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ABSTRACT. We establish uniform regularity estimates with respect to the Mach number for the three-dimensional free surface compressible Navier-Stokes system in the case of slightly well-prepared initial data in the sense that the acoustic components like the divergence of the velocity field are of size $\sqrt{\varepsilon}$, ε being the Mach number. These estimates allow us to justify the convergence towards the free surface incompressible Navier-Stokes system in the low Mach number limit. One of the main difficulties is the control of the regularity of the surface in presence of boundary layers with fast oscillations.

1. INTRODUCTION

We consider the motion of a slightly compressible viscous fluid with a free surface. It takes the following form:

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon w^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon w^\varepsilon) + \mathrm{div}(\rho^\varepsilon w^\varepsilon \otimes w^\varepsilon) - \mathrm{div}\mathcal{L}w^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega_t^\varepsilon, \\ \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon,\ w^\varepsilon|_{t=0} = w_0^\varepsilon, \end{cases} \tag{1.1}$$

where $\rho^\varepsilon > 0, w^\varepsilon \in \mathbb{R}^3$ are the density and the velocity of the fluid, $P(\rho^\varepsilon)$, a smooth function of ρ^ε, stands for the pressure. The viscous tensor $\mathcal{L}w^\varepsilon$ takes the form:

$$\mathcal{L}w^\varepsilon = 2\mu S w^\varepsilon + \lambda \mathrm{div} w^\varepsilon \mathrm{Id}, \quad S w^\varepsilon = \frac{1}{2}(\nabla w^\varepsilon + \nabla^t w^\varepsilon).$$

Here, μ, λ are the viscosity parameters that are assumed to be constant and to satisfy the conditions: $\mu > 0, 2\mu + 3\lambda > 0$. The parameter ε is the scaled Mach number which is assumed small, that is $\varepsilon \in (0,1]$. We focus on a fluid domain given by:

$$\Omega_t^\varepsilon = \{x = (y,z)|\ y \in \mathbb{R}^2, -1 < z < h^\varepsilon(t,y)\},$$

where the upper surface is free and the bottom is fixed. Here $h^\varepsilon(t,y)$, the surface of the fluid domain, is unknown and needs to be solved together with $(\rho^\varepsilon, w^\varepsilon)$. Since the fluid particles do not cross the surface, h^ε solves

$$\partial_t h^\varepsilon - w^\varepsilon(t,y,h^\varepsilon(t,y)) \cdot \mathbf{N}^\varepsilon = 0, \quad h^\varepsilon(0,y) = h_0^\varepsilon(y) \quad y \in \mathbb{R}^2 \tag{1.2}$$

where $\mathbf{N}^\varepsilon = (-\partial_1 h^\varepsilon, -\partial_2 h^\varepsilon, 1)^t$ denotes the outward normal vector to the surface $\Sigma_t^\varepsilon = \{x = (y,z), z = h^\varepsilon(t,y)\}$. We supplement the system (1.1) and (1.2) with the following physical conditions. At the upper boundary, the continuity of the stress tensor reads:

$$\mathcal{L}u^\varepsilon \mathbf{N}^\varepsilon = \frac{1}{\varepsilon^2}\big(P(\rho^\varepsilon) - P(\bar\rho)\big)\mathbf{N}^\varepsilon \quad \text{on} \quad \Sigma_t^\varepsilon \tag{1.3}$$

where $\bar\rho > 0$ is a reference constant density. At the bottom, we prescribe a slip boundary condition:

$$w_3^\varepsilon = 0, \quad \mu\partial_3 w_j^\varepsilon = a w_j^\varepsilon \quad (j = 1,2), \quad \text{on} \quad \{z = -1\}, \tag{1.4}$$

where a is a constant that quantifies the effects of the friction at the boundary (this can be easily generalized to a smooth function a, see [47]). The case of the Dirichlet boundary condition at the bottom raises other difficulties even without the presence of a free surface and is left for future work. Note that we could also consider the case of a strip with infinite depth, see Section 15.

The system (1.1) can be obtained from a suitable scaling of the original physical variables. Indeed, we get (1.1), (1.2) by performing the following scaling:

$$\tilde{\rho}(t,x) = \rho^\varepsilon(\varepsilon t, x),\ \tilde{w}(t,x) = \varepsilon w^\varepsilon(\varepsilon t, x),\ \tilde{h} = h^\varepsilon(\varepsilon t, x),\ \tilde{\mu} = \varepsilon\mu,\ \tilde{\lambda} = \varepsilon\lambda,$$

where $\tilde{\rho}, \tilde{u}, \tilde{h}$ satisfy:

$$\left\{\begin{array}{l} \partial_t\tilde{\rho} + \mathrm{div}(\tilde{\rho}\tilde{w}) = 0, \\ \partial_t(\tilde{\rho}\tilde{w}) + \mathrm{div}(\tilde{\rho}\tilde{w}\otimes\tilde{w}) - \mathrm{div}\tilde{\mathcal{L}}\tilde{w} + \nabla P(\tilde{\rho}) = 0, \\ \partial_t\tilde{h} + \tilde{w}(t,y,\tilde{h}(t,y))\cdot\tilde{\mathbf{N}} = 0, \end{array}\right. \tag{1.5}$$

where $\tilde{\mathcal{L}}\tilde{w} = 2\tilde{\mu}S\tilde{w} + \tilde{\lambda}\mathrm{div}\tilde{w}$.

The aim of this paper is to study the low Mach number limit problem, that is to study the behavior of (strong) solutions to (1.1) when ε tends to 0. Formally, due to the singular term $\frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2}$, the pressure (and hence the density ρ^ε) is expected to tend to a constant state in some suitable space, one thus expect that the limit of the solutions to (1.1), if it exists in a sufficiently strong sense, will be the solution to the following incompressible free surface Navier-Stokes system:

$$\left\{\begin{array}{ll} \bar{\rho}(\partial_t w^0 + w^0\cdot\nabla w^0) - 2\mu\mathrm{div}\, Sw^0 + \nabla\pi^0 = 0, & \\ \mathrm{div}\, w^0 = 0, & (t,x)\in\mathbb{R}_+\times\Omega_t^0, \\ w^0|_{t=0} = w_0^0,\ h^0|_{t=0} = h_0^0, & \end{array}\right. \tag{1.6}$$

supplemented with the boundary conditions:

$$\begin{aligned} \partial_t h^0 - w^0(t,y,h^0(t,y))\cdot\mathbf{N}^0 = 0, \quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^2, \\ Sw^0\mathbf{N}^0 = \pi^0\mathbf{N}^0 \quad \text{on} \quad \{z = h^0(t,y)\}, \\ w_3^0 = 0, \quad \partial_3 w_j^0 = a w_j^0\,(j=1,2) \quad \text{on} \quad \{z=-1\}, \end{aligned}$$

where $\mathbf{N}^0 = (-\partial_1 h^0, -\partial_2 h^0, 1)^t$.

The rigorous justification of the low Mach number limit has been studied extensively in different contexts depending on the generality of the system (isentropic or non-isentropic), the type of the system (Navier-Stokes or Euler), the type of solutions (strong solutions or weak solutions), the properties of the domain (without boundaries, with fixed or free boundaries), as well as the type of the initial data considered (well-prepared or ill-prepared). The mathematical justification of the low Mach number limit was initiated by Ebin [21], Klainerman-Majda [37, 38] for *local strong solutions* of compressible fluids (Euler or Navier-Stokes), in the whole space with well-prepared data ($\mathrm{div}\, u_0^\varepsilon = \mathcal{O}(\varepsilon), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon^2)$) and later, by Ukai [60] for ill-prepared data ($\mathrm{div}\, u_0^\varepsilon = \mathcal{O}(1), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon)$). These works are then extended by several authors in different settings. One can refer for instance to [3, 11, 49, 50] for the study of the non-isentropic (Euler or Navier-Stokes) equations under ill-prepared initial data whenever the domain is the whole space or the torus, and also [55, 36] for bounded domains with well-prepared initial data. There are also many other related works, one can see for example [2, 8, 15, 17, 19, 23, 30, 32, 33, 42, 43]. For more exhaustive information, one can refer for example to the well-written survey papers by Alazard [4], Danchin [16], Feireisl [24], Gallagher [26], Jiang-Masmoudi [35], Schochet [56].

The analysis of the low Mach number limit problem for the isentropic compressible Navier-Stokes (CNS) system in domains with fixed boundaries, which is more related to the interest of the current paper, has been done in two different directions. Roughly speaking, for (CNS) in fixed bounded domains, one can either justify the limit process directly from global weak solutions, or prove that local strong solutions exist on a time interval independent of the Mach number and use compactness arguments to pass to the limit. For the first case, Lions and Masmoudi [42] investigated the convergence of *weak solutions* to (CNS) in bounded domains with various boundary conditions. Later on, for the same problem in bounded domains with Dirichlet boundary conditions, the authors in [18, 34] noticed that under some geometric assumption on the domain, the acoustic waves are damped in a boundary layer so that local in time strong convergence ($L^2_{t,x}$) holds. One can also refer to [23] for the justification of the convergence towards a solution of the incompressible Navier-Stokes system in unbounded domains by using the local energy decay for the acoustic system. All these results hold true for ill-prepared initial data. Concerning the local strong solutions, uniform high order energy estimates are established in [36] with Dirichlet boundary conditions and in [52] with Navier-slip boundary conditions by assuming the initial data to be well-prepared. Recently, we established in [47] uniform high regularity estimates in bounded domains with Navier-slip boundary conditions and ill-prepared initial data. To match the boundary layer effects due to the fast oscillations and the ill-prepared initial data assumption, we proved uniform estimates in an anisotropic functional framework with only one normal derivative close to the boundary.

There are only a few works dealing with the low Mach number limit problem for systems in the presence of free boundaries. They deal with inviscid systems. In [41], Lindblad-Luo prove uniform a-priori estimates for the free boundary compressible Euler equations in the case of a bounded reference domain. More recently, this result is extended by Luo [44] for unbounded reference domains and by Disconzi-Luo [20] for a bounded reference domain but with surface tension. All these results are based on the assumption that the initial datum is sufficiently well-prepared in the sense that the time derivatives up to at least order two are bounded initially, an assumption which is stronger than the usual well-prepared data assumption which requires one time derivative to be bounded initially. Regarding viscous fluids, the author in [51] considered the 1d compressible Navier-Stokes system with free boundaries and established uniform estimates with respect to the Mach number and the Froude number for both well-prepared and ill-prepared initial data. Nevertheless, within our knowledge, there is no related work for multidimensional viscous systems. Indeed, in the multidimensional case, there are several difficulties that do not appear in the 1d case, as will be explained later, a boundary layer appears in the multidimensional case which will preclude the uniform control of higher order (≥ 2) normal derivatives of the solution. The aim of the current work is thus to investigate the low Mach number limit problem for 3d viscous fluids solving (1.1)-(1.4). For the simplicity of presentation (compared to the case of general bounded domains) we choose a channel with finite depth as the reference domain. Nevertheless, one can extend easily our analysis to the cases where the reference domain is the half space or a bounded domain, we shall explain more about this aspect in Section 15.

The core of the analysis in this paper is to establish some uniform high regularity estimates in order to get the existence of a local strong solution on a time interval independent of ε. Due to the presence of the diffusion term as well as the singular linear term, a boundary layer correction to the highly oscillating acoustic waves appears and creates unbounded high order normal derivatives of the velocity. Therefore, we need to work in a functional framework based on conormal Sobolev spaces that minimizes the use of normal derivatives near the boundary in the spirit of [22, 46, 48]. Note that in the current situation, we have to handle simultaneously fast oscillations in time and a boundary layer effect so that the difficulties and the analysis will be very different from the ones in [48], where compressible slightly viscous fluids are considered. Indeed, the energy estimates for conormal derivatives cannot be directly obtained since tangential vector fields do not commute with the singular part of the system. Moreover, to include only slightly well-prepared data (we will explain later what it means), it will be impossible for us to get uniform estimates for time derivatives. In

[47], we could establish uniform estimates for the isentropic compressible Navier-Stokes system with Navier boundary condition in smooth fixed domains and ill-prepared initial data. For free surface fluids, there are extra difficulties essentially related to the control of the regularity of the free surface. Indeed, because of the occurrence of the singular terms, the compressible part of the system behaves at time scale $\tau = t/\varepsilon$ like a small viscosity approximation of the acoustic system, we thus cannot obtain uniform extra regularity for the surface from the diffusion term. This is the main reason for which some kind of well-prepared assumption will be needed. We could nevertheless impose an assumption that we call slightly well-prepared which is weaker than the usual well-prepared assumption that requires one time derivative to be of order $\mathcal{O}(1)$ and thus much weaker than the assumption made for the free surface Euler system, for example in [41], where two derivatives of the solution are assumed to be $\mathcal{O}(1)$ initially. We only require the first time derivative of the solution to be of order $\varepsilon^{-\frac{1}{2}}$, this is thus intermediate between ill-prepared $\mathcal{O}(\varepsilon^{-1})$ and well-prepared $\mathcal{O}(1)$, see also Remark 1.2. The main heuristics is that despite the extra difficulties arising from the boundary layer effects (note that the presence of a boundary layer is a feature of the viscous problem and is absent in the inviscid case), the presence of the diffusion term can help us to gain some regularity of the surface (not necessarily uniform). It thus allows us to include more general data compared to the corresponding works on inviscid systems [41, 44, 20]. We shall explain more precisely below after the reformulation of the system and the statement of the main results.

1.1. **Reformulation of the system in a fixed domain.** Let us set

$$\varrho^\varepsilon = \frac{P(\rho) - P(\bar{\rho})}{\varepsilon},$$

the system (1.1) can be rewritten into the following symmetric form:

$$\begin{cases} g_1(\varepsilon\varrho^\varepsilon)\big(\partial_t \varrho^\varepsilon + w^\varepsilon \cdot \nabla \varrho^\varepsilon\big) + \dfrac{\mathrm{div} w^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\varrho^\varepsilon\big)(\partial_t w^\varepsilon + w^\varepsilon \cdot \nabla w^\varepsilon) - \mathrm{div}\mathcal{L}w^\varepsilon + \dfrac{\nabla \varrho^\varepsilon}{\varepsilon} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega_t^\varepsilon, \\ w^\varepsilon|_{t=0} = w_0^\varepsilon, \quad \varrho^\varepsilon|_{t=0} = \varrho_0^\varepsilon \end{cases} \tag{1.7}$$

where the scalar functions g_1, g_2 are defined by:

$$g_2(s) = \rho^\varepsilon = P^{-1}(P(\bar{\rho}) + s), \quad g_1(s) = (\ln g_2)'(s); \quad s > -\bar{P} = -P(\bar{\rho}). \tag{1.8}$$

Moreover, the boundary condition (1.3) is transformed into

$$\mathcal{L}u^\varepsilon \mathbf{N}^\varepsilon = \frac{\varrho^\varepsilon}{\varepsilon}\mathbf{N}^\varepsilon \quad \text{on} \quad \Sigma_t^\varepsilon. \tag{1.9}$$

In the following, we shall work on the system (1.7), (1.2) with boundary conditions (1.4), (1.9).

We then choose an appropriate change of coordinates to reduce the free-surface domain to a fixed one. One natural possibility is to use Lagrangian coordinates, nevertheless, since we shall consider the problem in the conormal Sobolev setting, the Lagrangian transformation would be also only bounded in the conormal setting, this would raise additional difficulties. Therefore, instead of using Lagrangian coordinates, we shall use the following smoothing diffeomorphism [40], where the map will enjoy the usual Sobolev regularity. Let us set $\mathcal{S} = \mathbb{R}^2 \times [-1, 0]$, and consider the map

$$\begin{aligned} &\Phi_t^\varepsilon : \mathcal{S} \to \Omega_t^\varepsilon \\ &(y,z) \mapsto \Phi^\varepsilon(t,y,z) = (y, \varphi^\varepsilon(t,y,z))^t \end{aligned} \tag{1.10}$$

where

$$\varphi^\varepsilon(t,y,z) = z + \eta^\varepsilon(t,y,z)(1+z). \tag{1.11}$$

Here η^ε is given by a smoothing extension

$$(\mathcal{F}\eta^\varepsilon)(t,\xi,z) = e^{-\delta_0(1+|\xi|^2)z^2}(\mathcal{F}h^\varepsilon)(t,\xi) \tag{1.12}$$

where $\mathcal{F}$ stands for the Fourier transform with respect to the horizontal variable $y \in \mathbb{R}^2$, δ_0 is a small parameter such that $\det(\mathrm{D}\Phi_0^\varepsilon) > 0$, which ensures that Φ_0^ε is a diffeomorphism. Note that

$$\det(\mathrm{D}\Phi_0^\varepsilon) = \partial_z\varphi^\varepsilon(0,x) = 1 + h^\varepsilon(0,x) + (\eta^\varepsilon - h^\varepsilon)(0,x) + \partial_z\eta^\varepsilon(0,x)(1+z) > 2c_0 > 0$$

as long as

$$1 + h^\varepsilon(0,x) \geq 3c_0 > 0,\ \forall x \in \mathcal{S}, \tag{1.13}$$

$$\|(\eta^\varepsilon - h^\varepsilon)(0)\|_{L^\infty(\mathcal{S})} + \|\partial_z\eta^\varepsilon(0)\|_{L^\infty(\mathcal{S})} < c_0, \tag{1.14}$$

where $c_0 > 0$ is a fixed constant. Let us notice that (1.14) holds if $\|h^\varepsilon(0)\|_{H^s(\mathbb{R}^2)} < +\infty$, for some $s > 2$ and δ_0 is chosen sufficiently small. Moreover, we have that

$$\|\nabla\varphi^\varepsilon(t)\|_{L^2(\mathcal{S})} \lesssim |h^\varepsilon(t)|_{H^{\frac{1}{2}}(\mathbb{R}^2)},$$

which means that we gain one half derivative.

Let us now set

$$u^\varepsilon(t,y,z) = w^\varepsilon(t,y,\Phi^\varepsilon(t,y,z)), \quad \sigma^\varepsilon = \varrho^\varepsilon(t,y,\Phi^\varepsilon(t,y,z))$$

where u^ε and σ^ε are defined in $\mathcal{S}$. Then we set, $\partial_j^{\varphi^\varepsilon} u^\varepsilon = (\partial_j w^\varepsilon)\circ\Phi^\varepsilon, \partial_j^{\varphi^\varepsilon}\sigma^\varepsilon = (\partial_j\varphi^\varepsilon)\circ\Phi^\varepsilon$, where $j = 0,1,2,3$ with $\partial_0 = \partial_t, \partial_3 = \partial_z$ which yields

$$\partial_i^{\varphi^\varepsilon} = \partial_i - \frac{\partial_i\varphi^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z, \quad i = 0,1,2, \quad \partial_z^{\varphi^\varepsilon} = \frac{1}{\partial_z\varphi^\varepsilon}\partial_z. \tag{1.15}$$

The equations (1.7), (1.2) and the boundary conditions (1.9), (1.4) are reformulated into the following systems:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t^{\varphi^\varepsilon}\sigma^\varepsilon + u^\varepsilon\cdot\nabla^{\varphi^\varepsilon}\sigma^\varepsilon\big) + \dfrac{\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\sigma^\varepsilon)(\partial_t^{\varphi^\varepsilon} u^\varepsilon + u^\varepsilon\cdot\nabla^{\varphi^\varepsilon} u^\varepsilon\big) - \mathrm{div}^{\varphi^\varepsilon}\mathcal{L}^{\varphi^\varepsilon} u^\varepsilon + \dfrac{\nabla^{\varphi^\varepsilon}\sigma^\varepsilon}{\varepsilon} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathcal{S} \\ u^\varepsilon|_{t=0} = w_0^\varepsilon(\Phi_0^\varepsilon(x)) := u_0^\varepsilon, \quad \sigma^\varepsilon|_{t=0} = \varrho_0^\varepsilon(\Phi_0^\varepsilon(x)) := \sigma_0^\varepsilon, \end{cases} \tag{1.16}$$

$$\partial_t h^\varepsilon - u^\varepsilon(t,y,h^\varepsilon(t,y))\cdot\mathbf{N}^\varepsilon = 0, \tag{1.17}$$

$$\mathcal{L}^{\varphi^\varepsilon} u^\varepsilon\mathbf{N}^\varepsilon = \frac{\sigma^\varepsilon}{\varepsilon}\mathbf{N}^\varepsilon \quad \text{on} \quad \{z=0\}, \tag{1.18}$$

$$u_3^\varepsilon = 0, \quad \mu\partial_z^{\varphi^\varepsilon} u_j^\varepsilon = a u_j^\varepsilon \quad (j=1,2), \quad \text{on} \quad \{z=-1\}. \tag{1.19}$$

1.2. **Conormal spaces and notations.** Before stating our results, we need to introduce some notations. We define the conormal vector fields:

$$Z_0 = \varepsilon\partial_t,\ Z_1 = \partial_{y_1},\ Z_2 = \partial_{y_2},\ Z_3 = \phi(z)\partial_z$$

where the weight function is $\phi(z) = z(1+z)/(2-z)^2$. We then introduce the space-time conormal space as follows, for $p = 2, +\infty$,

$$L_t^p H_{co}^m(\mathcal{S}) = \{f|\ Z^\alpha f \in L^p([0,t]; L^2(\mathcal{S})), |\alpha| \leq m\},$$

with the corresponding norms:

$$\|f\|_{L_t^p H_{co}^m} = \sum_{|\alpha|\leq m}\|Z^\alpha f\|_{L^p([0,t],L^2(\mathcal{S}))}, \tag{1.20}$$

where $\alpha = (\alpha_0, \alpha') = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4$. Moreover, we shall also use the $L^\infty_{t,x}$ type norm defined by:

$$\|\!|f|\!\|_{k,\infty,t} = \sum_{|\alpha|\le k} \|Z^\alpha f\|_{L^\infty([0,t]\times\mathcal{S})}. \tag{1.21}$$

To distinguish the number of time and space derivatives, we introduce also the norm:

$$\|f\|_{L^p_t\mathcal{H}^{j,l}} = \sum_{\alpha_0\le j, |\alpha'|\le l} \|Z^\alpha f\|_{L^p([0,t],L^2(\mathcal{S}))}, \tag{1.22}$$

and to simplify, we use

$$\mathcal{H}^j = \mathcal{H}^{j,0}. \tag{1.23}$$

To measure the regularity along the boundary, we use:

$$|f|_{L^p_t\tilde{H}^s} = \sum_{j=0}^{[s]} |(\varepsilon\partial_t)^j f|_{L^p_t H^{s-j}(\mathbb{R}^2)}, \quad |f|_{k,\infty,t} = \sum_{|\alpha|\le k, \alpha_3=0} |Z^\alpha f|_{L^\infty([0,t]\times\mathbb{R}^2)}. \tag{1.24}$$

Finally, to measure pointwise regularity at a given time t (in particular also with $t=0$), we shall use the semi-norms:

$$|f(t)|_{\tilde{H}^s} = \sum_{j=0}^{[s]} |(\varepsilon\partial_t)^j f(t)|_{H^{s-j}(\mathbb{R}^2)}, \tag{1.25}$$

$$\|f(t)\|_{H^m_{co}} := \sum_{|\alpha|\le m} \|(Z^\alpha f)(t)\|_{L^2(\mathcal{S})}, \quad \|f(t)\|_{\mathcal{H}^{j,l}} := \sum_{\alpha_0\le j, |\alpha'|\le l} \|(Z^\alpha f)(t)\|_{L^2(\mathcal{S})}, \tag{1.26}$$

$$\|f(t)\|_{k,\infty,\mathcal{S}} := \sum_{|\alpha|\le k} \|(Z^\alpha f)(t)\|_{L^\infty(\mathcal{S})}. \tag{1.27}$$

1.3. **Main results.** Before stating our main result, we first introduce the definition of the compatibility conditions which are necessary to obtain smooth enough solutions for the initial-boundary value problem of parabolic systems.

Definition 1.1 (Compatibility condition)**.** *We say that $(\sigma_0^\varepsilon, u_0^\varepsilon)$ satisfy the compatibility condition up to order m if for $j = 0, 1 \cdots m-1$,*

$$\begin{aligned}
&(\varepsilon\partial_t)^j \big(\mathcal{L}^{\varphi^\varepsilon} u^\varepsilon \boldsymbol{n}^\varepsilon\big)|_{t=0} = (\varepsilon\partial_t)^j (\sigma^\varepsilon/\varepsilon)\big|_{t=0}, \quad on \quad \{z=0\},\\
&\varepsilon^j \partial_t^{j+1} h^\varepsilon|_{t=0} = (\varepsilon\partial_t)^j (u^\varepsilon \cdot \mathbf{N}^\varepsilon)|_{t=0} \quad on \quad \{z=0\},\\
&\big((\varepsilon\partial_t)^j u_3^\varepsilon\big)\big|_{t=0} = 0, \quad \big((\varepsilon\partial_t)^j \partial_z^{\varphi^\varepsilon} u_j^\varepsilon\big)|_{t=0} = \frac{a}{\mu}(\varepsilon\partial_t)^j u_j^\varepsilon|_{t=0} \ (j=1,2) \quad on \quad \{z=-1\}.
\end{aligned} \tag{1.28}$$

Note that the restriction of time derivatives of the solution at the initial time is defined inductively by using the equations. For instance:

$$(\partial_t h^\varepsilon)|_{t=0} = u_0^\varepsilon|_{z=0} \cdot (-\nabla_y h_0^\varepsilon, 1)^t$$

$$(\varepsilon\partial_t u^\varepsilon)|_{t=0} = \frac{1}{g_2(\varepsilon\sigma_0^\varepsilon)}(-\varepsilon \underline{u}_0^\varepsilon \cdot \nabla u_0^\varepsilon + \varepsilon \mathrm{div}^{\varphi_0^\varepsilon} \mathcal{L}^{\varphi_0^\varepsilon} u_0^\varepsilon - \nabla^{\varphi_0^\varepsilon} \sigma_0^\varepsilon),$$

where

$$\underline{u}_0^\varepsilon = \big(u_{0,1}^\varepsilon, u_{0,2}^\varepsilon, (u_0^\varepsilon \cdot \mathbf{N}_0^\varepsilon - (\partial_t\varphi^\varepsilon)|_{t=0})/\partial_z\varphi_0^\varepsilon\big)^t, \quad \varphi_0^\varepsilon(\cdot) = \varphi^\varepsilon(0,\cdot) = z + \eta^\varepsilon(0,\cdot)(1+z).$$

We remark that $\partial_t\varphi^\varepsilon|_{t=0}, \partial_z\varphi_0^\varepsilon$ are determined by $(\partial_t h)^\varepsilon|_{t=0}$ and h_0^ε respectively through (1.11) and (1.12).

We now define the space for the initial data:

$$Y_m^\varepsilon = \left\{(\sigma_0^\varepsilon, u_0^\varepsilon, h_0^\varepsilon) \in H^3(\Omega_0)^4 \times H^{m-\frac{1}{2}}(\mathbb{R}^2) \middle| \begin{matrix} Y_m^\varepsilon(0) < +\infty, (\sigma_0^\varepsilon, u_0^\varepsilon, h_0^\varepsilon) \text{ satisfy} \\ \text{compatibility condition up to order } m \end{matrix}\right\}, \tag{1.29}$$

where

$$\begin{aligned} Y_m^\varepsilon(0) =: &|h_0^\varepsilon|_{H^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|h_0^\varepsilon|_{H^{m+\frac{1}{2}}} + \varepsilon^{-\frac{1}{2}}\|\nabla\sigma^\varepsilon(0)\|_{m-5,\infty,\mathcal{S}} + \|\nabla u^\varepsilon(0)\|_{1,\infty,\mathcal{S}} \\ &+ \varepsilon^{\frac{1}{2}}\|(\sigma_0^\varepsilon, u_0^\varepsilon)\|_{H^3(\mathcal{S})} + \varepsilon^{\frac{1}{2}}(\|(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H^m_{co}(\mathcal{S})} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H^{m-1}_{co}}) \\ &+ \|(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H^{m-1}_{co}} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H^{m-2}_{co}} + \varepsilon^{\frac{1}{2}}\|\partial_t(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H^{m-1}_{co}(\mathcal{S})} + \varepsilon^{\frac{1}{2}}\|\partial_t\omega^\varepsilon(0)\|_{H^{m-4}_{co}(\mathcal{S})}. \end{aligned} \tag{1.30}$$

In the above, expression, $\omega^\varepsilon = \mathrm{curl}^{\varphi^\varepsilon} u^\varepsilon$ stands for the vorticity.

To prove Theorem 1.1, we introduce the following quantities:

$$\mathcal{N}_{m,T}^\varepsilon = \mathcal{E}_{m,T}^\varepsilon + \mathcal{A}_{m,T}^\varepsilon =: \mathcal{E}_{low,T}^\varepsilon + \mathcal{E}_{high,m,T}^\varepsilon + \mathcal{A}_{m,T}^\varepsilon. \tag{1.31}$$

Here, $\mathcal{E}_{m,T}^\varepsilon$ is composed of the low order energy norms $\mathcal{E}_{low,T}^\varepsilon$ and the high order energy norms $\mathcal{E}_{high,m,T}^\varepsilon$:

$$\mathcal{E}_{low,T}^\varepsilon = \|\varepsilon^{\frac{1}{2}}\partial_t(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty L^2} + \varepsilon^{\frac{1}{2}}\|(\sigma^\varepsilon, u^\varepsilon)\|_{L_t^\infty H^3} + \varepsilon^{\frac{3}{2}}\|\nabla^4 u^\varepsilon\|_{L_t^2 L^2},$$

$$\begin{aligned} \mathcal{E}_{high,m,T}^\varepsilon =& \varepsilon^{\frac{1}{2}}|h^\varepsilon|_{L_t^\infty \tilde{H}^{m+\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H^m_{co}} \\ &+ \varepsilon^{\frac{1}{2}}\|\nabla u^\varepsilon\|_{L_T^\infty H^{m-1}_{co} \cap L_T^2 H^m_{co}} + \varepsilon^{\frac{1}{2}}\|\nabla^2 u^\varepsilon\|_{L_T^\infty H^{m-2}_{co} \cap L_T^2 \mathcal{H}^{m-1}} \\ &+ |h^\varepsilon|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\|\partial_t(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty \mathcal{H}^{m-1}} + \varepsilon^{\frac{1}{2}}\|\partial_t \nabla u^\varepsilon\|_{L_T^2 \mathcal{H}^{m-1} \cap L_T^2 H^{m-2}_{co} \cap L_T^\infty H^{m-4}_{co}} \\ &+ \varepsilon^{-\frac{1}{2}}\|(\nabla^{\varphi^\varepsilon}\sigma^\varepsilon, \mathrm{div}^{\varphi^\varepsilon} u^\varepsilon)\|_{L_T^\infty H^{m-2}_{co} \cap L_T^2 H^{m-1}_{co}} + \|(\sigma^\varepsilon, u^\varepsilon)\|_{L_t^\infty H^{m-1}_{co}} + \|\nabla u^\varepsilon\|_{L_T^\infty H^{m-4}_{co} \cap L_T^2 H^{m-1}_{co}} \end{aligned} \tag{1.32}$$

whereas $\mathcal{A}_{m,T}^\varepsilon$ contains the $L_{t,x}^\infty$ norms:

$$\begin{aligned} \mathcal{A}_{m,T}^\varepsilon = &\|\nabla u^\varepsilon\|_{1,\infty,T} + \|(\varepsilon^{\frac{1}{2}}\partial_t(\sigma^\varepsilon, u^\varepsilon), \varepsilon^{-\frac{1}{2}}(\nabla^{\varphi^\varepsilon}\sigma, \mathrm{div}^{\varphi^\varepsilon} u)\|_{m-5,\infty,T} + \|(\mathrm{Id}, \varepsilon\partial_t)(\sigma^\varepsilon, u^\varepsilon)\|_{m-4,\infty,T} \\ &+ \varepsilon^{\frac{1}{2}}\|\nabla u^\varepsilon\|_{m-3,\infty,T} + \varepsilon^{\frac{1}{2}}\|(\sigma^\varepsilon, u^\varepsilon)\|_{m-2,\infty,T} + |h^\varepsilon|_{m-2,\infty,T}. \end{aligned} \tag{1.33}$$

Our main result is the following:

Theorem 1.1 (Uniform estimates)**.** *Define* $0 < c_0 < \frac{1}{2}$ *such that*

$$\sup_{s\in[-3c_1\bar{P}, 3\bar{P}/c_1]} |(g_1, g_2)(s)| \in [c_0, 1/c_0]$$

where $0 < c_1 < \frac{1}{4}$ *is a fixed constant. Given* $m \geq 7$ *an integer, suppose that the initial data belongs to* Y_m^ε*, is such that*

$$1 + h_0^\varepsilon(x) \geq 3c_0 > 0, \quad \sup_{\varepsilon\in(0,1]} Y_m^\varepsilon(0) < +\infty,$$

$$-c_1\bar{P} \leq \varepsilon\sigma_0^\varepsilon(x) \leq \bar{P}/c_1, \quad \forall x \in \mathcal{S}, \quad \forall\varepsilon \in (0,1],$$

and δ_0 *(the parameter appearing in* (1.12)*) is chosen such that* (1.14) *holds for* $t = 0$ *so that*

$$\partial_z\varphi_0^\varepsilon \geq 2c_0, \quad \forall x \in \mathcal{S}, \quad \forall\varepsilon \in (0,1].$$

Moreover, (taking c_0 smaller if necessary), we can also assume that

$$|(\nabla\varphi_0^\varepsilon, \nabla^2\varphi_0^\varepsilon)(x)| \leq \frac{1}{2c_0}, \quad \forall x \in \mathcal{S}, \quad \forall \varepsilon \in (0,1].$$

Then there exist $T_0 > 0$, $\varepsilon_0 \in (0,1]$, such that for any $\varepsilon \in (0, \varepsilon_0]$, the system (1.16)-(1.19) has a unique solution which satisfies: $\mathcal{N}_{m,T_0}^\varepsilon(\sigma^\varepsilon, u^\varepsilon) < +\infty$. In particular, we have the uniform estimate

$$\sup_{0<\varepsilon\leq\varepsilon_0} \big(\|(\sigma^\varepsilon, u^\varepsilon)\|_{L^2_{T_0}H^m_{co}(\mathcal{S})\cap L^\infty_{T_0}H^{m-1}_{co}(\mathcal{S})} + \varepsilon^{-\frac{1}{2}} \|(\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon, \nabla\sigma^\varepsilon)\|_{L^\infty_{T_0}H^{m-2}_{co}(\mathcal{S})\cap L^2_{T_0}H^{m-1}_{co}}$$
$$+ \|\nabla u^\varepsilon\|_{1,\infty,T_0} + \varepsilon^{-\frac{1}{2}} \|(\nabla\sigma^\varepsilon, \mathrm{div}^{\varphi^\varepsilon} u^\varepsilon)\|_{m-5,\infty,T_0}\big) < +\infty.$$

Moreover, the following properties hold: for any $(t,x) \in [0,T_0] \times \mathcal{S}, \varepsilon \in (0, \varepsilon_0]$,

$$\partial_z\varphi^\varepsilon(t,x) \geq c_0, \quad |(\nabla\varphi^\varepsilon, \nabla^2\varphi^\varepsilon)(t,x)| \leq 1/c_0, \quad -2c_1\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 2\bar{P}/c_1. \tag{1.34}$$

Remark 1.2. *In view of the definition of Y_m^ε, we have assumed that the first time derivative of the solution is of size of order $\varepsilon^{-\frac{1}{2}}$, which is better than the usual well-prepared data case (where $\partial_t(\sigma^\varepsilon, u^\varepsilon)|_{t=0}$ is assumed to be order 1). This assumption is crucial in our analysis to control the regularity on the surface. We shall give more details in Subsection 1.5. Note that our assumption is thus much weaker than the one in [41, 44, 20] for the inviscid system where two time derivatives are assumed to be bounded initially.*

Remark 1.3. *It is also possible to prove the uniform estimates by imposing an alternative assumption on the size of the acoustic waves, we can assume them to be of order ε in a low regularity H^1_{co} space and of order 1 in a higher regularity H^m_{co} norm.*

Remark 1.4. *In view of the definition (1.32), one has three kinds of bounds for the solution. The first two lines of (1.32) only imply that the highest order norm with pointwise estimates in time $L^\infty_t H^m_{co}$ of $(\sigma^\varepsilon, u^\varepsilon)$ can be unbounded and has a size $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$. Nevertheless, in the two last lines of (1.32), we are able to get that the L^2_t type norm with maximal number of derivatives, $L^2_t H^m_{co}$ of $(\sigma^\varepsilon, u^\varepsilon)$ and the $L^\infty_t H^{m-1}_{co}$ norm (so with one less derivative) are uniformly bounded. Moreover, the first term in the fourth line of (1.32) shows that the compressible part of the remains of size $\mathcal{O}(\varepsilon^{\frac{1}{2}})$ in $L^\infty_T H^{m-2}_{co} \cap L^2_T H^{m-1}_{co}$.*

Theorem 1.5 (Convergence)**.** *Assuming that $(u_0^\varepsilon, h_0^\varepsilon)$ tends to (u_0^0, h_0^0) in $H^1(\mathcal{S}) \times L^2(\mathbb{R}^2)$ and the assumptions made in Theorem 1.1 hold. Let $(\sigma^\varepsilon, u^\varepsilon, h^\varepsilon)$ be the solution to (1.16)-(1.19). Then $(P(\bar{\rho})+\varepsilon\sigma^\varepsilon, u^\varepsilon, h^\varepsilon)$ converge in $C^\gamma([0,T_0]\times\mathcal{S}) \times C([0,T_0], L^2_{loc}(\mathcal{S})) \times C([0,T_0], H^s_{loc}(\mathbb{R}^2))$ to $(P(\bar{\rho}), u^0, h^0)$ where $0 \leq \gamma < \frac{1}{2}$ and $0 < s < m - 1/2$. Moreover, u^0 has the additional regularity:*

$$u^0 \in C\big([0,T_0], \mathcal{H}^{0,m-2}\big), \quad \nabla u^0 \in L^2([0,T_0], \mathcal{H}^{0,m-1}) \cap L^\infty([0,T_0]\times\mathcal{S}) \tag{1.35}$$

and one can find $\pi^0 \in L^2([0,T_0], \mathcal{H}^{0,m-1})$ such that (u^0, π^0, h^0) solves uniquely the following incompressible free-surface Navier-Stokes system:

$$\begin{cases} \bar{\rho}(\partial_t^{\varphi^0} u^0 + u^0 \cdot \nabla^{\varphi^0} u^0) - \mathrm{div}^{\varphi^0} S^{\varphi^0} u^0 + \nabla^{\varphi^0}\pi^0 = 0, \\ \mathrm{div}^{\varphi^0} u^0 = 0, \qquad\qquad (t,x) \in [0,T_0]\times\mathcal{S}, \\ u^0|_{t=0} = u_0^0, h^0|_{t=0} = h_0^0 \end{cases} \tag{1.36}$$

with boundary conditions:

$$\partial_t h^0 + u^0(t,y,0)\cdot\mathbf{N}^0 = 0, \tag{1.37}$$

$$S^{\varphi^0} u^0 \mathbf{N}^0 = \pi^0 \mathbf{N}^0 \quad \text{on } \{z = 0\}, \tag{1.38}$$

$$u_3^0 = 0, \quad \frac{\mu}{\partial_z\varphi_0}\partial_z u_j^0 = a u_j^0 \quad (j = 1,2) \quad \text{on } \{z = -1\}. \tag{1.39}$$

Here φ^0 is defined in (1.11) *(replacing h^ε by h^0),* $\mathbf{N}^0 = (-\partial_1 h^0, -\partial_2 h^0, 1)^t$.

