

Some new results in post-Pareto analysis

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South Pacific Optimization Meeting: SPOM¹³

Newcastle, February 9-12, 2013

Outline

- 1 **Introduction**
- 2 **Post-Pareto optimization in stochastic MOP**
 - Problems statement
 - Preliminaries
 - Main results in the decision space
 - Convergence in the outcome space
- 3 **Post-Pareto optimization in MO convex control problems in Hilbert spaces**
 - Grand coalition p -player cooperative differential game
 - Post Pareto optimization for MO control of parabolic systems

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- History : Edgeworth (XIX), Pareto (1906), Kuhn-Tucker (1951) . . .
- In a (MOP) we deal with several conflicting objectives. The solution set (called Pareto or efficient set) consists of the feasible solutions such that none of the objectives values can be improved further without deteriorating another.
- The Pareto set is often very large (infinite, and even unbounded), and technically speaking each Pareto solution is acceptable.
- Problem : how to choose one solution?
- One possible answer: optimize a *scalar* function f over the Pareto set associated with a (MOP), i.e. *post-Pareto analysis*.
- Practical applications : production planning and portfolio management.
- Solving this problem one may avoid generate all the Pareto set.

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Optimizing over the Pareto control set is a difficult problem!

Even in the simpler case of a **multi-objective mathematical programming problem** optimizing over the Pareto set is already very difficult because:

- the Pareto set is not described explicitly;
- the Pareto set is not convex even for linear multi-objective problem.

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History of the problem in the simpler case

Beginning with the work of Philip (1972) the problem of optimizing a scalar objective over the Pareto set of a multiobjective **mathematical programming problem** has been studied in the last decades by :

- Benson (1984 -->)
- Dauer (1991 -->)
- Craven (1991)
- HB (1993 -->)
- Fülöp (1994)
- An, Tao and Muu (1996)
- Horst and Thoai (1999)
- Yamamoto (2002) etc.

History of the problem in the simpler case

The results which I present in my talk seem to be the first approaches of the problem of optimizing over a Pareto set in **stochastic setting** as well as in **multiobjective control setting**.

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Stochastic optimization over Pareto in DS

This part of my talk is based on a very recent paper written jointly with my Ph.D. student Julien Collonge:

Stochastic Optimization over a Pareto Set Associated with a Stochastic Multi-Objective Optimization Problem

www.optimization-online.org/DB_HTML/2013/01/3733.html

Stochastic optimization over Pareto in DS

$$(SOOPD) \quad \min_{x \in E^w} \mathbb{E} \left[F^0(x, \xi(\cdot)) \right] \quad \text{where}$$

- $\xi : \Omega \rightarrow \mathbb{R}^d$ is a *random vector* defined on $(\Omega, \mathcal{F}, \mathbb{P})$,
- $x \in \mathbb{R}^n$ is a *deterministic vector*,
- $\mathbb{E} \left[F^0(x, \xi(\cdot)) \right]$ is the *expectation* of the r.v. $\omega \mapsto F^0(x, \xi(\omega))$
- E^w is the *weakly Pareto set* associated with

$$(SMOP) \quad \text{MIN}_{x \in S} \mathbb{E} \left[F(x, \xi(\cdot)) \right]$$

where the *feasible set* $S \subset \mathbb{R}^n$.

- The *multi-objectives* are given by

$$\mathbb{R}^n \times \Omega \ni (x, \omega) \mapsto F(x, \xi(\omega)) = \left(F^1(x, \xi(\omega)), \dots, F^r(x, \xi(\omega)) \right) \in \mathbb{R}^r$$

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Stochastic optimization over Pareto in DS

For (SMOP) the element x^* in S is said to be

- *Pareto* iff there is no element $x \in S$ satisfying

$$\forall j \in \{1, \dots, r\} \mathbb{E}_\xi[F^j(x, \cdot)] \leq \mathbb{E}_\xi[F^j(x^*, \cdot)]$$

and

$$\exists j_0 \in \{1, \dots, r\} \mathbb{E}_\xi[F^{j_0}(x, \cdot)] < \mathbb{E}_\xi[F^{j_0}(x^*, \cdot)]$$

- *weakly Pareto* iff there is no element $x \in S$ satisfying

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Stochastic optimization over Pareto in DS

$$(SOOPD) \quad \min_{x \in E^w} \mathbb{E} \left[F^0(x, \xi(\cdot)) \right]$$

Remark

For problem (SOOPD) we need to assume that (SMOP) is **strictly convex**.

Stochastic optimization over Pareto in OS

In this case the scalar function to be minimized over the properly Pareto set associated with (SMOP) **depends on the expectations of the objectives**, i.e.

$$(SOOPO) \quad \min_{x \in E} f\left(\mathbb{E}\left[F\left(x, \xi(\cdot)\right)\right]\right)$$

where $f : \mathbb{R}^r \rightarrow \mathbb{R}$ is a scalar deterministic continuous function and E is the Pareto set associated with the Stochastic Multi-Objective Optimization Problem (SMOP) defined above.

