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SEMI-CLASSICAL DEFECT MEASURE AND INTERNAL STABILIZATION FOR THE SEMILINEAR WAVE EQUATION SUBJECT TO ZAREMBA BOUNDARY CONDITIONS

MARCELO M. CAVALCANTI, PIERRE CORNILLEAU, VALÉRIA N. DOMINGOS CAVALCANTI, AND LUC ROBBIANO

ABSTRACT. In this article we exploite the uniform decay for damped linear wave equation with Zaremba boundary condition, obtained in a previous work, to treat the same problem in nonlinear context. We need a uniqueness assumption, usual for this type of nonlinear problem. The result is deduced from an observation estimate for nonlinear problem proved by a contradiction argument.

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1. INTRODUCTION

1.1. **Description of the Problem.** This article is devoted to the analysis of the exponential and uniform decay rates of solutions to the wave equation subject to a localized frictional damping and Zaremba boundary conditions:

$$\begin{cases} \partial_t^2 u - \Delta u + f(u) + a(x)\partial_t u = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega_D \times (0, +\infty), \\ \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega_N \times (0, +\infty), \\ u(x, 0) = u_0(x); \quad \partial_t u(x, 0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $meas(\partial\Omega_D) \neq 0$, $meas(\partial\Omega_N) \neq 0$, $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function with sub-critical growth which satisfies the sign condition $f(s)s \geq 0$, for all $s \in \mathbb{R}$ (see further assumptions (2.2) and (2.4)). Here, $M := (\overline{\Omega}, G)$ is a compact Riemannian manifold where we are inducing on Ω

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a Riemannian metric G, $\nabla \equiv \nabla_G$ is the associated Levi-Civita connection and Δ represents the Laplace Beltrami operator.

The following assumptions are made on the function $a(x)$, responsible for the localized dissipative effect of frictional type:

Assumption 1.1. *We assume that $a(\cdot) \in L^\infty(\Omega)$ is a nonnegative function. In addition, that ω geometrically controls Ω, i.e there exists $T_0 > 0$, such that every geodesic of the metric G, travelling with speed 1 and issued at $t = 0$, enters the set ω in a time $t < T_0$.*

Furthermore,

$$a(x) \geq a_0 > 0 \text{ a. e. in } \omega. \tag{1.2}$$

Setting

$$H^1_{\partial\Omega_D}(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_D\}$$

endowed, thanks to Poincaré inequality, with its natural topology

$$||u||^2_{H^1_{\partial\Omega_D}(\Omega)} := \int_\Omega |\nabla u|^2 \, dx,$$

let also assume the following unique continuation principle holds:

Assumption 1.2. *For every $T > 0$, the only solution v lying in the space $C(]0,T[; L^2(\Omega)) \cap C(]0,T[, H^{-1}_{\partial\Omega_D}(\Omega))$, to system*

$$\begin{cases} \partial_t^2 v - \Delta v + V(x,t)v = 0 \text{ in } \Omega \times (0,T), \\ v = 0 \text{ on } \omega, \end{cases} \tag{1.3}$$

where $V(x,t) \in L^\infty(\Omega \times (0,T)))$, is the trivial one $v \equiv 0$. Here, $H^{-1}_{\partial\Omega_D}(\Omega) = \left[H^1_{\partial\Omega_D}(\Omega)\right]'$.

1.2. **Previous Results, Main Goal and Methodology.** The contribution of the present paper is to introduce a new and a more general approach to obtain the exponential stability of problem (1.1), which generalizes the previous results, and, in addition, can be used for other equations as well regardless of the type of dissipation mechanism considered. In order to obtain the desired stability result for the wave equation subject to a frictional damping, we consider an approximate problem and we show that its solution decays exponentially to zero in the weak phase space. The method of proof combines an observability inequality, microlocal analysis tools and unique continuation properties. Then, passing to the limit, we recover the original model and prove its global existence as well as the exponential stability.

In what follows we are going to explain briefly the methodology we are going to use.

Setting

$$D(-\Delta) := \{v \in H^1_{\partial\Omega_D}(\Omega) : \Delta u \in L^2(\Omega)\},$$

and denoting $v = u_t$ we may rewrite problem (1.1) as the following Cauchy problem in $\mathcal{H} = H^1_{\partial\Omega_D}(\Omega) \times L^2(\Omega)$

$$\begin{cases} \dfrac{\partial}{\partial t}(u,v) = A(u,v) + \mathcal{F}(u,v) \\ \\ (u,v)(0) = (u_0, v_0), \end{cases} \tag{1.4}$$

where the linear unbounded operator $A : D(A) \to \mathcal{H}$ is given by

$$A(u,v) = (v, \Delta u - a(x)v), \tag{1.5}$$

with domain

$$D(A) = D(-\Delta) \times H^1_{\partial\Omega_D}(\Omega), \tag{1.6}$$

and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is the nonlinear operator

$$\mathcal{F}(u, v) = (0, -f(u)). \tag{1.7}$$

It is well known that the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ defined by (1.5) and (1.6) generates a C_0-semigroup of contractions e^{At} on the energy space $\mathcal{H}$ and $D(A)$ is dense in $\mathcal{H}$. For more details, see [26]. Thus, given $\{u_0, u_1\} \in H^1_{\partial\Omega_D}(\Omega) \times L^2(\Omega)$, consider a sequence $\{u_{0,k}, u_{1,k}\} \in D(A)$, satisfying

$$\{u_{0,k}, u_{1,k}\} \to \{u_0, u_1\} \text{ in } H^1_{\partial\Omega_D}(\Omega) \times L^2(\Omega). \tag{1.8}$$

Thus, instead of studying problem (1.1) directly, we shall study, for each $k \in \mathbb{N}$, the auxiliary problem

$$\begin{cases} \partial_t^2 u_k - \Delta u_k + f_k(u_k) + a(x)\partial_t u_k = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u_k = 0 \quad \text{on} \quad \partial\Omega_D \times (0, +\infty), \\ \partial_\nu u_k = 0 \quad \text{on} \quad \partial\Omega_N \times (0, +\infty), \\ u_k(x, 0) = u_{0,k}(x); \quad \partial_t u_k(x, 0) = u_{1,k}(x), \quad x \in \Omega, \end{cases} \tag{1.9}$$

where $f_k : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f_k(s) := \begin{cases} f(s), & |s| \le k, \\ f(k), & s > k, \\ f(-k), & s < -k. \end{cases} \tag{1.10}$$

Here, we use some ideas from Lasiecka and Tataru's work [18] adapted to the present context. The energy identity associated to problem (1.9) is given by

$$E_{u_k}(t) + \int_0^t \int_\Omega a(x)|\partial_t u_k(x, s)|^2 \, dxds = E_{u_k}(0), \text{ for all } t \in [0, +\infty) \text{ and } k \in \mathbb{N}, \tag{1.11}$$

where

$$E_{u_k}(t) := \frac{1}{2} \int_\Omega |\partial_t u_k(x, t)|^2 + |\nabla u_k(x, t)|^2 \, dx + \int_\Omega F_k(u_k(x, t)) \, dx, \tag{1.12}$$

with $F_k(s) := \int_0^s f_k(\lambda) \, d\lambda$. Furthermore, we will also prove the corresponding observability inequality to problem (1.9), that is, we shall prove that there exists a positive constant C which does not depend on k, verifying

$$E_{u_k}(0) \le C \int_0^T \int_\Omega a(x)|\partial_t u_k|^2 \, dx \, dt, \text{ for all } T \ge T_0. \tag{1.13}$$

Finally, passing to the limit in (1.11) and (1.13) as $k \to +\infty$, we achieve the energy identity and the observability inequality associated to problem (1.1), respectively, which are the necessary and sufficient ingredients to establish its exponential stability result. However, in order to established (1.13) we need two facts: (i) To prove the observability inequality associated to the linear problem:

$$\begin{cases} \partial_t^2 y - \Delta y = 0 \quad \text{in } \Omega \times (0, T), \\ y = 0 \quad \text{on} \quad \partial\Omega_D \times (0, T), \\ \partial_\nu y = 0 \quad \text{on} \quad \partial\Omega_N \times (0, T), \\ y(x, 0) = y_0(x) \in H^1_{\partial\Omega_D}(\Omega); \quad \partial_t y(x, 0) = y_1(x) \in L^2(\Omega), \quad x \in \Omega, \end{cases} \tag{1.14}$$

namely, there exists a constant $c > 0$ such that

$$E_y^L(0) \leq c \int_0^T \int_\omega |\partial_t y(x,t)|^2 \, dxdt, \tag{1.15}$$

for all $(y_0, y_1) \in H^1_{\partial\Omega_D}(\Omega) \times L^2(\Omega)$, where $E_y^L(t) := \frac{1}{2}\int_\Omega |\partial_t y(x,t)|^2 + |\nabla y(x,t)|^2 \, dx \, dx$. (ii) The second main ingredient in the proof is to consider the well known property which establishes the linear map $\{z_0, z_1, f\} \in H^1_{\partial_D}(\Omega) \times L^2(\Omega) \times L^1(0,T;L^2(\Omega)) \mapsto \{z, \partial_t z\} \in L^\infty(0,T;H^1_{\partial\Omega_D}(\Omega)) \times L^\infty(0,T;L^2(\Omega))$ associated to problem

$$\begin{cases} \partial_t^2 z - \Delta z = f \quad \text{in } \Omega \times (0,T), \\ z = 0 \quad \text{on} \quad \partial\Omega_D \times (0,T), \\ \partial_\nu z = 0 \quad \text{on} \quad \partial\Omega_N \times (0,T), \\ z(x,0) = z_0(x) \in H^1_{\partial\Omega_D}(\Omega); \quad \partial_t z(x,0) = z_1(x) \in L^2(\Omega), \quad x \in \Omega, \end{cases}$$

is continuous, that is,

$$\begin{aligned} &||z||^2_{L^\infty(0,T;H^1_{\partial\Omega_D}(\Omega))} + ||\partial_t z||^2_{L^\infty(0,T;L^2(\Omega))} \\ &\lesssim ||z_0||^2_{H^1_{\partial\Omega_D}(\Omega)} + ||z_1||^2_{L^\infty(0,T;L^2(\Omega))} + ||f||^2_{L^1(0,T;L^2(\Omega))}. \end{aligned} \tag{1.16}$$

