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Fuzzy Qualitative Reasoning with Words

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1. Introduction

Computing in its traditional sense involves manipulation of numbers for the most part. Symbolic computation has been developed in the meantime especially in Artificial Intelligence, but often without preserving the capability of interfacing the symbolic processing of information with numerical data. By contrast, humans employ mostly words and qualitative descriptions when providing assessments of situations, or reasoning about complex physical or human systems, even if a part of the data refers to numerical scales. For a long time, fuzzy sets have been advocated by Zadeh (1973) as a methodology for interfacing numerical data about the world (which are often imprecisely perceived), with linguistic categories or classes used in reasoning. In particular, it is the case for fuzzy rules or fuzzy algorithms where words are interpreted as the labels of fuzzy sets. There are different kinds of situations where the underlying reasoning process can be handled within the framework of a limited vocabulary. When developing his approach to approximate reasoning, Zadeh (1979) was already separating the combination/projection machinery acting at the levels of the fuzzy set membership functions, from the knowledge representation and linguistic approximation steps which respectively turn the linguistic input information into possibility distributions and restate the obtained conclusions into a prescribed vocabulary understandable by the user. The fuzzy set framework should supply the right tools for mixing a symbolic treatment to be performed at the level of the linguistic labels with facilities for permanently interfacing them with their fuzzy set semantics along the reasoning process. Indeed, fuzzy sets restore gradual transitions between categories on continuous universes, which would be lost if the interface between labels and data were provided by crisp sets only. In this paper, we investigate the feasibility of such a view on a particular type of reasoning: the order of magnitude reasoning in terms of closeness and negligibility relations. The same line of research has been explored on more general patterns of approximate reasoning by Dubois, Foulloy, Galichet and Prade (1997).

Qualitative reasoning, handled at the level of comparative relations, may be, viewed as a simple illustration of the idea of computing with words recently emphasized by Zadeh (1996, 1999). As the expression suggests, 'computing with words' is a methodology in which words are used in place of numbers for computing and reasoning. As pointed out by Zadeh

(1996), there are two major imperatives for computing with words. "First, computing with words is a necessity when the available information is too imprecise to justify the use of numbers, and second, when there is a tolerance for imprecision which can be exploited to achieve tractability, robustness, low solution cost, and better report with reality. Exploitation of the tolerance for imprecision is an issue of central importance in computing with words", Zadeh (1996) wrote. A key idea underlying computing with words is that it involves a fusion of symbolic processing and computation with fuzzy variables. The following summarizes Zadeh's main points. Computing with words is based on a granulation process, which generally serves as a way of achieving data compression. A granule g is a fuzzy set of points drawn together by similarity. A word w (e.g., young) is a label of granule g . In computing with words, a granule g which is a denotation of a word w is viewed as a fuzzy constraint on a variable. The rules of inference in fuzzy logic are employed to propagate fuzzy constraints from premises to conclusions. Thus, the techniques of computing with words underlie, in one way or another, almost all the apparatus of fuzzy logic.

This paper focuses on a particular type of reasoning where computation can be handled at the word level, as it will be shown: qualitative reasoning about relative orders of magnitude, where fuzzy constraints expressing closeness or negligibility relations have to be propagated. Building inference systems endowed with the ability to reason about physical systems in the same way as humans—or at least engineers—do, has led to the development of qualitative reasoning (Kuipers, 1994; MQ&D Project, 1995; Travé-Massuyès, Dague and Guerrin, 1997). The main aim of qualitative reasoning is both to address the need to deal with physical systems where some magnitudes are not easy to quantify (numerical data are not available), and to be able to reason at a qualitative or symbolic level (for example, reasoning directly in terms of orders of magnitude). The last decade has seen significant progress towards the development of formal methods for qualitative reasoning about the behavior of physical systems. The simplest formalism used in qualitative reasoning is based on the sign algebra $(-, 0, +)$; see for instance (Dormoy, 1989; Kuipers, 1994). Such models are enough to represent the sign of quantities and the direction in which each quantity affects another quantity. Information about magnitudes, or even relative orders of magnitude is not represented. As a consequence, the sign-based approach has too limited an expressive power in some practical cases to be widely applicable. A major limitation lies in the fact that often the sign of the result of an operation can be determined only if the order of magnitude of the involved parameters are considered. The only knowledge of the sign of the quantities involved in a mathematical expression often leaves completely indeterminate the sign of the result. Reasoning with relative orders of magnitude is an attempt towards overcoming these limitations. See (Raiman, 1991) for a general discussion.

This type of qualitative reasoning corresponds to a particular form of commonsense reasoning where ideas of closeness, comparability and negligibility are involved. The first attempt to formalize such reasoning appeared with the formal system FOG, proposed by Raiman (1986, 1989). FOG is based on three basic relations expressing the relations 'negligible in relation to' (Ne), 'close to' (Cl), and 'has the same sign and order of magnitude as', i.e., 'is comparable to' (Co). FOG includes one axiom and 31 inference rules, which allow

for deduction from pieces of knowledge expressed in terms of relative orders of magnitude. This set of rules was proved to be consistent by giving an interpretation to the three relations in the framework of Non-Standard Analysis. Note that FOG handles relative orders of magnitude through a purely symbolic computation process. Nevertheless, no numerical interpretation is provided for the symbolic computation of orders of magnitude in the FOG's approach which thus does not allow for the incorporation of numerical values. This drawback is remedied in the system O(M) proposed by Mavrovouniotis and Stephanopoulos (1987, 1988), where the primitive relations ('much smaller than', 'moderately smaller than', 'slightly smaller than', 'exactly equal', 'slightly larger than', 'moderately larger than', 'much larger than') are interpreted in terms of the location of the quotient of the two compared quantities in some interval. All these intervals are disjoint, and defined via a unique parameter which is fixed according to the application context. The fact that this parameter is fixed creates some problem since the result of the (repeated) composition of the primitive relations cannot always be expressed by a primitive relation if we want to perfectly adhere to the interval semantics. Indeed for instance, if $x/y \in [a, b]$ and $y/z \in [a, b]$, with $a > 0$, then $x/z \in [a^2, b^2]$, and $[a^2, b^2]$, or more generally $[a^n, b^n]$, are not necessarily included into an interval corresponding to the semantics of a primitive relation. Dague (1993) also provides an attempt to give a numerical interpretation to FOG-like relations manipulated in a formal way. However, FOG or O(M) do not take into account the fact that the satisfaction of relations such as Ne, Cl, or "much larger than", by two numerical values is often a matter of degree. Moreover FOG does not cope with the fact that it is also context-dependent. FOG does not acknowledge either that a relation of closeness (resp. negligibility) is usually transitive in a weak (resp. strong) sense (which will be made precise in the following), rather than just transitive (Dubois and Prade, 1989a). The same type of limitation applies to O(M) as well. The modeling of relations Ne, Cl and Co by means of fuzzy relations constitutes an appropriate framework for solving these problems.