Remark

In this case ***no convexity assumption is necessary***. Thus, for problem (SOOPO), we do not assume the convexity of (SMOP).

SAA approach

- If the expected value functions can be computed directly, the problem becomes a deterministic one.
- In most cases, the closed form of the expected values is very difficult to obtain. This is case considered here.
- In order to give approximations, we apply the well-known Sample Average Approximation (SAA-N, where N is the sample size) method.
- Under reasonable and suitable assumptions, we show that the SAA-N weakly Pareto sets converge w.r.t. the Hausdorff-Pompeiu distance to its true counterpart.
- Moreover, we show that the sequence of SAA-N optimal values converges to the true optimal value as the sample size increase, and any cluster point of any sequence of SAA-N optimal solutions is almost surely a true optimal solution.

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Some technical aspects and definitions

- The random vector $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}^d$ generates the **probability measure** \mathbb{P}_ξ on $(\mathbb{R}^d, \mathcal{B}_d)$ with $\mathbb{P}_\xi(B) = \mathbb{P}(\xi^{-1}(B))$, where \mathcal{B}_d is the Borel σ -field in \mathbb{R}^d .
- The **support** of ξ is the smallest closed set $\Xi \subset \mathbb{R}^d$ s.t. $\mathbb{P}_\xi(\Xi) = 1$.
- For each $x \in \mathbb{R}^n$, $i = 0, 1, \dots, r$, we suppose that the function

$$F_i(x, \cdot) : (\Xi, \mathcal{B}_\Xi, \mathbb{P}_\xi) \rightarrow (\mathbb{R}, \mathcal{B}_1)$$

is measurable, and we say that F_i is a **random function**.

- We have

$$\mathbb{E} \left[F_i(x, \xi(\cdot)) \right] = \mathbb{E}_\xi \left[F_i(x, \cdot) \right] := \int_\Xi F_i(x, \eta) d\mathbb{P}_\xi(\eta).$$

Some technical aspects and definitions

Let $\tilde{\Xi} = \prod_{N=1}^{\infty} \Xi$ and let $\tilde{\mathcal{B}} = \otimes_{N=1}^{\infty} \mathcal{B}_{\Xi}$ be the smallest σ -algebra on $\tilde{\Xi}$ generated by all sets of the form $B_1 \times B_2 \times \dots \times B_N \times \Xi \times \Xi \times \dots$, $B_k \in \mathcal{B}_{\Xi}$, $k = 1, \dots, N$, $N = 1, 2, \dots$. The next Theorem is well-known from General Measure Theory.

Theorem

There exists a unique probability $\tilde{\mathbb{P}}_{\xi}$ on $(\tilde{\Xi}, \tilde{\mathcal{B}})$ such that

$$\tilde{\mathbb{P}}_{\xi}(B_1 \times B_2 \times \dots \times B_N \times \Xi \times \Xi \times \dots) = \prod_{k=1}^N \mathbb{P}_{\xi}(B_k) \text{ for all } N = 1, 2, \dots$$

and $B_k \in \mathcal{B}_{\Xi}$ for all $k = 1, \dots, N$.

Some technical aspects and definitions

- For each $N \in \mathbb{N}^*$, $x \in \mathbb{R}^n$, $i = 0, 1, \dots, r$, consider the function

$$\hat{F}_N^i(x, \cdot) : (\tilde{\Xi}, \tilde{\mathcal{B}}, \tilde{\mathbb{P}}_\xi) \rightarrow \mathbb{R} \quad (1)$$

$$\tilde{\xi} = (\xi_1, \xi_2, \dots) \mapsto \frac{1}{N} \sum_{k=1}^N F_i(x, \xi_k)$$

called **N-sample average approximation (SAA-N function)**.

- Put

$$\hat{F}_N(x, \cdot) = (\hat{F}_N^1(x, \cdot), \dots, \hat{F}_N^r(x, \cdot)).$$

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Some technical aspects and definitions

For each $N \in \mathbb{N}^*$ and $\tilde{\xi} \in \tilde{\Xi}$, denote by $E_N^w(\tilde{\xi})$ the weakly-Pareto set associated with the N -Sample Average Approximation MOP

$$(MOP_N(\tilde{\xi})) \quad \text{MIN}_{x \in S} \mathbb{R}_+^r \hat{F}_N(x, \tilde{\xi})$$

Thus, for each $N \geq 1$, the SAA- N problem associated with our “true” problem (SOOPD) is

$$(OOPD_N(\tilde{\xi})) \quad \min_{x \in E_N^w(\tilde{\xi})} \hat{F}_N^0(x, \tilde{\xi})$$

Hypotheses

We assume the following:

- S is a non empty compact convex subset of \mathbb{R}^n ,
- the i.i.d property holds for the random process $\tilde{\xi} \in \tilde{\Xi}$,
- $\forall j = 0, \dots, r$, $x \mapsto F^j(x, \xi)$ is finite valued, strictly convex and continuous on S for a.e. $\xi \in \Xi$.
- $\forall j = 0, \dots, r$, F^j is dominated by an integrable function K^j , i.e.

