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# Bounded Real Lemma for Singular Linear Continuous-Time Fractional-Order Systems

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## Abstract

This paper proposes novel necessary and sufficient strict linear matrix inequalities ( $\mathcal{LMI}$ s), to characterize admissibility of singular fractional-order linear continuous-time systems with the fractional derivative of order  $\alpha$  belonging to  $1 \leq \alpha < 2$ . Then, the problem of the bounded real lemma corresponding to the  $\mathcal{H}_\infty$  norm computation is addressed involving additional variables. Necessary and sufficient conditions are established via a set of  $\mathcal{LMI}$ s that can be effectively used to design  $\mathcal{H}_\infty$  controllers. Based on the corresponding bounded real lemma, a state feedback control with a prescribed  $\mathcal{H}_\infty$  performance index for the underlying systems is proposed. Finally, numerical examples are provided to show effectiveness of the given results.

*Key words:* Fractional-order systems; singular systems; admissibility; bounded real lemma;  $\mathcal{LMI}$ s.

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## 1 Introduction

The history of fractional calculus is more than three centuries old. Background works on mathematical fractional calculus can be found in [20], [25], [26], [29]. Recently, fractional order control systems have attracted increasing interest.

Due to the fact that many real-world physical systems are better characterized with the help of fractional order differential equations, this approach has been successfully employed in many applications, such as electromagnetic systems [5], dielectric polarization [36], and viscoelastic [1], [30] systems. In [12], it is shown that an electrical circuit is a singular fractional order system if it contains at least one mesh consisting of branches with only ideal supercapacitors and voltage sources or at least one node with supercoil branches.

It is well-known that stability is fundamental for control systems. For linear fractional-order systems ( $FOS$ ), stability analysis has been performed firstly from an al-

gebraic point of view based on the localization of eigenvalues of the coefficient matrix [19], [32]; then, many  $\mathcal{LMI}$  stability conditions have been derived [31], [14]. Robustness of control systems to disturbances and uncertainties has always been the central issue in feedback control. Over the last years singular values have been developed as a tool to deal with this problem. In general terms, the controller should be chosen so that the closed loop transfer function matrix has certain characteristics that are derived from the specifications. An optimal design minimizes the maximum singular value of the discrepancy between the closed-loop transfer function matrix and the desired loop shape, subject to a closed-loop stability constraint. This is an  $\mathcal{H}_\infty$  optimization problem, for which a considerable mathematical theory is available [8], [7], [41].

The  $\mathcal{H}_\infty$  norm is also the research focus in the area of fractional-order systems. It has been studied in [21], [33], [15]. Some  $\mathcal{LMI}$ -based and Hamiltonian matrix-based methods for the  $\mathcal{H}_\infty$  norm computation have been proposed in [27], [2], [6]. Recently, using the bounded real lemma for  $FOS$ , the  $\mathcal{H}_\infty$  performance index has been characterized in [34], [23].

The key idea to derive  $\mathcal{LMI}$  conditions of the bounded real lemma for  $FOS$  is the generalized Kalman-Yakubovich-Popov (KYP) Lemma [10]. Nowadays, singular fractional order control systems, which are also

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called descriptor systems, implicit systems, or generalized state space systems, have been attracted increasing interest. Many theoretical results for regular systems have been extended to singular cases [13], [4], for example, the system analysis via transfer matrix for singular systems has been extensively studied [4], [37], [38].

It is well-known that for singular systems the admissibility is very important property for singular systems, so we need to consider, in addition to the stability, both the regularity and the absence of the impulses [24], [22], [35], [39], [3], [11], [17], [18], [16]. There are many other open difficult problems related to this class of systems, including the  $\mathcal{H}_\infty$  optimal control ones. Recently, the bounded real lemma problem in the sense of  $\mathcal{H}_\infty$  norm for singular fractional-order control systems has been studied in a few papers, (see, for example, [40]). However, the results of [40] yield  $\mathcal{LMI}$  conditions with equality constraints and, therefore, can not be directly used to design  $\mathcal{H}_\infty$  controllers or observers. In addition, the admissibility conditions are not guaranteed, since the obtained variable matrix  $P$  is not real.

Motivated by the discussions above, this paper derives a new formulation for condition of the admissibility conditions and the  $\mathcal{H}_\infty$  bounded real lemma (*BRL*) for linear singular fractional-order systems (*SFOS*) with fractional order  $\alpha \in (1, 2)$ , where free variables are introduced leading to more relaxed conditions for the  $\mathcal{H}_\infty$  design problems. The paper is organized as follows. Section 2 provides some preliminaries for *FOS* and *SFOS*. Section 3 presents a new formulation of the  $\mathcal{LMI}$  admissibility condition for *SFOS*. Then, necessary and sufficient  $\mathcal{LMI}$ s of the  $\mathcal{H}_\infty$  bounded real lemma are obtained. Finally, in Section 4, the obtained results are validated via numerical examples.

**Notations.** For matrix  $X$ , the transpose and complex conjugate transpose are denoted by  $X^T$  and  $X^*$ , respectively.  $z^*$  stands for the conjugate of a complex number  $z$ ,  $\text{Re}(z)$  for its real part, and  $\text{arg}(z)$  for its argument. The symbol  $\mathbb{H}_n$  denotes the set of  $n \times n$  Hermitian matrices. For any square complex matrix  $X$ ,  $\text{Her}(X)$  corresponds to  $X + X^*$ . The inequalities  $X < 0$  ( $\leq 0$ ) and  $X > 0$  ( $\geq 0$ ), with  $X$  being a Hermitian matrix, denote negative (semi)definiteness and positive (semi)definiteness, respectively. We note by  $I$  and  $0$  the identity and the null matrix of appropriate dimensions, respectively.  $\bar{\sigma}(X)$  is the maximum singular value of a matrix  $X$ .  $\text{spec}(X)$  denotes the set of the eigenvalues of the matrix  $X$ .  $\Theta = \begin{bmatrix} \sin(\alpha \frac{\pi}{2}) & -\cos(\alpha \frac{\pi}{2}) \\ \cos(\alpha \frac{\pi}{2}) & \sin(\alpha \frac{\pi}{2}) \end{bmatrix}$  and

$\theta = (1 - \alpha)\frac{\pi}{2}$ , where  $\alpha$  is a fractional-order. The upper or lower triangular elements in Hermitian matrices are denoted by the symbol  $*$  for brevity. Throughout the paper, all representations of systems are assumed minimum phase.

## 2 Preliminaries and background

### 2.1 Fractional-order systems

Consider the commensurate fractional-order linear time-invariant system described by the equations

$$\begin{cases} D^\alpha x(t) = Ax(t) + Bu(t), & 1 \leq \alpha < 2, \\ y(t) = Cx(t) + Du(t), \end{cases} \quad (1)$$

where  $\alpha$  is the fractional commensurate order,  $x(t) \in \mathbb{R}^n$  is the pseudo-state,  $u(t) \in \mathbb{R}^m$  is the input vector and  $y(t) \in \mathbb{R}^p$  is the output vector. The real-valued matrices  $A, B, C, D$  are assumed to be known and constant with appropriate dimensions.  $D^\alpha$  is the Caputo fractional-order differentiation operator with fractional order  $\alpha > 0$  defined in [29] by

$$D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(N - \alpha)} \int_0^t \frac{f^{(N)}(\tau)}{(t - \tau)^{\alpha + 1 - N}} d\tau,$$

where  $\alpha \in \mathbb{R}^+$  and  $N$  is an integer satisfying  $N - 1 \leq \alpha < N$ .  $\Gamma(\cdot)$  is the gamma function defined by the integral  $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$  for all  $z \in \mathbb{C}$  such that  $\text{Re}(z) > 0$ . The transfer function of (1) is given by

$$T(s) = C(s^\alpha I - A)^{-1} B + D. \quad (2)$$

The following definition, which is consistent with the traditional one [8], is introduced [33]:

**Definition 1** *The  $\mathcal{H}_\infty$  norm of transfer function  $T(s)$  in (2) is defined by  $\|T(s)\|_{\mathcal{H}_\infty} = \sup_{\text{Re}(s) \geq 0} \bar{\sigma}(T(s))$ .*

To deal with complex  $\mathcal{LMI}$ s, the complex version of Schur complement lemma is needed.