The paper is organized as follows. Section 2 gives the necessary background on the formal system FOG. Section 3 introduces fuzzy equality and inequality relations (such as 'approximately equal', 'much larger than') modelled in terms of difference, and presents a set of inference rules based on these relations. We show that the composition of such relations reduces to simple fuzzy arithmetic operations. However, a simple example suggests that the modelling in terms of difference may be more appropriate for temporal relations. In Section 4, we discuss the modeling of the basic relations of FOG using fuzzy relations in terms of ratios of values. New inference rules, based on relations Ne and Cl, are established. The composition of the fuzzy relations representing relations Ne and Cl also reduces to simple arithmetic operations on fuzzy numbers which play the role of parameters underlying the semantics of Ne and Cl. Results of inferences are always expressed in terms of the relations Ne and Cl, but based on fuzzy parameters which are symbolically computed from the fuzzy parameters underlying the relations appearing in the premises. This machinery contributes to provide a standard numerical semantics for the symbolic computation performed by the FOG's rules, as well as by the new inference rules which are proposed. Illustrative examples for solving equations, in an approximate way, are presented in Section 5, while the

concluding remarks point out some problems which are faced for the development and the efficient implementation of the approach.

2. The Formal System FOG

The FOG system is based on three relations, which capture closeness, negligibility and comparability and which are respectively denoted by Cl, Ne and Co. These three operators are:

x Ne y , which stands for x is negligible in relation to y ;

x Cl y , which stands for x is close to y , understood as

$(x - y)$ is negligible in relation to y ;

x Co y , which stands for x has the same sign and order of magnitude as y .

The underlying idea is that if x Ne z then y Ne z (when x Co y).

FOG has one axiom (A1: x Cl x) and 31 inference rules. Hereafter, we give some of these rules ($[x]$ denotes the sign (+, -, 0) of the quantity x):

$$1) x \text{ Cl } y \Rightarrow y \text{ Cl } x$$

$$2) x \text{ Cl } y \Rightarrow x \text{ Co } y$$

$$3) x \text{ Cl } y, y \text{ Cl } z \Rightarrow x \text{ Cl } z$$

$$4) x \text{ Cl } y, y \text{ Ne } z \Rightarrow x \text{ Ne } z$$

$$5) x \text{ Co } y, y \text{ Co } z \Rightarrow x \text{ Co } z$$

$$6) x \text{ Ne } y, y \text{ Ne } z \Rightarrow x \text{ Ne } z$$

$$7) x \text{ Co } y, y \text{ Cl } z \Rightarrow x \text{ Co } z$$

$$8) x \text{ Ne } y \Rightarrow (x+y) \text{ Cl } y$$

$$9) (x+y) \text{ Cl } z, y \text{ Ne } x \Rightarrow x \text{ Cl } z$$

$$10) x \text{ Ne } y, z \text{ Cl } t \Rightarrow x \cdot z \text{ Ne } y \cdot t$$

$$11) x \text{ Ne } y, z \text{ Ne } x \Rightarrow x \cdot z \text{ Ne } y \cdot t$$

$$12) x \cdot y \text{ Cl } z \cdot t, x \text{ Cl } z, [x] \neq 0 \Rightarrow y \text{ Cl } t$$

$$13) x \cdot y \text{ Cl } z \cdot t, x \text{ Ne } z, [z] \neq 0 \Rightarrow t \text{ Ne } y$$

$$14) x \cdot y \text{ Ne } z \cdot t, z \text{ Ne } x, [x] \neq 0 \Rightarrow y \text{ Ne } t.$$

Note that both Cl and Co are equivalence relations (i.e., reflexive, symmetric and transitive), with Co coarser than Cl. The FOG rules, which can be justified from the point of view of Non-Standard Analysis, describe how the three relations work together and allow for the propagation of initial qualitative information. Most of the rules have an explicit intuitive interpretation and need no explanation. Note that they are not independent (e.g., rules 2 and 5 \Rightarrow rule 7). Note that FOG has been used successfully in the DEDALE system of analog circuit diagnosis (Dague et al., 1987), and in macroeconomics (Bourguine and Raiman, 1986). Nevertheless, FOG has several limitations, as announced in the introduction:

- No standard numerical interpretation of the three relations Ne, Cl and Co is provided in this approach. It may create interface problems with genuine numerical values. It does not allow for the incorporation of quantitative information such as the assessment of numerical values. However, such information is often available and useful when solving engineering problems. FOG cannot use this type of information because it does not relate numbers to order of magnitude relations. Moreover FOG is unable to express a gradual change from one order of magnitude to another in the computation process, due to the absence of gradual transitions between the orders of magnitude (Dague, 1993).

- FOG does not take into account the fact that the extent to which two numerical values satisfy the relations Ne, Cl and Co is often a matter of degree and is also context-dependent (for example, the relation 'negligible' does not mean the same thing in a preliminary-design context and a detailed-design context).
- FOG cannot express that the relations are "weakly", or "strongly", transitive rather than just transitive in the usual sense. For instance, if 'a is close to b' and 'b close to c', it is not so true that we should conclude that 'a is close to c' in any case (at least with a too restrictive acceptance of "close"). This exemplifies the weakening of transitivity. In order to avoid undesirable transitivity effects and obtain results at least approximately valid with respect to a numerical semantics, Raiman is obliged to introduce arbitrary limitations on the chaining of rules (by means of control techniques).

The use of fuzzy relations for modeling relations Ne, Cl and Co can solve the above problems, and provide numerical semantics to the symbolic calculus of orders of magnitude performed by the FOG system.

3. Fuzzy Equalities and Inequalities

In this section, we consider particular fuzzy relations which are of interest in practice in some forms of qualitative reasoning. Approximate equalities and strong inequalities (e.g., "much greater than") are examples of binary fuzzy relations which can be easily handled using fuzzy arithmetic techniques. An approximate equality can be modelled by a fuzzy relation E of the form $\mu_E(x,y) = \mu_L(|x - y|)$. For instance:

$$\begin{aligned} \mu_E(x,y) &= 1 \text{ if } |x - y| \leq \delta; \\ \mu_E(x,y) &= 0 \text{ if } |x - y| \geq \delta + \varepsilon; \\ \mu_E(x,y) &= (\delta + \varepsilon - |x - y|) / \varepsilon \text{ otherwise} \end{aligned} \quad (1)$$

where δ and ε are respectively positive and strictly positive parameters which modulate the approximate equality. Then the approximate equality of quantities a and b (in the sense of E) will be denoted by:

$$a - b \in L \quad (2)$$

with the following intended meaning: the possible values of the difference $a - b$ are restricted by the fuzzy set L. Here L is a fuzzy set centered in 0, i.e., $\mu_L(d) = \mu_L(-d)$, or if we prefer $\mu_E(x,y) = \mu_E(y,x)$. See Figure1. It ensures the symmetry of the approximate equality relation. Classical equality is recovered for $\delta = 0$ and $\varepsilon \rightarrow 0$. We shall write equivalently $(a,b) \in E(\delta,\varepsilon)$ ($\Leftrightarrow (b,a) \in E(\delta,\varepsilon)$). Moreover $(a,b) \in E(\delta,\varepsilon) \Leftrightarrow (a - b, 0) \in E(\delta,\varepsilon)$.