$$\mathbb{E}_{\xi} \left[K^j(\cdot) \right] < \infty$$

$$\left| F^j(x, \xi) \right| \leq K^j(\xi) \quad \text{for all } x \in S \text{ and a.e. } \xi \in \Xi$$

Convergence results in the decision space

Theorem

For almost all $\tilde{\xi}$ in $\tilde{\Xi}$, the Hausdorff-Pompeiu distance^a between the SAA- N weakly Pareto sets $E_N^W(\tilde{\xi})$ and the true weakly Pareto set E^W tends to zero as N tends to infinity, i.e.

$$\tilde{\mathbb{P}}_{\xi} \left(\left\{ \tilde{\xi} \in \tilde{\Xi} \mid \lim_{N \rightarrow \infty} \mathbb{H} \left(E_N^W(\tilde{\xi}), E^W \right) = 0 \right\} \right) = 1$$

^aFor any bounded non empty sets $A, B \subset \mathbb{R}^n$ the Hausdorff-Pompeiu distance $\mathbb{H}(A, B) := \max(\mathbb{D}(A, B), \mathbb{D}(B, A))$, where the deviation $\mathbb{D}(A, B) := \sup_{x \in A} d(x, B)$.

Convergence results in the decision space

Theorem

For almost all $\tilde{\xi}$ in $\tilde{\Xi}$, the sequence of SAA- N optimal values

$$\left(V_N(\tilde{\xi}) := \min_{x \in E_N^w(\tilde{\xi})} \hat{F}_N^0(x, \tilde{\xi}) \right)_{N \geq 1}$$

converges to the true optimal value

$$V := \min_{x \in E^w} \mathbb{E}_{\xi} [F^0(x, \cdot)].$$

In particular, for almost all $\tilde{\xi}$ in $\tilde{\Xi}$, all cluster points of any sequence $(x_N^*)_{N \geq 1}$ in $\operatorname{argmin}_{x \in E_N^w(\tilde{\xi})} \hat{F}_N^0(x, \tilde{\xi})$ are optimal solutions of the true problem (SOOPD).

The problem in the outcome space

$$(SOOPO) \quad \min_{x \in E} f\left(\mathbb{E}\left[F\left(x, \xi(\cdot)\right)\right]\right)$$

where $f : \mathbb{R}^r \rightarrow \mathbb{R}$ is a scalar deterministic continuous function and E is the **Pareto set** associated with our Stochastic MOP

$$(SMOP) \quad \text{MIN}_{x \in S} \mathbb{E}_{\mathbb{R}_+^r} \left[F\left(x, \xi(\cdot)\right) \right]$$

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Main results

Theorem

For almost all $\tilde{\xi}$ in $\tilde{\Xi}$, the Hausdorff-Pompeiu distance between the SAA- N Pareto sets image $\hat{F}_N(E_N^p(\tilde{\xi}), \tilde{\xi})$ and the true Pareto set image $\mathbb{E}_\xi[F(E^p, \cdot)]$ tends to zero as N tends to infinity, i.e

$$\tilde{\mathbb{P}}_\xi \left(\left\{ \tilde{\xi} \in \tilde{\Xi} \mid \lim_{N \rightarrow \infty} \mathbb{H} \left(\hat{F}_N(E_N^p(\tilde{\xi}), \tilde{\xi}), \mathbb{E}_\xi[F(E^p, \cdot)] \right) = 0 \right\} \right) = 1$$

Main results

Theorem

For almost all $\tilde{\xi}$ in $\tilde{\Xi}$, the sequence of SAA- N optimal values
 $(V_N(\tilde{\xi})) := \inf_{x \in E_N^p(\tilde{\xi})} \hat{F}_N^0(x, \tilde{\xi})_{N \geq 1}$ converges to the true optimal value
 $V := \inf_{x \in E^p} f(\mathbb{E}_\xi[F^0(x, \cdot)])$.

Section

- 1 Introduction
- 2 Post-Pareto optimization in stochastic MOP
 - Problems statement
 - Preliminaries
 - Main results in the decision space
 - Convergence in the outcome space
- 3 **Post-Pareto optimization in MO convex control problems in Hilbert spaces**
 - Grand coalition p -player cooperative differential game
 - Post Pareto optimization for MO control of parabolic systems

This part of my talk is based on the paper

- *Post-Pareto Analysis for Multiobjective Parabolic Control Systems*, Mathematics and its Applications - Annals of AORS (in press), www.optimization-online.org/DB_HTML/2012/07/3545.html

and generalizes the paper

- H. Bonnel & C.Y. Kaya, *Optimization over the Efficient Set of Multi-objective Control Problems*, JOTA, **147**(1), 93-112, 2010.