**Lemma 1** [9] *A complex Hermitian matrix satisfies the*

$$\text{inequality } \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} < 0, \text{ if and only if}$$

$$(R < 0) \text{ and } (P - QR^{-1}Q^* < 0).$$

Note that the  $X + jY = (X + jY)^* > 0$ , holds if and only if

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}^T > 0,$$

where  $X$  and  $Y$  are real square matrices and  $j^2 = -1$ . The following lemmas, known in the literature as Kalman-Yakubovič-Popov (KYP) lemmas, are used [10].

**Lemma 2** Let matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $\Theta \in \mathbb{H}_{(n+m)}$ ,  $\Phi \in \mathbb{H}_2$ ,  $\Psi \in \mathbb{H}_2$ . The set  $\Lambda(\Phi, \Psi)$  is defined as

$$\Lambda(\Phi, \Psi) = \left\{ \lambda \in \mathbb{C} / \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Phi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \geq 0 \right\}. \quad (3)$$

When  $\Lambda(\Phi, \Psi)$  represents a curve in the complex plane, the following statements are equivalent:

i)  $\forall \lambda \in \Lambda(\Phi, \Psi)$ ,

$$\begin{bmatrix} (\lambda I - A)^{-1} B \\ I_m \end{bmatrix}^* \Theta \begin{bmatrix} (\lambda I - A)^{-1} B \\ I_m \end{bmatrix} < 0. \quad (4)$$

ii) There exist matrices  $P \in \mathbb{C}^{n \times n}$ ,  $Q \in \mathbb{C}^{n \times n}$ ,  $Q > 0$  such that

$$\begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix}^* (P \otimes \Phi + Q \otimes \Psi) \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} + \Theta < 0. \quad (5)$$

**Lemma 3** If the set  $\Lambda(\Phi, \Psi)$  in Lemma 2 is replaced by  $\Upsilon(\Phi, \Psi)$  defined as

$$\Upsilon(\Phi, \Psi) = \left\{ \lambda \in \mathbb{C} / \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Phi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \geq 0, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \geq 0 \right\} \quad (6)$$

then condition (4) holds for  $\forall \lambda \in \Upsilon(\Phi, \Psi)$ , if there exist matrices  $P > 0$ ,  $Q > 0$  such that  $\mathcal{LM}\mathcal{I}$  condition (5) holds.

Now, we can specify a proper region corresponding to  $\mathcal{H}_\infty$  norm. By taking  $\Phi = \begin{bmatrix} 0 & e^{-j\theta} \\ e^{j\theta} & 0 \end{bmatrix}$  with  $\theta = (1 - \alpha) \frac{\pi}{2}$ , we have

$$\begin{aligned} \Upsilon(\Phi, 0) &= \{ \lambda / \lambda \in \mathbb{C}, \lambda e^{j\theta} + \lambda^* e^{-j\theta} \geq 0 \}, \\ \bar{\Upsilon}(\Phi, 0) &= \{ \lambda / \lambda \in \mathbb{C}, \lambda^* e^{j\theta} + \lambda e^{-j\theta} \geq 0 \}, \end{aligned} \quad (7)$$

and it is easy to see that

$$\{s^\alpha / s \in \mathbb{C}, \Re(s) \geq 0\} = \Upsilon(\Phi, 0) \cup \bar{\Upsilon}(\Phi, 0), \quad (8)$$

where  $\bar{\Upsilon}(\Phi, 0)$  is the symmetric region of  $\Upsilon(\Phi, 0)$  with respect to the real axis. Then, we can write

$$\begin{aligned} \|T(s)\|_{\mathcal{H}_\infty} &= \sup_{\Re(s) \geq 0} \bar{\sigma}(T(s)) = \sup_{s \in \Upsilon(\Phi, 0)} \bar{\sigma}(T(s)) \\ &= \sup_{s \in \bar{\Upsilon}(\Phi, 0)} \bar{\sigma}(T(s)), \end{aligned} \quad (9)$$

which follows from  $\sigma(T(s)) = \sigma(T(\bar{s}))$ .

**Lemma 4** [7] If the transfer function  $T(s)$  is without poles in  $\Re(s) > 0$  and  $\|T(s)\|_{\mathcal{H}_\infty}$  is bounded, then  $\|T(s)\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(T(j\omega)) = \sup_{\omega \geq 0} \bar{\sigma}(T(j\omega))$ .

In the following, we recall some results on the  $\mathcal{H}_\infty$  bounded real lemma for FOS.

**Lemma 5** [15] Let  $\gamma$  be a positive scalar. Consider the system (1) with its transfer function (2). Then,  $\|T(s)\|_{\mathcal{H}_\infty} < \gamma$ , if and only if either one of the following equivalent statements holds:

(i) there exists a Hermitian positive definite matrix  $P \in \mathbb{C}^{n \times n}$  such that

$$\begin{bmatrix} \text{Her}(e^{-j\theta} AP) & * & * \\ e^{-j\theta} CP & -\gamma^2 I_p & * \\ B^T & D^T & -I_m \end{bmatrix} < 0. \quad (10)$$

(ii) there exists a Hermitian positive definite matrix  $P \in \mathbb{C}^{n \times n}$  such that

$$\begin{bmatrix} \text{Her}(e^{j\theta} A^T P) & * & * \\ e^{j\theta} B^T P & -\gamma^2 I_p & * \\ C & D & -I_m \end{bmatrix} < 0. \quad (11)$$

**Remark 1** Taking into account that  $\|T(s)\|_{\mathcal{H}_\infty} = \|T^T(s)\|_{\mathcal{H}_\infty}$  and for any Hermitian positive definite matrix  $H$  its conjugate  $\bar{H}$  is also positive definite, it suffices to replace in condition (i)  $(A, B, C, D)$  by  $(A^T, C^T, B^T, D^T)$  to obtain condition (ii).

## 2.2 Singular fractional-order systems

Now, consider the singular fractional-order system described by the equations:

$$\begin{cases} ED^\alpha x(t) = Ax(t) + Bu(t), & 1 \leq \alpha < 2, \\ y(t) = Cx(t), \end{cases} \quad (12)$$

where  $\alpha$  is the fractional commensurate order,  $x(t) \in \mathbb{R}^n$  is the pseudo-state,  $u(t) \in \mathbb{R}^m$  is the input vector and  $y(t) \in \mathbb{R}^p$  is the output vector. The real-valued matrices  $A, B, C$  are assumed to be known and constant with appropriate dimensions. The matrix  $E \in \mathbb{R}^{n \times n}$  is singular,  $\text{rank}(E) = r < n$ . The system (12) is assumed to be of minimal realization. The transfer function between  $u(t)$  and  $y(t)$  is

$$G_{uy}(s) = C(s^\alpha E - A)^{-1} B. \quad (13)$$

**Definition 2** [39] The unforced singular linear fractional-order system (12) is said to be admissible if it is regular, impulse free, and stable.

