FREE GROUP PRESENTATION OF TRIANGULAR PLANAR SHAPES AND THEIR BETTI NUMBERS

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Abstract. This paper introduces free group presentations of planar path triangular shapes and their corresponding Betti numbers. Paths are maps in homotopy theory. The edges of path triangles are homotopic paths. Main results in this paper are that (1)every path triangle has a free group presentation and (2) every path path triangle has a geometric realization.

Contents

1. Introduction

In this paper, the conventional planar triangle is replaced by a path triangle with 0-cells (vertices) attached to 1-cells (edges) in a cell complex. In homotopy theory, a **path** is a continuous map $h : I \to X$ from the unit interval to a space X. In the plane, a path triangle is geometrically realized as a 1-cycle. In effect, every path triangle results in a geometric realization of a simplified form of homotopy system, introduced by J.H.C. Whitehead [15, 16], namely:

Theorem [16, Theorem 2,§6]. A given homotopy system, ρ, has a geometric realization, if dim≤ 4.

In its simplest form, ρ is free group realized geometrically as a 1-cycle (sequences of 1-cells (edges) with no end vertex and with nonvoid interior). In the plane, each ρ is geometrically realized as a $|\rho|$, which is a new form of triangle in a path-connected

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cell complex. A **path** is a continuous mapping $h : I \to X$ from $h(0)$ to $h(1)$ [13, §2.1, p. 11].

In this form of triangulation, edges between vertexes are a geometric realization of paths in a finite, bounded cell complex in a CW (closure-finite weak) space K introduced by Whitehead [15, §5]. Briefly, a **CW space** K is Hausdorff (points in K reside in disjoint neighborhoods), the closure of every cell complex is K and every nerve complex (nonvoid intersection of cell complexes in K) is in K.

Path triangles lead to new forms of path cycles [8], Alexandrov nerves [7] as well as good covers [9] on a planar triangulated cell complex.

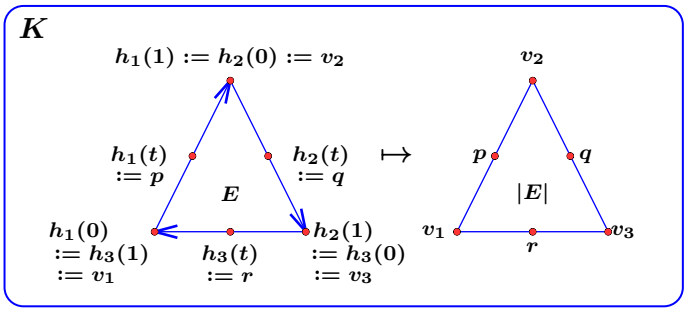

FIGURE 1. Three Paths that Construct the Sides of a Path Triangle $\boldsymbol{E}$ in space $\boldsymbol{K}$, leading to its geometric realization $\boldsymbol{|E|}$ as a 2-cell (triangle)

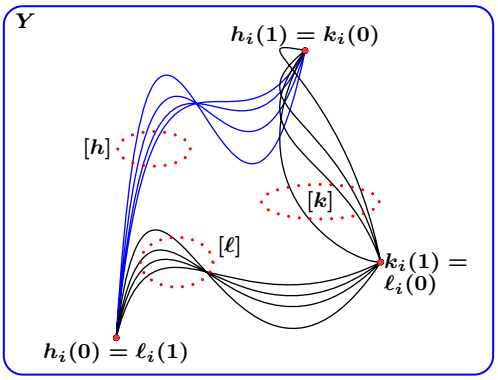

FIGURE 2. Curviliner, overlapping path triangle cycles

2. PRELIMINARIES

The basic building block in this work is a path triangle E (denoted by $h\triangle E$.

Definition 1. Path.
Let $I = [0, 1]$, the unit interval. A **path** *in a space X is a continuous map $h : I \to X$ with endpoints $h(0) = x_0, h(1) = x_1$ and $h(t) \in [h(0), h(1)]$, for all $t \in I$* [13, §2.1,p.11]. *The geometric realization of a path h is denoted $|h|$.* ■

Lemma 1. *Every path has a geometric realization as an edge.*

Proof. From Def. 1, a path $h : I \to X$ includes all points $h(t), t \in [0, 1]$, *i.e.*, we have the closed set of points

$$\{h(0), \ldots, h(t), \ldots, h(1)\}, \text{ for } 0 \leq t \leq 1,$$

which is geometrically realized as an edge $|h|$. □

From Lemma 1, the geometric realization of a path h is a 1-cell (edge) $|h|$ in a cell complex. In attaching paths together to construct a path triangle E, the end result is a path cycle with a geometric counterpart $|E|$ that is a 1-cycle.

