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Abstract. We show that plane bipolar posets (i.e., plane bipolar orientations with no transitive edge) and transversal structures can be set in correspondence to certain (weighted) models of quadrant walks, via suitable specializations of a bijection due to Kenyon, Miller, Sheffield and Wilson. We then derive exact and asymptotic counting results, and in particular we prove that the number t_n of transversal structures on $n+2$ vertices satisfies (for some $c > 0$) $t_n \sim c\,(27/2)^n n^{-1-\pi/\arccos(7/8)}$, which also ensures that the associated generating function is not D-finite.

Keywords: bijections, oriented planar maps, quadrant walks

1 Introduction

The combinatorics of planar maps (i.e., planar multigraphs endowed with an embedding on the sphere) has been a very active research topic ever since the early works of W.T. Tutte. In the last few years, after tremendous progress on the enumerative and probabilistic theory of maps, the focus has started to shift to planar maps endowed with *constrained orientations*. Indeed constrained orientations capture a rich variety of models [7] with connections to (among other) graph drawing, pattern-avoiding permutations, Liouville quantum gravity, or theoretical physics. From an enumerative perspective, these new families of maps are expected to depart (e.g. [6]) from the usual *algebraic generating function* pattern followed by many families of planar maps with local constraints [11]. From a probabilistic point of view, they lead to new models of random graphs and surfaces, as opposed to the universal Brownian map limit capturing earlier models. Both phenomena are first witnessed by the appearance of new critical exponents $\alpha \neq 5/2$ in the generic $\gamma^n n^{-\alpha}$ asymptotic formulas for the number of maps of size n.

A fruitful approach to oriented planar maps is through bijections (e.g. [1]) with walks with a specific step-set in the quadrant, or in a cone, up to shear transformations. We rely here on a recent such bijection [10] that encodes plane bipolar orientations by certain quadrant walks (so-called tandem walks): we show in Section 2 that it can be furthermore adapted to other models by introducing properly chosen weights. Building on these specializations, in Section 3 we obtain exact enumeration results for plane bipolar posets and transversal structures. In particular we show that the number b_n of plane bipolar posets on $n+2$ vertices is equal to the number of plane permutations of size n recently studied in [4], and that a reduction to small-steps quadrant walks models (which makes coefficient computation faster) can be performed for the number e_n of plane bipolar posets with n edges and the number t_n of transversal structures on $n+2$ vertices. In Section 4 we obtain asymptotic formulas for the coefficients b_n, e_n, t_n all of the form $c\gamma^n n^{-\alpha}$ with $c > 0$ and with $\gamma, \alpha \neq 5/2$ explicit, and by the approach of [3] we deduce from these estimates that the generating functions for e_n and t_n are not D-finite.

Note: An extended version on these results is available at `arXiv:2105.06955`.

2 Oriented planar maps and quadrant tandem walks

A *plane bipolar orientation* B is a planar map endowed with an acyclic orientation having a single source S and a single sink N, which both lie in the outer face, see Fig.1(a). It is known that the contour of each face f of B (including the outer one) splits into a left lateral path L_f and a right lateral path R_f (which share the same origin and end); the *type* of f is the pair (i, j) where $i+1$ (resp. $j+1$) is the length of L_f (resp. R_f). The *outer type* of B is the type of the outer face. The *pole-type* of B is the pair (p, q) such that $p+1$ is the degree of S and $q+1$ is the degree of N.

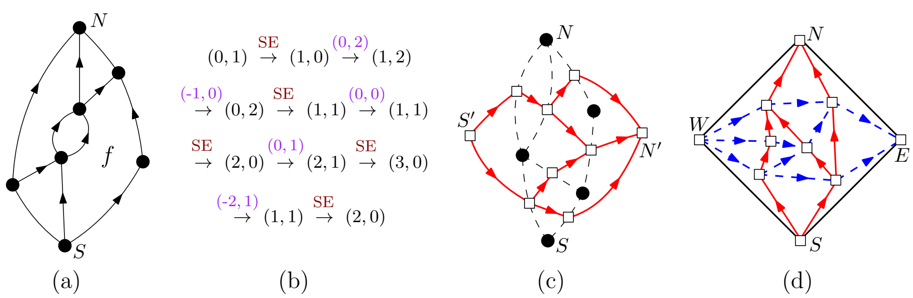

Fig. 1. (a) A plane bipolar orientation of outer type $(1, 2)$ (the marked inner face f has type $(2, 1)$). (b) A quadrant tandem walk from $(0, 1)$ to $(2, 0)$ (actually the one associated to (a) by the KMSW bijection). (c) From a plane bipolar orientation (round vertices) with n edges and $f+2$ vertices to one of the associated plane bipolar posets (square vertices) with $n+2$ vertices and f inner faces. (d) A 4-triangulation endowed with a transversal structure (blue edges are dashed).