Remark 1.6. *Due to the absence of estimate for the second order normal derivatives of the velocity u^0 (and thus for the strong trace of the normal derivative), the solution to* (1.36)-(1.39)*, must be interpreted in the following sense:* $\mathrm{div}^{\varphi^0} u^0 = 0$ *holds in* $L^2([0,T_0]\times\mathcal{S})$ *and for any vector field* $\psi = (\psi_1,\psi_2,\psi_3)^t \in \left[C_c^\infty\big(\overline{Q_{T_0}}\big)\right]^3$ *with* $\psi_3|_{z=-1} = 0$*, the following identity holds: for any* $0 < t \le T_0$*,*

$$
\begin{aligned}
&\bar\rho\int_{\mathcal{S}} u^0\cdot\psi(t,\cdot)\,\mathrm{d}\mathcal{V}_t^0 + 2\mu\int_0^t\int_{\mathcal{S}} S^{\varphi^0}u^0\cdot\nabla^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s + \bar\rho\int_0^t\int_{\mathcal{S}}(u^0\cdot\nabla^{\varphi^0}u^0)\cdot\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s\\
&= \bar\rho\int_{\mathcal{S}} u^0\cdot\psi(0,\cdot)\,\mathrm{d}\mathcal{V}_0^0 + \bar\rho\int_0^t\int_{\mathcal{S}} u^0\cdot\partial_t^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s + \int_0^t\int_{\mathcal{S}}\pi^0\mathrm{div}^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s\\
&\quad + \int_0^t\int_{z=0}(u^0\cdot\mathbf{N}^0)u^0\cdot\psi\,\mathrm{d}y\mathrm{d}s + a\int_0^t\int_{z=-1}(u_1^0\cdot\psi_1 + u_2^0\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s
\end{aligned}
\tag{1.40}
$$

where $\mathrm{d}\mathcal{V}_t^0 = \frac{1}{\partial_z\varphi^0}(t,\cdot)\,\mathrm{d}y\mathrm{d}z$.

Remark 1.7. *Note that we do not end up in the classical space of existence and uniqueness for the free boundary incompressible Navier-Stokes system, nevertheless, the uniqueness of the solution in our functional spaces can be obtained by taking benefits of the control of the Lipschitz norm of the solution. One can refer to subsection 14.1 for the proof.*

1.4. **Main difficulties, general strategies.** Due to the simultaneous presence of the singular term in the equation as well as the viscous term and boundaries, we are confronted with both difficulties resulting from boundary layer effects and fast time oscillations. These two phenomena are well understood when they occur separately, but some new difficulties occur when they occur at the same time. Indeed, on the one hand, regarding the vanishing viscosity limit problem (see for instance [46, 48]), one can estimate the high order tangential derivatives by direct energy estimates, and then use the vorticity to control the normal derivatives. Nevertheless, for the system with low Mach number, the tangential derivatives (∂_y) are not easy to control uniformly, since they do not commute with $\nabla^{\varphi^\varepsilon}, \mathrm{div}^{\varphi^\varepsilon}$ and thus create singular commutators. Without the a priori knowledge of the tangential derivatives, the estimate of the vorticity cannot be performed. On the other hand, for the compressible free boundary Euler system with a low Mach number, uniform estimates are established for example in [20, 41, 44]. Besides the difficulties arising from the Taylor sign condition and the regularity of the surface, the idea behind getting uniform estimates is to control first weighted time derivatives $(\varepsilon\partial_t)^k$ and then to recover space derivatives by using the equations and by direct energy estimates for the vorticity. Here, in the case of viscous fluids, the vorticity is not easy to estimate due to the lack of information on its trace on the boundaries. We shall explain more precisely in the following. For the sake of notational convenience, we will drop the $\varepsilon-$dependence of the solution.

Indeed, the vorticity $\omega = \mathrm{curl}^\varphi u$ solves a transport-diffusion equation with Dirichlet boundary condition (see (4.5), (4.8)) under the form

$$\omega|_{\partial\mathcal{S}} \approx \partial_y u + \mathrm{div}^\varphi u|_{\partial\mathcal{S}}. \tag{1.41}$$

Let us consider the simplest case, the heat equation with zero source and initial data but with nonhomogeneous Dirichlet condition in a half space :

$$\bar\rho\partial_t f - \mu\Delta f = 0, \quad f|_{t=0} = 0, \quad f|_{z=0} = f^{b,1}, \quad (t,x)\in[0,T]\times\mathbb{R}_-^3, \tag{1.42}$$

By using the heat kernel, we obtain

$$\|f\|_{L_t^2 H_{co}^{m-1}} \lesssim T^{\frac14}|f^{b,1}|_{L_t^2\tilde{H}^{m-1}}.$$

By applying this estimate to ω, we see that the boundary contribution when estimating $\|\omega\|_{L_t^2 H_{co}^{m-\frac{1}{2}}}$ is more or less $|(\partial_y u, \mathrm{div} u)|_{L_t^2 \tilde{H}^{m-1}}$, which requires the foreknowledge of the tangential derivatives and which indicates the loss of half derivative. One could also use the (tangential) smoothing effects of the heat equation to overcome this loss of derivative. Nevertheless, in this way, it seems impossible to extract the extra ε or T which are essential to close the estimate. More precisely, by using maximal regularity, one gets that

$$\begin{aligned}\|\omega\|_{L_t^2 H_{co}^{m-1}} &\leq C|(\partial_y u, \mathrm{div}^{\varphi} u)|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}} + \text{other terms} \\ &\leq C(\|\nabla u\|_{L_t^2 H_{co}^{m-1}} + \|\nabla \mathrm{div}^{\varphi} u\|_{L_t^2 H_{co}^{m-2}}) + \text{other terms}\end{aligned}$$

which does not gain anything. Note that the constant C is independent of T and ε.

To overcome these problems, we split the velocity u into a compressible part $\nabla^{\varphi}\Psi$ and an incompressible part v (see definition (5.2), (5.3)). On the one hand, the compressible part is governed by the elliptic equation $\Delta^{\varphi}\Psi = \mathrm{div}^{\varphi} u$ with mixed boundary conditions (with homogeneous Dirichlet boundary condition on the upper boundary and homogeneous Neumann boundary condition on the bottom). Hence the estimate for its gradient $\nabla^2\Psi$ can be deduced from the estimate of $\mathrm{div}^{\varphi} u$. We then use induction arguments and the equations to establish high-order estimates of $\mathrm{div}^{\varphi} u$. On the other hand, the incompressible part v, solves, up to the control of non-local commutators, a transport-diffusion equation and hence one can use direct energy estimates to get some suitable estimates (say $\|\partial_y^{m-1} v\|_{L_t^\infty L^2}$ and $\|\nabla v\|_{L_t^2 H_{co}^{m-1}}$), which together with the estimates on $\mathrm{div}^{\varphi} u$, lead to the uniform control of $\|\partial_y^{m-1} u\|_{L_t^\infty L^2(\mathcal{S})}$ and $\|\nabla u\|_{L_t^2 H_{co}^{m-1}}$. The final task is to estimate $\|\nabla v\|_{L_t^\infty H_{co}^{m-4}}$ which stems from a careful study on $\omega \times \mathbf{n}$. We remark that this strategy has been employed by the authors in [47] where uniform in low Mach number estimates are established in the case of smooth fixed bounded domain with Navier boundary condition and ill-prepared initial data. However, as will be explained in next subsection, there are various extra difficulties for the free boundary problem arising from the control of the regularity of the surface.

1.5. **Remarks on the slightly well-prepared data assumption.** In the free surface setting, a very sensitive part of the analysis is the control of the regularity of the surface. This is the reason why we have to allow the initial data to be slightly well-prepared. Indeed, since the incompressible part v^{ε} satisfies the boundary condition (see (1.51), (1.52))

$$(2\mu S^{\varphi} v - \pi \mathrm{Id})\mathbf{N}|_{z=0} = 2\mu(\mathrm{div}^{\varphi} u \mathrm{Id} - \nabla^{\varphi}\nabla^{\varphi}\Psi)\mathbf{N}|_{z=0},$$

in order to perform energy estimates for v at order $m-1$, it requires information on $\|\nabla^3\Psi\|_{L_t^2 H_{co}^{m-3}}$, which, by elliptic estimates, can be controlled by $\|\nabla \mathrm{div}^{\varphi} u\|_{L_t^2 H_{co}^{m-2}}$ and $|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$. Nevertheless, due to the fast oscillations, we cannot expect $|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$ (or alternatively $\|\nabla u\|_{L_t^2 H_{co}^m}$) to be uniformly bounded. A similar problem occurs when one recovers the $L_t^2 H_{co}^{m-1}$ norm of $\nabla^2\Psi$ from the one of $\mathrm{div}^{\varphi} u$ by elliptic estimates. To overcome this problem, we assume the data to be slightly well-prepared so that $\|\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^1}$ can be proved to be of order ε^{ϑ}, ($0 < \vartheta < 1$ to be chosen). This can make an extra ε^{ϑ} appear in front of $|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$ in the process of the elliptic estimates (one can refer to Step 3 of the following subsection for more details). In turn, to control uniformly the term $\varepsilon^{\vartheta}|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$, which reduces to the estimate of $\varepsilon^{\vartheta}\|\nabla u\|_{L_t^2 H_{co}^m}$, we must assume that the compressible part $(\mathrm{div}^{\varphi} u, \nabla\sigma)$ has the size of $\mathcal{O}(\varepsilon^{1-\vartheta})$ in $L_t^2 H_{co}^{m-1}$. Indeed, when performing the highest-order energy estimates, we need to be careful with the singular term

$$\varepsilon^{2\vartheta-1}\int_0^t\int_{\mathcal{S}} Z^\alpha\sigma\,\underbrace{[Z^\alpha,\mathrm{div}^\varphi]u}_{[Z^\alpha,\frac{\mathbf{N}}{\partial_z\varphi}\cdot\partial_z]u}+Z^\alpha u\cdot\underbrace{[Z^\alpha,\nabla^\varphi]\sigma}_{[Z^\alpha,\frac{\mathbf{N}}{\partial_z\varphi}\partial_z]\sigma}\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\quad |\alpha|=m. \tag{1.43}$$

By direct computations, these terms can be bounded by (up to other good terms and upon the foreknowledge of $|\varepsilon^\vartheta h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}$)

$$\varepsilon^{\vartheta-1}|\varepsilon^\vartheta h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}\big(\|Z\sigma\|_{L_t^2H_{co}^{m-1}}\|\nabla u\|_{0,\infty,t}+\|u\|_{L_t^2H_{co}^m}\|\nabla\sigma\|_{0,\infty,t}\big)\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big),$$

which can be uniformly bounded if

$$\|Z\sigma\|_{L_t^2H_{co}^{m-1}}=\mathcal{O}(\varepsilon^{1-\vartheta}),\quad \|\nabla\sigma\|_{0,\infty,t}=\mathcal{O}(\varepsilon^{1-\vartheta}).$$

By optimizing, $\vartheta=1-\vartheta$, we shall thus prove the uniform estimates by assuming that $(\nabla\sigma,\mathrm{div}^\varphi u)|_{t=0}=\mathcal{O}(\varepsilon^{\frac{1}{2}})$. By using the same ideas, it would be also possible to establish uniform estimates by assuming that the compressible part is of size at $\mathcal{O}(\varepsilon^\vartheta)$ ($\frac{1}{2}<\vartheta\le 1$) in a low regularity space (say H_{co}^1) and $\mathcal{O}(\varepsilon^{1-\vartheta})$ in a higher regularity space (say H_{co}^{m-1}).

One may wonder whether the introduction of the Alinhac good unknown which is used frequently in free boundary problems can help us to avoid to lose derivatives on the surface and to get uniform estimates without any size assumption on the data. However, this quantity does not seem useful here. Indeed, the use of the Alinhac good unknown would require the validity of the Taylor sign condition ($\partial_{\mathbf{n}}^\varphi\sigma|_{z=0}>0$), which seems out of reach for ill-prepared data since σ solves a transport equation with a source term of size of $\mathcal{O}(\varepsilon^{-1})$.

1.6. **Sketch of the proof.** Let us explain the main steps for the proof of Theorem 1.1. The uniform energy estimates will be established in the following steps:

Step 1: $\varepsilon-$dependent high-order energy estimates and $\varepsilon-$independent high-order time derivative estimates.

In this step, we aim to obtain two kinds of energy estimates. The first one is the estimate of $\varepsilon^{\frac{1}{2}}\|(\sigma,u)\|_{L_t^\infty H_{co}^m}$ and $\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u)\|_{L_t^\infty\mathcal{H}^{m-1}}$. Since the spatial conormal vector fields Z_1,Z_2,Z_3 do not commute with ∇^φ and div^φ, it seems hard to get the uniform estimate of $\|(\sigma,u)\|_{L_t^\infty H_{co}^m}$ by direct energy estimates. Nevertheless, it is easy to get an $\varepsilon-$dependent estimate involving the control of $\|\nabla(\sigma,u)\|_{L_t^2H_{co}^{m-1}}$. This can be done by applying $Z^\alpha(|\alpha|\le m)$ to the system (1.16) and then by performing standard energy estimates making use of the symmetric structure. We remark that at this stage we do not lose regularity on the surface. Indeed, besides the term listed in (1.43) (setting $\vartheta=\frac{1}{2}$), the possible most problematic commutator term is

$$\varepsilon\int_0^t\int_{\mathcal{S}} Z^\alpha\mathbf{N}\cdot\partial_z\mathcal{L}^\varphi u Z^\alpha u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\ \mathrm{d}\mathcal{V}_s=\frac{1}{\partial_z\varphi}\,\mathrm{d}y\mathrm{d}z$$

which can be bounded by: $\varepsilon^{\frac{1}{2}}|h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}\|u\|_{L_t^2H_{co}^m}\|\varepsilon^{\frac{1}{2}}\partial_z\mathcal{L}^\varphi u\|_{\infty,t}$. Note that the estimate of $\varepsilon^{\frac{1}{2}}|h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}$ is available owing to the control of $\varepsilon^{\frac{1}{2}}\|\nabla u\|_{L_t^2H_{co}^m}$ and $\|\varepsilon^{\frac{1}{2}}\partial_z\mathcal{L}^\varphi u\|_{\infty,t}$ by the terms appearing in $\mathcal{A}_{m,t}$, using the equation of the velocity.

The estimate of $\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u)\|_{L_t^\infty\mathcal{H}^{m-1}}$ can also be derived by straightforward energy estimates. The main observation is that: although the weighted time derivatives $\varepsilon^{\frac{1}{2}}(\varepsilon\partial_t)^k\partial_t$ do not commute with ∇^φ, their commutator can be uniformly controlled even for the singular term. Indeed, direct computation shows that for $k \le m-1$,

$$\varepsilon^{\frac12}\frac{[(\varepsilon\partial_t)^k\partial_t, \mathrm{div}^\varphi]u}{\varepsilon} = \varepsilon^{k-\frac12}\big[\partial_t^{k+1}, \frac{\mathbf{N}}{\partial_z\varphi}\big]\cdot\partial_z u$$

whose $L_t^2L^2(\mathcal{S})$ norm is uniformly controlled as long as $k \ge 1$ thanks to the boundedness of $|\varepsilon^{\frac12}\partial_t^2 h|_{L_t^2\tilde{H}^{m-\frac32}}$ (see (6.2)). We remark that in view of the definition (1.12), the boundedness of $\mathbf{N}$ can be derived from that of h. The case $k=0$ needs to be treated differently and is explained in the next step.

The second kind of estimate is for the terms $\varepsilon^{\frac12}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-1}}$, $\varepsilon^{\frac12}\|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}}$, which follows again from direct energy estimates, we thus do not detail more here.

Step 2. Uniform lower order energy estimates. In this step, we aim to show the boundedness of $\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^\infty L^2}$. We remark that a naive energy estimate fails due to bad commutators with the singular term. Actually, the $L_t^2L^2(\mathcal{S})$ norm of the term $\varepsilon^{-\frac12}[\partial_t, \mathrm{div}^\varphi]u = \varepsilon^{-\frac12}\partial_t(\mathbf{N}/\partial_z\varphi)\cdot\partial_z u$ is out of control. The trick to avoid this problem is to multiply $\partial_t^\varphi(1.16)_1$ by $\varepsilon\partial_t\sigma$ and multiply $\partial_t(1.16)_2$ by $\varepsilon\partial_t^\varphi u$. In this way, the singular term can be dealt with as:

$$\begin{aligned}&\int_0^t\int_{\mathcal{S}}\partial_t\nabla^\varphi\sigma\partial_t^\varphi u + \partial_t^\varphi\mathrm{div}^\varphi u\partial_t\sigma \,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ &= \int_0^t\int_{z=0}\partial_t^\varphi u\cdot\mathbf{N}\partial_t\sigma\,\mathrm{d}y\mathrm{d}s + \int_0^t\int_{\mathcal{S}}\partial_t^\varphi u[\partial_t,\nabla^\varphi]\sigma\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\end{aligned}\tag{1.44}$$

where $\mathrm{d}\mathcal{V}_s = \partial_z\varphi\,\mathrm{d}y\mathrm{d}z$. The first boundary term combined with another boundary term which comes from the integration by parts of the viscous term, result in a good term that can be controlled. Namely

$$\varepsilon\int_0^t\int_{z=0}\partial_t\big[-\mathcal{L}^\varphi u + \frac{\sigma}{\varepsilon}\mathrm{Id}\big]\mathbf{N}\cdot\partial_t^\varphi u\,\mathrm{d}y\mathrm{d}s = -\varepsilon\int_0^t\int_{z=0}(-\mathcal{L}^\varphi u + \frac{\sigma}{\varepsilon}\mathrm{Id})\partial_t\mathbf{N}\cdot\partial_t^\varphi u\,\mathrm{d}y\mathrm{d}s.$$

Note that the trace of $\frac{\sigma}{\varepsilon}$ on the upper boundary can be expressed as the spatial tangential derivatives of the velocity (see (4.1)) which can be easily treated by the trace inequality. The second term in (1.44) is also manageable since $\varepsilon^{-\frac12}\|[\partial_t,\nabla^\varphi]\sigma\|_{L_t^2L^2(\mathcal{S})}$ can be roughly bounded by $\|\varepsilon^{-\frac12}\nabla^\varphi\sigma\|_{L_t^2L^2(\mathcal{S})}$.

It should be mentioned that the above strategy does not apply for the control of $\varepsilon^{\frac12}\|(\partial_y, Z_3)\partial_t(\sigma,u)\|_{L_t^\infty L^2}$ due to the bad commutator terms. We thus use the strategy of the splitting mentioned before to deal with them in the following steps.

Step 3. Recovering high order spatial derivatives of $(\nabla\sigma, \nabla\nabla^\varphi\Psi)$ by induction. Denote by $\nabla^\varphi\Psi$ the compressible part of the velocity which is defined by the unique solution to the elliptic equation with mixed boundary conditions:

$$\begin{cases}-\mathrm{div}^\varphi\nabla^\varphi\Psi = -\mathrm{div}^\varphi u,\\ \Psi|_{z=0} = 0,\\ \partial_{\mathbf{n}}\Psi|_{z=-1} = 0.\end{cases}\tag{1.45}$$

In this step, we aim to control the $L_t^2H_{co}^{m-1}$ norm of $\nabla^\varphi(\sigma,\nabla^\varphi\Psi)$, which can be reduced to the control of $\varepsilon^{-\frac12}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^2H_{co}^{m-1}}$. We will use the equation and induction arguments to recover the latter. Indeed, let us rewrite the system (1.16) as follows:

$$\begin{cases}-\mathrm{div}^\varphi u = g_1\varepsilon\partial_t\sigma + \varepsilon g_1\underline{u}\cdot\nabla\sigma,\\ -\mu\varepsilon\,\mathrm{curl}^\varphi\omega - \nabla^\varphi\big(\sigma - (2\mu+\lambda)\varepsilon\mathrm{div}^\varphi u\big) = g_2\varepsilon\partial_t u + \varepsilon g_2\underline{u}\cdot\nabla u.\end{cases}\tag{1.46}$$

where

$$\underline{u} = (u_1, u_2, u_z) =: (u_1, u_2, \frac{u\cdot \mathbf{N} - \partial_t\varphi}{\partial_z\varphi}).$$

In view of (1.46), one wants to show that for $j+l\leq m-1$,

$$\varepsilon^{-\frac12}\|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{j,l}} \lesssim \|\varepsilon^{\frac12}\partial_t\sigma\|_{L_t^2\mathcal{H}^{j,l}} + \mathcal{O}(\varepsilon^{\frac12}) \lesssim \varepsilon^{-\frac12}\|\nabla^\varphi\sigma\|_{L_t^2\mathcal{H}^{j+1,l-1}} + \mathcal{O}(1), \tag{1.47}$$

$$\begin{aligned}\varepsilon^{-\frac12}\|\nabla^\varphi\sigma\|_{L_t^2\mathcal{H}^{j,l}} &\lesssim \|\varepsilon^{\frac12}\partial_t\nabla^\varphi\Psi\|_{L_t^2\mathcal{H}^{j+1,l}} + \mathcal{X}_{m,t} + \mathcal{O}(\varepsilon^{\frac12})\\ &\lesssim \varepsilon^{-\frac12}\|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{j+1,l-1}} + \mathcal{X}_{m,t} + \mathcal{O}(1),\end{aligned} \tag{1.48}$$

where

$$\mathcal{X}_{m,t} \approx \varepsilon^{\frac12}\|\nabla^\varphi \mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}} + \varepsilon^{\frac12}\|\nabla^\varphi u\|_{L_t^2H_{co}^m}$$

which has been controlled in the first step. These two inequalities in hand, we can conclude by induction arguments. Note that the inequality (1.47) results from the equality $(1.46)_1$ and the product estimate (3.8). To obtain (1.48), we take div^φ of the equation $(1.46)_2$ and use the boundary condition (1.18) to get the following elliptic equation:

$$\begin{cases}\Delta^\varphi(\varepsilon\theta) = \mathrm{div}^\varphi\big[\bar\rho\varepsilon\partial_t\nabla^\varphi\Psi + \varepsilon(\frac{g_2-\bar\rho}{\varepsilon}\varepsilon\partial_t + g_2\underline{u}\cdot\nabla)u\big] =: \mathrm{div}^\varphi\tilde{G}\\ \varepsilon\theta|_{z=0} = -2\varepsilon\mu(\partial_1u_1+\partial_2u_2) + \varepsilon(\omega\times\mathbf{N})_3|_{z=0}\\ \partial_{\mathbf{n}}\theta|_{z=-1} = \tilde{G}\cdot\mathbf{n} + \mu\varepsilon\,\mathrm{curl}^\varphi\,\omega\times\mathbf{n}|_{z=-1}.\end{cases} \tag{1.49}$$

where $\theta = \sigma/\varepsilon - (2\mu+\lambda)\mathrm{div}^\varphi u$. Inequality (1.48) is thus the consequence of the elliptic estimates in the conormal setting (see Section 5). We remark that the trace of $\omega\times\mathbf{N}$ involves only tangential derivatives of the velocity on the boundary (see (4.2)).

Now that $\mathrm{div}^\varphi u$ has been bounded, we can control the compressible part of the velocity $\nabla^2\Psi$ by again elliptic estimates. Nevertheless, there will be a loss of one derivative on the surface if no smallness condition is made on the compressible part. Indeed, as $\nabla^\varphi\Psi$ solves equation (1.45), we have by the elliptic estimates that

$$\|\nabla^2\Psi\|_{L_t^2\mathcal{H}^{0,m-1}} \lesssim (|h|_{L_t^2H^{m+\frac12}} + \|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{0,m-1}})\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big) \tag{1.50}$$

where Λ denotes a polynomial. This estimate involves more regularity of the surface than that we can afford since we have only the control of $|h|_{L_t^2H^{m-\frac12}}$. Nevertheless, checking the proof of the elliptic estimates for $\nabla^2\Psi$, we find that the main problematic term is indeed $\nabla\Psi Z^\alpha\nabla\mathbf{N}$ ($|\alpha| = m-1, \alpha_0 = 0$), whose $L_t^2L^2(\mathcal{S})$ norm can be bounded by

$$\|\nabla\Psi\|_{L_{t,x}^\infty}|h|_{L_t^2H^{m+\frac12}} \lesssim \Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{tan}^1}|h|_{L_t^2H^{m+\frac12}}.$$

The right hand side can be controlled if $\|\mathrm{div}^\varphi u\|_{L_t^2H_{tan}^1} = \mathcal{O}(\varepsilon^{\frac12})$. Hopefully, once assuming $\varepsilon^{\frac12}(\partial_t\sigma,\partial_t u)(0)$ to be bounded uniformly in $H_{co}^1(\mathcal{S})$, we can show that $\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^1} = \mathcal{O}(\varepsilon^{\frac12})$. This is one reason that we need the initial data to be slightly well-prepared.

Step 4. Uniform energy estimate of the incompressible part of the velocity. Set $v = u - \nabla^\varphi\Psi$ the incompressible part of the velocity. By the computations in Section 5, we find that v solves the following system:

$$\begin{cases}\bar\rho\partial_t^\varphi v - \mu\Delta^\varphi v + \nabla^\varphi\pi = -(f + \nabla^\varphi q + \bar\rho[\mathbb{P}_t, \partial_t^\varphi]u),\\ \mathrm{div}^\varphi v = 0,\\ (2\mu S^\varphi v - \pi\mathrm{Id})\mathbf{N}|_{z=0} = 2\mu(\mathrm{div}^\varphi u\mathrm{Id} - (\nabla^\varphi)^2\Psi)\mathbf{N}|_{z=0},\\ v_3|_{z=-1} = 0,\ \mu\partial_z^\varphi v_j = au_j|_{z=-1},\ j=1,2,\end{cases} \tag{1.51}$$

where $\mathbb{Q}_t, \mathbb{P}_t$ are time-dependent projectors defined in (5.2) (5.3) and

$$(1.52)\ \ f = (g_2 u \cdot \nabla^\varphi u + \frac{g_2 - \bar{\rho}}{\varepsilon} \varepsilon \partial_t^\varphi u), \nabla^\varphi q = -\mathbb{Q}_t(f - \mu \Delta^\varphi v), \nabla^\varphi \pi = \mathbb{P}_t[\nabla^\varphi (\frac{\sigma}{\varepsilon} - (2\mu + \lambda)\mathrm{div}^\varphi u)].$$

Note that $\nabla^\varphi \pi$ does not vanish identically since $\mathbb{Q}_t \nabla^\varphi \neq \nabla^\varphi$ and that $\nabla^\varphi \pi$ is actually not singular though it seems to involve σ/ε. Indeed by the definition of $\mathbb{Q}_t$ and the boundary conditions (4.1) (4.2), π solves the elliptic equation:

$$\begin{cases} \Delta^\varphi \pi = 0, \\ \pi|_{z=0} = -2\mu(\partial_1 u_1 + \partial_2 u_2) - 2\mu(\Pi(\partial_1 u \cdot \mathbf{N}, \partial_2 u \cdot \mathbf{N}, 0)^t)_3, \\ \partial_z^\varphi \pi|_{z=-1} = 0. \end{cases}$$

The key point is that the trace of π on the upper boundary can be uniformly bounded.

In view of (1.51), we expect to perform energy estimates to get a priori control of $\|v\|_{L_t^\infty H_{co}^{m-1}}, \|\varepsilon^{\frac{1}{2}} \partial_t v\|_{L_t^\infty H_{co}^{m-2}}$ and $\|\nabla^\varphi v\|_{L_t^2 H_{co}^{m-1}}, \|\varepsilon^{\frac{1}{2}} \partial_t \nabla v\|_{L_t^\infty H_{co}^{m-2}}$. Of course, due to the interaction with the compressible part through the boundary, their control rely also on the information for the compressible part $\nabla^\varphi \Psi$ and we cannot get higher order estimates.

Step 5. Control of the normal derivative of the velocity. We have obtained the estimates of $\|\nabla^\varphi u\|_{L_t^2 H_{co}^{m-1}}$ in Step 3 and Step 4. It remains to control $\varepsilon^{-\frac{1}{2}} \|(\nabla^\varphi \sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-2}}$ and $\|(\nabla v, \varepsilon^{\frac{1}{2}} \partial_t \nabla v)\|_{L_t^\infty H_{co}^{m-4}}$, which is useful to control the $L_{t,x}^\infty$ norm of the solution. The former quantity can be obtained again by induction arguments while the latter quantity can be deduced from that of $\omega \times \mathbf{n}$. Indeed, we have roughly the estimate:

$$\|(\nabla v, \varepsilon^{\frac{1}{2}} \partial_t \nabla v)\|_{L_t^\infty H_{co}^{m-4}} \lesssim \|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t)(\omega \times \mathbf{n})\|_{L_t^\infty H_{co}^{m-4}} + \|(v, \varepsilon^{\frac{1}{2}} \partial_t v)\|_{L_t^\infty H_{co}^{m-3}} + |(h, \varepsilon^{\frac{1}{2}} \partial_t h)|_{L_t^\infty \tilde{H}^{m-2}}.$$

Let us explain the estimate of $\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t)(\omega \times \mathbf{n})\|_{L_t^\infty H_{co}^{m-4}}$. Direct computations show that:

$$\omega \times \mathbf{n}|_{\partial \mathcal{S}} = -2\Pi(\partial_1 u \cdot \mathbf{n}, \partial_2 u \cdot \mathbf{n}, 0)^t \tag{1.53}$$

where $\Pi = \mathrm{Id}_{3\times 3} - \mathbf{n} \otimes \mathbf{n}$. We define the modified vorticity $\omega_{\mathbf{n}} = \omega \times \mathbf{n} + 2\Pi(\partial_1 v \cdot \mathbf{n}, \partial_2 v \cdot \mathbf{n}, 0)$, so that:

$$\omega_{\mathbf{n}}|_{\partial \mathcal{S}} = -2\Pi(\partial_1 \nabla^\varphi \Psi \cdot \mathbf{n}, \partial_2 \nabla^\varphi \Psi \cdot \mathbf{n}, 0)^t.$$

The advantage of working on $\omega_{\mathbf{n}}$ rather than $\omega \times \mathbf{n}$ is that the former one only involves the compressible part of velocity on the boundary, whose estimates have been established in Step 3. To estimate $\omega_{\mathbf{n}}$, we shall thus instead use a lifting of the boundary conditions by using the Green's function of the solution to the heat equation with non-homogenous boundary conditions and control the remainder by energy estimates. More precisely, let ω^h solves the heat equation (1.42) with boundary condition $\omega^{b,1}|_{z=0} = \omega_{\mathbf{n}}|_{z=0}$, we use (1.42) to get roughly that:

$$\begin{aligned} \|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t) \omega_{\mathbf{n}}^h\|_{L_t^\infty H_{co}^{m-4}} &\lesssim T^{\frac{1}{4}} \big(|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t) \nabla \Psi|_{L_t^\infty \tilde{H}^{m-3}} + |(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t) h|_{L_t^\infty \tilde{H}^{m-3}} \big) \\ &\lesssim T^{\frac{1}{4}} (\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t) \mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^{m-3}} + |(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t) h|_{L_t^\infty \tilde{H}^{m-3}}). \end{aligned}$$

The remainder $\omega_{\mathbf{n}} - \omega_{\mathbf{n}}^h$ can then be controlled by direct energy estimates.

Step 6. $L_{t,x}^\infty$ estimates. This final step is dedicated to the estimates of the $L_{t,x}^\infty$ type norms defined in $\mathcal{A}_{m,T}$. Most of them can be controlled thanks to the Sobolev embedding and the quantities appearing in $\mathcal{E}_{m,T}$. The estimate of the remaining terms $\varepsilon^{-\frac{1}{2}} \|\!|\nabla \sigma|\!\|_{m-5,\infty,T}$ and $\|\!|\nabla u|\!\|_{1,\infty,t}$ are obtained from the maximum principle of the damped transport equation satisfied by $\nabla \sigma$ and the estimate for the heat equation satisfied by ω.

Structure of the paper: We state the uniform a-priori estimates in Section 2, which are shown in the following sections. Some preliminaries (useful lemmas, identities, projections, and elliptic estimates) are first shown in Sections 3-5. The control of the energy norm $\mathcal{E}_{m,T}$ is achieved in Sections 6-Section 11. The $L^\infty_{t,x}$ type estimates are established in Section 12. Theorem 1.1 and Theorem 1.5 are then proved in Section 13 and Section 14 respectively. In Section 15, we explain how our results can be extended to the case when the reference domain is changed into a channel with infinite depth. Finally, one technical product estimate is presented in the appendix.

Further notations

- We denote $\Lambda(\cdot,\cdot)$ a polynomial that may differ from line to line but independent of $\varepsilon \in (0,1]$.
- The traces on the upper boundary {z=0} and lower boundary {z=-1} for a function $f \in H^1(\mathcal{S})$ are denoted by $f^{b,1}$ and $f^{b,2}$ respectively.
- We use the notation $\lesssim$ for $\leq C(1/c_0)$ for some number $C(1/c_0)$ that depends only on $1/c_0$.
- We use the notation $L^2_t L^2 = L^2([0,t]\times\mathcal{S})$.
- We denote $\|f\|_{E^k,t} = \|f\|_{L^2_t H^k_{co}} + \|\nabla f\|_{L^2_t H^{k-1}_{co}}$.

2. Uniform a-priori estimates

Our main a priori estimate is the following:

Theorem 2.1. *Let $c_0 \in (0,\frac{1}{2}]$ such that:*

$$\sup_{s\in[-3c_1\bar{P},3\bar{P}/c_1]} |(g_1,g_2)(s)| \in [c_0, 1/c_0] \tag{2.1}$$

where $0 < c_1 < \frac{1}{4}$ is a fixed constant. Suppose that for some $0 < T \leq 1$, for all $(t,x) \in [0,T]\times\mathcal{S}, \varepsilon \in [0,1]$, it holds that:

$$\partial_z\varphi^\varepsilon(t,x) \geq c_0, \quad |(\nabla\varphi^\varepsilon, \nabla^2\varphi^\varepsilon)(t,x)| \leq 1/c_0, \quad -3c_1\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 3\bar{P}/c_1. \tag{2.2}$$

Then there exist two continuous functions $P_1, P_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, and $\vartheta > 0$ which are independent of ε, such that the following estimate holds:

$$\mathcal{N}^\varepsilon_{m,T} \leq P_1\big(\frac{1}{c_0}, Y^\varepsilon_m(0)\big) + (T+\varepsilon)^\vartheta P_2\big(\frac{1}{c_0}, Y^\varepsilon_m(0) + \mathcal{N}^\varepsilon_{m,T}\big) \tag{2.3}$$

where $\mathcal{N}^\varepsilon_{m,T}$ is defined in (1.31).

This theorem is a direct consequence of the following two propositions.

Proposition 2.2. *Under the assumption of Theorem 2.1, there exist two $\varepsilon-$independent continuous functions $P_3, P_4 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, such that:*

$$\mathcal{E}^\varepsilon_{m,T} \leq P_3\big(\frac{1}{c_0}, Y^\varepsilon_m(0)\big) + (T+\varepsilon)^{\vartheta_1} P_4\big(\frac{1}{c_0}, Y^\varepsilon_m(0) + \mathcal{N}^\varepsilon_{m,T}\big). \tag{2.4}$$

Proof. This proposition is obtained by energy estimates, we split it into several sections (Section 6-11). By Lemma 6.1 for the estimate of the surface, Lemmas 7.1, 7.4, 10.1 for $\varepsilon-$dependent estimates to the highest order, Lemmas 9.1, 9.3, 11.1, 11.3, 11.10 for the uniform estimates, we can find two polynomials Λ_5, Λ_6 whose coefficients are independent of ε, such that:

$$(\mathcal{E}^\varepsilon_{high,m,T})^2 \leq \Lambda_5\big(\frac{1}{c_0}, |h^\varepsilon|^2_{L^\infty_T \tilde{H}^{m-\frac{1}{2}}} + Y^\varepsilon_m(0)^2\big) Y^\varepsilon_m(0)^2 + (T+\varepsilon)^{\frac{1}{4}} \Lambda_6\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big). \tag{2.5}$$

By Lemma 8.1, there exist polynomials Λ_7, Λ_8 whose coefficients are independent of ε, such that:

$$(\tilde{\mathcal{E}}_{low,T})^2 \lesssim \Lambda_7\big(\frac{1}{c_0}, |h^\varepsilon|^2_{3,\infty,T}\big)(Y^\varepsilon_m(0)^2 + (\mathcal{E}^\varepsilon_{high,m,T})^2) + (T+\varepsilon)^{\frac{1}{2}}\Lambda_8\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big).$$

By the Sobolev embedding $H^{\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$,

$$|h^\varepsilon|^2_{3,\infty,T} \lesssim |h^\varepsilon|^2_{L^\infty_T \tilde{H}^{m-\frac{1}{2}}},$$

we thus find two polynomials Λ_9 and Λ_{10} such that:

$$(\mathcal{E}^\varepsilon_{m,T})^2 \le \Lambda_9\big(\frac{1}{c_0}, |h^\varepsilon|^2_{L^\infty_T \tilde{H}^{m-\frac{1}{2}}} + Y^\varepsilon_m(0)^2\big)Y^\varepsilon_m(0)^2 + (T+\varepsilon)^{\frac{1}{4}}\Lambda_{10}\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big). \tag{2.6}$$

By (6.3), there exists a polynomial Λ_{11}, such that:

$$|h^\varepsilon|^2_{L^\infty_t \tilde{H}^{m-\frac{1}{2}}} \le Y^\varepsilon_m(0)^2 + T^{\frac{1}{2}}\Lambda_{11}\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big).$$

Plugging this inequality into (2.5), one finds two other polynomials $\Lambda_{12}, \Lambda_{13}$, and a constant $\vartheta_2 > 0$, such that:

$$(\mathcal{E}^\varepsilon_{high,m,T})^2 \lesssim \Lambda_{12}(\frac{1}{c_0}, Y^\varepsilon_m(0)^2) + (T+\varepsilon)^{\vartheta_2}\Lambda_{13}(\frac{1}{c_0}, Y^\varepsilon_m(0) + \mathcal{N}^\varepsilon_{m,T}).$$

We thus finish the proof by inserting the above inequality into (2.6). □

Proposition 2.3. *Assume that* (2.2) *holds, we have the a-priori estimate for the* $L^\infty_t L^\infty(\mathcal{S})$ *norms,*

$$\mathcal{A}^\varepsilon_{m,T} \le \Lambda\big(\frac{1}{c_0}, Y_m(0)\big) + \Lambda\big(\frac{1}{c_0}, |h^\varepsilon|_{3,\infty,t}\big)\tilde{\mathcal{E}}^\varepsilon_{m,T} + (\tilde{\mathcal{E}}^\varepsilon_{m,T})^4 + (T+\varepsilon^{\frac{1}{4}})\Lambda_{14}\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big). \tag{2.7}$$

where Λ_{14} *is a polynomial with* $\varepsilon-$*independent coefficients.*

Proof. Its proof is presented in Section 12. □

3. Preliminaries I: Useful lemmas.

In this section, we list some elementary lemmas which will be often used throughout this paper.

3.1. Product and commutator estimates.

We begin with the following product and commutator estimates in $\mathbb{R}^2$.

Lemma 3.1. *Let* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *belong to the spaces appearing in below. For any* $s \ge 1$,

$$|\Lambda^s(fg)|_{L^2(\mathbb{R}^2)} \lesssim |f|_{H^s(\mathbb{R}^2)}|g|_{L^\infty(\mathbb{R}^2)} + |g|_{H^s(\mathbb{R}^2)}|f|_{L^\infty(\mathbb{R}^2)} \tag{3.1}$$

$$|[\Lambda^s, f]g|_{L^2(\mathbb{R}^2)} \lesssim |f|_{H^{s-1}(\mathbb{R}^2)}|g|_{L^\infty(\mathbb{R}^2)} + |f|_{W^{1,\infty}(\mathbb{R}^2)}|g|_{H^{s-1}(\mathbb{R}^2)} \tag{3.2}$$

For any $-1 < s \le 1$,

$$|[\Lambda^s, g]f|_{L^2(\mathbb{R}^2)} \lesssim |f|_{H^{s-1}(\mathbb{R}^2)}|g|_{H^{2^+}(\mathbb{R}^2)}, \tag{3.3}$$

$$|fg|_{H^s(\mathbb{R}^2)} \lesssim |f|_{H^s(\mathbb{R}^2)} \min\{|g|_{H^{1^+}(\mathbb{R}^2)}, |g|_{W^{1,\infty}(\mathbb{R}^2)}\}. \tag{3.4}$$

where $(\Lambda^s f)(y) = \mathcal{F}^{-1}_{\xi\to y}\big((1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)\big)$, a^+ *denotes a real number that is larger but arbitrary close to* a.

The product estimate (3.1) and the commutator estimate (3.2) can be found in [13] for example, (3.3) is indeed a restatement of (A.6) in [9]. The proof of (3.4) is presented in the appendix.

