Statistical inference on stationary shot noise random fields

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Abstract. We study the asymptotic behaviour of a stationnary shot noise random field. We use the notion of association to prove the asymptotic normality of the moments and a multidimensional version for the correlation functions. The variance of the moment estimates is decreased by use of the isotropy property of the field. These results will be applied to the estimation of the model parameters in the case of a Gaussian kernel. The asymptotic normality is proved and a simulation study is carried out.

1. Introduction and notation

A spatial random field X is a family of random variables $\{X(t), t \in T\}$ indexed by $T = \mathbb{R}^2$ or $T = \mathbb{Z}^2$. Many models exist and have for instance applications in spatial modeling and image analysis (e.g. Bernardino and Duval, 2019). Most interesting random fields are smooth and therefore have a dependency structure and one may only adress their asymptotic normality by using for instance mixing properties, as can be found in Rosenblatt (1956) and Bolthausen (1982). Unfortunately, although the usual mixing definition are very useful for the processes, there are only few examples of a random field satisfying this mixing requirement. To obtain asymptotic results one may also use the concept of association introduced by Esary et al. (1967) for which Newman (1980, 1984), Burton et al. (1986) and more recently Bulinski and Shashkin (2007) proved asymptotic results. The logical connections between mixing and association have been established for some point process (Poinas et al., 2019) but it is more complicated for random fields. Several classical processes and models are associated (Last et al., 2019) and in particular Bulinski and Shashkin (2007) demonstrated the association for some shot noise random fields.
A shot noise is an elegant random field built from a point process and a kernel function; more precisely in this paper we will consider the following model

$$\forall x \in T, \quad X(x) = \sum_{(x_i, m_i) \in \phi_X} m_i g(x - x_i), \tag{1.1}$$

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where $g : \mathbb{R}^2 \to \mathbb{R}$ is a deterministic function that we will call kernel and ϕ_X is a marked Poisson point process on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $\lambda dx F(dm)$ i.e. the x_i are the points of an homogeneous Poisson point process with intensity λ in $\mathbb{R}^2$ and the m_i are the marks with common law F, independent of the Poisson point process.
Initially introduced in 1909 by Campbell (1909a,b), the one dimensional shot noise process was much studied in the 20th century to model physical phenomena with a first important theoretical paper by Rice (1944). More recently, the process has also been used in network theory (Baccelli, 2009) and finance (Schmidt, 2017). Generalization to higher dimensions is more recent with the important work of Heinrich and Schmidt (1985) proving asymptotic normality for high intensity and which can be seen as a link between the Gaussian setting and the discrete models of stochastic geometry such as the Boolean one (Chiu et al., 2013). We can also mention the recent works of Biermé and Desolneux (2020) and Lachièze-Rey (2016) studying geometric statistics of excursion sets. Shot noise random fields have also found their interest in Cox point processes (seeMø ller, 2003) to generalize cluster point processes model such as Neyman-scott processes.

By Campbell's theorem, the random sum (1.1) exists if and only if

$$\int_{\mathbb{R}^2 \times \mathbb{R}_+} \min(|mg(x)|, 1) \lambda \mathrm{d}x F(\mathrm{d}m) < \infty$$

which holds in particular if the distribution of the mark M given by F has a finite expectation and g is integrable. More precisely, in the rest of the paper we will consider the following set of assumptions **(H)**

(1) the mark M is positive almost surely and has finite moments of order 4 i.e. $\mathbb{E}[M^4] < \infty$;
(2) the kernel g is a non negative, continuous, bounded and Lebesgue integrable function on $\mathbb{R}^2$.

The paper is organized as follows. In Section 2 we propose an asymptotic study on the empirical moments : the first moment (the mean), the second moment and the second central moment (the variance). More specifically, we are interested in the mixed moment of second order, the central mixed moment of second order (the covariance) and the correlation. Due to the stationarity property, all those moments are functions on T. Our main aim is to prove the strong consistency and the asymptotic normality of those statistics. In Section 3, we deal with the case of isotropic kernels for which the mixed moments are given by functions on $\mathbb{R}_+$. By taking this property into account, we can construct a refinement of the estimates. Finally in the last part, these results are applied to estimate the parameters considering the case where the kernel g is Gaussian with variance σ^2. A specific application of this Gaussian kernel is the fundamental solution of the heat equation with initial data a point process, yielding a parameter σ depending on the time t. Moreover, in the Gaussian kernel case most of the computations can be performed in an explicit way.

2. Asymptotic behaviour of moments

Although moments do not fully characterise the law of a random variable, they provide a great amount of information. When considering a random field, it is not always easy to calculate the raw moments and it is even less easy to calculate the mixed moments. Under **(H)** and thanks to

Campbell theorem, since g is bounded and integrable, we can calculate all the moments up to order 4 of a shot noise random field using kernel dependent integrals. Let us start with a few notations

$$I_1 = \int_{\mathbb{R}^2} g(c)\mathrm{d}c \tag{2.1}$$

$$I_2(x) = \int_{\mathbb{R}^2} g(c-x)g(c)\mathrm{d}c \tag{2.2}$$

$$I_3(x,y) = \int_{\mathbb{R}^2} g(c-x)g(c-y)g(c)\mathrm{d}c \tag{2.3}$$

$$I_4(x,y,s) = \int_{\mathbb{R}^2} g(c-x)g(c-y)g(c-s)g(c)\mathrm{d}c \tag{2.4}$$

It is usually assumed without loss of generality that the kernel g has a total mass 1 (i.e. $I_1 = 1$) and recall that the 4 integrals are non negative since g is non negative.
By stationarity, for $k = 1, \dots, 4$, the moments $\mathbb{E}[X(x)^k]$ do not depend on the position $x \in T$ and the mixed moments of second order $\mathbb{E}[X(x)X(y)]$ depend only on $x - y$ for all $x, y \in T$. The first moments of a shot noise random fied are given in the following lemma which proof is postponed to the appendix.

Lemma 2.1. *Let $x \in T$ and $m_k = \mathbb{E}[M^k]$ for $k \in \{1, 2\}$ where M is a random variable with law F. Then under **(H)** and $I_1 = 1$ we have :*

- $M_1 := \mathbb{E}[X(0)] = \lambda m_1$;
- $h(x) := \mathbb{E}[X(x)X(0)] = \lambda m_2 I_2(x) + \lambda^2 m_1^2$;
- $f(x) := \mathrm{Cov}[X(x), X(0)] = \lambda m_2 I_2(x)$.

Note in particular that $M_2 := \mathbb{E}[X(0)^2] = h(0)$ and $V := \mathrm{Var}[X(0)] = f(0)$.
The limiting behaviour of a random field (or a sequence) of dependent random variables is of great interest. Roughly speaking, if the dependence between distant variables is small enough, then we will have our limit theorems such as the strong law of large numbers or the central limit theorem. The different notions of mixing (Rosenblatt, 1956; Bolthausen, 1982) are often used to characterise this weak dependance. However mixing properties are (in general) inaccessible for a shot noise process. The framework that will enable us to achieve results is that of the association that has been initially introduced by Esary et al. (1967) and for which a reference book is the one of Bulinski and Shashkin (2007). For $n \in \mathbb{N}$, we note $\mathcal{M}(n)$ the class of bounded coordinate-wise nondecreasing Borel function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathcal{M} = \cup_{n\in\mathbb{N}}\mathcal{M}(n)$. We note also $X_I = \{X(x_i),\ x_i \in I\}$ for $I \subset T$ with finite cardinal $|I|$.

Definition 2.2 (Associated random field)**.** A random field $X = \{X(x), x \in T\}$ is associated if for all finite set $I \subset T$ and $\phi, \psi \in \mathcal{M}(|I|)$, one has

$$\mathrm{Cov}(\phi(X_I), \psi(X_I)) \geq 0.$$

Note in particular that the covariance of 2 variables of the random field are necessarily non negative.

The starting point of this work is the following result due to Bulinski and Shashkin (2007). The proof is based on the property of independence of the underlying Poisson process.

Theorem 2.3. *Let X be a shot noise random field defined in (1.1) such that $M \geq 0$ a.s. and that g is a mesurable non negative function. Assume that $X(x) < \infty$ a.s. for any $x \in \mathbb{R}^2$.*
Then X is an associated random field.

Let us point out that Theorem 2.3 is true under our assumptions **(H)**. The finiteness condition is fulfilled because as M and g are integrable, our random field is also integrable.

The asymptotic results for associated random fields were mainly proved by Newman (1980, 1984) but we will rely on recent book of Bulinski and Shashkin (2007). The viewing window is equivalent to the sample of a traditional statistical study and asymptotic results for random field are based on the notion of growing set in the Van Hove sense (the VH-growing). For more details on this topic we refer to Bulinski and Shashkin (2007) but for example one may observe the random field on an image $U_n = [0, n-1]^2 \subset \mathbb{Z}^2$ with n^2 pixels.

We actually need to ascertain a multidimensional version of the association for two main reasons. The first one is that we want to study the correlation and the second one is that we are interested in results on several moments.

Definition 2.4 (Associated vector valued random field)**.** A random field $X = \{X(x), x \in T\}$ with values in $\mathbb{R}^k$ is associated if for all finite subset $I \subset T$ and $\phi, \psi \in \mathcal{M}(k|I|)$, one has

$$\mathrm{Cov}(\phi(X_I), \psi(X_I)) \geq 0.$$

The following result is given by Bulinski and Shashkin (2007) under the (BL,θ)-dependance assumption [1] but in the case where the random field is associated, square-integrable and check the finite susceptibility condition of Newman (1980) then it satisfies the (BL,θ)-dependance.

Theorem 2.5 (Corollary 1.13. of Bulinski and Shashkin (2007))**.** *Let $X = \{X(x), x \in \mathbb{Z}^2\}$ be an associated, strictly stationary, centered, square-integrable vector valued random field taking values in $\mathbb{R}^k$. Let us note*

$$C_{lm} = \sum_{z \in \mathbb{Z}^2} \mathrm{Cov}(X^l(0), X^m(z)), \qquad l, m = 1, \ldots, k;$$

all the cross-covariance and suppose that all series are convergent. Then, for any sequence of regularly growing sets $U_n \subset \mathbb{Z}^2$, as $n \to \infty$ one has

$$|U_n|^{-1/2} \sum_{x \in U_n} X(x) \xrightarrow{d} \mathcal{N}(0, C)$$

Here C is the $k \times k$ matrix with coefficients C_{lm}.

This is the conditions of convergence which are known as the finite susceptibility condition of Newman (1980).
Note that if X is an associated random field, then for all constant $c \in \mathbb{R}$ the random field $X + c$ is still associated and therefore the centering condition is not a problem. Indeed, for all $f \in \mathcal{M}(n)$ the function $f(\cdot + c)$ is still a bounded and coordinate-wise nondecreasing function. It is now possible to state our first theorem for which we introduce the following notation and recall that all these

[1] See Bulinski and Shashkin (2007) for the definition of (BL, θ)-dependance.

series have positive terms.

Notation: for all $w_1, w_2 \in \mathbb{Z}^2$ we write

$$S(2) \coloneqq \sum_{z\in\mathbb{Z}^2} I_2(z) \tag{2.5}$$

$$S(2,2,w_1) \coloneqq \sum_{z\in\mathbb{Z}^2} I_2(z)I_2(z+w_1) \tag{2.6}$$

$$S(3,w_1) \coloneqq \sum_{z\in\mathbb{Z}^2} I_3(z,w_1) \tag{2.7}$$

$$S(4,w_1,w_2) \coloneqq \sum_{z\in\mathbb{Z}^2} I_4(z,z+w_1,w_2) \tag{2.8}$$

Theorem 2.6. *Let X be a shot noise random field satisfying hypotheses **(H)** and the finite susceptibility condition $\sum_{z\in\mathbb{Z}^2} I_2(z) < \infty$. Let also $(U_n)_n$ be a sequence of regularly growing sets,*

$$\widehat{M_1} = \frac{1}{|U_n|}\sum_{x\in U_n} X(x) \tag{2.9}$$

be the estimators of the first moments and, for $w \in \mathbb{Z}^2$, let

$$\widehat{h_n(w)} = \frac{1}{|U_n \ominus w|}\sum_{x\in U_n\ominus w} X(x)X(x+w) \tag{2.10}$$

be the estimators of mixed moment of second order, with $U_n \ominus w = \{x \in U_n \ ; \ x+w \in U_n\}$. Then $\widehat{M_1}$ (resp. $\widehat{h_n(w)}$) are unbiased and strongly consistent estimators of M_1 (resp. $h(w)$).