Some connected results, but in a different setting, are presented in

- H. Bonnel, J. Morgan, *Semivectorial Bilevel Convex Optimal Control Problems: An Existence Result*, SIAM Journal on Control and Optimization, Vol. 50, No. 6, 3224-3241, 2012.
- H. Bonnel, J. Morgan, *Optimality Conditions for Semivectorial Bilevel Convex Optimal Control Problem*, Computational and Analytical Mathematics, Springer Proceedings in Mathematics in honor of Jonathan Borwein's 60th birthday, H. Bauschke et M. Théra eds., 30p. (in press), www.csef.it/WP/wp301.pdf

Grand coalition p -player cooperative differential game

Grand coalition p -player cooperative differential game \iff p -objective control problem

Grand coalition p -player cooperative differential game \iff p -objective control problem

Player i , ($i = 1, \dots, p$)

- interacts with the dynamics of the game

$$\dot{z}(t) = g(t, z(t), u_1(t), \dots, u_p(t)), \quad t \in [0, T], \quad z(0) = z_0$$

using his own control $u_i(t) \in U_i$ at each moment $t \in [0, T]$;

$z(t)$ denotes the state, and the initial state z_0 is specified.

- wants to minimise his own objective J_i

$$J_i(z, u_1, \dots, u_p) = l_i(z(T)) + \int_0^T L_i(t, z(t), u_1(t), \dots, u_p(t)) dt;$$

Grand coalition p -player cooperative differential game \iff p -objective control problem

Player i , ($i = 1, \dots, p$)

- **interacts** with the **dynamics** of the game

$$\dot{z}(t) = g(t, z(t), u_1(t), \dots, u_p(t)), \quad t \in [0, T], \quad z(0) = z_0$$

using his own **control** $u_i(t) \in U_i$ at each moment $t \in [0, T]$;

$z(t)$ denotes the **state**, and the initial state z_0 is specified.

- **wants to minimise** his own **objective** J_i

$$J_i(z, u_1, \dots, u_p) = l_i(z(T)) + \int_0^T L_i(t, z(t), u_1(t), \dots, u_p(t)) dt;$$

Grand coalition p -player cooperative differential game \iff p -objective control problem

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- wants to minimise his own objective J_i

$$J_i(z, u_1, \dots, u_p) = l_i(z(T)) + \int_0^T L_i(t, z(t), u_1(t), \dots, u_p(t)) dt;$$

Grand coalition p -player cooperative differential game \iff p -objective control problem

Denote

$$u(t) = (u_1(t), \dots, u_p(t)), \quad U = U_1 \times \dots \times U_p.$$

The p -player cooperative differential game can be written as a p -objective control problem

$$\text{MIN}_{\mathbb{R}_+^p} (J_1(x, u), \dots, J_p(x, u)) \quad \text{subject to}$$

$$\begin{cases} \dot{z}(t) = g(t, z(t), u(t)) & t \in [0, T], \\ u(t) \in U, & t \in [0, T], \\ z(0) = z_0 \end{cases}$$

Major drawback

The Pareto set is **large** (often it is infinite)



Major drawback

The Pareto set is **large** (often it is infinite)



Problem: **how to select a Pareto process ?**

Possible answer

Optimize a **scalar objective** over the Pareto set

The problem

$$(PPOCP) \quad \min J_0(z, u) \quad \text{s.t.}$$

(z, u) is a weakly (or properly) Pareto control process for the following multi-objective convex control optimization problem in Hilbert spaces

$$(MOCCOP) \quad \text{MIN}_{\mathbb{R}_+^p} [J_1(z, u), \dots, J_p(z, u)] \quad \text{s.t.}$$

$$\frac{dz}{dt}(t) + A(t)z(t) = B(t)u(t) \quad \text{a.e. on }]0, T[\quad (2)$$

$$u(t) \in U \quad \text{a.e. on }]0, T[\quad (3)$$

$$z(0) = z_0 \quad (4)$$

$$z(T) \in Z_T \quad (5)$$

Hypotheses

- $A(t) \in \mathcal{L}(V, V')$, ($0 < t < T$), V real Hilbert space, V' its topological dual.
- $B(t) \in \mathcal{L}(U, V')$, ($0 < t < T$), U real Hilbert space.
- There exists a real Hilbert space H s.t. $V \subset H$ with linear continuous and dense embedding. Then $H' \subset V'$ with linear continuous and dense embedding.
- We identify $H \equiv H'$ (Riesz' theorem), hence

$$V \subset H \subset V',$$

with linear continuous dense embeddings.

- For $v' \in V'$ and $v \in V$, we denote by $(v' | v)$ the value of the functional v' in v which coincide with the inner product of H (denoted in the same way) if $v' \in H$.
- The norm of H (respectively V and V') will be denoted by $|\cdot|$ (respectively by $\|\cdot\|$ and $\|\cdot\|_*$).

