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# Non uniform warping for composite beams. Theory and numerical applications

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## Abstract :

Started from the 3D solution of the original St Venant problem, a general non-uniform warping beam theory (NUW-BT) has recently been established for an arbitrary homogeneous isotropic and elastic cross-section [1]. Using now the solution of the extended St Venant problem for composite beam, NUW-BT may be rewritten in order to be extended for any composite cross-section. For the present work, which constitutes a first step of this extension, the cross-section is assumed to be symmetric and made by orthotropic materials ; however, Poisson's effects (called here in-plane warping) are also taken into account. Theoretical results are given for the structural behavior of the composite beam and for the expressions of the 3D stresses ; these ones, easy to compare with 3D St Venant stresses, make clear the additional contribution of the new internal forces induced by the non-uniformity of the (in and out) warpings. Numerical results are given for a torsion and a shear-bending of a cantilever sandwich beam.

## Résumé :

Basée sur la solution 3D du problème de St Venant, une théorie générale du gauchissement non uniforme a été récemment établie pour une poutre homogène isotrope et de section quelconque [1]. Cette théorie peut être étendue aux sections composites en s'appuyant sur la solution du problème de St Venant étendu aux sections composites. Cette extension se restreint, dans le présent papier, aux sections symétriques à phases orthotropes, mais introduit aussi la prise en compte des effets Poisson dans la déformation des sections. Les résultats théoriques concernent le comportement généralisé de la poutre et l'expression 3D des contraintes ; celles-ci, faciles à comparer à celles de St Venant, montrent explicitement la contribution de chacun des nouveaux efforts intérieurs. Les résultats numériques présentés analysent la torsion et la flexion-simple d'une poutre console de section sandwich.

**Mots clefs :** non-uniform warping, Poisson's effect, torsion, shear-bending, composite beam, orthotropic material, finite elements method

## 1 Introduction

For an homogeneous and isotropic beam, it is well known that (out-of-plane) warpings are due to the torsional moment and the shear forces. For an arbitrary composite cross-section, the situation is much more complex. Indeed, warpings may also be induced by axial force and bending moments, and several elastic couplings between extensional, flexural and torsional deformations may occur, even if the cross-section is symmetric. Laminated composite beams are known to exhibit such coupled deformations arising from the anisotropic nature of the laminae and from the stacking sequences. Further, in general cases of loading and boundary condition, these warpings are non-uniform along the beam axis. This leads to a beam mechanical behavior that may be enough different from that predicted by (*original or extended*) St Venant (SV) beam theory [2], or other simplified theories which are restricted to uniform warping.

To detect and describe the warping effects, different non-uniform warping theories have been proposed. The most part of these theories, called *high order* beam theories are based on kinematics including a non-uniform (out-of-plane) warping  $w$  with respect to the beam axis of the following shape  $w(x, \mathbf{X}) = \eta(x) \psi(\mathbf{X})$ , where  $x$  is the beam axis coordinate,  $\mathbf{X}$  the in-section vector position,  $\eta$  the warping parameter and  $\psi$  a warping mode associated to the torsion or to one of the shear forces. In each case, the mode  $\psi$  is supposed to represent the corresponding SV-warping-function, which is considered as *the reference* to describe the *natural warping* of a cross-section. Various assumptions are present in the literature, they concern principally the shape of the cross-sectional warping and the shear distribution (which are two connected points) and the choice of the warping parameter  $\eta$  as independent or linked to the cross-sectional strains (e.g. the twisting rate is used instead  $\eta$  in the non-uniform torsion theory of Vlasov). An another assumption, commonly made, is to maintain the shape of the cross-section ; this assumption, acceptable for an homogeneous section, should be reconsidered for a composite one, specially when the gap between the rigidities of the materials is important (indeed, Poisson's effects in

the plane of the section may generate significant stresses specially in the interfaces between the materials (e.g. the *free edge effect* in a laminated beam, which may induce the delamination)). Beside, there is a serious lack (particularly for composite sections) of experimental results and benchmark problems making us able to assess the validity or the relative importance of these various assumptions, which can obscure differences among theories [3]. The results provided by high order beam theories for non uniform warping are more or less in agreement for the structural behavior of the beam, but the situation is not so clear for the 3D stress field, specially close to the regions where the warping is restrained. The alternative to experimental validations is a full 3D finite elements computation (3DFEM) of the beam ; however, this resort is not so used.

Recently, a general non-uniform warping beam theory (denoted by NUW-BT) for homogeneous elastic and isotropic beam, including torsional and shear forces effects, has been proposed by El Fatmi [1, 4]. The theoretical development of this NUW-BT is completely based on the knowledge, a priori, of the 3D solution of the *original* SV-problem which provides, in particular, the SV-warping functions of the cross-section and their properties. This makes this theory free from the classical assumptions and valid for an arbitrary shape of cross-section [5]. According to the NUW-BT approach, and using now the 3D *extended* SV-solution established for straight composite beams [2], NUW-BT may be rewritten for any elastic composite cross-section. However, the problem being much more complex than for homogeneous and isotropic section, this extension will be, as a first step, restricted in this paper to the case of a symmetric cross-section (denoted by *spo*-CS) made by orthotropic materials for which the principal material coordinates coincide with those of the beam.