The following lemmas generalize the results for singular integer-order systems [4] to singular fractional-order ones.

**Lemma 6** [24] *The unforced singular linear fractional-order system (12) is admissible if and only if*

1) *the unforced singular linear fractional-order system (12) is regular, that is,  $\det(s^\alpha E - A)$  is not identically zero, where  $s \in \mathbb{C}$ .*

2) *the unforced singular linear fractional order system (12) is impulse-free, that is,  $\deg(\det(\lambda E - A)) = \text{rank}(E)$ , where  $\lambda \in \mathbb{C}$ .*

3) *The pair  $(E, A)$  is stable if and only if*

$$|\arg \text{spec}(E, A)| > \alpha \frac{\pi}{2},$$

where  $\text{spec}(E, A) = \{\lambda/\lambda \in \mathbb{C}, \lambda \text{ finite}, \det(\lambda E - A) = 0\}$ .

**PROOF.** The proof of 2) and 3) is given in [24]. For 1), the proof is as follows. Consider the system

$$\begin{cases} ED^\alpha x(t) = Ax(t), & 1 \leq \alpha < 2, \\ x(0) = x_0, \quad \dot{x}(0) = x_1. \end{cases} \quad (14)$$

The system (14) is regular, if there exists a unique solution  $x(t)$  for a given initial condition. Applying the Laplace transform to (14), we obtain

$$(s^\alpha E - A)X(s) = (s^{\alpha-1}x_0 + s^{\alpha-2}x_1)E, \quad (15)$$

where  $s \in \mathbb{C}$  is the Laplace parameter and  $X(s)$  the Laplace transform of  $x(t)$ . The solution of (15), and equivalently, that of (14), exists and is unique if and only if the matrix  $(s^\alpha E - A)$  is invertible, that is  $\det(s^\alpha E - A)$  is not identically zero.

**Lemma 7** [24] *Assume that the unforced singular linear fractional order system (12) is regular, then*

i) *there exist two invertible matrices  $M$  and  $N$  satisfying*

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & F \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad (16)$$

where  $A_1 \in \mathbb{R}^{r \times r}$  and  $F \in \mathbb{R}^{(n-r) \times (n-r)}$  is a nilpotent matrix.

ii) *if  $\deg(\det(\lambda E - A)) = \text{rank } E$ , then there exist two invertible matrices  $M$  and  $N$  satisfying*

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad (17)$$

where  $\lambda \in \mathbb{C}$ ,  $A_1 \in \mathbb{R}^{r \times r}$ .

iii) *The unforced singular linear fractional order system*

(12) *is admissible if and only if (i) and (ii) are satisfied and the finite modes of (12), which are the eigenvalues of the matrix  $A_1$ , satisfy*

$$|\arg(\text{spec}(A_1))| > \alpha \frac{\pi}{2}. \quad (18)$$

iv) *The unforced singular linear fractional order system (12) is admissible if and only if (i) and (ii) are satisfied and there exists matrix  $P_1 > 0$  such that*

$$\text{Her}(e^{-j\theta} A_1 P_1) < 0. \quad (19)$$

**Theorem 1** [17] *The unforced singular linear fractional-order system (12) is admissible if and only if one of the following equivalent statements holds:*

i) *there exist matrices  $X \in \mathbb{R}^{n \times n} > 0$  and  $Y \in \mathbb{R}^{(n-r) \times n}$  such that the following  $\mathcal{LMI}$  holds*

$$\text{Her}\{e^{-j\theta} A(XE^T + SY)\} < 0, \quad (20)$$

or, equivalently,

$$\text{Her}\{\Theta \otimes A(XE^T + SY)\} < 0, \quad (21)$$

where  $S \in \mathbb{R}^{n \times (n-r)}$  is any matrix of full column rank and verifies  $ES = 0$ .

ii) *there exists a matrix  $X \in \mathbb{R}^{n \times n}$  such that*

$$EX = X^T E^T \geq 0, \quad \text{Her}(e^{-j\theta} AX) < 0. \quad (22)$$

For a singular linear continuous-time system, with  $\alpha = 1$ , there exists a recent result on the bounded real lemma.

**Lemma 8** [3] *The singular system (12) with  $\alpha = 1$  is admissible and  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$  with  $\gamma > 0$ , if and only if there exist matrices  $X \in \mathbb{R}^{n \times n} > 0$ ,  $Y \in \mathbb{R}^{(n-r) \times n}$ ,  $\mathcal{V} \in \mathbb{R}^{(n+p) \times (n+p)}$  and  $\mathcal{U} \in \mathbb{R}^{(n+p) \times (n+p)}$  such that*

$$\begin{bmatrix} \text{Her}\{A_c \mathcal{U}\} + \mathcal{B} \mathcal{B}^T & * \\ (\mathcal{X} \mathcal{E}^T + \mathcal{S} \mathcal{Y}) + \mathcal{V}^T A_c^T - \mathcal{U}^T & -\mathcal{V} - \mathcal{V}^T \end{bmatrix} < 0, \quad (23)$$

with  $\mathcal{E} \mathcal{S} = 0$ , and

$$A_c = \begin{bmatrix} A & 0 \\ C & -I_p \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad (24a)$$

$$\mathcal{E} = \begin{bmatrix} E & 0 \\ 0 & \frac{\gamma^2}{2} I_p \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} X & 0 \\ 0 & I_p \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} Y & 0 \end{bmatrix}. \quad (24b)$$

**Remark 2** *Note that if  $A_c, \mathcal{U}, \mathcal{B}, \mathcal{X}, \mathcal{E}, \mathcal{S}, \mathcal{Y}, \mathcal{V}$  are replaced by  $A, U, B, X, S, Y, V$  respectively, the admissibility condition (9) in Theorem 1 of [3] is recovered.*

### 3 Main Results

#### 3.1 Admissibility for SFOS

The following theorem gives a new formulation of admissibility conditions for the unforced singular fractional-order system (12)

**Theorem 2** *The unforced singular fractional-order system (12) is admissible if and only if the following statement holds. There exist matrices  $X \in \mathbb{R}^{n \times n} > 0$ ,  $Y \in \mathbb{R}^{(n-r) \times n}$ ,  $V \in \mathbb{R}^{n \times n}$ , and  $U \in \mathbb{C}^{n \times n}$  such that:*

$$\begin{bmatrix} \text{Her}\{AU\} (e^{-j\theta}(XE^T + SY))^* + AV - U^* \\ (*) \quad \quad \quad -V - V^T \end{bmatrix} < 0, \quad (25)$$

where  $S \in \mathbb{R}^{n \times (n-r)}$  is any matrix of full column rank and verifies  $ES = 0$ .

