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NEW RESULTS FOR WITNESSES OF ROBIN'S CRITERION

YANNICK SAOUTER

ABSTRACT. In a seminal paper, Robin proved that the Riemann hypothesis holds if and only if the inequality $\sigma(n) = \sum_{d|n} d < e^\gamma n \log \log n$ holds for all integers greater than 5040, where γ is the Euler-Mascheroni constant. In this article, we prove new results on putative violations of Robin's criterion.

1. INTRODUCTION

In a seminal paper [1], Robin proved that the Riemann hypothesis holds if and only if the inequality $\sigma(n) = \sum_{d|n} d < e^\gamma n \log \log n$ holds for all integers greater than 5040, where γ is the Euler-Mascheroni constant. Since then, much work has been done to prove that Robin's criterion holds for wide families of integers. For instance, it is known that this is the case if n is

- odd and greater than 9 [2],
- square-free and greater than 30 [2],
- square-full and greater than 36 [2],
- the sum of two squares and greater than 720 [3],
- not divisible by the 5th [2] (resp. 7th [4], 11th [5], 20th [6]) power of a prime,
- smaller than $10^{10^{10}}$ [7] (resp. $10^{10^{13.099}}$ [6]).

In the following, we prove that a putative integer failing at Robin's criterion has at least 965 billion distinct prime factors. Moreover, details on valuations of small prime numbers in its factorization are derived.

2. DEFINITIONS AND PREREQUISITES

In the following, N will denote a generic putative counterexample. It will be said that N is a witness for Robin's criterion. Variables beginning by a symbol p or q will always refer to prime numbers. In particular, the list of prime integers in ascending order will be denoted $\{p_i\}$ with $p_1 = 2$. We also introduce the prime-counting function $\pi(x) = \sum_{p \leq x} 1$ as well as the Chebyshev function $\theta(x) = \sum_{p \leq x} \log p$ and the primorial values $P_n = \prod_{i=1}^n p_i$. The number of distinct prime factors of an integer n will be denoted $\omega(n) = \sum_{p|n} 1$. In the following, we will also denote $g(x) = \mathcal{O}^*(f(x))$ whenever $|g(x)| \leq |f(x)|$ for all x in the range under consideration.

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Theorem 2.1. *We have*

$$(2.1) \quad \theta(x) \underset{x \geq 3594641}{=} x + \mathcal{O}^*\left(\frac{0.2x}{\log^2 x}\right),$$

$$(2.2) \quad \pi(x) \underset{x \geq 4 \times 10^9}{=} \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \mathcal{O}^*\left(\frac{7.32}{\log^3 x}\right)\right),$$

$$(2.3) \quad \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \underset{x \geq 2278382}{=} \frac{e^{-\gamma}}{\log x} \left(1 + \mathcal{O}^*\left(\frac{0.2}{\log^3 x}\right)\right).$$

Proof. These equations are respectively Theorem 4.2, Theorem 5.1 and Theorem 5.9 in [8]. \square

3. ON THE NUMBER OF PRIME FACTORS OF A WITNESS

We set $K = \omega(N)$. Then we have

Lemma 3.1. *We have*

$$(3.1) \quad \frac{\sigma(N)}{N} \leq \prod_{p|N} \frac{p}{p-1} \leq \prod_{i=1}^K \frac{p_i}{p_i-1}.$$

Proof. The function σ is multiplicative and we have $\frac{\sigma(p^k)}{p^k} = \sum_{i=0}^k \frac{1}{p^i} = \frac{p-1/p^k}{p-1} \leq \frac{p}{p-1}$. Therefore the first inequality is obtained. The second comes from the fact that the function $x \rightarrow \frac{x}{x-1}$ is decreasing for $x > 1$. \square

By hypothesis, we have $\sigma(N)/N > \exp(\gamma) \log \log N$. Therefore, using the lower bound on N given in [6], we obtain

Corollary 3.1.1. *We have*

$$(3.2) \quad \sum_{i=1}^K \log \left(1 - \frac{1}{p_i}\right) \leq -4.01106074891310624828.$$

If the left-hand sum of (3.2) is computed with enough precision, lower bounds on the number of prime factors of a Robin's criterion witness can then be computed. Using (2.3), it is also possible to obtain a rough estimate. For $x \geq 2278382$, we have

$$(3.3) \quad -\gamma - \log \log x + \log \left(1 - \frac{0.2}{\log^3 x}\right) \leq \sum_{p \leq x} \log \left(1 - \frac{1}{p}\right).$$