Corollary 3.2. *Let $k \geq 2$ be an integer, one has the following estimates:*

$$|(fg)(t)|_{\tilde{H}^{k+\frac{1}{2}}} \lesssim |f(t)|_{\tilde{H}^{[\frac{k}{2}]+}} |g(t)|_{\tilde{H}^{k+\frac{1}{2}}} + |g(t)|_{\tilde{H}^{[\frac{k+1}{2}]+1+}} |f(t)|_{\tilde{H}^{k+\frac{1}{2}}}, \tag{3.5}$$

$$|[Z^\alpha, f]g(t)|_{H^{\frac{1}{2}}} \lesssim |f(t)|_{\tilde{H}^{[\frac{k}{2}]+}} |g(t)|_{\tilde{H}^{k-\frac{1}{2}}} + |g(t)|_{\tilde{H}^{[\frac{k+1}{2}]+1+}} |f(t)|_{\tilde{H}^{k+\frac{1}{2}}}, |\alpha| = k, \tag{3.6}$$

where $\tilde{H}^s$ is defined in (1.25)*, and commutator $[Z^\alpha, f]g = Z^\alpha(fg) - fZ^\alpha g$.*

Proof. For any $|\alpha| \leq k$, we write

$$Z^\alpha(fg)(t) = \Big(\sum_{|\beta| \leq [\frac{|\alpha|}{2}]-1} + \sum_{|\alpha-\beta| \leq [\frac{|\alpha|+1}{2}]} \Big) Z^\beta f(t) Z^{\alpha-\beta} g(t) \tag{3.7}$$

Inequality (3.5) can then be derived from product estimate (3.4). The proof of (3.6) follows in the same way. □

The following (crude) product estimates in $L_t^\infty \mathcal{H}^{j,l}$ will be useful for instance in the elliptic estimates.

Lemma 3.3. *Let $Z^\alpha = (\varepsilon \partial_t)^j \mathcal{Z}^{\alpha'}$ with $\mathcal{Z} = (Z_1, Z_2, Z_3), |\alpha'| \leq l = k - j, k \geq 2$. One has the crude estimates: for any integer $n \in [0, k-1]$*

$$\|(fg)(t)\|_{\mathcal{H}^{j,l}} \leq \|f(t)\|_{\mathcal{H}^{j,l}} \|g\|_{n,\infty,t} + \|g(t)\|_{\mathcal{H}^{j,l}} \|f\|_{k-n-1,\infty,t}, \tag{3.8}$$

$$\begin{aligned} \|[Z^\alpha, f]g(t)\|_{L^2(\mathcal{S})} \lesssim & \Big(\sum_{\substack{j' \leq j, l' \leq l, \\ j'+l' \leq k-n}} \|f(t)\|_{\mathcal{H}^{j',l'}} \Big) \|g\|_{n,\infty,t} \\ & + \big(\|g(t)\|_{\mathcal{H}^{j-1,l}} + \|g(t)\|_{\mathcal{H}^{j,l-1}} \big) \|f\|_{k-n-1,\infty,t}. \end{aligned} \tag{3.9}$$

We also have the following composition estimates:

Corollary 3.4. *Suppose that $\psi \in C^0(Q_t) \cap L_t^2 H_{co}^m$ with*

$$A_1 \leq \psi(t,x) \leq A_2, \quad \forall (t,x) \in Q_t.$$

Let $F(\cdot) : [A_1, A_2] \to \mathbb{R}$ be a smooth function satisfying

$$\sup_{s \in [A_1, A_2], j \leq m} |F^{(j)}|(s) \leq B.$$

Then we have the composition estimate:

$$\|F(\psi(\cdot,\cdot)) - F(0)\|_{L_t^p H_{co}^m} \leq \Lambda(B, \|\psi\|_{[\frac{m}{2}],\infty,t}) \|\psi\|_{L_t^p H_{co}^m}, \tag{3.10}$$

Corollary 3.5. *Let $g_1(\varepsilon\sigma), g_2(\varepsilon\sigma)$ defined in* (1.8) *and assume Property* (2.1) *and Assumption* (2.2) *hold. Then one has the following estimates: for $j = 1, 2$*

$$\|g_j(\varepsilon\sigma) - g_j(0)\|_{L_t^p H_{co}^m} \lesssim \varepsilon \Lambda\big(\frac{1}{c_0}, \|\sigma\|_{[\frac{m}{2}],\infty,t}\big) \|\sigma\|_{L_t^p H_{co}^m}. \tag{3.11}$$

$$\|Zg_j\|_{L_t^p \mathcal{H}^{m-1}} \leq \varepsilon \Lambda\big(\frac{1}{c_0}, \|\sigma\|_{[\frac{m}{2}],\infty,t}\big) \|(\sigma, Z\sigma)\|_{L_t^p \mathcal{H}^{m-1}}, \tag{3.12}$$

$$\|Zg_j\|_{L_t^p H_{co}^{m-1}} \leq \varepsilon \Lambda\big(\frac{1}{c_0}, \|\sigma\|_{[\frac{m}{2}],\infty,t}\big) \|\sigma\|_{L_t^p H_{co}^m}, \tag{3.13}$$

Proof. Inequality (3.11) is a direct consequence of the composition estimate (3.10). To get (3.12), (3.13), one can apply (3.8) for $n = [\frac{m-1}{2}] - 1$ and use again (3.10). □

The next lemma states the generalized product estimate and commutator estimate [28].

Lemma 3.6. *For $|\alpha| \le m, \alpha_0 = 0$ we have the product estimate and commutator estimates:*

$$\|Z^\alpha(fg)\|_{L_t^2L^2} \lesssim \|f\|_{L_t^2\mathcal{H}^{0,m}}\|g\|_{0,\infty,t} + \|g\|_{L_t^2\mathcal{H}^{0,m}}\|f\|_{0,\infty,t}, \tag{3.14}$$

$$\|[Z^\alpha, f]g\|_{L_t^2L^2} \lesssim \|f\|_{L_t^2\mathcal{H}^{0,m}}\|g\|_{0,\infty,t} + \|g\|_{L_t^2\mathcal{H}^{0,m-1}}\|f\|_{1,\infty,t}. \tag{3.15}$$

We finally state the following Sobolev embedding and trace inequalities whose proofs can be found in Proposition 2.2 of [46].

Lemma 3.7. *For each $t \in [0,T]$, we have:*

$$\|f(t)\|_{L^\infty(\mathcal{S})} \lesssim \|(f,\nabla f)(t)\|^{\frac{1}{2}}_{H^{s_1}_{tan}(\mathcal{S})}\|f(t)\|^{\frac{1}{2}}_{H^{s_2}_{tan}(\mathcal{S})}, \quad s_1+s_2>2, s_1,s_2\ge 0, \tag{3.16}$$

$$\begin{aligned} &|f(t,\cdot,0)|_{H^s(\mathbb{R}^2)} + |f(t,\cdot,-1)|_{H^s(\mathbb{R}^2)} \\ &\lesssim \|\partial_z f(t)\|^{\frac{1}{2}}_{H^{s-1/2}_{tan}(\mathcal{S})}\|f(t)\|^{1/2}_{H^{s+1/2}_{tan}(\mathcal{S})} + \|f(t)\|_{H^{s+1/2}_{tan}(\mathcal{S})}, \quad s \ge \frac{1}{2}. \end{aligned} \tag{3.17}$$

where we have used the notation $\|f(t)\|_{H^s_{tan}(\mathcal{S})} = \|\Lambda^s f(t)\|_{L^2(\mathcal{S})}$.

3.2. **Regularity of the extension and some further commutator estimates.** We first show that the diffeomorphism Φ has the same regularity as u in $\mathcal{S}$, which stems from the fact that the extension function φ gains half a space derivative with respect to h. Before stating the main estimates, let us recall that φ and η are defined in (1.11), (1.12).

Lemma 3.8. *For any integers $j,k\ge 0$, we have the following estimates:*

$$\|[(\varepsilon\partial_t)^j\nabla\varphi](t)\|_{H^k(\mathcal{S})} \lesssim |[(\varepsilon\partial_t)^j h](t,\cdot)|_{H^{k+\frac{1}{2}}(\mathbb{R}^2)}, \tag{3.18}$$

$$\|\nabla\varphi\|_{L_t^2\mathcal{H}^{j,k}(\mathcal{S})} \lesssim |h|_{L_t^2\tilde{H}^{k+j+\frac{1}{2}}(\mathbb{R}^2)}. \tag{3.19}$$

Moreover, we have the $L^\infty_{t,x}$ estimates for η :

$$\|[(\varepsilon\partial_t)^j\eta](t)\|_{W^{k,\infty}(\mathcal{S})} \lesssim |[(\varepsilon\partial_t)^j h](t)|_{W^{k,\infty}(\mathbb{R}^2)} \lesssim |h|_{k+j,\infty,t}. \tag{3.20}$$

Proof. These estimates can be deduced from Young's inequality and the following estimates:

$$\int_{-1}^0 e^{-2\delta_0 z^2\langle\xi\rangle^2}\mathrm{d}z \lesssim \delta_0^{-\frac{1}{2}}\langle\xi\rangle^{-1}; \quad \|\mathcal{F}^{-1}(e^{-\delta_0 z^2\langle\xi\rangle^2})\|_{L^\infty_z L^1_y} \lesssim 1.$$

One can refer to Proposition 3.1 of [46] for the detail of the case $j=0$. The case for $j>0$ follows from the observation that time derivatives commute with the actions $\varphi(h)$ and $\eta(h)$. □

Lemma 3.9. *Suppose that: $\partial_z\varphi(t,x)\ge c_0$ for $(t,x)\in[0,T]\times\mathcal{S}$. Then for any $k\in\mathbf{N}$,*

$$\Big\|\frac{f}{\partial_z\varphi}\Big\|_{L^p_tH^k_{co}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t} + \|f\|_{[\frac{k}{2}],\infty,t}\Big)\Big(\|f\|_{L^p_tH^k_{co}} + |h|_{L^p_t\tilde{H}^{k+\frac{1}{2}}}\Big), \quad p=2,+\infty. \tag{3.21}$$

Proof. Let us write:

$$\frac{f}{\partial_z\varphi} = \frac{f}{1+\eta+\partial_z\eta(1+z)} = f - f\frac{\eta+\partial_z\eta(1+z)}{1+\eta+\partial_z\eta(1+z)}.$$

Therefore, one obtains (3.21) by applying the product estimate (3.8) for $n=[\frac{k}{2}]$ and composition estimate (3.10) for $F(x)=\frac{x}{1+x}$ $(0<x<1)$. □

Remark 3.10. *Similar to* (3.21)*, under the same assumption as in Lemma 3.9, the following estimate also holds true,*

$$\|\frac{f}{\partial_z\varphi}\|_{L_t^p\mathcal{H}^{0,k}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{1,\infty,t} + \|\!|f|\!\|_{0,\infty,t}\big)(\|f\|_{L_t^p\mathcal{H}^{0,k}} + |h|_{L_t^p\tilde{H}^{k+\frac{1}{2}}}),\ p = 2, +\infty. \tag{3.22}$$

The next lemma contains useful commutator estimates.

Lemma 3.11. *Under the assumption* (2.2)*, the following commutator estimates hold, for* $j = 1, 2, 3, |\alpha| \le k$

$$\begin{aligned} \|[Z^\alpha, \partial_j^\varphi]f\|_{L_t^2L^2} \lesssim\ & \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h|_{L_t^2\tilde{H}^{k-n+\frac{1}{2}}}\|\!|\nabla f|\!\|_{n,\infty,t} \\ & + \Lambda\big(\frac{1}{c_0}, |h|_{k-n,\infty,t}\big)\|\nabla f\|_{L_t^2H_{co}^{k-1}} \quad (0 \le n \le k-1). \end{aligned} \tag{3.23}$$

If $\alpha_0 = 0$*, we have that:*

$$\|[Z^\alpha, \partial_j^\varphi]f\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{1,\infty,t}\big)\|\nabla f\|_{L_t^2H_{co}^{k-1}} + \Lambda\big(\frac{1}{c_0}, \|\!|\nabla f|\!\|_{0,\infty,t}\big)|h|_{L_t^2\tilde{H}^{k+\frac{1}{2}}}. \tag{3.24}$$

Moreover, for $k \ge 3$*,*

$$\begin{aligned} \|[Z_0^k\partial_t, \partial_j^\varphi]f\|_{L_t^2L^2} \lesssim\ & \Lambda\Big(\frac{1}{c_0}, \|\!|(\partial_z f, \varepsilon\partial_t\partial_z f)|\!\|_{0,\infty,t} + |(h, \partial_t h)|_{k-2,\infty,t} + \big(\int_0^t |\varepsilon\partial_t^2 h(s)|_{k-2,\infty}^2 \mathrm{d}s\big)^{\frac{1}{2}}\Big) \\ & \cdot \big(\varepsilon \sum_{l\le k-1} |Z_0^l\partial_t^2 h|_{L_t^2H^{\frac{1}{2}}} + \|Z_0\partial_z f\|_{L_t^2\mathcal{H}^{k-1}} + \|Z_0\partial_z f\|_{L_t^\infty\mathcal{H}^1}\big). \end{aligned} \tag{3.25}$$

Proof. By the definition (1.15) for ∇^φ,

$$[Z^\alpha, \partial_j^\varphi]f = [Z^\alpha, \mathbf{N}_j/\partial_z\varphi]\partial_z f + (\mathbf{N}_j/\partial_z\varphi)[Z^\alpha, \partial_z]f. \tag{3.26}$$

Moreover, there exist smooth functions $C_{\phi,\beta,\alpha}, C_{\phi,\gamma,\alpha}$ which depend on derivatives of ϕ such that:

$$[Z^\alpha, \partial_z] = \sum_{|\beta|\le|\alpha|-1} C_{\phi,\beta,\alpha}Z^\beta\partial_z = \sum_{|\gamma|\le|\beta|-1} C_{\phi,\gamma,\alpha}\partial_z Z^\gamma. \tag{3.27}$$

Therefore, we get (3.23) by (3.9), (3.21). and get (3.24) by (3.15), (3.22).

Next, for (3.25), we use the following direct expansion

$$[Z_0^k\partial_t, g]w = \Big(\sum_{0\le l\le 1} + \sum_{0\le k-l\le k-3}\Big)\big(C_{k,l}Z_0^{k-l}\partial_t g\, Z_0^l w\big) + C_{k,2}Z_0^{k-2}\partial_t g Z_0^2 w. \tag{3.28}$$

to obtain:

$$\begin{aligned} \|[Z_0^k\partial_t, g]w\|_{L^2L^2} \lesssim\ & \|Z_0\partial_t g\|_{L_t^2\mathcal{H}^{k-1}}\|\!|w|\!\|_{1,\infty,t} + \|Z_0 w\|_{L_t^2\mathcal{H}^{k-1}}\|\!|\partial_t g|\!\|_{k-3,\infty,t} \\ & + \|Z_0 w\|_{L_t^\infty\mathcal{H}^1}\|Z_0^{k-2}\partial_t g\|_{L_t^2L^\infty} \end{aligned} \tag{3.29}$$

Applying (3.29) with $g = \frac{\mathbf{N}_j}{\partial_z\varphi}, w = \partial_z f$, and using (3.18), we get (3.25). $\square$

3.3. **Energy identities and Korn inequality.** We now present some identities which will be often used in the energy estimates:

Lemma 3.12. *It holds that:*

$$\int_{\mathcal{S}} g_1(\partial_t^{\varphi^\varepsilon}+u\cdot\nabla^{\varphi^\varepsilon})\sigma(t)\cdot\sigma(t)\,\mathrm{d}\mathcal{V}_t=\frac{1}{2}\partial_t\int_{\mathcal{S}} g_1|\sigma|^2(t)\mathrm{d}\mathcal{V}_t-\frac{1}{2}\int_{\mathcal{S}}(\partial_t^{\varphi^\varepsilon}g_1+\mathrm{div}^{\varphi^\varepsilon}(g_1u))|\sigma|^2(t)\,\mathrm{d}\mathcal{V}_t, \tag{3.30}$$

$$\int_{\mathcal{S}} g_2(\partial_t^{\varphi^\varepsilon}+u\cdot\nabla^{\varphi^\varepsilon})u(t)\cdot u(t)\ \mathrm{d}\mathcal{V}_t=\frac{1}{2}\partial_t\int_{\mathcal{S}} g_2|u|^2(t)\ \mathrm{d}\mathcal{V}_t, \tag{3.31}$$

$$\begin{aligned}&\int_{\mathcal{S}}(-\mathrm{div}^{\varphi^\varepsilon}\mathcal{L}^{\varphi^\varepsilon}u+\nabla^{\varphi^\varepsilon}\sigma/\varepsilon)\cdot u(t)\ \mathrm{d}\mathcal{V}_t\\ &=\int_{\mathcal{S}}2\mu|S^{\varphi^\varepsilon}u(t)|^2+\lambda|\mathrm{div}^{\varphi^\varepsilon}u(t)|^2\ \mathrm{d}\mathcal{V}_t-\int_{\mathcal{S}}\sigma\mathrm{div}^{\varphi^\varepsilon}u(t)\ \mathrm{d}\mathcal{V}_t+a\int_{z=-1}|u_\tau|^2\ \mathrm{d}y.\end{aligned} \tag{3.32}$$

where $u_\tau=(u_1,u_2,0)^t$ *denotes the tangential components of* u, $\mathrm{d}\mathcal{V}_t=\partial_z\varphi\ \mathrm{d}y\mathrm{d}z$ *is the measure in* $\mathcal{S}$ *coming from the change of variable* (1.10).

Proof. By direct computations, one can obtain the following identities:

$$\int_{\mathcal{S}}\partial_j^{\varphi^\varepsilon}f(t)g(t)\,\mathrm{d}\mathcal{V}_t=-\int_{\mathcal{S}}f(t)\partial_j^{\varphi^\varepsilon}g(t)\,\mathrm{d}\mathcal{V}_t+\int_{\partial\mathcal{S}}f(t)g(t)\mathbf{N}_j\,\mathrm{d}y,\quad j=1,2,3$$

$$\int_{\mathcal{S}}\partial_t^{\varphi^\varepsilon}f(t)g(t)\,\mathrm{d}\mathcal{V}_t=\partial_t\int_{\mathcal{S}}fg(t)\,\mathrm{d}\mathcal{V}_t-\int_{\mathcal{S}}f(t)\partial_t^{\varphi^\varepsilon}g(t)\,\mathrm{d}\mathcal{V}_t+\int_{z=0}f(t)g(t)\partial_t h\,\mathrm{d}y,$$

which, along with the equation (1.17)-(1.19) lead to (3.30)-(3.32). Note that in the derivation of (3.31), we have used the fact that $\partial_t^{\varphi^\varepsilon}g_2+\mathrm{div}^{\varphi^\varepsilon}(g_2u)=0$ in $[0,t]\times\mathcal{S}$. □

The next lemma shows that one can control the gradient of the velocity by $S^\varphi u$.

Lemma 3.13 (Korn's inequality)**.** *Suppose that* (2.2) *is true, then there exists* $\Lambda_0(\frac{1}{c_0}),\Lambda_1(\frac{1}{c_0})$ *such that:*

$$\int_{\mathcal{S}}|\nabla u|^2(t)\,\mathrm{d}\mathcal{V}_t\leq\Lambda_0\big(\frac{1}{c_0}\big)\int_{\mathcal{S}}|\nabla^{\varphi^\varepsilon}u|^2(t)\,\mathrm{d}\mathcal{V}_t\leq\Lambda_1\big(\frac{1}{c_0}\big)\int_{\mathcal{S}}(|S^{\varphi^\varepsilon}u|^2+|u|^2)\,\mathrm{d}\mathcal{V}_t. \tag{3.33}$$

As a consequence, we have also:

$$\int_0^t\int_{\mathcal{S}}|\nabla u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\leq\Lambda_1\big(\frac{1}{c_0}\big)\int_0^t\int_{\mathcal{S}}(|S^{\varphi^\varepsilon}u|^2+|u|^2)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s. \tag{3.34}$$

These two inequalities can be shown similarly as in the proof of Proposition 2.9 in [46].

4. Preliminaries II: Reformulations of the boundary conditions

For notational convenience, from now on, we will skip the ε-dependence of the solution.

Proposition 4.1. *The following boundary condition on* $\{z=0\}$ *hold:*

$$\frac{\sigma}{\varepsilon}=(2\mu+\lambda)\mathrm{div}^\varphi u-2\mu(\partial_1u_1+\partial_2u_2)+\mu(\omega\times\mathbf{N})_3, \tag{4.1}$$

$$\omega\times\boldsymbol{n}=-2\Pi(\partial_1u\cdot\boldsymbol{n},\partial_2u\cdot\boldsymbol{n},0)^t, \tag{4.2}$$

$$\Pi(\partial_{\boldsymbol{n}}^\varphi u)=-\Pi(\partial_1u\cdot\boldsymbol{n},\partial_2u\cdot\boldsymbol{n},0)^t, \tag{4.3}$$

$$\begin{aligned}\partial_{\boldsymbol{n}}^{\varphi} u \cdot \boldsymbol{n} &= |\mathbf{N}|^2 \partial_z^{\varphi} u \cdot \boldsymbol{n} - (\boldsymbol{n}_1 \partial_1 u \cdot \boldsymbol{n} + \boldsymbol{n}_2 \partial_2 u \cdot \boldsymbol{n}) \\ &= |\mathbf{N}|(\mathrm{div}^{\varphi} u - \partial_1 u_1 - \partial_2 u_2) - (\boldsymbol{n}_1 \partial_1 u \cdot \boldsymbol{n} + \boldsymbol{n}_2 \partial_2 u \cdot \boldsymbol{n})\end{aligned} \tag{4.4}$$

where $\omega = \nabla^{\varphi} \times u, \Pi = Id_3 - \boldsymbol{n} \otimes \boldsymbol{n}$, *here* Id_3 *denotes the identity matrix of order 3.*

Proof. The first identity can be deduced from the boundary condition (1.18). Indeed, by taking the third component of (1.18), one gets that on the upper boundary $\{z = 0\}$,

$$\begin{aligned}\frac{\sigma}{\varepsilon} &= \lambda \mathrm{div}^{\varphi} u + 2\mu \partial_z^{\varphi} u \cdot \mathbf{N} + \mu\big[(\nabla^{\varphi} u - (\nabla^{\varphi} u)^t) \cdot \mathbf{N}\big]_3 \\ &= (2\mu + \lambda)\mathrm{div}^{\varphi} u - 2\mu(\partial_1 u_1 + \partial_2 u_2) + \mu(\omega \times \mathbf{N})_3.\end{aligned}$$

Note that we have used the identity

$$\partial_z^{\varphi} u \cdot \mathbf{N} = \mathrm{div}^{\varphi} u - \partial_1 u_1 - \partial_2 u_2 \tag{4.5}$$

which holds indeed in the whole domain $\mathcal{S}$. For the second identity (4.2), we have that on the upper boundary:

$$\begin{aligned}\mu\omega \times \mathbf{N} = \mu\Pi(\omega \times \mathbf{N}) &= 2\mu\Pi\big(- (\nabla^{\varphi} u)^t \mathbf{N} + S^{\varphi} u \mathbf{N}\big) \\ &= \Pi\big(- 2\mu(\nabla^{\varphi} u)^t \mathbf{N} + (\sigma/\varepsilon - \lambda \mathrm{div}^{\varphi} u)\mathbf{N}\big) \\ &= -2\mu\Pi(\partial_1 u \cdot \mathbf{N}, \partial_2 u \cdot \mathbf{N}, 0)^t.\end{aligned} \tag{4.6}$$

Note that $(\nabla^{\varphi} u)^t \cdot \mathbf{N} = (\partial_1 u \cdot \mathbf{N}, \partial_2 u \cdot \mathbf{N}, 0)^t + (\partial_z^{\varphi} u \cdot \mathbf{N})\mathbf{N}$. The inequality (4.3) can be derived in a similar way:

$$\mu\Pi(\partial_{\mathbf{n}}^{\varphi} u) = \mu\Pi(2S^{\varphi} u \mathbf{n} - (\nabla^{\varphi} u)^t \cdot \mathbf{n}) = -\mu\Pi\big((\nabla^{\varphi} u)^t \cdot \mathbf{n}\big). \tag{4.7}$$

The inequality (4.4) follows from direct computations and identity (4.5).

Remark 4.2. *By the identity:* $|\mathbf{N}|\partial_z^{\varphi} u = \partial_{\boldsymbol{n}}^{\varphi} u - \boldsymbol{n}_1 \partial_1 u - \boldsymbol{n}_2 \partial_2 u$, *we have also:*

$$|\mathbf{N}|\Pi\partial_z^{\varphi} u = \Pi(\partial_1 u \cdot \boldsymbol{n}, \partial_2 u \cdot \boldsymbol{n}, 0)^t - \Pi(\boldsymbol{n}_1 \partial_1 u + \boldsymbol{n}_2 \partial_2 u). \tag{4.8}$$

Remark 4.3. *In view of* (4.5), (4.8), *we have that* $\partial_z^{\varphi} u \approx \mathrm{div}^{\varphi} u + \partial_y u$ *on* $\{z = 0\}$, *so that:*

$$\begin{aligned}|(\nabla^{\varphi} u)^{b,1}|_{L_t^2 \tilde{H}^k} \lesssim \Lambda\Big(\frac{1}{c_0}, \|\mathrm{div}^{\varphi} u\|_{0,\infty,t} + \|u\|_{1,\infty,t} + |h|_{1,\infty,t}\Big) \\ \Big(|(\mathrm{div}^{\varphi} u)^{b,1}|_{L_t^2 \tilde{H}^k} + |u^{b,1}|_{L_t^2 \tilde{H}^{k+1}} + |h|_{L_t^2 \tilde{H}^{k+1}}\Big).\end{aligned} \tag{4.9}$$

Recall that we denote for any f, $f^{b,1} = f|_{z=0}$.

$\square$

5. Preliminaries III: Projection operators.

5.1. **Definition of the projection.** We define the projection operator $\mathbb{Q}_t$:

$$\begin{aligned}\mathbb{Q}_t : \quad L^2(\mathcal{S}, \mathrm{d}\mathcal{V}_t)^3 &\to L^2(\mathcal{S}, \mathrm{d}\mathcal{V}_t)^3 \\ f &\to \mathbb{Q}_t f = \nabla^{\varphi} \varrho\end{aligned} \tag{5.1}$$

where ϱ satisfies the elliptic equation with mixed boundary condition:

$$\begin{cases} -\Delta^{\varphi} \varrho = -\mathrm{div}^{\varphi} f \quad \text{in } \mathcal{S} \\ \varrho|_{z=0} = 0 \\ \partial_z^{\varphi} \varrho|_{z=-1} = f \cdot e_3 \end{cases} \tag{5.2}$$

where $e_3=(0,0,1)^t$. We define also the projection

$$\mathbb{P}_t = \mathrm{Id} - \mathbb{Q}_t. \tag{5.3}$$

Let us notice that $\mathbb{P}_t, \mathbb{Q}_t$ depends actually on $\varphi(t,\cdot)$, but we used a lighten notation.

Remark 5.1. *Let us notice that the definition of the projection $\mathbb{Q}_t$ is not the same as the standard Leary projection where only the Neumann boundary condition is involved. Nevertheless, the definition* (5.2) *is classical in free boundary problems, one can refer for example to* [7].

Remark 5.2. *We remark that these two projectors are time-dependent since φ depends on t. One also notes that in general, $\mathbb{P}_t\nabla^\varphi \neq 0, \mathbb{Q}_t\nabla^\varphi \neq \nabla^\varphi$. These facts will lead to some extra commutators when we act the projection to the equations* $(1.16)_2$.

Let us set $v=\mathbb{P}_t u, \nabla^\varphi\Psi = \mathbb{Q}_t u$. Applying the $\mathbb{P}_t$ projection on the velocity equation $(1.16)_2$, one gets:

$$\bar\rho\partial_t^\varphi v + \mathbb{P}_t\nabla^\varphi(\sigma/\varepsilon - 2(\mu+\lambda)\mathrm{div}^\varphi u) = -\mathbb{P}_t(f-\mu\Delta^\varphi v) - \bar\rho[\mathbb{P}_t, \partial_t^\varphi]u$$

where

$$f = \frac{g_2-\bar\rho}{\varepsilon}\varepsilon\partial_t^\varphi u + g_2 u\cdot\nabla^\varphi u.$$

By definition $\mathbb{P}_t\nabla^\varphi$ can be expressed as a gradient, we thus denote

$$\nabla^\varphi\pi = \mathbb{P}_t\nabla^\varphi(\sigma/\varepsilon - 2(\mu+\lambda)\mathrm{div}^\varphi u).$$

To shorten the notation, we denote further

$$\nabla^\varphi q = -\mathbb{Q}_t(f-\mu\Delta^\varphi v).$$

Therefore, the above equations read:

$$\bar\rho\partial_t^\varphi v - \mu\Delta^\varphi v + \nabla^\varphi\pi = -(f+\nabla^\varphi q + \bar\rho[\mathbb{P}_t, \partial_t^\varphi]u). \tag{5.4}$$

We are now in position to compute the boundary values of v. On the bottom, in light of (1.19) and the fact $\partial_z^\varphi\Psi = u_3$, we get that

$$v_3|_{z=-1}=0, \quad \partial_z^\varphi v_\tau|_{z=-1} = \partial_z^\varphi u_\tau|_{z=-1} - \nabla_\tau^\varphi\partial_z^\varphi\Psi|_{z=-1} = \frac{a}{\mu}u_\tau|_{z=-1}. \tag{5.5}$$

where $\nabla_\tau^\varphi = (\partial_1^\varphi, \partial_2^\varphi, 0)^t, f_\tau = (f_1,f_2,0)^t$. Note that $\nabla_\tau^\varphi = (\partial_1,\partial_2,0)^t$ on the boundary $\{z=-1\}$ since $\partial_\tau\varphi|_{z=-1}=0$.

On the upper boundary, one first notices that by definition, $\pi|_{z=0} = \sigma/\varepsilon - 2(\mu+\lambda)\mathrm{div}^\varphi u$. Therefore, with the aid of the condition (1.18), we find that:

$$(2\mu S^\varphi v - \pi\mathrm{Id}_3)\mathbf{N}|_{z=0} = 2\mu(\mathrm{div}^\varphi u\mathrm{Id}_3 - (\nabla^\varphi)^2\Psi)\mathbf{N}|_{z=0}. \tag{5.6}$$

5.2. **Elliptic estimates.** In this section, we establish some useful elliptic estimates in the conormal setting. We first consider the problem:

$$\begin{cases} -\Delta^\varphi\varrho = -\mathrm{div}^\varphi\tilde F \\ \varrho|_{z=0}=0 \\ \partial_z^\varphi\varrho|_{z=-1} = \tilde F\cdot e_3 + g \end{cases} \tag{5.7}$$

where $e_3=(0,0,1)^t$, $\tilde F, g$ are given source terms. To perform elliptic estimates, it would be convenient to write it in a more explicit way. By a straightforward calculation, one finds that:

$$\mathrm{div}^\varphi(\cdot) = \frac{1}{\partial_z\varphi}\mathrm{div}(P\cdot), \quad \nabla^\varphi = \frac{1}{\partial_z\varphi}P^*\nabla^\varphi, \quad \Delta^\varphi = \frac{1}{\partial_z\varphi}\mathrm{div}(E\nabla)$$

where

$$P = \begin{pmatrix} \partial_z\varphi & 0 & 0 \\ 0 & \partial_z\varphi & 0 \\ -\partial_1\varphi & -\partial_2\varphi & 1 \end{pmatrix}, \quad E = \frac{1}{\partial_z\varphi} P P^\star \tag{5.8}$$

Denote $F = P\tilde{F}$, the equation (5.7) is then equivalent to the following elliptic problem:

$$\begin{cases} -\mathrm{div}(E\nabla\varrho) = -\mathrm{div}F \\ \varrho|_{z=0} = 0 \\ (E\nabla\varrho\cdot e_3)|_{z=-1} = F_3^{b,2} + g \end{cases} \tag{5.9}$$

where $F_3^{b,2} = F^{b,2}\cdot e_3$. In this paragragh, we study the elliptic equations for a given time t.

Lemma 5.3 (Elliptic estimates)**.** *Suppose that $\|\nabla\varphi\|_{\infty,t} \leq 1/c_0, \partial_z\varphi \geq c_0$, we have the following estimates: for any $k \geq 0$,*

$$\|\nabla\varrho(t)\|_{H_{co}^{k+1}} + \|\nabla^2\varrho(t)\|_{H_{co}^k} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k+2,\infty,t}\big)\big(\|\mathrm{div}F(t)\|_{H_{co}^k} + |(F_3^{b,2}+g)(t)|_{\tilde{H}^{k+\frac12}}\big), \tag{5.10}$$

and for $j + l = k, l \geq 1, j \geq 0$,

$$\begin{aligned} \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac12}} \\ + \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|F(t)\|_{\mathcal{H}^{j,l}} + |g(t)|_{\tilde{H}^{k-\frac12}}\big), \end{aligned} \tag{5.11}$$

$$\begin{aligned} \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac12}} \\ + \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|\mathrm{div}F(t)\|_{\mathcal{H}^{j,l-1}} + |(F_3^{b,2}, g)(t)|_{\tilde{H}^{k-\frac12}}\big), \end{aligned} \tag{5.12}$$

$$\begin{aligned} \|\nabla^2\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+5}{2}],\infty,t}\big)\big(\|\mathrm{div}F(t)\|_{\mathcal{H}^{j,l}} + |(F_3^{b,2}, g)(t)|_{\tilde{H}^{k+\frac12}}\big) \\ + \Lambda\big(\frac{1}{c_0}\|\nabla\varrho\|_{[\frac{k-1}{2}],\infty,t} + |h|_{[\frac{k+5}{2}],\infty,t}\big)\big(\|\nabla\varrho\|_{0,\infty,t}|h(t)|_{\tilde{H}^{k+\frac32}} + |h(t)|_{\tilde{H}^{k+\frac12}}\big), \end{aligned} \tag{5.13}$$

$$\begin{aligned} \varepsilon^{\frac12}\|\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k+1,\infty,t}\big)\big(\|\varepsilon^{\frac12}\partial_t F(t)\|_{\mathcal{H}^{j,l}} + \varepsilon^{\frac12}|\partial_t g(t)|_{\tilde{H}^{k-\frac12}}\big) \\ + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{1,\infty,t} + |\partial_t h|_{k-1,\infty,t} + |h|_{k,\infty,t}\big)(|\partial_t h(t)|_{\tilde{H}^{k+\frac12}} + \|\nabla\varrho(t)\|_{H_{co}^k}), \end{aligned} \tag{5.14}$$

$$\begin{aligned} \|\varepsilon^{\frac12}\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|\varepsilon^{\frac12}\partial_t\mathrm{div}F(t)\|_{\mathcal{H}^{j,l-1}} + |\varepsilon^{\frac12}\partial_t(F_3^{b,2}, g)(t)|_{\tilde{H}^{k-\frac12}}\big) \\ + \Lambda\big(\frac{1}{c_0}, \|\varepsilon^{-\frac12}\nabla\varrho\|_{[\frac{k}{2}],\infty,t} + |(h, \partial_t h,)|_{[\frac{k+3}{2}],\infty,t}\big)(|(\varepsilon\partial_t h, h)(t)|_{\tilde{H}^{k+\frac12}} + \varepsilon^{\frac12}\|\nabla\varrho(t)\|_{H_{co}^k}). \end{aligned} \tag{5.15}$$

Remark 5.4. *We shall use* (5.14) *when $k \leq m-3$ since as will be seen later, $|h|_{m-2,\infty,t}$ can be uniformly controlled. The inequality* (5.15) *will be used when $m-3 \leq k \leq m-1$.*

Proof. We first notice that by using assumptions: $\|\nabla\varphi\|_{\infty,t} \leq 1/c_0, \partial_z\varphi \geq c_0$, E is uniformly elliptic, that is, one can find $\iota(1/c_0)$ such that for any vectors $X \in \mathbb{R}^3$, $EX\cdot X \geq \iota|X|^2$. The inequality (5.10) can be proved easily by the variational arguments and the use of Poincaré inequality:

$$\|\varrho(t)\|_{L^2(\mathcal{S})} \leq C\|\nabla\varrho(t)\|_{L^2(\mathcal{S})}.$$

Note that the generic constant C is independent of t and ε. More precisely, by testing (5.9) by $\varrho(t)$, we easily get that:

$$\delta\|\nabla\varrho(t)\|_{L^2(\mathcal{S})} \leq \int_{\mathcal{S}} E\nabla\varrho(t)\cdot\nabla\varrho(t)\,\mathrm{d}x = -\int_{\mathcal{S}}\varrho(t)\mathrm{div}F(t)\,\mathrm{d}x + \int_{z=-1}(F_3^{b,2}+g)(t)\varrho(t)\,\mathrm{d}y$$
$$\leq \frac{\delta}{2}\|\nabla\varrho(t)\|_{L^2(\mathcal{S})} + C_\delta(\|\mathrm{div}F(t)\|_{L^2(\mathcal{S})} + |(F_3^{b,2},g)(t)|_{H^{-\frac{1}{2}}}).$$

The estimates of the higher-order norms $\|\nabla\varrho(t)\|_{H^{k+1}}$ can be obtained again from variational arguments and commutator estimates. We skip them since they are essentially included in the proof of other inequalities (for instance (5.11) and (5.13)).

We now begin to prove (5.11). Let $\alpha = (j,\alpha')$, $Z^\alpha = (\varepsilon\partial_t)^j Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}$. If $\alpha_3 \neq 0$, taking Z^α derivatives on the equation shall destroy the divergence form. The trick to avoid this problem is to use another vector field $\tilde{Z}_3 = Z_3 + \partial_z\phi\mathrm{Id}$, such that: $\tilde{Z}_3\partial_z = \partial_z Z_3$. By induction, we have for any $\alpha_3 \geq 1$, $\tilde{Z}_3^{\alpha_3}\partial_z = \partial_z Z_3^{\alpha_3}$, which yields

$$\tilde{Z}^\alpha\partial_z =: (\varepsilon\partial_t)^j Z_1^{\alpha_1}Z_2^{\alpha_2}\tilde{Z}_3{}^{\alpha_3}\partial_z = \partial_z Z^\alpha.$$

It is useful to notice further that for any f,

$$\|(\tilde{Z}^\alpha - Z^\alpha)f(t)\|_{L^2(\mathcal{S})} \lesssim \|f(t)\|_{\mathcal{H}^{j,l-1}}. \tag{5.16}$$

Taking $\tilde{Z}^\alpha$ derivative on the equation (5.9), we find that:

$$\begin{cases} -\mathrm{div}\big(E(Z^\alpha\nabla\varrho)\big) = \mathrm{div}([Z^\alpha,E]\nabla\varrho - Z^\alpha F) + \mathrm{div}\big(\tilde{Z}^\alpha - Z^\alpha)[(E\nabla\varrho)_\tau - F_\tau]\big), \\ Z^\alpha\varrho|_{z=0} = 0, \\ Z^\alpha(E\nabla\varrho)\cdot e_3|_{z=-1} = \mathbb{I}_{\{\alpha_3=0\}}Z^\alpha(F_3^{b,2}+g). \end{cases} \tag{5.17}$$

Note that we denote by $X_\tau = (X_1,X_2,0)^t$ the horizontal components of a three dimensional vector X. Testing equation (5.17) by $Z^\alpha\varrho$, we obtain:

$$\begin{aligned} &\delta\|Z^\alpha\nabla\varrho\|_{L^2}^2 \leq \int_{\mathcal{S}} EZ^\alpha\nabla\varrho Z^\alpha\nabla\varrho\,\mathrm{d}x \\ &= \int_{\mathcal{S}} EZ^\alpha\nabla\varrho\cdot[Z^\alpha,\nabla]\varrho\,\mathrm{d}x - \int_{\mathcal{S}}[Z^\alpha,E]\nabla\varrho\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x \\ &- \int_{\mathcal{S}}(\tilde{Z}^\alpha - Z^\alpha)\big((E\nabla\varrho)_\tau - F_\tau\big)\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x + \int_{\mathcal{S}} Z^\alpha F\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x - \int_{z=-1}\mathbb{I}_{\{\alpha_3=0\}}Z^\alpha g Z^\alpha\varrho\,\mathrm{d}y. \end{aligned} \tag{5.18}$$

Combined with Young's inequality, property (5.16) and the trace inequality (3.17), this yields

$$\|Z^\alpha\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 \lesssim \|F(t)\|_{\mathcal{H}^{j,l}}^2 + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}}^2 + \|(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[Z^\alpha,E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2. \tag{5.19}$$

It follows from the product and commutator estimates (3.8), (3.9) that:

$$\begin{aligned} \|\nabla\varrho(t)\|_{\tilde{H}^{j,l}} \leq \Lambda(1/c_0)\big(&\|F(t)\|_{\mathcal{H}^{j,l}} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}} + \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}} \\ &+ \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\|E\|_{[\frac{k+1}{2}],\infty,t} + \|E(t)\|_{\tilde{H}^{j,l}}\|\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t}\big). \end{aligned} \tag{5.20}$$

By Lemma 3.8 and the expression of E in (5.8), we get

$$\|E\|_{n,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{n+1,\infty,t}\big), \quad \|E(t)\|_{\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}}. \tag{5.21}$$

Inserting (5.21) into (5.20), we arrive at:

$$
\begin{aligned}
\|\nabla\varrho(t)\|_{\tilde{H}^{j,l}} \le\ & \Lambda\big(\frac{1}{c_0}, \|\!|\nabla\varrho\|\!|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}} \\
&+ \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|F(t)\|_{\mathcal{H}^{j,l}} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}} + \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\big).
\end{aligned} \tag{5.22}
$$

The inequality (5.11) then follows by induction on j and l.