Similarly a more or less strong inequality can be modelled, for instance, by a relation I of the form:

$$\begin{aligned}
\mu_I(x,y) &= 1 \text{ if } x \geq y + \lambda + \rho; \\
\mu_I(x,y) &= 0 \text{ if } x \leq y + \lambda; \\
\mu_I(x,y) &= (x - y - \lambda) / \rho \text{ otherwise}
\end{aligned} \tag{3}$$

with $\lambda \geq 0$ and $\rho > 0$. A constraint of the form 'a is slightly / moderately /.../ much larger than b' (in the sense of I) can then be written, with $\mu_I(x,y) = \mu_K(x - y)$:

$$a - b \in K \tag{4}$$

where K is a fuzzy interval such that $K = [K, +\infty)$, where $\mu_{[K, +\infty)}(t) = \sup_{s \leq t} \mu_K(s)$. It means that K identifies itself with the set of values equal or greater than a value restricted by K . See Figure 1. According to the values of parameters λ and ρ , the modality which specifies how much a is larger than b may be linguistically expressed by "slightly", "moderately", "much",..., in a given context. When $\lambda = 0$ and $\rho \rightarrow 0$ the usual inequality relations '>' or ' \geq ' are recovered. We shall write $(a,b) \in I(\lambda,\rho)$ equivalently to $a - b \in K$.

The composition of two fuzzy relations E and I , with $\mu_I(x,y) = \mu_K(x - y)$ et $\mu_E(y,z) = \mu_L(y - z)$ is defined by:

$$\begin{aligned}
\forall x, \forall z, \mu_{E \circ I}(x,z) &= \sup_{y \in Y} \min(\mu_K(x - y), \mu_L(y - z)) \\
&= \sup_{\substack{s,t \\ x-z=s+t}} \min(\mu_K(s), \mu_L(t)) \\
&= \mu_{K \oplus L}(x - z),
\end{aligned} \tag{5}$$

where \oplus denotes the addition extended to fuzzy intervals (e.g., Dubois and Prade, 1987). Indeed arithmetic operations are extended to fuzzy quantities in the following way:

$$\mu_{K \odot L}(u) = \sup_{\substack{s,t \\ u=s*t}} \min(\mu_K(s), \mu_L(t)),$$

where \odot denotes the extension of the arithmetic operation $*$ to fuzzy sets of the real line.

Note that if K and L have trapezoidal membership functions, i.e., $K = (k1, k2, k3, k4)$ where $[k1, k4]$ is the support of μ_K and $[k2, k3]$ is the core of μ_K , and $L = (l1, l2, l3, l4)$, with a similar understanding of the components of the 4-tuple, then $K \oplus L = (k1 + l1, k2 + l2, k3 + l3, k4 + l4)$ et $K \dot{-} L = (k1 - l4, k2 - l3, k3 - l2, k4 - l1)$. In the Figure 1, $L = (-\delta - \epsilon, -\delta, \delta, \delta + \epsilon)$, $K = (\lambda, \lambda + \rho, +\infty, +\infty)$ and $L \oplus K = (\lambda - \delta - \epsilon, \lambda + \rho - \delta, +\infty, +\infty)$. For more details about fuzzy arithmetic, see (Dubois and Prade, 1987).

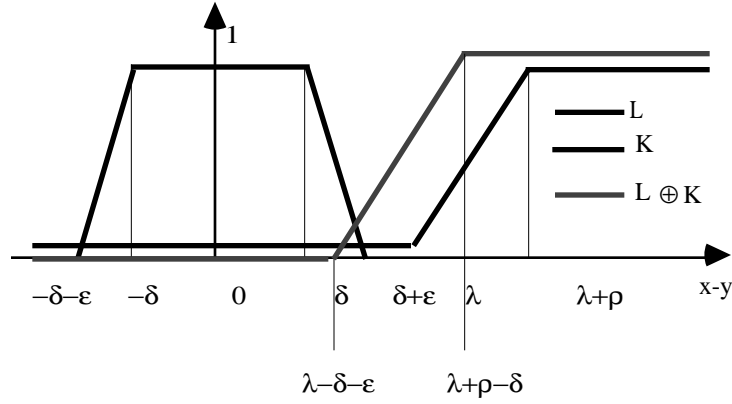


Figure 1

Let us take an example. If we know for instance that: 'a is approximately equal to b' (i.e., $a - b \in L$) and that 'b is much greater than c' (i.e., $b - c \in K$), using (5) we can deduce that $a - c \in L \oplus K$. This result is represented in Figure 1. Symbolically, this inference can be written:

$$(a,b) \in E(\delta,\varepsilon) \text{ and } (b,c) \in I(\lambda,\rho) \Rightarrow (a,c) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon) \quad (6)$$

which means that if a and b are approximately equal (in the sense of the approximate equality relation E defined with parameters δ et ε) and b is (more or less) larger than c (in the sense of the fuzzy inequality relation I defined with parameters λ and ρ) then we infer that a is (more or less) larger than c (in the sense of of the fuzzy relation I defined with parameters $\lambda - \delta - \varepsilon$ and $\rho + \varepsilon$). Note that $I(\lambda - \delta - \varepsilon, \rho + \varepsilon)$ means 'larger than' only if $\lambda - \delta - \varepsilon > 0$, strictly speaking. Otherwise if $\lambda - \delta - \varepsilon \leq 0$, it is at least somewhat possible that a is not greater than c, when $(a,c) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon)$; and it is fully possible that a is not greater than c, if $\lambda + \rho - \delta \leq 0$. Note also that $(a,b) \in I(\lambda,\rho) \Leftrightarrow (a - b, 0) \in I(\lambda,\rho)$.

In the same way, we can justify the following inference rules:

$$(a,b) \in E(\delta,\varepsilon) \text{ and } (b,c) \in E(\delta',\varepsilon') \Rightarrow (a,c) \in E(\delta + \delta', \varepsilon + \varepsilon'), \quad (7)$$

$$(a,b) \in I(\lambda,\rho) \text{ and } (b,c) \in I(\lambda',\rho') \Rightarrow (a,c) \in I(\lambda + \lambda', \rho + \rho'). \quad (8)$$

Other rules expressing the behavior of fuzzy relations E and I with respect to arithmetic operations can be easily derived. For instance, we have

$$(a,b) \in E(\delta,\varepsilon) \Rightarrow (a + c, b + c) \in E(\delta,\varepsilon) \quad (9)$$

$$(a,b) \in I(\lambda,\rho) \Rightarrow (a + c, b + c) \in I(\lambda,\rho) \quad (10)$$

$$(a,b) \in E(\delta,\varepsilon) \text{ and } (c,d) \in E(\delta',\varepsilon') \Rightarrow (a + c, b + d) \in E(\delta + \delta', \varepsilon + \varepsilon') \quad (11)$$

$$(a,b) \in I(\lambda,\rho) \text{ and } (c,d) \in I(\lambda',\rho') \Rightarrow (a + c, b + d) \in I(\lambda + \lambda', \rho + \rho') \quad (12)$$

$$(a,b) \in E(\delta,\varepsilon) \text{ and } (c,d) \in I(\lambda,\rho) \Rightarrow (a + c, b + d) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon) \quad (13)$$

$$(a + b, c) \in E(\delta,\varepsilon) \text{ and } (a,b) \in I(\lambda,\rho) \Rightarrow (a, c - a) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon) \quad (14A)$$

$$(a + b, c) \in E(\delta, \varepsilon) \text{ and } (a, b) \in I(\lambda, \rho) \Rightarrow (c - b, b) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon) \quad (14B)$$

$$(a + b, c) \in I(\lambda, \rho) \text{ and } (c, a) \in I(\lambda', \rho') \Rightarrow (b, 0) \in I(\lambda + \lambda', \rho + \rho') \quad (15)$$