This extended NUW-BT is based on a kinematics including the out-of-plane warpings due to the torsion and the shear forces and the in-plane warpings (i. e. Poisson's effect) due to the axial force and the bending moments. The study is carried out within the framework of small perturbations and for a straight elastic composite beam with any *spo*-CS. Starting from a displacement model including six independent warping parameters and using as (out or in) warping modes *the* SV-warping-functions of the composite section, the corresponding non-uniform warping beam theory is derived. It should be noted that the theoretical development of this theory is based on the knowledge of the SV-functions and their properties, but it is not possible, to specify these properties in this present short paper. Thus we just recall in the first section the expression of the 3D SV-solution for the particular case of a *spo*-CS.

## 2 The extended St Venant problem and its solution for a *spo*-CS

This extended problem is a 3D equilibrium elastic problem. The beam is along the  $x$  axis and is occupying a prismatic domain  $\Omega$  of a constant cross-section  $S$  and length  $L$ .  $S_{lat}$  is the lateral surface and  $S_0$  and  $S_L$  are the extremity cross-sections. The beam is in equilibrium only under surface force densities  $\mathbf{H}_0$  and  $\mathbf{H}_L$  acting on  $S_0$  and  $S_L$ , respectively. The cross-section is any *spo*-CS (body-CS, open/closed thin-walled-CS), where the materials are  $x$ - $y$ - $z$ -orthotropic and perfectly bonded together. We call SV-solution the solution that satisfies

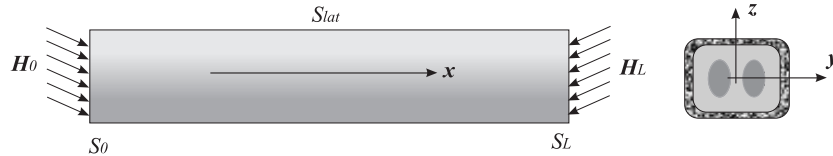


FIG. 1 – The 3D St Venant's problem for composite beam

all the equations of the linearized equilibrium problem, except the boundary conditions on  $(S_0, S_L)$  which are satisfied only in terms of resultant (force and moment). Let  $(\mathbf{R} = (N, T^y, T^z), \mathbf{M} = (M^x, M^y, M^z))$  denoting the cross-sectional stresses where  $(N, T^y, T^z, M^x, M^y, M^z)$  are the axial force the shear forces the torsional moment and the the bending moments, respectively. For the case of a *spo*-CS, St Venant displacement  $\bar{\xi}$  (the upperbar  $\bar{\cdot}$  denotes SV-quantities) is given by :

$$\bar{\xi} = \mathbf{u} + \boldsymbol{\omega} \wedge \mathbf{X} + \frac{N}{EA} \begin{bmatrix} 0 \\ V^x \\ W^x \end{bmatrix} + \frac{M^y}{EI_y} \begin{bmatrix} 0 \\ V^y \\ W^y \end{bmatrix} + \frac{M^z}{EI_z} \begin{bmatrix} 0 \\ V^z \\ W^z \end{bmatrix} + \frac{M^x}{GJ} \begin{bmatrix} \psi^x \\ 0 \\ 0 \end{bmatrix} + \frac{T^y}{GA_y} \begin{bmatrix} \psi^y \\ 0 \\ 0 \end{bmatrix} + \frac{T^z}{GA_z} \begin{bmatrix} \psi^z \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

where  $(\mathbf{u}, \boldsymbol{\omega})$  are the cross-sectional displacements, and  $(\widetilde{EA}, \widetilde{EI}_y, \widetilde{EI}_z, \widetilde{GJ}, \widetilde{GA}_y, \widetilde{GA}_z)$  the cross-sectional constants that depend only on the cross-section nature (shape and materials) and govern the structural elastic behavior of the beam.

$[V^x, W^x], [V^y, W^y], [V^z, W^z]$  are the in plane SV-warping functions (related to Poisson's effects) due to  $N, M^y$  and  $M^z$ , respectively ;  $(\phi^x, \phi^y, \phi^z)$  are the out of plane SV-warping functions due to  $M^x, T^y$  and  $T^z$ , respectively. All these SV-warpings are functions of  $(y, z)$  and are characteristics of the cross-section.