It is worth mentioning, according proved by Haraux [14], the equivalence between the exponential decay of solutions to the second order evolution equation:

$$\begin{cases} \partial_t^2 y - \Delta y + a(x)\partial_t y = 0 \quad \text{in } \Omega \times (0,T), \\ y = 0 \quad \text{on} \quad \partial\Omega_D \times (0,T), \\ \partial_\nu y = 0 \quad \text{on} \quad \partial\Omega_N \times (0,T), \\ y(x,0) = y_0(x) \in H^1_{\partial\Omega_D}(\Omega); \quad \partial_t y(x,0) = y_1(x) \in L^2(\Omega), \quad x \in \Omega, \end{cases} \tag{1.17}$$

(uniformly on bounded sets of $\mathcal{H}$), and the 'controllability property' given in (1.15) of the system governed by the undamped equation (1.14). As a consequence, instead of proving (1.15) it is sufficient to prove the exponential decay of weak solutions to problem (1.17). In order to do that, refined arguments of microlocal analysis will be considered jointly with the characterization given by [17] (Theorem 3), namely:

Theorem 1.1 (Gearhart–Prüss–Huang)**.** *Let e^{At} be a C_0-semigroup in a Hilbert space H and assume that there exists a positive constant $M > 0$ such that $|||e^{At}||| \leq M$ for all $t \geq 0$. Then e^{At} is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and*

$$\sup_{\mu \in \mathbb{R}} |||\left(A - i\mu I_d\right)^{-1}|||_{\mathcal{L}(\mathcal{H})} < +\infty. \tag{1.18}$$

2. Convergence of the auxiliary problem

2.1. **The limit process.** In this section we prove that the sequence $\{u_k\}_{k\in\mathbb{N}}$ of solutions to problem (1.9) converges to the unique solution to the problem (1.1).

The function f satisfies the following hypotheses:

Assumption 2.1. *$f : \mathbb{R} \to \mathbb{R}$ is a C^2 function with sub-critical growth; satisfying the sign condition $f(s)s \geq 0$, for all $s \in \mathbb{R}$, and*

$$f(0) = 0, \quad |f^{(j)}(s)| \leq k_0(1 + |s|)^{p-j}, \text{ for all } s \in \mathbb{R} \text{ and } j = 1, 2. \tag{2.1}$$

In particular, we obtain from (2.1),

$$|f(r)-f(s)| \leq c\left(1+|s|^{p-1}+|r|^{p-1}\right)|r-s|, \text{ for all } s,r \in \mathbb{R}, \tag{2.2}$$

for some $c>0$, with

$$1 \leq p \leq \frac{n+2}{n-2} \text{ if } n \geq 3 \text{ or } p \geq 1 \text{ if } n=1,2. \tag{2.3}$$

In addition,

$$0 \leq F(s) \leq f(s)s, \text{ for all } s \in \mathbb{R}, \tag{2.4}$$

where $F(\lambda) := \int_0^\lambda f(s)\,ds$.

We begin with some preliminary results.

Lemma 2.1. *The distributional derivative f_k' of the function defined in (1.10) is the essentially bounded function $g_k : \mathbb{R} \to \mathbb{R}$ given by*

$$g_k(s) := \begin{cases} f'(s), & |s| \leq k, \\ 0, & s > k, \\ 0, & s < -k. \end{cases} \tag{2.5}$$

Proof. Take $\varphi \in C_0^\infty(\mathbb{R})$. Once $f_k \in L^1_{loc}(\mathbb{R})$ we have

$$\begin{aligned}
&\langle f_k', \varphi\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} = -\int_{\mathbb{R}} f_k(s)\varphi'(s)\,ds \\
&= -\left[\int_{-\infty}^{-k} f_k(s)\varphi'(s)\,ds + \int_{-k}^{k} f_k(s)\varphi'(s)\,ds + \int_{k}^{+\infty} f_k(s)\varphi'(s)\,ds\right] \\
&= -\left[f(-k)\varphi(-k) + f(k)\varphi(k) - f(-k)\varphi(-k) - \int_{-k}^{k} f'(s)\varphi(s)\,ds - f(k)\varphi(k)\right] \\
&= \int_{-k}^{k} f'(s)\varphi(s)\,ds = \int_{\mathbb{R}} g(s)\varphi(s)\,ds.
\end{aligned}$$

□

Consider the following result which will be useful to the proof of Lemma 2.2.

Theorem 2.1. *Let $u \in W^{1,p}(I)$ with $1 \leq p \leq \infty$, where I is a bounded interval of $\mathbb{R}$. Then, there exists $\widetilde{u} \in C(\bar{I})$ such that*

$$u = \widetilde{u} \text{ a.e. in } I$$

and

$$\widetilde{u}(x) - \widetilde{u}(y) = \int_y^x u'(t)dt \text{ for all } x,y \in \bar{I}.$$

Proof. See Brezis [4], Theorem 8.2. □

Lemma 2.2. *For each $k \in \mathbb{N}$, there exists a positive constant C_k verifying*

$$|f_k(r) - f_k(s)| \leq C_k|r-s| \text{ for every} r,s \in \mathbb{R},$$

where f_k is the function defined in (1.10).

Proof. Consider $s, r \in \mathbb{R}$ with $s < r$. Applying Theorem 2.1 for $I =]s, r[$, it follows that

$$f_k(r) - f_k(s) = \int_s^r f_k'(\xi)\, d\xi.$$

Thus, Lemma 2.1 yields the following inequality:

$$|f_k(r) - f_k(s)| \leq \int_s^r |f_k'(\xi)|\, d\xi \leq \sup_{s\in[-k,k]} |g_k(s)|\, |r - s|, \tag{2.6}$$

which concludes the proof. □

From Lemma 2.2, for each $k \in \mathbb{N}$, standard arguments of Semigroup theory yield that problem (1.9) possesses an unique regular solution u_k in the class

$$C^0([0,\infty); D(-\Delta)) \cap C^1([0,\infty); H^1_{\partial\Omega_D}(\Omega)) \cap C^2([0,\infty); L^2(\Omega)).$$

Multiplying the first equation of (1.9) by $\partial_t u_k$ and performing integration by parts, it yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\partial_t u_k(t)||^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}||\nabla u_k(t)||^2_{L^2(\Omega)} + \frac{d}{dt}\int_\Omega F_k(u_k(x,t))\, dxdt \\ &+ \int_\Omega a(x)|\partial_t u_k(x,t)|^2\, dx = 0, \text{ for all } t \in [0,\infty), \end{aligned} \tag{2.7}$$

where

$$F_k(\lambda) = \int_0^\lambda f_k(s)\, ds. \tag{2.8}$$

Hence, taking (2.7) into account, we infer

$$E_{u_k}(t) \;+\; \int_0^t \int_\Omega a(x)|\partial_t u_k(x,s)|^2\, dxds = E_{u_k}(0), \text{ for all } t \in [0,+\infty) \text{ and } k \in \mathbb{N},$$

where

$$E_{u_k}(t) := \frac{1}{2}\int_\Omega |\partial_t u_k(x,t)|^2 + |\nabla u_k(x,t)|^2\, dx + \int_\Omega F_k(u_k(x,t))\, dx, \tag{2.9}$$

is the energy associated to problem (1.9).

We observe that from (1.10), the function defined in (2.8) is given by

$$F_k(s) := \begin{cases} \displaystyle\int_0^s f(\xi)\, d\xi, & |s| \leq k, \\ \displaystyle\int_0^k f(\xi)\, d\xi + f(k)[s-k], & s > k, \\ f(-k)[s+k] + \displaystyle\int_0^{-k} f(\xi)\, d\xi, & s < -k. \end{cases} \tag{2.10}$$

Since f satisfies the sign condition, it results that $F_k(s) \geq 0$ for all $s \in \mathbb{R}$ and $k \in \mathbb{N}$. In addition, from (2.2) and (2.4), we obtain, respectively, that $|f(s)| \leq c[|s| + |s|^p]$ and $0 \leq F(s) \leq f(s)\, s$ for all $s \in \mathbb{R}$. Then, we infer that

$$|F_k(s)| \leq c[|s|^2 + |s|^{p+1}], \text{ for all } s \in \mathbb{R} \text{ and } k \in \mathbb{N}. \tag{2.11}$$

Consequently,

$$\begin{aligned} \int_\Omega |F_k(u_{0,k})|\, dx &\leq c\int_\Omega \left[|u_{0,k}|^2 + |u_{0,k}|^{p+1}\right]\, dx \\ &\lesssim ||u_{0,k}||_{H^1_{\partial\Omega_D}(\Omega)}. \end{aligned} \tag{2.12}$$

Assuming that $p \geq 1$ is under conditions (2.3), we have for every dimension $n \geq 1$ that $H^1_{\partial\Omega_D}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, which implies that the RHS of (2.12) is bounded. So, estimates (2.9) (also called energy identity for the auxiliary problem (1.9) and (2.12) and convergence (1.8), yield a subsequence of $\{u_k\}$, reindexed again by $\{u_k\}$, such that

$$u_k \rightharpoonup u \text{ weakly * in } L^\infty(0,\infty; H^1_{\partial\Omega_D}(\Omega)), \tag{2.13}$$

$$\partial_t u_k \rightharpoonup \partial_t u \text{ weakly * in } L^\infty(0,\infty; L^2(\Omega)), \tag{2.14}$$

$$\sqrt{a(x)}\partial_t u_k \rightharpoonup \sqrt{a(x)}\partial_t u \text{ weakly in } L^2(0,\infty; L^2(\Omega)). \tag{2.15}$$