To get (5.12), it suffices to observe that the last three terms in (5.18) can indeed be replaced by:

$$
\int_{\mathcal{S}} Z^{\tilde{\alpha}}\mathrm{div}F\partial_y Z^\alpha\varrho\,\mathrm{d}x - \int_{z=-1} Z^\alpha(F_3+g)Z^\alpha\varrho\,\mathrm{d}y, \quad \text{if} \quad \alpha_3 = 0, Z^\alpha = \partial_y Z^{\tilde{\alpha}}.
$$

$$
-\int_{\mathcal{S}}(\tilde{Z}^\alpha - Z^\alpha)(E\nabla\varrho)_h\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x \int_{\mathcal{S}} Z^{\tilde{\alpha}}\mathrm{div}F(Z_3+\partial_z\phi)(Z^\alpha\varrho)\,\mathrm{d}x, \quad \text{if} \quad \alpha_3 \neq 0, Z^\alpha = Z_3 Z^{\tilde{\alpha}}.
$$

To prove (5.13), we first estimate $\|\partial_y\nabla\varrho(t)\|_{\mathcal{H}^{j,l}}$ and then use the equation itself to recover $\|\partial_z^2\varrho(t)\|_{\mathcal{H}^{j,l}}$. The estimate of $\|\partial_y\nabla\varrho(t)\|_{\mathcal{H}^{j,l}}$ is almost identical to that of (5.12). For this one, we only need to distinguish the highest derivatives hitting on E (or finally on h). Hence, when estimating the term $[Z^\alpha\partial_y, E]\nabla\varrho$, we write

$$
[Z^\alpha\partial_y, E]\nabla\varrho = (Z^\alpha\partial_y E)\nabla\varrho + \text{ other terms}
$$

and control the first term as

$$
\|(Z^\alpha\partial_y E)\nabla\varrho(t)\|_{L^2(\mathcal{S})} \lesssim \|\!|\nabla\varrho\|\!|_{0,\infty,t}\Lambda(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+2,\infty,t})|h(t)|_{\tilde{H}^{k+\frac{3}{2}}}.
$$

We now sketch the proof of (5.14) and (5.15). For (5.14), we first have the following inequality analogues to (5.19).

$$
\begin{aligned}
\|\varepsilon^{\frac{1}{2}}Z^\alpha\partial_t\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 \lesssim\ & \|\varepsilon^{\frac{1}{2}}\partial_t F(t)\|_{\mathcal{H}^{j,l}}^2 + |\varepsilon^{\frac{1}{2}}\partial_t g(t)|_{\tilde{H}^{k-\frac{1}{2}}}^2 \\
&+ \|\varepsilon^{\frac{1}{2}}\partial_t(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[\varepsilon^{\frac{1}{2}}\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2,
\end{aligned}
$$

where the last two terms can be bounded in a rather rough way:

$$
\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\partial_t(E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l}} \lesssim\ & \|\varepsilon^{\frac{1}{2}}\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}}\Lambda(\frac{1}{c_0}, |h|_{k,\infty,t}) \\
&+ \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \|\!|\nabla\varrho\|\!|_{0,\infty,t} + |\partial_t h|_{k-1,\infty,t}\big)(|\partial_t h(t)|_{\tilde{H}^{k-\frac{1}{2}}} + \|\nabla\varrho(t)\|_{H_{co}^{k-1}}),
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^{\frac{1}{2}}\|[\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})} &\le \varepsilon^{\frac{1}{2}}\|Z^\alpha(\partial_t E\nabla\varrho)(t)\|_{L^2(\mathcal{S})} + \varepsilon^{\frac{1}{2}}\|[Z^\alpha, E]\partial_t\nabla\varrho(t)\|_{L^2(\mathcal{S})} \\
&\lesssim \|\!|\varepsilon^{\frac{1}{2}}\partial_t\nabla\varrho\|\!|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\Lambda\big(\frac{1}{c_0}, |h|_{[k,\infty,t}\big) \\
&\quad + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \|\!|\nabla\varrho\|\!|_{1,\infty,t} + |\partial_t h|_{k-1,\infty,t}\big)(|\partial_t h(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \|\nabla\varrho(t)\|_{H_{co}^k}),
\end{aligned}
$$

The inequality (5.14) then follows from induction on j, l. For (5.15), similar to (5.12), we have:

$$
\begin{aligned}
\|\varepsilon^{\frac{1}{2}}Z^\alpha\partial_t\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 \lesssim\ & \|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}F(t)\|_{\mathcal{H}^{j,l}}^2 + |\varepsilon^{\frac{1}{2}}(F_3^{b,2}, \partial_t g)(t)|_{\tilde{H}^{k-\frac{1}{2}}}^2 \\
&+ \|\varepsilon^{\frac{1}{2}}\partial_t(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[\varepsilon^{\frac{1}{2}}\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2.
\end{aligned}
$$

The last two terms are bounded as

$$\begin{aligned}
&\|\varepsilon^{\frac12}\partial_t(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[\varepsilon^{\frac12}\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 \\
&\quad\lesssim \varepsilon^{\frac12}\|\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big) \\
&\quad + \Lambda\big(\frac{1}{c_0}, \|\varepsilon^{-\frac12}\nabla\varrho\|_{[\frac{k}{2}],\infty,t} + |(\partial_t h, h)|_{[\frac{k+3}{2}],\infty,t}\big)\big(|(\varepsilon\partial_t h, h)(t)|_{\tilde{H}^{k+\frac12}} + \varepsilon^{\frac12}\|\nabla\varrho(t)\|_{H^k_{co}}\big).
\end{aligned}$$

We obtain (5.15) again by induction on j and l.

□

Remark 5.5. *Similar to* (5.11), (5.15) *the following estimate also hold, for* $j+l=k\geq 3$,

$$\begin{aligned}
\|\nabla\varrho(t)\|_{H^k_{co}} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k-n,\infty,t}\big)\big(\|F(t)\|_{H^k_{co}} + |g(t)|_{\tilde{H}^{k-\frac12}}\big) \\
&\quad + \|\!|\nabla\varrho\|\!|_{n,\infty,t}\Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac12}}\ (n=0,1),
\end{aligned}\tag{5.23}$$

$$\begin{aligned}
&\varepsilon^{\frac12}\|\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} + \varepsilon^{\frac12}\|\partial_t\nabla^2\varrho(t)\|_{\mathcal{H}^{j,l-1}} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k,\infty,t}\big)\big(\|\varepsilon^{\frac12}\partial_t F(t)\|_{\mathcal{H}^j} + \|\varepsilon^{\frac12}\partial_t \mathrm{div} F(t)\|_{\mathcal{H}^{j,l-1}}\mathbb{I}_{\{l\geq1\}} + |\varepsilon^{\frac12}\partial_t(F_3^{b,2}, g)(t)|_{\tilde{H}^{k-\frac12}}\big) \\
&+ \Lambda\big(\frac{1}{c_0}, |h|_{k,\infty,t}\big)\|\!|\varepsilon^{\frac12}\partial_t\nabla\varrho\|\!|_{0,\infty,t}|h(t)|_{\tilde{H}^{k+\frac12}} \\
&+ \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \|\!|\varepsilon^{-\frac12}\nabla\varrho\|\!|_{[\frac{k+1}{2}],\infty,t} + |(h,\partial_t h)|_{[\frac{k+3}{2}],\infty,t}\big)\big(|\partial_t h(t)|_{\tilde{H}^{k+\frac12}} + \|\nabla\varrho(t)\|_{H^k_{co}}\big).
\end{aligned}\tag{5.24}$$

Corollary 5.6. *Let* $\nabla^\varphi\Psi = \mathbb{Q}_t u$ *be the compressible part of the velocity, we have the following two estimates:*

$$\|\nabla\nabla^\varphi\Psi\|_{L^2_t H^{m-1}_{co}} + \|\nabla^\varphi\Psi\|_{L^2_t H^m_{co}} \lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),\tag{5.25}$$

$$\|\varepsilon^{\frac12}\partial_t\nabla^\varphi\Psi\|_{L^2_t H^{m-1}_{co}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{L^\infty_t\tilde{H}^{m-\frac12}}\big)\|\varepsilon^{\frac12}\partial_t\mathrm{div}^\varphi u\|_{L^2_t H^{m-2}_{co}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),\tag{5.26}$$

$$\begin{aligned}
&\varepsilon^{\frac12}\|\partial_t\nabla^\varphi\Psi\|_{L^\infty_t H^{m-2}_{co}} + \varepsilon^{\frac12}\|\partial_t\nabla\nabla^\varphi\Psi\|_{L^\infty_t H^{m-3}_{co}} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{L^\infty_t\tilde{H}^{m-\frac12}}\big)\big(\|\varepsilon^{\frac12}\partial_t\mathrm{div}^\varphi u\|_{L^\infty_t H^{m-3}_{co}} + \|\varepsilon^{\frac12}\partial_t u\|_{L^\infty_t\mathcal{H}^{m-2}}\big) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}\tag{5.27}$$

Proof. We begin with the proof of (5.25). Let us detail the estimate of $\|\nabla\nabla^\varphi\Psi\|_{L^2_t H^{m-1}_{co}}$, the other term can be obtained by similar arguments. It suffices to show that:

$$\begin{aligned}
\|\nabla\nabla^\varphi\Psi\|_{L^2_t H^{m-1}_{co}} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\mathrm{div}^\varphi u\|_{L^2_t H^{m-1}_{co}} \\
&\quad + \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big)(|h|_{L^2_t\tilde{H}^{m-\frac12}} + |\varepsilon^{\frac12}h|_{L^2_t\tilde{H}^{m+\frac12}}).
\end{aligned}\tag{5.28}$$

which leads to (5.25). By definition, Ψ solves the elliptic equation:

$$\begin{cases}
\mathrm{div}(E\nabla\Psi) = \mathrm{div}(Pu), \\
\Psi|_{z=0} = 0, \\
\partial_{\mathbf{n}}\Psi|_{z=-1} = 0.
\end{cases}\tag{5.29}$$

We apply (5.13) for $F = Pu$, $\mathrm{div}F = \partial_z\varphi\,\mathrm{div}^\varphi u$, $F_3^{b,2} = g = 0$ to get:

$$\|\nabla^2\Psi\|_{L_t^2H_{co}^{m-1}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\partial_z\varphi\,\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}}$$
$$+ \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t} + \|\varepsilon^{-\frac{1}{2}}\nabla\Psi\|_{[\frac{m}{2}]-1,\infty,t}\big)(|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + |\varepsilon^{\frac{1}{2}}h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}).$$

By the product estimate (3.8), we find

$$\begin{aligned}\|\nabla\nabla^\varphi\Psi\|_{L_t^2H_{co}^{m-1}} &\lesssim \|\nabla^2\Psi\|_{L_t^2H_{co}^{m-1}} + \|\nabla\big(\frac{\mathbf{N}}{\partial_z\varphi}\partial_z\Psi\big)\|_{L_t^2H_{co}^{m-1}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}} + \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t} + \|\varepsilon^{-\frac{1}{2}}(\nabla\Psi, \mathrm{div}^\varphi u)\|_{[\frac{m}{2}]-1,\infty,t}\big)\cdot \\ &\qquad (|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}).\end{aligned} \tag{5.30}$$

Moreover, the Sobolev embedding (3.16) combined with the inequality (5.10) gives for $k \geq 0$,

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla\Psi\|_{[\frac{m}{2}]-1,\infty,t} &\lesssim \varepsilon^{-\frac{1}{2}}(\|\nabla^2\Psi\|_{L_t^\infty H_{co}^{[\frac{m}{2}]}} + \|\nabla\Psi\|_{L_t^\infty H_{co}^{[\frac{m}{2}]+1}}) \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\varepsilon^{-\frac{1}{2}}\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^{[\frac{m}{2}]}}.\end{aligned} \tag{5.31}$$

Plugging this inequality into (5.30), we arrive at (5.28).

Moreover, by applying (5.15), (5.24), (5.31) to the solution of (5.29), we get (5.26) and (5.27). □

Corollary 5.7. *Consider the elliptic system with nontrivial Dirichlet upper boundary condition:*

$$\begin{cases} -\mathrm{div}(E\nabla\varrho) = -\mathrm{div}F, \\ \varrho|_{z=0} = b, \\ (E\nabla\varrho)\cdot e_3|_{z=-1} = F_3^{b,2} + g. \end{cases} \tag{5.32}$$

The following estimates hold:

$$\|\nabla\varrho\|_{\infty,t} \lesssim \Lambda(\frac{1}{c_0}, |h|_{3,\infty,t})\big(\|\mathrm{div}F\|_{L_t^\infty H_{co}^1} + |b|_{L_t^\infty H^{\frac{5}{2}}} + |g|_{L_t^\infty H^{\frac{3}{2}}}\big), \tag{5.33}$$

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} &\lesssim \Lambda\big(\frac{1}{c_0}, \|\varepsilon^{-\frac{1}{2}}\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t} + \varepsilon^{-\frac{1}{2}}|b|_{L_t^\infty\tilde{H}^{[\frac{k}{2}]+1+}}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}} \\ &\quad + \varepsilon^{-\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|F(t)\|_{\mathcal{H}^{j,l}} + |b(t)|_{\tilde{H}^{k+\frac{1}{2}}} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big),\end{aligned} \tag{5.34}$$

$$\begin{aligned}\|\nabla\varrho(t)\|_{H_{co}^k} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k-j,\infty,t}\big)\big(\|F(t)\|_{H_{co}^k} + |b(t)|_{H^{k+\frac{1}{2}}} + |g(t)|_{L_t^\infty H^{k-\frac{1}{2}}}\big) \\ &\quad + \Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{j,\infty,t} + |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}},\ k \geq 2, j = 0 \text{ or } 1,\end{aligned} \tag{5.35}$$

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\partial_t\nabla\varrho(t)\|_{H_{co}^k} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k+1,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_tF(t)\|_{H_{co}^k} + |\varepsilon^{\frac{1}{2}}\partial_tb(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_tg(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big) \\ &\quad + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{1,\infty,t} + |\partial_th|_{k-1,\infty,t} + |h|_{k,\infty,t}\big)(|\partial_th(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \|\nabla\varrho(t)\|_{H_{co}^k}),\end{aligned} \tag{5.36}$$

Proof. We introduce the lifting:

$$\varrho^H(t,y,z) = \mathcal{F}_{\xi\to y}^{-1}(e^{-z^2\langle\xi\rangle^2}\hat{b}(t,\xi))(1+z),$$

and reformulate the problem as:

$$\begin{cases} -\mathrm{div}(E\nabla\varrho^L) = -\mathrm{div}(F - E\nabla\varrho^H) \\ \varrho^L|_{z=0} = 0 \\ \partial_z\varrho^L|_{z=-1} = (F - E\nabla\varrho^H)\cdot e_3 + g. \end{cases}$$

We apply Lemma 5.3 with $F - E\nabla\varrho^H$. Note that we use again the product estimate (3.8) to bound $E\nabla\varrho^H$. Moreover, Young's inequality and the definition of ϱ^H give:

$$\|\nabla\varrho^H(t)\|_{\mathcal{H}^{j,l}} \lesssim |b(t)|_{\tilde{H}^{j+l+\frac{1}{2}}}, \quad \|\nabla\varrho^H\|_{[\frac{k}{2}]-1,\infty,t} \lesssim |b|_{[\frac{k}{2}],\infty,t} \lesssim |b|_{L_t^\infty\tilde{H}^{[\frac{k}{2}]+1+}}.$$

□

6. Regularity of the surface

In this section, we prove some regularity properties for the surface h. Here and in the sequel, we will denote $m \geq 7$ an integer. We also recall that $\mathcal{N}_{m,T}, \mathcal{E}_{m,T}, \mathcal{A}_{m,T}$ are defined in (1.31).

Lemma 6.1. *The following regularity estimates hold: $0 < t \leq T$,*

$$|\partial_t h|_{L_t^\infty\tilde{H}^{m-\frac{3}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_t h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} \lesssim \mathcal{E}_{m,T} + \mathcal{E}_{m,T}^2, \tag{6.1}$$

$$\varepsilon^{\frac{1}{2}}|\partial_t^2 h|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} + |\varepsilon^{\frac{1}{2}}\partial_t^2 h|_{L_t^\infty\tilde{H}^{m-\frac{5}{2}}} + \sum_{k\leq m-1}|\varepsilon^{\frac{1}{2}}(\varepsilon\partial_t)^k\partial_t^2 h|_{L_t^2 H^{-\frac{1}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \tag{6.2}$$

$$|h|^2_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} + \varepsilon|h|^2_{L_t^\infty\tilde{H}^{m+\frac{1}{2}}} \lesssim Y_m^2(0) + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{6.3}$$

where Λ denotes a polynomial that may change according to the contexts.

Proof. Proof of (6.1): We have by using the equation (1.17), the product estimate (3.5) the trace inequality (3.17) and the definition of $\mathcal{E}_{m,T}$ that:

$$\begin{aligned} \varepsilon^{\frac{1}{2}}|\partial_t h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} &= \varepsilon|(u\cdot\mathbf{N})|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} \\ &\lesssim \big(1 + |u|_{L_t^\infty\tilde{H}^{[\frac{m-1}{2}]+\frac{1}{2}}} + |h|_{L_t^\infty\tilde{H}^{[\frac{m}{2}]+\frac{3}{2}}}\big)|\varepsilon^{\frac{1}{2}}(u, \nabla_y h)|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} \\ &\lesssim (1 + \mathcal{E}_{m,T})(\|\varepsilon^{\frac{1}{2}}(u, \nabla u)\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}|h|_{L_t^\infty\tilde{H}^{m+\frac{1}{2}}}) \lesssim \mathcal{E}_{m,T} + \mathcal{E}_{m,T}^2. \end{aligned}$$

Note that we have $[\frac{m-1}{2}] + 1 \leq m-2, [\frac{m}{2}] + \frac{3}{2} \leq m - \frac{1}{2}$ for $m \geq 5$. The quantity $|\partial_t h|_{L_t^\infty\tilde{H}^{m-\frac{3}{2}}}$ can be dealt with in the same way, we thus omit the proof.

Proof of (6.2): Let us detail the estimates of the first two terms, the last one can be controlled by similar calculations. Again, we use the equation (1.17) for h, the product estimate (3.5), the trace inequality (3.17) to obtain that

$$\begin{aligned} \varepsilon^{\frac{1}{2}}|\partial_t h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} &\lesssim |(\varepsilon^{\frac{1}{2}}\partial_t u\cdot\mathbf{N}, u\cdot\varepsilon^{\frac{1}{2}}\partial_t\mathbf{N})|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} \\ &\lesssim |\varepsilon^{\frac{1}{2}}\partial_t u|_{L_t^2\tilde{H}^{[\frac{m-1}{2}]+\frac{1}{2}}}|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} + (1 + |h|_{L_t^\infty\tilde{H}^{[\frac{m}{2}]+\frac{3}{2}}})|\varepsilon^{\frac{1}{2}}\partial_t u|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} \\ &\quad + |\varepsilon^{\frac{1}{2}}\partial_t h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}|u|_{L_t^2\tilde{H}^{[\frac{m}{2}]+\frac{3}{2}}} + |\varepsilon^{\frac{1}{2}}\partial_t h|_{L_t^\infty\tilde{H}^{[\frac{m}{2}]+\frac{1}{2}}}|u|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \end{aligned}$$

For the second term, we use Equation (1.17) and the trace inequality to get:

$$|\varepsilon^{\frac{1}{2}}\partial_t^2 h|_{L_t^\infty\tilde{H}^{m-\frac{5}{2}}} \lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\partial_z(u\cdot\mathbf{N})\|_{L_t^\infty H_{co}^{m-3}} + \|\varepsilon^{\frac{1}{2}}\partial_t(u\cdot\mathbf{N})\|_{L_t^\infty H_{co}^{m-2}}.$$

With the aid of identity (4.4) and the product estimate (3.8), we then find that:

$$
\begin{aligned}
&|\varepsilon^{\frac{1}{2}}\partial_t^2 h|_{L_t^\infty \tilde{H}^{m-\frac{5}{2}}}\\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t \mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^{m-3}} + \|(u, \varepsilon^{\frac{1}{2}}\partial_t u)\|_{L_t^\infty H_{co}^{m-2}} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^\infty \tilde{H}^{m-\frac{3}{2}}}\big)\\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

Proof of (6.3). We explain the estimate of $|h|_{L_t^\infty H^{m-\frac{1}{2}}}$, the control of $\varepsilon^{\frac{1}{2}}|h|_{L_t^\infty H^{m+\frac{1}{2}}}$ being similar. Acting $Z^\alpha \Lambda_y^{\frac{1}{2}} (|\alpha| \leq m-1, \alpha_3 = 0)$ on (1.17), one obtains:

$$
(\partial_t + u_y \partial_y)(Z^\alpha \Lambda_y^{\frac{1}{2}} h) - Z^\alpha \Lambda_y^{\frac{1}{2}} u_3 = f =: [\Lambda_y^{\frac{1}{2}}, u_y] Z^\alpha \partial_y h - \Lambda_y^{\frac{1}{2}}\big([Z^\alpha, u_y]\partial_y h\big).
$$

Multiplying this equation by $Z^\alpha \Lambda_y^{\frac{1}{2}} h$ and integrating in space and time, we get that:

$$
\begin{aligned}
|Z^\alpha \Lambda_y^{\frac{1}{2}} h(t)|_{L_y^2}^2 &\lesssim |Z^\alpha h(0)|_{H^{\frac{1}{2}}}^2\\
&+ T^{\frac{1}{2}} \Lambda(\|u\|_{1,\infty,t})\big(|u_3|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}^2 + |f|_{L_t^2 L_y^2}^2 + |h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}^2\big)
\end{aligned}
\tag{6.4}
$$

By the trace inequality (3.17),

$$
\big|u_3\big|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}^2 \lesssim \|(u, \nabla u)\|_{L_t^2 H_{co}^{m-1}}^2. \tag{6.5}
$$

To estimate the first term in f, we apply the commutator estimate (3.3) to get that:

$$
\begin{aligned}
|[\Lambda_y^{\frac{1}{2}}, u_y] Z^\alpha \partial_y h|_{L_t^2 L_y^2} &\lesssim |Z^\alpha \partial_y h|_{L_t^2 H^{-\frac{1}{2}}} |u_y|_{L_t^\infty H^{2.5}}\\
&\lesssim |h|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}} \|(u, \nabla u)\|_{L_t^\infty H^2(\mathcal{S})} \lesssim T^{\frac{1}{2}} \mathcal{E}_{m,T}^2.
\end{aligned}
\tag{6.6}
$$

For the second term in f, we have by the commutator estimate (3.6) and the trace inequality (3.17) that:

$$
|[Z^\alpha, u_y]\partial_y h|_{L_t^2 H^{\frac{1}{2}}} \lesssim |u|_{L_t^2 \tilde{H}^{[\frac{m}{2}]+\frac{1}{2}}} |h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + |h|_{L_t^\infty \tilde{H}^{[\frac{m}{2}]+\frac{5}{2}}} |u|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}} \lesssim \mathcal{E}_{m,T}^2. \tag{6.7}
$$

Inserting (6.5)-(6.7) into (6.4), we achieve (6.3). □

7. High order energy estimates

In this section, we prove two kinds of energy estimates, namely the $\varepsilon-$dependent high order conormal energy estimates involving at least one spatial derivative, and the higher order estimates when only the time derivatives are involved. These quantities we are going to bound appears in the definition of energy norms $\mathcal{E}_{high,m,T}$ in (1.32) and are necessary to prove the uniform estimates shown in Sections 10-12.

7.1. **Energy estimate I: Highest order energy estimates.**

Lemma 7.1. *Suppose that* (2.2) *holds for some* $T > 0$ *then for any* $0 < t \leq T$, *then we have the following energy estimates:*

$$
\varepsilon\|(\sigma, u)\|_{L_t^\infty H_{co}^m}^2 + \varepsilon\|\nabla u\|_{L_t^2 H_{co}^m}^2 \lesssim \varepsilon\|(\sigma, u)(0)\|_{H_{co}^m}^2 + (T+\varepsilon)^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.1}
$$

Proof. Let us start with (7.1) for $m=0$ which is standard. Performing direct energy estimates for (1.16) we get by identities (3.30)-(3.32) that:

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{S}}(g_1|\sigma|^2+g_2|u|^2)(t)\,\mathrm{d}\mathcal{V}_t+\int_0^t\int_{\mathcal{S}}2\mu|S^{\varphi}u|^2+\lambda|\mathrm{div}^{\varphi}u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\
&=\frac{1}{2}\int_{\mathcal{S}}(g_1|\sigma|^2+g_2|u|^2)(0)\,\mathrm{d}\mathcal{V}_0+\frac{1}{2}\int_{\mathcal{S}}(\partial_t^{\varphi}g_1+\mathrm{div}^{\varphi}(g_1u))|\sigma|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s-a\int_0^t\int_{z=-1}|u_\tau|^2\,\mathrm{d}y\mathrm{d}s
\end{aligned}
\tag{7.2}
$$

where $u_\tau=(u_1,u_2,0)^t$. Thanks to (2.1) and assumption (2.2), we have:

$$
\begin{aligned}
\|\partial_t^{\varphi}g_1+\mathrm{div}^{\varphi}(g_1u)\|_{0,\infty,t}&\leq\Lambda\big(\frac{1}{c_0},\|(\sigma,u)\|_{1,\infty,t}+\|\nabla(\sigma,u)\|_{0,\infty,t}+|h|_{1,\infty,t}\big)\\
&\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).
\end{aligned}
$$

In view of the Korn inequality (3.34), the trace inequality (3.17), one gets by using Young's inequality that:

$$
\begin{aligned}
\|(\sigma,u)\|_{L_t^\infty L^2}^2+\|\nabla u\|_{L_t^2L^2}^2&\lesssim\|(\sigma_0,u_0)\|_{L^2(\mathcal{S})}^2+\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\|(\sigma,u)\|_{L_t^2L^2}^2\\
&\lesssim\|(\sigma_0,u_0)\|_{L^2(\mathcal{S})}^2+T\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\|(\sigma,u)\|_{L_t^\infty L^2}^2.
\end{aligned}
\tag{7.3}
$$

We now detail the high order estimates in (7.1). Let α be a multi-index with $1\leq|\alpha|\leq m$, applying Z^α on the equation (1.16), and denoting $(\sigma^\alpha,u^\alpha)=Z^\alpha(\sigma,u)$, one obtains the system:

$$
\begin{cases}
g_1(\partial_t^{\varphi}+u\cdot\nabla^{\varphi})\sigma^\alpha+\frac{\mathrm{div}^{\varphi}u^\alpha}{\varepsilon}=\mathcal{C}_\sigma^\alpha-\frac{1}{\varepsilon}[Z^\alpha,\mathrm{div}^{\varphi}]u,\\
g_2(\partial_t^{\varphi}+u\cdot\nabla^{\varphi})u^\alpha-\mathrm{div}^{\varphi}Z^\alpha\mathcal{L}^{\varphi}u+\frac{\nabla^{\varphi}\sigma}{\varepsilon}=\mathcal{C}_u^\alpha-\frac{1}{\varepsilon}[Z^\alpha,\nabla^{\varphi}]\sigma+[Z^\alpha,\mathrm{div}^{\varphi}]\mathcal{L}^{\varphi}u.
\end{cases}
\tag{7.4}
$$

where the commutators are given by:

$$
\begin{aligned}
\mathcal{C}_\sigma^\alpha&=\big[Z^\alpha,\frac{g_1}{\varepsilon}\big]\varepsilon\partial_t\sigma+[Z^\alpha,g_1u_y]\nabla_y\sigma+[Z^\alpha,g_1U_z\partial_z]\sigma,\\
\mathcal{C}_u^\alpha&=\big[Z^\alpha,\frac{g_2}{\varepsilon}\big]\varepsilon\partial_tu+[Z^\alpha,g_2u_y]\nabla_yu+[Z^\alpha,g_2U_z\partial_z]u,
\end{aligned}
\tag{7.5}
$$

with

$$
U_z=\frac{u\cdot\mathbf{N}-\partial_t\varphi}{\partial_z\varphi}
\tag{7.6}
$$

Note that we have from (1.15) that

$$
\partial_t^{\varphi}+u\cdot\nabla^{\varphi}=\partial_t+u_y\nabla_y+U_z\partial_z.
\tag{7.7}
$$

The energy equality then reads:

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{S}}(g_1|\sigma^\alpha|^2+g_2|u^\alpha|^2)(t)\,\mathrm{d}\mathcal{V}_t+\int_0^t\int_{\mathcal{S}}2\mu|Z^\alpha S^{\varphi}u|^2+\lambda|Z^\alpha\mathrm{div}^{\varphi}u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\
&=F_0^\alpha+F_1^\alpha+\cdots+F_7^\alpha.
\end{aligned}
\tag{7.8}
$$

where

$$
\begin{aligned}
&F_0^\alpha=\frac{1}{2}\int_{\mathcal{S}}\big(g_1|\sigma^\alpha|^2+g_2|u^\alpha|^2\big)\,\mathrm{d}\mathcal{V}_0,\quad F_1^\alpha=\frac{1}{2}\int_0^t\int_{\mathcal{S}}\big(\partial_t^{\varphi}g_1+\mathrm{div}^{\varphi}(g_1u)\big)|\sigma^\alpha|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\
&F_2^\alpha=-\int_0^t\int_{z=0}[Z^\alpha,\mathbf{N}](\mathcal{L}^{\varphi}u-(\sigma/\varepsilon)\mathrm{Id})\cdot u^\alpha\,\mathrm{d}y\mathrm{d}s\ \mathbb{I}_{\{\alpha_3=0\}},\\
&F_3^\alpha=\int_0^t\int_{\mathcal{S}}Z^\alpha\mathcal{L}^{\varphi}u\cdot[Z^\alpha,\nabla^{\varphi}]u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\quad F_4^\alpha=-\int_0^t\int_{\mathcal{S}}[Z^\alpha,\mathrm{div}^{\varphi}]\mathcal{L}^{\varphi}u\cdot u^\alpha\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,
\end{aligned}
$$

$$F_5^\alpha = \int_0^t \int_{\mathcal{S}} \mathcal{C}_\sigma^\alpha \sigma^\alpha + \mathcal{C}_u^\alpha \cdot u^\alpha \, \mathrm{d}\mathcal{V}_s \mathrm{d}s, \quad F_6^\alpha = -\frac{1}{\varepsilon} \int_0^t \int_{\mathcal{S}} \sigma^\alpha [Z^\alpha, \mathrm{div}^\varphi] u + u^\alpha \cdot [Z^\alpha, \nabla^\varphi]\sigma \, \mathrm{d}\mathcal{V}_s \mathrm{d}s,$$
$$F_7^\alpha = -a \int_0^t \int_{z=-1} |Z^\alpha u_\tau|^2 \, \mathrm{d}y \mathrm{d}s.$$

The first two terms can be controlled directly by:

$$\varepsilon(|F_0^\alpha| + |F_1^\alpha|) \lesssim \varepsilon \|Z^\alpha(\sigma, u)(0)\|_{L^2(\mathcal{S})}^2 + T\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\varepsilon \|Z^\alpha \sigma\|_{L_t^\infty L^2(\mathcal{S})}^2. \tag{7.9}$$

For the boundary term F_2^α, which vanishes identically if $\alpha_3 = 0$, we split it as:

$$F_2^\alpha = -\int_0^t \int_{z=0} (\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id}) Z^\alpha \mathbf{N} \cdot u^\alpha + [Z^\alpha, (\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id}), \mathbf{N}] u^\alpha \, \mathrm{d}y \mathrm{d}s =: F_{21}^\alpha + F_{22}^\alpha.$$

By duality and (3.4), F_{21}^α can be bounded as:

$$|F_{21}^\alpha| \lesssim |(\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id})^{b,1}|_{L_t^\infty W_y^{1,\infty}} |(u^\alpha)^{b,1}|_{L_t^2 H_y^{\frac{1}{2}}} |Z^\alpha \mathbf{N}|_{L_t^2 H_y^{-\frac{1}{2}}}.$$

By the identities (4.1), (4.3), (4.4) and the definition (1.33), we have:

$$\begin{aligned} |(\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id})^{b,1}|_{[\frac{m}{2}]-1,\infty,t} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}],\infty,t} + \|\mathrm{div}^\varphi u\|_{[\frac{m}{2}]-1,\infty,t} + \|u\|_{[\frac{m}{2}],\infty,t}\big) \\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big). \end{aligned} \tag{7.10}$$

Hence, by the trace inequality and Young's inequality, we get that:

$$\varepsilon |F_{21}^\alpha| \leq \delta\varepsilon \|\nabla u\|_{L_t^2 H_{co}^m}^2 + \varepsilon\big(|Z^\alpha h|_{L_t^2 H^{\frac{1}{2}}}^2 + \|u^\alpha\|_{L_t^2 L^2}^2\big)\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big).$$

For F_{22}^α, we use successively the Cauchy-Schwarz inequality, the estimate (7.10) and the trace inequality (3.17) to get:

$$\begin{aligned} |F_{22}^\alpha| &\lesssim |(u^\alpha)^{b,1}|_{L_t^2 L_y^2} \big|[Z^\alpha, \mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id}, \mathbf{N}]\big|_{L_t^2 L_y^2} \\ &\lesssim |(u^\alpha)^{b,1}|_{L_t^2 L_y^2} \big(|(\mathcal{L}^\varphi u, \sigma/\varepsilon)|_{[\frac{m}{2}]-1,\infty,t} |h|_{L_t^2 \tilde{H}^m} + |(\mathcal{L}^\varphi u, \sigma/\varepsilon)|_{L_t^2 \tilde{H}^{m-1}} |\mathbf{N}|_{[\frac{m+1}{2}]+1,\infty,t}\big) \\ &\leq \delta \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_\delta \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|u\|_{E^m,t}^2 + \|\nabla \mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}} \|\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}} + |h|_{L_t^2 \tilde{H}^m}^2\big). \end{aligned}$$

To summarize, we can control εF_2^α as:

$$\varepsilon |F_2^\alpha| \leq 2\delta\varepsilon \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_\delta \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(T\varepsilon |h|_{L_t^\infty \tilde{H}^{m+\frac{1}{2}}}^2 + \varepsilon^{\frac{1}{2}}(\|u\|_{E^m,t}^2 + \varepsilon \|\nabla \mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}}^2)\big). \tag{7.11}$$

Let us detail the estimate of F_3^α. We use the estimate (3.23) for $n = 2$ and Young's inequality to get that:

$$\begin{aligned} |\varepsilon F_3^\alpha| &\leq \varepsilon \|Z^\alpha \mathcal{L}^\varphi u\|_{L_t^2 L^2} \big(\|\nabla u\|_{L_t^2 H_{co}^{m-1}} + |h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}\big)\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t} + \varepsilon^{\frac{1}{2}} \|\nabla u\|_{2,\infty,t}\big) \\ &\leq \delta\varepsilon \|\nabla u\|_{L_t^2 H_{co}^m}^2 + \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\varepsilon \|\nabla u\|_{L_t^2 H_{co}^{m-1}}^2 + T\varepsilon |h|_{L_t^\infty \tilde{H}^{m+\frac{1}{2}}}^2\big). \end{aligned} \tag{7.12}$$

Similarly, for F_4, by Hölder's inequality, the commutator estimate (3.23) and the definition (1.33), we find

$$
\begin{aligned}
|\varepsilon F_4^\alpha| &\le \varepsilon \|u^\alpha\|_{L_t^2L^2} \|[Z^\alpha, \mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L_t^2L^2} \\
&\lesssim \varepsilon^{\frac12}\|u^\alpha\|_{L_t^2L^2}\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t} + \varepsilon^{\frac12}\|\!|\nabla\mathcal{L}^\varphi u\|\!|_{2,\infty,t}\big)\big(|h|_{L_t^2\tilde{H}^{m+\frac12}} + \|\varepsilon^{\frac12}\nabla\mathcal{L}^\varphi u\|_{L_t^2H_{co}^{m-1}}\big) \\
&\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.
\end{aligned}
\tag{7.13}
$$

Next, we control F_5 as:

$$\varepsilon|F_5^\alpha| \le T^{\frac12}\|\varepsilon^{\frac12}(\sigma^\alpha, u^\alpha)\|_{L_t^\infty L^2}\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^\alpha, \mathcal{C}_u^\alpha)\|_{L_t^2L^2}.$$

It thus remains to estimate $(\mathcal{C}_\sigma^\alpha, \mathcal{C}_u^\alpha)$ defined in (7.5). Taking benefits of the commutator estimate (3.9) and the estimate (3.13) for g_1, g_2, we obtain:

$$\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^\alpha, \mathcal{C}_u^\alpha)\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|(\sigma,u)\|_{E^m,t} + \varepsilon^{\frac12}|h|_{L_t^2H^{m+\frac12}}\big).$$

Therefore, we obtain:

$$|\varepsilon F_5^\alpha| \lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2. \tag{7.14}$$

Let us split F_6^α as: $F_6^\alpha = F_{6,1}^\alpha + F_{6,2}^\alpha$ with

$$F_{6,1}^\alpha = -\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{S}} \sigma^\alpha[Z^\alpha, \mathrm{div}^\varphi]u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \qquad F_{6,2}^\alpha = -\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{S}} u^\alpha\cdot[Z^\alpha, \nabla^\varphi]\sigma\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.$$

For $F_{6,1}^\alpha$, thanks to the commutator estimate (3.23),

$$
\begin{aligned}
|\varepsilon F_{6,1}^\alpha| &\lesssim \|\varepsilon^{-\frac12}\sigma^\alpha\|_{L_t^2L^2}\varepsilon^{\frac12}\|[Z^\alpha, \mathrm{div}^\varphi]u\|_{L_t^2L^2} \\
&\lesssim (\|\varepsilon^{\frac12}\partial_t\sigma\|_{L_t^2\mathcal{H}^{m-1}} + \|\varepsilon^{-\frac12}\nabla\sigma\|_{L_t^2H_{co}^{m-1}})(\varepsilon^{\frac12}|h|_{L_t^2\tilde{H}^{m+\frac12}} + \varepsilon^{\frac12}\|\nabla u\|_{L_t^2H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) \\
&\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.
\end{aligned}
\tag{7.15}
$$

Similarly, by using the fact that (recall $m \ge 7$),

$$\varepsilon^{-\frac12}\|\!|\nabla\sigma\|\!|_{2,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big),$$

we finally find:

$$
\begin{aligned}
|\varepsilon F_{6,2}^\alpha| &\lesssim \|u\|_{L_t^2H_{co}^m}\big(\|\nabla\sigma\|_{L_t^2H_{co}^{m-1}} + |h|_{L_t^2\tilde{H}^{m+\frac12}}\|\!|\nabla\sigma\|\!|_{2,\infty,t}\big)\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \\
&\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.
\end{aligned}
\tag{7.16}
$$

Gathering (7.15) and (7.16), we find that:

$$|\varepsilon F_6^\alpha| \lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2. \tag{7.17}$$

Finally, for the boundary term F_7^α, we apply the trace inequality (3.17) and Young's inequality to get that:

$$\varepsilon|F_7^\alpha| \lesssim \delta\varepsilon\|\nabla Z^\alpha u_\tau\|_{L_t^2L^2}^2 + C_\delta T\varepsilon\|Z^\alpha u_\tau\|_{L_t^\infty L^2(\mathcal{S})}^2. \tag{7.18}$$

Collecting (7.9), (7.11), (7.12), (7.13), (7.14), (7.17), (7.18) and summing up for $|\alpha| \le m$, we find by Korn's inequality (3.34) and by choosing δ small enough,

$$\varepsilon\|(\sigma,u)\|^2_{L^\infty_t H^m_{co}} + \varepsilon\|\nabla u\|^2_{L^2_t H^m_{co}} \lesssim \varepsilon\|(\sigma,u)(0)\|^2_{H^m_{co}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.$$

□

Theorem 7.2 (Estimates for High-order time derivatives)**.** *Under the same assumption as in Lemma 7.1, we have the following estimates: for any* $0 < t \le T$,

$$\varepsilon\|\partial_t(\sigma,u)\|^2_{L^\infty_t \mathcal{H}^{m-1}} + \varepsilon\|\partial_t\nabla u\|^2_{L^2_t\mathcal{H}^{m-1}} \lesssim \varepsilon\|\partial_t(\sigma,u)(0)\|^2_{\mathcal{H}^{m-1}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.19}$$

Proof. Due to the singular terms in the system (1.16), we need to deal with the zero order and the higher order estimates for $\varepsilon^{\frac12}\partial_t(\sigma,u)$ differently. We will prove in (8.2) the zero order estimate:

$$\varepsilon\|\partial_t(\sigma,u)\|^2_{L^\infty_t L^2} + \varepsilon\|\partial_t\nabla u\|^2_{L^2_t L^2} \lesssim \varepsilon\|\partial_t(\sigma,u)(0)\|^2_{L^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Let us stress that this estimate does not depend on the higher order estimate to be shown here and vice versa.