If we set $x = 2.891 \times 10^{13}$ in (3.3), then the left-hand side evaluates to -4.011055 ± 10^{-6} and is therefore greater than the right-hand side of (3.2). As a consequence, we have $K \geq \pi(x)$. Using (2.2), we have then $K \geq 9.645 \times 10^{11}$. An explicit computation of the right-hand side of 3.3 could give a tighter lower bound for K . A direct evaluation is possible but would lead to a costly execution time. A great speedup can be obtained by the use of the Meissel-Lehmer summation method over primes. From the Taylor series with integral remainder, we have, for $0 < x < 1$:

$$(3.4) \quad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \mathcal{O}^*\left(\frac{x^4}{4(1-x)}\right).$$

Therefore, for K_0 being an integer such that $1 \leq K_0 \leq K$, we obtain

$$\begin{aligned}
(3.5) \quad S_K &= \sum_{i=1}^K \log \left(1 - \frac{1}{p_i} \right) \\
&= \sum_{i=1}^{K_0} \log \left(1 - \frac{1}{p_i} \right) - \sum_{i=K_0+1}^K \left(\frac{1}{p_i} + \frac{1}{2p_i^2} + \frac{1}{3p_i^3} \right) + \mathcal{O}^* \left(\frac{K}{2p_{K_0+1}^4} \right) \\
&= \sum_{i=1}^{K_0} \left(\log \left(1 - \frac{1}{p_i} \right) + \frac{1}{p_i} + \frac{1}{2p_i^2} + \frac{1}{3p_i^3} \right) \\
&\quad - \sum_{i=1}^K \left(\frac{1}{p_i} + \frac{1}{2p_i^2} + \frac{1}{3p_i^3} \right) + \mathcal{O}^* \left(\frac{K}{2p_{K_0+1}^4} \right).
\end{aligned}$$

In applications, K_0 will be chosen so that the remainder error term of (3.5) will be small enough. The first sum is then computed directly and we are left with computing sums of the form $T_j(K) = \sum_{i=1}^K 1/p_i^j$. The Meissel-Lehmer method [9] was initially designed to compute the prime-counting function $\pi(x) = \sum_{p \leq x} 1$. It was soon noted that this method could be adapted to evaluate sums of the form $\sum_{p \leq M} \lambda(p)$ where λ is a totally multiplicative function. In [10], such an implementation has been described to compute $T_1(M) = \sum_{p \leq M} 1/p$. In the present work, the method is extended for general sums of the form $T_j(M) = \sum_{p \leq M} 1/p^j$. At this point, new definitions are required. Let then λ be a totally multiplicative function, i.e. such that $\lambda(1) = 1$ and $\lambda(ab) = \lambda(a)\lambda(b)$ for any integers a and b . If $n > 1$ is an integer, we define $l(n)$ to be the least prime number dividing n with the additional convention that $l(1) = \infty$. If we suppose that the prime decomposition of n is of the form $n = q_1^{a_1} \cdots q_k^{a_k}$, we define $\omega^*(n) = a_1 + \cdots + a_k$. By convention, we set $\omega^*(1) = 0$. With the additional convention $p_0 = -\infty$, for any real value $x \geq 1$ and integers $j, k \geq 0$, the following three functions are introduced:

$$\Phi(x, j) = \sum_{\substack{1 \leq n \leq x \\ l(n) > p_j}} \lambda(n), \quad \Psi(x, j) = \sum_{\substack{1 \leq n \leq x \\ l(n) = p_j}} \lambda(n), \quad P_k(x, j) = \sum_{\substack{1 \leq n \leq x \\ l(n) > p_j \\ \omega^*(n) = k}} \lambda(n).$$

We have then obviously $\Phi(x, j) = \sum_{k \geq 0} P_k(x, j)$. For $0 \leq k \leq 2$, expressions $P_k(x, j)$ can be transformed and we have

$$\begin{aligned}
P_0(x, j) &= \lambda(1) = 1, \\
P_1(x, j) &= \sum_{p_j < p \leq x} \lambda(p) = P_1(x, 0) - P_1(p_j, 0), \\
P_2(x, j) &= \sum_{i=j+1}^{\pi(x^{1/2})} \sum_{l=i}^{\pi(x/p_i)} \lambda(p_i p_l) \\
&= \sum_{i=j+1}^{\pi(x^{1/2})} \lambda(p_i) \cdot \left[\sum_{l=1}^{\pi(x/p_i)} \lambda(p_l) - \sum_{l=1}^{i-1} \lambda(p_l) \right].
\end{aligned}$$