For an  $x$ - $y$ - $z$ -orthotropic material, Hooke's law  $\boldsymbol{\sigma} = \mathbf{K}\boldsymbol{\varepsilon}$  may be splitted to  $\boldsymbol{\sigma} = \mathbf{K}_1 \boldsymbol{\varepsilon}_1$  and  $\boldsymbol{\tau} = \mathbf{K}_2 \boldsymbol{\varepsilon}_2$  where

( $\sigma$ ,  $\tau$ ,  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\mathbf{K}_1$ ,  $\mathbf{K}_2$ ) are defined by

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_2 \end{bmatrix} \quad \mathbf{K}_1 = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 \\ K_{12} & K_{22} & K_{23} & 0 \\ K_{13} & K_{23} & K_{33} & 0 \\ 0 & 0 & 0 & G_{yz} \end{bmatrix} \quad \mathbf{K}_2 = \begin{bmatrix} G_{xy} & 0 \\ 0 & G_{xz} \end{bmatrix} \quad (2)$$

$$\sigma = \begin{bmatrix} \sigma \\ \tau \end{bmatrix} \quad \sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \end{bmatrix} \quad \tau = \begin{bmatrix} \tau_{xy} \\ \tau_{xz} \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \varepsilon_1 = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \end{bmatrix} \quad \varepsilon_2 = \begin{bmatrix} 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \end{bmatrix} \quad (3)$$

The 3D SV-stresses are linear with respect to the internal forces and their expressions may be splitted and written in the following compact form

$$\sigma = \bar{\sigma}^x N + \bar{\sigma}^y M^y + \bar{\sigma}^z M^z = \bar{\sigma} \begin{bmatrix} N \\ M^y \\ M^z \end{bmatrix} \quad \tau = \bar{\tau}^x M^x + \bar{\tau}^y T^y + \bar{\tau}^z T^z = \bar{\tau} \begin{bmatrix} M^x \\ T^y \\ T^z \end{bmatrix} \quad (4)$$

### 3 Non uniform warping beam theory (NUW-BT)

For the sake of simplicity the beam reference problem is taken similar to the 3D SV-problem defined in section-2. The kinematical modeling is the following St Venant-like displacement field (Eq.1) :

$$\xi(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\alpha}) = \mathbf{u} + \boldsymbol{\omega} \wedge \mathbf{X} + \alpha_x \begin{bmatrix} 0 \\ V^x \\ W^x \end{bmatrix} + \alpha_y \begin{bmatrix} 0 \\ V^y \\ W^y \end{bmatrix} + \alpha_z \begin{bmatrix} 0 \\ V^z \\ W^z \end{bmatrix} + \eta_x \begin{bmatrix} \psi^x \\ 0 \\ 0 \end{bmatrix} + \eta_y \begin{bmatrix} \psi^y \\ 0 \\ 0 \end{bmatrix} + \eta_z \begin{bmatrix} \psi^z \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

where, with  $i \in \{x, y, z\}$ , ( $\eta_i, \alpha_i$ ) are the out and in warping parameters and ( $\psi^i, V^i, W^i$ ) are the out and in plane SV-warping functions, respectively. At the abscissa  $x$ , each warping parameter measures the magnitude of the corresponding warping. The beam theory associated with this displacement, parametrized by ( $\mathbf{v}, \boldsymbol{\theta}, \boldsymbol{\eta}, \boldsymbol{\alpha}$ ), will be derived by the principle of virtual work.

Let us denote by  $\hat{\xi} = \xi(\hat{\mathbf{u}}, \hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\alpha}})$  a virtual displacement and  $\hat{\varepsilon} = \varepsilon(\hat{\xi})$  the corresponding strain tensor. The internal virtual work is  $W_i = - \int_L \langle \sigma : \hat{\varepsilon} \rangle dx$ . Using the displacement model Eq.(5),  $W_i$  takes the form :

$$\begin{aligned} W_i &= - \int_L (\mathbf{R} \cdot \hat{\boldsymbol{\gamma}} + \mathbf{M} \cdot \hat{\boldsymbol{\omega}} + \mathbf{M}_\psi \cdot \hat{\boldsymbol{\eta}}' + \mathbf{M}_s \cdot \hat{\boldsymbol{\eta}} + \mathbf{A}_\nu \cdot \hat{\boldsymbol{\alpha}}' + \mathbf{B}_s \cdot \hat{\boldsymbol{\alpha}}) dx \\ &= \int_L [\mathbf{R}' \cdot \hat{\mathbf{u}} + (\mathbf{M}' + \mathbf{x} \wedge \mathbf{R}) \cdot \hat{\boldsymbol{\omega}} + (\mathbf{M}'_\psi - \mathbf{M}) \cdot \hat{\boldsymbol{\eta}} + (\mathbf{A}_\nu' - \mathbf{B}) \cdot \hat{\boldsymbol{\alpha}}] dx \\ &\quad - [\mathbf{R} \cdot \hat{\mathbf{u}} + \mathbf{M} \cdot \hat{\boldsymbol{\omega}} + \mathbf{M}_\psi \cdot \hat{\boldsymbol{\eta}} + \mathbf{A}_\nu \cdot \hat{\boldsymbol{\alpha}}]_O^L \end{aligned} \quad (6)$$

where  $(\cdot)'$  is the derivative with respect to  $x$ , ( $\mathbf{R}, \mathbf{M}$ ) are the classical cross-sectional stresses and ( $\mathbf{M}_\psi, \mathbf{M}_s, \mathbf{A}_\nu, \mathbf{B}_s$ ) are the new (or additional) ones defined by