Definition 2. 1-Cycle.
In a CW space K [15], *a 1-cycle E (denoted by cycE) is a collection of path-connected vertexes (0-cells) on edges (1-cells) attached to each other with no end vertex and cycE has a nonvoid interior.* ■

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Definition 3. Path Triangle.
A **path triangle** *$h \triangle E$ is a sequence of three overlapping paths with no end path and with a nonvoid interior. In the plane, the geometric realization of $h \triangle E$ is a 2-cell (triangle) denoted by $|h \triangle E|$.* ■

Example 1. *A sample path triangle E and its geometric realization $|E|$ are shown in Fig. 1. From Lemma 1, we know that every path in E has a geometric realization as a 1-cell (edge). Here, triangle $\triangle v_1v_2v_3$ is a geometric realization path triangle $h \triangle E$ containing overlapping paths h_1, h_2, h_3,* i.e.*,*

$$\begin{aligned}
h_1, h_2, h_3 \in h \triangle E \mapsto\ & \textbf{edge } \widehat{v_1v_2} \in \triangle v_1v_2v_3, \textit{ with} \\
& h_1(0) := h_3(1) := v_1, h_1(1) := h_2(0) := v_2 \textit{ and} \\
& h_1(t) := \textbf{ vertex } p \in \textit{edge } \widehat{v_1v_2}, t \in [0, 1]. \\
h_1, h_2, h_3 \in h \triangle E \mapsto\ & \textbf{edge } \widehat{v_2v_3} \in \triangle v_1v_2v_3, \textit{ with} \\
& h_1(1) := h_2(0) := v_2, h_2(1) := h_3(0) := v_3 \textit{and} \\
& h_2(t) := \textbf{ vertex } q \in \textit{edge } \widehat{v_2v_3}, t \in [0, 1]. \\
h_1, h_2, h_3 \in h \triangle E \mapsto\ & \textbf{edge } \widehat{v_3v_1} \in \triangle v_1v_2v_3, \textit{ with} \\
& h_2(1) := h_3(0) := v_3, h_3(1) := h_1(0) := v_1 \textit{and} \\
& h_3(t) := \textbf{ vertex } r \in \textit{edge } \widehat{v_3v_1}, t \in [0, 1].
\end{aligned}$$ ■

Proposition 1. *Every path triangle has a geometric realization as a 1-cycle.*

Proof. From Def. 3, a path triangle $h \triangle E$ is a collection of path-connected vertexes with no end vertex and with nonvoid interior. From Lemma 1, the geometric realization of each path in $h \triangle E$ is an edge. Consequently, the sequence of paths in $h \triangle E$ has a geometric realization as a sequence of 1-cells (edges) attached to each other. Hence, from Def. 2, we obtain the geometric realization of $h \triangle E$ as a 1-cycle. □

3. Path Triangle vs. Veech Triangular Billiard

Path triangles resemble but are structurally different from the geometric representation of the motion of billiard ball inside the boundary of a polygon. Billiards in polygons in the Euclidean plane were introduced by A.N. Zemljakov and A.B. Katok [17]. A **billiard in a polygon** Q in $\mathbb{R}^2$ is a dynamical system produced by the frictionless motion of a point-sphere (billiard ball) inside Q with elastic reflection from the boundary of Q. Triangular billiards were introduced by V.A. Veech [14] as a part of a study of Delaunay partitions. A **triangular billiard** is an isosceles triangle containing edges that are finite billiard trajectories that begin and end in distinct vertices. By contrast, the edges in a path triangle are paths (from homotopy theory) having varying length edges with a particular direction indicated by arrows, geometrically realized as triangles.

4. Path Triangles the Conform to Curvilinear Shapes

So far, we have only considered path triangles geometrically realized as planar triangles with straight edges. To accommodate the need to cover curvilinear shapes with path triangles, we introduced curvilinear path triangles. This is made possible by defining paths (from homotopy theory) that have intermediate path prints that follow a curved path between vertices.