$$(a + b, c) \in E(\delta, \varepsilon) \text{ and } (a, c) \in E(\delta', \varepsilon') \Rightarrow (0, b) \in E(\delta + \delta', \varepsilon + \varepsilon') \quad (16)$$

$$(a + b, c) \in E(\delta, \varepsilon) \text{ and } (c, a) \in I(\lambda, \rho) \Rightarrow (b, 0) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon) \quad (17)$$

$$(a - c, b) \in E(\delta, \varepsilon) \text{ and } (c, a) \in E(\delta', \varepsilon') \Rightarrow (0, b) \in E(\delta + \delta', \varepsilon + \varepsilon') \quad (18)$$

$$(a + b, c + d) \in E(\delta, \varepsilon) \text{ and } (c, a) \in E(\delta', \varepsilon') \Rightarrow (b, d) \in E(\delta + \delta', \varepsilon + \varepsilon') \quad (19)$$

$$(a + b, c + d) \in I(\lambda, \rho) \text{ and } (c, a) \in E(\delta, \varepsilon) \Rightarrow (b, d) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon) \quad (20)$$

$$(a + b, c + d) \in I(\lambda, \rho) \text{ and } (c, a) \in I(\lambda', \rho') \Rightarrow (b, d) \in I(\lambda + \lambda', \rho + \rho') \quad (21)$$

Proof: Let us look at the proof of some of these inference rules.

- The left side of (11) can be written under the form, with $L' = (-\delta' - \varepsilon', -\delta', +\delta', \delta' + \varepsilon')$

$$\begin{aligned} & \sup_{\substack{x,y,z,t \\ u=x+z; v=y+t}} \min(\mu_{E(\delta, \varepsilon)}(x, y), \mu_{E(\delta', \varepsilon')}(z, t)) \\ &= \sup_{\substack{x,y,z,t \\ u-v=(x-y)+(z-t)}} \min(\mu_L(x - y), \mu_{L'}(z - t)) = \mu_{L \oplus L'}(u - v). \end{aligned}$$

This last result implies that $(a + c, b + d) \in E(\delta + \delta', \varepsilon + \varepsilon')$.

- Rule (17) can be established noticing that

$$\begin{aligned} & \sup_{x,z} \min(\mu_E(x + y, z), \mu_I(z, x)) \\ &= \sup_{x,z} \min(\mu_L(x + y - z), \mu_K(z - x)) \\ &= \mu_{L \oplus K}(y), \text{ observing that } y = (x + y - z) + (z - x). \end{aligned}$$

Thus, we conclude that $(b, 0) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon)$ when $(a + b, c) \in E(\delta, \varepsilon)$ and $(c, a) \in I(\lambda, \rho)$. Note that it is certain that $b > 0$ only if $\lambda - \delta - \varepsilon > 0$.

- Now, considering rule (20), the composition of relations $(a + b, c + d) \in I(\lambda, \rho)$ and $(c, a) \in E(\delta, \varepsilon)$ is given by

$$\begin{aligned} & \sup_{x,z} \min(\mu_I(x + y, z + t), \mu_E(z, x)) \\ &= \sup_{x,z} \min(\mu_K((x + y) - (z + t)), \mu_L(z - x)) \\ &= \mu_{K \oplus L}(y - t), \text{ observing that } y - t = [(x + y) - (z + t)] + (z - x). \end{aligned}$$

This means that $(b, d) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon)$.

The rules (9)-(21) are not independent. (9)-(10) are particular cases of (11)-(12) replacing (c, d) by (c, c) , noticing that $(c, c) \in E(0, 0)$ and $(c, c) \in I(0, 0)$. Similarly, (16) and (15) are particular cases of (19) and (21) respectively for $d = 0$. Rule (18) can be obtained from (16) by noticing that $(a - c, b) \in E(\delta, \varepsilon) \Leftrightarrow (a, b + c) \in E(\delta, \varepsilon)$ and $(a, c) \in E(\delta', \varepsilon') \Leftrightarrow (c, a) \in E(\delta', \varepsilon')$. Besides, rule (17) can be generalized into the rule

$$(a + b, c + d) \in E(\delta, \varepsilon) \text{ and } (c, a) \in I(\lambda, \rho) \Rightarrow (b, d) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon).$$

Then, letting $a + b = C$, $c + d = A$, $c = A + B$, $a = C + D$, we obtain

$$(C,A) \in E(\delta,\varepsilon) \text{ and } (A + B, C + D) \in I(\lambda,\rho) \Rightarrow (-D, -B) \in I(\lambda - \delta - \varepsilon, \rho + \varepsilon)$$

which is nothing but (20), noticing that $(-d, -b) \in I(\lambda',\rho') \Leftrightarrow (b,d) \in I(\lambda',\rho')$ always holds. y

As it can be seen, arithmetic computations on fuzzy numbers are mirrored at a symbolic level by inference rules which exhibit formal manipulations of the parameters underlying the approximate equality and inequality relations. The fuzzy number-based interpretation of these relations provides a natural semantics, and interfaces the symbolic level with the numerical one. From these inference rules, new conclusions can be symbolically drawn, still keeping the ability to interpret the obtained result numerically. For instance, from (7) and the equivalence $(a + b, c) \in E(\delta,\varepsilon) \Leftrightarrow (a, c - b) \in E(\delta,\varepsilon)$, we can conclude for instance that

$$\text{if } (a - c, b) \in E(\delta,\varepsilon) \text{ and } (d + b, c) \in E(\delta',\varepsilon') \text{ then } (a - c, c - d) \in E(\delta + \delta', \varepsilon + \varepsilon').$$

Lastly, let us recall that this kind of fuzzy relations (i.e., relations E and I) can offer a useful setting for representing and processing temporal knowledge (Dubois and Prade, 1989b). For instance

- Knowing that the date 'b' takes place "much after" the date 'a' is represented by $\pi_{(a,b)}(x,y) = \mu_I(y,x)$, where I is defined by (3) and where the meaning of "much" is modelled by means of the parameters λ and ρ . $\pi_{(a,b)}$ is the joint possibility distribution expressing the precedence relations between dates a and b.
- Knowing that dates 'a' and 'b' are "approximately equal" is represented by $\pi_{(a,b)}(x,y) = \mu_E(x,y)$, where E is defined by (1) and where the meaning of "approximately" is modelled by means of the parameters δ and ε .

Let us consider an example of reasoning involving temporal knowledge, where we have the following information about dates a, b, c and d:

- i) dates a and b almost coincide (in the sense of $E(\delta,\varepsilon)$);
- ii) dates b and c are "approximately equal" (in the sense of $E(\delta',\varepsilon')$);
- iii) date a + b takes place "much after" date c + d (in the sense of $I(\lambda,\rho)$).

Let us consider the question: What can be said about date b with respect to d? Information in (i)-(iii) can be represented respectively by the following relations:

- i) $(a,b) \in E(\delta,\varepsilon)$ or $(b,a) \in E(\delta,\varepsilon)$;
- ii) $(b,c) \in E(\delta',\varepsilon')$;
- iii) $(a + b, c + d) \in I(\lambda,\rho)$.