Moreover, if $x \leq p_a^k$, then $P_k(x, a) = 0$. Therefore, if we set $a = \pi(x^{1/3}) + 1$, we have

$$\begin{aligned}\Phi(x, a) &= 1 + P_1(x, a) + P_2(x, a) \\ &= 1 + P_1(x, 0) - P_1(p_a, 0) + P_2(x, a)\end{aligned}$$

so that

$$(3.6) \quad P_1(x, 0) = \Phi(x, a) + P_1(p_a, 0) - P_2(x, a) - 1.$$

Moreover, we have the following disjoint union:

$$\begin{aligned}\{1 \leq n \leq x \mid l(n) > p_j\} &= \{1 \leq n \leq x \mid l(n) > p_{j+1}\} \\ &\cup \{p_{j+1}.n' \mid 1 \leq n' \leq x/p_{j+1}, l(n') > p_j\}\end{aligned}$$

and we obtain

$$(3.7) \quad \Phi(x, j+1) = \Phi(x, j) - \lambda(p_{j+1})\Phi\left(\frac{x}{p_{j+1}}, j\right).$$

By definition, we have $T_k(M) = P_1(M, 0)$ for $\lambda(x) = 1/x^k$. This expression is first rewritten using (3.6). In the right-hand side, $P_1(p_a, 0)$ can be computed by a prime sieve up to p_a or recursively by using again the Meissel-Lehmer method. The value $P_2(M, a)$ can be computed by a prime sieve up to M/p_{a+1} , if accumulated values $\sum_{j=1}^{\pi(x/p_i)} \lambda(p_j)$ and $\sum_{j=1}^{i-1} \lambda(p_j)$ are kept in memory. Finally, the term $\Phi(M, a)$ is iteratively transformed by the recurrence (3.7) up to the point that it is rewritten as the sum and difference of terms of the form $\Phi(y, b)$ with either $b = 0$ and $y > M^{2/3}$, or $y \leq M^{2/3}$. We focus first on terms of the second form. In order to compute them, a sieve up to $M^{2/3}$ is performed. The interval $[1, M^{2/3}]$ is cut into successive blocks $B_k = [(k-1)N + 1, kN]$ with $N = \lfloor M^{1/3} \rfloor$. We suppose that, before sieving interval B_k , the values $\Phi(k(N-1), j)$ and $\Psi(k(N-1), j)$ with $0 \leq j \leq a$ have been computed and are available in memory. The sieve of interval B_k is performed in such a way to determine prime numbers of the interval B_k but also to keep in memory for all $n \in B_k$, the least prime number dividing n , i.e. $l(n)$. We compute then

$$(3.8) \quad \Phi(kN, 0) = \Phi((k-1)N, 0) + \sum_{n \in B_k} \lambda(n),$$

$$(3.9) \quad \Psi(kN, j) = \Psi((k-1)N, j) + \sum_{\substack{n \in B_k \\ l(n) = p_j}} \lambda(n)$$

and then, iteratively on j ,

$$(3.10) \quad \Phi(kN, j) = \sum_{\substack{1 \leq n \leq kN \\ l(n) > p_{j-1}}} \lambda(n) - \sum_{\substack{1 \leq n \leq kN \\ l(n) = p_j}} \lambda(n)$$

$$(3.11) \quad = \Phi(kN, j-1) - \Psi(kN, j).$$

Finally, for any value $\Phi(y, b)$ with $y \in B_k$ required in the summation, we first compute

$$(3.12) \quad \Phi(y, 0) = \Phi((k-1)N, 0) + \sum_{n=(k-1)N+1}^y \lambda(n)$$

and then, iteratively on j ,

$$(3.13) \quad \Phi(y, j) = \Phi(y, j-1) - \Psi(y, j)$$

$$(3.14) \quad = \Phi(y, j-1) - \Psi((k-1)N, j) - \sum_{\substack{(k-1)N+1 \leq n \leq y \\ l(n)=p_j}} \lambda(n).$$

The computed value $\Phi(y, b)$ is then saved in memory. At this point, we are left with terms of the first form, say $\Phi(y, 0) = \sum_{n=1}^y \lambda(n)$ with $y > M^{2/3}$. Computations by direct summations would lead to excessive computational cost. Following [10], the Euler-MacLaurin summation theorem was used instead.

Definition 3.2. The set of Bernoulli polynomials B_n with $n \geq 0$ is defined iteratively on n by $B_0 = 1$ and, for $n > 0$, $B'_n = nB_{n-1}$ with $\int_0^1 B_n(x)dx = 0$.