$$\begin{aligned} \mathbf{M}_\psi &= (M_\psi^x, M_\psi^y, M_\psi^z) \\ \mathbf{M}_s &= (M_s^x, T_s^y, T_s^z) \\ \mathbf{A}_\nu &= (A_\nu^x, A_\nu^y, A_\nu^z) \\ \mathbf{B}_s &= (N_s, M_s^y, M_s^z) \end{aligned} \quad \left\{ \begin{array}{l} M_\psi^x = \langle \sigma_{xx} \psi^x \rangle \\ M_\psi^y = \langle \sigma_{xx} \psi^y \rangle \\ M_\psi^z = \langle \sigma_{xx} \psi^z \rangle \\ M_s^x = \langle \tau_{xy} \psi_{,y}^x + \tau_{xz} \psi_{,z}^x \rangle \\ T_s^y = \langle \tau_{xy} \psi_{,y}^y + \tau_{xz} \psi_{,z}^y \rangle \\ T_s^z = \langle \tau_{xy} \psi_{,y}^z + \tau_{xz} \psi_{,z}^z \rangle \\ A_\nu^x = \langle \tau_{xy} V^x + \tau_{xz} W^x \rangle \\ A_\nu^y = \langle \tau_{xy} V^y + \tau_{xz} W^y \rangle \\ A_\nu^z = \langle \tau_{xy} V^z + \tau_{xz} W^z \rangle \\ N_s = \langle \sigma_{yy} V_{,y}^x + \sigma_{zz} W_{,y}^x + \tau_{yz} (V_{,z}^x + W_{,z}^x) \rangle \\ M_s^y = \langle \sigma_{yy} V_{,y}^y + \sigma_{zz} W_{,y}^y + \tau_{yz} (V_{,z}^y + W_{,z}^y) \rangle \\ M_s^z = \langle \sigma_{yy} V_{,y}^z + \sigma_{zz} W_{,y}^z + \tau_{yz} (V_{,z}^z + W_{,z}^z) \rangle \end{array} \right. \quad (7)$$

where  $\langle (\cdot) \rangle$  denotes  $\int_S (\cdot) dS$ .  $\mathbf{M}_\psi, \mathbf{M}_s$  introduced in [1] are called the bimoment vector and the secondary internal force vector, and they are both related to the out of plane warping. Similarly, it corresponds to the in plane warping two internal forces ( $\mathbf{A}_\nu, \mathbf{B}_s$ ) related to the Poisson's effects (the subscript  $(\cdot)_s$  as *secondary* is chosen to indicate that the components ( $N_s, M_s^y, M_s^z$ ) of  $\mathbf{B}_s$  may be seen as secondary axial force and secondary bending moments.

The external virtual work is  $W_e = \langle \mathbf{H}_O \cdot \hat{\boldsymbol{\xi}} \rangle_O + \langle \mathbf{H}_L \cdot \hat{\boldsymbol{\xi}} \rangle_L$  takes the form

$$W_e = \mathbf{P}_O \cdot \hat{\mathbf{u}}_O + \mathbf{C}_O \cdot \hat{\boldsymbol{\omega}}_O + \mathbf{Q}_O \cdot \hat{\boldsymbol{\eta}}_O + \mathbf{S}_O \cdot \hat{\boldsymbol{\alpha}}_O + \mathbf{P}_L \cdot \hat{\mathbf{u}}_L + \mathbf{C}_L \cdot \hat{\boldsymbol{\omega}}_L + \mathbf{Q}_L \cdot \hat{\boldsymbol{\eta}}_L + \mathbf{S}_L \cdot \hat{\boldsymbol{\alpha}}_L \quad (8)$$

where the onedimensional external forces ( $\mathbf{P}, \mathbf{C}, \mathbf{Q}, \mathbf{S}$ ) are defined by

$$\mathbf{P} = \langle \mathbf{H} \rangle \quad \mathbf{Q} = \langle H_x \psi^i \rangle \mathbf{x}_i \quad \mathbf{C} = \langle \mathbf{GM} \wedge \mathbf{H} \rangle \quad \mathbf{S} = \langle H_y V^i + H_z W^i \rangle \mathbf{x}_i \quad \mathbf{x}_i \in \{x, y, z\} \quad (9)$$

Thanks to the principle of virtual work, Eqs.6-8 are used to provide the equilibrium equations

$$\mathbf{R}' = \mathbf{0} \quad \mathbf{M}' + \mathbf{x} \wedge \mathbf{R} = \mathbf{0} \quad \mathbf{M}'_\psi - \mathbf{M} = \mathbf{0} \quad \mathbf{A}'_\nu - \mathbf{B} = \mathbf{0} \quad (10)$$

and the boundary conditions

$$\left. \begin{array}{l} x=0 \\ x=L \end{array} \right\} \begin{array}{l} (\mathbf{R}, \mathbf{M}, \mathbf{M}_\psi, \mathbf{A}_\nu) = -(\mathbf{P}_O, \mathbf{C}_O, \mathbf{Q}_O, \mathbf{S}_O) \\ (\mathbf{R}, \mathbf{M}, \mathbf{M}_\psi, \mathbf{A}_\nu) = (\mathbf{P}_L, \mathbf{C}_L, \mathbf{Q}_L, \mathbf{S}_L) \end{array} \quad (11)$$