Furthermore, for $k > 0$ and $w_1, \dots, w_k \in \mathbb{Z}^2$, noting $w_0 = 0$, as $n \to \infty$ we have :

$$\sqrt{|U_n|}\left(\widehat{M_1} - M_1, \widehat{h_n(w_0)} - h(w_0), \widehat{h_n(w_1)} - h(w_1), \dots, \widehat{h_n(w_k)} - h(w_k)\right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{1,h(w_0,\dots,w_k)})$$

where for $i,j = 1,\dots,k$ the coefficients $a_{i,j}$ of the matrix $\Sigma_{1,h(w_0,\dots,w_k)}$ are given by ;

- $a_{1,1} = \sigma_1^2 = \lambda m_2 S(2)$,
- $a_{1,i+2} = \sigma^2_{1,h(w_i)} = m_3\lambda S(3,w_i) + 2m_2m_1\lambda^2 S(2)$,
- $a_{i+2,j+2} = \sigma^2_{h(w_i),h(w_j)} = m_4\lambda S(4,w_j,w_i) + 2m_3m_1\lambda^2\Big(S(3,w_j) + S(3,w_i)\Big)$
$$+ m_2^2\lambda^2\Big(S(2,2,w_i+w_j) + S(2,2,w_i-w_j)\Big) + 4m_2m_1^2\lambda^3 S(2).$$

Proof: Let us start with the the computational but important lemma whose proof is postponed to Appendix and based on a repeated use of Campbell's theorem.

Lemma 2.7. *Let $x,y,s,t \in T$. Under **(H)** the third and fourth mixed moments of a shot noise random field X defined by (1.1) are respectively given by*

$$\mathbb{E}[X(x)X(y)X(s)] = m_3\lambda I_3(x-s,y-s) + m_2m_1\lambda^2(I_2(x-y) + I_2(x-s) + I_2(y-s)) + m_1^3\lambda^3, \tag{2.11}$$

and

$$\begin{aligned}
&\mathrm{Cov}(X(x)X(y), X(s)X(t)) = m_4\lambda I_4(x-t, y-t, s-t)\\
&+ m_3 m_1 \lambda^2 \Big(I_3(x-s, y-s) + I_3(x-t, y-t) + I_3(x-t, s-t) + I_3(y-t, s-t) \Big)\\
&+ m_2^2 \lambda^2 \Big(I_2(x-s)I_2(y-t) + I_2(x-t)I_2(y-s) \Big)\\
&+ m_2 m_1^2 \lambda^3 \Big(I_2(x-s) + I_2(x-t) + I_2(y-s) + I_2(y-t) \Big). \qquad (2.12)
\end{aligned}$$

Now return to the proof of Theorem 3. By Lemma 2.1 the convergence of $\sum_{z\in\mathbb{Z}^2} I_2(z)$ is equivalent to the convergence of $\sum_{z\in\mathbb{Z}^2} \mathrm{Cov}(X(0), X(z))$. By Theorem 3.2.1 of Guyon (1995) we have the consistency property of $\widehat{M_1}$. Let $w \in \mathbb{Z}^2$, we will prove the other convergences with the same theorem but this time we need the convergence of $\sum_{z\in\mathbb{Z}^2} \mathrm{Cov}(X(0)X(w), X(z)X(z+w))$. For this purpose, by Lemma 2.7 we have to check the convergence of $S(4, w, w)$, $S(3, w)$, $S(2, 2, 0)$ and $S(2, 2, 2w)$. To see the convergence of $S(4, w, w)$, just write $I_4(z, z+w, w) \leq ||g||_\infty^2 I_2(z)$ whose series is convergent. We have the same for the other series. The unbiased property is simply due to the stationarity of our random field.

To have the asymptotic normality we need to apply Theorem 2.5 and the following lemma :

Lemma 2.8. *Let $k > 0$, $w_0 = 0$ and $w_1, \dots, w_k \in T$. We denote also by Z^w the following vector valued random field*

$$Z^w(x) = (X(x), X(x)X(x+w_0), X(x)X(x+w_1), \dots, X(x)X(x+w_k))$$

Then Z^w is associated with values in $\mathbb{R}_+^{k+2}$.

Proof of Lemma 2.8: Let $I = \{x_1, \dots, x_n\} \in T$ be a finite set and $\phi, \psi \in \mathcal{M}((k+2)n)$. For $i = 1, \dots, n$ we note $X_{x_i:x_i+w_k} = (X(x_i), X(x_i+w_1), \dots, X(x_i+w_k))$ with values in $\mathbb{R}_+^{k+1}$. Then,

$$\begin{aligned}
\mathrm{Cov}(\phi(Z_I^w), \psi(Z_I^w)) &= \mathrm{Cov}\Big(\phi(Z^w(x_1), \dots, Z^w(x_n)), \psi(Z^w(x_1), \dots, Z^w(x_n)) \Big)\\
&= \mathrm{Cov}\Big(\phi \circ t_w(X_{x_1:x_1+w_k}, \dots, X_{x_n:x_n+w_k}), \psi \circ t_w(X_{x_1:x_1+w_k}, \dots, X_{x_n:x_n+w_k}) \Big)\\
&\geq 0
\end{aligned}$$

with

$$\begin{array}{rccc}
\tilde{t}_w & : & \mathbb{R}_+^{(k+1)} & \longrightarrow & \mathbb{R}^{(k+2)} \\
& & y = (y_i)_{1\leq i\leq k+1} & \mapsto & (y_1, y_1^2, y_2 y_1, \dots, y_{k+1} y_1)),
\end{array}$$

which we extend into a function on $\mathbb{R}^{(k+1)}$ with $\tilde{t}_w(y) = 0$ for all $y \notin \mathbb{R}_+^{(k+1)}$, and

$$\begin{array}{rccc}
t_w & : & \mathbb{R}^{(k+1)n} & \longrightarrow & \mathbb{R}^{(k+2)n} \\
& & y = (y^{(i)})_{1\leq i\leq n} & \mapsto & (\tilde{t}_w(y^{(i)}))_{1\leq i\leq n}.
\end{array}$$

The last line comes from the assocation of the random field X since the functions $\phi \circ t_w, \psi \circ t_w \in \mathcal{M}((k+1)n)$. Indeed, as ϕ and ψ are bounded, it is clear that those functions are bounded. Moreover, since t_w is coordinate-wise nondecreasing on $\mathbb{R}_+^{(k+1)n}$, these functions are also coordinate-wise nondecreasing on $\mathbb{R}_+^{(k+1)n}$, using the fact that X is positive almost surely this concludes the proof. □

Since X has a moments of order 4, Z^w has a moment of order 2 and Theorem 2.5 applies.
It remains to compute the asymptotic variances, where we mainly use Lemma 2.1. For the first one

$$\sigma_1^2 = \sum_{z\in\mathbb{Z}^2} \mathrm{Cov}(X(0), X(z)) = \sum_{z\in\mathbb{Z}^2} \lambda m_2 I_2(z) = \lambda m_2 S(2),$$

where $S(2)$ is defined by (2.5).
Moreover, for $i, j = 1, \dots, k$ we have :

$$\begin{aligned}
\sigma^2_{h(w_i),h(w_j)} &= \sum_{z\in\mathbb{Z}^2} \mathrm{Cov}(X(0)X(w_i), X(z)X(z+w_j)) \\
&= \sum_{z\in\mathbb{Z}^2} m_4\lambda I_4(z, z+w_j, w_i) \\
&+ \sum_{z\in\mathbb{Z}^2} m_3 m_1 \lambda^2 \Big(I_3(z-w_i, z+w_j-w_i) + I_3(z, z+w_j) + I_3(z, w_i) + I_3(z+w_j, w_i) \Big) \\
&+ \sum_{z\in\mathbb{Z}^2} m_2^2 \lambda^2 \Big(I_2(z-w_i) I_2(z+w_j) + I_2(z) I_2(z+w_j-w_i) \Big) \\
&+ \sum_{z\in\mathbb{Z}^2} m_2 m_1^2 \lambda^3 \Big(I_2(z-w_i) + I_2(z) + I_2(z+w_j-w_i) + I_2(z+w_j) \Big) \\
&= m_4\lambda S(4, w_j, w_i) + 2m_3 m_1 \lambda^2 \Big(S(3, w_j) + S(3, w_i) \Big) \\
&+ m_2^2\lambda^2 \Big(S(2, 2, w_i + w_j) + S(2, 2, w_i - w_j) \Big) + 4 m_2 m_1^2 \lambda^3 S(2),
\end{aligned}$$

using (2.5) - (2.8).
By first noticing that for $w_i, w_j \in \mathbb{Z}^2$ we have $I_3(z - w_i, z + w_j - w_i) = I_3(-z + w_i, w_j)$ and $I_3(z, z + w_j) = I_3(-z, w_j)$, it follows that

$$\begin{aligned}
&\sum_{z\in\mathbb{Z}^2} I_3(-z+w_i, w_j) = \sum_{z\in\mathbb{Z}^2} I_3(-z, w_j) = S(3, w_j); \\
&\sum_{z\in\mathbb{Z}^2} I_3(z, w_i) = \sum_{z\in\mathbb{Z}^2} I_3(z+w_j, w_i) = S(3, w_i); \\
&\sum_{z\in\mathbb{Z}^2} I_2(z-w_i) I_2(z+w_j) = \sum_{z\in\mathbb{Z}^2} I_2(z) I_2(z+w_i+w_j) = S(2, 2, w_i+w_j); \\
&\sum_{z\in\mathbb{Z}^2} I_2(z-w_i) = \sum_{z\in\mathbb{Z}^2} I_2(z) = \sum_{z\in\mathbb{Z}^2} I_2(z+w_j-w_i) = \sum_{z\in\mathbb{Z}^2} I_2(z+w_j) = S(2).
\end{aligned}$$

The computation for $\sigma^2_{1,h(w_i)}$ is similar. □

These estimators are important because they allow us to introduce other estimators such as $\widehat{V} = \widehat{h(0)} - \widehat{M_1}^2$, the variance estimator, and $\widehat{f_n(w)} = \widehat{h_n(w)} - \widehat{M_1}^2$, the covariance estimator. All

these estimators are biased because $\mathbb{E}[\widehat{M_1}^2] \neq M_1^2$, due to the correlation between variables, but asymptotically unbiased. Their strong consistency is straightforward since they are transformations of strongly consistent estimators by a continuous function. We also have the following proposition.

Proposition 2.9. *Let X be a shot noise random field satisfying hypotheses **(H)** and the finite susceptibility condition $\sum_{z\in\mathbb{Z}^2} I_2(z) < \infty$. Let also $(U_n)_n$ be a sequence of regularly growing sets, $w_0 = 0$ and $w_1, \dots, w_k \in \mathbb{Z}^2$ with $k > 0$. Then as $n \to \infty$ we have :*

$$\sqrt{|U_n|}\left(\widehat{M_1} - M_1, \widehat{f_n(w_0)} - f(w_0), \widehat{f_n(w_1)} - f(w_1), \dots, \widehat{f_n(w_k)} - f(w_k)\right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{1,f(w_0),\dots,f(w_k)})$$

where for $i, j = 1, \dots, k$ the coefficients $a_{i,j}$ of the matrix $\Sigma_{1,f(w_0),\dots,f(w_k)}$ are given by :

- $a_{1,i+2} = \sigma^2_{1,f(w_i)} = m_3\lambda S(3, w_i)$
- $a_{i+2,j+2} = \sigma^2_{f(w_i),f(w_j)} = m_4\lambda S(4, w_i, w_j) + m_2^2\lambda^2\Big(S(2, 2, w_i - w_j) + S(2, 2, w_i + w_j)\Big)$

Proof: Let ϕ be the smooth function defined by

$$\begin{array}{rccc} \phi & : & \mathbb{R}^{k+2} & \longrightarrow & \mathbb{R}^{k+2} \\ & & (x, z_0, z_1, \dots, z_k) & \mapsto & (x, z_0 - x^2, z_1 - x^2, \dots, z_k - x^2) \end{array}$$

then

$$(\widehat{M_1}, \widehat{f_n(w_0)}, \widehat{f_n(w_1)}, \dots, \widehat{f_n(w_k)}) = \phi(\widehat{M_1}, \widehat{h_n(w_0)}, \widehat{h_n(w_1)}, \dots, \widehat{h(w_k)})$$

and

$$(M_1, f(w_0), f(w_1), \dots, f(w_k)) = \phi(M_1, h(w_0), h(w_1), \dots, h(w_k)).$$

The jacobian matrix of ϕ is

$$\mathrm{D}_\phi(x, z_0, z_1, \dots, z_k) = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ -2x & 1 & 0 & \dots & \dots & 0 \\ -2x & 0 & 1 & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -2x & 0 & \dots & 0 & 1 & 0 \\ -2x & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