Hypotheses

- There exist $\alpha \in \mathbb{R}$ and $\omega > 0$ s.t., for all $v \in V$, $t \in]0, T[$,

$$(A(t)v | v) + \alpha |v|^2 \geq \omega \|v\|^2.$$

- For all $v, w \in V$, the function $t \mapsto (A(t)v | w)$ is measurable on $]0, T[$, and there is a constant $c > 0$, such that

$$\|A(t)\|_{\mathcal{L}(V, V')} \leq c \quad \text{a.e. on }]0, T[.$$

- for any $u \in L^2(0, T; \mathcal{U})$, the function $t \mapsto B(t)u(t)$ is measurable on $]0, T[$ and

$$\|B(t)\|_{\mathcal{L}(\mathcal{U}, V')} \leq c \quad \text{a.e. on }]0, T[.$$

- $z_0 \in H$, U is a nonempty closed convex subset of \mathcal{U} . The “target set” Z_T is a nonempty closed convex subset of H .

Hypotheses

- $J_i : L^2(0, T; V) \cap C(0, T; H) \times L^2(0, T; \mathcal{U}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is given by

$$J_i(z, u) = l_i(z(T)) + \int_0^T L_i(t, z(t), u(t)) dt,$$

where $L_i :]0, T[\times V \times \mathcal{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a Borel function such that for each $t \in]0, T[$, the function $L_i(t, \cdot, \cdot)$ is lower semicontinuous and proper, $l_i : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is supposed proper, lower semicontinuous, and there are some real numbers β_i, γ_i and $a \in L^1(0, T)$ such that

$$\forall (v, u) \in V \times \mathcal{U} \quad L_i(t, v, u) \geq a_i(t) + \beta_i \|v\|^2 + \gamma_i \|u\|_{\mathcal{U}}^2, \quad t \in]0, T[$$

$$i = 0, 1, \dots, p.$$

- the objectives of (MOCCOP) problem are supposed convex, i.e. for all $i = 1, \dots, p$, and $t \in]0, T[$, the functions $L_i(t, \cdot, \cdot)$, l_i are convex. J_0 is not necessarily convex.

Example 1

$\text{MIN}_{\mathbb{R}_+^p} [J_1(z, u), \dots, J_p(z, u)] \quad \text{s.t. } (z, u) \text{ verifies}$

$$\frac{\partial z}{\partial t} - \text{div}_x(k(x)\nabla_x z) - q(x)z = 0 \quad \text{a.e. in } Q \quad (6)$$

$$\frac{\partial z}{\partial n} + \rho(x)z = u \quad \text{a.e. in } \Sigma \quad (7)$$

$$z(x, 0) = z_0(x) \quad \text{a.e. in } \Omega \quad (8)$$

$$u(t) \in U \quad \text{a.e. in }]0, T[\quad (9)$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set, its boundary Γ is of class C^1 ,

$$Q = \Omega \times]0, T[, \quad \Sigma = \Gamma \times]0, T[,$$

$$k \in C^1(\bar{\Omega}), k(x) > 0, \forall x \in \bar{\Omega}, q \in C(\bar{\Omega}), \rho \in C(\Gamma), \rho \geq 0.$$

Example 1

The function $z = z(x, t) : \Omega \times [0, T] \rightarrow \mathbb{R}$ is the *state*, and the function $u(t) \in L^2(\Gamma)$ is the (boundary) *control* at the moment $t \in]0, T[$, supposed square integrable, i.e. $u \in L^2(0, T; L^2(\Gamma))$. The initial value $z_0 \in L^2(\Omega)$ is specified.

Put $V = H^1(\Omega)$, $H = L^2(\Omega)$, $\mathcal{U} = L^2(\Gamma)$, and define $A(t) \equiv A \in \mathcal{L}(V, V')$, $B(t) \equiv B \in \mathcal{L}(\mathcal{U}, V')$ by

$$\forall z, w \in V \quad (Az | w) = \int_{\Omega} (k \nabla z \cdot \nabla w - qzw) dx + \int_{\Gamma} k_{\rho} z w d\sigma$$

$$\forall u \in \mathcal{U}, w \in V, \quad (Bu | w) = \int_{\Gamma} k u w d\sigma.$$

Note that the last boundary integral is well defined since for each element w of $H^1(\Omega)$ its *trace* on Γ , $w|_{\Gamma}$ is well defined and belongs to $L^2(\Gamma)$

Example 1

It is easy to see that using Green formula, the variational formulation this problem can be written in the abstract form (MOCCOP), and A, B satisfy all the hypotheses.

Suppose we have p captors, the i^{th} captor being located on the boundary in a measurable zone $\Gamma_i \subset \Gamma$, $i = 1, \dots, p$, and the **desirable state** is $z_d \in L^2(0, T; V)$. Suppose that the sets $(\Gamma_i)_{1 \leq i \leq p}$ are mutually disjoint and the values of the desired state are known only on the boundary zone Γ_i . Consider $l_i = 0$, and L_i described by

$$\forall (t, z, u) \in]0, T[\times V \times \mathcal{U} \quad L_i(t, z, u) = \int_{\Gamma_i} (z - z_d(t))^2 d\sigma + \langle R_i u, u \rangle_{\mathcal{U}},$$

where $R_i \in \mathcal{L}(\mathcal{U})$ is a nonnegative symmetric operator. Finally, let us consider $L_0 = 0$, $\forall x_1 \in H$, $l_0(x_1) = \|x_1\|$.