Definition 4. Curvilinear Path Triangle.
A **curvilinear path triangle** *is a path triangle $h\triangle E$ whose edges have intermediate values that follow a curvilinear path,* i.e.*, for a path $h \in h \triangle E$, the edge traced by $h(t), 0 < t < 1$ is curilinear yielding a curved edge instead of the usual straight edge in a conventional path triangle.* ■

We use the notation $[h]$ for a **path class**, which is a collection of paths that identical end points.

Example 2. Overlapping Path Triangle Classes. *A sample collection of overlapping curvilinear path triangle classes $[h], [k], [\ell]$ is shown in Fig. 2.* ■

The overlapping path triangle classes in Example 2 provide an exotic example of an Alexandrov nerve complex. Notice that each path triangle in Fig. 2 is an example of a cell complex.

Definition 5. Alexandrov Nerve Complex.
A nerve complex NrvE in a CW space is a collection of nonempty cell complexes with nonvoid intersection. This is an example of an Alexandrov nerve [1, §4.3,p. 39]. ■

A vertex with a collection of triangles attached to it, is called the **nucleus** of the attached triangles (also called the nucleus of an Alexandrov-Hopf nerve complex or **Nerv** [2]).

Lemma 2. [7] *Every vertex in the triangulation of the vertices in a CW space is the nucleus of an Alexandrov nerve complex.*

Theorem 1. [7] *A CW space containing n path triangle vertexes contains n Alexandrov-Hopf nerve complexes.*

Proof. Immediate from Lemma 2. □

5. Path Triangle Cycles

Path triangles lead to path cycles, which are geometrically realized as 1-cycles.

Proposition 2. *Every path triangle has a geometric realization as a 1-cycle.*

Proof. This is a direct consequence of Lemma 1 and Def. 2. □

Lemma 3. *Every collection of overlapping path class triangles has a geometric realization as a collection of overlapping 1-cycles.*

Proof. Immediate from Prop. 2. □

Theorem 2. *Every collection of overlapping path class triangles is a collection of Alexandrov nerves with geometric realization as overlapping 1-cycles.*

Proof. Immediate from Def. 5 and Lemma 3. □

The result in Theorem 2 have important implications in the study of overlapping vector fields, which is a common occurrence in the formation of vortexes in trade winds [12], electromagnetic vector field shapes [3] and in sequences of video frame shapes [10, 4, 6].

Definition 6. Path-Connected.
A pair of 0-cells v, v' in a cell complex E is **path-connected**, *provided there is a sequence of paths $h_1, \ldots, h_k$, starting with $h_1(0) := v$ and ending with $h_{k-1}(1) = h_k(0) := v'$.* ■

Lemma 4. *Every pair of vertexes in a path triangle is path-connected.*

Proof. Let $h \triangle E$ (briefly, E) be a path triangle in a triangulated cell complex and let v, v' be vertexes in E. We want to move from v to v' in this path triangle. Also let h, h' be paths in E Let $h \in E$ with $h(0) := v$ and $h(1) := h'(0)$. If $h(1) = v'$, then h, h' is the desired path. Otherwise, we follow path h' until we reach vertex v'. □

6. Free Group Presentation of Path Triangles

A finite group G is free, provided every element $x \in G$ is a linear combination of its basis elements (called generators) [5, §1.4, p. 21]. We write $\mathcal{B}$ to denote a nonempty basis set of generators $\{g_1, \ldots, g_{|\mathcal{B}|}\}$ and $G(\mathcal{B}, +)$ to denote the free group with binary operation $+$.

Definition 7. Rotman Presentation[11, p.239]
Let $X = \{g_1, \ldots\}, \triangle = \{v = \sum kg_i, v \in groupG, g_i \in X\}$ be a set of generators of members of a nonempty set X and set of relations between members of G and the generators in X. A mapping of the form $\{X, \triangle\} \to G$, a free group, is called a presentation of G. ■

We write $G(V, +)$ to denote a group G on a nonvoid set V with a binary operation $+$. For a group $G(V, +)$ presentable as a collection of linear combinations of members of a basis set $\mathcal{B} \subseteq V$, we write $G(\mathcal{B}, +)$.

