbb{R}^{N \times q} \rightarrow \mathbb{R}, \mathbf{X} \rightarrow \frac{1}{N} \mathbf{X}^\top \mathbf{X}$ is pseudo-Lipschitz of order 2. A straightforward calculation shows that, for any $\mathbf{Z} \sim \mathbf{N}(0, \mathbf{G} \otimes \mathbf{I}_N)$, we have $\mathbb{E}[\Phi(\mathbf{Z})] = \mathbf{G}$. Then :

$$\mathbb{P}(\|\Phi(\mathbf{A}\mathbf{U}) - \mathbb{E}[\Phi(\mathbf{Z})]\|_F \geq \epsilon) \leq \mathbb{P}(\|\Phi(\mathbf{A}\mathbf{U}) - \Phi(\mathbf{Z})\|_F \geq \epsilon) + \mathbb{P}(\|\Phi(\mathbf{Z}) - \mathbb{E}[\Phi(\mathbf{Z})]\|_F \geq \epsilon) \quad (162)$$

the second term on the right-hand side vanishes as $N \rightarrow \infty$ using the Gaussian concentration of matrix-valued pseudo-Lipschitz functions Lemma 16, and the first term vanishes using the definition of pseudo-Lipschitz function and the statement (c) proven above. This concludes the proof of statement (d). \square

E Detailed mappings for the asymmetric AMP

The purpose of this appendix is to provide intuition on two important points for practical applications of Theorem 1: how is an AMP iteration related to the solution of a given inference or optimization problem, and how the SE equations formulated here can be used to exactly recover those originally derived (heuristically) in the literature. Although the first point in its full generality goes beyond the scope of this paper, we propose a straightforward, self-contained way of finding the update functions of GAMP [Ran11] from an optimization point of view, giving a concrete example where pseudo-Lipschitz regularity conditions are met and the relation between AMP iterates and an estimator are explicit. Regarding the second point, the results of [JM13] already prove the SE equations derived heuristically in [Ran11], although the exact mapping is not given. In particular, it is not straightforward to understand from the main result of [JM13] how the SE equations may be used to evaluate the reconstruction performance of AMP iterates w.r.t. a given generative model (or *teacher*). We propose a decomposition that allows to explicitly recover the equations from [Ran11] and contains intuition adaptable to more complex cases.

E.1 Asymmetric AMP for convex generalized linear modelling

Recall the convex GLM problem with a design matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$:

$$\hat{\mathbf{x}} \in \arg \min_{\mathbf{x} \in \mathbb{R}^N} g(\mathbf{A}\mathbf{x}, \mathbf{y}) + f(\mathbf{x}),$$

where \mathbf{y} contains given labels and f, g are convex functions. The optimality condition for this problem reads:

$$\mathbf{A}^\top \partial g(\mathbf{A}\mathbf{x}, \mathbf{y}) + \partial f(\mathbf{x}) = 0.$$

For any strictly positive constants $\gamma_1, \gamma_2, \gamma_3$, we can equivalently write:

$$\begin{aligned} \gamma_2 \mathbf{A}^\top \mathbf{A}\mathbf{x} + \gamma_3 \mathbf{x} + \gamma_1 \left(\mathbf{A}^\top \partial g(\mathbf{A}\mathbf{x}, \mathbf{y}) + \partial f(\mathbf{x}) \right) &= \gamma_2 \mathbf{A}^\top \mathbf{A}\mathbf{x} + \gamma_3 \mathbf{x}, \\ \iff \gamma_2 \mathbf{A}^\top \left(\mathbf{A}\mathbf{x} + \frac{\gamma_1}{\gamma_2} \partial g(\mathbf{A}\mathbf{x}, \mathbf{y}) \right) + \gamma_3 \left(\mathbf{x} + \frac{\gamma_1}{\gamma_3} \partial f(\mathbf{x}) \right) &= \gamma_2 \mathbf{A}^\top \mathbf{A}\mathbf{x} + \gamma_3 \mathbf{x}. \end{aligned}$$