Employing the standard compactness result (see Simon [29]) we also deduce that

$$u_k \to u \text{ strongly in } L^\infty(0,T; L^{2^*-\eta}(\Omega)); \text{ for all } T > 0, \tag{2.16}$$

where $2^* := \frac{2n}{n-2}$ and $\eta > 0$ is small enough. In addition, from (2.16), we obtain

$$u_k \to u \text{ a. e. in } \Omega \times (0,T), \text{ for all } T > 0. \tag{2.17}$$

On the other hand, from (2.2), (2.3), (2.13) and once $H^1_{\partial\Omega_D}(\Omega) \hookrightarrow L^{p+1}(\Omega) \hookrightarrow L^{\frac{p+1}{p}}(\Omega)$ the following estimate holds:

$$\begin{aligned}
\|f_k(u_k)\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}} &= \int_0^T \int_\Omega |f_k(u_k(x,t))|^{\frac{p+1}{p}}\, dxdt \\
&\lesssim \int_0^T \int_\Omega |u_k|^{\frac{p+1}{p}}\, dxdt + \int_0^T \int_\Omega |u_k|^{p+1}\, dxdt \\
&= \int_0^T \|u_k\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\Omega)}\, dt + \int_0^T \|u_k\|^{p+1}_{L^{p+1}(\Omega)}\, dt \\
&\lesssim \int_0^T \|u_k\|^{\frac{p+1}{p}}_{H^1_{\partial\Omega_D}(\Omega)}\, dt + \int_0^T \|u_k\|^{p+1}_{H^1_{\partial\Omega_D}(\Omega)}\, dt \\
&\lesssim \|u_k\|^{\frac{p+1}{p}}_{L^\infty(0,T;H^1_{\partial\Omega_D}(\Omega))} + \|u_k\|^{p+1}_{L^\infty(0,T;H^1_{\partial\Omega_D}(\Omega))} \\
&\leq c < +\infty, \quad \text{for all } t \geq 0.
\end{aligned} \tag{2.18}$$

It is easy to see that

$$f(u) \in L^\infty(0,\infty; L^{\frac{p+1}{p}}(\Omega)). \tag{2.19}$$

Indeed,

$$\begin{aligned}
\int_\Omega |f(u(x,t))|^{\frac{p+1}{p}}\, dx &\lesssim \int_\Omega |u(x,t)|^{\frac{p+1}{p}}\, dx + \int_\Omega |u(x,t)|^{p+1}\, dx \\
&\lesssim \|u(\cdot,t)\|^{\frac{p+1}{p}}_{H^1_{\partial\Omega_D}(\Omega)} + \|u(\cdot,t)\|^{p+1}_{H^1_{\partial\Omega_D}(\Omega)} \\
&\lesssim \|u\|^{\frac{p+1}{p}}_{L^\infty(0,T;H^1_{\partial\Omega_D}(\Omega))} + \|u\|^{p+1}_{L^\infty(0,T;H^1_{\partial\Omega_D}(\Omega))} < +\infty, \quad \text{for all } t \geq 0.
\end{aligned} \tag{2.20}$$

From (2.20) and the definition of essential supremum we obtain (2.19).

In addition, from (2.17) and the continuity of the function f, we get

$$f_k(u_k) \to f(u) \text{ a. e. in } \Omega \times (0,T), \text{ for all } T > 0. \tag{2.21}$$

Indeed, the convergence (2.17) guarantees the existence of set $Z_T \subset \Omega \times (0,T)$ with $\mathrm{meas}(Z_T) = 0$ such that $u_k(x,t) \to u(x,t)$ for all $(x,t) \in \Omega \times (0,T) \setminus Z_T$ when $k \to \infty$. Therefore, for all

$(x,t) \in \Omega \times (0,T) \setminus Z_T$ there exists a positive constant $L = L(x,t) > 0$ verifying $|u_k(x,t)| < L$, for all $k \in \mathbb{N}$. Then, using the definition of f_k, we obtain that

$$\text{if } |u_k(x,t)| < L, \text{ for all } k \in \mathbb{N} \text{ then } f_k(u_k(x,t)) = f(u_k(x,t)), \text{ for all } k \geq L, \tag{2.22}$$

that is,

$$f_k(u_k(x,t)) - f(u_k(x,t)) \to 0 \text{ when } k \to \infty \text{ for all } (x,t) \in \Omega \times (0,T) \setminus Z_T. \tag{2.23}$$

On the other hand, employing the continuity of f it follows that

$$f(u_k(x,t)) - f(u(x,t)) \to 0 \text{ when } k \to \infty \text{ for all } (x,t) \in \Omega \times (0,T) \setminus Z_T. \tag{2.24}$$

From (2.23) and (2.24) the convergence (2.21) holds.

Lemma 2.3 (Strauss)**.** *Let $\mathcal{O}$ be an open and bounded subset of $\mathbb{R}^N$, $N \geq 1$, $1 < q < +\infty$ and $\{u_n\}_{n\in\mathbb{N}}$ a sequence which is bounded in $L^q(\mathcal{O})$. If $u_n \to u$ a.e. in $\mathcal{O}$, then $u \in L^q(\mathcal{O})$ and $u_n \rightharpoonup u$ weakly in $L^q(\mathcal{O})$. In addition, if $1 \leq r < q$ we also have $u_n \to u$ strongly in $L^r(\mathcal{O})$.*

Proof. See [4] (Exercise 4.16) or [30]. □

Gathering together (2.18), (2.19) and Lions' Lemma, we deduce that

$$f_k(u_k) \rightharpoonup f(u) \text{ weakly in } L^{\frac{p+1}{p}}(\Omega \times (0,T)). \tag{2.25}$$

Going back to problem (1.9), multiplying by $\varphi\,\theta$, where $\varphi \in C_0^\infty(\Omega), \theta \in C_0^\infty(0,T)$ and performing integration by parts, we obtain

$$\begin{aligned} &-\int_0^T \theta'(t) \int_\Omega \partial_t u_k(x,t)\,\varphi(x)\,dxdt + \int_0^T \theta(t) \int_\Omega \nabla u_k(x,t) \cdot \nabla\varphi(x)\,dxdt \\ &+ \int_0^T \theta(t) \int_\Omega f_k(u_k(x,t))\,\varphi(x)\,dxdt + \int_0^T \theta(t) \int_\Omega a(x)\partial_t u_k(x,t)\varphi(x)\,dxdt = 0. \end{aligned} \tag{2.26}$$

Passing to the limit in (2.26) and observing convergences (2.13)-(2.15) and (2.25), we get

$$\begin{aligned} &-\int_0^T \theta'(t) \int_\Omega \partial_t u(x,t)\,\varphi(x)\,dxdt + \int_0^T \theta(t) \int_\Omega \nabla u(x,t) \cdot \nabla\varphi(x)\,dxdt \\ &+ \int_0^T \theta(t) \int_\Omega f(u(x,t))\,\varphi(x)\,dxdt + \int_0^T \theta(t) \int_\Omega a(x)\partial_t u(x,t)\varphi(x)\,dxdt = 0, \end{aligned} \tag{2.27}$$

for all $\varphi \in C_0^\infty(\Omega)$ and $\theta \in C_0^\infty(0,T)$. We conclude that

$$\partial_t^2 u - \Delta u + f(u) + a(x)\partial_t u = 0 \text{ in } \mathcal{D}'(\Omega \times (0,T)), \tag{2.28}$$

and since

$$a(\cdot)\partial_t u \in L^\infty(0,T;L^2(\Omega)),\ \Delta u \in L^\infty(0,T;H^{-1}_{\partial\Omega_D}(\Omega)),\ (\text{here } H^{-1}_{\partial\Omega_D}(\Omega) = (H^1_{\partial\Omega_D}(\Omega))'),$$

$$a(x)\partial_t u \in L^2(0,T;L^2(\Omega)) \text{ and } f(u) \in L^\infty(0,T;L^{\frac{p+1}{p}}(\Omega)),$$

we deduce that $\partial_t^2 u \in L^2(0,T;H^{-1}_{\partial\Omega_D}(\Omega))$ and

$$\partial_t^2 u - \Delta u + f(u) + a(x)\partial_t u = 0 \text{ in } L^2(0,T;H^{-1}_{\partial\Omega_D}(\Omega)). \tag{2.29}$$

Applying Lemma 8.1 of Lions-Magenes [22], we deduce that

$$u \in C_w(0,T;H^1_{\partial\Omega_D}(\Omega)) \text{ and } \partial_t u \in C_w(0,T;L^2(\Omega)), \tag{2.30}$$

where $C_w(0,T;Y) =$ space of functions $f \in L^\infty(0,T;Y)$ whose mappings $[0,T] \mapsto Y$ are weakly continuous, that is, $t \mapsto \langle y', f(t)\rangle_{Y',Y}$ is continuous in $[0,T]$ for all $y' \in Y'$, dual of Y.

Our first result reads as follows:

Theorem 2.2. *Assume that $a \in L^\infty(\Omega)$ and $f \in C^1(\mathbb{R})$ satisfies $f(s)s \geq 0$ for all $s \in \mathbb{R}$. In addition, suppose that assumptions (2.2), (2.3) and (2.4) are in place. Then, problem (1.1) has at least a global solution in the class*

$$u \in C_w(0,T;H^1_{\partial\Omega_D}(\Omega)),\ \partial_t u \in C_w(0,T;L^2(\Omega)),\ \partial_t^2 u \in L^2(0,T;H^{-1}_{\partial\Omega_D}(\Omega)),$$

provided that $\{u_0,u_1\} \in H^1_{\partial\Omega_D}(\Omega) \times L^2(\Omega)$. Furthermore, assuming that $1 \leq p \leq \frac{n}{n-2}, n \geq 3$ or $p \geq 1, n = 1,2$, we have the uniqueness of solution.