*Beam structural behavior.* Let  $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2) = ((\mathcal{D}_1^0, \mathcal{D}_1^w), (\mathcal{D}_2^0, \mathcal{D}_2^w))$  denote the generalized strain vector and  $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) = ((\mathcal{T}_1^p, \mathcal{T}_1^w), (\mathcal{T}_2^p, \mathcal{T}_2^w))$  the corresponding generalized force vector defined by :

$$\begin{array}{l} \mathcal{D}_1^0 = \begin{bmatrix} \chi_x \\ \chi_y \\ \chi_z \end{bmatrix} \\ \mathcal{D}_1^w = \begin{bmatrix} \eta'_x \\ \eta'_y \\ \eta'_z \\ \alpha_x - \gamma_x \\ \alpha_y - \chi_y \\ \alpha_z - \chi_z \end{bmatrix} \\ \mathcal{T}_1^p = \begin{bmatrix} N_p \\ M_p^y \\ M_p^z \\ M_\psi^x \\ M_\psi^y \\ M_\psi^z \\ N_s \\ M_s^y \\ M_s^z \end{bmatrix} \\ \mathcal{T}_1^w = \begin{bmatrix} M_\psi^x \\ M_\psi^y \\ M_\psi^z \\ N_s \\ M_s^y \\ M_s^z \end{bmatrix} \\ \mathcal{D}_2^0 = \begin{bmatrix} \chi_x \\ \gamma_y \\ \gamma_z \end{bmatrix} \\ \mathcal{D}_2^w = \begin{bmatrix} \eta_x - \chi_x \\ \eta_y - \gamma_y \\ \eta_z - \gamma_z \\ \alpha'_z \\ \alpha'_y - \tilde{R}_z \gamma_z \\ \alpha'_z + \tilde{R}_y \gamma_y \end{bmatrix} \\ \mathcal{T}_2^p = \begin{bmatrix} M_p^x \\ T_p^y \\ T_p^z \\ M_s^x \\ T_s^y \\ T_s^z \\ A_\nu^x \\ A_\nu^y \\ A_\nu^z \end{bmatrix} \\ \mathcal{T}_2^w = \begin{bmatrix} M_s^x \\ T_s^y \\ T_s^z \\ A_\nu^x \\ A_\nu^y \\ A_\nu^z \end{bmatrix} \end{array} \quad (12)$$

where  $(N_p, M_p^y, M_p^z, M_p^x, T_p^y, T_p^z)$ , called the *primary* internal forces are defined by

$$\left. \begin{array}{l} N_p = N + N_s \\ M_p^y = M^y + M_s^y \\ M_p^z = M^z + M_s^z \\ M_p^x = M^x + M_s^x \\ T_p^y = T^y + T_s^y - \tilde{R}_y A_\nu^z \\ T_p^z = T^z + T_s^z + \tilde{R}_z A_\nu^y \end{array} \right\} \quad (13)$$

where  $\tilde{R}_y = \frac{\widetilde{GA}_y}{EI_z}$ ;  $\tilde{R}_z = \frac{\widetilde{GA}_z}{EI_y}$ . Using all the properties of the SV-solution and in particular the mathematical properties of the SV-functions ( $V^i, W^i, \psi^i$ ), we can show that the generalized 1D behavior, defined by  $\mathcal{T} = \Gamma \mathcal{D}$ , is given by the operator  $\Gamma$  which has the following form :

$$\Gamma = \begin{bmatrix} \bar{\Gamma}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Gamma_1^w & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\Gamma}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Gamma_2^w \end{bmatrix}; \bar{\Gamma}_1 = \begin{bmatrix} \widetilde{EA} & 0 & 0 \\ 0 & \widetilde{EI}_y & 0 \\ 0 & 0 & \widetilde{EI}_z \end{bmatrix}; \bar{\Gamma}_2 = \begin{bmatrix} \widetilde{GJ} & 0 & 0 \\ 0 & \widetilde{GA}_y & 0 \\ 0 & 0 & \widetilde{GA}_z \end{bmatrix} \quad (14)$$

$$\Gamma_1^w = \begin{bmatrix} \widetilde{KI}_\psi^x & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{KI}_\psi^y & 0 & 0 & 0 & a_y \\ 0 & 0 & \widetilde{KI}_\psi^z & 0 & a_z & 0 \\ 0 & 0 & 0 & \langle K_{11} \rangle - \widetilde{EA} & 0 & 0 \\ 0 & 0 & a_z & 0 & \langle z^2 K_{11} \rangle - \widetilde{EI}_y & 0 \\ 0 & a_y & 0 & 0 & 0 & \langle y^2 K_{11} \rangle - \widetilde{EI}_z \end{bmatrix} \quad (15)$$