Theorem 3.3. Let $m, n \in \mathbb{N}$ with $m < n$. Let $r \in \mathbb{N}^*$ such that $f \in \mathcal{C}^r([m, n])$. Then we have

$$(3.15) \quad \begin{aligned} \sum_{k=m}^n f(k) &= \int_m^n f(t)dt + \frac{1}{2}(f(m) + f(n)) \\ &+ \sum_{p=1}^{\lfloor r/2 \rfloor} \frac{b_{2p}}{(2p)!} (f^{(2p-1)}(n) - f^{(2p-1)}(m)) \\ &+ \frac{(-1)^{r+1}}{r!} \int_m^n \tilde{B}_r(t) f^{(r)}(t) dt \end{aligned}$$

where $b_n = B_n(1)$ and $\tilde{B}_r(t)$ is the period 1 function coinciding with the polynomial B_n on the interval $[0, 1]$.

We have also

Proposition 3.4. For all $k \in \mathbb{N}^*$, $b_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k)$. Moreover, for all $x \in \mathbb{R}$, $|\tilde{B}_{2k}(x)| \leq |b_{2k}|$ and $|\tilde{B}_{2k+1}(x)| \leq (k + 1/2)|b_{2k}|$.

Therefore, with Theorem 3.3, it is possible to compute expressions of the form $\sum_{k=m}^n 1/k^j$ and using Proposition 3.4 the size of the last integral of the right-hand side of (3.15) can be estimated. In applications, this integral will be considered as an error term. Setting $r = 4$ in Theorem 3.3, we have

| k | p_k | $\sum_{i=1}^k \log(1 - 1/p_i)$ |
|--------------|----------------|--|
| 7726 | 78781 | $-3.00000622762836686699 \pm 10^{-20}$ |
| 22692 | 258197 | $-3.10000361628611903743 \pm 10^{-20}$ |
| 75516 | 958547 | $-3.20000065709966032135 \pm 10^{-20}$ |
| 288592 | 4082053 | $-3.30000001038312813469 \pm 10^{-20}$ |
| 1285140 | 20244503 | $-3.40000003613662278743 \pm 10^{-20}$ |
| 6776871 | 118793447 | $-3.50000000838882920349 \pm 10^{-20}$ |
| 43073879 | 839559299 | $-3.60000000043396596749 \pm 10^{-20}$ |
| 336502549 | 7288129661 | $-3.70000000002492940295 \pm 10^{-20}$ |
| 3301587824 | 79410773459 | $-3.80000000000855532965 \pm 10^{-20}$ |
| 41665102302 | 1112324163331 | $-3.90000000000058896005 \pm 10^{-20}$ |
| 694360371976 | 20565764172307 | $-4.0000000000000806678 \pm 10^{-20}$ |
| 965335869632 | 28921153575169 | $-4.01106074891308670632 \pm 10^{-20}$ |
| 965335869633 | 28921153575211 | $-4.01106074891312128309 \pm 10^{-20}$ |

TABLE 1. Values for $\sum_{i=1}^k \log(1 - 1/p_i)$

Lemma 3.5. *Let $m, n \in \mathbb{N}^*$ with $m < n$. Then, we have*

$$\begin{aligned}
\sum_{k=m}^n \frac{1}{k} &= \log(n) - \log(m) + \frac{1}{2n} + \frac{1}{2m} - \frac{1}{12n^2} + \frac{1}{12m^2} \\
&\quad + \frac{1}{120n^4} - \frac{1}{120m^4} + n\vartheta_1(m), \\
\sum_{k=m}^n \frac{1}{k^2} &= \frac{1}{m} - \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{2m^2} - \frac{1}{6n^3} + \frac{1}{6m^3} \\
&\quad + \frac{1}{30n^5} - \frac{1}{30m^5} + n\vartheta_2(m), \\
\sum_{k=m}^n \frac{1}{k^3} &= \frac{1}{2m^2} - \frac{1}{2n^2} + \frac{1}{2n^3} + \frac{1}{2m^3} - \frac{1}{4n^4} + \frac{1}{4m^4} \\
&\quad + \frac{1}{12n^6} - \frac{1}{12m^6} + n\vartheta_3(m)
\end{aligned}
\tag{3.16}$$

with

$$|\vartheta_1(m)| \leq \frac{1}{30m^5}, \quad |\vartheta_2(m)| \leq \frac{5}{m^6}, \quad |\vartheta_3(m)| \leq \frac{12}{m^7}.
\tag{3.17}$$

For applications, computations have to be done with at least 20 digits after the decimal point. They could be performed using the 128 bits `_float128 C` type. However, truncations done with functions related to this type are not provided to the user and taking the worst cases systematically for all evaluations would give an excessive final error term. Therefore, the interval arithmetic library *MPFI* [11] was used instead with a precision of 192 bits. Errors implied by Lemma 3.5 can be directly included into the interval structures and the final result can then be directly interpreted.