**PROOF.** We have to prove the equivalence between this result and (20) in Theorem 1.

**Necessity.** Assume that the unforced system (12) is admissible, so (20) holds for some matrices  $X > 0, Y$ . Then, a nonsingular matrix  $V$  can always be found such that

$$\begin{bmatrix} \text{Her}\{e^{-j\theta}A(XE^T + SY)\} & 0 \\ 0 & -V - V^T \end{bmatrix} < 0. \quad (26)$$

Pre-multiplying and post-multiplying (26) by  $\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}$

and  $\begin{bmatrix} I & 0 \\ -A^T & I \end{bmatrix}$ , respectively, we obtain

$$\begin{bmatrix} \text{Her}\{e^{-j\theta}A(XE^T + SY) - AVA^T\} & AV + AV^T \\ VA^T + V^T A^T & -V - V^T \end{bmatrix} < 0. \quad (27)$$

Then by setting  $U = e^{-j\theta}(XE^T + SY) - VA^T$ , we get

$$\begin{bmatrix} AU + U^* A^T & AV + AV^T \\ VA^T + V^T A^T & -V - V^T \end{bmatrix} < 0 \quad (28)$$

and, from  $AV^T = (e^{-j\theta}(XE^T + SY))^* - U^*$ , (25) is obtained.

**Sufficiency.** To obtain (20), pre-multiply and post-multiply (25) by  $\begin{bmatrix} I & A \\ & I \end{bmatrix}$  and  $\begin{bmatrix} I & A \\ & I \end{bmatrix}^T$ , respectively. This means that (25) implies the admissibility of the system (12). This ends the proof.

**Remark 3** *Note that the introduction of matrix variables  $U$  and  $V$  benefits to the control design, since the gain matrix depends on  $U$  and not on  $X$ . This new representation is more suitable for the  $\mathcal{H}_\infty$  control design in comparison to the conditions (21),(22) in Theorem 1, which cannot be directly used to design controllers, since they are numerically intractable due to involving the inversion of the matrix  $XE^T + SY$ . Also, for  $\alpha = 1$ , the condition (25) coincides with the condition (9) in Theorem 1 in [3].*

#### 3.2 Bounded Real Lemma for SFOS

Singular fractional linear continuous systems performance can be evaluated through different norms of transfer function  $G_{uy}(s)$ . In this section, we consider the  $\mathcal{H}_\infty$  control problem. The corresponding versions of the bounded real lemma are proposed in terms of  $\mathcal{LMI}$ s. Note that for  $FOS$ , the bounded real lemma characterizes only the  $\mathcal{H}_\infty$  norm and has nothing to do with stability. This is different from the  $SFOS$  case, where the transfer function is well defined if and only if the system is regular. For these reasons, the admissibility conditions are added to the statement of the bounded real lemma for  $SFOS$ .

**Theorem 3** *Let the SFOS (12) be admissible and  $\gamma > 0$ . The following statements are equivalent.*

(i) *The SFOS (12) with its transfer function  $G_{uy}(s)$  in (13) verifies  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$ .*

(ii) *There exist matrices  $P \in \mathbb{C}^{n \times n}$  such that*

$$EP = P^* E^T \geq 0 \quad (29a)$$

$$\begin{bmatrix} \text{Her}\{e^{-j\theta}AP\} & * & * \\ e^{-j\theta}CP & -\gamma^2 I_p & * \\ B^T & 0 & -I_m \end{bmatrix} < 0. \quad (29b)$$

(iii) *There exist matrices  $P \in \mathbb{C}^{n \times n} > 0$  and  $Q \in \mathbb{C}^{(n-r) \times n}$  such that the following  $\mathcal{LMI}$  holds:*

$$\begin{bmatrix} \text{Her}\{e^{-j\theta}A(PE^T + SQ)\} & * & * \\ e^{-j\theta}C(PE^T + SQ) & -\gamma^2 I_p & * \\ B^T & 0 & -I_m \end{bmatrix} < 0, \quad (30)$$

where  $S \in \mathbb{R}^{n \times (n-r)}$  is any matrix with full column rank and satisfies  $ES = 0$ .

(iv) *There exist matrices  $P \in \mathbb{C}^{n \times n} > 0$ ,  $Q \in \mathbb{C}^{(n-r) \times n}$ ,  $\mathcal{V} \in \mathbb{R}^{(n+p) \times (n+p)}$ , and  $\mathcal{U} \in \mathbb{C}^{(n+p) \times (n+p)}$  such that*

$$\begin{bmatrix} \text{Her}\{e^{-j\theta}A_c \mathcal{U}\} + \mathcal{B} \mathcal{B}^T & * \\ \mathcal{P}_c \mathcal{E}^T + \mathcal{S} \mathcal{Q}_c + e^{j\theta} \mathcal{V}^T A_c^T - \mathcal{U} & -\mathcal{V} - \mathcal{V}^T \end{bmatrix} < 0, \quad (31)$$

with  $\mathcal{E}_c \mathcal{S} = 0$ , and

$$\mathcal{E}_c = \begin{bmatrix} E & 0_{n \times p} \\ 0_{p \times n} & \frac{\gamma^2}{2} I_p \end{bmatrix}, \mathcal{A}_c = \begin{bmatrix} A & 0_{n \times p} \\ C & -I_p \end{bmatrix}, \mathcal{B} = \begin{bmatrix} B \\ 0_{p \times m} \end{bmatrix}, \quad (32a)$$

$$\mathcal{P}_c = \begin{bmatrix} P & 0_{n \times p} \\ 0_{p \times n} & e^{j\theta} I_p \end{bmatrix}, \mathcal{Q}_c = \begin{bmatrix} Q & 0_{(n-r) \times p} \end{bmatrix}. \quad (32b)$$

**PROOF.** We start demonstrating the equivalence between (i) and (ii).

**Necessity.** Since the system (12) is admissible, then, by Lemma 7, nonsingular matrices  $M$  and  $N$  can always be found such that

$$E = M \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N, \quad A = M \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} N. \quad (33)$$

We also write

$$B = M \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} N, \quad (34)$$

where the partition is compatible with that of  $A$  in (33). Then, it is easy to see that the transfer function (13) is represented as

$$G_{uy}(s) = C_1 (s^\alpha I_r - A_1)^{-1} B_1 - C_2 B_2. \quad (35)$$

Then, from (35) and Theorem 5, the inequality  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$  implies that there exists a matrix  $P_1 \in H_r > 0$  such that

$$\begin{bmatrix} \text{Her}(e^{-j\theta} A_1 P_1) & * & * \\ e^{-j\theta} C_1 P_1 & -\gamma^2 I_p & * \\ B_1^T & -B_2^T C_2^T & -I_m \end{bmatrix} < 0. \quad (36)$$

Applying Schur complement to (36), we obtain

$$\begin{bmatrix} \text{Her}(e^{-j\theta} A_1 P_1) + B_1 B_1^T & * \\ e^{-j\theta} C_1 P_1 - C_2 B_2 B_1^T & C_2 B_2 B_2^T C_2^T - \gamma^2 I_p \end{bmatrix} < 0.$$

Then a sufficiently small scalar  $\delta > 0$  can always be found such that

$$\begin{bmatrix} \text{Her}(e^{-j\theta} A_1 P_1) + B_1 B_1^T & * \\ e^{-j\theta} C_1 P_1 - C_2 B_2 B_1^T & C_2 (B_2 B_2^T + \delta I) C_2^T - \gamma^2 I_p \end{bmatrix} < 0. \quad (37)$$

Let  $\Psi = B_2 B_2^T + \delta I$ , then  $\Psi > 0$ , and by (37) we have

$$\begin{bmatrix} \text{Her}(e^{-j\theta} A_1 P_1) + B_1 B_1^T & * \\ e^{-j\theta} C_1 P_1 - C_2 B_2 B_1^T & C_2 \Psi C_2^T - \gamma^2 I_p \end{bmatrix} < 0, \quad (38)$$