$$\Gamma_2^w = \begin{bmatrix} \widetilde{GI}_x - \widetilde{GJ} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \langle G_{xy} \rangle - \widetilde{GA}_y - \tilde{R}_y (f_y - e_y) & 0 & 0 & 0 & 0 & e_y \\ 0 & 0 & 0 & \langle G_{xz} \rangle - \widetilde{GA}_z + \tilde{R}_z (f_z - e_z) & 0 & e_z & 0 \\ 0 & 0 & 0 & 0 & 0 & g_x & 0 \\ 0 & 0 & 0 & 0 & e_z & 0 & g_y \\ 0 & e_y & 0 & 0 & 0 & 0 & g_z \end{bmatrix} \quad (16)$$

where

$$\begin{array}{l} \widetilde{KI}_\psi^x = \langle K_{11} (\psi^x)^2 \rangle \\ \widetilde{KI}_\psi^y = \langle K_{11} (\psi^y)^2 \rangle \\ \widetilde{KI}_\psi^z = \langle K_{11} (\psi^z)^2 \rangle \\ e_y = \langle G_{xy} \psi_{,y}^y V^z + G_{xz} \psi_{,z}^y W^z \rangle \\ g_x = \langle G_{xy} (V^x)^2 + G_{xz} (W^x)^2 \rangle \\ g_y = \langle G_{xy} (V^y)^2 + G_{xz} (W^y)^2 \rangle \\ g_z = \langle G_{xy} (V^z)^2 + G_{xz} (W^z)^2 \rangle \\ e_z = \langle G_{xy} \psi_{,y}^z V^y + G_{xz} \psi_{,z}^z W^y \rangle \\ a_y = \langle y K_{11} \psi^y \rangle \\ a_z = \langle -z K_{11} \psi^z \rangle \\ f_y = \langle G_{xy} V^z \rangle \\ f_z = \langle G_{xz} W^y \rangle \end{array} \quad (17)$$

*Comments.* The components of  $\mathbf{\Gamma}$  define the cross-sectional constants of the composite beam. The six constants ( $\widetilde{EA}$ ,  $\widetilde{EI}_y$ ,  $\widetilde{EI}_z$ ,  $\widetilde{GJ}$ ,  $\widetilde{GA}_y$ ,  $\widetilde{GA}_z$ ) are known, and the new ones are : six constants  $\langle K_{11} \rangle$ ,  $\langle K_{11}y^2 \rangle$ ,  $\langle K_{11}z^2 \rangle$ ,  $\widetilde{GI}_x = \langle G_{xy}z^2 + G_{xz}y^2 \rangle$ ,  $\langle G_{xy} \rangle$ ,  $\langle G_{xz} \rangle$  related to the shape and the materials of the section ; six *warping rigidities* ( $[\widetilde{KI}_\psi^x, \widetilde{KI}_\psi^y, \widetilde{KI}_\psi^z], [g_x, g_y, g_z]$ ) related to the out and in-plane warpings ; and six constants ( $a_y, a_z, e_y, e_z, f_y, f_z$ ) which express the coupling between the in and out warpings related to the bending moments ( $M^y, M^z$ ) and the shear forces ( $T^y, T^z$ ).

We can also deduce from  $\mathbf{\Gamma}$  the following constitutive relation for the primary internal forces

$$N_p = \widetilde{EA}\gamma_x \quad M_p^y = \widetilde{EI}_y\chi_y \quad M_p^z = \widetilde{EI}_z\chi_z \quad M_p^x = \widetilde{GJ}\chi_x \quad T_p^y = \widetilde{GA}_y\gamma_y \quad T_p^z = \widetilde{GA}_z\gamma_z \quad (18)$$

Note that these constitutive relations coincide with those of St Venant beam theory.

*Stresses.* Using Hooke's law, we can establish that the stresses  $\boldsymbol{\sigma}$  and  $\boldsymbol{\tau}$  may be written in the following compact form

$$\boldsymbol{\sigma} = \underbrace{\bar{\boldsymbol{\sigma}} \begin{bmatrix} N \\ M^y \\ M^z \end{bmatrix}}_{sv} + \bar{\boldsymbol{\sigma}} \begin{bmatrix} N_s \\ M_s^y \\ M_s^z \end{bmatrix} + \mathbf{K}_1 \begin{bmatrix} \psi^x & \psi^y & \psi^z & 0 & 0 & 0 \\ 0 & 0 & 0 & V^x & V^y & V^z \\ 0 & 0 & 0 & W^x & W^y & W^z \\ 0 & 0 & 0 & (V^x+W^x) & (V^y+W^y) & (V^z+W^z) \end{bmatrix} (\mathbf{\Gamma}_1^w)^{-1} \begin{bmatrix} M_\psi^x \\ M_\psi^y \\ M_\psi^z \\ N_s \\ M_s^y \\ M_s^z \end{bmatrix}$$

$$\boldsymbol{\tau} = \underbrace{\bar{\boldsymbol{\tau}} \begin{bmatrix} M^x \\ T^y \\ T^z \end{bmatrix}}_{sv} + \bar{\boldsymbol{\tau}} \begin{bmatrix} M_s^x \\ T_s^y - \widetilde{R}_y A_\nu^z \\ T_s^z + \widetilde{R}_z A_\nu^y \end{bmatrix} + \mathbf{K}_2 \begin{bmatrix} \psi_{,y}^x & \psi_{,y}^y & \psi_{,y}^z & V^x & V^y & V^z \\ \psi_{,z}^x & \psi_{,z}^y & \psi_{,z}^z & W^x & W^y & W^z \end{bmatrix} (\mathbf{\Gamma}_2^w)^{-1} \begin{bmatrix} M_s^x \\ T_s^y \\ T_s^z \\ A_\nu^x \\ A_\nu^y \\ A_\nu^z \end{bmatrix} \quad (19)$$

*Remark.* Compared with SV-stresses, the expression of the stresses make clear the additional contribution of the new internal forces induced by the non-uniformity of the warpings.