The cutting parameter K_0 involved in the sums S_K of (3.5) was set to $K_0 = \pi(10^8)$ and we obtain

$$(3.18) \quad \sum_{i=1}^{K_0} \left(\log \left(1 - \frac{1}{p_i} \right) + \frac{1}{p_i} + \frac{1}{2p_i^2} + \frac{1}{3p_i^3} \right) = \\ - 0.031340528933542815457 \pm 10^{-20}.$$

Values for the logarithm of the product over primes have been computed for various parameter values. Results of computations are summarized in Table 1 and we have

Lemma 3.6. *If N fails for the Robin's criterion, then the number of distinct prime factors of N is greater than 965335869632.*

Using our implementation of the Meissel-Lehmer method, for $p_k = 28921153575-169$, the execution of the three summations took 307000 seconds in time on a recent Intel-i7 Linux computer, i.e. 85 hours and 27 minutes. This could be easily improved since our implementation was not very optimized. Moreover, summations of $\sum 1/p^2$ and $\sum 1/p^3$ could have been stopped below p_k , since high index terms do not contribute much to the final result. A comparison has been made with direct computation. To this end, another program was written to sieve prime numbers and accumulate the sum $\sum_{p \leq x} \log(1 - 1/p)$. The reference implementation

of the Atkin-Bernstein sieve [12] and the library *MPFI* were used. For $x = 10^{11}$, with the same computer, the direct computation took 16882 seconds where 386 seconds were used by the Atkin-Bernstein sieve. This sieve has a time complexity of $\mathcal{O}(x/\log \log x)$. The time required for the accumulation is proportional to the number of prime numbers below x and is therefore of order $\mathcal{O}(x/\log x)$. Given this information, it is possible to estimate the required time of a direct evaluation for $x = p_k$. The global time is then estimated to approximately 4×10^6 seconds with 10^5 seconds required for the sieve. The expected time is then about 1100 hours and thus the Meissel-Lehmer method provides a speedup by almost a factor of 13. This quite modest value is explained by the moderate size of arguments. However, the asymptotic complexity of the Meissel-Lehmer method is $\mathcal{O}(x^{2/3} \log x)$ while that of direct evaluation is $\mathcal{O}(x/\log \log x)$.

4. ON THE VALUATION OF SMALL PRIME FACTORS

Champion values n for $\sigma(n)/n$ are integer values such that $\sigma(n)/n > \sigma(m)/m$ for all integers m such that $1 \leq m < n$. It has been remarked by Ramanujan in [13] that champion values n for $\sigma(n)/n$ are necessarily of the form $n = 2^{a_1} \cdot 3^{a_2} \cdots p_k^{a_k}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. These numbers were called Hardy-Ramanujan numbers and from [2]

Lemma 4.1. *The putative least Robin's witness N' is a Hardy-Ramanujan number.*

This lemma is proven by application of the following two lemmas.

Lemma 4.2. *Suppose $M = q_1^{a_1} \cdot q_2^{a_2} \cdots q_l^{a_l}$ with $a_1 \geq a_2 \geq \cdots \geq a_l \geq 1$. Let $\{q'_i | 1 \leq i \leq l\}$ be the ascending sequence of the prime numbers $\{q_i | 1 \leq i \leq l\}$ and let $M_1 = q_1^{a_1} \cdot q_2^{a_2} \cdots q_l^{a_l}$. Then we have $M_1 \leq M$ and $\sigma(M_1)/M_1 \geq \sigma(M)/M$.*