Using the Schur complement, we get

$$\begin{bmatrix} \text{Her}(e^{-j\theta} A_1 P_1) + B_1 B_1^T & * & * \\ e^{-j\theta} C_1 P_1 - C_2 B_2 B_1^T & -\gamma^2 I_p & * \\ 0 & \Psi C_2^T & -\Psi - \delta I \end{bmatrix} < 0. \quad (39)$$

Replacing  $\delta I = \Psi - B_2 B_2^T$ , (39) is equivalent to

$$\begin{bmatrix} \text{Her}(e^{-j\theta} A_1 P_1) + B_1 B_1^T & * & * \\ 0 & B_2 B_2^T - 2\Psi & * \\ e^{-j\theta} C_1 P_1 - C_2 B_2 B_1^T & -C_2 \Psi & -\gamma^2 I_p \end{bmatrix} < 0. \quad (40)$$

Taking into account

$$P = N^{-1} \begin{bmatrix} P_1 & 0 \\ -e^{j\theta} B_2 B_1^T & -e^{j\theta} \Psi \end{bmatrix} M^T, \quad (41)$$

we get  $EP = M \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} M^T \geq 0$ , then (29a) holds.

Now, to verify that (29b) holds, it suffices to show

$$\begin{bmatrix} \text{Her}(e^{-j\theta} AP) + BB^T & * \\ e^{-j\theta} CP & -\gamma^2 I_p \end{bmatrix} < 0. \quad (42)$$

Noting that  $\Sigma = \begin{bmatrix} \text{Her}(e^{-j\theta} AP) + BB^T & * \\ e^{-j\theta} CP & -\gamma^2 I_p \end{bmatrix}$  and using (33), (34), and (41), after a few transformations, we obtain  $\Sigma = \begin{bmatrix} M & 0 \\ 0 & I_p \end{bmatrix}$

$$\begin{aligned} & \times \begin{bmatrix} \text{Her}(e^{-j\theta} A_1 P_1) + B_1 B_1^T & * & * \\ 0 & B_2 B_2^T - 2\Psi & * \\ e^{-j\theta} C_1 P_1 - C_2 B_2 B_1^T & -C_2 \Psi & -\gamma^2 I_p \end{bmatrix} \\ & \times \begin{bmatrix} M^T & 0 \\ 0 & I_p \end{bmatrix}. \end{aligned}$$

Using (40), it follows that (42) is true and the condition (29b) holds for this  $P$ .

**Sufficiency.** Assume that there exists a matrix  $P \in \mathbb{C}^{n \times n}$  such that conditions in (ii) are satisfied. In the following, we show that the transfer function  $G_{uy}(s)$  in (13) satisfies  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$ . To this end, we apply

Schur complement to (29b) and obtain

$$\text{Her}(e^{-j\theta}AP) + BB^T + \frac{1}{\gamma^2}P^*C^T CP < 0. \quad (43)$$

Then, a matrix  $w > 0$  can always be found such that from (7), (29a), and (43), we get for all  $\lambda \in \Upsilon(\Phi, 0)$

$$(\lambda e^{j\theta} + \lambda^* e^{-j\theta})EP - \text{Her}(e^{-j\theta}AP) - BB^T - \Omega > 0, \quad (44)$$

where

$$\Omega = \frac{1}{\gamma^2}P^*C^T CP + w. \quad (45)$$

The inequality (44) can be written as

$$e^{j\theta}P^*((\lambda^*)^*E^T - A^T) + e^{-j\theta}(\lambda^*E - A)P - BB^T - \Omega > 0. \quad (46)$$

Since  $\lambda \in \Upsilon(\Phi, 0)$ , then  $\lambda^* \in \bar{\Upsilon}(\Phi, 0)$  and, taking (8) into account, we put  $\lambda^* = s^\alpha$ . Therefore, inequality (46) becomes

$$e^{j\theta}P^*(s^\alpha E - A)^* + e^{-j\theta}(s^\alpha E - A)P - BB^T - \Omega > 0. \quad (47)$$

The matrix  $(s^\alpha E - A)^{-1}$  is well defined, since the system (12) is admissible. Thus, pre- and post-multiplying the inequality (47) by  $C(s^\alpha E - A)^{-1}$  and  $[C(s^\alpha E - A)^{-1}]^*$ , respectively, we obtain

$$\begin{aligned} & -C(s^\alpha E - A)^{-1}B \left[ C(s^\alpha E - A)^{-1}B \right]^* \\ & \geq C(s^\alpha E - A)^{-1}\Omega \left[ C(s^\alpha E - A)^{-1} \right]^* \end{aligned}$$

$-e^{j\theta}C(s^\alpha E - A)^{-1}P^*C^T - e^{-j\theta}CP \left[ C(s^\alpha E - A)^{-1} \right]^*$ .  
By adding  $\gamma^2 I_p$ , we get

$$\begin{aligned} & \gamma^2 I_p - C(s^\alpha E - A)^{-1}B \left[ C(s^\alpha E - A)^{-1}B \right]^* \\ & \geq \gamma^2 I_p + C(s^\alpha E - A)^{-1}\Omega \left[ C(s^\alpha E - A)^{-1} \right]^* \\ & - e^{j\theta}C(s^\alpha E - A)^{-1}P^*C^T - e^{-j\theta}CP \left[ C(s^\alpha E - A)^{-1} \right]^*, \end{aligned} \quad (48)$$

From  $\Omega = \frac{1}{\gamma^2}P^*C^T CP + w > 0$  and (48), the following inequality

$$\gamma^2 I_p - G(s)G^*(s) \geq \gamma^2 I_p - CP\Omega^{-1}P^*C^T \quad (49)$$

is valid. Note that  $w = \Omega - \frac{1}{\gamma^2}P^*C^T CP > 0$  is equivalent to  $\gamma^2 I_p - CP\Omega^{-1}P^*C^T > 0$ . Then, from (49) we get  $\gamma^2 I_p - G(s)G^*(s) > 0$ , which means that for all elements of the set  $\{s^\alpha/s \in \mathbb{C}, \text{Re}(s) \geq 0\}$ ,  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$ . Now, we demonstrate the equivalence between (ii) and (iii).

**Sufficiency.** Assume that the  $\mathcal{LMZ}$  (30) in (iii) is satisfied for some matrices  $P \in \mathbb{C}^{n \times n} > 0$  and  $Q \in \mathbb{C}^{(n-r) \times n}$ . Let  $X = PE^T + SQ$ , then we have  $EX = X^*E^T = EPE^T \geq 0$ ,

which means that the conditions (29a) in (ii) are satisfied. By replacing  $PE^T + SQ$  by  $X$  in (30), that the condition (29b) in (ii) is also satisfied. We conclude then that the transfer function of the system (12) verifies  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$ .

**Necessity.** Assume that the system (12) is admissible and its transfer function (13) verifies  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$  for some scalar  $\gamma > 0$ . Since the system (12) is admissible, then, without loss the generality, the matrix  $E$  is

assumed to have the form  $E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ . Assume that

there exists matrix  $X \in \mathbb{C}^{n \times n}$  that verifies (ii). From (29a), we deduce that the matrix  $X$  takes the form:

$$X = \begin{bmatrix} P_1 & 0 \\ Q_3 & Q_4 \end{bmatrix}, P_1 > 0. \text{ We can then write } X \text{ as:}$$

$$X = PE^T + SQ, \text{ with } P = \begin{bmatrix} P_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, S =$$

$$\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}, Q = \begin{bmatrix} Q_3 & Q_4 \end{bmatrix}. \text{ It is clear that } ES = 0 \text{ and}$$

the matrix  $P$  is positive definite. By (29b), the condition (30) in (iii) is satisfied, so the equivalence between (ii) and (iii) is proven.