By the delta method (see Wasserman, 2010) and Theorem 2.6 we have

$$\Sigma_{1,f(w_0),\dots,f(w_k)} = \mathrm{D}_\phi(M_1, h(w_0, \dots, w_k))\Sigma_{1,h(w_0),\dots,h(w_k)}\mathrm{D}_\phi(M_1, h(w_0), \dots, h(w_k))^t$$

which is too cumbersome to be detailed. Nevertheless, to know a sub-matrix such as $\Sigma_{f(w_i),f(w_j)}$ we only need the sub-matrix $\Sigma_{1,h(w_i),h(w_j)}$ that leads to :

$$
\begin{aligned}
&\Sigma_{f(w_i),f(w_j)}\\
&= \begin{pmatrix} -2M_1 & 1 & 0 \\ -2M_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{1,h(w_i)}^2 & \sigma_{1,h(w_j)}^2 \\ \sigma_{1,h(w_i)}^2 & \sigma_{h(w_i)}^2 & \sigma_{h(w_i),h(w_j)}^2 \\ \sigma_{1,h(w_j)}^2 & \sigma_{h(w_i),h(w_j)}^2 & \sigma_{h(w_j)}^2 \end{pmatrix} \begin{pmatrix} -2M_1 & -2M_1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\\
&= \begin{pmatrix} -2M_1 & 1 & 0 \\ -2M_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2M_1\sigma_1^2 + \sigma_{1,h(w_i)}^2 & -2M_1\sigma_1^2 + \sigma_{1,h(w_j)}^2 \\ -2M_1\sigma_{1,h(w_i)}^2 + \sigma_{h(w_i)}^2 & -2M_1\sigma_{1,h(w_i)}^2 + \sigma_{h(w_i),h(w_j)}^2 \\ -2M_1\sigma_{1,h(w_j)}^2 + \sigma_{h(w_i),h(w_j)}^2 & -2M_1\sigma_{1,h(w_j)}^2 + \sigma_{h(w_j)}^2 \end{pmatrix}\\
&= \begin{pmatrix} 4M_1^2\sigma_1^2 - 4M_1\sigma_{1,h(w_i)}^2 + \sigma_{h(w_i)}^2 & 4M_1^2\sigma_1^2 - 2M_1(\sigma_{1,h(w_i)}^2 + \sigma_{1,h(w_j)}^2) + \sigma_{h(w_i),h(w_j)}^2 \\ 4M_1^2\sigma_1^2 - 2M_1(\sigma_{1,h(w_i)}^2 + \sigma_{1,h(w_j)}^2) + \sigma_{h(w_i),h(w_j)}^2 & 4M_1^2\sigma_1^2 - 4M_1\sigma_{1,h(w_j)}^2 + \sigma_{h(w_j)}^2. \end{pmatrix}
\end{aligned}
$$

By developing the calculations we then have

$$
\begin{aligned}
\sigma_{f(w_i),f(w_j)}^2 &= 4M_1^2\sigma_1^2 - 2M_1(\sigma_{1,h(w_i)}^2 + \sigma_{1,h(w_j)}^2) + \sigma_{h(w_i),h(w_j)}^2\\
&= 4\lambda^2 m_1^2 m_2 \lambda S(2) - 2\lambda m_1(m_3\lambda S(3,w_i) + 2m_2 m_1\lambda^2 S(2) + m_3\lambda S(3,w_j) + 2m_2 m_1\lambda^2 S(2))\\
&\quad + m_4\lambda S(4,w_j,w_i) + 2m_3 m_1\lambda^2\Big(S(3,w_j) + S(3,w_i)\Big)\\
&\quad + m_2^2\lambda^2\Big(S(2,2,w_i-w_j) + S(2,2,w_i+w_j)\Big) + 4m_2 m_1^2\lambda^3 S(2)\\
&= m_4\lambda S(4,w_i,w_j) + m_2^2\lambda^2\Big(S(2,2,w_i-w_j) + S(2,2,w_i+w_j)\Big).
\end{aligned}
$$

□

Another very important function for this model is the correlation function, *i.e.* the covariance function normalized by the variance, that we note $\rho(w) = \frac{f(w)}{f(0)}$. The important point is that this function does not depend on the Poisson process (neither on λ, nor on the mark) because $\rho(w) = \frac{I_2(w)}{I_2(0)}$ by Lemma 2.1. It is therefore a very useful statistic to infer about the kernel itself. This function is estimated by $\widehat{\rho(w)} = \frac{\widehat{f(w)}}{\widehat{f(0)}}$ which is strongly consistent since it is a transformation of strongly consistent estimators by a continuous function and we also have the following central limit theorem (CLT).

Proposition 2.10. *Let X be a shot noise random field satisfying hypotheses **(H)** and the finite susceptibility condition $\sum_{z\in\mathbb{Z}^2} I_2(z) < \infty$. Let also $(U_n)_n$ be a sequence of regularly growing sets and $w_0 = 0$. Then as $n \to \infty$ we have :*

$$
\sqrt{|U_n|}\left(\widehat{M_1} - M_1, \widehat{f_n(w_0)} - f(w_0), \widehat{\rho_n(w_1)} - \rho(w_1), \dots, \widehat{\rho_n(w_k)} - \rho(w_k)\right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{1,f(w_0),\rho(w_1),\dots,\rho(w_k)})
$$

where for $i,j = 1,\dots,k$ the coefficients $a_{i,j}$ of the matrix $\Sigma_{1,f(w_0),\rho(w_1),\dots,\rho(w_k)}$ are given by :

- $a_{1,i+2} = \sigma_{1,\rho(w_i)}^2 = \frac{m_3}{m_2 I_2(0)}(S(3,w_i) - S(3,0)\frac{I_2(w_i)}{I_2(0)})$

- $a_{2,i+2} = \sigma^2_{f(w_0),\rho(w_i)} = \frac{2m_2\lambda}{I_2(0)}(S(2,2,w_i) - S(2,2,0)\frac{I_2(w_i)}{I_2(0)})$
$$+ \frac{m_4}{m_2 I_2(0)}(S(4,0,w_i) - S(4,0,0)\frac{I_2(w_i)}{I_2(0)})$$
- $a_{i+2,j+2} = \sigma^2_{\rho(w_i),\rho(w_j)}$
$$= (m_4\lambda S(4,0,0) + 2m_2^2\lambda^2 S(2,2,0))\frac{I_2(w_i)I_2(w_j)}{\lambda^2 m_2^2 I_2(0)^4}$$
$$- (m_4\lambda S(4,0,w_i) + 2m_2^2\lambda^2 S(2,2,w_i))\frac{I_2(w_j)}{\lambda^2 m_2^2 I_2(0)^3}$$
$$- (m_4\lambda S(4,0,w_j) + 2m_2^2\lambda^2 S(2,2,w_j))\frac{I_2(w_i)}{\lambda^2 m_2^2 I_2(0)^3}$$
$$+ (m_4\lambda S(4,w_i,w_j) + m_2^2\lambda^2\Big(S(2,2,w_i - w_j) + S(2,2,w_i + w_j)\Big))\frac{1}{\lambda^2 m_2^2 I_2(0)^2}$$

Proof: We have the asymptotic normality by the same argument as for Proposition 2.9. To know a sub-matrix $\Sigma^2_{\rho(w_i),\rho(w_j)}$ we only need the sub-matrix

$$\Sigma_{f(0),f(w_i),f(w_j)} = \begin{pmatrix} \sigma^2_{f(0)} & \sigma^2_{f(0),f(w_i)} & \sigma^2_{f(0),f(w_j)} \\ \sigma^2_{f(0),f(w_i)} & \sigma^2_{f(w_i)} & \sigma^2_{f(w_i),f(w_j)} \\ \sigma^2_{f(0),f(w_j)} & \sigma^2_{f(w_i),f(w_j)} & \sigma^2_{f(w_j)}. \end{pmatrix}$$

Let ϕ be the smooth function defined by

$$\begin{array}{rccc} \phi & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & & (x,y,z) & \mapsto & (\frac{y}{x},\frac{z}{x}) \end{array}$$

then by the delta method we have

$$\begin{aligned}
&\Sigma_{\rho(w_1),\rho(w_2)} \\
&= \mathrm{D}_\phi(f(0),f(w_i),f(w_j))\Sigma_{f(0),f(w_i),f(w_j)}\mathrm{D}_\phi(f(0),f(w_i),f(w_j))^t \\
&= \begin{pmatrix} \frac{-f(w_i)}{f(0)^2} & \frac{1}{f(0)} & 0 \\ \frac{-f(w_j)}{f(0)^2} & 0 & \frac{1}{f(0)} \end{pmatrix} \begin{pmatrix} \sigma^2_{f(0)} & \sigma^2_{f(0),f(w_i)} & \sigma^2_{f(0),f(w_j)} \\ \sigma^2_{f(0),f(w_i)} & \sigma^2_{f(w_i)} & \sigma^2_{f(w_i),f(w_j)} \\ \sigma^2_{f(0),f(w_j)} & \sigma^2_{f(w_i),f(w_j)} & \sigma^2_{f(w_j)} \end{pmatrix} \begin{pmatrix} \frac{-f(w_i)}{f(0)^2} & \frac{-f(w_j)}{f(0)^2} \\ \frac{1}{f(0)} & 0 \\ 0 & \frac{1}{f(0)} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-f(w_i)}{f(0)^2} & \frac{1}{f(0)} & 0 \\ \frac{-f(w_j)}{f(0)^2} & 0 & \frac{1}{f(0)} \end{pmatrix} \begin{pmatrix} \frac{-f(w_i)}{f(0)^2}\sigma^2_{f(0)} + \frac{1}{f(0)}\sigma^2_{f(0),f(w_i)} & \frac{-f(w_j)}{f(0)^2}\sigma^2_{f(0)} + \frac{1}{f(0)}\sigma^2_{f(0),f(w_j)} \\ \frac{-f(w_i)}{f(0)^2}\sigma^2_{f(0),f(w_i)} + \frac{1}{f(0)}\sigma^2_{f(w_i)} & \frac{-f(w_j)}{f(0)^2}\sigma^2_{f(0),f(w_i)} + \frac{1}{f(0)}\sigma^2_{f(w_i),f(w_j)} \\ \frac{-f(w_i)}{f(0)^2}\sigma^2_{f(0),f(w_j)} + \frac{1}{f(0)}\sigma^2_{f(w_i),f(w_j)} & \frac{-f(w_j)}{f(0)^2}\sigma^2_{f(0),f(w_j)} + \frac{1}{f(0)}\sigma^2_{f(w_j)} \end{pmatrix} \\
&= \begin{pmatrix} \sigma^2_{\rho(w_i)} & \sigma^2_{\rho(w_i),\rho(w_j)} \\ \sigma^2_{\rho(w_i),\rho(w_j)} & \sigma^2_{\rho(w_j)} \end{pmatrix},
\end{aligned}$$

with

$$\begin{aligned}\sigma^2_{\rho(w_i),\rho(w_j)} &= \frac{f(w_i)f(w_j)}{f(0)^4}\sigma^2_{f(0)} - \frac{f(w_j)}{f(0)^3}\sigma^2_{f(0),f(w_i)} - \frac{f(w_i)}{f(0)^3}\sigma^2_{f(0),f(w_j)} + \frac{1}{f(0)^2}\sigma^2_{f(w_i),f(w_j)}\\ &= (m_4\lambda S(4,0,0) + 2m_2^2\lambda^2 S(2,2,0))\frac{I_2(w_i)I_2(w_j)}{\lambda^2 m_2^2 I_2(0)^4}\\ &- (m_4\lambda S(4,0,w_i) + 2m_2^2\lambda^2 S(2,2,w_i))\frac{I_2(w_j)}{\lambda^2 m_2^2 I_2(0)^3}\\ &- (m_4\lambda S(4,0,w_j) + 2m_2^2\lambda^2 S(2,2,w_j))\frac{I_2(w_i)}{\lambda^2 m_2^2 I_2(0)^3}\\ &+ (m_4\lambda S(4,w_i,w_j) + m_2^2\lambda^2\Big(S(2,2,w_i-w_j) + S(2,2,w_i+w_j)\Big))\frac{1}{\lambda^2 m_2^2 I_2(0)^2}.\end{aligned}$$

□

3. Isotropic shot noise random field

In this section, we consider the case where the kernel g is isotropic. We immediately note that this implies that the random field X is isotropic: let $n \geq 1$, $x_1, \ldots, x_n \in T$ and R a rotation matrix, then by Campbell's theorem (see Kingman, 1967) we have for all $(u_1, \ldots, u_n) \in \mathbb{R}^n$

$$\begin{aligned}\mathbb{E}\Big[e^{iu_1X(x_1)+\ldots+iu_nX(x_n)}\Big] &= \exp\Big(-\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}[1-exp(i\sum_{l=1}^n u_l m g(x_l-c))]\lambda \mathrm{d}c\mathrm{d}F(m)\Big)\\ &= \exp\Big(-\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}[1-exp(i\sum_{l=1}^n u_l m g(Rx_l-Rc))]\lambda \mathrm{d}c\mathrm{d}F(m)\Big)\\ &= \exp\Big(-\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}[1-exp(i\sum_{l=1}^n u_l m g(Rx_l-c'))]\lambda \mathrm{d}c'\mathrm{d}F(m)\Big)\\ &= \mathbb{E}\Big[e^{iu_1X(Rx_1)+\ldots+iu_nX(Rx_n)}\Big],\end{aligned}$$

and therefore $X \circ R \overset{fdd}{=} X$.