Proof. The uniqueness of solution as well as to prove that $u(0) = u_0$ and $\partial_t u(0) = u_1$ follow the same ideas used in Lions [21] (Theorem 1.2). □

2.2. **Recovering the regularity in time for the range $1 \leq p < \frac{n}{n-2}$, $n \geq 3$.** When $p \geq 1$, $n = 1,2$, the result is trivially verified and it will be omitted.

The goal of this subsection is to prove that if $1 \leq p < \frac{n}{n-2}$, $n \geq 3$, the related solutions to problem (1.1) are in the class

$$u \in C^0([0,T];H^1_{\Omega_D}(\Omega)),\ \partial_t u \in C^0([0,T];L^2(\Omega))$$

and, in addition, one has

$$\{u_k, \partial_t u_k\} \to \{u, \partial_t u\} \text{ in } C^0([0,T];H^1_{\Omega_D}(\Omega)) \times C^0([0,T];L^2(\Omega)).$$

To prove the above statements, we need to prove that

$$f_k(u_k) \to f(u) \text{ strongly in } L^2(\Omega \times (0,T)). \tag{2.31}$$

In fact, first we observe that

$$\begin{aligned} &\int_0^T \int_\Omega |f_k(u_k) - f(u)|^2\, dxdt \\ &\lesssim \int_0^T \int_\Omega |f_k(u_k) - f(u_k)|^2\, dxdt + \int_0^T \int_\Omega |f(u_k) - f(u)|^2\, dxdt. \end{aligned} \tag{2.32}$$

In view of (2.2) one has

$$\begin{aligned} \int_\Omega |f(u_k) - f(u)|^2\, dx &\lesssim \int_\Omega |u_k - u|^2\, dx + \int_\Omega |u_k|^{2(p-1)}|u_k - u|^2\, dx \\ &\quad + \int_\Omega |u|^{2(p-1)}|u_k - u|^2\, dx \\ &= I_{1,k} + I_{2,k} + I_{3,k} \end{aligned}$$

We observe that since $\frac{p-1}{p} + \frac{1}{p} = 1$, Hölder inequality yields

$$I_{2,k} \leq \left(\int_\Omega |u_k|^{2p}\right)^{\frac{p-1}{p}} \left(\int_\Omega |u_k - u|^{2p}\right)^{\frac{1}{p}}.$$

Choosing $p < \frac{n}{n-2}$ it implies that $2p < \frac{2n}{n-2} = 2^*$ and, consequently, from (2.13) and (2.16) we deduce that $I_{2,k} \to 0$ as $k \to +\infty$. Analogously, we also deduce that $I_{3,k} \to 0$ as $k \to +\infty$. We trivially obtain that $I_{1,k} \to 0$ as $k \to +\infty$. Then,

$$\int_0^T \int_\Omega |f(u_k) - f(u)|^2\, dxdt \to 0 \text{ as } k \to \infty. \tag{2.33}$$

From (2.32) it remains to prove that

$$\int_0^T \int_\Omega |f_k(u_k) - f(u_k)|^2 \, dxdt \to 0 \text{ as } k \to \infty. \tag{2.34}$$

Let us consider, initially, $t \in [0, T]$ fixed and define

$$\Omega_k^t := \{x \in \Omega : |u_k(x,t)| > k\}.$$

Observing that

$$f_k(u_k) - f(u_k) = 0, \text{ if } |u_k(x,t)| \leq k,$$

we have

$$\begin{aligned} \int_\Omega |f_k(u_k) - f(u_k)|^2 \, dx &= \int_{\Omega_k^t} |f_k(u_k) - f(u_k)|^2 \, dx \\ &\lesssim \left[\int_{\Omega_k^t} |f(u_k)|^2 \, dx + \int_{\Omega_k^t} |f(-k)|^2 \, dx + \int_{\Omega_k^t} |f(k)|^2 \, dx \right] \\ &\lesssim \left[\int_{\Omega_k^t} [|u_k|^2 + |u_k|^{2p}] \, dx + \int_{\Omega_k^t} [|k|^2 + |k|^{2p}] \, dx \right] \\ &\lesssim \left[\int_{\Omega_k^t} |u_k|^{2p} \, dx + \int_{\Omega_k^t} |k|^{2p} \, dx \right] \\ &\lesssim \int_{\Omega_k^t} |u_k|^{2p} \, dx. \end{aligned} \tag{2.35}$$

Before analyzing the term on the RHS of (2.35) we note that since $H^1_{\Omega_D}(\Omega) \hookrightarrow L^{\frac{2n-\frac{1}{2}}{n-2}}(\Omega)$ and the convergence (1.8) are in place, we obtain

$$\begin{aligned} \left(\int_{\Omega_k^t} k^{\frac{2n-\frac{1}{2}}{n-2}} \, dx \right) &\lesssim \left(\int_{\Omega_k^t} |u_k|^{\frac{2n-\frac{1}{2}}{n-2}} \, dx \right) \\ &= ||u_k(t)||^{\frac{2n-\frac{1}{2}}{n-2}}_{L^{\frac{2n-\frac{1}{2}}{n-2}}(\Omega_k^t)} \lesssim ||u_k(t)||^{\frac{2n-\frac{1}{2}}{n-2}}_{H^1_{\Omega_D}(\Omega)} \lesssim [E_{u_k}(0)]^{\frac{2n-\frac{1}{2}}{n-2}} \leq C, \end{aligned} \tag{2.36}$$

for all $t \in [0, T]$, where C is a positive constant which does not depend on k and t. Thus, it yields

$$\text{meas}(\Omega_k^t) \lesssim k^{\frac{-2n+\frac{1}{2}}{n-2}}, \text{ for all } t \in [0, T]. \tag{2.37}$$

Let $\beta := \frac{2n}{(2p)(n-2)}$, for $n \geq 3$. Observe that we have the following inequalities:

$$p < \frac{n}{n-2} \Leftrightarrow 2n > (2p)(n-2) \Leftrightarrow 2p < \frac{2n}{n-2} = 2^* \Leftrightarrow \beta > 1.$$

Setting $\alpha > 0$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we deduce that $\alpha = \frac{2n}{2n-(2p)(n-2)}$ and using Hölder inequality we get

$$\begin{aligned} \int_{\Omega_k^t} |u_k|^{2p} \, dx &\leq (\text{meas}(\Omega_k))^{\frac{2n-(2p)(n-2)}{2n}} \left(\int_{\Omega_k^t} |u_k|^{\frac{2n}{n-2}} \right)^{\frac{(2p)(n-2)}{2n}} \\ &= (\text{meas}(\Omega_k))^{\frac{2n-(2p)(n-2)}{2n}} \, ||u_k(t)||^{2p}_{L^{\frac{2n}{n-2}}(\Omega)}. \end{aligned} \tag{2.38}$$

Thus, from (2.37) and (2.38) we conclude

$$\begin{aligned}\int_0^T\int_{\Omega_k^t}|u_k|^{2p}\,dx &\leq k^{\left(\frac{-2n+\frac{1}{2}}{n-2}\right)\left(\frac{2n-(2p)(n-2)}{2n}\right)}\int_0^T||u_k(t)||^{2p}_{L^{\frac{2n}{n-2}}(\Omega)}\,dt\\ &\lesssim k^{\left(\frac{-2n+\frac{1}{2}}{n-2}\right)\left(\frac{2n-(2p)(n-2)}{2n}\right)}\int_0^T||u_k(t)||^{2p}_{H^1_{\Omega_D}(\Omega)}\,dt\\ &\lesssim k^{\left(\frac{-2n+\frac{1}{2}}{n-2}\right)\left(\frac{2n-(2p)(n-2)}{2n}\right)}[E_{u_k}(0)]^p,\end{aligned}\tag{2.39}$$

Employing the fact that $E_{u_k}(0)\leq C$ for all $k\in\mathbb{N}$ and $\left(\frac{-2n+\frac{1}{2}}{n-2}\right)\left(\frac{2n-(2p)(n-2)}{2n}\right)<0$, in light of inequality (2.39) , we prove that

$$\int_0^T\int_{\Omega_k^t}|u_k|^{2p}\,dx\to 0,\ \text{ as } k\to+\infty.\tag{2.40}$$

Gathering (2.35) and (2.40) together, we conclude (2.34) which proves (2.31).

Now, we define the sequence $z_{\mu,\sigma}=u_\mu-u_\sigma$, $\mu,\sigma\in\mathbb{N}$, and from (1.9) we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{||\partial_t z_{\mu,\sigma}(t)||^2_{L^2(\Omega)}+||\nabla z_{\mu,\sigma}(t)||^2_{L^2(\Omega)}\right\}+\int_\Omega a(x)|\partial_t z_{\mu,\sigma}|^2\,dx\\ &=\int_\Omega\left(f_\mu(u_\mu)-f_\sigma(u_\sigma)\right)\left(\partial_t u_\mu-\partial_t u_\sigma\right)dx.\end{aligned}\tag{2.41}$$

Integrating (2.41) over $(0,t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\left\{||\partial_t z_{\mu,\sigma}(t)||^2_{L^2(\Omega)}+||\nabla z_{\mu,\sigma}(t)||^2_{L^2(\Omega)}\right\}+\int_0^t\int_\Omega a(x)|\partial_t z_{\mu,\sigma}|^2\,dxds\\ &\leq\frac{1}{2}\left\{||u_{1,\mu}-u_{1,\sigma}||^2_{L^2(\Omega)}+||\nabla u_{0,\mu}-\nabla u_{0,\sigma}||^2_{L^2(\Omega)}\right\}\\ &+\int_0^t\int_\Omega\left(f_\mu(u_\mu)-f_\sigma(u_\sigma)\right)\left(\partial_t u_\mu-\partial_t u_\sigma\right)dxds.\end{aligned}\tag{2.42}$$

The convergences (1.8), (2.14) and (2.31) imply that the terms on the RHS of the (2.42) converges to zero as $\mu,\sigma\to+\infty$. Thus, we deduce that

$$u_\mu\to u \text{ in } C^0([0,T];H^1_{\Omega_D}(\Omega))\cap C^1([0,T];L^2(\Omega)),\tag{2.43}$$

$$\lim_{\mu\to+\infty}\int_0^T\int_\Omega a(x)|\partial_t u_\mu|^2dx\,ds=\int_0^T\int_\Omega a(x)|\partial_t u|^2dx\,ds,\tag{2.44}$$

for all $T>0$.