In the following, we demonstrate that the statements (iii) and (vi) are equivalent.

**Sufficiency.** By pre-multiplying and post-multiplying (31) by  $[I \ e^{-j\theta}\mathcal{A}_c]$  and  $[I \ e^{-j\theta}\mathcal{A}_c]^*$ , respectively, we obtain

$$e^{-j\theta}\mathcal{A}_c(\mathcal{P}_c\mathcal{E}_c^T + \mathcal{S}Q_c) + e^{j\theta}(\mathcal{P}_c\mathcal{E}_c^T + \mathcal{S}Q_c)^*\mathcal{A}_c^* + \mathcal{B}\mathcal{B}^T < 0. \quad (50)$$

Setting  $\mathcal{S} = \begin{bmatrix} S \\ 0 \end{bmatrix}$ , where  $S$  is any matrix of full column rank and verifies  $ES = 0$ , implies  $\mathcal{E}_c\mathcal{S} = 0$ . Along with (32), (50), this yields (30).

**Necessity.** The condition (30) can be written as

$$e^{-j\theta}\mathcal{A}_c\Omega + e^{j\theta}\Omega^*\mathcal{A}_c^* + \mathcal{B}\mathcal{B}^T < 0, \quad (51)$$

where  $\Omega = \begin{bmatrix} PE^T + SQ & 0 \\ 0 & e^{j\theta}\frac{\gamma^2}{2}I_p \end{bmatrix}$ , with  $\mathcal{A}_c$  and  $\mathcal{B}$  are

defined in (32). Therefore, there exists a scalar  $\epsilon > 0$  such that:  $e^{-j\theta}\mathcal{A}_c\Omega + e^{j\theta}\Omega^*\mathcal{A}_c^* + \mathcal{B}\mathcal{B}^T < -\epsilon\mathcal{A}_c\mathcal{A}_c^T$ .

By Shur complement, the last inequality is equivalent to

$$\begin{bmatrix} e^{-j\theta}\mathcal{A}_c\Omega + e^{j\theta}\Omega^*\mathcal{A}_c^* + \mathcal{B}\mathcal{B}^T & \epsilon e^{-j\theta}\mathcal{A}_c \\ \epsilon e^{j\theta}\mathcal{A}_c^T & -\epsilon I_{n+p} \end{bmatrix} < 0. \quad (52)$$

Since  $\epsilon e^{-j\theta}\mathcal{A}_c$  can be written as  $\epsilon e^{-j\theta}\mathcal{A}_c = e^{-j\theta}\mathcal{A}_c\mathcal{V}^T + e^{-j\theta}\mathcal{A}_c\mathcal{V}$  with  $\mathcal{V} = \frac{\epsilon}{2}I_{n+p}$ , and taking into account that



$\mathcal{U} = \Omega - e^{j\theta} \mathcal{V} \mathcal{A}_c^T$ , the inequality (52) is equivalent to

$$\begin{bmatrix} e^{-j\theta} \mathcal{A}_c \mathcal{U} + e^{j\theta} \mathcal{U}^* \mathcal{A}_c^* + \mathcal{B} \mathcal{B}^T + 2\mathcal{A}_c \mathcal{V} \mathcal{A}_c^T & * \\ \Omega - \mathcal{U} + e^{j\theta} \mathcal{V}^T \mathcal{A}_c^T & -\mathcal{V} - \mathcal{V}^T \end{bmatrix} < 0. \quad (53)$$

By expanding the matrix  $\Omega$  as  $\Omega = \mathcal{P}_c \mathcal{E}_c^T + \mathcal{S} \mathcal{Q}_c$  with  $\mathcal{S} = \begin{bmatrix} \mathcal{S}^T & 0 \end{bmatrix}^T$  and using (32) and  $2\mathcal{A}_c \mathcal{V} \mathcal{A}_c^T \geq 0$ , the inequality (53) implies the condition (31). This proves the equivalence between (iii) and (iv) and completes the proof.

**Remark 4** *The statements of Theorem 3 summarizes the existing results on the bounded real lemma for a larger class of dynamic linear systems. Indeed,*

- For  $\alpha = 1$  and  $E = I_n$ , the  $\mathcal{LMI}$  condition for the  $\mathcal{H}_\infty$  norm (31) coincides with the  $\mathcal{LMI}$  in [28].
- For  $\alpha = 1$ , both  $\mathcal{LMI}$ s conditions for the  $\mathcal{H}_\infty$  norm, (30) and (31), in Theorem 3 coincide with the condition (5.20) for singular linear systems in [37] and with the condition (19) in [3] respectively.
- For  $E = I_n$ , the  $\mathcal{LMI}$  condition for the  $\mathcal{H}_\infty$  norm (30) in Theorem 3 coincides with the condition (14) in [15].

This confirms that results of Theorem 3 are more general.

By Theorem 3 and the duality criterion, we obtain the following corollary.

**Corollary 1** *Let the SFOS (12) be admissible and  $\gamma > 0$ . The transfer function  $G_{uy}(s)$  in (13) verifies  $\|G_{uy}(s)\|_{\mathcal{H}_\infty} < \gamma$ , if and only if the following equivalent statements hold.*

(i) *There exist matrices  $X \in \mathbb{C}^{n \times n}$  such that*

$$E^T X = X^* E \geq 0, \quad (54a)$$

$$\begin{bmatrix} \text{Her}\{(e^{-j\theta} A)^* X\} & * & * \\ (e^{-j\theta} B)^* X & -\gamma^2 I_m & * \\ C & 0 & -I_p \end{bmatrix} < 0. \quad (54b)$$

(ii) *There exist matrices  $P \in \mathbb{C}^{n \times n} > 0$  and  $Q \in \mathbb{C}^{(n-r) \times n}$  such that the following  $\mathcal{LMI}$  holds*

$$\begin{bmatrix} \text{Her}\{(e^{-j\theta} A)^* (PE + SQ)\} & * & * \\ (e^{-j\theta} B)^* (PE + SQ) & -\gamma^2 I_m & * \\ C & 0 & -I_p \end{bmatrix} < 0, \quad (55)$$

where  $S \in \mathbb{R}^{n \times (n-r)}$  is any matrix with full column rank and satisfies  $E^T S = 0$ .