On the other hand, a *tandem walk* (see Fig.1(b)) is defined as a walk on $\mathbb{Z}^2$ with steps in $SE \cup \{(-i, j), i, j \geq 0\}$; it is a *quadrant* walk if it stays in $\mathbb{N}^2$ all along. Every step $(-i, j)$ in such a walk is called a *face-step*, and the pair (i, j) is called its *type*. We will crucially rely on the following bijective result:

Theorem 1 (KMSW bijection [10]). *Plane bipolar orientations of outer type* (a, b) *with* $n + 1$ *edges are in bijection with quadrant tandem walks of length* n *from* $(0, a)$ *to* $(b, 0)$*. Every non-pole vertex corresponds to a SE-step, and every inner face corresponds to a face-step, of the same type.*

An edge $e = (u, v) \in B$ is called *transitive* if there is a path from u to v avoiding e. If B has no transitive edge it is called a *plane bipolar poset*.

Remark 1. Let B be a plane bipolar orientation. Then B is a plane bipolar poset iff it has no inner face whose type has a zero entry. Hence the KMSW bijection specializes into a bijection (with same parameter-correspondence) between plane bipolar posets of outer type (a, b) and quadrant tandem walks from $(0, a)$ to $(b, 0)$ such that the type of every face-step has no zero-entry.

In Remark 1 the primary parameter of the poset (the one corresponding to the walk length) is the number of edges (minus 1). We will see below another way to relate plane bipolar posets to (weighted) quadrant tandem walks, this time with the number of vertices as the primary parameter. Other oriented maps to be related below to weighted quadrant tandem walks are transversal structures [8]. A *4-triangulation* is a map whose outer face contour is a (simple) 4-cycle and whose inner faces are triangles; the outer vertices are denoted W, N, E, S in clockwise order, and V denotes the set of inner vertices. A *transversal structure* on such a map (see Fig.1(d)) is an orientation and bicoloration of its inner edges (in blue or red) so that red (resp. blue) edges form a bipolar poset with V as the set of non-pole vertices and (S, N) (resp. (W, E)) as the pair (source,sink), and moreover any intersection of a blue path with a red path is a crossing where the blue path arrives from the left side of the red path.

For w a function from $\mathbb{N}^2$ to $\mathbb{N}$, a *w-weighted* plane bipolar orientation is a bipolar orientation where every inner face f carries an integer $\iota(f)$ in $[1..w(i, j)]$ with (i, j) the type of f. A *w-weighted* tandem walk is a tandem walk where every face-step s carries an integer $\iota(s)$ in $[1..w(i, j)]$ with (i, j) the type of s.

Proposition 1. *For* $w : (i, j) \to \binom{i+j}{i}$*, plane bipolar posets of pole-type* (p, q)*, with* $n + 2$ *vertices and* f *inner faces, are in bijection with* w*-weighted plane bipolar orientations of outer type* (p, q)*, with* n *edges and* $f + 2$ *vertices. These correspond (via KMSW) to* w*-weighted quadrant tandem walks of length* $n - 1$ *from* $(0, p)$ *to* $(q, 0)$ *with* f *SE-steps.*

For $w : (i, j) \to \binom{i+j-2}{i-1}$ *(with* $w(i, j) = 0$ *if* $i = 0$ *or* $j = 0$*), transversal structures having* n *inner vertices and* m *blue edges are in bijection with* w*-weighted plane bipolar posets of outer type* $(1, 1)$ *having* $n+4$ *vertices and* $m+4$ *edges. These correspond (via KMSW) to* w*-weighted quadrant tandem walks from* $(0, 1)$ *to* $(1, 0)$ *of length* $m + 3$ *with* $n + 2$ *SE-steps.*