Due to isotropy, the mixed moment function h is reduced to a function $\tilde{h}$ defined on $\mathbb{R}_+$ such that for all $w \in T$ we have $h(w) = \tilde{h}(||w||)$, and as for the variance reduction techniques by antithetic variables, the estimators of mixed moment can be improved taking into account isotropy. Concretely, noting (e_1, e_2) the canonical basis of $\mathbb{R}^2$, for all $r \in \mathbb{Z}_+$ there are 2 vectors $w_1 = re_1$ and $w_2 = re_2$ in $\mathbb{Z}^2$ such that $||w_1|| = ||w_2|| = r$ so we use the isotropic estimator

$$\widehat{\tilde{h}_n(r)} = \frac{1}{2}(\widehat{h_n(re_1)} + \widehat{h_n(re_2)}). \tag{3.1}$$

We do not count $-re_1$ and $-re_2$ because even without isotropy $h(-re_1) = h(re_1)$ and $h(-re_2) = h(re_2)$. Anyway since we have $\widehat{h_n(re_1)} = \widehat{h_n(-re_1)}$ and $\widehat{h_n(re_2)} = \widehat{h_n(-re_2)}$, it would not bring anything more. Note that there are other values of $r \notin \mathbb{Z}_+$ such that there are several $w \in \mathbb{Z}^2$ with norm r but for this article, estimating the second order moments on the integers is more than enough. Let us also note that for some values of r there are more than 4 elements of $\mathbb{Z}^2$ with norm

r but for this paper we did not go in that way because simplifications are only possible when there are only two orthogonal vectors $w \in \mathbb{Z}^2$.

Proposition 3.1. *Let X be a shot noise random field satisfying hypotheses **(H)** and the finite susceptibility condition $\sum_{z\in\mathbb{Z}^2} I_2(z) < \infty$. Let also $(U_n)_n$ be a sequence of regularly growing sets, $r_0 = 0$ $k > 0$ and $r_1, \dots, r_k \in \mathbb{N}$. Then considering $\tilde{h}_n$ given by 3.1 the vector*

$$\left(\widehat{M_1} - M_1, \widehat{\tilde{h}_n(r_0)} - \tilde{h}(r_0), \widehat{\tilde{h}_n(r_1)} - \tilde{h}(r_1), \dots, \widehat{\tilde{h}_n(r_k)} - \tilde{h}(r_k)\right)$$

is asymptotically normal with asymptotic variance given by :

- $\sigma^2_{1,\tilde{h}(r_i)} = \sigma^2_{1,h(r_i e_1)}$
- $\sigma^2_{\tilde{h}(r_i),\tilde{h}(r_j)} = 2m_3 m_1 \lambda^2 \left(S(3, r_j e_1) + S(3, r_i e_1)\right) + 4m_2 m_1^2 \lambda^3 S(2)$
$$+ \frac{1}{4}\sum_{p=1}^{2}\sum_{q=1}^{2} m_4 \lambda S(4, r_j e_q, r_i e_p) + m_2^2 \lambda^2 \left(S(2, 2, -r_i e_p, r_j e_q) + S(2, 2, r_i e_p, r_j e_q)\right)$$

and the asymptotic variance of $\widehat{\tilde{h}_n(r)}$ is smaller than that obtained in the case of directional estimation in Theorem 2.6.

Proof: The asymptotic normality comes from the asymptotic normality of $(\widehat{M_1}, \widehat{h_n(0)}, \dots, \widehat{h_n(w_{2k})})$ in Theorem 2.6 where the $2k$ elements $w_1, \dots, w_{2k}$ of $\mathbb{Z}^2$ are $r_1 e_1, r_1 e_2, \dots, r_i e_1, r_i e_2, \dots, r_k e_1, r_k e_2$. Before computing asymptotics variances, let us give the following lemma :

Lemma 3.2. *Let g be an isotropic kernel and $r \in \mathbb{Z}_+$. Then we have the following identities*

$$\begin{aligned}
S(4, re_1, re_1) &= S(4, re_2, re_2)\\
S(4, re_1, re_2) &= S(4, re_2, -re_1)\\
S(4, 0, re_1) &= S(4, 0, re_2)\\
S(3, re_1) &= S(3, re_2)\\
S(2, 2, re_1) &= S(2, 2, re_2)\\
S(2, 2, re_1 - re_2) &= S(2, 2, re_1 + re_2).
\end{aligned}$$

Proof: Let R be the rotation matrix such that $R(e_1) = e_2$. Then

$$\begin{aligned}
S(4, re_1, re_1) &= \sum_{z\in\mathbb{Z}^2} \int_{\mathbb{R}^2} g(c - z) g(c - z - re_1) g(c - re_1) g(c) \mathrm{d}c\\
&= \sum_{z\in\mathbb{Z}^2} \int_{\mathbb{R}^2} g(Rc - Rz) g(Rc - Rz - re_2) g(Rc - re_2) g(Rc) \mathrm{d}c\\
&= \sum_{z'\in\mathbb{Z}^2} \int_{\mathbb{R}^2} g(a - z') g(a - z' - re_2) g(a - re_2) g(a) \mathrm{d}a\\
&= S(4, re_2, re_2)
\end{aligned}$$

by isotropy of g and by changes of variables $a = Rc$, $z' = Rz$. Let us note here that for this rotation matrix R we have got $z' \in \mathbb{Z}^2$. Similar computations lead to the other equalities. □

Now by this lemma, since the asymptotic variance $\sigma^2_{1,h(w)}$ depends only on $S(3,w)$ and $S(2)$ we can directly affirm that

$$\sigma^2_{1,\tilde{h}(r_i)} = \frac{1}{2}\Big(\sigma^2_{1,h(r_ie_1)} + \sigma^2_{1,h(r_ie_2)}\Big) = \sigma^2_{1,h(r_ie_1)}$$

we also have

$$\begin{aligned}
\sigma^2_{\tilde{h}(r_i),\tilde{h}(r_j)} &= \frac{1}{4}\sum_{p=1}^{2}\sum_{q=1}^{2}\sigma^2_{h(r_ie_p),h(r_je_q)} \\
&= \frac{1}{4}\sum_{p=1}^{2}\sum_{q=1}^{2} m_4\lambda S(4, r_je_q, r_ie_p) + 2m_3m_1\lambda^2\Big(S(3,r_je_q) + S(3,r_ie_p)\Big) \\
&+ m_2^2\lambda^2\Big(S(2,2,r_ie_p - r_je_q) + S(2,2,r_ie_p - r_je_q)\Big) + 4m_2m_1^2\lambda^3 S(2) \\
&= 2m_3m_1\lambda^2\Big(S(3,r_je_1) + S(3,r_ie_1)\Big) + 4m_2m_1^2\lambda^3 S(2) \\
&+ \frac{1}{4}\sum_{p=1}^{2}\sum_{q=1}^{2} m_4\lambda S(4, r_je_q, r_ie_p) + m_2^2\lambda^2\Big(S(2,2,r_ie_p + r_je_q) + S(2,2,r_ie_p - r_je_q)\Big)
\end{aligned}$$

then $\sigma^2_{\tilde{h}(r_i)}$ is then obtained as a special case. To see the benefit on the asymptotic variance directly in this formula is not simple but as $\forall r \in \mathbb{Z}_+$ we have

$$\mathrm{Cov}(\widehat{h_n(re_1)}, \widehat{h_n(re_2)}) \leq \sqrt{\mathrm{Var}(\widehat{h_n(re_1)})\mathrm{Var}(\widehat{h_n(re_2)})} = \mathrm{Var}(\widehat{h_n(re_1)})$$

we can write

$$\mathrm{Var}(\widehat{\tilde{h}_n(r)}) = \frac{1}{4}(\mathrm{Var}(\widehat{h_n(re_1)}) + \mathrm{Var}(\widehat{h_n(re_2)}) + 2\mathrm{Cov}(\widehat{h_n(re_1)}, \widehat{h_n(re_2)}) \leq \mathrm{Var}(\widehat{h_n(re_1)})$$

□

Let us observe that the gain on the variance is more and more important with increasing r, up to a factor of 2 when $r \to \infty$.

The same is true considering the covariance function $\tilde{f}$ and corelation function $\tilde{\rho}$ which are now functions defined on $\mathbb{Z}_+$. The isotropic estimator of $\tilde{f}(r)$ is given by

$$\widehat{\tilde{f}_n(r)} = \frac{1}{2}(\widehat{f_n(re_1)} + \widehat{f_n(re_2)}) \tag{3.2}$$

and that one for $\tilde{\rho}(r)$ is

$$\widehat{\tilde{\rho}_n(r)} = \frac{1}{2}(\widehat{\rho_n(re_1)} + \widehat{\rho_n(re_2)}) \tag{3.3}$$

These isotropic versions have a smaller asymptotic variance than their directional counterpart. More precisely, the asymptotic variance of $\widehat{\tilde{f}_n(r)}$ is

$$\begin{aligned}
\sigma^2_{\tilde{f}(r)} &= \frac{m_4\lambda}{2}\Big(S(4, re_1, re_1) + S(4, re_1, re_2)\Big) \\
&+ \frac{m_2^2\lambda^2}{2}\Big(S(2,2,0) + S(2,2,2re_1) + 2S(2,2,re_1 - re_2)\Big)
\end{aligned}$$

and that of $\widehat{\tilde{\rho}_n(r)}$ is

$$\sigma^2_{\widehat{\tilde{\rho}(r)}} = \frac{m_4}{\lambda m_2^2 I_2(0)^2}(S(4,0,0)\frac{I_2(r)^2}{I_2(0)^2} - 2S(4,0,re_1)\frac{I_2(r)}{I_2(0)} + \frac{1}{2}S(4,re_1,re_1) + \frac{1}{2}S(4,re_1,re_2))$$
$$+ \frac{1}{I_2(0)^2}(2S(2,2,0)\frac{I_2(r)^2}{I_2(0)^2} - 2S(2,2,re_1)\frac{I_2(r)}{I_2(0)} + \frac{1}{2}S(2,2,0) + \frac{1}{2}S(2,2,re_1) + S(2,2,re_1 - re_2)) \tag{3.4}$$

4. Parameters estimation on Gaussian shot noise random field

In this section we want to apply the previous results to the estimation of the parameters of a spatial shot noise random field defined in (1.1) by

$$\forall x \in T, \quad X(x) = \sum_{(x_i,m_i)\in\phi_X} m_i g(x - x_i), \tag{4.1}$$

and we will focus exclusively on the isotropic case with a Gaussian kernel

$$g(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{||x||^2}{2\sigma^2}} \tag{4.2}$$

with standard deviation σ. This is an example of a smooth random field since it is of class $\mathcal{C}^\infty$ almost surely. The main motivation of this kernel is that by considering σ^2 as a temporal variable t, the spatio-temporal random field is a fundamental solution of the heat equation

$$\frac{\partial X}{\partial t} = \Delta X,$$

where $\Delta = \partial^2_{11} + \partial^2_{22}$ is the classical 2d Laplace operator. This can be seen by comparing the images on the right where $\sigma = 5$ with the one on the left where $\sigma = 50$ in Figure 4.1. In this space-time model, the points of the Poisson process are only drawn once at t_0 and do not move. At first the points are isolated and then over time the Gaussians become more and more spread out and meet each other. It is the heat that spreads and the texture becomes more structured.
Thanks to Campbell's Theorem we know the marginal law of a shot noise random field in integral form, but in general we can't explicitly compute it. One exception is when considering this Gaussian kernel with an exponential mark of parameter μ, then the marginal is of Gamma law $\Gamma(2\pi\lambda\sigma^2, 2\pi\mu\sigma^2)$. This model satisfies **(H)** and we note it $X_{\lambda,\mu,\sigma}$. To illustrate this model we have simulated with Matlab four shot noise random fields with different parameters (see Figure 4.1).

With a simple application of Campbell's theorem, we also have an interesting scale invariance property. More precisely, for $k > 0$, let $X_{\lambda k^2,\mu k^2,\frac{\sigma}{k}}$ be a Gaussian shot noise random fiel with exponential mark of parameter $(\lambda k^2, \mu k^2, \frac{\sigma}{k})$ and $X_{\lambda,\mu,\sigma}$ be a Gaussian shot noise random field with exponential mark of parameter (λ, μ, σ). Then we have

$$\{X_{\lambda k^2,\mu k^2,\frac{\sigma}{k}}(x) \mid x \in \mathbb{R}^2\} \overset{fdd}{=} \{X_{\lambda,\mu,\sigma}(kx) \mid x \in \mathbb{R}^2\}$$

This property can be seen as a zoom on a part of an image where the k plays the role of the discretization step.

