



HAL
open science

ABOUT THE NEGATIVE DIGITAL ROOT AND SOME OF ITS PROPERTIES RELATED TO THE MODULAR ARITHMETICS.

Paul Francisco Marrero Romero, Eduardo J. Acuña T

► **To cite this version:**

Paul Francisco Marrero Romero, Eduardo J. Acuña T. ABOUT THE NEGATIVE DIGITAL ROOT AND SOME OF ITS PROPERTIES RELATED TO THE MODULAR ARITHMETICS.. 2021. hal-03387683

HAL Id: hal-03387683

<https://hal.science/hal-03387683>

Preprint submitted on 20 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ABOUT THE NEGATIVE DIGITAL ROOT AND SOME OF ITS PROPERTIES RELATED TO MODULAR ARITHMETIC.

Paul Francisco Marrero R.^a, Eduardo J. Acuña Tarazona.^b

Abstract: In this paper we propose a general formula for calculating the digital root of any non-positive integer, and express this formula in two forms including one related to the floor function. We also present some properties of congruences that manifest themselves when working with negative digital roots.

Key Words: Negative Digital Root, Formula, Proof

1 INTRODUCTION.

The digital root of a non-negative integer is the digit obtained by an iterative process, which involves the addition of its digits until a consistent single-digit value is obtained.

In this paper we will be focusing more specifically on the direct formulas of the negative digit roots, which use modular arithmetic as a support. In the following we present the known formulas for the positive part:

1.1 Digital Root.

Directly for a base $b > 1$, we have that

$$\Phi_{b-1}^+(n) : \mathbb{N}_0 \rightarrow \mathbb{N}_0.$$

$$\Phi_{b-1}^+(n) = \begin{cases} 0, & \text{if } n = 0, \\ b - 1, & \text{if } n \neq 0, n \equiv 0 \pmod{b-1} \\ n \bmod (b-1), & \text{if } n \not\equiv 0 \pmod{b-1}. \end{cases}$$

^aInvestigador, Facultad de Matemática. Universidad Nacional Abierta de Venezuela

^bInvestigador. Universidad de Carabobo.

Another way in which we can define the formula in base b is:

- (1). $\Phi_{b-1}^+(n) = 1 + ((n - 1) \bmod (b - 1))$ if $n \neq 0$
- (2). $\Phi_{b-1}^+(n) = n - (b + 1) \lfloor \frac{n-1}{b-1} \rfloor$

Formulas proposed by Hall (1980).

2 NEGATIVE DIGITAL ROOT

2.1 Definition 1:

Directly for an integer base $b < -1$, and one integer $n \leq 0$, we have that $\Phi_{b+1}^-(n)$ will be given by:

$$\Phi_{b+1}^-(n) : \mathbb{Z}_0^- \rightarrow \mathbb{Z}_0^-.$$

$$\Phi_{b+1}^-(n) = \begin{cases} 0, & \text{if } n = 0, \\ b + 1, & \text{if } n \neq 0, n \equiv 0 \pmod{b + 1} \\ n \bmod (b + 1), & \text{if } n \not\equiv 0 \pmod{b + 1}. \end{cases}$$

2.2 Definition 2.

Considering the above, we can finally define the following as $\Phi_{b+1}^-(n)$ as follows:

$$\Phi_{b+1}^-(n) = ((n + 1) \bmod (b + 1)) - 1 \tag{1}$$

with b, n integers such that $b < -1$, and $n \leq 0$.

2.3 Definition 3: NEGATIVE DIGITAL ROOT NOVEN.

Let $b = -10$, for an integer $n \leq 0$, we have that:

$$\mathcal{U}_{-9}(n) = ((n + 1) \bmod (-9)) - 1 \tag{2}$$

For the case of negative roots, that is those roots that operate under a basis $b < -1 / b \in \mathbb{Z}^-$, we will work with the modular value $\text{mod } (b + 1)$, because, by definition of congruence, we have that $b < -1 \text{ mod } (b + 1)$, by property of it we know that $b + 1 \mid b - (-1)$. Otherwise, we can also say that for $b < -1$.

$$b^k \equiv -1^k \equiv -1 \text{ mod } (b + 1)$$

For an integer k such that $k \geq 0$.

The negative digital root can be expressed in relation to the floor function, as defined below.