By the transitivity rule (7), we obtain $(a,c) \in E(\delta + \delta', \varepsilon + \varepsilon')$. Since E is symmetric, this last result implies that $(c,a) \in E(\delta + \delta', \varepsilon + \varepsilon')$. Rule (20), applied to this last relation and to the relation in (iii), enables us to conclude that

$$(b,d) \in I[\lambda - (\delta + \delta') - (\varepsilon + \varepsilon'), \rho + (\varepsilon + \varepsilon')].$$

This means that date b is "much after" date d , where the meaning of "much" is modelled by means of parameters $\lambda' = \lambda - (\delta + \delta') - (\varepsilon + \varepsilon')$ and $\rho' = \rho + (\varepsilon + \varepsilon')$, provided that $\lambda' > 0$.

4. Symbolic reasoning with fuzzy negligibility and closeness relations

4.1. Modeling closeness and negligibility

In the previous section, we have seen that some relational order of magnitude information can be expressed in practice by fuzzy relations whose membership functions depend on difference of values. If we prefer, we can also work in terms of the ratio x/y , rather than in terms of the difference $x - y$, especially when both x and y are positive. Then an approximate equality is modelled by a fuzzy relation $\mu_E(x,y) = \mu_M(x / y)$ where M is a fuzzy interval such as $\mu_M(1) = 1$, i.e., M represents a "fuzzy one". More or less strong inequality relations can be obtained by substituting x / y to $x - y$ in definition (3) of μ_K , i.e., $\mu_I(x,y) = \mu_K(x / y)$. The use of ratios look natural for the modeling of negligibility relations. When μ_K and μ_L are functions of x / y (instead of $x - y$), we obtain $\mu_{IOE}(x,z) = \mu_{K \otimes L}(x / z)$ instead of (5), where \otimes denotes the product extended to fuzzy numbers. In the following, we shall omit ' \otimes ' when writing products.

As pointed out by Dubois and Prade (1989a), the idea of relative closeness (Cl) can be captured by the following approximate equality relation between numbers x and y :

$$\mu_{Cl}(x,y) = \mu_M(x / y) \quad (22)$$

where the characteristic function μ_M is such that $\mu_M(1) = 1$ (since x is close to x), $\mu_M(t) = 0$ if $t \leq 0$ (since two numbers which are close should have the same sign), and $\mu_M(t) = \mu_M(1 / t)$ in order to ensure that $\mu_{Cl}(x,y) = \mu_{Cl}(y,x)$. In other words, M is a fuzzy interval which restricts values around 1 and which is equal to its "inverse", M^{-1} which is defined by $\mu_{M^{-1}}(t) = \sup_{s=1/s} \mu_M(s) = \mu_M(1 / t)$. Thus, we have the semantic equivalence

$$(a,b) \in Cl \Leftrightarrow (b,a) \in Cl. \quad (23)$$

Strict equality is recovered for $M = 1$, i.e., $\mu_1(x/y) = 1$ if $x = y$ and $\mu_1(x/y) = 0$ otherwise.

Raiman, in his inference system FOG, relates the ideas of closeness and of negligibility in the following way: 'a is close to b' is equivalent to 'a - b is negligible in comparison with b'. In other words, 'a is negligible in comparison with b' if and only if 'a + b is close to b' (see rule 8 of FOG). It leads to define the extent to which x is negligible in comparison with y , by

$$\mu_{Ne}(x,y) = \mu_M((x+y)/y) = \mu_M(1+x/y). \quad (24)$$

By definition, if $a, b, c > 0$ then the following equivalence hold, $\forall c$:

$$\begin{aligned} (a,b) \in Cl &\Leftrightarrow (ac,bc) \in Cl, \\ (a,b) \in Ne &\Leftrightarrow (ac,bc) \in Ne. \end{aligned}$$

The third relation Co is such that (see rule 2 of FOG): $a Cl b \Rightarrow a Co b$, which means that comparability is a less restrictive notion than closeness. Note that it is desirable to have not only the fuzzy relation inclusion $Cl \subseteq Co$ but also to express that a and b are no longer comparable (in the sense of Co) when a becomes negligible in comparison with b , or b becomes negligible in comparison to a . This leads to the definition:

$$\begin{aligned} &((a,b) \in Co \Leftrightarrow \neg [(a,b) \in Ne \text{ or } (b,a) \in Ne]) \\ \text{i.e.,} \quad \mu_{Co}(x,y) &= 1 - \max(\mu_{Ne}(x,y), \mu_{Ne}(y,x)) \\ &= \min(1 - \mu_M(1+x/y), 1 - \mu_M(1+y/x)). \end{aligned} \quad (25)$$

However, in the rest of the paper, we only consider rules involving the more basic relations Cl and Ne .

4.2. Inference rules with Cl and Ne relations

In the following we show that it is possible to develop inference rules where the fuzzy sets which parametrize the closeness and negligibility relations are manipulated at a symbolic level. The use of fuzzy relations can provide, as we shall see, an appropriate numerical semantics to the order of magnitude symbolic calculus performed by the formal system FOG. From now on, we use the notations $Cl[M]$ and $Ne[M]$ in order to express that Cl (resp. Ne) is defined from a membership function μ_M by (22) (resp. (24)). Using fuzzy arithmetic, the following inference rules can be established (Dubois and Prade, 1989a, 1991):

$$(a,b) \in Ne[M] \Leftrightarrow (a+b, b) \in Cl[M] \quad (26)$$

$$(a,b) \in Cl[M] \Leftrightarrow (a-b, b) \in Ne[M] \quad (27)$$

$$(a,b) \in Ne[M] \Leftrightarrow (-a,b) \in Ne[2 \dot{A} M] \quad (28)$$

$$(a, a+b) \in Ne[M] \Leftrightarrow (b, a+b) \in Cl[2 \dot{A} M] \quad (29)$$

$$(a,b) \in Cl[M] \text{ and } (b,c) \in Cl[N] \Rightarrow (a,c) \in Cl[MN] \quad (30)$$

$$(a,b) \in Ne[M] \text{ and } (b,c) \in Ne[N] \Rightarrow (a,c) \in Ne[(M \dot{A} 1)(N \dot{A} 1) \oplus 1] \quad (31)$$

$$(a,b) \in Cl[M] \text{ and } (b,c) \in Ne[N] \Rightarrow (a,c) \in Ne[M(N \dot{A} 1) \oplus 1] \quad (32)$$

$$(a,c) \in Cl[M] \text{ and } (b,a) \in Ne[N] \Rightarrow (a+b, c) \in Cl[MN] \quad (33)$$

$$(a,c) \in Cl[M] \text{ and } (b,c) \in Ne[N] \Rightarrow (a+b, c) \in Cl[M \oplus N \dot{A} 1] \quad (34)$$

$$(a,c) \in Ne[M] \text{ and } (b,c) \in Ne[N] \Rightarrow (a+b, c) \in Ne[M \oplus N \dot{A} 1] \quad (35)$$

$$(a,b) \in Cl[M] \text{ and } (c,d) \in Ne[N] \Rightarrow (a \cdot c, b \cdot d) \in Ne[M(N \dot{A} 1) \oplus 1] \quad (36)$$

$$(a,b) \in Ne[M] \text{ and } (c,d) \in Ne[N] \Rightarrow (a \cdot c, b \cdot d) \in Ne[(M \dot{A} 1)(N \dot{A} 1) \oplus 1] \quad (37)$$

$$(a,b) \in Cl[M] \text{ and } (c,d) \in Cl[N] \Rightarrow (a \cdot c, b \cdot d) \in Cl[MN] \quad (38)$$

$$(a + b, c) \in \text{Cl}[M] \text{ and } (b, a) \in \text{Ne}[N] \Rightarrow (a, c) \in \text{Cl}[MN]. \quad (39)$$