Proof. The proof is made by recurrence on l . If $l = 1$, Lemma 4.2 is a tautology. If $l = 2$ then $M = q_1^{a_1} \cdot q_2^{a_2}$. If $q_1 < q_2$, we have again a tautology. Therefore, let us

suppose that $q_1 > q_2$. We have then:

$$(4.1) \quad \begin{aligned} M &= q_1^{a_1} \cdot q_2^{a_2} = q_1^{a_2} \cdot q_1^{a_1 - a_2} \cdot q_2^{a_2} \\ &\geq q_1^{a_2} \cdot q_2^{a_1 - a_2} \cdot q_2^{a_2} = q_2^{a_1} \cdot q_1^{a_2} = M_1 \end{aligned}$$

and the first property is proven. Moreover, we have:

$$(4.2) \quad \begin{aligned} \frac{\sigma(M)}{M} &= \frac{\sigma(q_1^{a_1})}{q_1^{a_1}} \frac{\sigma(q_2^{a_2})}{q_2^{a_2}} = \left[\sum_{i=0}^{a_1} \frac{1}{q_1^i} \right] \left[\sum_{i=0}^{a_2} \frac{1}{q_2^i} \right] \\ &= \left[\sum_{0 \leq i, j \leq a_2} \frac{1}{q_1^i q_2^j} \right] + \left[\sum_{\substack{a_2+1 \leq i \leq a_1 \\ 0 \leq j \leq a_2}} \frac{1}{q_1^i q_2^j} \right]. \end{aligned}$$

If $a_2 + 1 \leq i \leq a_1$ and $0 \leq j \leq a_2$, then $j \leq i$ and the first property proven above for $l = 2$ gives $q_1^j q_2^i \geq q_1^i q_2^j$ and then

$$(4.3) \quad \begin{aligned} \frac{\sigma(M)}{M} &\leq \left[\sum_{0 \leq i, j \leq a_2} \frac{1}{q_1^i q_2^j} \right] + \left[\sum_{\substack{a_2+1 \leq i \leq a_1 \\ 0 \leq j \leq a_2}} \frac{1}{q_2^j q_1^i} \right] \\ &= \left[\sum_{i=0}^{a_2} \frac{1}{q_1^i} \right] \left[\sum_{i=0}^{a_1} \frac{1}{q_2^i} \right] = \frac{\sigma(M_1)}{M_1}. \end{aligned}$$

Suppose now that the result is true for l and suppose that $M = q_1^{a_1} \cdot q_2^{a_2} \cdots q_{l+1}^{a_{l+1}}$. Let m be such that $1 \leq m \leq l+1$ and $q_m \leq q_i$ for all $1 \leq i \leq l+1$. We have then $q_1' = q_m$. Suppose first that $m = 1$ and set $L = q_2^{a_2} \cdot q_3^{a_3} \cdots q_{l+1}^{a_{l+1}}$ and apply the recurrence hypothesis to L . We obtain then L_1 with $L_1 \leq L$ and $\sigma(L_1)/L_1 \geq \sigma(L)/L$. Now set $M_1 = q_1^{a_1} \cdot L_1$. Then M_1 is of the prescribed form and $M_1 \leq M$ as well as $\sigma(M_1)/M_1 \geq \sigma(M)/M$. Suppose then that $m > 1$ and rewrite M as

$M = q_1^{a_1} \cdot q_m^{a_m} \cdot \prod_{\substack{i=2 \\ i \neq m}}^{l+1} q_i^{a_i}$. From the recurrence hypothesis for $l = 2$, we have then

$q_m^{a_1} q_1^{a_m} \leq q_1^{a_1} q_m^{a_m}$ and $\sigma(q_m^{a_1} q_1^{a_m}) / (q_m^{a_1} q_1^{a_m}) \geq \sigma(q_1^{a_1} q_m^{a_m}) / (q_1^{a_1} q_m^{a_m})$. Therefore, if

we set $L = q_1^{a_m} \cdot \prod_{\substack{i=2 \\ i \neq m}}^{l+1} q_i^{a_i}$, we have $q_1^{a_1} \cdot L \geq M$ and $\sigma(q_1^{a_1} \cdot L) / (q_1^{a_1} \cdot L) \geq \sigma(M)/M$,

since σ is a multiplicative function. Apply then the recurrence hypothesis to obtain L_1 from L , set $M_1 = q_1^{a_1} \cdot L_1$ and conclude as previously. \square

Lemma 4.3. *Let R be a set of prime numbers, possibly empty. Suppose $L = q_1^{a_1} \cdot q_2^{a_2} \cdots q_l^{a_l}$ with $a_1 \geq a_2 \geq \cdots \geq a_l \geq 1$, $q_1 \leq q_2 \leq \cdots \leq q_l$ and $q_i \notin R$, for $1 \leq i \leq l$. Let $\{q'_i | 1 \leq i \leq l\}$ be the ascending sequence of the first l prime numbers avoiding R . Then if we set $L_1 = q_1'^{a_1} \cdot q_2'^{a_2} \cdots q_l'^{a_l}$, we have $L_1 \leq L$ and $\sigma(L_1)/L_1 \geq \sigma(L)/L$.*