Define the variables

$$\tilde{\mathbf{u}} = \mathbf{x} + \frac{\gamma_1}{\gamma_3} \partial f(\mathbf{x}) \quad \tilde{\mathbf{v}} = \mathbf{A}\mathbf{x} + \frac{\gamma_1}{\gamma_2} \partial g(\mathbf{A}\mathbf{x}, \mathbf{y}), \quad (163)$$

$$\text{such that } \mathbf{x} = \text{prox}_{\frac{\gamma_1}{\gamma_3} f}(\tilde{\mathbf{u}}) \quad \mathbf{A}\mathbf{x} = \text{prox}_{\frac{\gamma_1}{\gamma_2} g}(\tilde{\mathbf{v}}), \quad (164)$$

which are one-to-one mappings, see [BC⁺11]. This leads to the following optimality condition

$$\gamma_2 \mathbf{A}^\top \left(\text{prox}_{\frac{\gamma_1}{\gamma_2} g} - \mathbf{I}_M \right) (\tilde{\mathbf{v}}) = \gamma_3 \left(\mathbf{I}_N - \text{prox}_{\frac{\gamma_1}{\gamma_3} f}(\cdot) \right) (\tilde{\mathbf{u}}), \quad (165)$$

$$\mathbf{A} \text{prox}_{\frac{\gamma_1}{\gamma_3} f}(\tilde{\mathbf{u}}) = \text{prox}_{\frac{\gamma_1}{\gamma_2} g}(\tilde{\mathbf{v}}), \quad (166)$$

Now recall the asymmetric AMP iteration, using the notations of [BMN20] discussed in Appendix A,

$$\begin{aligned}\mathbf{u}^{t+1} &= \mathbf{A}^\top \mathbf{h}^t(\mathbf{v}^t) - d^t \mathbf{e}_t(\mathbf{u}^t), \\ \mathbf{v}^t &= \mathbf{A} \mathbf{e}^t(\mathbf{u}^t) - b^t \mathbf{h}^{t-1}(\mathbf{v}^{t-1}),\end{aligned}$$

the fixed point of which reads

$$(\mathbf{I}_N + d\mathbf{e}(\cdot))(\mathbf{u}) = \mathbf{A}^\top \mathbf{h}(\mathbf{v}), \quad (167)$$

$$(\mathbf{I}_M + b\mathbf{h}(\cdot))(\mathbf{v}) = \mathbf{A} \mathbf{e}(\mathbf{u}), \quad (168)$$

where we omit the time indices for simplicity. Matching Eq.(165)-(166) to Eq.(167)-(168) prescribes:

$$\begin{aligned}h(\mathbf{v}) &= \gamma_2 \left(\text{prox}_{\frac{\gamma_2}{b}g}(\cdot) - \mathbf{I}_N \right) (\tilde{\mathbf{v}}), \\ e(\mathbf{u}) &= \frac{1}{d} \left(\gamma_3 \left(Id - \text{prox}_{\frac{\gamma_3}{d}f}(\cdot) \right) (\tilde{\mathbf{u}}) - \tilde{\mathbf{u}} \right), \\ e(\mathbf{u}) &= \text{prox}_{\frac{\gamma_3}{d}f}(\tilde{\mathbf{u}}), \\ h(\mathbf{v}) &= \frac{1}{b} \left(\text{prox}_{\frac{\gamma_2}{b}g}(\cdot) - \mathbf{I}_M \right) (\tilde{\mathbf{v}}),\end{aligned}$$

Taking $\gamma_2 = \frac{1}{b}, \gamma_3 = -d, \mathbf{v} = \tilde{\mathbf{v}}, \mathbf{u} = \gamma_3 \tilde{\mathbf{u}}$:

$$\begin{aligned}\mathbf{h}(\mathbf{v}) &= \frac{1}{b} \left(\text{prox}_{bg(\cdot, \mathbf{y})}(\cdot) - \mathbf{I}_M \right) (\mathbf{v}), \\ \mathbf{e}(\mathbf{u}) &= \text{prox}_{\frac{-\gamma_3}{d}f}\left(\frac{-\mathbf{u}}{d}\right),\end{aligned}$$