2.3. **Estimating $F_k(u_k)$.** Inequality (2.11) gives

$$|F_k(s)|\leq c[|s|^2+|s|^{p+1}],$$

for all $s\in\mathbb{R}$ and $k\in\mathbb{N}$.

Since $1 \leq p < \frac{n}{n-2}$ if $n \geq 3$ and $\frac{n}{n-2} < \frac{n+2}{n-2}$ we obtain $2 \leq p+1 < \frac{2n}{n-2} = 2^*$. Consequently, there exists $\varepsilon > 0$ such that $p+1+\varepsilon = 2^*$. Then, $H^1_{\Omega_D}(\Omega) \hookrightarrow L^{p+1+\varepsilon}(\Omega)$ and, consequently,

$$\begin{aligned} \int_\Omega |F_k(u_{0,k})|^{\frac{p+1+\varepsilon}{p+1}}\,dx &\leq c\int_\Omega |u_{0,k}|^{\frac{2(p+1+\varepsilon)}{p+1}} + |u_{0,k}|^{p+1+\varepsilon}\,dx \\ &\lesssim ||u_{0,k}||^{p+1+\varepsilon}_{H^1_{\Omega_D}(\Omega)} \leq C. \end{aligned} \tag{2.45}$$

Analogously,

$$\int_\Omega |F_k(u_k(x,t_0))|^{\frac{p+1+\varepsilon}{p+1}}\,dx \lesssim ||u_k(\cdot,t_0)||^{p+1+\varepsilon}_{H^1_{\Omega_D}(\Omega)} \leq CE_{u_k}(0)^{p+1+\varepsilon}, \tag{2.46}$$

for all $t_0 \in [0,T]$. The boundedness of $E_{u_k}(0)$ implies that there exists $\chi \in L^{\frac{2^*}{p+1}}(\Omega)$ verifying the following convergence:

$$F_k(u_k(\cdot,t_0)) \rightharpoonup \chi \text{ weakly in } L^{\frac{2^*}{p+1}}(\Omega), \text{ as } k \to +\infty. \tag{2.47}$$

In what follows we are going to prove that $\chi = F(u(\cdot,t_0))$. Indeed, from (2.43) we obtain $u_k(\cdot,t_0) \to u(\cdot,t_0)$ strongly in $L^2(\Omega)$. Thus,

$$u_k(x,t_0) \to u(x,t_0) \text{ a. e. in } \Omega. \tag{2.48}$$

Note that,

$$\begin{aligned} &|F_k(u_k(x,t_0)) - F(u(x,t_0))| \\ &\leq |F_k(u_k(x,t_0)) - F(u_k(x,t_0))| + |F(u_k(x,t_0)) - F(u(x,t_0))|. \end{aligned} \tag{2.49}$$

The convergence (2.48) and the continuity of F imply

$$F(u_k(x,t_0)) \to F(u(x,t_0)) \text{ a. e. in } \Omega. \tag{2.50}$$

In light of inequality (2.49), to prove that

$$F_k(u_k(x,t_0)) \to F(u(x,t_0)) \text{ a. e. in } \Omega. \tag{2.51}$$

it remains to prove that

$$F_k(u_k(x,t_0)) - F(u_k(x,t_0)) \to 0 \text{ a. e. in } \Omega,$$

In fact, from (2.22), there exists a positive constant $L = L(x,t) > 0$ such that

$$\begin{aligned} |F_k(u_k(x,t_0)) - F(u_k(x,t_0))| &= \left|\int_0^{u_k(x,t_0)} f_k(s)ds - \int_0^{u_k(x,t_0)} f(s)ds\right| \\ &\leq \int_{-L}^{L} |f_k(s) - f(s)|ds = 0, \text{ if } k \geq L. \end{aligned} \tag{2.52}$$

Therefore, combining (2.49), (2.50) and (2.52), we obtain (2.51). Thus, from (2.46) and Lions Lemma we deduce that

$$F_k(u_k(\cdot,t_0)) \rightharpoonup F(u(\cdot,t_0)) \text{ weakly in } L^{\frac{2^*}{p+1}}(\Omega), \text{ as } k \to +\infty, \tag{2.53}$$

proving that $\chi = F(u(\cdot,t_0))$.

In addition, employing Strauss Lemma we also deduce that

$$F_k(u_k(\cdot,t_0)) \to F(u(\cdot,t_0)) \text{ strongly in } L^r(\Omega), \text{ as } k \to +\infty, \tag{2.54}$$

for all $1 \leq r < \frac{2^*}{p+1}$ and $t_0 \in [0,T]$.

Now we are in a position to establish the following result:

Theorem 2.3. *Assume that $a \in L^\infty(\Omega)$ is a nonnegative function and $f \in C^1(\mathbb{R})$ satisfies $f(s)s \geq 0$ for all $s \in \mathbb{R}$. In addition, suppose that f verifies assumption (2.2) with $1 \leq p < \frac{n}{n-2}, n \geq 3$ and $p \geq 1, n = 1, 2$ and assumption (2.4). Then, given $\{u_0, u_1\} \in H^1_{\Omega_D}(\Omega) \times L^2(\Omega)$ problem (1.1) has an unique global solution in the class*

$$u \in C^0([0,T]; H^1_{\Omega_D}(\Omega)),\ \partial_t u \in C^0([0,T]; L^2(\Omega)),\ \partial_t^2 u \in L^2(0,T; H^{-1}_{\partial\Omega_D}(\Omega)).$$

In addition, the energy identity is verified, namely

$$E_u(t_2) + \int_{t_1}^{t_2} \int_\Omega a(x)|a(x)\partial_t u(x,t)|^2\, dxdt = E_u(t_1),\ 0 \leq t_1 \leq t_2 < +\infty, \text{ where} \tag{2.55}$$

$$E_u(t) := \frac{1}{2}\int_\Omega |\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2\, dxdt + \int_\Omega F(u(x,t))\, dxdt.$$

3. Exponential Decay to Problem (1.1)

Throughout this section we will assume that $1 \leq p < \frac{n}{n-2}$ if $n \geq 3$ and $p \geq 1$ if $n = 1, 2$. Under these conditions we have the following embeddings:

$$H^1_{\Omega_D}(\Omega) \hookrightarrow L^{2p}(\Omega) \hookrightarrow L^p(\Omega). \tag{3.1}$$

Consider the auxiliary problem

$$\begin{cases} \partial_t^2 u_k - \Delta u_k + f_k(u_k) + a(x)\partial_t u_k = 0 \quad \text{in } \Omega \times (0,+\infty), \\ u_k = 0 \quad \text{on} \quad \partial\Omega_D \times (0,+\infty), \\ \partial_\nu u_k = 0 \quad \text{on} \quad \partial\Omega_N \times (0,+\infty), \\ u_k(x,0) = u_{0,k}(x); \quad \partial_t u_k(x,0) = u_{1,k}(x), \quad x \in \Omega, \end{cases} \tag{3.2}$$

whose associated energy functional is given by

$$E_{u_k}(t) := \frac{1}{2}\int_\Omega |\partial_t u_k(x,t)|^2 + |\nabla u_k(x,t)|^2\, dxdt + \int_\Omega F_k(u_k(x,t))\, dxdt, \tag{3.3}$$

where $F_k(\lambda) = \int_0^\lambda F_k(s)\, ds$ and the energy identity reads as follows

$$E_{u_k}(t_2) - E_{u_k}(t_1) = -\int_{t_1}^{t_2}\int_\Omega a(x)|\partial_t u_k|^2\, dxdt, \tag{3.4}$$

for all $0 \leq t_1 \leq t_2 < +\infty$.

Let $T_0 > 0$ be associated to the geometric control condition, that is, every ray of the geometric optics enters ω in a time $T^* < T_0$. Thus, our goal is to prove the observability inequality established in the following lemma.

Lemma 3.1. *There exists $k_0 \geq 1$ such that for every $k \geq k_0$, the corresponding solution u_k of* (3.2) *satisfies the inequality*

$$E_{u_k}(0) \leq C\int_0^T \int_\Omega a(x)|\partial_t u_k|^2\, dxdt\, dxdt, \tag{3.5}$$

for all $T > T_0$ and for some positive constant $C = C(||\{u_0, u_1\}||_{H^1_0(\Omega)\times L^2(\Omega)})$.

Proof. The initial datum $\{u_0, u_1\} \in H^1_{\Omega_D}(\Omega) \times L^2(\Omega)$ in the original problem (1.1) is either zero or not zero.

In the first case, when $\{u_0, u_1\} = (0,0)$ and, observing (1.8), we can consider $\{u_{0,k}, u_{1,k}\} = (0,0)$ for all $k \geq 1$ and the corresponding unique solution to the auxiliary problem (1.9) will be $u_k \equiv 0$. Then, (3.5) is verified.