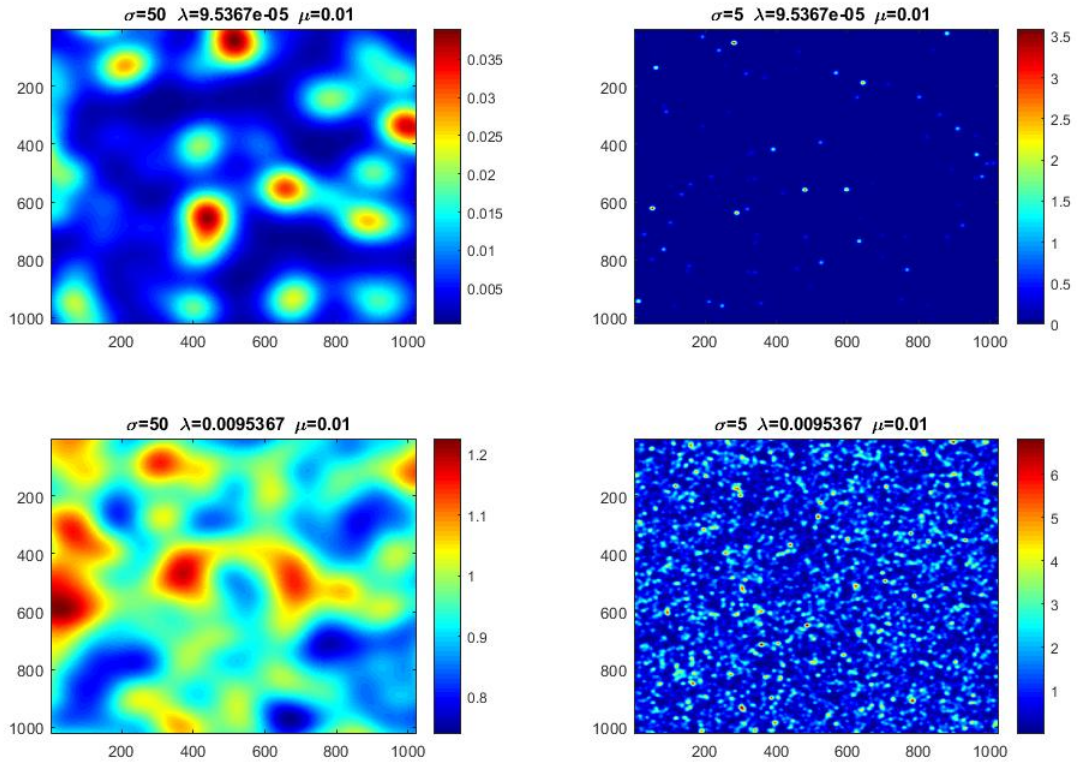

FIGURE 4.1. Four images with 1024x1024 pixels. On the first line, both images have the same $\lambda = \frac{1000}{1024^2}$ which represents an average of 1000 points on the image. On the second line, we have chosen λ to have an average of 100000 points. On the left, both images have the same $\sigma = 50$ while $\sigma = 5$ on the right. Marks are exponential laws of parameter $\mu = \frac{1}{100}$ which represents a average of 100.

We have that $I_2(x) = \frac{1}{4\pi\sigma^2} e^{-\frac{1}{4\pi\sigma^2}||x||^2}$ (proof in Appendix of Lemma 4.2) and consequently the finite susceptibility condition is checked. Let us recall that the moments of an exponential law are $m_k = \frac{k!}{\mu^k}$, thus the moments of the random field are given for $x \in \mathbb{R}^2$ by :

- $M_1 = \frac{\lambda}{\mu}$;
- $h(x) = \frac{\lambda}{2\pi\mu^2\sigma^2} e^{-\frac{1}{4\sigma^2}||x||^2} + \frac{\lambda^2}{\mu^2}$;
- $f(x) = \frac{\lambda}{2\pi\mu^2\sigma^2} e^{-\frac{1}{4\sigma^2}||x||^2}$;
- $\rho(x) = e^{-\frac{1}{4\sigma^2}||x||^2}$.

The main aim of this section is to estimate the triplet of parameters (λ, μ, σ) of this model. Let's start by assuming we know σ, then a simple inversion of

$$\begin{cases} M_1 &= \dfrac{\lambda}{\mu} \\ f(0) &= \dfrac{\lambda}{2\pi\mu^2\sigma^2} \end{cases}$$

allow us to estimate (λ, μ) by

$$\begin{cases} \widehat{\lambda} &= \dfrac{\widehat{M_1}^2}{2\pi\sigma^2\widehat{f(0)}} \\ \widehat{\mu} &= \dfrac{\widehat{M_1}}{2\pi\sigma^2\widehat{f(0)}} \end{cases}$$

Estimating higher-order raw moments does not allow us to have 3 independent equations. To capture σ, it is necessary to take into account the mixed moment of second order, and more particularly the correlation that allows to isolate σ. As $\tilde{\rho}(r) = e^{-\frac{r^2}{4\sigma^2}}$ we can get an estimate of σ for different values of $r \in \mathbb{Z}_+^*$ by considering

$$\widehat{\sigma_r} = \sqrt{\frac{-r^2}{4\log(\widehat{\tilde{\rho}_n(r)})}}.$$

Note that we can consider this estimator only if $\widehat{\rho_n(r)}$ is stricty between 0 and 1. Once σ is estimated, we keep the same estimators $\widehat{\lambda}$ and $\widehat{\mu}$ as if σ was known. We then propose:

$$\begin{cases} \widehat{\lambda_r} &= \dfrac{-\widehat{M_1}^2 2\log(\widehat{\tilde{\rho}_n(r)})}{\pi r^2(\widehat{M_2} - \widehat{M_1}^2)} \\ \widehat{\mu_r} &= \dfrac{-\widehat{M_1} 2\log(\widehat{\tilde{\rho}_n(r)})}{\pi r^2(\widehat{M_2} - \widehat{M_1}^2)} \\ \widehat{\sigma_r} &= \sqrt{\dfrac{-r^2}{4\log(\widehat{\tilde{\rho}_n(r)})}} \end{cases} \tag{4.3}$$

Theorem 4.1. *Let $X_{\lambda,\mu\sigma}$ be a shot noise random field with exponential mark et and Gaussian kernel. Then $\forall r \in \mathbb{Z}_+^*$, the estimators $(\widehat{\lambda_r}, \widehat{\mu_r}, \widehat{\sigma_r})$ given by (4.3) are consistent estimators of (λ, μ, σ). Moreover, for $(U_n)_n$ a sequence of regularly growing sets, as $n \to \infty$ one has*

$$\sqrt{|U_n|}\begin{pmatrix} \widehat{\lambda_r} - \lambda \\ \widehat{\mu_r} - \mu \\ \widehat{\sigma_r} - \sigma \end{pmatrix} \to \mathcal{N}(0, \Sigma_{\lambda_r,\mu_r,\sigma_r})$$

with

$$\Sigma_{\mu,\lambda,\sigma(r)} = \mathrm{D}_\phi(M_1, V, \tilde{\rho}(r))\Sigma_{1,V,\tilde{\rho}(r)}\mathrm{D}_\phi(M_1, V, \tilde{\rho}(r))^t$$

where $D_\phi(x,y,z) = \begin{pmatrix} \frac{-4x\log(z)}{\pi r^2 y} & \frac{2x^2\log(z)}{\pi r^2 y^2} & \frac{-2x^2}{\pi r^2 yz} \\ \frac{-2\log(z)}{\pi r^2 y} & \frac{2x\log(z)}{\pi r^2 y^2} & \frac{-2x}{\pi r^2 yz} \\ 0 & 0 & \frac{r\sqrt{-\log(z)}}{4z\log(z)^2} \end{pmatrix}$.

The asymptotic variance of $\widehat{\sigma_r}$ is given by

$$\sigma^2_{\sigma_r} = \sigma^2_{\tilde{\rho}(r)}\frac{4\sigma^6}{r^4}\,\mathrm{e}^{\frac{r^2}{2\sigma^2}}. \tag{4.4}$$

Proof: The random field $X_{\lambda,\mu\sigma}$ check **(H)** and the finite susceptibility condition $\sum_{z\in\mathbb{Z}^2} I_2(z)\infty$ so we can apply Proposition 2.10 and its isotropic version. Now we use a delta method with $\phi : \mathbb{R}^3 \mapsto \mathbb{R} \times \mathbb{R}_+^* \times (0,1)$, $(x,y,z) \mapsto (\frac{-2x^2\log(z)}{\pi r^2 y}, \frac{-2x\log(z)}{\pi r^2 y}, \sqrt{\frac{-r^2}{4\log(z)}})$. Then we have $(\widehat{\lambda_r}, \widehat{\mu_r}, \widehat{\sigma_r}) = \phi(\widehat{M_1}, \widehat{V}, \widehat{\tilde{\rho}_n(r)})$

and $(\lambda, \mu, \sigma) = \phi(M_1, V, \tilde{\rho}(r))$.

The jacobian of ϕ is $D_\phi(x,y,z) = \begin{pmatrix} \frac{-4x\log(z)}{\pi r^2 y} & \frac{2x^2\log(z)}{\pi r^2 y^2} & \frac{-2x^2}{\pi r^2 yz} \\ \frac{-2\log(z)}{\pi r^2 y} & \frac{2x\log(z)}{\pi r^2 y^2} & \frac{-2x}{\pi r^2 yz} \\ 0 & 0 & \frac{r\sqrt{-\log(z)}}{4z\log(z)^2} \end{pmatrix}$ so we have

$$\Sigma_{\mu_r,\lambda_r,\sigma_r} = \mathrm{D}_\phi(M_1, V, \tilde{\rho}(r)) \begin{pmatrix} \sigma_1^2 & \sigma_{1,V}^2 & \sigma_{1,\tilde{\rho}(r)}^2 \\ \sigma_{1,V}^2 & \sigma_V^2 & \sigma_{V,\tilde{\rho}(r)}^2 \\ \sigma_{1,\tilde{\rho}(r)}^2 & \sigma_{V,\tilde{\rho}(r)}^2 & \sigma_{\tilde{\rho}(r)}^2 \end{pmatrix} \mathrm{D}_\phi(M_1, V, \tilde{\rho}(r))^t.$$

In the same spirit as above, we note $\Sigma_{\lambda_r,\mu_r,\sigma_r} = \begin{pmatrix} \sigma_{\lambda_r}^2 & \sigma_{\lambda_r,\mu_r}^2 & \sigma_{\lambda_r,\sigma_r}^2 \\ \sigma_{\lambda_r,\mu_r}^2 & \sigma_{\mu_r}^2 & \sigma_{\mu_r,\sigma_r}^2 \\ \sigma_{\lambda_r,\sigma_r}^2 & \sigma_{\mu_r,\sigma_r}^2 & \sigma_{\sigma_r}^2 \end{pmatrix}$

To specify the asymptotic variance we just use a delta method with $\psi\colon (0,1) \mapsto \mathbb{R}$, $x \mapsto \sqrt{\frac{-r^2}{4\ln(x)}}$.

We have $\psi'(x) = \frac{r\sqrt{-\ln(x)}}{4x\ln(x)^2}$ and hence $\psi'(\tilde{\rho}(r)) = \frac{2\sigma^3 e^{\frac{r^2}{4\sigma^2}}}{r^2}$. Thus $\sigma_{\sigma_r}^2 = \sigma_{\tilde{\rho}(r)}^2 \psi'(\tilde{\rho}(r))^2 = \sigma_{\tilde{\rho}(r)}^2 \frac{4\sigma^6}{r^4} e^{\frac{r^2}{2\sigma^2}}$. □

parameter	theoretical value	relative bias	relative MSE
σ	10	$1,5e^{-2}$	$2,2e^{-3}$
μ	0.01	$-5,5e^{-2}$	$3,1e^{-5}$
λ	$9.5e^{-4}$	$-3,1e^{-2}$	$9,5e^{-8}$

TABLE 4.1. Bias and MSE for the parameters estimators of the example. Here the observation windows is $[0, 3000]^2$.

To illustrate Theorem 4.1, see Figure 4.2 where the size of the window increases. We also note that the asymptotic variances of the parameter estimators with isotropy are smaller than without isotropy. For the parameters of this example we have with isotropy :

$$\sigma_{\lambda_{10}}^2 = 0.005;\ \sigma_{\mu_{10}}^2 = 0,5454;\ \sigma_{\sigma_{10}}^2 = 19049$$

whereas if we estimate ρ in only one direction $w = (10, 0)$:

$$\sigma_{\lambda_w}^2 = 0.0113;\ \sigma_{\mu_w}^2 = 1,2381;\ \sigma_{\sigma_w}^2 = 192230$$

The bias and mean square error are excellent as shown in Table 4.1.

Moreover, by the interesting convolution properties of the Gaussian kernel, we have the following lemma whose proof is postponed to appendix.