which are the functions appearing in [Ran11] when applying GAMP to a convex optimization problem, see also [AKLZ20]. Using the Lipschitz regularity of the proximal operators, the state evolution equations on the iterates $\mathbf{u}^t, \mathbf{v}^t$ and the mappings Eq.(163-164), we can recover the high-dimensional statistical properties of the estimator $\hat{\mathbf{x}}^t$ given by the AMP iteration at each time-step, notably the state evolution equation on the overlap $m = \frac{1}{d} \mathbf{x}_0^\top \hat{\mathbf{x}}$. We develop on this in the next subsection.

E.2 Conditional decomposition on the planted model

We now show how to deal with *planted models* involving a data or measurement matrix appearing in both the generative model and the learner, i.e.,

$$\begin{aligned}\hat{\mathbf{x}} &\in \arg \min_{\mathbf{x} \in \mathbb{R}^N} g(\mathbf{A}\mathbf{x}, \mathbf{y}) + f(\mathbf{x}), \\ &\text{where } \mathbf{y} = \phi(\mathbf{A}\mathbf{x}_0).\end{aligned}$$

An additional decomposition must be performed to recover the usual state evolution equations found in, e.g., [Ran11, KMS⁺12, ZK16]. We perform a conditional decomposition on the matrix \mathbf{A} to isolate components independent on \mathbf{y} , which gives

$$\mathbf{A} = \mathbb{E}[\mathbf{A}|\mathbf{y}] + \mathbf{A} - \mathbb{E}[\mathbf{A}|\mathbf{y}] = \mathbb{E}[\mathbf{A}|\mathbf{A}\mathbf{x}_0] + (\mathbf{A} - \mathbb{E}[\mathbf{A}|\mathbf{A}\mathbf{x}_0]), \quad (169)$$

since conditioning on \mathbf{y} is equivalent to conditioning on $\mathbf{A}\mathbf{x}_0$. Using the same expression for conditional Gaussian decompositions as used in Lemma 3, we can write:

$$\mathbb{E}[\mathbf{A}|\mathbf{A}\mathbf{x}_0] = \mathbf{A}\mathbf{P}_{\mathbf{x}_0} \quad \mathbf{A} - \mathbb{E}[\mathbf{A}|\mathbf{y}] = \tilde{\mathbf{A}}\mathbf{P}_{\mathbf{x}_0}^\perp, \quad (170)$$

$$\text{where } \mathbf{P}_{\mathbf{x}_0} = \frac{\mathbf{x}_0\mathbf{x}_0^\top}{\|\mathbf{x}_0\|_2^2} \text{ and } \mathbf{P}_{\mathbf{x}_0}^\perp = \mathbf{I}_N - \mathbf{P}_{\mathbf{x}_0}, \quad (171)$$

and $\tilde{\mathbf{A}}$ is a copy of \mathbf{A} independent of \mathbf{y} . Introducing the variable (*order parameter* or *overlap* in statistical physics) $m = \frac{1}{d}\mathbf{x}_0^\top \mathbf{x}$ and $\rho = \frac{1}{d}\|\mathbf{x}_0\|_2^2$, we get the reformulation

$$\inf_{\mathbf{x}, m} g\left(\frac{m}{\sqrt{\rho}}\mathbf{s} + \tilde{\mathbf{A}}\mathbf{P}_{\mathbf{x}_0}^\perp \mathbf{x}\right) + f\left(\frac{m}{\rho}\mathbf{x}_0 + \mathbf{P}_{\mathbf{x}_0}^\perp \mathbf{x}\right), \quad (172)$$

where $\mathbf{s} = \mathbf{A}\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}$ is an i.i.d. $\mathbf{N}(0, 1)$ vector. Introducing the variable $\tilde{\mathbf{x}} = \mathbf{P}_{\mathbf{x}_0}^\perp \mathbf{x}$ and a Lagrange multiplier ν to enforce the orthogonality constraint, the problem can be equivalently reformulated as

$$\inf_{\mathbf{x} \in \mathbb{R}^N, m} \sup_{\nu} g\left(\frac{m}{\sqrt{\rho}}\mathbf{s} + \tilde{\mathbf{A}}\mathbf{P}_{\mathbf{x}_0}^\perp \mathbf{x}\right) + f\left(\frac{m}{\rho}\mathbf{x}_0 + \tilde{\mathbf{x}}\right) - \nu \mathbf{x}_0^\top \tilde{\mathbf{x}}, \quad (173)$$