In the second case, there exists a positive number $R > 0$ such that

$$0 < ||\{u_0, u_1\}||_{H^1_{\Omega_D}(\Omega)\times L^2(\Omega)} < R,$$

consider, for instance $R = 2||\{u_0, u_1\}||_{H^1_{\Omega_D}(\Omega)\times L^2(\Omega)}$.

Therefore, there exists, $k_0 \geq 1$ such that for all $k \geq k_0$, $\{u_{0,k}, u_{1,k}\}$ satisfies

$$||\{u_{0,k}, u_{1,k}\}||_{H^1_{\Omega_D}(\Omega)\times L^2(\Omega)} < R. \tag{3.6}$$

We are going to prove that under condition (3.6) on the initial datum, the corresponding solution u_k to (1.9) satisfies (3.5). Our proof relies on contradiction arguments. So, if (3.5) is false, then there exists $T > T_0$ such that for every $k \geq 1$ and every constant $C > 0$, there exists an initial datum $\{u^C_{0,k}, u^C_{1,k}\}$ verifying (3.6), whose corresponding solution u^C_k violates (3.5).

In particular, for every $k \geq 1$ and $C = m \in \mathbb{N}$, we obtain the existence of an initial datum $\{u^m_{0,k}, u^m_{1,k}\}$ verifying (3.6) and whose corresponding solution u^m_k satisfies

$$E_{u^m_k}(0) > m \int_0^T \int_\Omega a(x)|\partial_t u^m_k|^2 \, dxdt. \tag{3.7}$$

Then, we obtain a sequence $\{u^m_k\}_{m\in\mathbb{N}}$ of solutions to problem (1.9) such that

$$\lim_{m\to+\infty} \frac{E_{u^m_k}(0)}{\int_0^T \int_\Omega a(x)|\partial_t u^m_k|^2 \, dxdt} = +\infty.$$

Equivalently

$$\lim_{m\to+\infty} \frac{\int_0^T \int_\Omega a(x)|\partial_t u^m_k|^2 \, dxdt}{E_{u^m_k}(0)} = 0. \tag{3.8}$$

Since $E_{u^m_k}(0)$ is bounded, (3.8) yields

$$\lim_{m\to+\infty} \int_0^T \int_\Omega a(x)|\partial_t u^m_k|^2 \, dxdt = 0. \tag{3.9}$$

Furthermore, there exists a subsequence of $\{u^m_k\}_{m\in\mathbb{N}}$, still denoted by $\{u^m_k\}$, verifying the following convergences:

$$u^m_k \rightharpoonup u_k \text{ weakly-star in } L^\infty(0,T; H^1_{\Omega_D}(\Omega)), \quad \text{as } m \to +\infty, \tag{3.10}$$

$$\partial_t u^m_k \rightharpoonup \partial_t u_k \text{ weakly-star in } L^\infty(0,T; L^2(\Omega)), \quad \text{as } m \to +\infty, \tag{3.11}$$

$$u^m_k \to u_k \text{ strongly in } L^\infty(0,T; L^q(\Omega)), \quad \text{as } m \to +\infty, \quad \text{for all } q \in \left[2, \frac{2n}{n-2}\right), \tag{3.12}$$

where the last convergence is obtained using Aubin-Lions-Simon Theorem (see [29]). The proof is divided into two distinguished cases: $u_k \neq 0$ and $u_k = 0$.

Case (a): $u_k \neq 0$.

For $m \in \mathbb{N}$, u^m_k is the solution to the problem

$$\begin{cases} \partial_t^2 u^m_k - \Delta u^m_k + f_k(u^m_k) + a(x)\partial_t u^m_k = 0 \quad \text{in } \Omega \times (0,T), \\ u^m_k = 0 \quad \text{on} \quad \partial\Omega_D \times (0,T), \\ \partial_\nu u^m_k = 0 \quad \text{on} \quad \partial\Omega_N \times (0,T), \\ u^m_k(x,0) = u^m_{0,k}(x); \quad \partial_t u^m_k(x,0) = u^m_{1,k}(x), \quad x \in \Omega. \end{cases}$$

Taking (3.9)-(3.12) into consideration we obtain

$$\begin{cases} \partial_t^2 u_k - \Delta u_k + f_k(u_k) = 0 \quad \text{in } \Omega \times (0,T), \\ u_k = 0 \quad \text{on} \quad \partial\Omega_D \times (0,T), \\ \partial_\nu u_k = 0 \quad \text{on} \quad \partial\Omega_N \times (0,T), \\ \partial_t u_k = 0 \text{ a.e. in } \omega. \end{cases} \tag{3.13}$$

Defining $y_k = \partial_t u_k$, the above problem yields

$$\begin{cases} \partial_t^2 y_k - \Delta y_k + f_k'(u_k) y_k = 0 \quad \text{in } \Omega \times (0,+\infty), \\ y_k = 0 \quad \text{on} \quad \partial\Omega_D \times (0,+\infty), \\ \partial_\nu y_k = 0 \quad \text{on} \quad \partial\Omega_N \times (0,+\infty), \\ y_k = 0 \text{ a.e. in } \omega. \end{cases}$$

Once $f_k'(u_k) \in L^\infty(\Omega \times (0,T))$ since f_k is globally Lipschitz, for each $k \in m \in \mathbb{N}$, we deduce from Assumption 1.2 that $y_k = \partial_t u_k \equiv 0$. Returning to (3.13) we conclude that $u_k \equiv 0$ as well and we obtain the desired contradiction.

Case (b): $u_k = 0$.

Setting

$$\alpha_m := \sqrt{E_{u_k^m}(0)}, \text{ and } v_k^m := \frac{u_k^m}{\alpha_m}, \tag{3.14}$$

in light of (3.8), we obtain

$$\lim_{m\to+\infty} \int_0^T \int_\Omega a(x) |\partial_t v_k^m|^2 \, dxdt = 0. \tag{3.15}$$

According to (3.14), the sequence $\{v_k^m\}_{m\in\mathbb{N}}$ is the solution to the following problem:

$$\begin{cases} \partial_t^2 v_k^m - \Delta v_k^m + \dfrac{1}{\alpha_m} f_k(u_k^m) + a(x)\partial_t v_k^m = 0 \quad \text{in } \Omega \times (0,T), \\ v_k^m = 0 \quad \text{on} \quad \partial\Omega_D \times (0,T), \\ \partial_\nu v_k^m = 0 \quad \text{on} \quad \partial\Omega_N \times (0,T), \\ v_k^m(x,0) = \dfrac{u_{0,k}^m}{\alpha_m}; \quad \partial_t v_k^m(x,0) = \dfrac{u_{1,k}^m}{\alpha_m} \end{cases} \tag{3.16}$$

and the associated energy functional is given by

$$E_{v_k^m}(t) = \frac{1}{2} \int_\Omega \left(|\partial_t v_k^m|^2 + |\nabla v_k^m|^2 \right) dx + \frac{1}{\alpha_m^2} \int_\Omega F_k(u_k^m) \, dx,$$

since

$$\frac{1}{\alpha_m} \int_\Omega f_k(u_k^m) \partial_t v_k^m \, dx = \frac{1}{\alpha_m^2} \frac{d}{dt} \int_\Omega F(u_k^m) \, dx.$$

Note that $E_{v_k^m}(t) = \frac{1}{\alpha_m^2} E_{u_k^m}(t)$ for all $t \geq 0$ and, in particular, for $t = 0$

$$E_{v_k^m}(0) = \frac{1}{\alpha_m^2} E_{u_k^m}(0) = 1, \text{ for all } m \in \mathbb{N}. \tag{3.17}$$

In order to achieve the contradiction we are going to prove that

$$\lim_{m\to+\infty} E_{v_k^m}(0) = 0. \tag{3.18}$$

Indeed, initially, we observe that (3.17) yields the existence of a subsequence of $\{v_k^m\}_{m\in\mathbb{N}}$, reindexed again by $\{v_k^m\}$, such that

$$v_k^m \rightharpoonup v_k \text{ weakly-star in } L^\infty(0,T;H^1_{\Omega_D}(\Omega)), \text{ as } m\to+\infty, \tag{3.19}$$

$$\partial_t v_k^m \rightharpoonup \partial_t v_k \text{ weakly-star in } L^\infty(0,T;L^2(\Omega)), \text{ as } m\to+\infty, \tag{3.20}$$

$$v_k^m \to v_k \text{ strongly in } L^\infty(0,T;L^q(\Omega)), \text{ as } m\to+\infty, \text{ for all } q\in\left[2,\frac{2n}{n-2}\right). \tag{3.21}$$

For some eventual subsequence, we have that $\alpha_m\to\alpha$ with $\alpha\geq 0$.

If $\alpha>0$, thus, passing to the limit in (3.16) and considering convergences (3.15), (3.19) - (3.21), we deduce

$$\begin{cases} \partial_t^2 v_k - \Delta v_k + \dfrac{1}{\alpha} f_k(u_k) = 0 \quad \text{in} \quad \Omega\times(0,T),\\ v_k = 0 \quad \text{on} \quad \partial\Omega_D\times(0,T),\\ \partial_\nu v_k = 0 \quad \text{on} \quad \partial\Omega_N\times(0,T),\\ \partial_t v_k = 0 \text{ a.e. in } \omega. \end{cases} \tag{3.22}$$

The above problem yields, for $w_k=\partial_t v_k$, in the distributional sense,

$$\begin{cases} \partial_t^2 w_k - \Delta w_k + \dfrac{1}{\alpha} f_k'(u_k) w_k = 0 \quad \text{in} \quad \Omega\times(0,T),\\ w_k = 0 \quad \text{on} \quad \partial\Omega_D\times(0,T),\\ \partial_\nu w_k = 0 \quad \text{on} \quad \partial\Omega_N\times(0,T),\\ w_k = 0 \text{ a.e. in } \omega. \end{cases} \tag{3.23}$$

Once $\frac{1}{\alpha}f_k'(u_k)\in L^\infty(\Omega\times(0,T))$, using again Assumption (1.2) we conclude that $w_k=\partial_t v_k\equiv 0$, and, therefore, returning to (3.22) we deduce that $v_k\equiv 0$.