Lemma 4.2. *All the integrals appearing in the asymptotic variances are explicit:*

- $I_1 = 1$
- $I_2(x) = \frac{1}{4\pi\sigma^2} e^{-\frac{1}{4\sigma^2}||x||^2}$
- $I_3(x,y) = \frac{1}{12\pi^2\sigma^4} e^{-\frac{1}{6\sigma^2}(||x-y||^2+||x||^2+||y||^2)}$
- $I_4(x,y,s) = \frac{1}{32\pi^3\sigma^6} e^{-\frac{1}{8\sigma^2}(||x-y||^2+||x-s||^2+||y-s||^2+||x||^2+||y||^2+||s||^2)}$

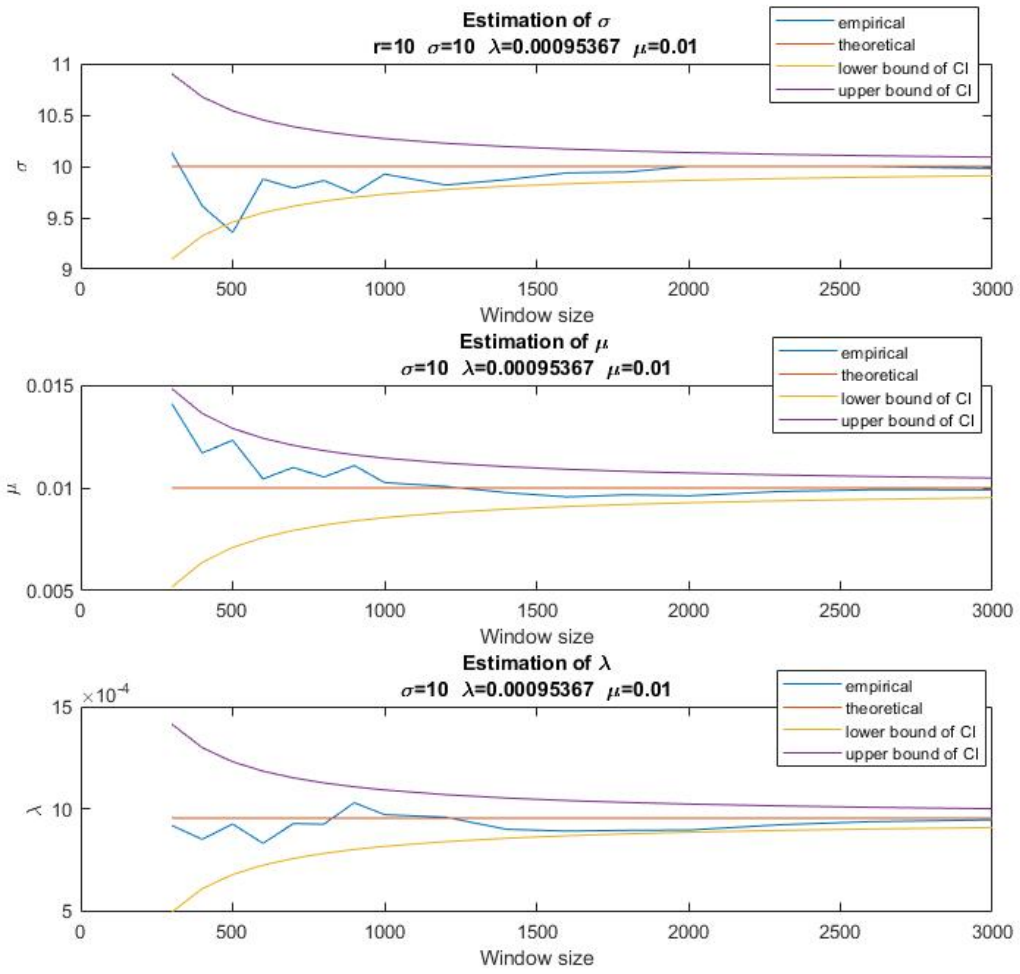

FIGURE 4.2. Parameters estimations of a shot noise random fiel with Gaussian kernel and exponential marks. Parameters were estimated from $\rho(5)$. On the x-axis, the observation windows size that are increasing. Lower and upper bound of the 95% confidence interval are also displayed.

and the series can be simplified as

$$S(2) = \frac{1}{4\pi\sigma^2} \sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{4\sigma^2}||z||^2} \tag{4.5}$$

$$S(2,2,w_1-w_2) = \frac{1}{16\pi^2\sigma^4} e^{-\frac{1}{8\sigma^2}||w_1-w_2||^2} \sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{2\sigma^2}||z+\frac{w_1+w_2}{2}||^2} \tag{4.6}$$

$$S(3,w_1) = \frac{1}{12\pi^2\sigma^4} e^{-\frac{1}{4\sigma^2}||w_1||^2} \sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{3\sigma^2}||z-\frac{w_1}{2}||^2} \tag{4.7}$$

$$S(4,w_1,w_2) = \frac{1}{32\pi^3\sigma^6} e^{-\frac{1}{4\sigma^2}(||w_1||^2+||w_2||^2)} \sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{2\sigma^2}||z+\frac{w_1-w_2}{2}||^2} \tag{4.8}$$

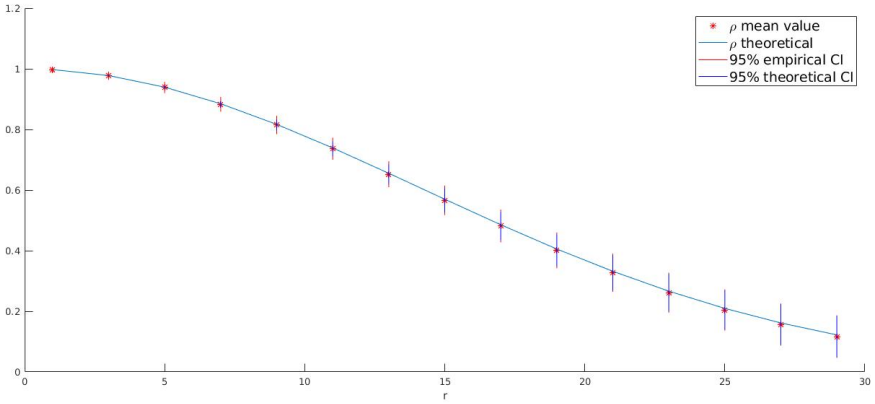

FIGURE 4.3. Theoretical curve and estimation of the correlation function. The mean value and empirical variance are compute with a sample of $D = 300$ random field simulated on the windows $[0, 500]^2$. The parameter of the model are $\sigma = 10$, $\lambda \approx 10^{-3}$ and $\mu = \frac{1}{100}$.

When σ is not too small, let's say greater than 1, we can approach (4.5) - (4.8) by a series/integral comparison leading to :

- $S(2) \approx 1$
- $S(2, 2, w_1 - w_2) \approx \frac{1}{8\pi\sigma^2} e^{-\frac{1}{8\sigma^2}||w_1 - w_2||^2}$
- $S(3, w_1) \approx \frac{1}{4\pi\sigma^2} e^{-\frac{1}{4\sigma^2}||w_1||^2}$
- $S(4, w_1, w_2) \approx \frac{1}{16\pi^2\sigma^4} e^{-\frac{1}{4\sigma^2}(||w_1||^2 + ||w_2||^2)}$.

Thus the asymptotic variance of $\widehat{\tilde{\rho}_n(r)}$ given by (3.4) is

$$\begin{aligned}
\sigma^2_{\tilde{\rho}(r)} = \frac{m_4 e^{-\frac{r^2}{2\sigma^2}}}{2\pi\sigma^2\lambda m_2^2} \sum_{z\in\mathbb{Z}^2} \Bigg(& e^{-\frac{1}{2\sigma^2}||z||^2} - 2e^{-\frac{1}{2\sigma^2}||z+\frac{re_1}{2}||^2} + \frac{1}{2}e^{-\frac{1}{2\sigma^2}||z||^2} + \frac{1}{2}e^{-\frac{1}{2\sigma^2}||z+\frac{re_1-re_2}{2}||^2} \Bigg) \\
& + \sum_{z\in\mathbb{Z}^2} \Bigg(2e^{-\frac{r^2}{2\sigma^2}} e^{-\frac{1}{2\sigma^2}||z||^2} - 4e^{-\frac{3r^2}{8\sigma^2}} e^{-\frac{1}{2\sigma^2}||z+\frac{re_1}{2}||^2} \\
& + \frac{1}{2}e^{-\frac{1}{2\sigma^2}||z||^2} + \frac{1}{2}e^{-\frac{-r^2}{2\sigma^2}} e^{-\frac{1}{2\sigma^2}||z||^2} + e^{-\frac{-r^2}{4\sigma^2}} e^{-\frac{1}{2\sigma^2}||z+\frac{re_1+re_2}{2}||^2} \Bigg) \\
& \approx \sum_{z\in\mathbb{Z}^2} \Bigg(2e^{-\frac{r^2}{2\sigma^2}} e^{-\frac{1}{2\sigma^2}||z||^2} - 4e^{-\frac{3r^2}{8\sigma^2}} e^{-\frac{1}{2\sigma^2}||z+\frac{re_1}{2}||^2} \\
& + \frac{1}{2}e^{-\frac{1}{2\sigma^2}||z||^2} + \frac{1}{2}e^{-\frac{-r^2}{2\sigma^2}} e^{-\frac{1}{2\sigma^2}||z||^2} + e^{-\frac{-r^2}{4\sigma^2}} e^{-\frac{1}{2\sigma^2}||z+\frac{re_1+re_2}{2}||^2} \Bigg),
\end{aligned}$$

where the approximation is a very small approximation coming from the fact that $\frac{re_1}{2}$ and $\frac{re_1-re_2}{2}$ are not always in $\mathbb{Z}^2$. We put aside this small approximation. The asymptotic variance of $\tilde{\rho}$ does not depend on λ and μ. If we make the series/integral approximation we have the simple formula

$$\sigma^2_{\tilde{\rho}(r)} \approx \pi\sigma^2 \left(1 + 2e^{-\frac{r^2}{4\sigma^2}} - 8e^{-\frac{3r^2}{8\sigma^2}} + 5e^{-\frac{r^2}{2\sigma^2}} \right)$$

We observe that for all $\sigma \geq 0$, the function $r \mapsto \sigma^2_{\tilde{\rho}(r)}$ starts at 0 then grows to its limit $\pi\sigma^2$. To minimize variance, it would be useful to estimate this function for small r but for practical observation windows, inconsistencies are often observed with an estimate greater than 1 for small r. We observe also (see the Table 4.2) that for all $\sigma \geq 0$, the function $r \mapsto \sigma^2_{\sigma_r}$ starts at $\frac{\pi\sigma^4}{2}$ then grows slowly (almost constant) then rapidly to infinity. As for $\widehat{\tilde{\rho}(r)}$, to minimize variance, it would be useful to estimate this function for small r.

σ \ r	1	2	3	4	5	10	15	20	30
1	1,90	3,64	13,86	149,03	5e+3	6e+18	1e+45	5e+82	4,19e+190
2	26,34	30,47	39,40	58,25	102,05	8e+4	1,04e+11	1,04e+20	2,86e+46
3	129,92	138,43	154,29	180,63	223,53	2e+3	4e+5	2e+9	5,27e+20
4	406,87	421,56	447,59	487,64	546,15	1e+3	1e+4	1e+6	1e+12
5	989,14	1e+3	1e+3	1e+3	1e+3	2e+3	8e+3	9e+4	4e+8
10	1e+4	1e+4	1e+4	1e+4	1e+4	2e+4	2e+4	3e+4	10e+4
15	7e+4	8e+4	8e+4	8e+4	8e+4	8e+4	9e+4	11e+4	18e+4
20	2e+5	2e+5	2e+5	2e+5	2e+5	2e+5	2e+5	3e+5	4e+5

TABLE 4.2. Theoritical asymptotic variance of $\widehat{\sigma_r}$ given by 4.4

Thus, to estimate σ, it seems more interesting to take into account the correlation function on small r. But looking at Table 4.3, although a small r gives a small variance, the function ρ is so close to 1 (especially by increasing σ) that the estimate is rough. The exponential form of the correlation function (it starts from 1, decreases slowly then strongly then slowly to 0) does not allow to distinguish the first values since it is slightly decreasing. This table reinforces us in the idea of not using the large r values. A first idea would be to make an average of several estimated σ_r for some specifically chosen r's, when the scope of the ρ function is maximal and therefore allows us to distinguish these values. For example, for $\sigma = 1$ we recommend using $r = 1$ to 3, for $\sigma = 5$ we recommend using $r = 3$ to 7, for $\sigma = 10$ we recommend using $r = 7$ to 13.