Definition 8. Free Group Presentation of a Cell Complex.
Let 2^K be the collection of cell complexes in a CW space K, $E \in 2^K$ containing n vertexes, $G(E, +)$ a group on nonvoid set E with binary operation $+$, $\triangle =$

$\{v = \sum kg_i, v \in E, g_i \in E\}$ *be a set of generators of members in* E*, set of relations between members of* E *and the generators* $\mathcal{B} \subset E$*,* $g_i \in \mathcal{B}, v = h_{i mod\ n}(0) \in K$*,* k_i *the* i^{th} *integer coeficient modn in a linear combination* $\sum_{i,j} k_i g_j$ *of generating elements* $g_j = h_j(0) \in \mathcal{B}$*. A free group* **presentation** *of* G *is a continuous map* $f : 2^K \times \triangle \to 2^K$ *defined by*

$$f(\mathcal{B}, \triangle) = \left\{ v := \sum_{i,j} k_i g_j \in \triangle : v \in E, g_j \in \mathcal{B}, k_i \in \mathbb{Z} \right\}$$

$$\mathcal{B} \times \triangle \mapsto \textbf{free group } \boldsymbol{G} \textbf{ presentation of } G(E,+)$$

$$= \overbrace{\boldsymbol{G}(\{\boldsymbol{g_1}, \ldots, \boldsymbol{g}_{|\mathcal{B}|}\}, +)}. \qquad \blacksquare$$

Lemma 5. [8]
Every path cycle in a CW space has a free group presentation.

Proof. Let E be a set of vertices in the range of the paths $h : I \to E$ in a path cycle hCyc$E$ in a CW space K, G(hCyc$E, +$) a group on path vertexes of hCyc$E$ with move operation $+$, and let

$$\mathcal{B} \subset \text{hCyc}E,$$
$$\triangle = \left\{ v = \sum kh_i(0), v, h_i(0) \in \text{hCyc}E \right\}.$$

be the basis $\mathcal{B}$ set of members of hCyc$E$ and set of relations $\triangle$ between members of hCyc$E$ and the generators in basis $\mathcal{B}$. Then the free group presentation of G is a mapping $f : 2^K \times \triangle \to 2^K$ defined by

$$f(\mathcal{B}, \triangle) = \left\{ v := \sum_{i,j} k_i h_j(0) \in \triangle : v \in \text{hCyc}E, h_j \in \mathcal{B}, k_i \in \mathbb{Z} \right\}$$

$$\mathcal{B} \times \triangle \mapsto \textbf{free group presentation of } G(\textbf{hCyc}E, +)$$

$$= \overbrace{\boldsymbol{G}(\{\boldsymbol{h_1}, \ldots, \boldsymbol{h}_{|\mathcal{B}|}\}, +)}.$$

□

Proposition 3. *Every path cycle in a path triangle has a free group presentation.*

Proof. Immediate from Lemma 5. □

Example 3. *From Prop. 3, each of the path cycles the path class triangles in Example 2 has a free group presentation.* ■

Lemma 6. *Every free group presentation of a path cycle has a geometric realization.*

Proposition 4. .
Every path class triangle has a geometric realization.

Proof. The result follows from from Prop. 6, since every path class triangle is a collection of overlapping path triangles with the same beginning vertex and the same ending vertex. □

Theorem 3. *Every path triangle has free group presentation.*

Proof. This results is an immediate consequence of Lemma 5, since every path triangle contains a path cycle. □

A simplified view of path triangles results is a consequence of the fact that every free group has a Betti number. A **Betti number** is a count of the number generators in a free group [5, §4,p. 24]. The Betti number of a free group G is also the **rank** of G.

Proposition 5. *Every path triangle has a Betti number.*

Proof. Let $h \triangle E$ be a path triangle. From Lemma 5, $h \triangle E$ has a free group presentation $G(\beta, +)$, since $h \triangle E$ is also a path cycle. Hence, the rank of G (*i.e.*, $|\beta|$) is the Betti number of $h \triangle E$. □

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Computational Intelligence Laboratory, University of Manitoba, WPG, MB, R3T 5V6, Canada and Department of Mathematics, Faculty of Arts and Sciences, Adïyaman University, 02040 Adïyaman, Turkey

Email address: james.peters3@umanitoba.ca