We now focus on the higher order estimates. Substituting Z^α by $\varepsilon^{\frac12}Z_0^k\partial_t$ $(1 \le k \le m-1)$ in (7.8), we find that:

$$\begin{aligned}&\frac{\varepsilon}{2}\int_{\mathcal{S}}(g_1|Z_0^k\partial_t\sigma|^2 + g_2|Z_0^k\partial_t u|^2)(t)\,\mathrm{d}\mathcal{V}_t + \varepsilon\int_0^t\int_{\mathcal{S}} 2\mu|Z_0^k\partial_t S^\varphi u|^2 + \lambda|Z_0^k\partial_t \mathrm{div}^\varphi u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ &= F_0^k + F_1^k + \cdots + F_7^k.\end{aligned} \tag{7.20}$$

where $F_0^k - F_7^k$ are defined in the same way as $F_0^\alpha - F_7^\alpha$ (defined in (7.8)) by changing Z^α into $\varepsilon^{\frac12}Z_0^k\partial_t$. Our following task is to control $F_0^k - F_7^k$ one by one. The first two terms can be controlled by:

$$|F_0^k + F_1^k| \lesssim \varepsilon\|\partial_t(\sigma,u)\|^2_{L^2_t\mathcal{H}^k} + T\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\varepsilon\|\partial_t\sigma\|^2_{L^\infty_t\mathcal{H}^k}. \tag{7.21}$$

Now, for the term

$$F_2^k = -\varepsilon\int_0^t\int_{z=0}[Z_0^k\partial_t, \mathbf{N}](\mathcal{L}^\varphi u - \frac{\sigma}{\varepsilon}\mathrm{Id})Z_0^k\partial_t u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$

we first use the duality $\langle\cdot\rangle_{H^{\frac12}\times H^{-\frac12}}$, Cauchy-Schwarz inequality and the estimate (7.10) to control it as:

$$|F_2^k| \lesssim |\varepsilon^{\frac12}Z_0^k\partial_t u|_{L^2_t H^{\frac12}}|\varepsilon^{\frac12}Z_0^k\partial_t h|_{L^2_t H^{\frac12}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) + |\varepsilon^{\frac12}Z_0^k\partial_t u|_{L^2_t L^2_y}\big|\varepsilon^{\frac12}[Z_0^k\partial_t, \mathbf{N}, (\mathcal{L}^\varphi u - \frac{\sigma}{\varepsilon}\mathrm{Id})]\big|_{L^2_t L^2_y}$$

By (6.1), the trace inequality (3.17) and Young's inequality, the first term in the right hand side of the above inequality is bounded by:

$$\delta\|\varepsilon^{\frac12}Z_0^k\nabla^\varphi\partial_t u\|^2_{L^2_t L^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.$$

Moreover, we use the expansion (3.28), the estimates (4.9), (6.1), the trace inequality (3.17) successively to control the second one as:

$$\begin{aligned}&C|\varepsilon^{\frac12}Z_0^k\partial_t u|_{L^2_t L^2_y}\big(\varepsilon^{\frac12}|\partial_t h|_{L^2_t\tilde{H}^k}|(\mathcal{L}^\varphi u, \sigma/\varepsilon)^{b,1}|_{[\frac{k-1}{2}],\infty,t} + \varepsilon^{\frac12}|(\mathcal{L}^\varphi u, \sigma/\varepsilon)^{b,1}|_{L^2_t\tilde{H}^k}|\partial_t h|_{[\frac{k}{2}],\infty,t}\big)\\ &\le \delta\|\varepsilon^{\frac12}Z_0^k\nabla^\varphi\partial_t u\|^2_{L^2_t L^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.\end{aligned}$$

Note that by (4.1), (4.3), (4.4), one has that:

$$|\varepsilon^{\frac{1}{2}}(\mathcal{L}^\varphi u,\sigma)^{b,1}|_{L_t^2\tilde{H}^{m-1}} \lesssim (|\varepsilon^{\frac{1}{2}}(\partial_y u, \mathrm{div}^\varphi u)^{b,1}|_{L_t^2\tilde{H}^{m-1}} + \varepsilon^{\frac{1}{2}}|h|_{L_t^2\tilde{H}^m})\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)$$
$$\lesssim \varepsilon^{\frac{1}{4}}\big(\|\nabla u\|_{L_t^2H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\|\nabla u\|_{L_t^2H_{co}^m} + \varepsilon^{\frac{1}{2}}\|\nabla \mathrm{div} u\|_{L_t^2H_{co}^{m-1}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big) + T^{\frac{1}{2}}|\varepsilon^{\frac{1}{2}}h|_{L_t^\infty\tilde{H}^m}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)$$
$$\lesssim (\varepsilon^{\frac{1}{4}}+T^{\frac{1}{2}})\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

We thus find that:

$$|F_2^k| \lesssim 2\delta\|\varepsilon^{\frac{1}{2}}Z_0^k\nabla^\varphi\partial_t u\|_{L_t^2L^2}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2. \tag{7.22}$$

Next, with the aid of the commutator estimate (3.25) and the estimate (6.2), we can control the commutator $[Z_0^k\partial_t,\nabla^\varphi]u$ as:

$$\begin{aligned}\|[Z_0^k\partial_t,\nabla^\varphi]u\|_{L_t^2L^2} \lesssim &\big(|\varepsilon\partial_t^2 h|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} + \|\varepsilon\partial_t\partial_z u\|_{L_t^2\mathcal{H}^{m-2}\cap L_t^\infty\mathcal{H}^1}\big)\cdot\\ &\Lambda\big(\frac{1}{c_0}, \|\partial_z u\|_{1,\infty,t} + |(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t} + (\int_0^t|\varepsilon^{\frac{1}{2}}\partial_t^2 h(s)|_{m-2,\infty}\mathrm{d}s)^{\frac{1}{2}}\big)\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, we bound the term

$$F_3^k = \varepsilon\int_0^t\int_{\mathcal{S}} Z_0^k\partial_t\mathcal{L}^\varphi u\cdot[Z_0^k\partial_t,\nabla^\varphi]u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s$$

by using Young's inequality and the assumption $k\leq m-1$,

$$|F_3^k| \leq \delta\varepsilon\|Z_0^k\partial_t\nabla^\varphi u\|_{L_t^2L^2}^2 + \varepsilon\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.23}$$

We proceed to estimate

$$F_4^k = -\varepsilon\int_0^t\int_{\mathcal{S}}[Z_0^k\partial_t,\mathrm{div}^\varphi]\mathcal{L}^\varphi u\cdot Z_0^k\partial_t u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.$$

By the expansion (7.29), the estimate (6.2), and the assumption $k\leq m-1$, we obtain:

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}[Z_0^k\partial_t,\mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L_t^2L^2} \lesssim &(|\varepsilon\partial_t^2 h|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} + \varepsilon^{\frac{1}{2}}\|\partial_z\mathcal{L}^\varphi u\|_{L_t^2\mathcal{H}^{m-1}})\cdot\\ &\Lambda\big(\frac{1}{c_0},\varepsilon^{\frac{1}{2}}\|\partial_z\mathcal{L}^\varphi u\|_{[\frac{m}{2}]-1,\infty,t} + |\partial_t h|_{[\frac{m-1}{2}],\infty,t} + |h|_{[\frac{m+1}{2}],\infty,t}\big)\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

We thus control F_4^k by the Cauchy-Schwarz inequality:

$$\begin{aligned}|F_4^k| &\leq T^{\frac{1}{2}}\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty\mathcal{H}^k}\|\varepsilon^{\frac{1}{2}}[Z_0^k\partial_t,\mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L_t^2L^2}\\ &\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned} \tag{7.24}$$

The next term F_5^k is defined by

$$F_5^k = \varepsilon\int_0^t\int_{\mathcal{S}} C_\sigma^k Z_0^k\partial_t\sigma + C_u^k\cdot Z_0^k\partial_t u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.$$

To continue, we need the following proposition to control the commutators $\varepsilon^{\frac{1}{2}}(\mathcal{C}_\sigma^k,\mathcal{C}_u^k)$:

Proposition 7.3. *For commutators*

$$
\begin{aligned}
\mathcal{C}_\sigma^k &= \big[Z_0^k\partial_t, \frac{g_1}{\varepsilon}\big]\varepsilon\partial_t\sigma + [Z_0^k\partial_t, g_1 u_y]\nabla_y\sigma + [Z_0^k\partial_t, g_1 U_z\partial_z]\sigma,\\
\mathcal{C}_u^k &= \big[Z_0^k\partial_t, \frac{g_1}{\varepsilon}\big]\varepsilon\partial_t u + [Z_0^k\partial_t, g_1 u_y]\nabla_y u + [Z_0^k\partial_t, g_1 U_z\partial_z]u.
\end{aligned} \tag{7.25}
$$

we have the estimate: for $k \le m-1$

$$
\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^k, \mathcal{C}_u^k)\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
$$

We will postpone the proof of this proposition and continue to estimate the remaining terms $F_5^k - F_7^k$. By using Proposition 7.3, F_5^k can be estimated as:

$$
|F_5^k| \lesssim T^{\frac12}\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^\infty\mathcal{H}^{m-1}}\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^k, \mathcal{C}_u^k)\|_{L_t^2L^2} \lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.26}
$$

For the term

$$
F_6^k = -\int_0^t\int_{\mathcal{S}} Z_0^k\partial_t\sigma\cdot[Z_0^k\partial_t, \mathrm{div}^\varphi]u + Z_0^k\partial_t u\cdot[Z_0^k\partial_t, \nabla^\varphi]\sigma \,\mathrm{d}\mathcal{V}_s\mathrm{d}s,
$$

we can apply commutator estimate (3.25) to obtain:

$$
\begin{aligned}
&\varepsilon^{-\frac12}(\|[Z_0^k\partial_t, \mathrm{div}^\varphi]u\|_{L_t^2L^2} + \|[Z_0^k\partial_t, \nabla^\varphi]\sigma\|_{L_t^2L^2})\\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big)\big(|\varepsilon^{\frac12}\partial_t^2 h|_{L_t^2\tilde{H}^{m-\frac32}} + \|\varepsilon^{\frac12}\partial_t\nabla(\sigma,u)\|_{L_t^2\mathcal{H}^{m-2}\cap L_t^\infty\mathcal{H}^1}\big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

This estimate, combined with the Cauchy-Schwarz inequality, yields:

$$
|F_6^k| \lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.27}
$$

Finally, we control the last term

$$
F_7^k = -a\varepsilon\int_0^t\int_{z=-1}|Z_0^k\partial_t u_\tau|^2\,\mathrm{d}y\mathrm{d}s
$$

by the trace inequality (3.17) and Young's inequality:

$$
F_7^k \le \delta\varepsilon\int_0^t\int_{\mathcal{S}}|Z_0^k\partial_t\nabla^\varphi u|^2\mathrm{d}\mathcal{V}_s\mathrm{d}s + (T+\varepsilon)\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2. \tag{7.28}
$$

Collecting (7.21)-(7.28), summing up for $k \le m-1$ and choosing δ small enough, we find (7.19). $\square$

We now give the proof of Proposition 7.3.

Proof of Proposition 7.3. We use the following two expansions

$$
\varepsilon^{\frac12}[Z_0^{m-1}\partial_t, f]g = \sum_{0\le l\le[\frac{m}{2}]-1}(C_m^l Z_0^l g Z_0^{m-1-l}\varepsilon^{\frac12}\partial_t f) + \sum_{[\frac{m}{2}]\le l\le m-1}(C_m^l Z_0^{l-1}\varepsilon^{\frac12}\partial_t g Z_0^{m-l} f) \tag{7.29}
$$

$$
\begin{aligned}
\varepsilon^{\frac12}[Z_0^{m-1}\partial_t, f]g = &\sum_{0\le l\le 1}(C_m^l Z_0^l g Z_0^{m-1-l}\varepsilon^{\frac12}\partial_t f) + C_m^2 Z_0^1\varepsilon^{\frac12}\partial_t g Z_0^{m-2} f\\
&+ \sum_{3\le l\le m-1}(C_m^l Z_0^{l-1}\varepsilon^{\frac12}\partial_t g Z_0^{m-l} f).
\end{aligned}
$$

In light of the second expansion, we control the last term in $\mathcal{C}_u^{m-1}$ as follows:

$$\begin{aligned}
&\varepsilon^{\frac12}\|[Z_0^{m-1}\partial_t, g_1U_z]\partial_z u\|_{L_t^2L^2} \lesssim \|\!|\partial_z u|\!\|_{1,\infty,t}\|\varepsilon^{\frac12}\partial_t(g_1U_z)\|_{L_t^2\mathcal{H}^{m-1}}\\
&\qquad + \|\varepsilon^{\frac12}\partial_t\partial_z u\|_{L_t^\infty\mathcal{H}^1}\|Z_0^{m-2}(g_1U_z)\|_{L_t^\infty L^2} + \|\varepsilon^{\frac12}\partial_t\partial_z u\|_{L_t^2\mathcal{H}^{m-1}}\|\!|g_1U_z|\!\|_{m-3,\infty,t}\\
&\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}$$

The remaining terms appearing in $\mathcal{C}_\sigma^{m-1},\mathcal{C}_u^{m-1}$ can be estimated by using the first expansion:

$$\begin{aligned}
&\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^{m-1},\mathcal{C}_u^{m-1}-[Z_0^{m-1}\partial_t, g_1U_z]\partial_z u)\|_{L_t^2L^2}\\
&\lesssim \sum_{j=1}^{2}\big[\|\varepsilon^{\frac12}\partial_t(g_j/\varepsilon, g_ju_y, g_jU_z)\|_{L_t^2\mathcal{H}^{m-1}}(\|\!|(\sigma,u)|\!\|_{[\frac{m}{2}],\infty,t}+\|\!|\nabla\sigma|\!\|_{[\frac{m}{2}]-1,\infty,t})\\
&\qquad + \|\!|\varepsilon\partial_t(g_j/\varepsilon, g_ju_y, g_jU_z)|\!\|_{[\frac{m-1}{2}],\infty,t}\|\varepsilon^{\frac12}\partial_t(Z_0,\nabla)(\sigma,u)\|_{L_t^2\mathcal{H}^{m-2}}\big]\\
&\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}$$

□

7.2. **Energy estimates II: High-order energy estimate for the compressible part of the system.** In this step, we estimate the compressible part $(\nabla^\varphi\sigma,\mathrm{div}^\varphi u)$:

Lemma 7.4. *Under the same assumption as in Lemma 7.1, the following estimates hold:*

$$\begin{aligned}
&\varepsilon(\|(\nabla^\varphi\sigma,\mathrm{div}^\varphi u)\|^2_{L_t^\infty H_{co}^{m-1}}+\|\nabla^\varphi\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-1}})\\
&\lesssim \Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)Y_m^2(0)+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}\tag{7.30}$$

Proof. Let β be a multi-index satisfying $|\beta|\le m-1$. Applying $Z^\beta\nabla^\varphi$ (resp. Z^β) to the equation for σ (resp. u), we find that:

$$\begin{cases} g_1(\partial_t^\varphi+u\cdot\nabla^\varphi)Z^\beta\nabla^\varphi\sigma+\frac{1}{\varepsilon}Z^\beta\nabla^\varphi\mathrm{div}^\varphi u=\mathcal{R}_\sigma^\beta\\ g_2(\partial_t^\varphi+u\cdot\nabla^\varphi)Z^\beta u+\mu\,\mathrm{curl}^\varphi Z^\beta\omega-(2\mu+\lambda)\nabla^\varphi Z^\beta\mathrm{div}^\varphi u+\frac{1}{\varepsilon}Z^\beta\nabla^\varphi\sigma=\mathcal{R}_u^\beta\end{cases}\tag{7.31}$$

where

$$\mathcal{R}_\sigma^\beta=\mathcal{R}_{\sigma,1}^\beta+\mathcal{R}_{\sigma,2}^\beta+\mathcal{R}_{\sigma,3}^\beta,\quad \mathcal{R}_u^\beta=\mathcal{R}_{u,1}^\beta+\cdots\mathcal{R}_{u,3}^\beta,\tag{7.32}$$

with

$$\begin{aligned}
&\mathcal{R}_{\sigma,1}^\beta=Z^\beta(\nabla^\varphi g_1\partial_t^\varphi\sigma+\nabla^\varphi(g_1u)\cdot\nabla^\varphi\sigma), &&\mathcal{R}_{u,1}^\beta=[Z^\beta,g_2/\varepsilon]\varepsilon\partial_t u+[Z^\beta,g_1u_y]\nabla_y u,\\
&\mathcal{R}_{\sigma,2}^\beta=[Z^\beta,g_1/\varepsilon]\varepsilon\partial_t\nabla^\varphi\sigma+[Z^\beta,g_1u_y]\nabla_y\nabla^\varphi\sigma, &&\mathcal{R}_{u,2}^\beta=[Z^\beta,g_1U_z\partial_z]u,\\
&\mathcal{R}_{\sigma,3}^\beta=[Z^\beta,g_1U_z\partial_z]\nabla^\varphi\sigma, &&\mathcal{R}_{u,3}^\beta=-\mu[Z^\beta,\mathrm{curl}^\varphi]\omega+(2\mu+\lambda)[Z^\beta,\nabla^\varphi]\mathrm{div}^\varphi u
\end{aligned}$$

and U_z is defined in (7.6). Taking the scalar product of (7.31) by $(Z^\beta\nabla^\varphi\sigma,-\nabla^\varphi Z^\beta\mathrm{div}^\varphi u)^t$ and by integrating in space and time, one gets the following energy identity:

$$\begin{aligned}
&\frac12\int_{\mathcal{S}}\big(g_1|Z^\beta\nabla^\varphi\sigma|^2+g_2|Z^\beta\mathrm{div}^\varphi u|^2\big)(t)\,\mathrm{d}\mathcal{V}_t+(2\mu+\lambda)\|\nabla^\varphi Z^\beta\mathrm{div}^\varphi u\|^2_{L_t^2L^2}\\
&=J_0^\beta+J_1^\beta+\cdots J_7^\beta
\end{aligned}\tag{7.33}$$

with:

$$J_0^\beta = \frac{1}{2}\int_{\mathcal{S}} \big(g_1|Z^\beta\nabla^\varphi\sigma|^2 + g_2|Z^\beta \mathrm{div}^\varphi u|^2\big)(0)\,\mathrm{d}\mathcal{V}_0,$$
$$J_1^\beta = \frac{1}{2}\int_0^t\int_{\mathcal{S}} (\partial_t^\varphi g_1 + \mathrm{div}^\varphi(g_1 u))|Z^\beta\nabla^\varphi\sigma|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$
$$J_2^\beta = \int_0^t\int_{\mathcal{S}} \big(\nabla^\varphi g_2\cdot\partial_t^\varphi Z^\beta u + \nabla^\varphi(g_2 u)\otimes\nabla^\varphi Z^\beta u\big)Z^\beta \mathrm{div}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$
$$J_3^\beta = \int_0^t\int_{\mathcal{S}} g_2(\partial_t^\varphi + u\cdot\nabla^\varphi)([Z^\beta, \mathrm{div}^\varphi]u)Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$
$$J_4^\beta = \int_0^t\int_{z=0} g_2(\partial_t + u_y\partial_y)Z^\beta u\cdot\mathbf{N}Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}y\mathrm{d}s\mathbb{I}_{\{\beta_3=0\}},$$
$$J_5^\beta = -\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{S}} Z^\beta\nabla^\varphi\sigma[Z^\beta,\nabla^\varphi]\mathrm{div}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$
$$J_6^\beta = \mu\int_0^t\int_{\mathcal{S}} \mathrm{curl}^\varphi\, Z^\beta\omega\cdot\nabla^\varphi Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$
$$J_7^\beta = \int_0^t\int_{\mathcal{S}} \mathcal{R}_\sigma^\beta\cdot Z^\beta\nabla^\varphi\sigma + \mathcal{R}_u^\beta\cdot\nabla^\varphi Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.$$

The first three terms can be controlled directly:

$$\varepsilon J_0^\beta \le \varepsilon\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)(0)\|_{H_{co}^{m-1}}^2, \tag{7.34}$$

$$\varepsilon(J_1^\beta + J_2^\beta) \lesssim \varepsilon(\|(\sigma,u)\|_{E^m,t}^2 + |h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}^2). \tag{7.35}$$

In order to bound J_3^β, we need to control $(\partial_t^\varphi + u\cdot\nabla^\varphi)[Z^\beta, \mathrm{div}^\varphi]u$. By the identity (7.7), we can write

$$\partial_t^\varphi + u\cdot\nabla^\varphi = \partial_t + u_1\partial_1 + u_2\partial_2 + \frac{U_z}{\phi}Z_3.$$

Since $U_z|_{\partial\mathcal{S}} = \frac{u\cdot\mathbf{N}-\partial_t\varphi}{\partial_z\varphi}\big|_{\partial S} = 0$, we have by the fundamental theorem of calculus and (3.20) that:

$$\|U_z/\phi\|_{0,\infty,t} \lesssim \|(U_z, \partial_z U_z)\|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, \|(u,\nabla u)\|_{0,\infty,t} + |h|_{2,\infty,t}\big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big). \tag{7.36}$$

Therefore, we see that:

$$\|(\partial_t^\varphi + \underline{u}\cdot\nabla^\varphi)[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2} \lesssim \frac{1}{\varepsilon}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\|(\varepsilon\partial_t, \varepsilon Z)[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2}. \tag{7.37}$$

Let us first consider:

$$\varepsilon\partial_t[Z^\beta, \mathrm{div}^\varphi]u = \varepsilon\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}[Z^\beta,\partial_z]u\big) + \big[Z^\beta, \varepsilon\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\big]\partial_z u + \big[Z^\beta, \frac{\mathbf{N}}{\partial_z\varphi}\big]\varepsilon\partial_t\partial_z u.$$

In view of Lemma 3.9, the identity (3.27) and the commutator estimate (3.9), the first two terms in the right hand side of the above identity can be bounded by:

$$\big(\|\nabla u\|_{L_t^2H_{co}^{m-1}} + |(h,\varepsilon\partial_t h)|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big)\Lambda\big(\frac{1}{c_0}, \|\nabla u\|_{1,\infty,t} + |h|_{[\frac{m}{2}]+2,\infty,t}\big).$$

For the third one, we control it as:

$$\big\|[Z^\beta, \frac{\mathbf{N}}{\partial_z\varphi}]\varepsilon\partial_t\partial_z u\big\|_{L_t^2L^2} \lesssim \big\|\frac{\mathbf{N}}{\partial_z\varphi}\big\|_{L_t^2H_{co}^{m-1}}\|\nabla u\|_{1,\infty,t} + \|\varepsilon^{\frac12}(\frac{\mathbf{N}}{\partial_z\varphi})\|_{m-2,\infty,t}\|\varepsilon^{\frac12}\partial_t\partial_z u\|_{L_t^2H_{co}^{m-2}}$$
$$\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

Gathering the previous two estimates, we find that:

$$\|\varepsilon\partial_t[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

In a similar way, we have:

$$\|\varepsilon Z[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

Plugging the above two estimates into (7.37), we can then control J_3^β as:

$$\begin{aligned}\varepsilon J_3^\beta &\lesssim \varepsilon^{\frac12}\|\mathrm{div}^\varphi u/\varepsilon^{\frac12}\|_{L_t^2H_{co}^{m-1}}\|\varepsilon(\partial_t^\varphi + \underline{u}\cdot\nabla^\varphi)[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2} \\ &\lesssim \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.\end{aligned} \tag{7.38}$$

We now switch to estimate J_4^β. On the one hand, if $Z^\beta = Z_0^k, k \leq m-1$, we have by the trace inequality (3.17) that:

$$\begin{aligned}&\varepsilon^{\frac12}|(\partial_t + u_y\partial_y)Z_0^k u|_{L_t^2L_y^2} \\ &\lesssim (\|\varepsilon^{\frac12}\partial_t(u,\nabla u)\|_{L_t^2\mathcal{H}^{m-1}} + \|\varepsilon^{\frac12}\nabla u\|_{L_t^2H_{co}^m})\Lambda(\|u\|_{0,\infty,t}) \lesssim \Lambda(\|u\|_{0,\infty,t})\mathcal{E}_{m,t}.\end{aligned}$$

Therefore, by the trace inequality (3.17), we get that in this case:

$$\begin{aligned}\varepsilon J_4^\beta &\lesssim \varepsilon^{\frac12}|Z_0^k\mathrm{div}^\varphi u|_{L_t^2L_y^2}|\varepsilon^{\frac12}(\partial_t + u_y\partial_y)Z_0^k u|_{L_t^2L_y^2}|\mathbf{N}|_{0,\infty,t} \\ &\lesssim \varepsilon^{\frac12}(\|\mathrm{div}^\varphi u/\varepsilon^{\frac12}\|_{L_t^2\mathcal{H}^{m-1}}^2 + \|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{m-1}}^2 + \mathcal{E}_{m,t}^2)\Lambda(\|u\|_{0,\infty,t} + |h|_{1,\infty,t}) \\ &\lesssim \varepsilon^{\frac12}\Lambda(\|u\|_{0,\infty,t} + |h|_{1,\infty,t})\mathcal{E}_{m,t}^2.\end{aligned} \tag{7.39}$$

On the other hand, if Z^β contains at least one spatial tangential derivatives $\partial_{y_1}, \partial_{y_2}$, we control εJ_3^β as follows. By the equation $(1.16)_2$ and the identity (4.1), we can express $(\mathrm{div}^\varphi u)$ on the boundary $\{z=0\}$ as:

$$\mathrm{div}^\varphi u = \varepsilon g_1(\partial_t + u_y\partial_y)\big(\varepsilon\mathrm{div}^\varphi u + 2\mu\varepsilon(\partial_1 u_1 + \partial_2 u_2) - \mu\varepsilon(\omega\times\mathbf{N})_3\big) \text{ on } \{z=0\}.$$

This, together with the product estimate (3.14), the identity (4.2) and the trace inequality (3.17) yields that:

$$\begin{aligned}&|(Z^\beta\mathrm{div}^\varphi u)^{b,1}|_{L_t^2H^{-\frac12}} \\ &\lesssim |(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2\tilde{H}^{m-\frac32}} \lesssim \varepsilon\big|\big((\mathrm{div}^\varphi u)^{b,1}, \partial_y u^{b,1}, (\omega\times\mathbf{N})_3\big)\big|_{L_t^2\tilde{H}^{m-\frac12}} \\ &\lesssim \varepsilon^{\frac12}\big(\varepsilon^{\frac12}(|h|_{L_t^2\tilde{H}^{m+\frac12}} + \|\nabla u\|_{L_t^2H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) + \varepsilon^{\frac12}\|\nabla\mathrm{div} u\|_{L_t^2H_{co}^{m-1}} \\ &\qquad + \varepsilon^{\frac12}\|\nabla u\|_{L_t^2H_{co}^m}\Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big)\big),\end{aligned}$$

which, combined with the Young's inequality, allows us to control εJ_4^β as:

$$\begin{aligned}|\varepsilon J_4^\beta| &\lesssim \Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)|\varepsilon^{\frac12}(\varepsilon\partial_t,\varepsilon Z)Z^\beta u|_{L_t^2H^{\frac12}}|\varepsilon^{-\frac12}Z^\beta \mathrm{div}^\varphi u|_{L_t^2H^{-\frac12}}\\ &\leq \delta\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-1}}+C_\delta\|\varepsilon^{\frac12}\nabla u\|^2_{L_t^2H_{co}^m}\Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)+T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.\end{aligned} \tag{7.40}$$

In view of (7.39) and (7.40), we find that:

$$|\varepsilon J_4^\beta|\leq \delta\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-1}}+C_\delta\|\varepsilon^{\frac12}\nabla u\|^2_{L_t^2H_{co}^m}\Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}. \tag{7.41}$$

Next, thanks to (3.23), J_5^β can be bounded by:

$$\begin{aligned}\varepsilon J_5^\beta &\lesssim \varepsilon^{\frac12}\|\nabla\sigma/\varepsilon^{\frac12}\|_{L_t^2H_{co}^{m-1}}\big(\|\nabla\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-2}}+|h|_{L_t^2\tilde H^{m-\frac12}}\big)\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+\|\partial_z\mathrm{div}^\varphi u\|_{1,\infty,t}\big)\\ &\lesssim \varepsilon^{\frac12}\big(\|\nabla\sigma/\varepsilon^{\frac12}\|^2_{L_t^2H_{co}^{m-1}}+\|\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-2}}+|h|^2_{L_t^2\tilde H^{m-\frac12}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\lesssim \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.\end{aligned} \tag{7.42}$$

Note that by the equation $(1.16)_1$, we have $\partial_z\mathrm{div}^\varphi u=\partial_z(g_1\varepsilon\partial_t+\varepsilon u_y\partial_y+\varepsilon U_z\partial_z)\sigma$, we thus get that

$$\|\partial_z\mathrm{div}^\varphi u\|_{1,\infty,t}\lesssim \Lambda\big(1/c_0,\|(\sigma,\nabla\sigma)\|_{2,\infty,t}+\|(u,\nabla u)\|_{1,\infty,t}+|h|_{3,\infty,t}\big)\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).$$

For the next term J_6^β, we assume $\beta_3=0$, since otherwise it vanishes identically. It follows from integration by parts that:

$$\begin{aligned}J_6^\beta &= \mu\int_0^t\int_{z=0}(Z^\beta\omega\times\mathbf{n})\Pi\nabla^\varphi Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}y\mathrm{d}s+\mu\int_0^t\int_{z=-1}Z^\beta(\omega_2,-\omega_1,0)^t\cdot(\partial_y,0)Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}y\mathrm{d}s\\ &= J_{6,1}^\beta+J_{6,2}^\beta\end{aligned}$$

where $\omega=\nabla^\varphi\times u=(\omega_1,\omega_2,\omega_3)^t$. In light of the boundary condition (1.19), we have by integration by parts along the boundary and the trace inequality (3.17) that:

$$\begin{aligned}\varepsilon J_{6,2}^\beta \lesssim \varepsilon|u^{b,2}|_{L_t^2\tilde H^m}|Z^\beta(\mathrm{div}^\varphi u)^{b,2}|_{L_t^2L^2} &\lesssim \varepsilon^{\frac12}(\|u\|^{\frac12}_{L_t^2H_{co}^m}\|\varepsilon^{\frac12}\nabla u\|^{\frac12}_{L_t^2H_{co}^m}+\|u\|_{L_t^2H_{co}^m})\cdot\\ &\quad(\|\mathrm{div}^\varphi u\|^{\frac12}_{L_t^2H_{co}^m}\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^{\frac12}_{L_t^2H_{co}^{m-1}}+\|\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}})\\ &\lesssim\varepsilon^{\frac12}\mathcal{E}^2_{m,t}.\end{aligned} \tag{7.43}$$

For $J_{6,1}^\beta$, since $\Pi\nabla^\varphi=\Pi(\partial_1,\partial_2,0)^t$, we also integrate by parts along the boundary to get:

$$\begin{aligned}\varepsilon J_{6,1}^\beta\lesssim \varepsilon\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\Big(&\big|\big(Z^\beta(\omega^{b,1}\times\mathbf{n}),Z^\beta\mathbf{n}\big)\big|_{L_t^2H^{\frac12}}|Z^\beta(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2H^{\frac12}}\\ &+\big|\big(Z^\beta\omega^{b,1},[\partial_yZ^\beta,\mathbf{n},\omega^{b,1}]\big)\big|_{L_t^2L_y^2}|Z^\beta(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2L_y^2}\Big).\end{aligned}$$

Thanks to the boundary condition (4.2), we have that

$$|Z^\beta(\omega^{b,1}\times\mathbf{n})|_{L_t^2H^{\frac12}}\lesssim|u^{b,1}|_{L_t^2\tilde H^{m+\frac12}}\Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)+(|u^{b,1}|_{L_t^2\tilde H^{m-\frac12}}+|h|_{L_t^2\tilde H^{m+\frac12}})\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).$$

Moreover, by (4.5) (4.8), we have:

$$|Z^\beta \omega^{b,1}|_{L_t^2 L_y^2} \lesssim \big(|(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2 \tilde{H}^{m-1}} + |(u^{b,1}, h)|_{L_t^2 \tilde{H}^m}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big),$$

$$\begin{aligned}\big|[\partial_y Z^\beta, \mathbf{n}, \omega^{b,1}]\big|_{L_t^2 L_y^2} &\lesssim (|\omega^{b,1}|_{L_t^2 \tilde{H}^{m-1}} + |h|_{L_t^2 \tilde{H}^m})\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) \\ &\lesssim \big(|(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2 \tilde{H}^{m-1}} + |(u^{b,1}, h)|_{L_t^2 \tilde{H}^m}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big).\end{aligned}$$

Hence, by the trace inequality and Young's inequality, we end up with:

$$\begin{aligned}\varepsilon J_{6,1}^\beta \leq \delta\varepsilon \|\nabla^\varphi \mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}}^2 + \varepsilon \|\nabla u\|_{L_t^2 H_{co}^m}^2 \Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big) \\ + \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)(\varepsilon \|\nabla \mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-2}}^2 + \varepsilon |h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}^2 + \varepsilon \|u\|_{E^m,t}).\end{aligned} \tag{7.44}$$

Summing up (7.43) and (7.44), and using (9.4), we obtain:

$$\varepsilon J_6^\beta \leq 2\delta\varepsilon^2 \|\nabla^\varphi \mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}}^2 + C_\delta \Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}} \nabla u\|_{L_t^2 H_{co}^m}^2) + (T+\varepsilon)^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2. \tag{7.45}$$

Finally, for J_7^β, by Young's inequality,

$$\varepsilon J_7^\beta \leq \delta\varepsilon \|\nabla^\varphi Z^\beta \mathrm{div}^\varphi u\|_{L_t^2 L^2}^2 + C_\delta \varepsilon \|\mathcal{R}_u^\beta\|_{L_t^2 L^2}^2 + \varepsilon^{\frac{1}{2}}(\|\nabla^\varphi \sigma / \varepsilon^{\frac{1}{2}}\|_{L_t^2 H_{co}^{m-1}}^2 + \varepsilon \|\mathcal{R}_\sigma^\beta\|_{L_t^2 L^2}^2). \tag{7.46}$$

Hence, it suffices to control $\varepsilon^{\frac{1}{2}}\|(\mathcal{R}_\sigma^\beta, \mathcal{R}_u^\beta)\|_{L_t^2 L^2}$. Let us first see the estimate of $\varepsilon \mathcal{R}_\sigma^\beta$. In view of the definition (7.32), we have by the product estimate (3.8) and Corollary 3.5 that

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma,1}^\beta\|_{L_t^2 L^2} \lesssim \varepsilon^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|u\|_{E^m,t} + \|(\varepsilon^{\frac{1}{2}} \partial_t \sigma, \varepsilon^{-\frac{1}{2}} \nabla \sigma)\|_{L_t^2 H_{co}^{m-1}} + |h|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}\big). \tag{7.47}$$

Similarly, by the commutator estimate (3.9) and Corollary 3.5, we have that:

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma,2}^\beta\|_{L_t^2 L^2} \lesssim \varepsilon^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|u\|_{E^m,t} + \|\varepsilon^{-\frac{1}{2}} \sigma\|_{E^m,t} + |h|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}\big). \tag{7.48}$$

For $\mathcal{R}_{\sigma,3}^\beta$, we split it as:

$$\mathcal{R}_{\sigma,3}^\beta = [Z^\beta, g_1 U_z/\phi] Z_3 \nabla^\varphi \sigma + (g_1 U_z/\phi)[Z^\beta, \phi]\partial_z \nabla^\varphi \sigma + g_1 U_z [Z^\beta, \partial_z]\nabla^\varphi \sigma =: (1) + (2) + (3). \tag{7.49}$$

Thanks to the commutator estimate (3.9), we have:

$$\varepsilon^{\frac{1}{2}}\|(1)\|_{L_t^2 L^2} \lesssim \varepsilon^{\frac{1}{2}}\|\nabla^\varphi \sigma\|_{L_t^2 H_{co}^{m-1}} \|g_1 U_z/\phi\|_{[\frac{m+1}{2}],\infty,t} + \varepsilon^{\frac{1}{2}}\|g_1 U_z/\phi\|_{L_t^2 H_{co}^{m-1}} \|\nabla^\varphi \sigma\|_{[\frac{m}{2}]-1,\infty,t}.$$

Note that as U_z vanishes on the boundary, we have by Hardy's inequality,

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|g_1 U_z/\phi\|_{L_t^2 H_{co}^{m-1}} &\lesssim \varepsilon^{\frac{1}{2}}\|\partial_z(g_1 U_z)\|_{L_t^2 H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\|g_1 U_z\|_{L_t^2 H_{co}^{m-1}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)(\|(\sigma, u, \nabla\sigma, \mathrm{div} u)\|_{L_t^2 H_{co}^{m-1}} + |\varepsilon^{\frac{1}{2}} h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}} + |\varepsilon^{\frac{1}{2}} \partial_t h|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}).\end{aligned}$$

Moreover, as for (7.36), the fundamental theorem of calculus leads to:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|g_1 U_z/\phi\|_{[\frac{m+1}{2}],\infty,t} &\lesssim \varepsilon^{\frac{1}{2}}\|(U_z, \partial_z U_z)\|_{[\frac{m+1}{2}],\infty,t}(1 + \|Z g_1\|_{[\frac{m-1}{2}],\infty,t}) \\ &\lesssim \Lambda\big(\frac{1}{c_0}, \varepsilon^{\frac{1}{2}}\|(\sigma, u)\|_{[\frac{m+3}{2}],\infty,t} + \varepsilon^{\frac{1}{2}}\|\mathrm{div}^\varphi u)\|_{[\frac{m+1}{2}],\infty,t} \\ &\quad + |\varepsilon^{\frac{1}{2}} h|_{[\frac{m+5}{2}],\infty,t} + |\varepsilon^{\frac{1}{2}} \partial_t h|_{[\frac{m+3}{2}],\infty,t} + \|(\sigma, u)\|_{[\frac{m}{2}],\infty,t} + |(h, \partial_t h)|_{[\frac{m}{2}]+1}\big).\end{aligned}$$

In view of Equation (1.17) and the definition (1.33), we conclude:

$$\varepsilon^{\frac{1}{2}}\|g_1 U_z/\phi\|_{[\frac{m+1}{2}],\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.50}$$

We thus obtain that:

$$\varepsilon^{\frac{1}{2}}\|(1)\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.51}$$

It remains to estimate $(2),(3)$ in (7.49). By induction, one has up to some smooth function which depends only on ϕ and its derivatives,

$$[Z^\beta,\phi]=\sum_{\gamma<\beta,|\gamma|\leq|\beta|-1}*_{\beta,\gamma}Z^\gamma\phi,$$

The above identity, combined with (3.27), (7.50) yields:

$$\varepsilon^{\frac{1}{2}}\|(2)+(3)\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\|\nabla^\varphi\sigma/\varepsilon^{\frac{1}{2}}\|_{L_t^2H_{co}^{m-1}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).$$

To summarize, we have obtained:

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma,3}^\beta\|_{L_t^2L^2}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\varepsilon^{\frac{1}{2}}\mathcal{E}_{m,t}. \tag{7.52}$$

Collecting (7.47)-(7.52), we thus arrive at:

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma}^\beta\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.53}$$

To finish the estimates of the right hand side of (7.46), it remains to control $\mathcal{R}_u^\beta$ which is defined in (7.32). We first find, in a similar way as for the control of $\mathcal{R}_\sigma^\beta$, that:

$$\varepsilon^{\frac{1}{2}}\|(\mathcal{R}_{u,1}^\beta+\mathcal{R}_{u,2}^\beta)\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{7.54}$$

From the identities:

$$[Z^\beta,\mathrm{curl}^\varphi]\omega=[Z^\beta,\frac{\mathbf{N}}{\partial_z\varphi}\partial_z]\times\omega,\quad [Z^\beta,\nabla^\varphi]\mathrm{div}^\varphi u=[Z^\beta,\frac{\mathbf{N}}{\partial_z\varphi}\partial_z]\mathrm{div}^\varphi u,$$

$\mathcal{R}_{u,3}^\beta$ can be treated thanks to (3.23) as:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{u,3}^\beta\|_{L_t^2L^2}&\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+\varepsilon^{\frac{1}{2}}\|\partial_z(\omega,\mathrm{div}^\varphi u)\|_{1,\infty,t}\big)\big(\varepsilon^{\frac{1}{2}}\|\partial_z(\omega,\mathrm{div}^\varphi u)\|_{L_t^2H_{co}^{m-2}}+|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big)\\&\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\big(\varepsilon^{\frac{1}{2}}\|\nabla^2u\|_{L_t^2H_{co}^{m-2}}+\varepsilon^{\frac{1}{2}}\|u\|_{E^m,t}+|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big).\end{aligned} \tag{7.55}$$

Combining (7.54) and (7.55), one finds that:

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_u^\beta\|_{L_t^2L^2}\lesssim(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{7.56}$$

Plugging (7.53) and (7.56) into (7.46), we finally get that:

$$\varepsilon|J_7^\beta|\leq\delta\varepsilon^2\|\nabla^\varphi Z^\beta\mathrm{div}^\varphi u\|_{L_t^2L^2}^2+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2. \tag{7.57}$$

Collecting (7.34)-(7.42), (7.45), (7.57), and summing up for $k\leq m-1$, we find that by choosing δ small enough,

$$\varepsilon\|(\nabla^\varphi\sigma,\mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-1}}^2+\varepsilon\|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}}^2$$

$$\lesssim \varepsilon\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)(0)\|_{H^{m-1}_{co}}^2 + \Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla^\varphi u\|_{L^2_t H^m_{co}}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

This inequality, combined with (7.1) leads to (7.30).