3 NEGATIVE DIGITAL ROOT AS FLOOR FUNCTION.

Following the parameters given in the **Definition 1**, we have to:

$$\Phi_{b+1}^-(n) = n - (b + 1) \left\lfloor \frac{n + 1}{b + 1} \right\rfloor \quad (3)$$

Before providing a proof to support the above result, let us introduce the following definitions involving the floor function.

If x and y are both integers, then we define the following binary operations:

$$x \text{ mod } y = x - y \left\lfloor \frac{x}{y} \right\rfloor, \text{ if } y \neq 0 ; \quad (4)$$

According to Knuth (1972) we have that consequently:

- a) If $y > 0$, then $0 \leq x \text{ mod } y < y$;
- b) If $y < 0$, then $0 \geq x \text{ mod } > y$.

4 NEGATIVE DIGITAL ROOT THEOREM.

The negative digital root, under a basis $b < -1 / b \in \mathbb{Z}^-$, with an integer $n \leq 0$ can be expressed as the equation **(3)**.

5 PROOF:

According to the equation **(4)**, let's say that $r = x - y \left\lfloor \frac{x}{y} \right\rfloor$.

Then, let's say that $x = n + 1$ & $y = b + 1$, for integers b, n such that $b < -1$ and $n \leq 0$, then, substituting the given values of x and y , we have that:

$$r = (n + 1) - (b + 1) \left\lfloor \frac{n + 1}{b + 1} \right\rfloor. \quad (5)$$

Now since $r = x \bmod y$, we can also say that $r = (n + 1) \bmod (b + 1)$, and then substituting into the equation (1), it follows that.

$$\Phi_{b+1}^-(n) = r - 1 \quad (6)$$

Finally, taking (5) and substituting in (6), we are left with the following expression:

$$\begin{aligned} \Phi_{b+1}^-(n) &= (n + 1) - (b + 1) \left\lfloor \frac{n+1}{b+1} \right\rfloor - 1 \\ &= n - (b + 1) \left\lfloor \frac{n+1}{b+1} \right\rfloor. \end{aligned}$$

And the latter is consistent with the definitions related to the floor function (y), which implies that if y is a negative integer, then y is a negative integer:

$$0 \geq x \bmod y > y,$$

Ergo for our case it is satisfied that:

$$0 \geq (n + 1) \bmod (b + 1) > b + 1. \quad \blacksquare$$

6 SOME PROPERTIES OF THE NEGATIVE DIGITAL ROOT.

For some n, n', b non-positive integers, with $b < -1$, it follows that:

6.1 Property 1.

$$\mathbf{A.} \quad \Phi_{b+1}^-(n) + \Phi_{b+1}^-(n') \equiv \Phi_{b+1}^-(n + n') \bmod (b + 1).$$

6.1.1 Proof:

As we know, by definition of negative digital root, it is valid to say that:

$$\Phi_{b+1}^-(n) = n - (b + 1) \left\lfloor \frac{n + 1}{b + 1} \right\rfloor, \text{ and } \Phi_{b+1}^-(n') = n' - (b + 1) \left\lfloor \frac{n' + 1}{b + 1} \right\rfloor.$$

For n, n' both non-positive integers and $b < -1$. So now, if we add $\Phi_{b+1}^-(n)$ with $\Phi_{b+1}^-(n')$, we get the following:

$$\begin{aligned} \Phi_{b+1}^-(n) + \Phi_{b+1}^-(n') &= \left[n - (b + 1) \left\lfloor \frac{n+1}{b+1} \right\rfloor \right] + \left[n' - (b + 1) \left\lfloor \frac{n'+1}{b+1} \right\rfloor \right] \\ &= n + n' - (b + 1) \left[\left\lfloor \frac{n+1}{b+1} \right\rfloor + \left\lfloor \frac{n'+1}{b+1} \right\rfloor \right] \end{aligned}$$

Therefore, if:

$$\Phi_{b+1}^-(n') = (n + n') - (b + 1) \left\lfloor \frac{(n + n') + 1}{b + 1} \right\rfloor$$