Most of these rules have counterparts in the FOG system. For instance, the rules (30)-(32) and (39) are the counterparts of the following rules of FOG:

$$\begin{aligned} (a, b) \in \text{Cl} \text{ and } (b, c) \in \text{Cl} &\Rightarrow (a, c) \in \text{Cl}, \\ (a, b) \in \text{Ne} \text{ and } (b, c) \in \text{Ne} &\Rightarrow (a, c) \in \text{Ne}, \\ (a, b) \in \text{Cl} \text{ and } (b, c) \in \text{Ne} &\Rightarrow (a, c) \in \text{Ne}, \\ (a + b, c) \in \text{Cl} \text{ and } (b, a) \in \text{Ne} &\Rightarrow (a, c) \in \text{Cl}. \end{aligned}$$

The fuzzy relations approach shows that these rules are only "qualitatively valid". The repeated use of some of these rules without control can lead to dubious conclusions. Fuzzy relations enable us to acknowledge that the relation Cl is "weakly transitive" (since for $M = N$, $\text{Cl}[M] \subseteq \text{Cl}[M^2]$, i.e., 'a may be less close to c' than 'a is close to b'), and that the relation Ne is "strongly transitive" (since for $M = N$, $\text{Ne}[M] \supseteq \text{Ne}[(M \dot{-} 1)(M \dot{-} 1) \oplus 1]$, i.e., 'a is more negligible in comparison with c' than 'a in comparison with b', and 'b with c').

In order to equip all the rules of FOG with numerical semantics, we propose new inference rules (not studied in Dubois and Prade (1989a)) involving the product of the considered quantities:

$$(a \cdot b, c \cdot d) \in \text{Ne}[M] \text{ and } (c, a) \in \text{Ne}[N] \Rightarrow (b, d) \in \text{Ne}[(M \dot{-} 1)(N \dot{-} 1) \oplus 1] \quad (40)$$

$$(a \cdot b, c \cdot d) \in \text{Cl}[M] \text{ and } (c, a) \in \text{Cl}[N] \Rightarrow (b, d) \in \text{Cl}[MN] \quad (41)$$

$$(a \cdot b, c \cdot d) \in \text{Cl}[M] \text{ and } (a, c) \in \text{Ne}[N] \Rightarrow (d, b) \in \text{Ne}[M(N \dot{-} 1) \oplus 1] \quad (42)$$

Proof:

- The composition of relations $(a \cdot b, c \cdot d) \in \text{Ne}[M]$ and $(c, a) \in \text{Ne}[N]$ can be written under the form:

$$\begin{aligned} &\sup_{x, z} \min(\mu_{\text{Ne}[M]}(x \cdot y, z \cdot t), \mu_{\text{Ne}[N]}(z, x)) \\ &= \sup_{x, z} \min(\mu_M(1 + [x \cdot y / z \cdot t]), \mu_N(1 + z / x)) \\ &= \sup_{u, v} \min(\mu_M(u), \mu_N(v)), \text{ letting } u=1+[x \cdot y / z \cdot t] \text{ and } v=1+z/x \\ &\quad (u-1)(v-1)+1=1+y/t \\ &= \mu_{[(M \dot{-} 1)(N \dot{-} 1) \oplus 1]}(1 + y / t). \end{aligned}$$

This implies that $(b, d) \in \text{Ne}[(M \dot{-} 1)(N \dot{-} 1) \oplus 1]$.

- In the same way, we can establish rules (41) and (42). Considering the rule (42), the composition of relations $(a \cdot b, c \cdot d) \in \text{Cl}[M]$ and $(a, c) \in \text{Ne}[N]$ can be written under the form:

$$\begin{aligned} &\sup_{x, z} \min(\mu_{\text{Cl}[M]}(x \cdot y, z \cdot t), \mu_{\text{Ne}[N]}(x, z)) \\ &= \sup_{x, z} \min(\mu_M(x \cdot y / z \cdot t), \mu_N(1 + x / z)) \\ &= \sup_{x, z} \min(\mu_M(z \cdot t / x \cdot y), \mu_N(1 + x / z)) \\ &= \sup_{u, v} \min(\mu_M(u), \mu_N(v)), \text{ letting } u=z \cdot t / x \cdot y \text{ and } v=1+x/z \end{aligned}$$

$$\begin{aligned} & u(v-1)+1=1+t/y \\ & = \mu_{[M(N\acute{A}1)\oplus 1]}(1 + t / y). \end{aligned}$$

This latter result implies that $(d,b) \in \text{Ne}[M(N\acute{A} 1) \oplus 1]$. y

The syntactic approximations of rules (40), (41) and (42) are given by the following FOG's rules:

$$\begin{aligned} (a \cdot b, c \cdot d) \in \text{Ne} \text{ and } (c,a) \in \text{Ne} &\Rightarrow (b,d) \in \text{Ne} \\ (a \cdot b, c \cdot d) \in \text{Cl} \text{ and } (a,c) \in \text{Cl} &\Rightarrow (b,d) \in \text{Cl} \\ (a \cdot b, c \cdot d) \in \text{Cl} \text{ and } (a,c) \in \text{Ne} &\Rightarrow (d,b) \in \text{Ne}. \end{aligned}$$

In the following, we show that it is possible to derive other symbolic inference rules which are not explicit in the formal system FOG. First, we start with simple rules whose premises are not compound, namely

$$(a, a \cdot b) \in \text{Ne}[M] \Rightarrow (1 + b, b) \in \text{Cl}[M] \text{ if } a \neq 0 \quad (43)$$

$$(a, a \cdot b) \in \text{Cl}[M] \Rightarrow (1,b) \in \text{Cl}[M] \text{ if } a \neq 0 \quad (44)$$

$$(a - b, a) \in \text{Ne}[M] \Rightarrow (b,a) \in \text{Cl}[2 \acute{A} M] \quad (45)$$

$$(a - b, a) \in \text{Cl}[M] \Rightarrow (b,a) \in \text{Ne}[2 \acute{A} M]. \quad (46)$$

These rules can be easily proved by using the definitions of relations Cl and Ne given in (22) and (24) respectively.