□

8. Control of the low-order energy norms

This section is devoted to the control of the lower order term $\mathcal{E}_{low,T}$. and

$$\mathcal{E}_{low,T} = \varepsilon^{\frac{1}{2}}\|\partial_t(\sigma,u)\|_{L^\infty_t L^2} + \varepsilon^{\frac{1}{2}}\|(\sigma,u)\|_{L^\infty_t H^3} + \varepsilon^{\frac{3}{2}}\|\nabla^4 u\|_{L^2_t L^2}. \tag{8.1}$$

Except the first norm, the other norms appearing in $\mathcal{E}_{low,T}$ are indeed not crucial to get an estimate uniformly in ε. Nevertheless, their presence allows us to take benefit of the known local existence results [57, 58, 64] (see Theorem 13.1 in Section 13).

Lemma 8.1. *Under the assumption* (2.2)*, the following estimate holds:*

$$\mathcal{E}_{low,T}^2 \leq \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,T}^2\big)(Y_m^2(0) + \mathcal{E}_{high,m,T}^2) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{8.2}$$

Proof. This lemma is the consequence of the following three lemmas. □

The first term in $\mathcal{E}_{low,T}$ is estimated in the next lemma. Before stating the result, it is convenient to introduce the notation:

$$\Lambda_{2,\infty,t} = \Lambda\big(\frac{1}{c_0}, \|\!|(\sigma,u)\|\!|_{2,\infty,t} + \varepsilon^{-\frac{1}{2}}\|\!|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|\!|_{1,\infty,t} + \varepsilon^{\frac{1}{2}}\|\!|\nabla^2 u\|\!|_{0,\infty,t} + |h|_{3,\infty,t}\big), \tag{8.3}$$

where Λ denotes a polynomial that may differ from line to line. Note that by the equation for h (1.17), we have:

$$|\partial_t h|_{2,\infty,t} \lesssim \Lambda_{2,\infty,t}. \tag{8.4}$$

Lemma 8.2. *Assuming that* (2.2) *holds true, then for every* $0 < t \leq T$*, we have the following estimate,*

$$\varepsilon\|\partial_t(\sigma,u)\|_{L^\infty_t L^2}^2 + \varepsilon\|\nabla\partial_t u\|_{L^2_t L^2}^2 \lesssim \varepsilon\|\partial_t(\sigma,u)(0)\|_{L^2(\mathcal{S})}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda_{2,\infty,T}\mathcal{E}_{m,T}^2. \tag{8.5}$$

Proof. Denote $Z_0 = \varepsilon\partial_t$. Applying ∂_t^φ (resp. ∂_t) on $(1.16)_1$ (resp.$(1.16)_2$), one gets that:

$$\begin{cases} g_1(\partial_t^\varphi + \underline{u}\cdot\nabla)(\partial_t\sigma) + \dfrac{1}{\varepsilon}\partial_t^\varphi \mathrm{div}^\varphi u = \mathcal{T}_\sigma \\ g_2(\partial_t + \underline{u}\cdot\nabla)(\partial_t^\varphi u) + \dfrac{1}{\varepsilon}\partial_t\nabla^\varphi\sigma - \mathrm{div}^\varphi(\partial_t\mathcal{L}^\varphi u) = \mathcal{T}_u \end{cases} \tag{8.6}$$

where

$$\mathcal{T}_\sigma = \mathcal{T}_\sigma^1 + \mathcal{T}_\sigma^2 + \mathcal{T}_\sigma^3, \quad \mathcal{T}_u = \mathcal{T}_u^1 + \mathcal{T}_u^2 + \mathcal{T}_u^3 + \mathcal{T}_u^4 \tag{8.7}$$

with the following definitions:

$$\mathcal{T}_\sigma^1 = \big(\frac{\partial_t^\varphi g_1}{\varepsilon}\big)(\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)\sigma, \quad \mathcal{T}_\sigma^2 = g_1[\partial_t, \underline{u}\cdot\nabla]\sigma, \quad \mathcal{T}_\sigma^3 = -\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z(\underline{u}\cdot\nabla\sigma),$$

$$\mathcal{T}_u^1 = \big(\frac{\partial_t g_2}{\varepsilon}\big)(\partial_t + \underline{u}\cdot\nabla)u, \quad \mathcal{T}_u^2 = g_2\partial_t\underline{u}\cdot\nabla u,$$

$$\mathcal{T}_u^3 = [\partial_t, \mathrm{div}^\varphi]\mathcal{L}^\varphi u, \quad \mathcal{T}_u^4 = -g_2(\partial_t + \underline{u}\cdot\nabla)\big(\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z u\big).$$

where $\underline{u} = (u_1, u_2, U_z)$ and U_z is defined in (7.6). Taking the scalar product of (8.6) and $\varepsilon\big(\partial_t\sigma, \partial_t^\varphi u\big)^t$, integrating in space and time, we get by using Lemma 3.12 that

$$\begin{aligned}\frac{\varepsilon}{2}\int_{\mathcal{S}} g_1|\partial_t\sigma|^2(t) + g_2|\partial_t^\varphi u|^2(t)\,\mathrm{d}\mathcal{V}_t - \varepsilon\int_0^t\int_{\mathcal{S}} \mathrm{div}^\varphi(\partial_t\mathcal{L}^\varphi u)\partial_t^\varphi u(s)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s \\ = I_0 + I_1 + \cdots I_4\end{aligned} \tag{8.8}$$

where

$$\begin{aligned} I_0 &= \frac{\varepsilon}{2}\int_{\mathcal{S}} g_1|\partial_t\sigma|^2(0) + g_2|\partial_t^\varphi u|^2(0)\,\mathrm{d}\mathcal{V}_0, \quad I_1 = \int_0^t\int_{\mathcal{S}} \partial_t\sigma\partial_t^\varphi \mathrm{div}^\varphi u + \partial_t\nabla^\varphi\sigma\cdot\partial_t^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\ I_2 &= \frac{\varepsilon}{2}\int_0^t\int_{z=0} g_1\partial_t h|\partial_t\sigma|^2\,\mathrm{d}y\mathrm{d}s, \quad I_3 = \frac{\varepsilon}{2}\int_0^t\int_{\mathcal{S}}\big(\partial_t^\varphi g_1 + \frac{1}{\partial_z\varphi}\mathrm{div}(g_1\underline{u}\partial_z\varphi)\big)|\partial_t\sigma|^2(s)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\ I_4 &= \varepsilon\int_0^t\int_{\mathcal{S}}\partial_t\sigma\mathcal{T}_\sigma + \partial_t^\varphi u\cdot\mathcal{T}_u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.\end{aligned}$$

We focus on the control of $I_1 - I_4$ in the following. Let us with I_1, which is the most involved one and explains why we need to perform energy estimate in this non-standard way. Let us integrate by parts in space to get:

$$I_1 = \int_0^t\int_{\mathcal{S}}\partial_t^\varphi u\cdot[\partial_t, \nabla^\varphi]\sigma\,\mathrm{d}\mathcal{V}_s\mathrm{d}s + \int_0^t\int_{z=0}\partial_t\sigma\partial_t^\varphi u\cdot\mathbf{N}\,\mathrm{d}y\mathrm{d}s =: I_{11} + B_1.$$

Since $[\partial_t, \nabla^\varphi]\sigma = [\partial_t, \frac{\mathbf{N}}{\partial_z\varphi}]\partial_z\sigma$, it follows from the Cauchy-Schwarz inequality that:

$$\begin{aligned}|I_{11}| &\lesssim \|\partial_t^\varphi u\|_{L_t^2L^2}\big\|\partial_z\sigma\big\|_{L_t^2L^2}\|\!|\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\|\!|_{0,\infty,t}\\ &\lesssim T^{\frac{1}{2}}\Lambda_{2,\infty,t}\varepsilon^{\frac{1}{2}}\|(\partial_t u, \nabla u)\|_{L_t^\infty L^2}\big\|\varepsilon^{-\frac{1}{2}}\nabla\sigma\big\|_{L_t^2L^2}.\end{aligned} \tag{8.9}$$

Note that $\Lambda_{2,\infty,t}$ is defined in (8.3). The boundary term B_1 combined with the boundary term arising from the integration by parts of the viscous term (in the right hand-side of (8.8)), lead to some cancellations, we thus first rewrite the viscous term:

$$\begin{aligned}-\varepsilon\int_0^t\int_{\mathcal{S}}\mathrm{div}^\varphi(\partial_t\mathcal{L}^\varphi u)\cdot\partial_t^\varphi u(s)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s &= \varepsilon\int_0^t\int_{\mathcal{S}}\partial_t\mathcal{L}^\varphi u\cdot\nabla^\varphi\partial_t^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ + \varepsilon a\int_0^t\int_{z=-1}|\partial_t u_\tau|^2\,\mathrm{d}y\mathrm{d}s &- \underbrace{\varepsilon\int_0^t\int_{z=0}\partial_t\mathcal{L}^\varphi u\,\mathbf{N}\cdot\partial_t^\varphi u\,\mathrm{d}y\mathrm{d}s}_{=:\,B_2}.\end{aligned} \tag{8.10}$$

In view of the boundary condition (1.18), the identities (4.9), (4.1) as well as the trace inequality (3.17), we have:

$$\begin{aligned}B_1 + B_2 &= -\varepsilon\int_0^t\int_{z=0}\partial_t^\varphi u\cdot\big(\mathcal{L}^\varphi u - \frac{\sigma}{\varepsilon}\mathrm{Id}_3\big)\partial_t\mathbf{N}\,\mathrm{d}y\mathrm{d}s\\ &\lesssim \varepsilon\big|\big(\mathcal{L}^\varphi u - \frac{\sigma}{\varepsilon}\mathrm{Id}_3\big)\partial_t\mathbf{N}\big|_{L_t^2L_y^2}|\partial_t^\varphi u|_{L_t^2L_y^2}\\ &\lesssim \varepsilon^{\frac{1}{2}}\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^2H^1}^2 + \|\varepsilon^{\frac{1}{2}}u\|_{L_t^2H^2}^2 + \|\nabla u\|_{L_t^2H_{co}^1}^2 + \|\nabla\mathrm{div}u\|_{L_t^2L^2}^2\big)\Lambda\big(\frac{1}{c_0}, |\partial_t h|_{0,\infty,t} + |h|_{1,\infty,t}\big)\\ &\lesssim \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.\end{aligned}$$

We can also estimate the first two terms in the right hand side of (8.10). By using Young's inequality and the fact $[\nabla^\varphi, \partial_t^\varphi]=0$,

$$
\begin{aligned}
&\varepsilon\int_0^t\int_{\mathcal{S}}\partial_t\mathcal{L}^\varphi u\cdot\nabla^\varphi\partial_t^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\
=&\varepsilon\int_0^t\int_{\mathcal{S}}\partial_t\mathcal{L}^\varphi u\cdot\big(\partial_t\nabla^\varphi u-\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\nabla^\varphi u)\mathrm{d}\mathcal{V}_s\mathrm{d}s\\
\geq&\varepsilon\int_0^t\int_{\mathcal{S}}2\mu|\partial_t S^\varphi u|^2+\lambda|\partial_t\mathrm{div}^\varphi u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s-\Lambda_{2,\infty,t}\|\varepsilon^{\frac12}\partial_t\mathcal{L}^\varphi u\|_{L_t^2L^2}\|\varepsilon^{\frac12}\partial_z\nabla^\varphi u\|_{L_t^2L^2}\\
\geq&\varepsilon\int_0^t\int_{\mathcal{S}}\mu|\partial_t S^\varphi u|^2+\frac{\lambda}{2}|\partial_t\mathrm{div}^\varphi u|^2\mathrm{d}\mathcal{V}_s\mathrm{d}s-C_{\mu,\lambda}T\Lambda_{2,\infty,t}\|\varepsilon^{\frac12}\nabla^\varphi u\|_{L_t^\infty H^1}^2
\end{aligned}
\tag{8.11}
$$

Moreover, by the trace inequality, we have

$$
\varepsilon\int_0^t\int_{z=-1}|\partial_t u_\tau|^2\,\mathrm{d}y\mathrm{d}s\leq\delta\varepsilon\|\partial_t\nabla^\varphi u\|_{L_t^2L^2}^2+TC_\delta\varepsilon\|(\partial_t u_\tau,\nabla u_\tau)\|_{L_t^\infty L^2}^2\Lambda_{2,\infty,t}. \tag{8.12}
$$

Therefore, we get by collecting (8.9)-(8.12) that:

$$
\begin{aligned}
&I_1+\varepsilon\int_0^t\int_{\mathcal{S}}\mathrm{div}^\varphi(\partial_t\mathcal{L}^\varphi u)\cdot\partial_t^\varphi u(s)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\
&\leq-\varepsilon\int_0^t\int_{\mathcal{S}}\mu|\partial_t S^\varphi u|^2+\frac{\lambda}{2}|\partial_t\mathrm{div}^\varphi u|^2\mathrm{d}\mathcal{V}_s\mathrm{d}s+\delta\varepsilon\|\partial_t\nabla^\varphi u\|_{L_t^2L^2}^2+(T+\varepsilon)^{\frac12}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.
\end{aligned}
\tag{8.13}
$$

We are now left to control I_2-I_4. The estimates of I_2, I_3 are direct, we write

$$
|I_2|\lesssim\varepsilon|\partial_t h|_{\infty,t}|\partial_t\sigma|_{z=0}|_{L_t^2L_y^2}^2,
$$

$$
|I_3|\lesssim\Lambda(\|\nabla(\sigma,u)\|_{\infty,t}+\|(\sigma,u)\|_{1,\infty,t}+|h|_{2,\infty,t})\|\varepsilon^{\frac12}\partial_t\sigma\|_{L_t^2L^2}^2. \tag{8.14}
$$

We remark that in view of the boundary condition (4.1), one has

$$
\partial_t\sigma|_{z=0}=\partial_t(\sigma|_{z=0})=\varepsilon\partial_t\big((2\mu+\lambda)\mathrm{div}^\varphi u-2\mu(\partial_1 u_1+\partial_2 u_2)+\mu(\omega\times\mathbf{N})_3|_{z=0}\big).
$$

Therefore, by the trace inequality (3.17), we have:

$$
\begin{aligned}
|I_2|\lesssim\varepsilon|\partial_t h|_{\infty,t}|\partial_t\sigma|_{L_t^2L_y^2}^2&\lesssim\varepsilon\big(\|\nabla\mathrm{div}^\varphi u\|_{L_t^2H_{co}^1}+\|(u,\nabla u)\|_{L_t^2H_{co}^2}+|h|_{L_t^2\tilde{H}^2}\big)\Lambda_{2,\infty,t}\\
&\lesssim\varepsilon\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.
\end{aligned}
\tag{8.15}
$$

As for the term I_4, it can be bounded directly by

$$
|I_4|\lesssim T^{\frac12}\big(\|\varepsilon^{\frac12}\mathcal{T}_\sigma\|_{L_t^2L^2}\|\varepsilon^{\frac12}\partial_t\sigma\|_{L_t^\infty L^2}+\|\varepsilon^{\frac12}\mathcal{T}_u\|_{L_t^2L^2}\|\varepsilon^{\frac12}\partial_t u\|_{L_t^\infty L^2}\big). \tag{8.16}
$$

It thus remains to control the commutators $\mathcal{T}_\sigma, \mathcal{T}_u$ defined in (8.7). By the explicit expression of $\mathcal{T}_\sigma, \mathcal{T}_u$, we can obtain that:

$$
\varepsilon^{\frac12}\|(\mathcal{T}_\sigma,\mathcal{T}_u)\|_{L_t^2L^2}\lesssim\Lambda_{2,\infty,t}(\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^2L^2}+\|\nabla(\sigma,u)\|_{L_t^2H_{co}^1})\lesssim\Lambda_{2,\infty,t}\mathcal{E}_{m,t}. \tag{8.17}
$$

For instance, since we have:

$$
\varepsilon^{\frac12}\mathcal{T}_\sigma^1=\varepsilon^{\frac12}\partial_t^\varphi(g_1/\varepsilon)(\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)\sigma,\quad \varepsilon^{\frac12}\mathcal{T}_u^1=\varepsilon^{\frac12}\partial_t(g_2/\varepsilon)(\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)u
$$

by:

$$
\|\varepsilon^{\frac12}(\mathcal{T}_\sigma^1+\mathcal{T}_u^1)\|\lesssim\Lambda_{1,\infty,t}(\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^2L^2}+\|\nabla(\sigma,u)\|_{L_t^2L^2})\lesssim\Lambda_{2,\infty,t}\mathcal{E}_{m,t}.
$$

Collecting (8.16)-(8.17), we obtain that

$$
|I_4|\leq T^{\frac12}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2. \tag{8.18}
$$

Now, in view of the estimates: (8.14)-(8.15), (8.13) (8.18), we get by choosing δ small enough, that

$$\begin{aligned}\frac{1}{2}\varepsilon\int_{\mathcal{S}} g_1|\partial_t\sigma|^2(t)+g_2|\partial_t^{\varphi}u|^2(t)\,\mathrm{d}\mathcal{V}_t+\varepsilon\int_0^t\int_{\mathcal{S}}\mu|\partial_t S^{\varphi}u|^2+\frac{\lambda}{2}|\partial_t\mathrm{div}^{\varphi}u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ \leq\frac{1}{2}\varepsilon\int g_1|\partial_t\sigma|^2(0)+g_2|\partial_t^{\varphi}u|^2(0)\,\mathrm{d}\mathcal{V}_0+\delta\varepsilon\|\nabla^{\varphi}\partial_t u\|_{L_t^2L^2}^2+(T+\varepsilon)^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.\end{aligned}\tag{8.19}$$

From an explicit commutator, We can write that:

$$\begin{aligned}&\int_0^t\int_{\mathcal{S}}\mu|\partial_t S^{\varphi}u|^2+\frac{\lambda}{2}|\partial_t\mathrm{div}^{\varphi}u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ &\geq\int_0^t\int_{\mathcal{S}}\mu|S^{\varphi}\partial_t u|^2+\frac{\lambda}{2}|\mathrm{div}^{\varphi}\partial_t u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s-\Lambda_{2,\infty,t}T^{\frac{1}{2}}\|\nabla u\|_{L_t^\infty L^2}^2.\end{aligned}$$

Hence, by using Korn's inequality (3.34) and by choosing δ small enough, we finally obtain (8.5). $\square$

The following two lemmas are devoted to the estimates of the other norms appearing in $\mathcal{E}_{low,T}$, for the proof of Lemma 8.1.

Lemma 8.3. *Suppose that* (2.2) *are holds, then we have for any* $0<t\leq T$,

$$\varepsilon\|\nabla^3\sigma\|_{L_t^\infty L^2}^2+\varepsilon^{-1}\|\nabla^3\sigma\|_{L_t^2L^2}^2+\varepsilon\|\nabla^2\sigma\|_{L_t^\infty H_{co}^1}^2+\varepsilon^{-1}\|\nabla^2\sigma\|_{L_t^2H_{co}^1}^2\lesssim Y_m^2(0)+(T+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.\tag{8.20}$$

Proof. By applying $\varepsilon^2\nabla^{\varphi}$ to the equation $(1.16)_1$ and expressing the term $\varepsilon\nabla^{\varphi}\mathrm{div}^{\varphi}u$ by using the velocity equations $(1.16)_2$, we find that $\nabla^{\varphi}\sigma$ solves

$$\varepsilon^2 g_1(\partial_t+\underline{u}\cdot\nabla)\nabla^{\varphi}\sigma+\frac{1}{2\mu+\lambda}\nabla^{\varphi}\sigma=\mathcal{Q}_1\tag{8.21}$$

where

$$\mathcal{Q}_1=-\varepsilon^2g_1'\nabla^{\varphi}\sigma(\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)\sigma-\varepsilon^2g_1\nabla^{\varphi}u\cdot\nabla^{\varphi}\sigma-\frac{\mu\varepsilon}{2\mu+\lambda}\,\mathrm{curl}^{\varphi}\,\omega-\frac{1}{2\mu+\lambda}g_2(\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)u.$$

Next, by taking div^{φ} of the equation (8.21), we find that $\Delta^{\varphi}\sigma$ solves:

$$\begin{aligned}\varepsilon^2g_1(\partial_t+\underline{u}\cdot\nabla)\Delta^{\varphi}\sigma+\frac{1}{2\mu+\lambda}\Delta^{\varphi}\sigma&=\mathrm{div}^{\varphi}\mathcal{Q}_1-\varepsilon^2g_1'\nabla^{\varphi}\sigma\cdot\varepsilon\partial_t\nabla^{\varphi}\sigma-\varepsilon^2\nabla^{\varphi}(g_1\underline{u})\cdot\nabla\nabla^{\varphi}\sigma\\ &=:\mathcal{H}\end{aligned}\tag{8.22}$$

Standard energy estimates for (8.22) yield:

$$\begin{aligned}&\varepsilon\|\Delta^{\varphi}\sigma\|_{L_t^\infty H_{co}^1}^2+\varepsilon^{-1}\|\Delta^{\varphi}\sigma\|_{L_t^2H_{co}^1}^2\\ &\lesssim\varepsilon\|\Delta^{\varphi}\sigma(0)\|_{H_{co}^1}^2+T\Lambda_{1,\infty,t}\varepsilon\|\Delta^{\varphi}\sigma\|_{L_t^\infty H_{co}^1}^2+T^{\frac{1}{2}}\|\varepsilon^{-\frac{1}{2}}\Delta^{\varphi}\sigma\|_{L_t^2H_{co}^1}(\|\varepsilon^{-\frac{1}{2}}\mathcal{H}\|_{L_t^\infty H_{co}^1}+\varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t})\\ &\lesssim T\Lambda_{1,\infty,t}\mathcal{E}_{low,t}^2+T^{\frac{1}{2}}(\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^1}+\varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t})\|\varepsilon^{-\frac{1}{2}}\Delta^{\varphi}\sigma\|_{L_t^2H_{co}^1}\end{aligned}$$

It thus follows from Young's inequality that

$$\varepsilon\|\Delta^{\varphi}\sigma\|_{L_t^\infty H_{co}^1}^2+\varepsilon^{-1}\|\Delta^{\varphi}\sigma\|_{L_t^2H_{co}^1}^2\lesssim Y_m^2(0)+T\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.$$

Moreover, we can get also that:

$$\begin{aligned}&\varepsilon\|\partial_z\Delta^{\varphi}\sigma\|_{L_t^\infty L^2}^2+\varepsilon^{-1}\|\partial_z\Delta^{\varphi}\sigma\|_{L_t^2L^2}^2\\ &\lesssim\varepsilon\|\partial_z\Delta^{\varphi}\sigma(0)\|_{L^2}^2+T\Lambda_{2,\infty,t}\big(\|\varepsilon\nabla^3\sigma\|_{L_t^\infty L^2}^2+\|\varepsilon^{\frac{1}{2}}\partial_t\nabla\mathrm{div}u\|_{L_t^\infty L^2}^2+\varepsilon\mathcal{E}_{m,t}^2\big)\\ &\lesssim Y_m^2(0)+T\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.\end{aligned}$$

Next, we see that:

$$\varepsilon\|\nabla^2\sigma\|^2_{L^\infty_t H^1_{co}} \lesssim \varepsilon\|\nabla\sigma\|^2_{L^\infty_t H^2_{co}} + \varepsilon\|\partial_z^2\sigma\|^2_{L^\infty_t L^2}$$

By the expressions of $\Delta^\varphi\sigma$,

$$\Delta^\varphi\sigma = \frac{|\mathbf{N}|^2}{\partial_z\varphi}\partial_z^2\sigma + \Delta_y\sigma + \partial_1(\mathbf{N}_1\partial_z^\varphi\sigma) + \partial_2(\mathbf{N}_2\partial_z^\varphi\sigma) + \mathbf{N}_1\partial_z^\varphi\partial_1\sigma + \mathbf{N}_2\partial_z^\varphi\partial_2\sigma + \frac{1}{2}\partial_z\sigma\partial_z\Big|\frac{\mathbf{N}}{\partial_z\varphi}\Big|^2, \tag{8.23}$$

Therefore,

$$\begin{aligned}\varepsilon\|\nabla^2\sigma\|^2_{L^\infty_t H^1_{co}} &\lesssim \varepsilon\Lambda(1/c_0, |h|_{3,\infty,t})\|\nabla\sigma\|^2_{L^\infty_t H^2_{co}} + \varepsilon\|\Delta^\varphi\sigma\|^2_{L^\infty_t H^1_{co}} \\ &\lesssim Y_m^2(0) + (T+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.\end{aligned}$$

Note that $|h|_{3,\infty,t}$ is included in the definition of $\Lambda_{2,\infty,t}$ (8.3). We have further that:

$$\varepsilon\|\nabla^3\sigma\|^2_{L^\infty_t L^2} \lesssim \varepsilon\|\partial_z\Delta^\varphi\sigma\|^2_{L^\infty_t L^2} + \varepsilon\Lambda_{2,\infty,t}\|\nabla^2\sigma\|^2_{L^\infty_t H^1_{co}} \lesssim Y_m^2(0) + (T+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.$$

In a similar way, the following estimate holds also:

$$\varepsilon^{-1}\|\nabla^3\sigma\|^2_{L^2_t L^2} + \varepsilon^{-1}\|\nabla^2\sigma\|^2_{L^2_t H^1_{co}} \lesssim Y_m^2(0) + T\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.$$

The proof of (8.20) is now finished. □

Remark 8.4. *In a similar way, one can also show that:*

$$\|\nabla^3\sigma\|_{L^2_t H^1_{co}} \lesssim Y_m(0) + (T+\varepsilon)^{\frac{1}{2}}\mathcal{E}_{m,t}. \tag{8.24}$$

Lemma 8.5. *Assume that* (2.2) *holds, then we have for any* $0 < t \le T$:

$$\begin{aligned}&\varepsilon^{-1}\|\nabla^2\sigma\|^2_{L^\infty_t L^2} + \varepsilon\|\nabla^3 u\|^2_{L^\infty_t L^2} + \varepsilon^3\|\nabla^4 u\|^2_{L^2_t L^2} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{3,\infty,t}\big)(\varepsilon\|\nabla^2\sigma\|^2_{L^\infty_t H^1_{co}} + \mathcal{E}^2_{high,m,t}) + (T+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.\end{aligned} \tag{8.25}$$

Proof. By taking div^φ on $(1.16)_2$, we see that σ solves the following elliptic problem:

$$\begin{cases} -\Delta^\varphi(\sigma/\varepsilon) = \mathrm{div}^\varphi G, \\ \sigma/\varepsilon = (2\mu+\lambda)\mathrm{div}^\varphi u - 2\mu(\partial_1 u_1 + \partial_2 u_2) - \mu(\omega\times\mathbf{N})_3 \quad \text{on} \quad \{z=0\}, \\ \partial_z^\varphi\sigma/\varepsilon = -G\cdot e_3 + \mu\,\mathrm{curl}^\varphi\,\omega\cdot e_3 \quad \text{on} \quad \{z=-1\}, \end{cases} \tag{8.26}$$

where

$$G = \bar\rho\partial_t^\varphi u + g_2 u\cdot\nabla^\varphi u + \frac{g_2-\bar\rho}{\varepsilon}\varepsilon\partial_t^\varphi u - (2\mu+\lambda)\nabla^\varphi\mathrm{div}^\varphi u. \tag{8.27}$$

Note that on the upper boundary we have boundary identity (4.2) for $\omega\times\mathbf{N}$ and on the bottom, we have

$$\mu\,\mathrm{curl}^\varphi\,\omega\times e_3 = \mu(\partial_1^\varphi\omega_2 - \partial_2^\varphi\omega_1) = a(\partial_1 u_1 + \partial_2 u_2). \tag{8.28}$$

Applying the elliptic estimate (5.10), we find that:

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla^2\sigma\|_{L^\infty_t L^2} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\big(\varepsilon^{\frac{1}{2}}\|(\mathrm{div}^\varphi G, G)\|_{L^\infty L^2} + |\varepsilon^{-\frac{1}{2}}\sigma^{b,1}|_{L^\infty_t H^{\frac{3}{2}}} + |\varepsilon^{-\frac{1}{2}}(\partial_{\mathbf{n}}\sigma)^{b,2}|_{L^\infty_t H^{\frac{1}{2}}}\big) \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)(\varepsilon^{\frac{1}{2}}\|\mathrm{div}^\varphi u\|_{L^\infty_t H^2} + \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L^\infty_t H^1_{co}} + \|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^\varphi u\|_{L^\infty_t L^2}) + \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)(\varepsilon^{\frac{1}{2}}\|\nabla^2\sigma\|_{L^\infty_t H^1_{co}} + \tilde{\mathcal{E}}_{m,t}) + \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t},\end{aligned}$$

where G is defined in (8.27). Note that by $(1.16)_1$ and the definition of $\mathcal{E}_{m,t}$,

$$\varepsilon^{\frac{1}{2}}\|\mathrm{div}^\varphi u\|_{L^\infty_t H^2} \lesssim \varepsilon^{\frac{1}{2}}\|\nabla^2\sigma\|_{L^\infty_t H^1_{co}} + \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}.$$

Next, we get by the equation of velocity (1.16) that:

$$\varepsilon\mu\Delta^{\varphi}u = g_2(\varepsilon\partial_t + \underline{u}\cdot\nabla)u - (\mu+\lambda)\nabla^{\varphi}\mathrm{div}^{\varphi}u + \nabla\sigma$$

Moreover, a direct computation shows that:

$$\Delta^{\varphi}u = \frac{|\mathbf{N}|^2}{|\partial_z\varphi|^2}\partial_z^2 u + \Delta_y u + \partial_1(\mathbf{N}_1\partial_z^{\varphi}u) + \partial_2(\mathbf{N}_2\partial_z^{\varphi}u) + \mathbf{N}_1\partial_z^{\varphi}\partial_1 u + \mathbf{N}_2\partial_z^{\varphi}\partial_2 u + \frac{1}{2}\partial_z u\partial_z\Big|\frac{\mathbf{N}}{\partial_z\varphi}\Big|^2. \tag{8.29}$$

By using the previous two identities successively, we find the following two estimates

$$\begin{aligned}\varepsilon^{\frac12}\|\nabla^3 u\|_{L_t^\infty L^2} &\lesssim \varepsilon^{\frac12}\|\partial_z\Delta^{\varphi}u\|_{L_t^\infty L^2} + \varepsilon^{\frac12}\|\nabla^2 u\|_{L_t^\infty H_{co}^1}\Lambda_{2,\infty,t}\\ &\lesssim \varepsilon^{-\frac12}\|\nabla\sigma\|_{L_t^\infty H^1} + \varepsilon^{\frac12}\|\nabla^2\sigma\|_{L_t^\infty H_{co}^1} + |h|_{L_t^\infty \tilde{H}^{\frac32}} + \|\varepsilon^{\frac12}\partial_t u\|_{L_t^\infty H^1} + \varepsilon^{\frac12}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}.\end{aligned}$$

and

$$\begin{aligned}\varepsilon^{\frac32}\|\nabla^4 u\|_{L_t^2 L^2} &\lesssim \varepsilon^{\frac32}\|\nabla^2\Delta^{\varphi}u\|_{L_t^2L^2} + \varepsilon^{\frac32}\big(\|\nabla^3 u\|_{L_t^2 H_{co}^1} + |h|_{L_t^2\tilde{H}^{\frac72}}\big)\Lambda_{2,\infty,t}\\ &\lesssim \varepsilon^{\frac12}\|\nabla^3\sigma\|_{L_t^2L^2} + \varepsilon^{\frac12}\|\nabla^2 u\|_{L_t^2\mathcal{H}^1} + \varepsilon^{\frac32}\|\nabla^3(\sigma,u)\|_{L_t^2H_{co}^1}\Lambda_{2,\infty,t} + \varepsilon\Lambda_{2,\infty,t}\mathcal{E}_{m,t}\\ &\lesssim \varepsilon^{\frac12}\|\partial_t u\|_{L_t^2\mathcal{H}^1} + \varepsilon^{\frac32}\|\nabla^3\sigma\|_{L_t^2H_{co}^1} + (T^{\frac12}+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}_{m,t}\\ &\lesssim Y_m(0) + (T+\varepsilon)^{\frac12}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}.\end{aligned}$$

Note that in the second estimate, (8.24) has been used in the derivation of the last inequality. □

As stated in the beginning, we can now finish the proof of Lemma 8.1 since gathering (8.5), (8.20) and (8.25) we finally obtain (8.2).

In the following several sections (Sections 9-11), we aim to show the estimate of high order norms $\mathcal{E}_{high,m,T}$ defined in (1.32).

9. Uniform control of high order energy norms-I

In this section, we focus on the uniform $L_t^2H_{co}^{m-1}$ estimates for $\nabla^{\varphi}(\sigma,u)$. We first bound the higher order norms for $(\nabla^{\varphi}\sigma, \mathrm{div}^{\varphi}u)$ by using elliptic estimates for σ and the equations to recover spatial derivatives from time derivatives iteratively. Then, we perform direct energy estimates for the incompressible part v ($v = \mathbb{P}_t u$ solves (5.4)) to get the uniform control for $\|\nabla^{\varphi}v\|_{L_t^2H_{co}^{m-1}}$ (and also $\|v\|_{L_t^\infty H_{co}^{m-1}}$ as a by-product).

9.1. Uniform estimates for the compressible part.

In this subsection, we focus on the uniform estimates of the compressible part of the solution. More precisely, we shall establish the estimate of $\|(\nabla^{\varphi}\sigma, \mathrm{div}^{\varphi}u)\|_{L_t^2H_{co}^{m-1}}$.