If $\alpha=0$, first, observe that hypothesis (2.6) yields

$$\frac{1}{\alpha_m^2}|f_k(u_k^m)|^2 \leq c\frac{1}{\alpha_m^2}|u_k^m|^2 = c\frac{1}{\alpha_m^2}\alpha_m^2|v_k^m|^2,$$

and

$$\frac{1}{\alpha_m^2}\int_0^T\int_\Omega |f_k(u_k^m)|^2\,dxdt \leq c\int_0^T\int_\Omega |v_k^m|^2\,dxdt. \tag{3.24}$$

We are going to prove that

$$\frac{1}{\alpha_m} f_k(\alpha_m v_k^m) \rightharpoonup f'(0)v_k \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } m\to\infty. \tag{3.25}$$

Since

$$\frac{1}{\alpha_m} f_k(\alpha_m v_k^m) - f'(0)v_k = \frac{1}{\alpha_m} f_k(\alpha_m v_k^m) - \frac{1}{\alpha_m} f(\alpha_m v_k^m) + \frac{1}{\alpha_m} f(\alpha_m v_k^m) - f'(0)v_k,$$

if we prove that

$$\frac{1}{\alpha_m} f_k(\alpha_m v_k^m) - \frac{1}{\alpha_m} f(\alpha_m v_k^m) \to 0 \text{ in } L^2(0,T;L^2(\Omega)) \tag{3.26}$$

and

$$\frac{1}{\alpha_m} f(\alpha_m v_k^m) - f'(0)v_k \rightharpoonup 0 \text{ in } L^2(0,T;L^2(\Omega)), \tag{3.27}$$

as $m\to\infty$, we prove (3.25).

To prove (3.26), let's consider

$$\Omega_m^t = \{x \in \Omega : |u_k^m(x,t)| > k\}.$$

Employing definition (1.10), $|f_k(\alpha_m v_k^m) - f(\alpha_m v_k^m)| = 0$ in $\Omega \setminus \Omega_m^t$. Then, hypotheses (2.1) and (2.2) yield

$$\begin{aligned}
&\left\| \frac{1}{\alpha_m} f_k(\alpha_m v_k^m) - \frac{1}{\alpha_m} f(\alpha_m v_k^m) \right\|^2_{L^2(0,T;L^2(\Omega))} \\
&= \int_0^T \int_{\Omega_m^t} \left| \frac{1}{\alpha_m} f_k(\alpha_m v_k^m) - \frac{1}{\alpha_m} f(\alpha_m v_k^m) \right|^2 dxdt \\
&= \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |f_k(\alpha_m v_k^m) - f(\alpha_m v_k^m)|^2 \, dxdt \\
&\lesssim \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |f_k(\alpha_m v_k^m)|^2 \, dxdt + \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |f(\alpha_m v_k^m)|^2 \, dxdt \\
&\lesssim \int_0^T \int_{\Omega_m^t} |f(k)|^2 + |f(-k)|^2 \, dxdt + \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |\alpha_m v_k^m|^2 + |\alpha_m v_k^m|^{2p} \, dxdt \\
&\lesssim \int_0^T \int_{\Omega_m^t} |k|^2 + |k|^{2p} \, dxdt + \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |\alpha_m v_k^m|^2 + |\alpha_m v_k^m|^{2p} \, dxdt \\
&\lesssim \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |u_k^m|^2 + |u_k^m|^{2p} \, dxdt + \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |\alpha_m v_k^m|^2 + |\alpha_m v_k^m|^{2p} \, dxdt.
\end{aligned}$$

Since $p > 1$, $k \geq 1$ and $k < |u_k^m| = |\alpha_m v_k^m|$ in Ω_m^t, we obtain

$$\begin{aligned}
\left\| \frac{1}{\alpha_m} f_k(\alpha_m v_k^m) - \frac{1}{\alpha_m} f(\alpha_m v_k^m) \right\|^2_{L^2(0,T;L^2(\Omega))} &\lesssim \frac{1}{\alpha_m^2} \int_0^T \int_{\Omega_m^t} |\alpha_m v_k^m|^{2p} \, dxdt \\
&\lesssim \alpha_m^{2(p-1)} \|v_k^m\|^{2p}_{L^{2p}(0,T;L^{2p}(\Omega))} \to 0, \text{ as } m \to \infty,
\end{aligned}$$

which proves the convergence (3.26).

On the other hand, $f \in C^2(\mathbb{R})$ and, consequently, from Taylor's Theorem and (2.1) we have

$$f(s) = f'(0)s + R(s), \text{ where } |R(s)| \leq C(|s|^2 + |s|^p). \tag{3.28}$$

Hence

$$\frac{1}{\alpha_m} f(\alpha_m v_k^m) = f'(0) v_k^m + \frac{R(\alpha_m v_k^m)}{\alpha_m} \tag{3.29}$$

and

$$\left| \frac{R(\alpha_m v_k^m)}{\alpha_m} \right| \leq C \left(\alpha_m |v_k^m|^2 + |\alpha_m|^{p-1} |v_k^m|^p \right). \tag{3.30}$$

In light of identity (3.28), we establish $\frac{R(\alpha_m v_k^m)}{\alpha_m} = \frac{f(\alpha_m v_k^m)}{\alpha_m} - f'(0) v_k^m$ and hypotheses (2.1) and (2.2) imply that $|f(\alpha_m v_k^m)| \lesssim |\alpha_m v_k^m| + |\alpha_m v_k^m|^p$. Then, we deduce that

$$\left\| \frac{R(\alpha_m v_k^m)}{\alpha_m} \right\|^2_{L^2(0,T;L^2(\Omega))} \lesssim \|v_k^m\|^2_{L^2(0,T;L^2(\Omega))} + |\alpha_m|^{2(p-1)} \|v_k^m\|^{2p}_{L^{2p}(0,T;L^{2p}(\Omega))} \leq C,$$

for some constant $C > 0$. We obtain a subsequence of $\frac{R(\alpha_m v_k^m)}{\alpha_m}$ and $\gamma \in L^2(0,T;L^2(\Omega))$ such that

$$\frac{R(\alpha_m v_k^m)}{\alpha_m} \rightharpoonup \gamma \text{ in } L^2(0,T;L^2(\Omega)). \tag{3.31}$$

Besides, employing inequality (3.30) and observing (3.1), we get

$$\begin{aligned}\left|\left|\frac{R(\alpha_m v_k^m)}{\alpha_m}\right|\right|_{L^1(0,T;L^1(\Omega))} &\lesssim \int_0^T\int_\Omega \alpha_m|v_k^m|^2\,dxdt + \int_0^T\int_\Omega \alpha_m^{p-1}|v_k^m|^p\,dxdt\\ &= \alpha_m\int_0^T \|v_k^m\|^2_{L^2(\Omega)}\,dt + \alpha_m^{p-1}\int_0^T \|v_k^m\|^p_{L^p(\Omega)}\,dt\\ &= \alpha_m||v_k^m||^2_{L^2(0,T;L^2(\Omega))} + \alpha_m^{p-1}||v_k^m||^p_{L^p(0,T;L^p(\Omega))} \to 0. \end{aligned} \tag{3.32}$$

From (3.31) and (3.32) we conclude that

$$\frac{R(\alpha_m v_k^m)}{\alpha_m} \rightharpoonup 0 \text{ in } L^2(0,T;L^2(\Omega)). \tag{3.33}$$

Observing (3.21), (3.29) and (3.33), the convergence (3.27) is proved.

Remark 3.1. *The case $p = 1$ is trivially contemplated once the truncation is not necessary.*

Since convergences (3.26) and (3.27) are proved, we conclude convergence (3.25).
Passing to the limit in (3.16) as $m \to +\infty$, we obtain

$$\begin{cases} \partial_t^2 v_k - \Delta v_k + f'(0)v_k = 0 \quad \text{in} \quad \Omega\times(0,T),\\ v_k = 0 \quad \text{on} \quad \partial\Omega_D\times(0,T),\\ v_k = 0 \quad \text{on} \quad \partial\Omega_N\times(0,T),\\ \partial_t v_k = 0 \text{ a.e. in } \omega, \end{cases} \tag{3.34}$$

and defining $w_k = \partial_t v_k$, it satisfies the following problem:

$$\begin{cases} \partial_t^2 w_k - \Delta w_k + f'(0)w_k = 0 \quad \text{in} \quad \Omega\times(0,T),\\ w_k = 0 \quad \text{on} \quad \partial\Omega_D\times(0,T),\\ w_k = 0 \quad \text{on} \quad \partial\Omega_N\times(0,T),\\ w_k = 0 \text{ a.e. in } \omega. \end{cases} \tag{3.35}$$

Using Assumption (1.2) we obtain that $w_k = \partial_t v_k \equiv 0$ and returning to (3.34) we deduce that $v_k \equiv 0$.