## 4 Numerical applications

In order to apply this NUW-BT, it is imperative to previously know, for any given cross-section, all its constants and in particular its SV-warping-functions. This is achieved by using the software tool designated by *SECOPE* which is available within the finite elements code *CASTEM*. *SECOPE* has been developed conforming to the numerical method proposed by [6] for the computation of the 3D St Venant solution within the framework of the *exact* beam theory [2]. This NUW-BT is applied to analyse the elastic behavior of a cantilever sandwich beam subjected to a tip torsional moment ( $C^x = 1$ ) or a tip transversal force ( $F^z = 1$ ). The geometry of the cross-section and the mechanical constants of the materials (isotropic) are given in Fig-2. The SV-warping-functions are depicted in Fig-3. We present some numerical results for the 1D-behavior and the 3D stresses. For the stresses comparisons are made with a full 3DFEM computations.

*Torsion.* Fig-4 compares the variations of  $\omega_x$  and  $\eta_x$  along the span with those of SV-BT and, starting from the built-in section, Fig-6 compares, for the point B<sub>1</sub>, the  $x$ -variation of the axial stress  $\sigma_{xx}$  with that obtained by a 3DFEM computation.

*Shear-bending.* Fig-5 compares the variations of  $\alpha_y$ ,  $\eta_z$  and  $u_z$  along the span with those of SV-BT and, starting from the built-in section, Fig-7 compares, for the point A<sub>2</sub>, the  $x$ -variation of the axial stress  $\sigma_{xx}$  with that obtained by a 3DFEM computation.

## Références

- [1] El Fatmi R. Non-uniform warping including the effects of torsion and shear forces. part-I : A general beam theory. *International Journal of Solids and Structures*, 44, 5912–5929, 2007.
- [2] Ladevèze P. and Simmonds J. New concepts for linear beam theory with arbitrary geometry and loading. *European Journal of Mechanics*, 17(3), 377–402, 1998.
- [3] Volovoi V. V., Hodges D. H., Cesnik C. E. S., and Popescu B. Assessment of beam modeling methods for rotor blade applications. *Mathematical and computer modelling*, 33, 1099–1112, 2001.
- [4] El Fatmi R. Non-uniform warping theory for beams. *Compte Rendu de l'Académie des Sciences, C. R. Mécanique*, 335, 467–474, 2007.
- [5] El Fatmi R. Non-uniform warping including the effects of torsion and shear forces. part-II : Analytical and numerical applications. *International Journal of Solids and Structures*, 44, 5930–5952, 2007.

[6] El Fatmi R. and Zenzri H. A numerical method for the exact elastic beam theory. Applications to homogeneous and composite beams. International Journal of Solids and Structures, 41, 2521–2537, 2004.