(iii) *There exist matrices  $P > 0$ ,  $Q$ ,  $\mathcal{V}$  and  $\mathcal{U}$  such that*

$$\begin{bmatrix} \text{Her}\{(e^{-j\theta} \mathcal{A}_b)^* \mathcal{U}^*\} + \mathcal{C}^T \mathcal{C} & * \\ \mathcal{P}_b \mathcal{E}_b + \mathcal{S} \mathcal{Q}_b + e^{-j\theta} \mathcal{V} \mathcal{A}_b - \mathcal{U}^* & -\mathcal{V} - \mathcal{V}^T \end{bmatrix} < 0, \quad (56)$$

with  $\mathcal{E}_b^T \mathcal{S} = 0$ , and

$$\mathcal{E}_b = \begin{bmatrix} E & 0_{n \times m} \\ 0_{m \times n} & \frac{\gamma^2}{2} I_m \end{bmatrix}, \mathcal{A}_b = \begin{bmatrix} A & B \\ 0_{m \times n} & -I_m \end{bmatrix}, \quad (57a)$$

$$\mathcal{P}_b = \begin{bmatrix} P & 0_{n \times m} \\ 0_{m \times n} & e^{j\theta} I_m \end{bmatrix}, \mathcal{Q}_b = \begin{bmatrix} Q & 0_{(n-r) \times m} \end{bmatrix}, \quad (57b)$$

$$\mathcal{C} = \begin{bmatrix} C & 0_{p \times m} \end{bmatrix}. \quad (57c)$$

#### 4 $\mathcal{H}_\infty$ Controller Synthesis for SFOS

The bounded real lemma can be applied to the controller synthesis. Since the matrices in  $\mathcal{LMI}$  (30) are complex and gain control gains must be real, a novel method is proposed to overcome this difficulty. Let us consider the following SFOS:

$$\begin{cases} ED^\alpha x(t) = Ax(t) + Bu(t) + Fw(t), & 1 \leq \alpha < 2, \\ y(t) = Cx(t), \end{cases} \quad (58)$$

where  $x(t) \in \mathbb{R}^n$  is the descriptor variable,  $u(t) \in \mathbb{R}^m$  is the control input,  $w(t) \in \mathbb{R}^q$  is the exogenous input,  $y(t) \in \mathbb{R}^p$  is the output,  $E, A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $F \in \mathbb{R}^{n \times q}$ ,  $C \in \mathbb{R}^{p \times n}$ . The control input is given by the pseudo-state feedback

$$u(t) = Kx(t), \quad (59)$$

where  $K \in \mathbb{R}^{m \times n}$  is the gain matrix to be determined. The system (58) becomes

$$\begin{cases} ED^\alpha x(t) = (A + BK)x(t) + Fw(t), \\ y(t) = Cx(t), \end{cases} \quad (60)$$

and its transfer function has the form

$$G_{wy}^K(s) = C(s^\alpha E - (A + BK))^{-1} F. \quad (61)$$

The problem is to design the gain matrix  $K$ , such that  $\|G_{wy}^K(s)\|_{\mathcal{H}_\infty} < \gamma$ , where  $\gamma$  is a known positive scalar.

**Theorem 4** *Let the singular fractional order continuous system (60) with  $w(t) = 0$  be admissible and  $\gamma > 0$ . For  $w(t) \neq 0$ ,  $\|G_{wy}^K\|_{\mathcal{H}_\infty} < \gamma$  if and only if there exist matrices  $P \in \mathbb{C}^{n \times n} > 0$ ,  $Q \in \mathbb{C}^{(n-r) \times n}$ ,  $\mathcal{U} \in \mathbb{C}^{(n+p) \times (n+p)}$ ,  $V \in \mathbb{R}^{(n+p) \times (n+p)}$  and  $K \in \mathbb{R}^{m \times n}$  such that*

$$\begin{bmatrix} \text{Her}\{e^{-j\theta} \mathcal{A}_c \mathcal{U}\} + \mathcal{F} \mathcal{F}^T & * \\ \mathcal{P}_c \mathcal{E}_c^T + \mathcal{S} \mathcal{Q}_c + e^{j\theta} \mathcal{V}^T \mathcal{A}_c^T - \mathcal{U} & -\mathcal{V} - \mathcal{V}^T \end{bmatrix} < 0, \quad (62)$$

with  $\mathcal{E}_c \mathcal{S} = 0$ ,  $\mathcal{F} = \begin{bmatrix} F \\ 0_{p \times q} \end{bmatrix}$ , and definitions (32), where  $A$  is replaced by  $A + BK$  in  $\mathcal{A}_c$ .

**PROOF.** The proof is obtained directly by applying Theorem 3 to the system (60).

**Remark 5** Theorem 4 gives a necessary and sufficient condition for the system (60) to satisfy an  $\mathcal{H}_\infty$  performance index  $\gamma$ . However, the condition (62) is given by Bilinear Matrix Inequalities (BMI). The following result proposes a sufficient condition in terms of LMI.

**Theorem 5** The singular fractional order continuous system (60) is admissible with an  $\mathcal{H}_\infty$  performance index  $\gamma$  if there exist matrices  $P \in \mathbb{R}^{n \times n} > 0$ ,  $Q \in \mathbb{R}^{(n-r) \times n}$ ,  $U_1 \in \mathbb{R}^{n \times n}$ ,  $U_3 \in \mathbb{C}^{p \times n}$ ,  $V_3 \in \mathbb{R}^{p \times n}$ ,  $U_4 \in \mathbb{C}^{p \times p}$ , and  $V_4 \in \mathbb{R}^{p \times p}$  such that the following LMI holds:

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ * & * & \Psi_{33} & \Psi_{34} \\ * & * & * & \Psi_{44} \end{bmatrix} < 0, \quad (63)$$

where  $S \in \mathbb{R}^{n \times (n-r)}$  is any matrix of full column rank satisfying  $ES = 0_{n \times (n-r)}$ ,  $\mathcal{J} \in \mathbb{C}^{n \times p}$  is known matrix, and  $\Psi_{11} = \text{Her} \{e^{-j\theta}(AU_1 + BM)\} + FF^T$ ,  $\Psi_{12} = e^{j\theta}(U_1^T C^T - U_3^*) + e^{-j\theta}(AU_1 \mathcal{J} + BM \mathcal{J})$ ,  $\Psi_{13} = (PE^T + SQ)^T + e^{-j\theta}(AU_1 + BM) - U_1^T$ ,  $\Psi_{14} = e^{-j\theta}(AU_1 \mathcal{J} + BM \mathcal{J}) - U_3^*$ ,  $\Psi_{22} = \text{Her} \{e^{-j\theta}(CU_1 \mathcal{J} - U_4)\}$ ,  $\Psi_{23} = e^{-j\theta}(CU_1 - V_3) - \mathcal{J}^T U_1^T$ ,  $\Psi_{24} = e^{-j\theta}(\frac{\gamma^2}{2} I_p + CU_1 \mathcal{J} - V_4) - U_4^*$ ,  $\Psi_{33} = -U_1 - U_1^T$ ,  $\Psi_{34} = -U_1 \mathcal{J} - V_3^T$ ,  $\Psi_{44} = -V_4 - V_4^T$ . The gain matrix  $K$  of the controller (59) can be designed as  $K = MU_1^{-1}$ .

**PROOF.** Assume that the LMI (63) holds for some matrices  $P, Q, U_1, U_3, U_4, V_3, V_4, M$  satisfying the conditions in the statement of Theorem 5. Set the matrices  $\mathcal{U}, \mathcal{V}$  in Theorem 4 to

$$\mathcal{U} = \begin{bmatrix} U_1 & U_1 \mathcal{J} \\ U_3 & U_4 \end{bmatrix}, \mathcal{V} = \begin{bmatrix} U_1 & U_1 \mathcal{J} \\ V_3 & V_4 \end{bmatrix}. \quad (64)$$

Then, let  $M = KU_1$ , which yields the inequality (62).

**Remark 6** • The matrix  $\mathcal{J}$  is usually real but can also be assigned complex or null.

- It is possible to assign distinct matrices  $\mathcal{J}_u$  and  $\mathcal{J}_v$  in place of  $\mathcal{J}$ , i.e., define the matrices  $U$  and  $V$  in (64) as  $U_2 = U_1 \mathcal{J}_u$  and  $V_2 = U_1 \mathcal{J}_v$ .