Under mild conditions, this problem is feasible and strong duality can be used to invert the order of the inf-sup, to obtain

$$\inf_m \sup_{\nu} \inf_{\tilde{\mathbf{x}} \in \mathbb{R}^N} g\left(\frac{m}{\sqrt{\rho}}\mathbf{s} + \tilde{\mathbf{A}}\mathbf{P}_{\mathbf{x}_0}^\perp \mathbf{x}\right) + f\left(\frac{m}{\rho}\mathbf{x}_0 + \tilde{\mathbf{x}}\right) - \nu \mathbf{x}_0^\top \tilde{\mathbf{x}}. \quad (174)$$

What we have gained from these manipulations is a formulation in terms of a matrix $\tilde{\mathbf{A}}$ that is independent on the other variables of the problem, allowing a fully rigorous and explicit application of the state evolution theorem. Focusing on the optimization problem on $\tilde{\mathbf{x}}$, we write

$$\inf_{\tilde{\mathbf{x}} \in \mathbb{R}^N} \tilde{g}(\tilde{\mathbf{A}}\tilde{\mathbf{x}}) + \tilde{f}(\tilde{\mathbf{x}}), \quad (175)$$

where we defined $\tilde{g}(\cdot, \mathbf{y}) = g\left(\frac{m}{\sqrt{\rho}}\mathbf{s} + \cdot, \mathbf{y}\right)$ and $\tilde{f}(\cdot) = f\left(\frac{m}{\rho}\mathbf{x}_0 + \cdot\right) - \nu \mathbf{x}_0^\top \cdot$, which are still convex functions. Working on the minimization problem defining the proximal operators, we get:

$$\text{prox}_{\gamma_1 \tilde{g}(\cdot)}(\mathbf{u}) = \text{prox}_{\gamma_1 g(\cdot)}\left(\frac{m}{\sqrt{\rho}}\mathbf{s} + \mathbf{u}\right) - \frac{m}{\sqrt{\rho}}\mathbf{s}, \quad (176)$$

$$\text{prox}_{\gamma_2 \tilde{f}(\cdot)}(\mathbf{v}) = \text{prox}_{\gamma_2 f(\cdot)}\left(\left(\frac{m}{\rho} + \gamma_2 \nu\right)\mathbf{x}_0 + \mathbf{v}\right) - \frac{m}{\rho}\mathbf{x}_0, \quad (177)$$

for any strictly positive constants (γ_1, γ_2) . The algorithm then reads [Ran11, ZK16]

Initialize: $\beta^0, \mathbf{u}^0, e_0(\cdot), \mathbf{h}^{-1} = 0,$

Iterate:

$$\begin{aligned} \alpha^t &= -\frac{1}{\langle \mathbf{h}'_t(\mathbf{v}^t) \rangle} \quad \beta^t = \langle \mathbf{e}'_{t-1}(\mathbf{u}^{t-1}) \rangle, \\ \mathbf{h}^t(\cdot) &= \frac{1}{\beta^t} \left(\text{prox}_{\beta^t \tilde{g}(\cdot, \mathbf{y})} - Id \right) (\cdot) \quad \mathbf{e}^t(\cdot) = \text{prox}_{\alpha^t \tilde{f}(\cdot)}(\alpha^t \cdot), \\ \mathbf{u}^{t+1} &= \mathbf{A}^\top \mathbf{h}^t(\mathbf{v}^t) - \langle (\mathbf{h}^t)' \rangle \mathbf{e}^t(\mathbf{u}^t), \\ \mathbf{v}^t &= \mathbf{A} \mathbf{e}^t(\mathbf{u}^t) - \langle (\mathbf{e}^t)' \rangle \mathbf{h}_{t-1}(\mathbf{v}^{t-1}), \end{aligned}$$