Then we can subtract both

$$\begin{aligned} (\Phi_{b+1}^-(n)) + \Phi_{b+1}^-(n') - \Phi_{b+1}^-(n + n') &= n + n' - (b + 1) \left(\left\lfloor \frac{n+1}{b+1} \right\rfloor + \left\lfloor \frac{n'+1}{b+1} \right\rfloor \right) - \left[(n + n') - (b + 1) \left\lfloor \frac{(n+n')+1}{b+1} \right\rfloor \right] \\ &= -(b + 1) \left(\left\lfloor \frac{n+1}{b+1} \right\rfloor + \left\lfloor \frac{n'+1}{b+1} \right\rfloor \right) + (b + 1) \left\lfloor \frac{(n+n')+1}{b+1} \right\rfloor, \\ &= (b + 1) \left[\left\lfloor \frac{(n+n')+1}{b+1} \right\rfloor - \left(\left\lfloor \frac{n+1}{b+1} \right\rfloor + \left\lfloor \frac{n'+1}{b+1} \right\rfloor \right) \right] \end{aligned}$$

and of the latter we can say that:

$$\left\lfloor \frac{n + (n')}{b + 1} \right\rfloor = q; \quad \left(\left\lfloor \frac{n + 1}{b + 1} \right\rfloor + \left\lfloor \frac{n' + 1}{b + 1} \right\rfloor \right) = t,$$

where q, t are both integers. Moreover, substituting we are finally left with:

$$(\Phi_{b+1}^-(n) + \Phi_{b+1}^-(n')) - \Phi_{b+1}^-(n + n') = (b + 1)[q - t],$$

\therefore

$$(b + 1) \mid (\Phi_{b+1}^-(n) + \Phi_{b+1}^-(n')) - \Phi_{b+1}^-(n + n'). \blacksquare$$

6.2 Property 2.

$$\mathbf{B.} \quad [\Phi_{b+1}^-(n) - \Phi_{b+1}^-(n')] + (b + 1) \equiv \Phi_{b+1}^-(n - n') + (b + 1) \pmod{b + 1}.$$

6.2.1 Proof:

After checking the proof of **Property 1**, we will consider that this proof for **Property 2** is self-evident.

6.3 Property 3.

$$\mathbf{C.} \quad \Phi_{b+1}^-(n) \cdot \Phi_{b+1}^-(n') \equiv \Phi_{|b|-1}^+(n \cdot n') \pmod{b + 1}.$$

6.3.1 Proof:

By definition, we know that for the non-positive integers n, n' and $b < -1$, we have that:

$$\begin{aligned}
\Phi_{b+1}^-(n) \cdot \Phi_{b+1}^-(n') &= \left[n - (b+1) \left\lfloor \frac{n+1}{b+1} \right\rfloor \right] \cdot \left[n' - (b+1) \left\lfloor \frac{n'+1}{b+1} \right\rfloor \right] \\
&= n \cdot n' - n(b+1) \left\lfloor \frac{n'+1}{b+1} \right\rfloor - n'(b+1) \left\lfloor \frac{n+1}{b+1} \right\rfloor + (b+1)^2 \left\lfloor \frac{n+1}{b+1} \right\rfloor \left\lfloor \frac{n'+1}{b+1} \right\rfloor \\
&= n \cdot n' + (b+1) \left[(b+1) \left\lfloor \frac{n+1}{b+1} \right\rfloor \left\lfloor \frac{n'+1}{b+1} \right\rfloor - n \left\lfloor \frac{n'+1}{b+1} \right\rfloor - n' \left\lfloor \frac{n+1}{b+1} \right\rfloor \right]. \quad (*)
\end{aligned}$$

From (*) we can state that:

$$\left[(b+1) \left\lfloor \frac{n+1}{b+1} \right\rfloor \left\lfloor \frac{n'+1}{b+1} \right\rfloor - n \left\lfloor \frac{n'+1}{b+1} \right\rfloor - n' \left\lfloor \frac{n+1}{b+1} \right\rfloor \right] = t,$$

Such that $t \in \mathbb{Z}$.

Now, we also know by definition of digital root.

$$\Phi_{|b|-1}^+(n \cdot n') = (n \cdot n') - (|b| - 1) \left\lfloor \frac{(n \cdot n') - 1}{|b| - 1} \right\rfloor. \quad (**)$$

If we substitute t in (*) and then subtract (*) with (**) we have as a result that:

$$[\Phi_{b+1}^-(n) \cdot \Phi_{b+1}^-(n')] - [\Phi_{|b|-1}^+(n \cdot n')] = [(b+1)t + n \cdot n'] - \left[(n \cdot n') - (|b| - 1) \left\lfloor \frac{(n \cdot n') - 1}{|b| - 1} \right\rfloor \right].$$

Then as we know by property of the absolute value $|b| = -b$ if $b < 0$. therefore.