Lemma 9.1. *Suppose that* (2.2) *is true, we can find some polynomial* Λ*, such that, for any* $0<t\le T$,

$$\begin{aligned}&\varepsilon^{-1}\|(\nabla^{\varphi}\sigma,\mathrm{div}^{\varphi}u)\|^2_{L_t^2H_{co}^{m-1}} + \varepsilon^{-1}\|\nabla\mathrm{div}^{\varphi}u\|^2_{L_t^2H_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_T^\infty\tilde{H}^{m-\frac12}}\big)Y_m^2(0) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned} \tag{9.1}$$

More precisely, we have for any j,l *with* $j+l\le m-1$,

$$\begin{aligned}\varepsilon^{-\frac12}\|(\nabla^{\varphi}\sigma,\mathrm{div}^{\varphi}u)\|_{L_t^2\mathcal{H}^{j,l}} &\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\ &+ \big(\varepsilon^{\frac12}\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}} + \varepsilon^{\frac12}\|\nabla^{\varphi}u\|_{L_t^2H_{co}^m} + \varepsilon^{\frac12}\|\partial_t(\sigma,u)\|_{L_t^2\mathcal{H}^{m-1}}\big)\Lambda\big(\frac{1}{c_0},|h|_{L_T^\infty\tilde{H}^{m-\frac12}}\big).\end{aligned} \tag{9.2}$$

Proof. By using the equation $(1.16)_1$ for σ, we have:

$$\nabla \mathrm{div}^{\varphi} u = g_1(0)\varepsilon\partial_t\nabla\sigma + \varepsilon\nabla\big((\frac{g_1-g_1(0)}{\varepsilon}\varepsilon\partial_t\sigma) + g_1\underline{u}\cdot\nabla\sigma\big), \tag{9.3}$$

combined with the product estimate (3.8), this yields:

$$\varepsilon^{-\frac{1}{2}}\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-2}} \lesssim \varepsilon^{-\frac{1}{2}}\|\nabla\sigma\|_{L_t^2H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{9.4}$$

By (7.1), (7.19), (7.30), (9.4), we can derive (9.1) from (9.2). In what follows, we shall establish (9.2) by induction on the number of conormal spatial derivatives. Firstly, let us rewrite the equation $(1.16)_1$ as:

$$\mathrm{div}^{\varphi}u = g_1(0)\varepsilon\partial_t\sigma + \varepsilon\big(\frac{g_1-g_1(0)}{\varepsilon}\varepsilon\partial_t\sigma + g_1u\cdot\nabla\sigma\big), \tag{9.5}$$

By the product estimate (3.8), we obtain:

$$\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{m-1}} \lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^2\mathcal{H}^{m-1}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

Moreover, as σ solves by the elliptic problem (8.26), we can apply the elliptic estimate (5.34) with $b=\sigma^{b,1}$, $g=(\varepsilon\mu\,\mathrm{curl}^{\varphi}\,\omega\cdot e_3)^{b,2}\,F=\varepsilon PG$ (the vector G is defined in (8.27), the matrix P is defined in (5.3)) and the identity (8.28) to get:

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{m-1}} \lesssim{}& \Lambda\big(\frac{1}{c_0},|h|_{[\frac{m}{2}]+1,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}G\|_{L_t^2\mathcal{H}^{m-1}} + |\varepsilon^{-\frac{1}{2}}\sigma^{b,1}|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_y u^{b,2}|_{L_t^2\tilde{H}^{m-\frac{3}{2}}}\big)\\ &+\Lambda\big(\frac{1}{c_0},|h|_{[\frac{m}{2}]+1,\infty,t} + \|(\varepsilon^{-\frac{1}{2}}\nabla\sigma,\varepsilon^{\frac{1}{2}}G)\|_{[\frac{m}{2}]-1,\infty,t}\big)|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}.\end{aligned}$$

By the definition (8.27) of G and the product estimate (3.8),

$$\|\varepsilon^{\frac{1}{2}}G\|_{[\frac{m}{2}]-1,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big),$$

$$\|\varepsilon^{\frac{1}{2}}G\|_{L_t^2\mathcal{H}^{m-1}} \lesssim \varepsilon^{\frac{1}{2}}\big(\|\partial_t u\|_{L_t^2\mathcal{H}^{m-1}} + \|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}}\big) + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

Moreover, thanks to the identity (4.1) and the trace inequality (3.17), we have that:

$$\begin{aligned}&|\varepsilon^{-\frac{1}{2}}\sigma^{b,1}|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_y u^{b,2}|_{L_t^2\tilde{H}^{m-\frac{3}{2}}}\\ &\lesssim \Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}\big)\varepsilon^{\frac{1}{2}}(\|\nabla u\|_{L_t^2H_{co}^m} + \|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}}) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Gathering the previous four inequalities, we get (9.2) for $j\le m-1, l=0$. For a given integer l $(1\le l\le m-1)$, assuming now that (9.2) holds for $(j,l-1)$ with $j+l\le m-1$ we then prove that it is also true for (j,l) with $j+l\le m-1$. By equation (9.5) and the product estimate (3.8), we get:

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{j,l}} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^2\mathcal{H}^{j,l}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\ &\lesssim \|\varepsilon^{-\frac{1}{2}}\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{j+1,l-1}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big) \lesssim \text{R.H.S of (9.2)}.\end{aligned}$$

For the estimate of $\nabla^{\varphi}\sigma$, we first remark that in the elliptic equation (8.26), G (defined in (8.27)) can be simplified slightly by changing $\partial_t^{\varphi}u$ into $\partial_t^{\varphi}\nabla\Psi$, since $\mathrm{div}^{\varphi}v=0, \partial_t^{\varphi}v_3|_{z=-1}=0$. Denote thus

$$\tilde{G} = \bar{\rho}\partial_t^{\varphi}\nabla^{\varphi}\Psi + g_2u\cdot\nabla^{\varphi}u + \frac{g_2-\bar{\rho}}{\varepsilon}\varepsilon\partial_t^{\varphi}u - (2\mu+\lambda)\nabla^{\varphi}\mathrm{div}^{\varphi}u.$$

We can use again the elliptic estimate (5.34) to get that:

$$
\begin{aligned}
\varepsilon^{-\frac12}\|\nabla^\varphi\sigma\|_{L_t^2\mathcal{H}^{j,l}} &\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\
&\quad+\Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac12}}\big)\big(\|\varepsilon^{\frac12}\tilde{G}\|_{L_t^2\mathcal{H}^{j,l}}+\varepsilon^{\frac12}(\|\nabla u\|_{L_t^2H_{co}^m}+\|\nabla\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}})\big)\\
&\lesssim \Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac12}}\big)\varepsilon^{\frac12}\big(\|\partial_t\nabla^\varphi\Psi\|_{L_t^2\mathcal{H}^{j,l}}+\|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}}+\|\nabla^\varphi u\|_{L_t^2H_{co}^m}\big)+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Since Ψ solves the elliptic problem (5.29), we can apply the elliptic estimate (5.15) and the estimate (5.31) to get that:

$$
\|\varepsilon^{\frac12}\partial_t\nabla^\varphi\Psi\|_{L_t^2\mathcal{H}^{j,l}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{[\frac{m}{2}]+1,\infty,t}\big)\varepsilon^{-\frac12}\|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{j+1,l-1}}+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
$$

Combining the two previous inequalities and using the induction assumption to estimate $\|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{j+1,l-1}}$, one finds:

$$
\varepsilon^{-\frac12}\|\nabla^\varphi\sigma\|_{L_t^2\mathcal{H}^{j,l}}\lesssim \text{R.H.S of (9.2)}.
$$

□

9.2. **Energy estimates: Incompressible part.** In this subsection, we focus on the analysis of the incompressible part of the velocity $v=\mathbb{P}_t u$ whose estimates can be obtained from direct energy estimates. By (5.4)-(5.6), v solves the following system:

$$
\begin{cases}
\bar\rho\partial_t^\varphi v-\mu\Delta^\varphi v+\nabla^\varphi\pi=-(f+\nabla^\varphi q+\bar\rho[\mathbb{P}_t,\partial_t^\varphi]u),\\
(2\mu S^\varphi v-\pi Id)\mathbf{N}|_{z=0}=2\mu(\mathrm{div}^\varphi u\mathrm{Id}-(\nabla^\varphi)^2\Psi)\mathbf{N}|_{z=0},\\
v_3|_{z=-1}=0,\quad \mu\partial_z^\varphi v_j|_{z=-1}=au_j|_{z=-1},\quad j=1,2.
\end{cases}\tag{9.6}
$$

where

$$
\nabla^\varphi\pi=\mathbb{P}_t\nabla^\varphi(\sigma/\varepsilon-2(\mu+\lambda)\mathrm{div}^\varphi u)=:\mathbb{P}_t\nabla^\varphi\theta,\tag{9.7}
$$

$$
f=\frac{g_2-\bar\rho}{\varepsilon}(\varepsilon\partial_t^\varphi u+\varepsilon u\cdot\nabla^\varphi u)+\bar\rho u\cdot\nabla^\varphi u,\qquad \nabla^\varphi q=-\mathbb{Q}_t(f-\mu\Delta^\varphi v).\tag{9.8}
$$

Before stating the main result for v, it is useful to establish some auxiliary estimates for $\nabla^\varphi\pi, f, \nabla^\varphi q$.

Proposition 9.2. *Under the assumption* (2.2), *the following $L_t^2L^2(\mathcal{S})$ type estimates hold: for any $m\geq 7$,*

$$
\|f\|_{L_t^2H_{co}^{m-1}}+\|\mathrm{div}^\varphi f\|_{L_t^2H_{co}^{m-2}}+\varepsilon^{\frac12}\|\partial_t f\|_{L_t^2H_{co}^{m-2}}+\varepsilon^{\frac12}\|f\|_{L_t^\infty H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),\tag{9.9}
$$

$$
\|\nabla q\|_{L_t^2H_{co}^{m-1}}+\varepsilon^{\frac12}\|\nabla^\varphi q\|_{L_t^\infty H_{co}^{m-2}}+\varepsilon^{\frac12}\|\partial_t\nabla^\varphi q\|_{L_t^2H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),\tag{9.10}
$$

$$
\|\nabla\pi\|_{1,\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{5,\infty,t}\big)\mathcal{E}_{m,T},\tag{9.11}
$$

$$
\|\nabla\pi\|_{L_t^2H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\nabla u\|_{L_t^2H_{co}^{m-1}}+T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),\tag{9.12}
$$

$$
\varepsilon^{\frac12}\|\nabla^\varphi\pi\|_{L_t^\infty H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\nabla u\|_{L_t^\infty H_{co}^{m-1}}+\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),\tag{9.13}
$$

$$
\varepsilon^{\frac12}\|\partial_t\nabla\pi\|_{L_t^2H_{co}^{m-3}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\partial_t(u,\nabla u)\|_{L_t^2H_{co}^{m-2}}+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),\tag{9.14}
$$

$$(9.15)\qquad \|[\mathbb{P}_t, \partial_t^\varphi]u\|_{L_t^2 H_{co}^{m-1}} + \|[\mathbb{P}_t, \partial_t^\varphi]u\|_{L_t^\infty H_{co}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t[\mathbb{P}_t, \partial_t^\varphi]u\|_{L_t^2 H_{co}^{m-2}} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Proof. Proof of (9.9). In view of definition of f in (9.8), we give details for the estimate of $u\cdot\nabla^\varphi u$ and $\mathrm{div}^\varphi(u\cdot\nabla^\varphi u)$, the other terms can be controlled in a similar way. First, for the $L_t^\infty H_{co}^{m-2}$ norm, we have thanks to the product estimate (3.8) that:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|u\cdot\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-2}} &\lesssim \Lambda\big(\frac{1}{c_0}, \|u\|_{[\frac{m}{2}],\infty,t} + \varepsilon^{\frac{1}{2}}\|\nabla u\|_{[\frac{m}{2}]-1,\infty,t}\big)\|(u, \varepsilon^{\frac{1}{2}}\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}_{m,T}.\end{aligned}$$

For the first three norms in the left-hand side of (9.9), we first have by the product estimate (3.14),

$$\begin{aligned}&\|u\cdot\nabla^\varphi u\|_{L_t^2\mathcal{H}^{0,m-1}} + \|\mathrm{div}^\varphi(u\cdot\nabla u)\|_{L_t^2\mathcal{H}^{0,m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \|(u,\nabla^\varphi u)\|_{0,\infty,t} + \|\nabla\mathrm{div}^\varphi u\|_{1,\infty,t}\big)\big(\|(u,\nabla^\varphi u)\|_{L_t^2\mathcal{H}^{0,m-1}} + \|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{0,m-2}}\big).\end{aligned}$$

It remains to control $\|\varepsilon\partial_t\mathrm{div}^\varphi(u\cdot\nabla^\varphi u)\|_{L_t^2 H_{co}^{m-3}}$ and $\varepsilon^{\frac{1}{2}}\|\partial_t(u\cdot\nabla^\varphi u)\|_{L_t^2 H_{co}^{m-2}}$. We can estimate them in a rather rough way:

$$\begin{aligned}&\|\varepsilon\partial_t\mathrm{div}^\varphi(u\cdot\nabla^\varphi u)\|_{L_t^2 H_{co}^{m-3}} \lesssim \|(\varepsilon\partial_t\nabla^\varphi u\cdot\nabla^\varphi u, \varepsilon\partial_t(u\cdot\nabla^\varphi\mathrm{div}^\varphi u))\|_{L_t^2 H_{co}^{m-3}}\\ &\lesssim \|\nabla^\varphi u\|_{0,\infty,t}\|\nabla^\varphi u\|_{L_t^2 H_{co}^{m-2}} + \|\varepsilon\partial_t\nabla^\varphi u\|_{0,\infty,t}\|\nabla^\varphi u\|_{L_t^2 H_{co}^{m-3}}\\ &\quad + \varepsilon^{\frac{1}{2}}\|\partial_t\nabla^\varphi u\|_{L_t^2 H_{co}^{m-4}}\|\varepsilon^{\frac{1}{2}}\nabla^\varphi u\|_{m-4,\infty,t}\\ &\quad + \|\nabla^\varphi\mathrm{div}^\varphi u\|_{[\frac{m}{2}]-2,\infty,t}\|u\|_{L_t^2 H_{co}^{m-2}} + \|u\|_{[\frac{m-1}{2}],\infty,t}\|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,T},\end{aligned}$$

$$\begin{aligned}&\varepsilon^{\frac{1}{2}}\|\partial_t(u\cdot\nabla^\varphi u)\|_{L_t^2 H_{co}^{m-2}} \lesssim \|(u\cdot\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi u, \varepsilon^{\frac{1}{2}}\partial_t u\cdot\nabla^\varphi u)\|_{L_t^2 H_{co}^{m-2}}\\ &\lesssim \|u\|_{1,\infty,t}\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi u\|_{L_t^2 H_{co}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-4}}\Big(\int_0^t \|u(s)\|_{m-2,\infty}^2\mathrm{d}s\Big)^{\frac{1}{2}}\\ &\quad + \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^2 H_{co}^{m-2}}\|\nabla^\varphi u\|_{0,\infty,t} + \|\nabla^\varphi u\|_{L_t^2 H_{co}^{m-2}}\Big(\int_0^t \|\varepsilon^{\frac{1}{2}}\partial_t u(s)\|_{m-3,\infty}\mathrm{d}s\Big)^{\frac{1}{2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

Proof of (9.10) Let us now show the estimate (9.10) for q. By the definition of $\mathbb{Q}_t$ in (5.2) and the fact that $\mathrm{div}^\varphi\Delta^\varphi v = 0$, q solves the elliptic problem:

$$\begin{cases}\mathrm{div}(E\nabla q) = -\mathrm{div}(Pf),\\ q|_{z=0} = 0,\\ \partial_z^\varphi q|_{z=-1} = -f\cdot e_3|_{z=-1} + g\end{cases}$$

where P and E are defined in (5.8) and $g = (\Delta^\varphi v_3)^{b,2} = \Delta^\varphi v_3|_{z=-1}$. Applying the elliptic estimate (5.35), (5.10) for $F = f$, we find:

$$(9.16)\qquad \begin{aligned}\|\nabla q\|_{L_t^2 H_{co}^{m-1}} \lesssim \Lambda\big(&\frac{1}{c_0}, |h|_{m-2,\infty,t} + \|\mathrm{div}^\varphi f\|_{L_t^\infty H_{tan}^2} + \big|(\Delta^\varphi v_3)^{b,2}\big|_{L_t^\infty H_{tan}^{\frac{5}{2}}}\big)\\ &\big(\|f\|_{L_t^2 H_{co}^{m-1}} + |h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + \big|(\Delta^\varphi v_3)^{b,2}\big|_{L_t^2\tilde{H}^{m-\frac{3}{2}}}\big),\end{aligned}$$

$$\varepsilon^{\frac{1}{2}}\|\nabla q\|_{L_t^\infty H_{co}^{m-2}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t} + \|\mathrm{div}^\varphi f\|_{L_t^\infty H_{tan}^1} + \big|(\Delta^\varphi v_3)^{b,2}\big|_{L_t^\infty H_{tan}^{\frac{3}{2}}}\big)$$
$$(\varepsilon^{\frac{1}{2}}\|f\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}|(\Delta^\varphi v)^{b,2}|_{L_t^\infty \tilde{H}^{m-\frac{5}{2}}} + \varepsilon^{\frac{1}{2}}|h|_{L_t^\infty \tilde{H}^{m-\frac{3}{2}}}), \tag{9.17}$$

$$\begin{aligned}&\varepsilon^{\frac{1}{2}}\|\partial_t \nabla q\|_{L_t^2 H_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t} + \|\varepsilon^{-\frac{1}{2}}\mathrm{div}^\varphi f\|_{L_t^\infty H_{co}^2} + \big|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)(\Delta^\varphi v_3)^{b,2}\big|_{L_t^\infty H_{tan}^{\frac{5}{2}}}\big)\cdot\\ &\quad(\varepsilon^{\frac{1}{2}}\|\partial_t f\|_{L_t^2 H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}|\partial_t(\Delta^\varphi v)^{b,2}|_{L_t^\infty \tilde{H}^{m-\frac{3}{2}}} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}} + \|\nabla q\|_{L_t^2 H_{co}^{m-2}}).\end{aligned} \tag{9.18}$$

It follows from direct computations that:

$$\Delta^\varphi v_3 = \Delta^\varphi u_3 - \partial_z^\varphi \mathrm{div}^\varphi u = (\partial_1^\varphi)^2 u_3 + (\partial_2^\varphi)^2 u_3 - (\partial_1^\varphi \partial_z^\varphi u_1 + \partial_2^\varphi \partial_z^\varphi u_2).$$

This, combined with the identities

$$\partial_1^\varphi|_{z=-1} = \partial_1, \quad \partial_2^\varphi|_{z=-1} = \partial_2$$

as well as the boundary condition (1.19), yields:

$$(\Delta^\varphi v_3)^{b,2} = -\frac{a}{\mu}(\partial_1 u_1 + \partial_2 u_2)^{b,2}. \tag{9.19}$$

In light of (9.9), (9.16)-(9.18), (9.19), we find (9.10) by the trace inequality (3.17).

Proof of (9.12)-(9.14). Let us switch to the estimate of π. By definition, π satisfies the following elliptic problem:

$$\begin{cases} \mathrm{div}(E\nabla\pi) = 0,\\ \pi|_{z=0} = \theta^{b,1},\\ \partial_z^\varphi \pi|_{z=-1} = 0. \end{cases}$$

where $\theta^{b,1} = \theta|_{z=0}$. Therefore, to prove (9.11), we apply (5.33) to get that:

$$\begin{aligned}\|\nabla\pi\|_{1,\infty,t} &\lesssim \|\nabla^2\pi\|_{L_t^\infty H_{tan}^2} + \|\nabla\pi\|_{L_t^\infty H_{tan}^3}\\ &\lesssim \Lambda(\frac{1}{c_0}, |h|_{4,\infty,t})|\theta^{b,1}|_{L_t^\infty H^{\frac{7}{2}}}.\end{aligned}$$

By using the boundary conditions (4.1) (4.2), we have that on the upper boundary,

$$\theta = -2\mu(\partial_1 u_1 + \partial_2 u_2) - 2\mu(\Pi(\partial_1 u \cdot \mathbf{N}, \partial_2 u \cdot \mathbf{N}, 0)^t)_3, \tag{9.20}$$

hence, by the product estimate (3.4) and the trace inequality (3.17), we get:

$$|\theta^{b,1}|_{L_t^\infty H^{\frac{7}{2}}} \lesssim (\|\nabla u\|_{L_t^\infty H_{co}^4} + \|u\|_{L_t^\infty H_{co}^5})\Lambda(\frac{1}{c_0}, |h|_{5,\infty,t}).$$

This ends the proof of (9.11).

Now, we can apply (5.35) and (9.11) to get that for $p = 2, +\infty$,

$$\|\nabla^\varphi \pi\|_{L_t^p H_{co}^{m-2}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)|\theta^{b,1}|_{L_t^p \tilde{H}^{m-\frac{3}{2}}} + |h|_{L_t^p \tilde{H}^{m-\frac{3}{2}}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \tag{9.21}$$

In view of (9.20), one has by the product estimate (3.4) and the trace inequality (3.17) that

$$|\theta^{b,1}|_{L_t^p \tilde{H}^{m-k+\frac{1}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+1,\infty,t}\big)\|\nabla u\|_{L_t^p H_{co}^{m-1}} + \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)|h|_{L_t^p \tilde{H}^{m-\frac{1}{2}}}, \tag{9.22}$$

which, combined with (9.21), yields (9.12)-(9.13). Finally, for the estimate of (9.14), we use the elliptic estimate (5.36) to obtain that:

$$\varepsilon^{\frac{1}{2}}\|\partial_t\nabla^\varphi\pi\|_{L_t^2H_{co}^{m-3}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\big(|\varepsilon^{\frac{1}{2}}\partial_t\theta^{b,1}|_{L_t^2\tilde{H}^{m-\frac{5}{2}}}+|\varepsilon^{\frac{1}{2}}(\Delta^\varphi v)^{b,2}|_{L_t^2\tilde{H}^{m-\frac{7}{2}}}\big)+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),$$

we thus obtain (9.14) by observing that:

$$|\varepsilon^{\frac{1}{2}}\partial_t\theta^{b,1}|_{L_t^2\tilde{H}^{m-\frac{7}{2}}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\varepsilon^{\frac{1}{2}}\|\partial_t(u,\nabla^\varphi u)\|_{L_t^2H_{co}^{m-2}}+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

Proof of (9.15). Finally, we estimate the commutator between the projection and the time derivative. Set $\nabla^\varphi\Psi_1=\mathbb{Q}_t\partial_t^\varphi u$, then

$$[\mathbb{P}_t,\partial_t^\varphi]=-[\mathbb{Q}_t,\partial_t^\varphi]=\nabla^\varphi(\Psi_1-\Psi).$$

By definition, $\Psi_1-\Psi$ solves the elliptic problem:

$$\Delta^\varphi(\Psi_1-\partial_t^\varphi\Psi)=0,\quad(\Psi_1-\partial_t^\varphi\Psi)|_{z=0}=\frac{\partial_t h}{\partial_z\varphi}\partial_z\Psi,\quad\partial_z^\varphi(\Psi_1-\partial_t^\varphi\Psi)|_{z=-1}=0.$$

It follows from (5.23) and the product estimate (3.14) that:

$$\begin{aligned}&\|\nabla^\varphi(\Psi_1-\partial_t^\varphi\Psi)\|_{L_t^2H_{co}^{m-1}}\\&\lesssim\Lambda\bigg(\frac{1}{c_0},|h|_{m-2,\infty,t}+\bigg|\frac{\partial_t h}{\partial_z\varphi}\partial_z\pi_1\bigg|_{L_t^\infty\tilde{H}^{\frac{5}{2}}}\bigg)\bigg|(h,\frac{\partial_t h}{\partial_z\varphi}\partial_z\Psi)\bigg|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\\&\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+|\partial_t h|_{3,\infty,t}+\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^2}\big)\\&\qquad\big(|\partial_t h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\|\nabla\Psi\|_{3,\infty,t}+\|(\nabla\Psi,\nabla^2\Psi)\|_{L_t^2H_{co}^{m-1}}|\partial_t h|_{m-3,\infty,t}\big).\end{aligned}\tag{9.23}$$

Combined with (5.25), (5.27), (5.31), this yields the control of the first quantity in (9.15). The second quantity can be controlled in a similar way, we omit the proof. □

Lemma 9.3. *Suppose that $m\geq 7$ and (2.2) holds, then we have the following high order energy estimate for v: for every $0<t\leq T$,*

$$\|v\|^2_{L_t^\infty H_{co}^{m-1}}+\|\nabla^\varphi v\|^2_{L_t^2H_{co}^{m-1}}\leq\Lambda\big(\frac{1}{c_0},|h|^2_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}+Y_m^2(0)\big)Y_m^2(0)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\tag{9.24}$$

Remark 9.4. *By using the elliptic estimates (5.11) and (5.31), we have:*

$$\|\nabla^\varphi\Psi(0)\|_{H_{co}^{m-1}}\lesssim\Lambda\big(\frac{1}{c_0},\tilde{Y}_{[\frac{m}{2}]}(0)\big)\big(\|u(0)\|_{H_{co}^{m-1}}+|h(0)|_{\tilde{H}^{m-\frac{1}{2}}}\big)$$

where $\tilde{Y}_{[\frac{m}{2}]}(0)=\|(\mathrm{div}^\varphi u)(0)\|_{H_{co}^{[\frac{m}{2}]}(\mathcal{S})}+\sum_{|\alpha|\leq[\frac{m}{2}]+1}|(Z^\alpha h)(0)|_{L^\infty(\mathbb{R}^2)}\lesssim Y_m(0)$.

Since $v=u-\nabla^\varphi\Psi$, we thus get:

$$\|(v,\nabla^\varphi\Psi)(0)\|_{H_{co}^{m-1}}\lesssim\Lambda\big(\frac{1}{c_0},Y_m(0)\big)Y_m(0).$$

Remark 9.5. *By the control of normal derivative of the compressible part (5.25), (9.1) and of the incompressible part (9.24), one deduces that:*

$$\|\nabla^\varphi u\|^2_{L_t^2H_{co}^{m-1}}\lesssim\Lambda\big(\frac{1}{c_0},|h|^2_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}\big)Y_m^2(0)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\tag{9.25}$$

Proof. Let $\alpha=(\alpha_0,\alpha'), |\alpha|=k\le m-1$. We can assume that Z^α contains at least one spatial vector field (ie. $|\alpha'|\neq 0$), since $\|v\|_{L_t^\infty\mathcal{H}^{m-1}}$ and $\|\nabla^\varphi v\|_{L_t^2\mathcal{H}^{m-1}}$ can be derived directly from the norms that have been bounded. Indeed, one has by elliptic estimates (5.23) and (5.13) that

$$\|v\|_{L_t^\infty\mathcal{H}^{m-1}}\lesssim\|(u,\nabla^\varphi\Psi)\|_{L_t^\infty\mathcal{H}^{m-1}}\lesssim\|u\|_{L_t^\infty\mathcal{H}^{m-1}}\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

$$\|\nabla v\|_{L_t^2\mathcal{H}^{m-1}}\lesssim\|(u,\nabla^\varphi\Psi)\|_{L_t^2\mathcal{H}^{m-1}}\lesssim\Lambda(\frac{1}{c_0},|h|_{m-2,\infty,t})\|\nabla u\|_{L_t^2\mathcal{H}^{m-1}}+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

Applying Z^α to $(9.6)_1$, we obtain:

$$\begin{aligned}&\bar\rho\partial_t^\varphi Z^\alpha v-2\mu\mathrm{div}^\varphi Z^\alpha S^\varphi v+\nabla^\varphi Z^\alpha\pi\\&=-Z^\alpha(f+\nabla^\varphi q+\bar\rho[\mathbb{P}_t,\partial_t^\varphi]u)-[Z^\alpha,\nabla^\varphi]\pi+2\mu[Z^\alpha,\mathrm{div}^\varphi]S^\varphi u-\bar\rho[Z^\alpha,\partial_t^\varphi]v.\end{aligned}$$

Performing standard energy estimates, we obtain the energy identity:

$$\begin{aligned}&\frac12\bar\rho\int_{\mathcal{S}}|Z^\alpha v|^2(t)\mathrm{d}\mathcal{V}_t+2\mu\int_0^t\int_{\mathcal{S}}|Z^\alpha S^\varphi v|^2\mathrm{d}\mathcal{V}_s\mathrm{d}s+a\int_0^t\int_{z=-1}|Z^\alpha v_\tau|^2\mathrm{d}y\mathrm{d}s\\&=:\mathcal{K}_0+\mathcal{K}_1+\cdots\mathcal{K}_8,\end{aligned}\tag{9.26}$$

where

$$\begin{aligned}&\mathcal{K}_0=\frac12\bar\rho\int_{\mathcal{S}}|Z^\alpha v|^2(0)\,\mathrm{d}\mathcal{V}_0, &&\mathcal{K}_1=\frac12\bar\rho\int_0^t\int_{z=0}\partial_t h|Z^\alpha v|^2\,\mathrm{d}y\mathrm{d}s,\\&\mathcal{K}_2=2\mu\int_0^t\int_{\mathcal{S}}Z^\alpha S^\varphi v\cdot[Z^\alpha,\nabla^\varphi]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, &&\mathcal{K}_3=\int_0^t\int_{z=0}Z^\alpha(2\mu S^\varphi v-\pi\mathrm{Id})\mathbf{N}\cdot Z^\alpha v\,\mathrm{d}y\mathrm{d}s,\\&\mathcal{K}_4=\int_0^t\int_{\mathcal{S}}Z^\alpha\pi[\mathrm{div}^\varphi,Z^\alpha]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, &&\mathcal{K}_5=-\int_0^t\int_{\mathcal{S}}Z^\alpha v\cdot[Z^\alpha,\nabla^\varphi]\pi\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\&\mathcal{K}_6=-\bar\rho\int_0^t\int_{\mathcal{S}}Z^\alpha v\cdot[Z^\alpha,\partial_t^\varphi]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, &&\mathcal{K}_7=2\mu\int_0^t\int_{\mathcal{S}}[Z^\alpha,\mathrm{div}^\varphi]S^\varphi v\cdot Z^\alpha v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\&\mathcal{K}_8=-\int_0^t\int_{\mathcal{S}}Z^\alpha v\cdot\big(Z^\alpha(f+\nabla^\varphi q+\bar\rho[\mathbb{P}_t,\partial_t^\varphi]u)\big)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.\end{aligned}$$

By the trace inequality,

$$a\int_0^t\int_{z=-1}|Z^\alpha v_\tau|^2\mathrm{d}y\mathrm{d}s\ge-\delta\|\nabla v\|^2_{L_t^2H_{co}^k}-C_\delta(\|\nabla v\|^2_{L_t^2H_{co}^{k-1}}+\|v\|^2_{L_t^2H_{co}^k}),\tag{9.27}$$

we will choose δ sufficiently small in the end. Our following task is to estimate $\mathcal{K}_0-\mathcal{K}_8$ one by one. By Remark 9.4, we get that:

$$\mathcal{K}_0\lesssim\Lambda\big(\frac{1}{c_0},Y_m^2(0)\big)Y_m^2(0).\tag{9.28}$$

Thanks to the trace inequality and Young's inequality, $\mathcal{K}_1$ can be treated as:

$$\begin{aligned}\mathcal{K}_1&\lesssim|\partial_t h|_{0,\infty,t}(\|\nabla Z^\alpha v\|_{L_t^2L^2}\|Z^\alpha v\|_{L_t^2L^2}+\|Z^\alpha v\|^2_{L_t^2L^2})\\&\le\delta\|\nabla v\|^2_{L_t^2H_{co}^k}+C_\delta\|\nabla v\|^2_{L_t^2H_{co}^{k-1}}+\Lambda\big(\frac{1}{c_0},|\partial_t h|_{0,\infty,t}\big)\|v\|^2_{L_t^2H_{co}^k}.\end{aligned}\tag{9.29}$$

For the term $\mathcal{K}_2$, to deal with the commutator term $[Z^\alpha, \nabla^\varphi]v$, we apply (3.24) if $\alpha_0 = 0$ and (3.23) if $\alpha_0 \geq 1$ and find that:

$$\begin{aligned}
\|[Z^\alpha, \nabla^\varphi]v\|_{L^2} &\lesssim \Lambda\big(\frac{1}{c_0}, \|\nabla v\|_{1,\infty,t} + |(h, \varepsilon\partial_t h)|_{m-2,\infty,t}\big)(|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + |\varepsilon h|_{L_t^2\tilde{H}^{m-\frac{3}{2}}}) \\
&\quad + \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon\partial_t h)|_{m-2,\infty,t}\big)\|\nabla v\|_{L_t^2 H_{co}^{m-2}} \\
&\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon\partial_t h)|_{m-2,\infty,t}\big)\|\nabla v\|_{L_t^2 H_{co}^{m-2}}.
\end{aligned} \tag{9.30}$$

Note that by the estimate (5.31), we have:

$$\begin{aligned}
\|\nabla v\|_{1,\infty,t} &\lesssim \|\nabla(u, \nabla^\varphi \Psi)\|_{1\infty,t} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, \|\nabla u\|_{1\infty,t} + |h|_{4,\infty,t} + \|\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^2}\big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}$$

Therefore, by Young's inequality, one can control $\mathcal{K}_2$ by:

$$\mathcal{K}_2 \leq \delta\|\nabla v\|_{L_t^2 H_{co}^k}^2 + \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon\partial_t h)|_{m-2,\infty,t}\big)\|\nabla v\|_{L_t^2 H_{co}^{k-1}}^2 + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{9.31}$$

For the boundary term $\mathcal{K}_3$, we use the boundary condition $(9.6)_2$ to split it into two terms:

$$\begin{aligned}
\mathcal{K}_3 &= \int_0^t \int_{z=0} Z^\alpha\big(2\mu(\mathrm{div}^\varphi u\mathrm{Id} - \nabla^\varphi\nabla^\varphi\Psi)\mathbf{N}\big) \cdot Z^\alpha v - [Z^\alpha, \mathbf{N}](2\mu S^\varphi v - \pi\mathrm{Id}) \cdot Z^\alpha v \,\mathrm{d}y\mathrm{d}s \\
&=: \mathcal{K}_{31} + \mathcal{K}_{32}.
\end{aligned}$$

Since κ_3 vanishes if $\alpha_3 \neq 0$, we may assume that $Z^\alpha = \partial_y Z^{\tilde{\alpha}}$. It then follows by duality that:

$$\mathcal{K}_{31} \lesssim |Z^\alpha v|_{L_t^2 H^{\frac{1}{2}}} |Z^{\tilde{\alpha}}\big(2\mu(\mathrm{div}^\varphi u\mathrm{Id} - \nabla^\varphi\nabla^\varphi\Psi)\mathbf{N}\big)|_{L_t^2 H^{\frac{1}{2}}}$$

Thanks to product estimate (3.5), we obtain for $k \leq m-1$,

$$\begin{aligned}
|Z^{\tilde{\alpha}}\big(2\mu(\mathrm{div}^\varphi u\mathrm{Id} - (\nabla^\varphi)^2)\mathbf{N}\big)|_{L_t^2 H^{\frac{1}{2}}} &\lesssim |(\mathrm{div}^\varphi u, (\nabla^\varphi)^2\Psi)|_{L_t^2\tilde{H}^{k-\frac{1}{2}}} |h|_{L_t^\infty \tilde{H}^{[\frac{k-1}{2}]+2^+}} \\
&\quad + |h|_{L_t^2\tilde{H}^{k+\frac{1}{2}}} |(\mathrm{div}^\varphi u, (\nabla^\varphi)^2\Psi)|_{L_t^\infty\tilde{H}^{[\frac{k-1}{2}]+1^+}} \\
&\lesssim (\|\nabla\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-2}} + \|\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0}, |h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} + \|\nabla\Psi\|_{2,\infty,t}\big) \\
&\quad + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big)\big(|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|h|_{L_t^\infty\tilde{H}^{m+\frac{1}{2}}}\big) \\
&\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}$$

We remark that by the estimate (5.31), one has that for $l \leq [\frac{k-1}{2}] + 1^+ \leq [\frac{m}{2}]^+ \leq m-3$ (since $k \leq m-1, m \geq 7$),

$$\begin{aligned}
|(\nabla^\varphi)^2\Psi|_{L_t^\infty\tilde{H}^l} &\lesssim \|\nabla(\nabla^\varphi)^2\Psi\|_{L_t^\infty\tilde{H}^l} + \|(\nabla^\varphi)^2\Psi\|_{L_t^\infty\tilde{H}^l} \\
&\lesssim \big(\|(\nabla\mathrm{div}^\varphi u, \mathrm{div}^\varphi u)\|_{L_t^\infty\tilde{H}^l} + |h|_{L_t^\infty\tilde{H}^{l+5/2}}\big)\Lambda\big(\frac{1}{c_0}, \|\nabla^\varphi\Psi\|_{[\frac{m}{2}]-1,\infty,t} + |h|_{[\frac{m}{2}]+2,\infty,t}\big) \\
&\lesssim \Lambda(\frac{1}{c_0}, \mathcal{N}_{m,T}).
\end{aligned}$$

Therefore, by the trace inequality and Young's inequality, we get:

$$\mathcal{K}_{31} \leq \delta\|\nabla v\|_{L_t^2 H_{co}^k}^2 + C_\delta\|\nabla v\|_{L_t^2 H_{co}^{k-1}}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{9.32}$$

For $\mathcal{K}_{32}$, in order not to involve too many derivatives on the surface, we write it further as:

$$\begin{aligned}\mathcal{K}_{32} &= -\int_0^t \int_{z=0} (2\mu S^\varphi v - \pi \mathrm{Id}) Z^\alpha \mathbf{N} \cdot Z^\alpha v + [Z^\alpha, (S^\varphi v - \pi \mathrm{Id}), \mathbf{N}] Z^\alpha v \,\mathrm{d}y\mathrm{d}s \\ &=: \mathcal{K}_{321} + \mathcal{K}_{322}.\end{aligned}$$

By the definition (9.7) for π we have that on the upper boundary,

$$\pi = \theta = -2\mu(\partial_1 u_1 + \partial_2 u_2) - 2\mu(\Pi(\partial_1 u \cdot \mathbf{N}, \partial_2 u \cdot \mathbf{N}, 0)^t)_3. \tag{9.33}$$

Moreover, thanks to the boundary condition (4.8), we can indeed express $\partial_z^\varphi v$ on the upper boundary. On the one hand, we have the identity:

$$\partial_z^\varphi v \cdot \mathbf{N} = \mathrm{div}^\varphi v - \partial_1 v_1 - \partial_2 v_2 = -(\partial_1 v_1 + \partial_2 v_2). \tag{9.34}$$

On the other hand, by the identity (4.8), one deduces:

$$\begin{aligned}|\mathbf{N}|\Pi \partial_z^\varphi v &= |\mathbf{N}|\Pi \partial_z^\varphi u - |\mathbf{N}|\Pi \nabla^\varphi \partial_z^\varphi \Psi \\ &= \Pi(\partial_1 u \cdot \mathbf{n}, \partial_2 u \cdot \mathbf{n}, 0)^t - \Pi(\mathbf{n}_1 \partial_1 u + \mathbf{n}_2 \partial_2 u) - |\mathbf{N}|\Pi(\partial_1, \partial_2, 0)^t \partial_z^\varphi \Psi,\end{aligned} \tag{9.35}$$

One thus has that:

$$|(S^\varphi v, \pi)^{b,1}|_{1,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, \|(v, \nabla^\varphi \Psi)\|_{2,\infty,t} + |h|_{2,\infty,t}\big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Therefore, by duality and the trace inequality (3.17), we obtain

$$\begin{aligned}\mathcal{K}_{321} &\le |2\mu S^\varphi v - \pi \mathrm{Id}|_{\infty,t} |Z^\alpha \mathbf{N}|_{L_t^2 H^{-\frac12}} |Z^\alpha v|_{L_t^2 H^{\frac12}} \\ &\le \delta \|\nabla v\|^2_{L_t^2 H_{co}^k} + C_\delta \|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} + (\|v\|^2_{L_t^2 H_{co}^{m-1}} + T|h|^2_{L_t^\infty \tilde{H}^{m-\frac12}}) \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{9.36}$$

Next, we can control $\mathcal{K}_{322}$, in the following way:

$$\mathcal{K}_{322} \lesssim |Z^\alpha v|_{L_t^2 L_y^2} \big(|h|_{L_t^2 \tilde{H}^{m-1}} |(S^\varphi v, \pi)|_{1,\infty,t} + |(S^\varphi v, \pi)|_{L_t^2 \tilde{H}^{k-1}} |h|_{m-2,\infty,t}\big).$$

By virtue of the boundary conditions (9.33)-(9.35), we obtain that:

$$|(S^\varphi v, \pi)|_{L_t^2 \tilde{H}^{k-1}} \lesssim \Lambda\big(|h|_{m-2,\infty,t} + \|(v, \nabla^\varphi \Psi)\|_{2,\infty,t}\big)\big(|(v, \nabla^\varphi \Psi)|_{L_t^2 \tilde{H}^k} + |h|_{L_t^2 \tilde{H}^k}\big).$$

Combined with the trace inequality (3.17), Young's inequality and the elliptic estimate (5.25), we find:

$$\mathcal{K}_{322} \le \delta \|\nabla v\|^2_{L_t^2 H_{co}^k} + C_\delta \|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} + \big(\|v\|^2_{L_t^2 H_{co}^{m-1}} + (T+\varepsilon)^{\frac12}\big) \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

This estimate, together with (9.36), (9.32), gives (with possibly another C_δ)

$$\mathcal{K}_3 \le 3\delta \|\nabla v\|^2_{L_t^2 H_{co}^k} + C_\delta \|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} + \big(\|v\|^2_{L_t^2 H_{co}^k} + (T+\varepsilon)^{\frac12}\big) \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{9.37}$$

For the term $\mathcal{K}_4$, since Z^α contains at least one spatial derivative, we can estimate it as:

$$\mathcal{K}_4 \lesssim \|\nabla \pi\|_{L_t^2 H_{co}^{k-1}} \Big(\|\nabla v\|_{L_t^2 H_{co}^{k-1}} \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) + |h|_{L_t^2 \tilde{H}^{k+\frac12}} \Lambda\big(\frac{1}{c_0}, \|\nabla v\|_{1,\infty,t} + |h|_{m-2,\infty,t}\big)\Big).$$

We then apply (9.12) and the elliptic estimate (5.25) to estimate $\nabla^\varphi \pi$ as:

$$\begin{aligned}\|\nabla^\varphi \pi\|_{L_t^2 H_{co}^{k-1}} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \|\nabla u\|_{L_t^2 H_{co}^k} + T^{\frac12} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \|\nabla v\|_{L_t^2 H_{co}^k} + (T+\varepsilon)^{\frac12} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, by Young's inequality, we get:

$$\mathcal{K}_4 \leq \delta\|\nabla v\|^2_{L^2_t H^k_{co}} + \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L^2_t H^{k-1}_{co}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Similarly, for $\mathcal{K}_5$, by applying (3.21), (9.12), (9.11), we obtain:

$$\begin{aligned}\mathcal{K}_5 &\lesssim \|v\|_{L^2_t H^k_{co}}\bigg(\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\nabla \pi\|_{L^2_t H^k_{co}} + \Lambda\big(\frac{1}{c_0}, \|\nabla\pi\|_{1,\infty,t} + |h|_{m-2,\infty,t}\big)|h|_{L^2_t \tilde{H}^{m-\frac{1}{2}}}\bigg)\\ &\lesssim \|v\|_{L^2_t H^k_{co}}\bigg(\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\nabla v\|_{L^2_t H^k_{co}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big)\bigg).\end{aligned}$$

Combined with the Young's inequality, this yields:

$$\mathcal{K}_5 \leq \delta\|\nabla v\|^2_{L^2_t H^k_{co}} + C_\delta\Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|v\|^2_{L^2_t H^k_{co}} + (T+\varepsilon)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{9.38}$$

For the term $\mathcal{K}_6$, we use similar arguments as in (9.30) to deal with the commutator term:

$$\big\|\big[Z^\alpha, \frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\big]v\big\|_{L^2_t L^2} \lesssim \big(\|\nabla v\|_{L^2_t H^{k-1}_{co}} + (T+\varepsilon)^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Therefore, we control $\mathcal{K}_6$ by the Cauchy-Schwarz inequality to get:

$$\begin{aligned}\mathcal{K}_6 &\leq \|Z^\alpha v\|_{L^2_t L^2}\big\|\big[Z^\alpha, \frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\big]v\big\|_{L^2_t L^2}\\ &\lesssim \|\nabla v\|^2_{L^2_t H^{k-1}_{co}} + \big(\|v\|^2_{L^2_t H^{m-1}_{co}} + (T+\varepsilon)^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{9.39}$$

We are now ready to estimate $\mathcal{K}_7$. In order not to lose normal derivative, we split it into three terms:

$$\mathcal{K}_7 = \mathcal{K}_{71} + \mathcal{K}_{72} + \mathcal{K}_{73}.$$

with

$$\begin{aligned}\mathcal{K}_{71} &= 2\mu\int_0^t\int_{\mathcal{S}}[Z^\alpha, \partial_z]\big(\frac{1}{\partial_z\varphi}S^\varphi v\mathbf{N}\big)\cdot Z^\alpha v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\ \mathcal{K}_{72} &= 2\mu\int_0^t\int_{\mathcal{S}}\bigg(\partial_z Z^\alpha\big(\frac{1}{\partial_z\varphi}S^\varphi v\mathbf{N}\big) - \big(\partial_z Z^\alpha S^\varphi v\big)\frac{\mathbf{N}}{\partial_z\varphi}\bigg)\cdot Z^\alpha v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\ \mathcal{K}_{73} &= -2\mu\int_0^t\int_{\mathcal{S}} Z^\alpha\bigg(S^\varphi v\partial_z\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\bigg)\cdot Z^\alpha v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.\end{aligned}$$

To deal with $\mathcal{K}_{71}$, we can use the identity (3.27) to integrate by parts in space. By doing so, we are led to control the following type of terms (up to some smooth functions that depends only on ϕ and its derivatives)

$$\int_0^t\int_{\mathcal{S}} Z^\gamma\big(\frac{1}{\partial_z\varphi}S^\varphi v\mathbf{N}\big)\partial_z(Z^\alpha v\partial_z\varphi)\,\mathrm{d}x\mathrm{d}s, \quad \int_0^t\int_{\partial\mathcal{S}} Z^\gamma\big(\frac{1}{\partial_z\varphi}S^\varphi v\mathbf{N}\big)Z^\alpha v\partial_z\varphi\,\mathrm{d}y\mathrm{d}s, \quad |\gamma| \leq k-1.$$

The first type of term can be controlled easily by:

$$\delta\|\nabla v\|^2_{L^2_t H^k_{co}} + C_\delta\Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L^2_t H^{k-1}_{co}} + T\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),$$

while the second type of terms can be bounded by:

$$
\begin{aligned}
&|v|_{L_t^2\tilde{H}^k}\big(|S^\varphi v|_{L_t^2\tilde{H}^{k-1}}+T^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\
&\lesssim |v|_{L_t^2\tilde{H}^k}\big(|(v,\nabla^\varphi\Psi)|_{L_t^2\tilde{H}^k}+T^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\
&\leq \delta\|\nabla v\|_{L_t^2H_{co}^k}^2+\big(\|v\|_{L_t^2H_{co}^k}^2+(T+\varepsilon)^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Hence, we get that:

$$
\begin{aligned}
\mathcal{K}_{71}\leq 2\delta\|\nabla v\|_{L_t^2H_{co}^k}^2+C_\delta\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}^2\big)\|\nabla v\|_{L_t^2H_{co}^{k-1}}^2&\\
+\big(\|v\|_{L_t^2H_{co}^k}^2+(T+\varepsilon)^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).&
\end{aligned}
\tag{9.40}
$$

For $\mathcal{K}_{72}$, we use again integration by parts to split it into three terms: $\mathcal{K}_{72}=\mathcal{K}_{721}+\mathcal{K}_{722}+\mathcal{K}_{723}$, with

$$
\begin{aligned}
\mathcal{K}_{721}&=-2\mu\int_0^t\int_{\mathcal{S}}\big[Z^\alpha,\frac{\mathbf{N}}{\partial_z\varphi}\big]S^\varphi v\cdot\partial_z(Z^\alpha v\partial_z\varphi)\,\mathrm{d}x\mathrm{d}s,\\
\mathcal{K}_{722}&=2\mu\int_0^t\int_{\mathcal{S}}Z^\alpha S^\varphi v\cdot\partial_z\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)Z^\alpha v\,\mathrm{d}x\mathrm{d}s,\\
\mathcal{K}_{723}&=2\mu\int_0^t\int_{\partial\mathcal{S}}\big[Z^\alpha,\frac{\mathbf{N}}{\partial_z\varphi}\big]S^\varphi v\cdot Z^\alpha v\partial_z\varphi\,\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

In view of the expressions of these three terms, one can show by the commutator estimate (3.9) that

$$
\begin{aligned}
\mathcal{K}_{72}\leq \delta\|\nabla v\|_{L_t^2H_{co}^k}^2+C_\delta\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}^2\big)\|\nabla v\|_{L_t^2H_{co}^{k-1}}^2&\\
+\big(\|v\|_{L_t^2H_{co}^k}^2+(T+\varepsilon)^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).&
\end{aligned}
\tag{9.41}
$$

Note that the boundary term $\mathcal{K}_{723}$ can be controlled in a similar way as $\mathcal{K}_{32}$. We thus skip the details.