Then, in both cases $\alpha = 0$ and $\alpha > 0$, we obtain that $v_k \equiv 0$. Consequently, inequality (3.24) and convergence (3.21) yield that

$$\frac{1}{\alpha_m^2}\int_0^T\int_\Omega |f_k(u_k^m)|^2\,dxdt \to 0 \text{ in } L^2(0,T,L^2(\Omega)) \text{ as } m\to+\infty. \tag{3.36}$$

In order to achieve a contradiction we need to prove that $E_{v_k^m}(0) \to 0$ as $m \to +\infty$. In fact, from (3.16), we can write $v_k^m = y_k^m + z_k^m$ such that y_k^m and v_k^m are, respectively, solutions of the following problems:

$$\begin{cases} \partial_t y_k^m - \Delta y_k^m = 0 \text{ in } \Omega\times(0,T),\\ y_k^m = 0 \text{ on } \partial\Omega_D\times(0,T),\\ \partial_\nu y_k^m = 0 \text{ on } \partial\Omega_N\times(0,T),\\ y_k^m(0) = v_k^m(0),\ \partial_t y_k^m(0) = \partial_t v_k^m(0), \end{cases}$$

and

$$\begin{cases} \partial_t z_k^m - \Delta z_k^m = -\dfrac{1}{\alpha_m} f_k(u_k^m) + a(x)\partial_t u_k^m \ \text{ in } \ \Omega \times (0,T), \\ z_k^m = 0 \ \text{ on } \partial\Omega_D \times (0,T), \\ \partial_\nu z_k^m = 0 \ \text{ on } \partial\Omega_N \times (0,T), \\ z_k^m(0) = 0, \ \partial_t z_k^m(0) = 0. \end{cases}$$

Setting

$$E^L_{v_k^m}(t) := \int_\Omega \left(|\partial_t v_k^m(x,t)|^2 + |\nabla v_k^m(x,t)|^2 \right) \, dx,$$

the linear part associated with energy $E_{v_k^m}(t)$, then we can write

$$E_{v_k^m}(t) = E^L_{v_k^m}(t) + \frac{1}{\alpha_m^2} \int_\Omega F_k(u_k((x,t))) \, dx. \tag{3.37}$$

In the sequel, let us estimates the nonlinear term of the RHS of (3.37) in terms of $E^L_{v^m}(t)$.

Estimate for $I_1 := \frac{1}{\alpha_m^2} \int_\Omega F_k(u_k(x,t)) \, dx$.

Taking (2.4) into account, one has

$$\begin{aligned} |I_1| &\leq \frac{1}{\alpha_m^2} \int_\Omega \left[|u_k^m|^2 + |u_k^m|^{p+1} \right] \, dx \\ &= \frac{1}{\alpha_m^2} \left[||u_k^m(t)||^2_{L^2(\Omega)} + ||u_k^m(t)||^{p+1}_{L^{p+1}(\Omega)} \right]. \end{aligned}$$

If $p = 1$, it follows, in view of (3.14), that

$$|I_1| \leq \frac{2}{\alpha_m^2} ||u^m(t)||^2_{L^2(\Omega)} = 2||v_k^m(t)||^2_{L^2(\Omega)} \lesssim E^L_{v_k^m}(t).$$

Now, if $p > 1$ then $p+1 > 2$ and, therefore $p+1 = 2+\varepsilon$ for some $\varepsilon > 0$, Thus, having in mind that the map $t \mapsto E_{v^m}$ is non increasing and $E_{v^m}(0) = 1$, we infer

$$\begin{aligned} |I_1| &\leq \left[||v_k^m(t)||^2_{L^2(\Omega)} + \alpha_m^{p-1} ||v_k^m(t)||^{p+1}_{L^2(\Omega)} \right] \\ &= \left[||v_k^m(t)||^2_{L^2(\Omega)} + \alpha_m^{p-1} ||v_k^m(t)||^{2+\varepsilon}_{L^2(\Omega)} \right] \\ &\lesssim E^L_{v_k^m}(t) + \alpha_m^{p-1} [E^L_{v_k^m}(t)][E_{v_k^m}(0)]^{\frac{\varepsilon}{2}} \\ &\lesssim [1 + \alpha_m^{p-1}] E^L_{v_k^m}(t). \end{aligned}$$

In any case, we deduce

$$|I_1| \lesssim E^L_{v_k^m}(t). \tag{3.38}$$

So, combining (3.37) and (3.38) we obtain

$$E_{v_k^m}(t) \lesssim E^L_{v_k^m}(t), \ \text{ for all } t \in [0,T]. \tag{3.39}$$

Now, employing the observability given in (1.15) and having in mind that $E^L_{v_k^m}(0) \equiv E_{y_k^m}(0)$, we deduce from (3.39) that

$$E_{v_k^m}(0) \ \lesssim \ E^L_{v_k^m}(0) = E_{y_k^m}(0) \leq c \int_0^T \int_\omega |\partial_t y_k^m(x,t)|^2 \, dxdt. \tag{3.40}$$

From (3.40), observing that $a(x) \geq a_0 > 0$ in ω and since $v_k^m = y_k^m + z_k^m$, we obtain

$$E_{v_k^m}(0) \lesssim \int_0^T \int_\Omega a(x)|\partial_t v_k^m(x,t)|^2 \, dxdt + \int_0^T \int_\Omega |\partial_t z_k^m(x,t)|^2 \, dxdt. \tag{3.41}$$

On the other hand, using the well-known result which establishes that the map $\{z_0, z_1, f\} \mapsto \{z, \partial_t z\} \in L^\infty(0,T; H^1_{\partial\Omega_D}(\Omega)) \times L^\infty(0,T; L^2(\Omega))$ associating the initial data $\{z_0, z_1, f\} \in H^1_{\partial\Omega_D}(\Omega) \times L^2(\Omega) \times L^1(0,T; L^2(\Omega))$ to the unique solution to the linear problem

$$\begin{cases} \partial_t^2 z - \Delta z = f \ \text{ in } \Omega \times (0,T) \\ z = 0 \ \text{ on } \partial\Omega_D \times (0,T), \\ \partial_\nu z = 0 \ \text{ on } \partial\Omega_N \times (0,T), \\ z(0) = z_0, \ \partial_t z(0) = z_1 \end{cases} \tag{3.42}$$

is linear and continuous; we obtain, from (3.41), and, in particular, considering $z_0 = z_1 = 0$ and $f := -\frac{1}{\alpha_m} f_k(u_k^m) - a(x)\partial_t u_k^m$, that

$$E_{v_k^m}(0) \lesssim \int_0^T \int_\Omega a(x)|\partial_t v_k^m(x,t)|^2 \, dxdt + \frac{1}{\alpha_m^2} \int_0^T \int_\Omega |f_k(u_k^m)|^2 \, dxdt. \tag{3.43}$$

Thus, from (3.15), (3.36) and (3.43) we deduce that $E_{v_k^m}(0) \to 0$ as $m \to +\infty$ as desire to prove in (3.18).

□

In what follows, we are going to conclude the exponential stability to the problem (1.1).

Thanks to inequality (3.5), the auxiliary problem (1.9) satisfies the following observability inequality:

$$E_{u_k}(0) \leq C \int_0^T \int_\Omega a(x)|\partial_t u_k|^2 \, dx \, dt, \text{ for all } T \geq T_0, \text{ and } k \in \mathbb{N}, \ k \geq k_0, \tag{3.44}$$

where C is a positive constant which does not depend on $k \in \mathbb{N}$.

Passing to the limit as $k \to +\infty$ and observing convergences (2.43), (2.44) and (2.54), the above inequality yields the observability inequality associated to the original problem (1.1), that is,

$$E_u(0) \leq C \int_0^T \int_\Omega a(x)|\partial_t u|^2 \, dx \, dt, \text{ for all } T \geq T_0. \tag{3.45}$$

On the other hand, passing to the limit as $k \to +\infty$ and considering the same convergences (2.43), (2.44) and (2.54), identity (2.9) yields the identity associated to the original problem (1.1), namely,

$$E_u(t_2) - E_u(t_1) + \int_{t_1}^{t_2} \int_\Omega a(x)|\partial_t u|^2 \, dx \, dt = 0, \text{ for all } 0 \leq t_1 < t_2 < +\infty. \tag{3.46}$$

Gathering together (3.45), (3.46), and since the map $t \mapsto E_u(t)$ is a non-increasing function, we obtain

$$\begin{aligned} E_u(T_0) &\leq C \int_0^{T_0} \int_\Omega \left(a(x)|\partial_t u|^2\right) \, dx \, dt \\ &= C \left(E_u(0) - E_u(T_0)\right), \end{aligned} \tag{3.47}$$

that is,

$$E_u(T_0) \leq \left(\frac{C}{1+C}\right) E_u(0). \tag{3.48}$$

Repeating the same steps for mT_0, $m \in \mathbb{N}, m \geq 1$, we deduce

$$E_u(mT_0) \leq \frac{1}{(1+\hat{C})^m} E_u(0),$$

where $\hat{C} = C^{-1}$. Consider $t \geq T_0$ and $t = mT_0 + r$, $0 \leq r < T_0$. Thus,

$$E_u(t) \leq E_u(t-r) = E_u(mT_0) \leq \frac{1}{(1+\hat{C})^m} E_u(0) = \frac{1}{(1+\hat{C})^{\frac{t-r}{T_0}}} E_u(0).$$

Defining $C := \mathrm{e}^{\frac{r}{T_0}\ln(1+\hat{C})}$ and $\lambda_0 := \frac{\ln(1+\hat{C})}{T_0} > 0$, we obtain

$$E_u(t) \leq C\,\mathrm{e}^{-\lambda_0 t} E_u(0) \text{ for all } t \geq T_0, \tag{3.49}$$

which proves the exponential decay to problem (1.1) and we prove the following result.

Theorem 3.1. *Under the assumptions of Theorem 2.3 and Assumptions 1.1 and 1.2 there exist positive constants C and γ such that the following exponential decay holds*

$$E_u(t) \leq C\, e^{-\lambda_0 t} E_u(0), \textit{ for all } t \geq T_0. \tag{3.50}$$

for every solution to problem (1.1), provided that the initial data are taken in bounded sets of the phase-space $\mathcal{H} := H^1_{\partial\Omega_D}(\Omega) \times L^2(\Omega)$.

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