Example 1

Roughly speaking, our problem of optimizing $J_0(z, u)$ over the set of weakly (or properly) Pareto processes of the multi-objective control problem presented in this example means that amongst all the (weakly or properly) Pareto controls, i.e., amongst all the controls which are such that we cannot improve an objective J_i ($i \geq 1$) without deteriorating further another objective J_k , ($k \geq 1$), we are looking for the control which realizes the minimal final state norm.

Example 2: grand coalition p -player cooperative differential game

In this example (MOCCOP) problem is stated as a *grand coalition of a p -player cooperative differential game*:

- \mathcal{U} is a product of p Hilbert spaces $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_p$
- $U = U_1 \times \cdots \times U_p$, $u(t) = (u_1(t), \dots, u_p(t))$.
- $B(t)u(t) = B_1(t)u_1(t) + \cdots + B_p(t)u_p(t)$, with $B_i(t) \in \mathcal{L}(\mathcal{U}, V')$.
- The player i has the objective J_i and *interacts with the system* with the control $u_i \in L^2(0, T; \mathcal{U}_i)$.
- Consider that a *“supervisor”* of the game has its own objective J_0 .

Thus, amongst all the controls which are such that no player can improve his objective without further deteriorating the performance of another player, the supervisor chooses the control which optimizes his objective.

Example 2: grand coalition p -player cooperative differential game

Suppose we have the same diffusion process as in previous example, but the boundary control is different :

$$\frac{\partial z}{\partial t} - \operatorname{div}_x(k(x)\nabla_x z) - q(x)z = 0 \quad \text{a.e. in } Q \quad (10)$$

$$\frac{\partial z}{\partial n} + \rho(x)z = \sum_{i=1}^p u_i \quad \text{a.e. in } \Sigma \quad (11)$$

$$z(x, 0) = z_0(x) \quad \text{a.e. in } \Omega \quad (12)$$

$$u(t) \in U \quad \text{a.e. in }]0, T[\quad (13)$$

Example 2: grand coalition p -player cooperative differential game

The functional spaces are the same except that we take $\mathcal{U} = \prod_{i=1}^p \mathcal{U}_i$ where $\mathcal{U}_i = L^2(\Gamma)$, $i = 1, \dots, p$, and $U_i = \{u_i \in L^2(\Gamma) \mid \text{supp } u_i \subset \Gamma_i\}$ where $\Gamma_i \subset \Gamma$ is a closed subset of Γ representing the **zone where player (agent) i interacts with the system**. Now the control is of the form $u(t) = (u_1(t), \dots, u_p(t))$, $U = U_1 \times \dots \times U_p$. The operator A is the same, but B is now given by

$$\forall u = (u_1, \dots, u_p) \in \mathcal{U} \quad Bu = \sum_{i=1}^p B_i u_i,$$

where

$$\forall w \in V \quad B_i u_i = \int_{\Gamma_i} k u_i w d\sigma.$$

Suppose that Ω is sufficiently smooth such that the state at each moment belongs to $H^2(\Omega)$, and $n \leq 3$, hence $z(t, \cdot) \in C(\bar{\Omega})$

Example 2: grand coalition ρ -player cooperative differential game

- The player i observes the systems in some points (point sensors) $x_k^{(i)} \in \bar{\Omega}$, $k = 1, \dots, m_i$.
- Each player wants to minimize his energy and the square of the deviation from the desired state z_d in his points of observation i.e.

$$J_i(z, u) = \int_0^1 \left(\sum_{k=1}^{m_i} |z(t, x_k^{(i)}) - z_d(t, x_k^{(i)})|^2 + \|u_i(t)\|_{\mathcal{U}_i}^2 \right) dt,$$

- The supervisor wants to minimize the final state global deviation, i.e.

$$J_0(z, u) = \|z(T) - z_d(T)\|_{L^2(\Omega)}.$$

Preliminary results

Lemma

For each $z_0 \in H$ and $u \in L^2(0, T; \mathcal{U})$, there exists a unique function $z_u : [0, T] \rightarrow H$ such that $z_u \in L^2(0, T; V) \cap C(0, T; H)$, $\frac{dz_u}{dt} \in L^2(0, T; V')$ verifying the abstract Cauchy problem

$$\begin{aligned} \frac{dz}{dt}(t) + A(t)z(t) &= B(t)u(t) \quad \text{a.e. on }]0, T[\\ z(0) &= z_0 \end{aligned}$$

Moreover, the correspondence $u \mapsto z_u$ is an affine continuous operator from $L^2(0, T; \mathcal{U})$ to $L^2(0, T; V)$, and from $L^2(0, T; \mathcal{U})$ to $C(0, T; H)$.