Proof. The first correspondence (see Fig.1(c)) is adapted from [9]. Starting from a plane bipolar orientation B, insert a square vertex in the middle of each edge (these are to be the non-pole vertices of the bipolar poset). Then in each inner face f, with (i, j) its type, insert $i + j + 1$ non-crossing edges from the square vertices on $L(f)$ to the square vertices on $R(f)$; there are precisely $w(i,j) = \binom{i+j}{i}$ ways to do so (so the chosen way can be encoded by an integer $\iota(f) \in [1..w(i,j)]$). Finally create a square vertex S' (resp. N') in the left (resp. right) outer face and connect it to all square vertices on the left (resp. right) lateral path of B. Then the bipolar poset is obtained by erasing the vertices and edges of B in the obtained figure.

The second correspondence relies on the fact that a transversal structure is completely encoded by its red bipolar poset (augmented by the 4 outer edges oriented from S to N) and the knowledge of how each inner face is transversally triangulated by blue edges: if the face has type (i,j) then there are precisely $\binom{i+j-2}{i-1}$ ways to do so.

3 Exact counting results

Let $P_a^w(x,y)$ denote the generating series of w-weighted quadrant tandem walks starting in position $(0,a)$, with respect to the number of steps (variable t), end positions (variables x and y) and number of SE steps (variable u). A last step decomposition immediately yields the following *master equation* in the ring $\mathbb{Q}((\bar{x}))[[y,t]]$ of formal power series in t and y with coefficients that are Laurent series in $\bar{x} = 1/x$, where $W_k(\bar{x},y) = \sum_{i\geq k, j\geq 0} w(i,j)\frac{y^j}{x^i}$:

$$P_a^w(x,y) = y^a + tu\frac{x}{y}\left(P_a^w(x,y) - P_a^w(x,0)\right) + tW_0(\bar{x},y)P_a^w(x,y) \\ - t\sum_{k\geq 0} W_{k+1}(\bar{x},y)x^k[x^k]P_a^w(x,y).$$

In the case of plane bipolar posets enumerated by vertices, we have (cf Proposition 1) $w(i,j) = \binom{i+j}{i}$ for $i,j\geq 0$, so that $W_k(\bar{x},y) = \frac{1}{1-(\bar{x}+y)}\frac{\bar{x}^k}{(1-y)^k}$ in $\mathbb{Q}[[y,\bar{x}]]$. For $B(x,y) \equiv P_0^w(x,y)$ the master equation then rewrites

$$B(x,y) = 1 + t\frac{x}{y}\left(B(x,y) - B(x,0)\right) + \frac{t}{1-y}\frac{1}{x-\frac{1}{1-y}}\left(xB(x,y) - \frac{1}{1-y}B\left(\frac{1}{1-y},y\right)\right).$$

Let b_n denote the number of plane bipolar posets with $n+2$ vertices. It is also, by adding a new sink of degree 1 (connected to the former sink), the number of plane bipolar posets of pole-type $(0,b)$ with $n+3$ vertices and arbitrary $b\geq 0$, so that $b_n = [t^n]B(1,0)$. Then we prove[3] that b_n is also the number of plane permutations of size n which are studied in [4]: to see this let $S(u,v) := x(B(x,y)-1)$ under the change of variable relation $\{y = 1-\bar{u}, x = v\}$ (note that $B(1,0) = 1+S(1,1)$), and observe that the equation for S derived from the above equation for B is exactly [4, Eq. (2)] (they use (x,y,z) for our (t,u,v)). Furthermore $B(1,0) = 1+S(1,1)$ is D-finite [4, Prop 13], and there are single sum expressions for b_n [4, Thm 14]).

[3] Our proof relies on generating function manipulations, but a similar bijective approach as in [2] also applies, as detailed in the extended version.