More complex rules expressing the behavior of the relations Cl[M] and Ne[N] with respect to arithmetic operations (addition, subtraction and product) either in the premise or in the conclusion, can be derived as well

$$(a,c) \in \text{Ne}[M] \text{ and } (b,a) \in \text{Ne}[N] \Rightarrow (a + b, c) \in \text{Ne}[(M \acute{A} 1)N \oplus 1] \quad (47)$$

$$(a,b) \in \text{Ne}[M] \text{ and } (c,b) \in \text{Ne}[N] \Rightarrow (a, b + c) \in \text{Ne}[(M \acute{A} 1)N \oplus 1] \quad (48)$$

$$(a \cdot b, c \cdot d) \in \text{Ne}[M] \text{ and } (a,c) \in \text{Cl}[N] \Rightarrow (b,d) \in \text{Ne}[(M \acute{A} 1)N \oplus 1] \quad (49)$$

$$(a + b, c) \in \text{Cl}[M] \text{ and } (b,c) \in \text{Cl}[N] \Rightarrow (a,c) \in \text{Ne}[M \acute{A} N \oplus 1] \quad (50)$$

$$(a + b, c) \in \text{Cl}[M] \text{ and } (b,c) \in \text{Ne}[N] \Rightarrow (a,c) \in \text{Cl}[M \acute{A} N \oplus 1] \quad (51)$$

$$(a + b, c) \in \text{Ne}[M] \text{ and } (b,c) \in \text{Ne}[N] \Rightarrow (a,c) \in \text{Ne}[M \acute{A} N \oplus 1] \quad (52)$$

$$(a - b, c) \in \text{Cl}[M] \text{ and } (b,a) \in \text{Ne}[N] \Rightarrow (c,a) \in \text{Cl}[M(2 \acute{A} N)] \quad (53)$$

$$(a - b, c) \in \text{Cl}[M] \text{ and } (a,c) \in \text{Cl}[N] \Rightarrow (b,c) \in \text{Ne}[N \acute{A} M \oplus 1] \quad (54)$$

$$(a - b, c) \in \text{Ne}[M] \text{ and } (a,c) \in \text{Cl}[N] \Rightarrow (b,c) \in \text{Cl}[N \acute{A} M \oplus 1] \quad (55)$$

$$(a - b, c) \in \text{Cl}[M] \text{ and } (b,c) \in \text{Ne}[N] \Rightarrow (a,c) \in \text{Cl}[M \oplus N \acute{A} 1]. \quad (56)$$

Proof: Let us look at the proof of some of these rules:

- Rule (47) can be established noticing that

$$\begin{aligned} & \sup_{\substack{x,y \\ x+y=w}} \min(\mu_{\text{Ne}[M]}(x,z), \mu_{\text{Ne}[N]}(y,x)) \\ & = \sup_{x,y} \min(\mu_M(1 + x / z), \mu_N(1 + y / x)) \end{aligned}$$

$$\begin{aligned}
& \stackrel{x+y=w}{=} \sup_{u,v} \min(\mu_M(u), \mu_N(v)), \text{ letting } u=1+x/z \text{ and } v=1+y/x \\
& \quad (u-1)v+1=1+w/z \\
& = \mu_{(M \dot{\wedge} 1)N \oplus 1}(1 + w / z).
\end{aligned}$$

This means that $(a + b, c) \in \text{Ne}[(M \dot{\wedge} 1)N \oplus 1]$.

Rule (48) can be obtained similarly noticing that $\left(\left(1 + \frac{x}{y} \right) - 1 \right) \frac{y}{y+z} + 1 = 1 + \frac{x}{y+z}$ and using $\mu_N \left(1 + \frac{z}{y} \right) = \mu_N \left(\frac{y}{y+z} \right)$.

- The result of rule (50) can be easily obtained noticing also that

$$\begin{aligned}
& \sup_y \min(\mu_{\text{Cl}[M]}(x + y, z), \mu_{\text{Cl}[N]}(y, z)) \\
& = \sup_y \min(\mu_M((x + y) / z), \mu_N(y/x)) \\
& = \mu_{[M \dot{\wedge} N \oplus 1]}(1 + x / z), \text{ observing that } 1+(x/z)=[(x+y)/z]-(y/z)+1.
\end{aligned}$$

Then we conclude that $(a, c) \in \text{Ne}[M \dot{\wedge} N \oplus 1]$.

- In rule (53), the composition of relations $(a - b, c) \in \text{Cl}[M]$ and $(b, a) \in \text{Ne}[N]$ leads to

$$\begin{aligned}
& \sup_y \min(\mu_{\text{Cl}[M]}(x - y, z), \mu_{\text{Ne}[N]}(y, x)) \\
& = \sup_y \min(\mu_M((x - y) / z), \mu_N(1 + y / x)) \\
& = \sup_y \min(\mu_M(z / (x - y)), \mu_N(1 + y / x)) \\
& = \sup_{u,v} \min(\mu_M(u), \mu_N(v)), \text{ letting } u=z/(x-y) \text{ and } v=1+y/x \\
& \quad u(2-v)=z/x \\
& = \mu_{[M(2 \dot{\wedge} N)]}(z/x).
\end{aligned}$$

This expresses that $(c, a) \in \text{Cl}[M(2 \dot{\wedge} N)]$.

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5. Propagation of Relative Orders of Magnitude: Some Examples

In this section, we show that the previous inference rules can be useful for solving equations in an approximate way. Such equations involve arithmetic expressions and order of magnitude relations (expressed in terms of parametrized relations Cl and Ne).

Example 1: The following relations are supposed to be known to hold:

$$(c \cdot d, a \cdot b) \in \text{Cl}[M]; (a + c, c) \in \text{Cl}[N]; (d / b, f / e) \in \text{Cl}[P].$$

The problem is to compare the orders of magnitude of quantities e and f. The following relations can be easily obtained:

$$(a \cdot b, c \cdot d) \in \text{Cl}[M] \text{ (since the relation Cl is symmetrical)}$$

$$(a, c) \in \text{Ne}[N] \text{ (using the definition of Ne in terms of Cl, i.e., rule (27))}$$

By applying rule (42), we can deduce that $(d,b) \in \text{Ne}[M(N \dot{\wedge} 1) \oplus 1]$. The relation $(d/b, f/e) \in \text{Cl}[P]$ implies that $(d \cdot e, b \cdot f) \in \text{Cl}[P]$. Rule (42), based on this latter relation and $(d,b) \in \text{Ne}[Q]$ with $Q = M(N \dot{\wedge} 1) \oplus 1$, enables us to conclude that $(f,e) \in \text{Ne}[PM(N \dot{\wedge} 1) \oplus 1]$. If we start with classical equalities, i.e., $a \cdot b = c \cdot d$ and $d/b = f/e$, then letting $M = 1$ and $P = 1$, it leads to $(f,e) \in \text{Ne}[N]$. Now, if the parameter $e = 100$, then we can come up with an estimate of the parameter f under the form of a fuzzy number F , namely:

$$\mu_F(x) = \mu_{\text{Ne}[N]}(x,100) = \mu_N(1 + x/100) = \mu_{[100(N \dot{\wedge} 1)]}(x)$$

i.e., $F = 100(N \dot{\wedge} 1)$, which can be easily computed. This illustrates that fuzzy relations can provide a natural framework for interfacing symbolic information and numerical data.

Example 2: (taken from Mavrovouniotis and Stephanopoulos (1988))

$$\begin{aligned} (d,b) \in \text{Ne}[M]; (b,c) \in \text{Ne}[N]; (e,g) \in \text{Ne}[P]; \\ a - b - c - d = 0; a \cdot e \cdot f - c \cdot g \cdot h = 0. \end{aligned}$$

The question now is to compare the values of f and h . By applying rule (26), we get $(d + b, b) \in \text{Cl}[M]$. Substituting $a - c$ to $d + b$ (because $a - b - c - d = 0$), we obtain $(a - c, b) \in \text{Cl}[M]$. Rule (32), based on this latter relation and $(b, c) \in \text{Ne}[N]$, enables us to deduce that $(a - c, c) \in \text{Ne}[M(N \dot{\wedge} 1) \oplus 1]$. This relation implies that $(a,c) \in \text{Cl}[M(N \dot{\wedge} 1) \oplus 1]$. Using rule (36), we get $(a \cdot e, c \cdot g) \in \text{Ne}[Q(P \dot{\wedge} 1) \oplus 1]$, with $Q = M(N \dot{\wedge} 1) \oplus 1$. Since $(a \cdot e / c \cdot g) = h / f$, the pair (h,f) should satisfy the same fuzzy relation, i.e., $(h,f) \in \text{Ne}[Q(P \dot{\wedge} 1) \oplus 1]$. This example can be solved using other inference rules. For example, the chaining of rules (31), (35) and (27) enables us to obtain $(a,c) \in \text{Cl}[R \oplus N]$, with $R = (M \dot{\wedge} 1)(N \dot{\wedge} 1)$; then using rule (36) we get $(a \cdot e, c \cdot g) \in \text{Ne}[(R \oplus N)(P \dot{\wedge} 1) \oplus 1]$ and finally $(h,f) \in \text{Ne}[(R \oplus N)(P \dot{\wedge} 1) \oplus 1]$.