The state evolution theorem then prescribes the Gaussian fields describing the distribution of the iterates $\mathbf{u}^t, \mathbf{v}^t$ in the pseudo-Lipschitz sense. This gives four equations for the variances and Onsager reaction terms, and a fifth one for m^t using its definition. To get the explicit expression for ν^t from the last expression, we use the property of the proximal operator

$$\forall \gamma > 0 \quad \text{prox}_{\gamma f}(\mathbf{x}) + \gamma \partial f(\text{prox}_{\gamma f}(\mathbf{x})) = \mathbf{x}, \quad (178)$$

to obtain

$$\frac{m^t}{\rho} + \alpha^t \nu^t = \frac{m^t}{\rho \alpha^t} - \frac{\mathbf{s}^\top}{\sqrt{\rho} \beta^t} \left(\left(\frac{m^t}{\rho} \mathbf{s} + \sqrt{\kappa_1^t} \mathbf{h} \right) - \text{prox}_{\beta^t g(\cdot)} \left(\frac{m^t}{\sqrt{\rho}} \mathbf{s} + \sqrt{\kappa_1^t} \mathbf{g} \right) \right), \quad (179)$$

Rewriting $\tilde{\nu}^t = \frac{m^t}{\rho} + \alpha^t \nu^t$ and remembering the definitions of the update functions, we get:

$$\begin{cases} \kappa_1^{t+1} = \frac{1}{d} \mathbb{E} \left[\|\mathbf{e}^t(\tilde{\nu}^t \mathbf{x}_0 + \sqrt{\kappa_2^t} \mathbf{h})\|_2^2 \right], & \kappa_2^t = \frac{1}{d} \mathbb{E} \left[\|\mathbf{h}^t \left(\frac{m^t}{\sqrt{\rho}} \mathbf{s} + \sqrt{\kappa_1^t} \mathbf{g} \right)\|_2^2 \right], \\ m^{t+1} = \frac{1}{d} \mathbb{E} \left[\mathbf{x}_0^\top \text{prox}_{\alpha^t f(\cdot)}^t \left(\alpha^t \left(\tilde{\nu}^t \mathbf{x}_0 + \sqrt{\kappa_2^t} \mathbf{h} \right) \right) \right], & \tilde{\nu}^t = \frac{m^t}{\rho \alpha^t} + \frac{1}{d} \mathbb{E} \left[\frac{\mathbf{s}^\top}{\sqrt{\rho}} \mathbf{h}^t \left(\frac{m^t}{\sqrt{\rho}} \mathbf{s} + \sqrt{\kappa_1^t} \mathbf{g} \right) \right], \\ \beta^{t+1} = \frac{1}{d} \mathbb{E} \left[(\mathbf{e}^t)^\top (\tilde{\nu}^t \mathbf{x}_0 + \sqrt{\kappa_2^t} \mathbf{h}) \right], & \alpha^t = -\frac{1}{d} \mathbb{E} \left[\frac{1}{\beta^t} (\mathbf{h}^t)^\top \left(\frac{m^t}{\sqrt{\rho}} \mathbf{s} + \sqrt{\kappa_1^t} \mathbf{g} \right) \right] \end{cases}$$

which are the usual six SE equations appearing in GAMP [Ran11, ZK16, AKLZ20], when applied to the convex optimization case. As mentioned before, the theorem of [JM13] is (more than) enough to prove the result, however the explicit mapping detailing the isolation of independent components was not given. Provided a converging trajectory of the appropriate AMP can be systematically found, the fixed point of the state evolution equations characterize the performance of the estimator, see, e.g., [BM11b, DM16] or the more recent [LSG⁺21].

Comment on matrix-valued variables All the manipulations done above can be reproduced with matrix-valued variables, matrix Lagrange multipliers and matrix proximal operators. For more detail on how to appropriately map the matrix update functions to proximity operators, see the Appendix of [LSG⁺21].