Preliminary results

Proposition

Let us define for all $u \in L^2(0, T; \mathcal{U})$ and $i = 0, 1, \dots, p$

$$\hat{J}_i(u) := J_i(z_u, u),$$

where the map $u \mapsto z_u$ has been introduced in the previous Lemma.
Then the function $\hat{J}_i : L^2(0, T; \mathcal{U}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

An equivalent form of (PPOCP)

Consider the set

$$U_{ad} := \{u \in L^2(0, T; \mathcal{U}) \mid u(t) \in U \text{ a.e. on }]0, T[, \quad z_u(T) \in Z_T\}.$$

which is *closed and convex* in $L^2(0, T; \mathcal{U})$.

We assume that

(A) $U_{ad} \neq \emptyset.$

(B) *the functionals \hat{J}_i take finite values on U_{ad} , ($1 \leq i \leq p$).*

(PPOCP) can be written equivalently as

$$(PPO) \quad \min \hat{J}_0(u) \quad \text{s.t.}$$

u is a (weakly or properly) Pareto solution to

$$\text{MIN}_{\mathbb{R}_+^p} [\hat{J}_1(u), \dots, \hat{J}_p(u)] \quad \text{s.t. } u \in U_{ad}.$$

Scalarization theorem

Denote $\hat{J} = (\hat{J}_1, \dots, \hat{J}_p) : U_{ad} \rightarrow \mathbb{R}^p$. Let the symbol σ stands for “weak” ($\sigma = w$) or “proper” ($\sigma = p$). Denote

$$\Theta_\sigma = \begin{cases} \mathbb{R}_+^p \setminus \{0\} & \text{if } \sigma = w \\ \mathbb{R}_{++}^p & \text{if } \sigma = p \end{cases}$$

Then the scalarization theorem can be written as

$$\sigma\text{-ARGMIN}_{u \in U_{ad}} \mathbb{R}_+^p \hat{J}(u) = \bigcup_{\theta \in \Theta_\sigma} \operatorname{argmin}_{u \in U_{ad}} \langle \theta, \hat{J}(u) \rangle_{\mathbb{R}_+^p},$$

A useful set valued function

Let $P_\sigma : \mathbb{R}^p \rightarrow 2^{L^2(0,T;U)}$ be the set-valued map given by

$$P_\sigma(\theta) := \begin{cases} \underset{u \in U_{ad}}{\operatorname{argmin}} \langle \theta, \widehat{J}(u) \rangle & \text{if } \theta \in \Theta_\sigma \\ \emptyset & \text{if } \theta \in \mathbb{R}^p \setminus \Theta_\sigma. \end{cases}$$

It is obvious that P_σ has convex closed values which are subsets of U_{ad} . Moreover the scalarization theorem can be written as

$$\sigma\text{-ARGMIN}_{\substack{\mathbb{R}_+^p \\ u \in U_{ad}}} \widehat{J}(u) = P_\sigma(\Theta_\sigma).$$

Consider the following **scalar set-valued minimization problem**

$$(\text{SSVM}_\sigma) \quad \min_{\theta \in \Theta_\sigma} \widehat{J}_0 \circ P_\sigma(\theta).$$

A **solution** to this problem is an element $(\tilde{\theta}, \tilde{y}) \in \operatorname{Gr}(\widehat{J}_0 \circ P_\sigma)$ such that

$$\tilde{y} = \min(\widehat{J}_0 \circ P_\sigma)(\Theta_\sigma).$$

A useful set valued function

Proposition. Problem (PPO) is equivalent to problem $(SSVM_\sigma)$ in the following sense

- If \tilde{u} solves (PPO), then $P_\sigma^-(\{\tilde{u}\}) \neq \emptyset$, and for each $\tilde{\theta} \in P_\sigma^-(\{\tilde{u}\})$ we have that $(\tilde{\theta}, \hat{J}_0(\tilde{u}))$ is a solution to problem $(SSVM_\sigma)$.
- Conversely, if $(\tilde{\theta}, \tilde{y})$ is a solution to problem $(SSVM_\sigma)$, then there exists $\tilde{u} \in P_\sigma(\tilde{\theta})$ such that \tilde{u} solves problem (PPO) and $\tilde{y} = \hat{J}_0(\tilde{u})$.

Optimality conditions for (PPOCP)

Theorem. Suppose that \hat{J}_0 is Fréchet differentiable on an open set containing U_{ad} . Let \tilde{u} be such that $(\tilde{u}, z_{\tilde{u}})$ solves problem (PPOCP). Then $P_{\sigma}^{-}(\tilde{u}) \neq \emptyset$, and for each $\tilde{\theta} \in P_{\sigma}^{-}(\tilde{u})$

$$\forall \theta \in \mathbb{R}^p, \quad \forall u \in DP_{\sigma}(\tilde{\theta}, \tilde{u})(\theta) \quad \langle \nabla \hat{J}_0(\tilde{u}), u \rangle_{L^2(0, T; \mathcal{U})} \geq 0.$$

where $DP_{\sigma}(\tilde{\theta}, \tilde{u}) : \mathbb{R}^p \rightarrow 2^{L^2(0, T; \mathcal{U})}$ is the the **contingent derivative of P_{σ}** at $(\tilde{\theta}, \tilde{u}) \in \text{Gr}(P_{\sigma})$, and $\nabla \hat{J}_0(\tilde{u}) \in L^2(0, T; \mathcal{U})$ stands for the Fréchet derivative of \hat{J}_0 at \tilde{u} .