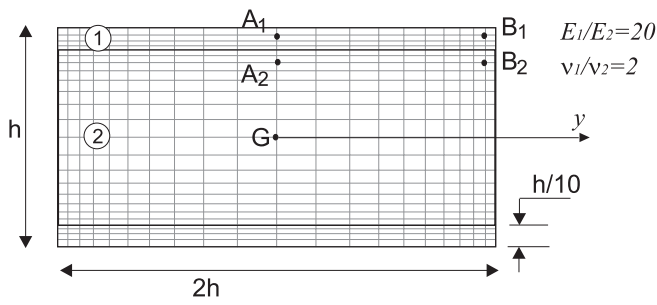


Fig-2. Composite section

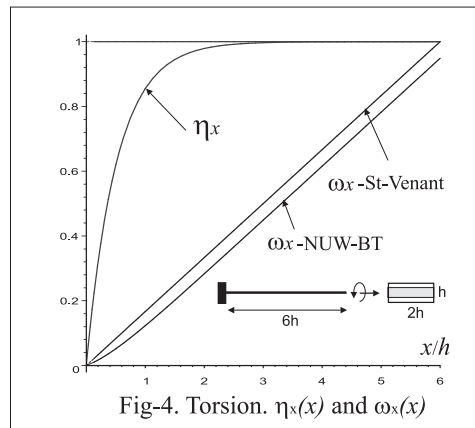


Fig-4. Torsion.  $\eta_x(x)$  and  $\omega_x(x)$

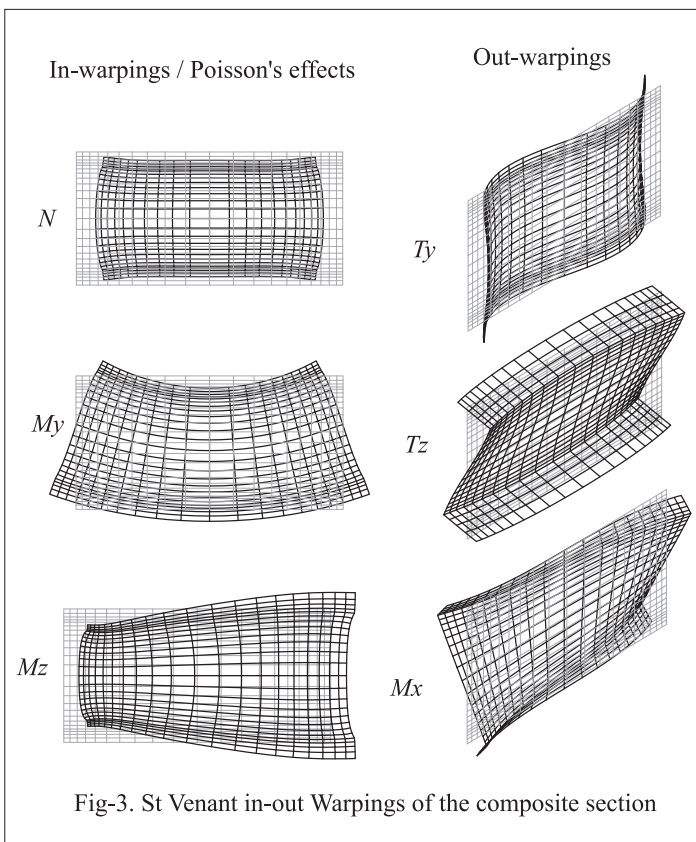


Fig-3. St Venant in-out Warpings of the composite section

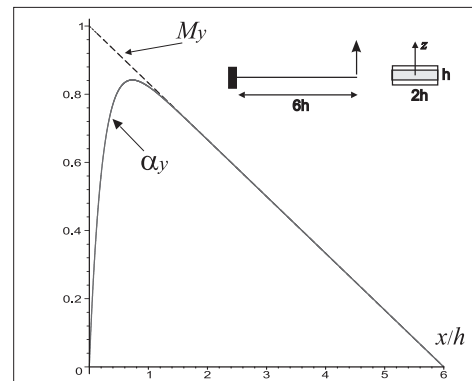


Fig-5. Shear-bending.  $\alpha_y(x), \eta_z(x)$  and  $u_z(x)$

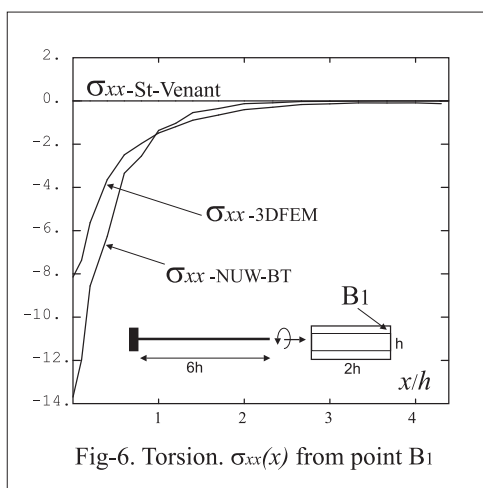
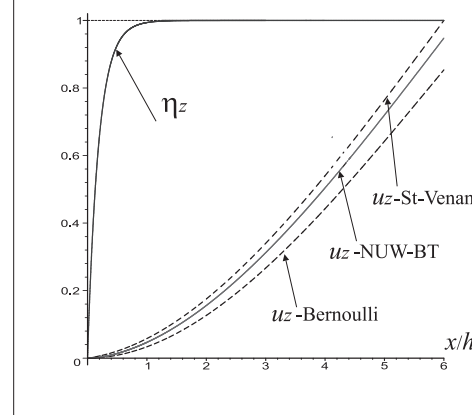


Fig-6. Torsion.  $\sigma_{xx}(x)$  from point B1

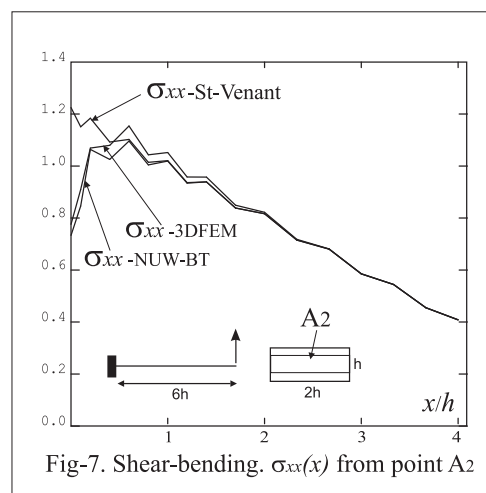


Fig-7. Shear-bending.  $\sigma_{xx}(x)$  from point A2