Generally, different results (some less restrictive than others) can be obtained from different chainings of inference rules, as suggested by the above example. The conjunctive combination of these partial results restricting the possible values of a ratio of interest enables us to refine the conclusion. Note that the most restrictive result would be obtained by applying the combination/projection method (Zadeh, 1979), because in this approach all the existing constraints are exploited. Nevertheless, this approach can be computationally untractable in practice, and no explanation can be provided for the obtained results using such a global handling of the constraints.

Example 3: Let us now consider a practical example of a counter-current heat exchanger as described in (Mavrovouniotis and Stephanopoulos, 1988). The important parameters in the analysis of the device are the molar-heat KH and the molar-flowrate FH of the hot stream, and the molar-heat KC and the molar-flowrate FC of the cold stream. Four temperature differences are defined: DTH is the temperature drop of the hot stream, DTC is the temperature rise of the cold stream, $DT1$ is the driving force at the left end of the device, and $DT2$ is the driving force at the right end of the device. The two following equations hold

$$(e1) \quad DTH - DT1 - DTC + DT2 = 0$$

$$(e2) \quad DTH \cdot KH \cdot FH = DTC \cdot KC \cdot FC.$$

The first one is a consequence of the definition of the temperature differences, and the second one is the energy balance of the device. Let us take the following assumptions expressed as orders of magnitude relations

- (i) $(DT2, DT1) \in Cl[M]$
- (ii) $(DT1, DTH) \in Ne[N]$
- (iii) $(KC, KH) \in Cl[P]$.

The problem now is to deduce the relation between the parameters FC and FH. Rule (32), based on the relations (i) and (ii) enables us to obtain

$$(DT2, DTH) \in Ne[M(N \acute{A} 1) \oplus 1].$$

The following inferences are easily obtained as well

$$\begin{aligned} (DT1, DTH) \in Ne[N] &\Rightarrow (-DT1, DTH) \in Ne[2 \acute{A} N] && \text{(by rule (28))} \\ (DT2, DTH) \in Ne[M(N \acute{A} 1) \oplus 1] &\text{ and } (-DT1, DTH) \in Ne[2 \acute{A} N] \\ &\Rightarrow (DT2 - DT1, DTH) \in Ne[M(N \acute{A} 1) \oplus 2 \acute{A} N] && \text{(by rule (35)).} \end{aligned}$$

In the latter result, we can substitute $DTC - DTH$ to $DT2 - DT1$ using equation (e1), we get

$$(DTC - DTH, DTH) \in Ne[M(N \acute{A} 1) \oplus 2 \acute{A} N].$$

Then applying rule (27), we get

$$(DTC, DTH) \in Cl[M(N \acute{A} 1) \oplus 2 \acute{A} N].$$

Rule (38), based on this relation and relation (iii) enables us to conclude that

$$(DTC \cdot KC, DTH \cdot KH) \in Cl[(M(N \acute{A} 1) \oplus 2 \acute{A} N)P]$$

and finally since $DTC \cdot KC / DTH \cdot KH = FH / FC$, see equation (e2), the pair (FH, FC) should satisfy the same fuzzy relation, i.e.,

$$(FH, FC) \in Cl[(M(N \acute{A} 1) \oplus 2 \acute{A} M)P].$$

6. Conclusion

Fuzzy relations can provide an appropriate semantics for inference rules about relative orders of magnitude. Each rule has a precise interpretation in terms of fuzzy sets. The composition of relations $Ne[M]$ and $Cl[N]$ reduces to simple arithmetic operations on the fuzzy numbers M and N . Moreover, the fuzzy relations calculus enables us to reason about closeness and negligibility in a rigorous way without being obliged to introduce arbitrary limitations on the chaining of rules (by means of control techniques) in order to avoid undesirable transitivity effects for instance. The approach is faithful to the gradual nature of comparators. Another clear advantage of the use of fuzzy relations is that it enables us to interface numbers with symbolic information in a convenient way as hinted in example 1. Thus, it provides an example of computing with the words 'negligible' and 'close to' in a

symbolic manner, in full agreement with the numerical semantics underlying these words. As briefly suggested in Section 3, a similar approach based on other fuzzy relations could be also developed in temporal reasoning (where approximate equalities or fuzzy ordering relations can apply to dates), or even in spatial reasoning.

A noticeable feature of the approach is the separation of the symbolic processing of the labels of the inferred relations from the expression of their semantics in terms of their membership functions. It is why it is important to always check that the produced conclusions are semantically meaningful, e.g. that the obtained relation of closeness is not too permissive. If a relation $\text{Ne}[P]$ of negligibility is rather obtained, we should make sure that it is in agreement with the intuitive semantics underlying this notion (e.g., $\mu_P(2) = 0$, otherwise x would be negligible with respect to itself, since $\mu_P(1 + x / x) = \mu_P(2)$). Indeed even if we start with relations having semantically meaningful parameters, it may happen that after several chainings of the rules, the result is no longer a closeness or a negligibility relation semantically speaking, in spite of its appearance at the syntactic level.

Future works should deal especially with: i) the conception of a complete system of inference rules (for particular types of equations systems, for example, linear equations involving only addition and subtraction operations, and product by a constant); ii) the choice of a set of formal computation rules which enables us to write equations in a suitable way when necessary (since the form of equations can interfere with the possibility of applying inference rules); iii) to implement this system and the set of computation rules in order to obtain automated strategies for simplifying equations involving relative orders of magnitude relations and arithmetic expressions. This should be useful for solving numerical equations, in an approximate way, for physico-chemical systems, or for the qualitative calculus of probabilities (Dubois and Prade, 1991; Dubois et al., 1993).

Finally, it should be also emphasized that reasoning with relative orders of magnitude can be seen as a particular type of fuzzy Constraint Satisfaction Problems (Dubois, Fargier and Prade, 1996), which may provide a source of inspiration for the implementation. Thus, there is a balance to find between a minimal complete set of rules and a larger set of rules exhibiting some redundancy, but offering more possibilities of being applicable to a given set of constraints. See (Hadj-Ali et al., 1998; Dubois et al., 1999) for the study of a reduced set of rules from which all the rules provided in Section 4 can be derived. Moreover it is possible to approximate all these rules using symmetrical fuzzy numbers as parameters (e.g. in terms of MN , if both M and N are symmetrical); a method for establishing the inconsistency of a set of hypotheses w.r.t. a set of equations, is also hinted (see Hadj-Ali et al., 1999).

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Appendix

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