When P_σ is single-valued with $\text{dom}(P_\sigma) = \Theta_\sigma$

Hypotheses:

- ① (coercivity) $\forall i \in \{1, \dots, p\}$ L_i is bounded from below and L_i verifies for some $\gamma_i \geq 0$ and $a_i \in L^1(0, T)$

$$\forall (v, u) \in V \times \mathcal{U} \quad L_i(t, v, u) \geq a_i(t) + \gamma_i \|u\|_{\mathcal{U}}^2, \quad t \in]0, T[$$

Moreover

$$\begin{cases} \forall i \in \{1, \dots, p\} \gamma_i > 0 & \text{if } \sigma = w \\ \exists i \in \{1, \dots, p\} \gamma_i > 0 & \text{if } \sigma = p. \end{cases}$$

- ② (strict convexity)

$$\begin{cases} \forall i \in \{1, \dots, p\} & L_i \text{ is strictly convex if } \sigma = w \\ \exists i \in \{1, \dots, p\} & L_i \text{ is strictly convex if } \sigma = p. \end{cases}$$

When P_σ is single-valued with $\text{dom}(P_\sigma) = \Theta_\sigma$

Theorem. Suppose that at least one of the following holds :

- (i) coercivity hypothesis;
- (ii) U is bounded;

Then

$$\text{dom}(P_\sigma) = \Theta_\sigma,$$

i.e. for each $\theta \in \Theta_\sigma$, the scalarized problem $\min_{u \in U_{ad}} \langle \theta, \hat{J}(u) \rangle$ admits at least a solution.

Moreover, if in addition the strict convexity hypothesis holds, then for each $\theta \in \Theta_\sigma$ the set $P_\sigma(\theta)$ is a singleton.

When P_σ is single-valued with $\text{dom}(P_\sigma) = \Theta_\sigma$

Proposition

Suppose that for each $\theta \in \Theta_\sigma$, $P_\sigma(\theta)$ is a singleton denoted $\{\tilde{u}(\theta)\}$.
Then $P_\sigma : \theta \mapsto \tilde{u}(\theta)$ is a **surjection** from Θ_σ to $\sigma\text{-ARGMIN}_{u \in U_{ad}} \hat{J}_+(u)$.

Corollary

Under the hypothesis of the previous Proposition, the problem (PPOCP) is equivalent to the following **finite dimensional scalar minimization problem**

$$(SMFD) \quad \min_{\theta \in \Theta_\sigma} \hat{J}_0(\tilde{u}(\theta))$$

in the sense that

(z, u) is a solution to (PPOCP) if, and only if, there exists a solution θ to (SMFD) such that $u = \tilde{u}(\theta)$ and $z = z_u$.

When P_σ is single-valued with $\text{dom}(P_\sigma) = \Theta_\sigma$

Proposition

Suppose that for each $\theta \in \Theta_\sigma$, $P_\sigma(\theta)$ is a singleton denoted $\{\tilde{u}(\theta)\}$.
Then $P_\sigma : \theta \mapsto \tilde{u}(\theta)$ is a **surjection** from Θ_σ to $\sigma\text{-ARGMIN}_{u \in U_{ad}} \hat{J}_{\mathbb{R}_+}(u)$.

Corollary

Under the hypothesis of the previous Proposition, the problem (PPOCP) is equivalent to the following **finite dimensional scalar minimization problem**

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in the sense that

(z, u) is a solution to (PPOCP) if, and only if, there exists a solution θ to (SMFD) such that $u = \tilde{u}(\theta)$ and $z = z_u$.

When P_σ is single-valued with $\text{dom}(P_\sigma) = \Theta_\sigma$

Remarks. Therefore the main practical problem is to be able to *find in closed form* (or at least to have the maximum of information about) the map $\theta \mapsto \tilde{u}(\theta)$. In the case when the (MOCCOP) is a **linear-quadratic multiobjective control problem in Hilbert spaces** some related results are presented in my paper

Post-Pareto Analysis for Multiobjective Parabolic Control Systems

www.optimization-online.org/DB_HTML/2012/07/3545.html

In the very particular case when **the system is governed by ODE**, i.e. $V = H = \mathbb{R}^n$, $u = \mathbb{R}^m$, hence $A(t)$, $B(t)$ are **matrices**, some practical issues are discussed in the papers

- H. Bonnel & C.Y. Kaya, *Optimization over the Efficient Set of Multi-objective Control Problems*, JOTA, **147**(1), 93-112, 2010
- H. Bonnel & N.S. Pham, *Nonsmooth Optimization over the (Weakly or Properly) Pareto Set of a Linear-Quadratic Multi-objective Control Problem : Explicit Optimality Conditions* JIMO, **7**(4), 789-809, 2011.

Thank you!