Actually, these estimators are often identical and averaging them does not yield a significant gain. Thus, for the sake of simplicity, we will consider only one well-chosen value σ_r to estimate σ. In practice, when you have no idea of the value of σ it is also necessary to estimate a good r. A good way to choose it is

$$r^* = \underset{r\in\mathbb{N}^*}{\operatorname{argmin}} |\widehat{\tilde{\rho}_n(r)} - \exp(-\frac{1}{2})| \tag{4.9}$$

because as $\tilde{\rho}''(r) = \frac{e^{-\frac{r^2}{4\sigma^2}}}{2\sigma^2}(\frac{r^2}{2\sigma^2} - 1)$ the function is the most decreasing at $r = \sqrt{2}\sigma$. Observe the correct estimation of r^* in Table 4.4. Remark that the estimation is worse for large σ but if the size of the image increases then the estimation is improved. The σ estimator we consider is then

$$\widehat{\sigma_{r^*}} = \sqrt{\frac{-{r^*}^2}{4\log(\widehat{\tilde{\rho}_n(r^*)})}} \tag{4.10}$$

σ \ r	1	2	3	4	5	10	15	20	30
2	1,95	1,98	1,98	1,98	1,99	2,26	NA	NA	NA
3	3,19	3,05	3,02	3,00	2,99	2,89	2,65	5,23	NA
4	3,36	3,70	3,80	3,83	3,86	3,90	3,86	NA	NA
5	6,24	5,28	5,13	5,07	5,04	4,98	4,91	4,65	NA
10	NA	11,78	10,53	10,19	10,05	9,89	9,87	9,87	9,84
15	NA	19,42	16,99	16,20	15,81	15,16	14,95	14,82	14,58
20	9,96	14,29	16,38	17,54	18,27	19,76	20,24	20,47	20,65

TABLE 4.3. Value of $\widehat{\sigma_r}$ with different σ and r estimating on one Gaussian shot noise random field simulated on the windows $[0, 500]^2$. The two other parameters are $\lambda = 1000/1024^2$ and $\mu = 1/100$. The NA values are complex number.

$\sqrt{2}\sigma$	σ \ r	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2.83	2	300	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7.07	5	0	0	0	0	300	0	0	0	0	0	0	0	0	0	0	0	0	0
11.31	8	0	0	0	0	0	0	0	1	214	85	0	0	0	0	0	0	0	0
15.55	11	0	0	0	0	0	0	0	0	0	0	0	20	134	146	0	0	0	0
19.80	14	0	0	0	0	0	0	0	0	0	0	0	0	0	2	6	55	237	0

TABLE 4.4. Number of $\widehat{r^*}$ for different σ. The sample size is $D = 300$ on the windows $F = 500$. The two others parameters are $\lambda = \frac{1000}{500^2}$ and $\mu = \frac{1}{100}$.

Given a random field realization X, it is now sufficient to implement the following steps :

(1) estimate the correlation function ρ with (3.3)
(2) calculate r^* with (4.9)
(3) estimate the mean and variance with (2.9) and (2.10)
(4) estimate (λ, μ, σ) with (4.3)

Let us finish with some illustration. In order not to study a single parameter triplet, the parameters in the two following figures evolve in such a way that the parameter range is not too restrictive. Figure 4.4 represents bias and variance of the 3 parameter estimators for several σ values on a sample of size $D = 300$. In comparison with the table 4.1, let us look at our estimation on a small image size like $F = 500$. There is a very good estimate with a small bias and an acceptable variance for this window size. Again, the estimators become a bit worse for large σ values. Figures 4.5 represents bias and variance of the 3 parameter estimators for several λ values on a sample of size $D = 300$. There again, the statistical properties are suitable for this window size. This time, there is no consequence on the estimation of the three parameters when we consider larger λ values (to see it for λ, just consider the relative bias and relative variance).

5. Conclusion and perspective

Thank to Theorem 2.6 we have a solid foundation of moment estimates for a large class of shot noise random field. The study of the case of the Gaussian kernel and the exponential mark in the last part is an application to parameter estimation that could be extended to other marks very easily. Other kernels whose possibe computations of (2.1)-(2.4) can aslo be used. The estimation of the parameters of this model can allow applications in model validation and the calculation of other statistics such as the geometry of the excursion levels (see Biermé and Desolneux, 2016). Finally, a shot noise can be used for physical modeling such as heat propagation phenomena or also in biology, so applications to practical cases can be considered.

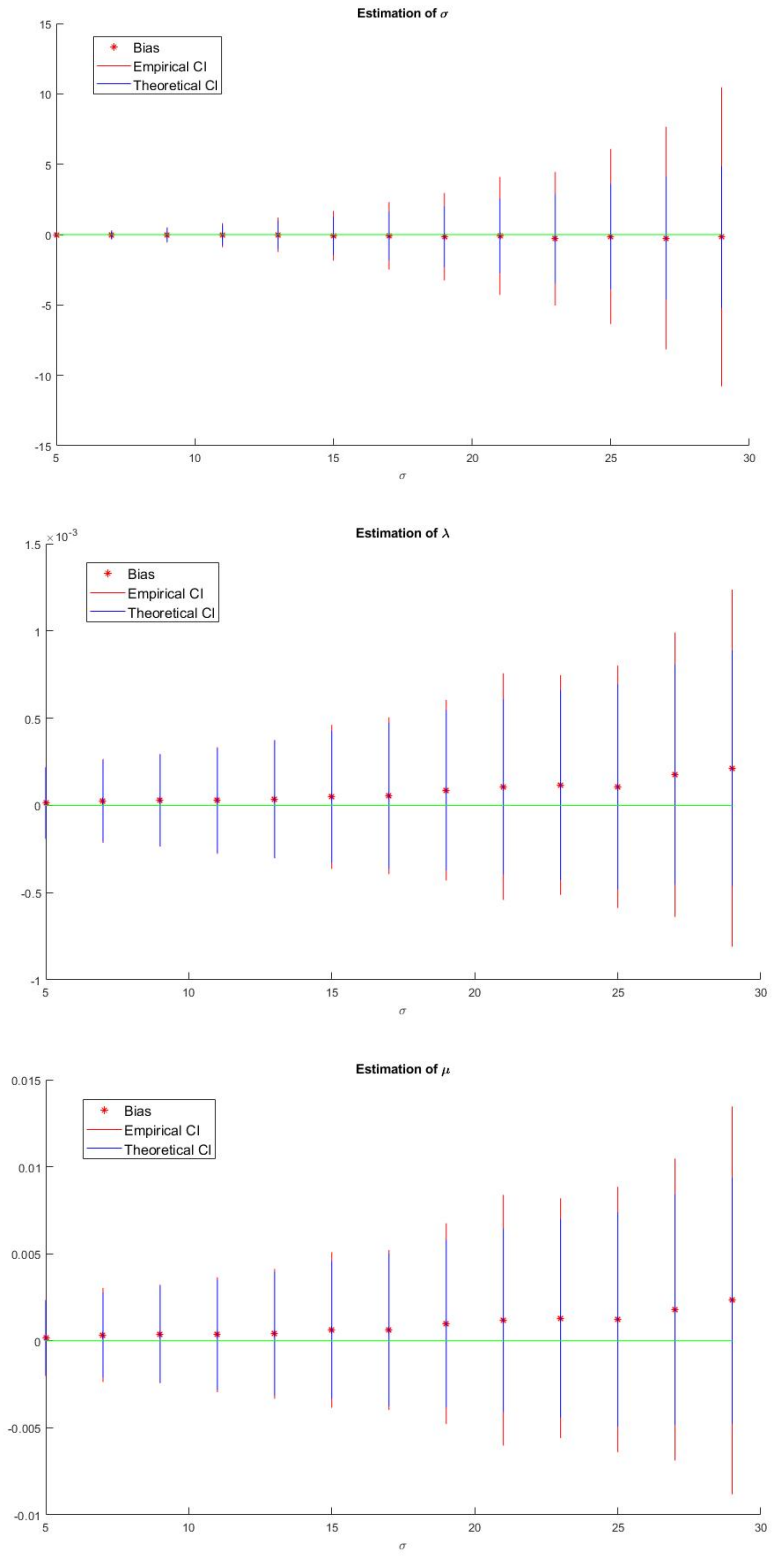

FIGURE 4.4. Bias and 95% CI of the three parameters for several σ values. Each are calculated on a sample of $D = 300$ random field simulated on the windows $[0, 500]^2$. The two others parameters are $\lambda \approx 10^{-3}$ and $\mu = \frac{1}{100}$.

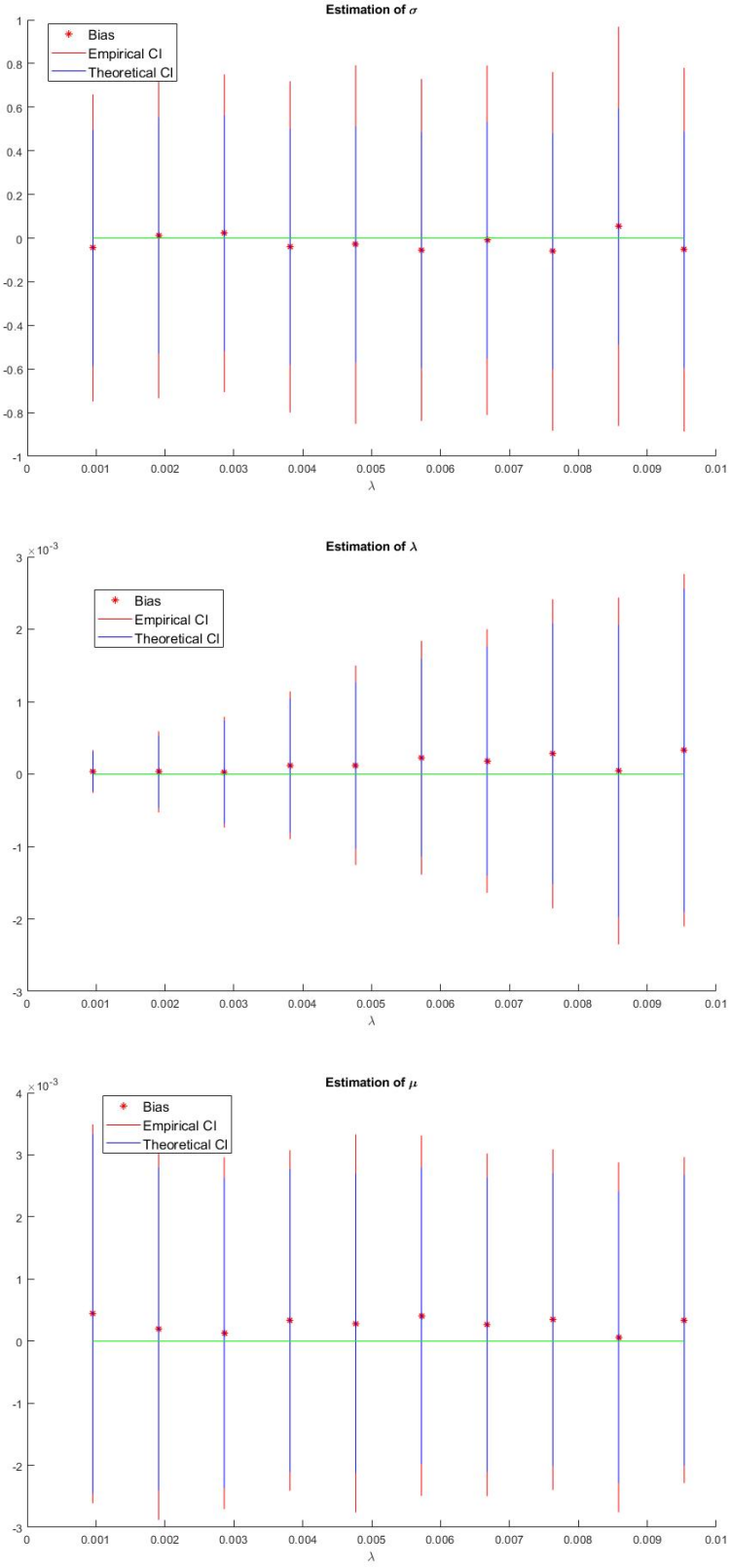

FIGURE 4.5. Bias and 95% CI of the three parameters for several λ values. Each are calculated on a sample of $D = 300$ random field simulated on the windows $[0, 500]^2$. The two others parameters are $\sigma = 10$ and $\mu = \frac{1}{100}$.