- Note that choosing the matrix  $U_1$  real leads to a real control gain  $K \in \mathbb{R}^{m \times n}$ .
- Taking  $P$  real allows one to ensure the admissibility and the  $\mathcal{H}_\infty$  performance.

## 5 Examples

In this section, numerical examples are given to demonstrate effectiveness of the proposed results.

### 5.1 Admissibility and Bounded real lemma

Consider the system (12) with the fractional order derivative  $\alpha = 1.2$  and system matrices

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}.$$

The unforced system is admissible, since the LMI of Theorem 2 is found feasible, where

$$P = \begin{bmatrix} 285.6883 & -62.1939 & 13.0596 \\ * & 98.8670 & -62.2977 \\ * & * & 52.6148 \end{bmatrix}, \\ Q = \begin{bmatrix} 18.2907 & 94.9916 & 55.7583 \\ 35.7051 & 20.8486 & 69.5772 \end{bmatrix}, \\ V = \begin{bmatrix} 58.5614 & -13.2158 & 29.7990 \\ -13.2158 & 16.9121 & 0.9530 \\ 29.7990 & 0.9530 & 59.4204 \end{bmatrix}, \\ U = \begin{bmatrix} 83.7300 & -16.9338 & 17.6469 \\ -16.9338 & 51.9385 & 8.0815 \\ 17.6469 & 8.0815 & 56.4907 \end{bmatrix}.$$

Solving the LMI (30) in Theorem 3 for  $\gamma = 1.5$ , we obtain

$$P = \begin{bmatrix} 327.1931 & -60.4637 & -25.6958 \\ -60.4637 & 102.2384 & -56.7679 \\ -25.6958 & -56.7679 & 46.9763 \end{bmatrix}, \\ Q = \begin{bmatrix} 17.2931 & 35.1381 & 23.2448 \\ 22.4470 & 13.7534 & 20.3702 \end{bmatrix}.$$

The transfer function is  $G(s) = \left[ \frac{-14s^{1.2}-4}{19s^{1.2}+10}, \frac{-14s^{1.2}-2}{19s^{1.2}+10} \right]$ . Since  $\|G(s)\|_{\mathcal{H}_\infty} < \infty$ , according to Lemma 4, we have

$\|G(s)\|_{\mathcal{H}_\infty} = \sup_{\omega \geq 0} \bar{\sigma}(G(j\omega)) = \sup_{\omega \geq 0} \|G(j\omega)\|_2$ . The computation of  $\|G(j\omega)\|_2$  yields

$$\|G(j\omega)\|_2 = \sqrt{\frac{392\omega^{2.4} - 51.9149\omega^{1.2} + 20}{361\omega^{2.4} - 117.4260\omega^{1.2} + 100}}.$$

We can show that  $\sup_{\omega \geq 0} \|G(j\omega)\|_2 = 1.0653$ . Thus, all the

maximal singular values of the transfer function are less than  $\gamma$ , which demonstrates validity of the obtained results. Note that the admissibility of the system in Example 1 in [40] is not guaranteed by solving  $\mathcal{LMIs}$  (i) and (ii) of Theorem 1 in [40], as stated in that paper. This point was discussed in several papers (see, for example, [34]).

## 5.2 State feedback control

This example deals with the  $\mathcal{H}_\infty$  state feedback control for the admissible singular system (58) with the following data:

$$E = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, A = \begin{bmatrix} -5 & 3 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, F = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where  $\alpha = 1.12$ . The transfer function of the open-loop system is equal to  $G_o(s) = (s^\alpha E - A)^{-1}F$ , and the transfer function of the closed-loop system (51) is given by  $G_c(s) = (s^\alpha E - A - BK)^{-1}F$ .

Since  $\det(s^\alpha E - A) = 20s^\alpha + 9$ , the pair  $(E, A)$  is admissible. As observed from Fig. 1,  $\|G_o(j\omega)\|_{\mathcal{H}_\infty} = 2.7759$ , which means that  $\|G_o(j\omega)\|_{\mathcal{H}_\infty}$  is bounded.

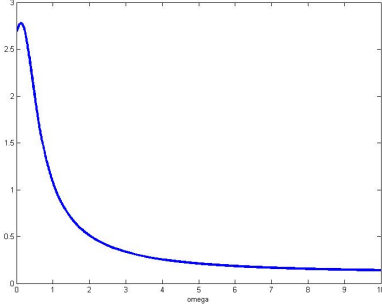


Fig. 1. Modulus of  $G_o(j\omega)$ .

The objective of this example is to minimize the influence of  $w(t)$  on the system (58), satisfying the condition  $\|G_c(j\omega)\|_{\mathcal{H}_\infty} < \|G_o(j\omega)\|_{\mathcal{H}_\infty}$ .

Applying Theorem 5 with the matrix  $\mathcal{J} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , the

following results are obtained:

$$P = \begin{bmatrix} 15.4960 & -6.9670 \\ -6.9670 & 5.0455 \end{bmatrix}, Q = \begin{bmatrix} -1.3319 & 0.3691 \end{bmatrix},$$

$$\text{and } M = \begin{bmatrix} -0.6510 & -1.0317 \end{bmatrix}.$$

The obtained  $\mathcal{H}_\infty$  controller gain is

$$K = \begin{bmatrix} -1.0400 & -4.6891 \end{bmatrix}.$$

Therefore, the system  $(E, A + BK)$  is admissible, its controlled states are plotted in Fig. 2, and  $\|G_c(s)\|_{\mathcal{H}_\infty}$  is bounded. As shown in Fig. 3, it can be obtained that

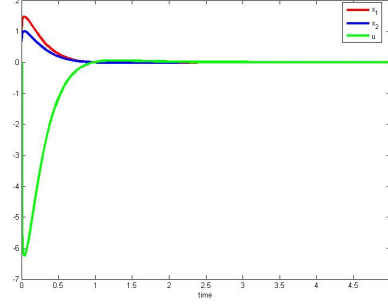


Fig. 2. Time histories of system states with  $u(t) = Kx(t)$ .

$$\|G_c(s)\|_{\mathcal{H}_\infty} = \sup_{\omega \geq 0} \|G_c(j\omega)\|_2 = 0.1286.$$

This example demonstrates effectiveness of Theorem 5.

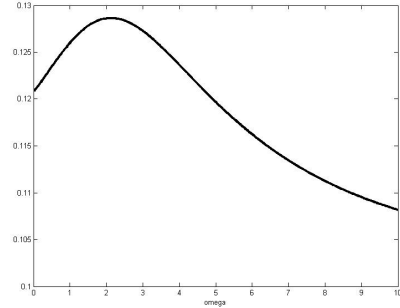


Fig. 3. Modulus of  $G_c(j\omega)$ .

## 6 Conclusions

In this paper, novel necessary and sufficient conditions in terms of  $\mathcal{LMIs}$  are proposed for the admissibility of singular linear fractional-order systems. Then, the  $\mathcal{H}_\infty$  control problem has been solved for this class of systems with the fractional order  $\alpha$  satisfying  $1 \leq \alpha < 2$ . By introducing additional variables, the obtained conditions are more suitable for the  $\mathcal{H}_\infty$  control design. All the presented results on the bounded real lemma are necessary and sufficient in the forms of non-strict and strict  $\mathcal{LMIs}$ , which can be solved by the existing software. Numerical examples have been provided to validate the obtained results. Further research will be focused on the  $\mathcal{H}_\infty$  design problems for uncertain continuous-time singular fractional-order systems.

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