Proof. We have $q'_i \leq q_i$ for $1 \leq i \leq l$ and thus $L_1 \leq L$. Moreover we have $\sigma(q^a)/q^a = \sum_{i=0}^a 1/q^i$ and is thus a decreasing function for parameter q . Therefore we have $\sigma(L_1)/L_1 \geq \sigma(L)/L$. \square

With the two preceding lemmas, it is then possible to prove Lemma 4.1. Indeed, if N is an arbitrary Robin's witness, applying Lemma 4.2 for $M = N$ and then Lemma 4.3 for $L = M_1$ and R empty, we obtain a Hardy-Ramanujan integer $N' = L_1$ such that $N' \leq N$ and $\sigma(N')/N' \geq \sigma(N)/N$. Moreover $\log \log N' \leq \log \log N$

and thus N' is a Robin's witness. As a consequence, if it exists, the least Robin's witness is necessarily a Hardy-Ramanujan number. In order to study the valuation of small prime numbers in Robin's witness factorizations, we will use the following lemma.

Lemma 4.4. *Suppose that $N = p^k M$ with $k \geq 0$, $M = q_1^{a_1} q_2^{a_2} \cdots q_l^{a_l}$ with $a_1 \geq a_2 \geq \cdots \geq a_l \geq 1$ and $q_i \neq p$ for $1 \leq i \leq l$. Let $\{q'_i | 1 \leq i \leq l\}$ be the ascending sequence of the first l prime numbers excluding p and let $N' = p^k M'$ with $M' = q_1'^{a_1} q_2'^{a_2} \cdots q_l'^{a_l}$. Then we have $N' \leq N$ and $\sigma(N')/N' \geq \sigma(N)/N$. Moreover if N is a Robin's witness, then N' is also a Robin's witness.*

Proof. The proof is similar to that of Lemma 4.1 given in the previous paragraph with $R = \{p\}$. \square

Let then N be an arbitrary Robin's witness such that $p^k | N$ and $(N/p^k, p) = 1$ with $k \geq 0$ and $p \leq p_K$ and let N' be the corresponding Robin's witness obtained by application of Lemma 4.4. From Table 1, we have $K \geq 9.65 \times 10^{11}$ and $p_K \geq 2.89 \times 10^{13}$. We have then

$$\begin{aligned}
 \frac{\sigma(N')}{N'} &= \frac{\sigma(p^k)}{p^k} \frac{\sigma(M')}{M'} \leq \frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}} \prod_{i=1}^{K-1} \frac{q'_i}{q'_i - 1} \\
 (4.4) \quad &\leq \frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}} \frac{p-1}{p} \prod_{i=1}^K \frac{p_i}{p_i - 1} \\
 &\leq \left(1 - \frac{1}{p^{k+1}}\right) e^\gamma \log p_K \left(1 - \frac{0.2}{\log^3 p_K}\right)^{-1}.
 \end{aligned}$$

On the other hand, we have $\omega(M') \geq K - 1$ and $p \nmid M'$, so that $M' \geq P_K/p$ since $p \leq p_K$. Therefore we have $N' \geq p^{k-1} P_K$ and by hypothesis, we obtain

$$(4.5) \quad \log((k-1) \log p + \theta(p_K)) \leq \left(1 - \frac{1}{p^{k+1}}\right) \log p_K \left(1 - \frac{0.2}{\log^3 p_K}\right)^{-1}.$$

In the following, we will use that for $-1/2 \leq x \leq 1/2$, $x - x^2 \leq \log(1+x)$. From the lower bound on p_K , we have $(1 - 0.2/\log^3 p_K)^{-1} < 1.000006717$. We have $|\theta(x) - x| \leq 0.2x/\log^2 x$ for $x \geq 3594641$, so that $\theta(p_K) \geq 2.8893 \times 10^{13}$. If we suppose, for example, that $|k-1| \log p < 10^3$, then $|(k-1) \log(p)/\theta(p_K)| \leq 1/2$. Moreover, if we suppose that $\log(p) \leq 20$, we obtain

$$\begin{aligned}
 \log(\theta(p_K) + (k-1) \log p) &= \log(\theta(p_K)) + \log\left(1 + \frac{(k-1) \log p}{\theta(p_K)}\right) \\
 (4.6) \quad &\geq \log(\theta(p_K)) + \frac{(k-1) \log p}{\theta(p_K)} - \left(\frac{(k-1) \log p}{\theta(p_K)}\right)^2 \\
 &\geq \log(\theta(p_K)) - \frac{\log p}{\theta(p_K)} - \frac{10^6}{\theta(p_K)^2} \\
 &\geq \log(\theta(p_K)) - 6.9205 \times 10^{-13}
 \end{aligned}$$

so that

$$(4.7) \quad \log(\theta(p_K)) - 6.9205 \times 10^{-13} \leq 1.000006717 \left(1 - \frac{1}{p^{k+1}}\right) \log p_K.$$