6. Proof of results

Proof of Lemma 2.1 and 2.7: Let X a Poisson point process on $\mathbb{R}^n$ with intenstity measure Λ and f a non-negative bounded measurable function defined on $\mathbb{R}^d$, then we have :

$$\mathbb{E}\Big[\sum_{(x_1,\dots,x_n)\in X}^{\neq} f(x_1,\dots,x_n)\Big] = \int_{\mathbb{R}^{kn}} f(x_1,\dots,x_n)\Lambda(\mathrm{d}x_1)\dots\Lambda(\mathrm{d}x_n) \tag{6.1}$$

Previously called Campbell's Theorem due to Campbell's work on shot noise processes (Campbell, 1909a), the kind of equation such as 6.1 is now a common way to directly define the kth order moment measure. We can now calculate the moments of a shot noise random field. Let $S = \mathbb{R}^2 \times \mathbb{R}_+$, the first one is simply given by

$$\mathbb{E}[X(0)] = \mathbb{E}\Big[\sum_{(x_i,m_i)\in\Phi_X} m_i g(x_i)\Big] = \int_S mg(c)\lambda \mathrm{d}c F(\mathrm{d}m) = \lambda m_1$$

To compute the others moments, we need to split the sum such that the points are different then simply use the Campbelle formula several times. Let $x \in T$ then :

$$\begin{aligned}
\mathbb{E}[X(x)X(0)] &= \mathbb{E}\Big[\Big(\sum_{(x_i,m_i)\in\Phi_X} m_i g(x_i - x)\Big)\Big(\sum_{(x_j,m_j)\in\Phi_X} m_j g(x_j)\Big)\Big] \\
&= \mathbb{E}\Big[\Big(\sum_{(x_i,m_i)\in\Phi_X} m_i g(x_i - x)\Big)\Big(\sum_{(x_j,m_j)\in\Phi_X}^{i\neq j} m_j g(x_j)\Big) + \sum_{(x_i,m_i)\in\Phi_X} m_i^2 g(x_i - x)g(x_i)\Big] \\
&= \int_{S^2} mg(c-x)ng(a)\lambda^2 \mathrm{d}c F(\mathrm{d}m)\mathrm{d}a F(\mathrm{d}n) + \int_S m^2 g(c-x)g(c)\lambda \mathrm{d}c F(\mathrm{d}m) \\
&= \lambda^2 m_1^2 + \lambda m_2 I_2(x)
\end{aligned}$$

Let $x, y, s \in T$, to calculate

$$\mathbb{E}[X(x)X(y)X(s)] = \mathbb{E}\Big[\Big(\sum_{i\in\Phi_X} m_i g(i-x)\Big)\Big(\sum_{j\in\Phi_X} m_j g(j-y)\Big)\Big(\sum_{k\in\Phi_X} m_k g(k-s)\Big)\Big] \tag{6.2}$$

it is required to divide the sum into 5 as shown in Figure 6.6.

In order, the 5 terms are :

- $\mathbb{E}\Big[\sum_{i\in\Phi_X} m_i^3 g(i-x)g(i-y)g(i-s)\Big] = \int_S m^3 g(x-c)g(y-c)g(s-c)\lambda \mathrm{d}c F(\mathrm{d}m)$
$= m_3 \lambda I_3(x-s, y-s)$
- $\mathbb{E}\Big[\sum_{i,k\in\Phi_X}^{\neq} m_i^2 m_l g(i-x)g(i-y)g(k-s)\Big] = \int_{S^2} m^2 ng(x-c)g(y-c)g(s-a)\lambda^2 \mathrm{d}c F(\mathrm{d}m)\mathrm{d}a F(\mathrm{d}n)$
$= m_2 m_1 \lambda^2 I_2(x-y)$
- $\mathbb{E}\Big[\sum_{i,j\in\Phi_X}^{\neq} m_i^2 m_k g(i-x)g(j-y)g(i-s)\Big] = \int_{S^2} m^2 ng(x-c)g(y-a)g(s-c)\lambda^2 \mathrm{d}c F(\mathrm{d}m)\mathrm{d}a F(\mathrm{d}n)$
$= m_2 m_1 \lambda^2 I_2(x-s)$

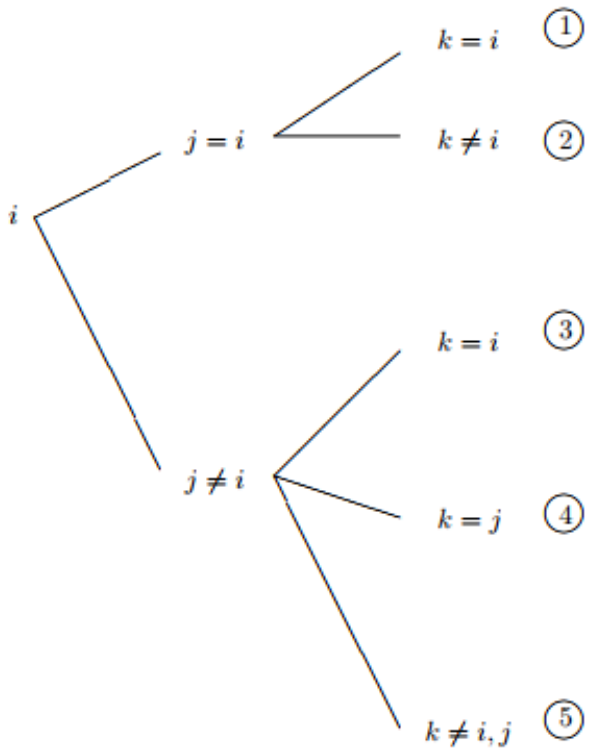

FIGURE 6.6. Decomposition of the triple summation

- $\mathbb{E}\Big[\sum_{i,j\in\Phi_X}^{\neq} m_i m_j^2 g(i-x)g(j-y)g(j-s)\Big] = \int_{S^2} mn^2 g(x-c)g(y-a)g(s-a)\lambda^2 \mathrm{d}cF(\mathrm{d}m)\mathrm{d}aF(\mathrm{d}n)$
$= m_2 m_1 \lambda^2 I_2(y-s)$
- $\mathbb{E}\Big[\sum_{i,j,k\in\Phi_X}^{\neq} m_i m_j m_k g(i-x)g(j-y)g(k-s)\Big] = \int_{S^3} mnpg(x-c)g(y-a)g(s-b)\lambda^3 \mathrm{d}cF(\mathrm{d}m)\mathrm{d}aF(\mathrm{d}n)\mathrm{d}bF(\mathrm{d}$
$= m_1^3 \lambda^3$

Hence

$$\mathbb{E}[X(x)X(y)X(s)] = m_3\lambda I_3(x-s, y-s) + m_2 m_1 \lambda^2 \big(I_2(x-y) + I_2(x-s) + I_2(y-s)\big) + m_1^3\lambda^3.$$

To calculate $\mathbb{E}[X(x)X(y)X(s)X(t)]$, the sum must be divided into 15 terms as shown in Figure 6.7.

The calculations are similar to the ones perfomed for $\mathbb{E}[X(x)X(y)X(s)]$. □

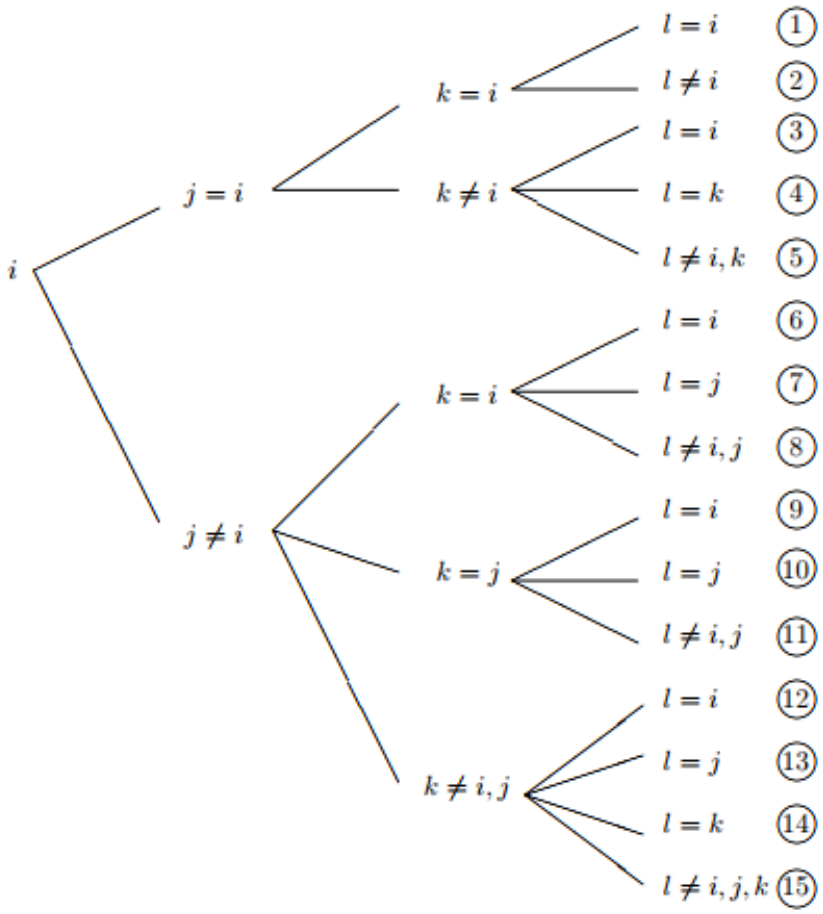

FIGURE 6.7. Decomposition of the quadruple summation

Proof of lemma 4.2: Let $x, y, s \in T$.

$$\begin{aligned}
I_3(x,y) &= \frac{1}{(2\pi\sigma^2)^3}\int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}\left(||c-x||^2+||c-y||^2+||c||^2\right)}\mathrm{d}c \\
&= \frac{1}{(2\pi\sigma^2)^3} e^{-\frac{||x-y||^2+||x||^2+||y||^2}{6\sigma^2}}\int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}||\sqrt{3}c-\frac{x+y}{\sqrt{3}}||^2}\mathrm{d}c \\
&= \frac{1}{(2\pi\sigma^2)^3} e^{-\frac{||x-y||^2+||x||^2+||y||^2}{6\sigma^2}}\int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}||a||^2}\frac{1}{3}\mathrm{d}a \\
&= \frac{1}{(2\pi\sigma^2)^3} e^{-\frac{||x-y||^2+||x||^2+||y||^2}{6\sigma^2}}\frac{2\pi\sigma^2}{3} \\
&= \frac{1}{12\pi^2\sigma^4} e^{-\frac{1}{6\sigma^2}(||x-y||^2+||x||^2+||y||^2)}
\end{aligned}$$

The calculation of the other integrals are very similar, for exemple for $I_4(x,y,s)$ juste write

$$||c-x||^2+||c-y||^2+||c-s||^2+||c||^2 = ||2c-\frac{x+y+s}{2}||^2+\frac{1}{4}\left(||x-y||^2+||x-s||^2+||y-s||^2+||x||^2+||y||^2+||s||^2\right)$$

□

Proof of 4.5, 4.6, 4.7 and 4.8: By Lemma 4.2 we have $S(2) = \frac{1}{4\pi\sigma^2}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{4\sigma^2}||z||^2}$. For the others series, we can reduce as follows :

$$\begin{aligned}
S(2,2,w_1,w_2) &= \sum_{z\in\mathbb{Z}^2} I_2(z+w_1)I_2(z+w_2) \\
&= \frac{1}{16\pi^2\sigma^4}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{4\sigma^2}(||z+w_1||^2+||z+w_2||^2)} \\
&= \frac{1}{16\pi^2\sigma^4}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{4\sigma^2}||\sqrt{2}z+\frac{w_1+w_2}{\sqrt{2}}||^2} e^{-\frac{1}{4\sigma^2}||\frac{w_1-w_2}{\sqrt{2}}||^2} \\
&= \frac{1}{16\pi^2\sigma^4} e^{-\frac{1}{8\sigma^2}||w_1-w_2||^2}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{2\sigma^2}||z+\frac{w_1+w_2}{2}||^2}
\end{aligned}$$

$$\begin{aligned}
S(3,w_1) &= \sum_{z\in\mathbb{Z}^2} I_3(z,w_1) \\
&= e^{-\frac{1}{6\sigma^2}||w_1||^2}\frac{1}{12\pi^2\sigma^4}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{6\sigma^2}(||z-w_1||^2+||z||^2)} \\
&= e^{-\frac{1}{6\sigma^2}||w_1||^2}\frac{1}{12\pi^2\sigma^4}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{6\sigma^2}||\sqrt{2}z-\frac{w_1}{\sqrt{2}}||^2} e^{-\frac{1}{6\sigma^2}||\frac{w_1}{\sqrt{2}}||^2} \\
&= e^{-\frac{1}{4\sigma^2}||w_1||^2}\frac{1}{12\pi^2\sigma^4}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{3\sigma^2}||z-\frac{w_1}{2}||^2}
\end{aligned}$$

$$\begin{aligned}
S(4,w_1,w_2) &= \sum_{z\in\mathbb{Z}^2} I_4(z,z+w_1,w_2) \\
&= e^{-\frac{1}{8\sigma^2}(||w_1||^2+||w_2||^2)}\frac{1}{32\pi^3\sigma^6}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{8\sigma^2}(||z-w_2||^2+||z+w_1-w_2||^2+||z+w_1||^2+||z||^2)} \\
&= e^{-\frac{1}{8\sigma^2}(||w_1||^2+||w_2||^2)}\frac{1}{32\pi^3\sigma^6}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{8\sigma^2}||2z+w_1-w_2||^2} e^{-\frac{1}{8\sigma^2}(||w_1||^2+||w_2||^2)} \\
&= e^{-\frac{1}{4\sigma^2}(||w_1||^2+||w_2||^2)}\frac{1}{32\pi^3\sigma^6}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{2\sigma^2}||z+\frac{w_1-w_2}{2}||^2}
\end{aligned}$$

□

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