The case of bipolar posets counted by edges corresponds to having $w(i,j) = \mathbf{1}_{i\neq 0, j\neq 0}$ (cf Remark 1). By some manipulations on the functional equation in that case, we can show that the number e_n of plane bipolar orientations with n edges coincides with the number of quadrant excursions of length $n-1$ with steps in $\{0, E, S, NW, SE\}$. While the series $\sum_n e_n t^n$ is non D-finite (as discussed in the next section) the reduction to a quadrant walk model with small steps allows to compute the sequence $e_1, \ldots, e_n$ with time complexity $O(n^4)$ using $O(n^3)$ bit space. The sequence starts as $1, 1, 1, 2, 5, 12, 32, 93, 279, 872, 2830, \ldots$.

The case of transversal structures corresponds to having $w(i,j) = \binom{i+j-2}{i-1}$ for $i, j \geq 1$, 0 otherwise. The corresponding weighted quadrant walks can be turned into unweighted quadrant walks with small steps (see the extended version for details), ensuring that the number t_n of transversal structures on $n+2$ vertices is equal to the coefficient $d_{3n-2}(1,0)$, where $d_n(i,j)$ and $u_n(i,j)$ are coefficients specified by the recurrence

$$\begin{cases} d_n(i,j) = d_{n-1}(i-1,j+1) + u_{n-1}(i-1,j+1), \\ u_n(i,j) = d_{n-2}(i+1,j-1) + u_{n-2}(i+1,j-1) + u_{n-1}(i+1,j) + u_{n-1}(i,j-1), \end{cases}$$

with boundary conditions $d_n(i,j) = u_n(i,j) = 0$ for any (n,i,j) with $n \leq 0$ or $i < 0$ or $j < 0$, with the exception (initial condition) of $d_0(0,1) = 1$. The recurrence allows us again to compute the sequence $t_3, \ldots, t_n$ with $O(n^4)$ bit operations using $O(n^3)$ bit space, giving an alternative to the recurrence in [12] (again the series of t_n is non D-finite). The sequence starts as $1, 2, 6, 24, 116, 642, 3938, \ldots$.

4 Asymptotic counting results

We adopt here the method by Bostan, Raschel and Salvy [3] (itself relying on results by Denisov and Wachtel [5]) to obtain asymptotic estimates for the counting coefficients of plane bipolar posets (by vertices and by edges) and transversal structures (by vertices). Let $\mathcal{S} = SE \cup \{(-i,j),\ i,j \geq 0\}$ be the tandem step-set. Let $w : \mathbb{N}^2 \to \mathbb{R}_+$ satisfying the symmetry property $w(i,j) = w(j,i)$. The induced weight-assignment on $\mathcal{S}$ is $w(s) = 1$ for $s = SE$ and $w(s) = w(i,j)$ for $s = (-i,j)$. Let $a_n^{(w)}$ be the weighted number (i.e., each walk σ is counted with weight $\prod_{s\in\sigma} w(s)$) of quadrant tandem walks of length n, for some fixed starting and ending points. Let $S(z;x,y) := \frac{x}{y} z^{-2} + \sum_{i,j\geq 0} w(i,j) \frac{y^j}{x^i} z^{i+j}$, let $S(z) := S(z;1,1)$, and let ρ be the radius of convergence (assumed here to be strictly positive) of $S(z) - z^{-2}$. Let $\widetilde{w}(s) := \frac{1}{\gamma} w(s) z_0^{y(s)-x(s)}$ be the modified weight-distribution where $\gamma, z_0 > 0$ are adjusted so that $\widetilde{w}(s)$ is a probability distribution (i.e. $\sum_{s\in\mathcal{S}} \widetilde{w}(s) = 1$) and the drift is zero, which is here equivalent to having $z = z_0 \in (0,\rho)$ solution of $S'(z) = 0$ (one solves first for z_0 and then takes $\gamma = S(z_0)$). Then according to [3] we have, for some $c > 0$, $\boxed{a_n^{(w)} \sim c\,\gamma^n\, n^{-\alpha},\ \text{where } \alpha = 1 + \pi/\arccos(\xi),\ \text{with } \xi = -\frac{\partial_x\partial_y S(z_0;1,1)}{\partial_x\partial_x S(z_0;1,1)}}$.