Moreover, from (2.1), we have

$$\begin{aligned}
(4.8) \quad \log(\theta(p_K)) &\geq \log\left(p_K - \frac{0.2p_K}{\log^2 p_K}\right) \\
&\geq \log(p_K) + \log\left(1 - \frac{0.2}{\log^2 p_K}\right) \\
&\geq \log(p_K) - 0.0002082071737.
\end{aligned}$$

We obtain then successively

$$\begin{aligned}
(4.9) \quad \log(p_K) - 0.000208207173 &\leq 1.000006717\left(1 - \frac{1}{p^{k+1}}\right) \log p_K \\
1 - \frac{0.0002081638727}{\log p_K} &\leq 1.000006717\left(1 - \frac{1}{p^{k+1}}\right) \\
0.9999932825 &\leq 1.000006717\left(1 - \frac{1}{p^{k+1}}\right) \\
0.9999865655 &\leq \left(1 - \frac{1}{p^{k+1}}\right) \\
p^{k+1} &> 74435.
\end{aligned}$$

In the preceding derivations, the value $k = 0$ is admissible. In this case, it means that p does not divide N . Therefore p does not divide the corresponding integer N' either. However, upon existence of N' , (4.9) permits us to claim that $p > 74435$. Therefore, by contradiction, if $p \leq 74435$, neither such N nor N' can exist. As a consequence, a putative Robin's witness is divisible by all the prime numbers up to 74419. Appealing again to (4.9) gives the following lemma:

Lemma 4.5. *Suppose N is a witness for Robin's criterion. Then N is divisible by p if $2 \leq p \leq 74419$, by p^2 if $2 \leq p \leq 271$, by p^3 if $2 \leq p \leq 41$, by p^4 if $2 \leq p \leq 13$ and by 7^5 , 5^6 , 3^{10} and 2^{16} .*

An equivalent formulation is the following one.

Lemma 4.6. *If N is a witness for Robin's criterion, then N is divisible by $2^{11} \cdot 3^5 \cdot 5 \cdot P_7 \cdot P_{13} \cdot P_{41} \cdot P_{271} \cdot P_{74419}$.*

These results can still be improved. Indeed, we have

Lemma 4.7. *For $599 < x < 2.169 \times 10^{25}$, we have*

$$(4.10) \quad \theta(x) = x + \mathcal{O}^*\left(\frac{\sqrt{x}}{8\pi} \log^2 x\right).$$

Proof. This result is a direct application of Theorem 2 of [14], knowing that the Riemann hypothesis has been verified up to height $T = 3 \times 10^{12}$ [6, 15]. \square

We have then, for $2.8921 \times 10^{13} \leq p_K \leq 2.169 \times 10^{25}$,

$$\begin{aligned}
(4.11) \quad \log(\theta(p_K)) &\geq \log(p_K) - \frac{\log^2(p_K)}{8\pi\sqrt{p_K}} - \left(\frac{\log^2(p_K)}{8\pi\sqrt{p_K}}\right)^2 \\
&\geq \log(p_K) - 7.110395218 \times 10^{-6}.
\end{aligned}$$

For $p_K \geq 2.169 \times 10^{25}$, (4.8) gives

$$(4.12) \quad \log(\theta(p_K)) \geq \log(p_K) - 0.0000587660267.$$

Using this latter equation for $p_K \geq 2.8921 \times 10^{13}$, derivations similar to those of (4.9) give then

$$(4.13) \quad p^{k+1} > 116103.$$

5. CONCLUSION

In this paper, we have obtained new results about the factorization of potential integers violating the Robin's criterion. These results could be numerically improved if the lower bound of possible counterexamples is pushed further. From an analytical point of view, it is also most probable that (2.3) could also be tightened if density results for nontrivial zeros of the Riemann zeta function [16] are used. However, in regard to the size of considered integers, as well as of the delicacy of Robin's criterion, it seems quite unlikely that the Riemann hypothesis could be disproven by finding an explicit counterexample.

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