$$\begin{aligned}
[\Phi_{b+1}^-(n) \cdot \Phi_{b+1}^-(n')] - [\Phi_{|b|-1}^+(n \cdot n')] &= [(b+1)t + n \cdot n'] - \left[(n \cdot n') - (-b - 1) \left\lfloor \frac{(n \cdot n') - 1}{|b| - 1} \right\rfloor \right] \\
&= (b+1)t + n \cdot n' - (n \cdot n') - \left((b+1) \left\lfloor \frac{(n \cdot n') - 1}{|b| - 1} \right\rfloor \right) \\
&= (b+1)t - (b+1) \left\lfloor \frac{(n \cdot n') - 1}{|b| - 1} \right\rfloor \\
&= (b+1) \left[t - \left\lfloor \frac{(n \cdot n') - 1}{|b| - 1} \right\rfloor \right]
\end{aligned}$$

∴

$$(b+1) \mid [\Phi_{b+1}^-(n) \cdot \Phi_{b+1}^-(n')] - [\Phi_{|b|-1}^+(n \cdot n')]. \quad \blacksquare$$

7 SOME EXAMPLES.

- 1) Let $n = -267$, now calculate its negative digital root.

Solution.

This negative digital root will be calculated using the equation **(3)**. We will work for this case with a negative base $b = -10$ (novem case). Now calculating we have that:

$$\begin{aligned}
 \Phi_{-9}^{-}(-267) &= -267 - (-9) \left\lfloor \frac{-267+1}{-9} \right\rfloor \\
 &= -267 - (-9) \left\lfloor \frac{-266}{-9} \right\rfloor \\
 &= -267 - (-9)[29.5] \\
 &= -267 - (-261) \\
 &= -6.
 \end{aligned}$$

Therefore

$$\Phi_{-9}^{-}(-267) = -6.$$

- (2)** Let there be two integers such that $n = -17$ and $n' = -11$. Use a base $b = -10$, and then add the negative digital roots of n with n' .

Solution.

Calculating we have that

$$\Phi_{-9}^{-}(-17) = -8$$

and

$$\Phi_{-9}^{-}(-11) = -2$$

then adding the two together we are left with

$$\begin{aligned}
 \Phi_{-9}^{-}(-17) + \Phi_{-9}^{-}(-11) &= -8 + (-2) \\
 &= -10.
 \end{aligned}$$

On the other hand, if we first add n to n' and then calculate its negative digit root, we find that

$$\begin{aligned}
 \Phi_{-9}^{-}(-17 + (-11)) &= \Phi_{-9}^{-}(-28) \\
 &= -1.
 \end{aligned}$$

Finally, we can see that **property 1** is satisfied, since

$$\Phi_{-9}^{-}(-17) + \Phi_{-9}^{-}(-11) \equiv \Phi_{-9}^{-}(-28) \pmod{-9}.$$

(3) Using the same values for n , n' and b from the previous question; let's check property 3.

Solution.

Now calculating

$$\begin{aligned} \Phi_{-9}^{-}(-17) \cdot \Phi_{-9}^{-}(-11) &= (-8) \cdot (-2) \\ &= 16. \end{aligned}$$

Then we perform the following calculation

$$\Phi_{|-10|_{-1}}^{+}((-17) \cdot (-11)) = \Phi_9^{+}(187)$$

Now applying the positive digital root formula, we can see that

$$\begin{aligned} \Phi_9^{+}(187) &= 187 - (9) \lfloor \frac{186}{9} \rfloor \\ &= 187 - 9(20) \\ &= 187 - 180 \\ &= 7. \end{aligned}$$

Therefore, in accordance with the above, we can verify the fulfillment of **property 3**, since

$$16 \equiv 7 \pmod{-9}.$$

References

- José de Jesús Camacho Medina (2020). Fórmulas para calcular la Raíz Digital y la Suma Digital de un Número Entero Positivo [En línea]. Disponible en MasScience: <https://www.masscience.com/2020/06/12/formulas-para-calcular-la-raiz-digital-y-la-suma-digital-de-un-numero-entero-positivo/>
- Hall, F. M. (1980), An Introduction into Abstract Algebra, 1 (2nd ed.), Cambridge, U.K.: CUP Archive, p. 101
- D. E. Knuth. The Art of Computer Programming, Volume 1, Fascicle 1: MMIX—A RISC Computer for the New Millennium. Addison-Wesley Professional, 1972.