Plane bipolar posets counted by vertices correspond to $w(i,j) = \binom{i+j}{i}$, giving $S(z;x,y) = \frac{x}{y} z^{-2} + \frac{1}{1 - z/x - zy}$, $z_0 = \frac{3-\sqrt{5}}{2} \approx 0.38$, $\boxed{\gamma = \frac{1}{2}(11 + 5\sqrt{5}) \approx 11.09}$,

$\boxed{\xi = \frac{1}{4}(1+\sqrt{5}) \approx 0.81}$, and $\boxed{\alpha = 6}$. We recover, as expected in view of the previous section, the asymptotic constants γ and α for plane permutations, which were obtained in [4] (where c was also explicitly computed).

Plane bipolar posets counted by edges correspond to taking $w(i,j) = \mathbf{1}_{i\neq 0, j\neq 0}$, which gives $S(z;x,y) = \frac{x}{y}z^{-2} + \frac{z/x}{1-z/x}\frac{zy}{1-zy}$. We find that $z_0 \approx 0.54$ is the unique positive root of $z^4+z^3-3z^2+3z-1$, $\boxed{\gamma = 5z_0^3 + 7z_0^2 - 13z_0 + 9 \approx 4.80}$, $\boxed{\xi = 1 - z_0/2 \approx 0.73}$, and $\boxed{\alpha \approx 5.14}$. With the method in [3] one can also check that α is irrational (this amounts to checking that the minimal polynomial $P(X)$ of ξ is such that no prime factor of $P(\frac{1}{2}(X+1/X))$ is cyclotomic) so the generating function of plane bipolar posets by edges is not D-finite.

Finally for **transversal structures** we take $w(i,j) = \binom{i+j-2}{i-1}$ but to count by vertices we aggregate the steps into groups formed by a SE step followed by a (possibly empty) sequence of non-SE steps. The series for one (aggregated) step is $S(z;x,y) = \frac{xy^{-1}z^{-2}}{1-yx^{-1}z^2/(1-zx^{-1}-zy)}$, which gives $z_0 = 1/3$, $\boxed{\gamma = 27/2}$, $\boxed{\xi = 7/8}$, and $\boxed{\alpha \approx 7.21}$. Again the method of [3] ensures that the associated series is not D-finite. Another consequence of our estimate is that the coding procedure in [12] can be made asymptotically optimal, as it yields the bound $\gamma \leq 27/2$.

References

1. O. Bernardi and N. Bonichon. Intervals in Catalan lattices and realizers of triangulations. *Journal of Combinatorial Theory, Series A*, 116(1):55–75, 2009.
2. N. Bonichon, M. Bousquet-Mélou, and É. Fusy. Baxter permutations and plane bipolar orientations. *Séminaire Lotharingien de Combinatoire*, 61:B61Ah, 2010.
3. A. Bostan, K. Raschel, and B. Salvy. Non-D-finite excursions in the quarter plane. *Journal of Combinatorial Theory, Series A*, 121:45–63, January 2014.
4. M. Bouvel, V. Guerrini, A. Rechnitzer, and S. Rinaldi. Semi-Baxter and strong-Baxter: two relatives of the Baxter sequence. *SIAM Journal on Discrete Mathematics*, 32(4):2795–2819, 2018.
5. D. Denisov and V. Wachtel. Random walks in cones. *Annals of Probability*, 43(3):992–1044, 2015.
6. A. Elvey Price and M. Bousquet-Mélou. The generating function of planar Eulerian orientations. *Journal of Combinatorial Theory, Series A*, 172:105183, 2020. 37 pp.
7. D. Eppstein. Regular labelings and geometric structures. In *Algorithms and Computation*. Springer Berlin Heidelberg, 2010.
8. É. Fusy. Transversal structures on triangulations: A combinatorial study and straight-line drawings. *Discrete Mathematics*, 309(7):1870–1894, 2009.
9. É. Fusy. New bijective links on planar maps via orientations. *European Journal of Combinatorics*, 31(1):145–160, 2010.
10. R. Kenyon, J. Miller, S. Sheffield, and DB Wilson. Bipolar orientations on planar maps and SLE_{12}. *Annals of Probability*, 47(3):1240–1269, 2019.
11. G. Schaeffer. Planar maps. In *Handbook of Enumerative Combinatorics*. CRC Press, 2015.
12. T. Takahashi, R. Fujimaki, and Y. Inoue. A (4n-4)-bit representation of a rectangular drawing or floorplan. In *International Computing and Combinatorics Conference*, pages 47–55. Springer, 2009.