For $\mathcal{K}_{73}$, to avoid losing regularity on the surface, we use the assumption that $|\alpha'|\geq 1$ to integrate by parts in space. By doing so, we find that it can be bounded as:

$$
\mathcal{K}_{73}\leq \delta\|\nabla v\|_{L_t^2H_{co}^k}^2+C_\delta\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}^2\big)\|\nabla v\|_{L_t^2H_{co}^{k-1}}^2+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\tag{9.42}
$$

We remark that there is no boundary contribution in the process of integration by parts since the spatial vector fields are tangent to the boundary. Collecting (9.40)-(9.42), we finally find that:

$$
\begin{aligned}
\mathcal{K}_{7}\leq 4\delta\|\nabla v\|_{L_t^2H_{co}^k}^2+C_\delta\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}^2\big)\|\nabla v\|_{L_t^2H_{co}^{k-1}}^2&\\
+\big(\|v\|_{L_t^2H_{co}^{m-1}}^2+(T+\varepsilon)^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).&
\end{aligned}
\tag{9.43}
$$

It remains to treat the last term $\mathcal{K}_8$. By (9.9) (9.10), (9.15), we have:

$$
\begin{aligned}
\mathcal{K}_8&\lesssim \|v\|_{L_t^2H_{co}^k}(\|(f,\nabla^\varphi q)\|_{L_t^2H_{co}^k}+\|[\mathbb{P}_t,\partial_t^\varphi]u\|_{L_t^2H_{co}^k})\\
&\lesssim \|v\|_{L_t^2H_{co}^{m-1}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
\tag{9.44}
$$

Gathering (9.27)-(9.31), (9.37)-(9.39), (9.43), (9.44), we find by using Korn's inequality (3.34) and by choosing δ small enough that for any $0 \le |\alpha| = k \le m-1$,

$$\|v\|^2_{L_t^\infty H^k_{co}} + \|\nabla^\varphi v\|^2_{L_t^2 H^k_{co}} \lesssim Y_m^2(0) + \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla^\varphi v\|^2_{L_t^2 H^{k-1}_{co}} + (\|v\|_{L_t^2 H^{m-1}_{co}} + (T+\varepsilon)^{\frac{1}{2}})\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Therefore, by induction (on k), we get (up to changing possibly the polynomial)

$$\begin{aligned}\|v\|^2_{L_t^\infty H^{m-1}_{co}} + \|\nabla^\varphi v\|^2_{L_t^2 H^{m-1}_{co}} &\lesssim (Y_m^2(0) + \|\nabla v\|^2_{L_t^2 L^2})\Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}\big) \\ &\quad + (\|v\|_{L_t^2 H^{m-1}_{co}} + (T+\varepsilon)^{\frac{1}{2}})\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{9.45}$$

By (5.27), we can extract an extra $T^{\frac{1}{2}}$ from $\|v\|_{L_t^2 H^{m-1}_{co}}$. More precisely, we obtain:

$$\|v\|_{L_t^2 H^{m-1}_{co}} \lesssim \|(u, \nabla^\varphi \Psi)\|_{L_t^2 H^{m-1}_{co}} \lesssim T^{\frac{1}{2}}\|(u, \nabla^\varphi \Psi)\|_{L_t^\infty H^{m-1}_{co}} \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Moreover, thanks to the elliptic estimate (5.10) and the definition $v = \mathbb{P}_t u = u - \nabla^\varphi \Psi$, we also have:

$$\|\nabla v\|_{L_t^2 L^2} \le \|\nabla u\|_{L_t^2 L^2}\Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big).$$

Inserting the above two estimates and (7.19) into (9.45), we finally arrive at (9.24). $\Box$

In the following lemma, we prove some estimates for $\varepsilon^{\frac{1}{2}}\partial_t v$, which is useful to the estimate for $\varepsilon^{\frac{1}{2}}\partial_t u$ later.

Lemma 9.6. *Under the assumption* (2.2)*, the following estimate for v holds:*

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t v\|^2_{L_t^\infty H^{m-2}_{co}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla v\|^2_{L_t^2 H^{m-2}_{co}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + Y_m^2(0)\big)Y_m^2(0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{9.46}$$

Proof. The proof of this Lemma is very similar to the previous one, we thus only sketch its proof. We have by the elliptic estimate (5.15) that:

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t v\|_{L_t^\infty \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla v\|_{L_t^2 \mathcal{H}^{m-2}} \\ &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t (u, \nabla^\varphi \Psi)\|_{L_t^\infty \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla (u, \nabla^\varphi \Psi)\|_{L_t^2 \mathcal{H}^{m-2}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla u\|_{L_t^2 \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \mathrm{div}^\varphi u\|_{L_t^2 \mathcal{H}^{m-2}}\big) \\ &\quad + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(|\partial_t h|_{L_t^\infty \tilde{H}^{m-\frac{3}{2}}} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}\big).\end{aligned}$$

For any multi-index β with $|\beta| = k \le m-2$, direct energy estimates for v yield:

$$\begin{aligned}&\frac{1}{2}\bar{\rho}\varepsilon\int_{\mathcal{S}} |Z^\beta \partial_t v|^2(t)\,\mathrm{d}\mathcal{V}_t + 2\mu\varepsilon\int_0^t\int_{\mathcal{S}} |Z^\beta \partial_t S^\varphi v|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s + a\varepsilon\int_0^t\int_{z=-1} |Z^\beta \partial_t v_\tau|^2\,\mathrm{d}y\mathrm{d}s \\ &=: \tilde{\mathcal{K}}_0 + \tilde{\mathcal{K}}_1 + \cdots \tilde{\mathcal{K}}_8,\end{aligned} \tag{9.47}$$

where $\tilde{\mathcal{K}}_0 - \tilde{\mathcal{K}}_8$ are terms analogues to $\mathcal{K}_0 - \mathcal{K}_8$ defined in (9.26) in which Z^α is replaced by $\varepsilon^{\frac{1}{2}}Z^\beta\partial_t$.

At first, thanks to the trace inequality (3.17), Korn's inequality (3.34) and Young's inequality, we have:

$$a\varepsilon\int_0^t\int_{z=-1}|Z^\beta\partial_t v_\tau|^2\,\mathrm{d}y\mathrm{d}s\geq-\delta\varepsilon\|Z^\beta\partial_t S^\varphi v\|^2_{L^2_tL^2}-C_\delta(\varepsilon\|\partial_t\nabla v\|^2_{L^2_tH^{m-3}_{co}}+\varepsilon\|\partial_t v\|^2_{L^2_tH^{m-2}_{co}}). \tag{9.48}$$

The remaining task is thus to estimate $\tilde{\mathcal{K}}_1-\tilde{\mathcal{K}}_8$. We assume that Z^β contains at least one spatial conormal derivative $Z_i (i=1,2,3)$.

$\underline{\tilde{\mathcal{K}}_1}$: Similar to the proof of (9.48), we have by the trace inequality (3.17), Young's inequality and Korn's inequality (3.34) that:

$$\begin{aligned}\tilde{\mathcal{K}}_1&=\frac{1}{2}\varepsilon\int_0^t\int_{z=0}\partial_t h|Z^\beta\partial_t v|^2\,\mathrm{d}y\mathrm{d}s\\&\leq\delta\varepsilon\|Z^\beta\partial_t S^\varphi v\|^2_{L^2_tL^2}+C_\delta\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)(T\varepsilon\|Z^\beta\partial_t v\|^2_{L^\infty_tL^2}+\varepsilon\|\partial_t\nabla v\|^2_{L^2_tH^{k-1}_{co}}).\end{aligned}\tag{9.49}$$

$\underline{\tilde{\mathcal{K}}_2}$: By Young's inequality, $\mathcal{K}_2$ can be controlled similarly:

$$\tilde{\mathcal{K}}_2=2\mu\varepsilon\int_0^t\int_{\mathcal{S}}Z^\beta\partial_t S^\varphi v\cdot[Z^\beta\partial_t,\nabla^\varphi]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\leq\delta\varepsilon\|Z^\beta\partial_t S^\varphi v\|^2_{L^2_tL^2}+C_\delta\varepsilon\|[Z^\beta\partial_t,\nabla^\varphi]v\|^2_{L^2_tL^2}$$

Since

$$[Z^\beta\partial_t,\partial_j^\varphi]f=Z^\beta\big(\partial_t(\frac{\mathbf{N}_j}{\partial_z\varphi})\cdot\partial_z f\big)+\big[Z^\beta,\frac{\mathbf{N}_j}{\partial_z\varphi}\big]\partial_t\partial_z f+\frac{\mathbf{N}_j}{\partial_z\varphi}[Z^\beta,\partial_z]\partial_t\partial_z f,\ j=1,2,3,$$

we can use the fact that $|\beta|=k\leq m-2$ to get that:

$$\begin{aligned}&\varepsilon^{\frac12}\|[Z^\beta\partial_t,\partial_j^\varphi]f\|_{L^2_tL^2}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\varepsilon^{\frac12}\|\partial_t\nabla f\|_{L^2_tH^{k-1}_{co}}\\&+\Lambda\big(\frac{1}{c_0},\|(\mathrm{Id},\varepsilon^{\frac12}\partial_t)\partial_z f\|_{0,\infty,t}+|(h,\partial_t h)|_{m-3,\infty,t}\big)(\varepsilon^{\frac12}\|\partial_z f\|_{L^2_tH^k_{co}}+|(h,\varepsilon^{\frac12}\partial_t h)|_{L^2_t\tilde{H}^{k+\frac12}}).\end{aligned}\tag{9.50}$$

We thus obtain that:

$$\tilde{\mathcal{K}}_2\leq\delta\varepsilon\|Z^\beta\partial_t S^\varphi v\|^2_{L^2_tL^2}+C_\delta\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\partial_t\nabla v\|_{L^2_tH^{k-1}_{co}}+(T+\varepsilon)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

$\underline{\tilde{\mathcal{K}}_3}$: Regarding the estimate of

$$\tilde{\mathcal{K}}_3=\varepsilon\int_0^t\int_{z=0}Z^\beta\partial_t(2\mu S^\varphi v-\pi\mathrm{Id})\mathbf{N}\cdot Z^\beta\partial_t v\,\mathrm{d}y\mathrm{d}s,$$

as we did for $\mathcal{K}_3$, we write:

$$\begin{aligned}&Z^\beta\partial_t(2\mu S^\varphi v-\pi\mathrm{Id})\mathbf{N}=2\mu Z^\beta\partial_t\big((\mathrm{div}^\varphi u\mathrm{Id}-(\nabla^\varphi)^2\Psi)\mathbf{N}\big)+[Z^\beta\partial_t,\mathbf{N}](2\mu S^\varphi v-\pi\mathrm{Id})\\&=2\mu Z^\beta\partial_t(\mathrm{div}^\varphi u\mathrm{Id}-(\nabla^\varphi)^2\Psi)\mathbf{N}+\varepsilon^{\frac12}[Z^\beta\partial_t,\mathbf{N}](2\mu(\mathrm{div}^\varphi u\mathrm{Id}-(\nabla^\varphi)^2\Psi)+2\mu S^\varphi v-\pi\mathrm{Id}).\end{aligned}$$

By using the trace inequality (3.17) and Lemma 5.3, we get in a similar way as for (9.50) that:

$$\begin{aligned}&\varepsilon^{\frac12}\big|Z^\beta\partial_t(\mathrm{div}^\varphi u\mathrm{Id}-(\nabla^\varphi)^2\Psi)\mathbf{N}\big|_{L^2_tH^{-\frac12}_y}\\&\lesssim|h|_{2,\infty,t}\|\varepsilon^{\frac12}\partial_t(\mathrm{div}^\varphi u,(\nabla^\varphi)^2\Psi)\|_{L^2_tH^{m-2}_{co}}+\|\varepsilon^{\frac12}\partial_t\nabla(\mathrm{div}^\varphi u,(\nabla^\varphi)^2\Psi)\|_{L^2_tH^{m-3}_{co}}\\&\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac12}\partial_t\nabla\mathrm{div}^\varphi u\|_{L^2_tH^{m-3}_{co}}+\|\varepsilon^{\frac12}\partial_t\mathrm{div}^\varphi u\|_{L^2_tH^{m-2}_{co}}\big)+\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Moreover, by the boundary conditions (9.33)-(9.35), we have

$$
\begin{aligned}
&\varepsilon^{\frac{1}{2}}\big|[Z^\beta\partial_t,\mathbf{N}](2\mu(\mathrm{div}^\varphi u\mathrm{Id}-(\nabla^\varphi)^2\Psi)+2\mu S^\varphi v-\pi\mathrm{Id})\big|_{L_t^2L_y^2}\\
&\lesssim|\varepsilon^{\frac{1}{2}}\partial_t(S^\varphi v,\pi,\mathrm{div}^\varphi u,(\nabla^\varphi)^2\Psi)^{b,1}|_{L_t^2\tilde{H}^{k-1}}\Lambda\big(\frac{1}{c_0},|h|_{k,\infty,t}\big)\\
&+\Lambda\big(\frac{1}{c_0},|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,Z)(S^\varphi v,\pi,\mathrm{div}^\varphi u,(\nabla)^2\Psi)^{b,1}|_{0,\infty,t}+|\partial_t h|_{k-1,\infty,t}\big)\cdot\\
&\qquad\big(\varepsilon^{\frac{1}{2}}|(S^\varphi v,\pi,\mathrm{div}^\varphi u,(\nabla)^2\Psi)|_{L_t^2\tilde{H}^k}+|(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^2\tilde{H}^{k+1}}\big)\\
&\lesssim\big(|\varepsilon^{\frac{1}{2}}\partial_t v|_{L_t^2\tilde{H}^k}+\|\varepsilon^{\frac{1}{2}}\partial_t(\mathrm{div}^\varphi u,\nabla\mathrm{div}^\varphi u)\|_{L_t^2H_{co}^{k-1}}\big)\Lambda\big(\frac{1}{c_0},|h|_{k,\infty,t}\big)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Therefore, by duality, the Cauchy-Schwarz inequality and Young's inequality (3.34), we obtain that:

$$
\begin{aligned}
\tilde{\mathcal{K}}_3&\le\delta\|\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\nabla v\|_{L_t^2H_{co}^k}^2+(T+\varepsilon)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\\
&\quad+C_\delta\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}^2\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t(v,\mathrm{div}^\varphi u)\|_{L_t^2H_{co}^{m-2}}^2+\|\varepsilon^{\frac{1}{2}}\partial_t\nabla\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-3}}^2\big).
\end{aligned}
\tag{9.51}
$$

$\underline{\tilde{\mathcal{K}}_4:}$ $\tilde{\mathcal{K}}_4$: has the following expression:

$$
\tilde{\mathcal{K}}_4=\varepsilon\int_0^t\int_{\mathcal{S}}Z^\beta\partial_t\pi[\mathrm{div}^\varphi,Z^\beta\partial_t]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s
$$

By Hölder inequality, the estimate (9.14) for $\varepsilon^{\frac{1}{2}}\nabla\partial_t\pi$, the Korn inequality (3.34) and the commutator estimate (9.50) we get:

$$
\begin{aligned}
\tilde{\mathcal{K}}_4&\le\|\varepsilon^{\frac{1}{2}}\nabla\partial_t\pi\|_{L_t^2H_{co}^{k-1}}\|\varepsilon^{\frac{1}{2}}[\mathrm{div}^\varphi,Z^\beta\partial_t]v\|_{L_t^2L^2}\le\delta\|\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|_{L_t^2H_{co}^k}^2\\
&\quad+C_\delta\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}^2\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi v\|_{L_t^2H_{co}^{k-1}}^2+\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-2}}^2\big)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

$\underline{\tilde{\mathcal{K}}_5.}$ By the Cauchy-Schwarz inequality and estimates (9.12), (9.14), we obtain:

$$
\begin{aligned}
\tilde{\mathcal{K}}_5&\lesssim\|\varepsilon^{\frac{1}{2}}Z^\beta\partial_t v\|_{L_t^2L^2}\|\varepsilon^{\frac{1}{2}}[Z^\beta\partial_t,\nabla^\varphi]\pi\|_{L_t^2L^2}\\
&\lesssim\|\varepsilon^{\frac{1}{2}}\partial_t v\|_{L_t^2H_{co}^k}\big(\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\nabla\pi\|_{L_t^2H_{co}^{k-1}}+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\big)\\
&\le\delta\|\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|_{L_t^2H_{co}^k}^2+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
\tag{9.52}
$$

$\underline{\tilde{\mathcal{K}}_6,\tilde{\mathcal{K}}_8:}$ By (9.9), (9.10), (9.15), we have:

$$
\varepsilon^{\frac{1}{2}}\|\partial_t(f+\nabla^\varphi q+\partial_t[\mathbb{P}_t,\partial_t^\varphi]u)\|_{L_t^2H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
$$

In addition, since $|\beta|=k\le m-2$, the following estimate holds:

$$
\begin{aligned}
\varepsilon^{\frac{1}{2}}\|\big[Z^\beta\partial_t,\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\big]v\|_{L_t^2L^2}&\lesssim\Lambda\big(\frac{1}{c_0},|(h,\partial_t h)|_{m-3,\infty,t}+\|\nabla v,\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|_{0,\infty,t}\big)\\
&\qquad\big(\|(\varepsilon^{\frac{1}{2}}\partial_t\nabla v,\nabla v)\|_{L_t^2H_{co}^{m-3}}\|(\varepsilon^{\frac{1}{2}}\partial_t\nabla v,\nabla v)\|_{L_t^\infty H_{co}^{m-4}}|(\partial_t h,\varepsilon^{\frac{1}{2}}\partial_t^2 h)|_{L_t^2\tilde{H}^{m-3}}\big)\\
&\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Therefore, we control $\tilde{\mathcal{K}}_6+\tilde{\mathcal{K}}_8$ as:

(9.53)
$$\begin{aligned}\tilde{\mathcal{K}}_6+\tilde{\mathcal{K}}_8 &\lesssim \|\varepsilon^{\frac12}Z^\beta\partial_t v\|_{L_t^2L^2}\Big(\varepsilon^{\frac12}\big\|\big[Z^\beta\partial_t,\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\big]v\big\|_{L_t^2L^2}+\varepsilon^{\frac12}\big\|Z^\beta\partial_t(f+\nabla^\varphi q+[\mathbb{P}_t,\partial_t^\varphi]u)\big\|_{L_t^2L^2}\Big)\\ &\lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

$\underline{\tilde{\mathcal{K}}_7.}$ For this term, one needs to integrate by parts to avoid losing normal derivatives. By following the same lines as the control of $\mathcal{K}_7$ in Lemma 9.3, we find that:

$$\tilde{\mathcal{K}}_7\leq\delta\|\varepsilon^{\frac12}\partial_t\nabla v\|^2_{L_t^2H_{co}^k}+\Lambda\big(\frac{1}{c_0},|h|^2_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\partial_t\nabla^\varphi v\|^2_{L_t^2H_{co}^{k-1}}+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\tag{9.54}$$

Plugging (9.49)-(9.54) into (9.47), we get by choosing δ small enough and by using Korn inequality (3.34) that for any $0\leq k\leq m-2$,

$$\begin{aligned}&\|\varepsilon^{\frac12}\partial_t v\|^2_{L_t^\infty H_{co}^k}+\|\varepsilon^{\frac12}\partial_t\nabla v\|^2_{L_t^2H_{co}^k}\lesssim\|\varepsilon^{\frac12}\partial_t v(0)\|^2_{H_{co}^{m-2}(\mathcal{S})}+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\ &+\Lambda\big(\frac{1}{c_0},|h|^2_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac12}\partial_t\nabla^\varphi v\|^2_{L_t^2H_{co}^{k-1}}+\|\varepsilon^{\frac12}\partial_t\nabla^\varphi\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-3}}+\|\varepsilon^{\frac12}\partial_t\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-2}}\big),\end{aligned}$$

where we have used the convention that $\|\cdot\|_{H_{co}^l}=0$, if $l<0$. This estimate, combined with (6.3), (8.5), (9.1) and the induction on k yields (9.46). $\Box$

10. $\varepsilon-$DEPENDENT HIGH ORDER ENERGY ESTIMATE-II

In this subsection, we aim to control $\varepsilon^{\frac12}\|\nabla u\|_{L_t^\infty H_{co}^{m-1}}$, which is useful for the control of L^∞ type norms.

Lemma 10.1. *Under the assumption* (2.2), *we have for any* $0<t\leq T$,

$$\varepsilon\|\nabla u\|^2_{L_t^\infty H_{co}^{m-1}}\lesssim\Lambda\big(\frac{1}{c_0},|h|^2_{L_t^\infty\tilde{H}^{m-\frac12}}+Y_m^2(0)\big)Y_m^2(0)+(T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\tag{10.1}$$

Proof. We will prove the following estimates:

(10.2)
$$\begin{aligned}&\|\varepsilon^{\frac12}\nabla u\|^2_{L_t^\infty H_{co}^{m-1}}\lesssim Y_m^2(0)+(T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\ &+\Lambda\big(\frac{1}{c_0},|h|^2_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac12}\nabla u\|^2_{L_t^2H_{co}^m}+\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-1}}+\|\varepsilon^{\frac12}\partial_t(u,\nabla u)\|^2_{L_t^2\mathcal{H}^{m-1}\cap L_t^2H_{co}^{m-2}}\big).\end{aligned}$$

By (7.1), (7.19), (7.30), (9.46), we can then find a polynomial Λ, such that (10.1) holds.

The inequality (10.2) can be obtained by direct energy estimates. Applying $Z^\alpha,|\alpha|\leq m-1$ to $(1.16)_2$, taking the scalar product with $-\varepsilon^2Z^\alpha(\mathrm{div}^\varphi\mathcal{L}^\varphi u)$ and integrating in space and time, we get by integration by parts that:

$$\begin{aligned}&\varepsilon\mu\int_{\mathcal{S}}|Z^\alpha S^\varphi u|^2(t)\,\mathrm{d}\mathcal{V}_t+\frac12\varepsilon\lambda\int_{\mathcal{S}}|Z^\alpha\mathrm{div}^\varphi u|^2(t)\,\mathrm{d}\mathcal{V}_t+\varepsilon\|Z^\alpha\mathrm{div}^\varphi\mathcal{L}^\varphi u\|^2_{L_t^2L^2}\\ &+\frac{a}{2}\varepsilon\int_{z=1}|Z^\alpha u_\tau|^2(t)\,\mathrm{d}y=K_0+K_1+\cdots+K_5,\end{aligned}\tag{10.3}$$

where

$$K_0=\varepsilon\mu\int_{\mathcal{S}}|Z^\alpha S^\varphi u|^2(0)\,\mathrm{d}\mathcal{V}_0+\frac{1}{2}\varepsilon\lambda\int_{\mathcal{S}}|Z^\alpha \mathrm{div}^\varphi u|^2(0)\,\mathrm{d}\mathcal{V}_0+\frac{a}{2}\varepsilon\int_{z=1}|Z^\alpha u_\tau|^2(0)\,\mathrm{d}y,$$

$$K_1=-\varepsilon\int_0^t\int_{\mathcal{S}}\big(\partial_t[\nabla^\varphi,Z^\alpha]u+[\nabla^\varphi,\partial_t]Z^\alpha u\big)\cdot Z^\alpha\mathcal{L}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$

$$K_2=\varepsilon\int_0^t\int_{\mathcal{S}}\partial_t Z^\alpha u\cdot[Z^\alpha,\mathrm{div}^\varphi]\mathcal{L}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\quad K_3=\varepsilon\int_0^t\int_{\mathcal{S}}Z^\alpha(\frac{g_2-1}{\varepsilon}\varepsilon\partial_t u)\cdot Z^\alpha\mathrm{div}^\varphi\mathcal{L}^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$

$$K_4=\varepsilon\int_0^t\int_{\mathcal{S}}Z^\alpha(\nabla^\varphi\sigma)Z^\alpha(\mathrm{div}^\varphi\mathcal{L}^\varphi u)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\quad K_5=-\varepsilon\int_0^t\int_{\partial\mathcal{S}}Z^\alpha\mathcal{L}^\varphi u\mathbf{N}\cdot\partial_t Z^\alpha u\,\mathrm{d}y\mathrm{d}s.$$

At first, by the trace inequality (3.17):

$$\frac{a}{2}\varepsilon\int_{z=1}|Z^\alpha u_\tau|^2(t)\,\mathrm{d}y\geq-\delta\|\varepsilon^{\frac12}\nabla u(t)\|^2_{H^{m-1}_{co}}-C_\delta\varepsilon\|u\|^2_{L^\infty_t H^{m-1}_{co}}.\tag{10.4}$$

Next, for the term K_1, we use (3.23) to find that:

$$K_1\lesssim\|\nabla^\varphi u\|_{L^2_tH^{m-1}_{co}}\bigg(\varepsilon\Lambda\big(\frac{1}{c_0},|\partial_t h|_{0,\infty,t}\big)\|\nabla u\|_{L^2_tH^{m-1}_{co}}+\|\varepsilon\partial_t[\nabla^\varphi,Z^\alpha]u\|_{L^2_tL^2}\bigg).$$

By using the identity (3.26), we find that:

$$\varepsilon\partial_t[Z^\alpha,\nabla^\varphi]u=\varepsilon^{\frac12}\big[Z^\alpha,\varepsilon^{\frac12}\partial_t(\frac{\mathbf{N}}{\partial_z\varphi})\big]\partial_z u+\varepsilon\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}[Z^\alpha,\partial_z]u\big)+\varepsilon^{\frac12}\big[Z^\alpha,\frac{\mathbf{N}}{\partial_z\varphi}\big]\varepsilon^{\frac12}\partial_t\partial_z u.$$

The $L^2_tL^2$ norm of the first two terms in the right hand side can be controlled by:

$$\begin{aligned}&\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},|(h,\varepsilon^{\frac12}\partial_t h)|_{m-2,\infty,t}+\|\nabla u\|_{1,\infty,t}\big)\big(\|\nabla u\|_{L^2_tH^{m-2}_{co}}+|(h,\varepsilon^{\frac12}\partial_t h)|_{L^2_t\tilde{H}^{m-\frac12}}\big)\\&\lesssim\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Moreover, the third term can be bounded as:

$$\begin{aligned}\big\|\varepsilon^{\frac12}\big[Z^\alpha,\frac{\mathbf{N}}{\partial_z\varphi}\big]\varepsilon^{\frac12}\partial_t\partial_z u\big\|_{L^2_tL^2}&\lesssim T^{\frac12}\|\varepsilon^{\frac12}\partial_t\nabla u\|_{L^2_tH^1_{co}}\big|\varepsilon^{\frac12}\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\big|_{m-2,\infty,t}\\&\quad+\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}\|\varepsilon^{\frac12}\partial_t\nabla u\|_{0,\infty,t}+|h|_{m-2,\infty,t}\big)(\|\varepsilon^{\frac12}\partial_t\nabla u\|_{L^2_tH^{m-2}_{co}}+|h|_{L^2_t\tilde{H}^{m-\frac12}})\\&\lesssim(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

The previous two estimates then lead to:

$$\|\varepsilon\partial_t[Z^\alpha,\nabla^\varphi]u\|_{L^2_tL^2}\lesssim(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),$$

from which we find that:

$$K_1\lesssim(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\tag{10.5}$$

Thanks to the commutator estimate (3.23), we control the term $\varepsilon^{\frac12}[Z^\alpha,\mathrm{div}^\varphi]\mathcal{L}^\varphi u$ in the term K_2 as follows:

$$\begin{aligned}\varepsilon^{\frac12}\|[Z^\alpha,\mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L^2_tL^2}&\lesssim\Lambda\big(\frac{1}{c_0},\|\varepsilon^{\frac12}\partial_z\mathcal{L}^\varphi u\|_{1,\infty,t}+|h|_{m-2,\infty,t}\big)|h|_{L^2_t\tilde{H}^{m-\frac12}}\\&\quad+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\nabla\mathcal{L}^\varphi u\|_{L^2_tH^{m-2}_{co}}\lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, by Cauchy-Schwarz inequality, K_2 can be bounded by:

$$K_2 \lesssim \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^2 H_{co}^{m-1}} \|\varepsilon^{\frac{1}{2}}[Z^\alpha, \mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L_t^2 L^2} \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{10.6}$$

Moreover, by the product estimate (3.8), we obtain:

$$K_3 + K_4 \leq \delta\|\varepsilon^{\frac{1}{2}}Z^\alpha \mathrm{div}^\varphi \mathcal{L}^\varphi u\|_{L_t^2 L^2}^2 + C_\delta \varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\|(\sigma, u)\|_{E^m, t}. \tag{10.7}$$

For the term K_5, we use the boundary condition (1.18) to split it as :

$$K_5 = -\varepsilon\int_0^t\int_{z=0} Z^\alpha(\sigma/\varepsilon)\partial_t Z^\alpha u\cdot \mathbf{N} + [Z^\alpha, \mathbf{N}]\mathcal{L}^\varphi u\cdot\partial_t Z^\alpha u\,\mathrm{d}y\mathrm{d}s =: K_{51} + K_{52}.$$

Thanks to the trace inequality (3.17) and the boundary conditions (4.5), (4.8), K_{52} can be bounded as:

$$\begin{aligned}
K_{52} &\lesssim |\varepsilon^{\frac{1}{2}}\partial_t Z^\alpha u|_{L_t^2 H^{-\frac{1}{2}}} |\varepsilon^{\frac{1}{2}}[Z^\alpha, \mathbf{N}]\mathcal{L}^\varphi u|_{L_t^2 H^{\frac{1}{2}}} \\
&\lesssim \big(\|\varepsilon^{\frac{1}{2}}\partial_t(u, \nabla u)\|_{L_t^2\mathcal{H}^{m-1}} + \|\varepsilon^{\frac{1}{2}}\partial_t\nabla u\|_{L_t^2 H_{co}^{m-2}}\big)\big(\varepsilon^{\frac{1}{2}}|h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big)\big) \\
&\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.
\end{aligned}$$

For K_{51}, we take benefits of the boundary condition (4.1) and the trace inequality (3.17) to find that, if $Z^\alpha = (\varepsilon\partial_t)^j, j \leq m-1$,

$$\begin{aligned}
K_{51} &\lesssim \varepsilon^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|(\varepsilon^{\frac{1}{2}}\partial_t(u, \nabla u), \mathrm{div}^\varphi u, \varepsilon^{\frac{1}{2}}\nabla\mathrm{div}^\varphi u)\|_{L_t^2\mathcal{H}^{m-1}}^2 + \|\nabla u\|_{L_t^2 H_{co}^{m-1}}^2 + |h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}^2\big) \\
&\lesssim \varepsilon^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2,
\end{aligned}$$

if $Z^\alpha = Z_3 Z^{\tilde{\alpha}}$, this term vanishes on the boundary and if $Z^\alpha = \partial_y Z^{\tilde{\alpha}}$,

$$\begin{aligned}
K_{51} &\lesssim |\varepsilon^{\frac{1}{2}}Z^\alpha(\sigma/\varepsilon)|_{L_t^2 H_y^{\frac{1}{2}}} |\varepsilon^{\frac{1}{2}}\partial_t Z^{\tilde{\alpha}} u|_{L_t^2 H_y^{\frac{1}{2}}} |h|_{2,\infty,t} \\
&\lesssim \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\big(\|\varepsilon^{\frac{1}{2}}\nabla\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}}^2 + \|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^2 H_{co}^m}^2 + \|\varepsilon^{\frac{1}{2}}\partial_t(u, \nabla u)\|_{L_t^2 H_{co}^{m-2}}^2\big) \\
&\quad + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}.
\end{aligned}$$

The previous three inequalities yield:

$$\begin{aligned}
K_5 \lesssim{}& (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2 + \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\cdot \\
&\big(\|\varepsilon^{\frac{1}{2}}\nabla\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}}^2 + \|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^2 H_{co}^m}^2 + \|\varepsilon^{\frac{1}{2}}\partial_t(u, \nabla u)\|_{L_t^2 H_{co}^{m-2}\cap L_t^2\mathcal{H}^{m-1}}^2\big).
\end{aligned}$$

Inserting this inequality and (10.4)-(10.7) into (10.3), using Korn's inequality (3.33) and choosing δ small enough, we obtain (10.2). □

11. Uniform control of high order energy norms-II

11.1. $L_t^\infty L^2$ **type norm for the compressible part.** In this section, we aim to get the a-priori estimates for $\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-2}}$. This is mainly done by induction arguments.

Lemma 11.1. *Suppose that (2.2) is true, we have for any $0 < t \le T, m \ge 7$,*

$$\begin{aligned} &\varepsilon^{-1}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-2}}^2 \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{L_t^\infty \tilde{H}^{m-\frac12}}^2 + Y_m^2(0)\big)Y_m^2(0) + (\varepsilon+T)^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \end{aligned} \tag{11.1}$$

Proof. We shall prove for for $j + l \le m - 2$ that:

$$\begin{aligned} &\varepsilon^{-\frac12}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty \mathcal{H}^{j,l}} \\ &\lesssim (T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\partial_t \mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac32}} \\ &\quad + \Lambda\big(\frac{1}{c_0}, |h|_{L_t^\infty \tilde{H}^{m-\frac12}}\big)(\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^\infty \mathcal{H}^{m-2,0}} + \|\varepsilon^{\frac12}\nabla(\sigma,u)\|_{L_t^\infty H_{co}^{m-1}}), \end{aligned} \tag{11.2}$$

and also:

$$\|\varepsilon^{\frac12}\partial_t(\mathrm{div}^\varphi u, \nabla^\varphi\sigma)\|_{L_t^\infty H_{co}^1} \lesssim \|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^\infty \mathcal{H}^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.3}$$

These two inequalities, together with (7.19), (7.30) and (10.1) lead to (9.1). Indeed, thanks to the estimate (7.19), we derive that:

$$\|\varepsilon^{\frac12}\partial_t(\mathrm{div}^\varphi u, \nabla^\varphi\sigma)\|_{L_t^\infty H_{co}^1} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)Y_m(0) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Inserting this inequality into (11.2), and using the estimate (7.19), (7.30), (10.1), we find (11.1).

We present the proof of (11.2). First of all, for any non-negative integers j, l such that $j + l \le m - 2$, it follows from the equation (9.5) that:

$$\begin{aligned} \varepsilon^{-\frac12}\|\mathrm{div}^\varphi u\|_{L_t^\infty \mathcal{H}^{j,l}} &\lesssim \|\varepsilon^{\frac12}\partial_t\sigma\|_{L_t^\infty \mathcal{H}^{j,l}} + \varepsilon^{\frac12}\|\big(\frac{g_1 - g_1(0)}{\varepsilon}\varepsilon\partial_t + g_1\underline{u}\cdot\nabla\big)\sigma\|_{L_t^\infty \mathcal{H}^{j,l}} \\ &\lesssim \|\varepsilon^{\frac12}\partial_t\sigma\|_{L_t^\infty \mathcal{H}^{m-2,0}} + \|\varepsilon^{-\frac12}\nabla^\varphi\sigma\|_{L_t^\infty \mathcal{H}^{j+1,l-1}}\mathbb{I}_{\{l\ge1\}} + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \end{aligned} \tag{11.4}$$

Let us control $\|\nabla^\varphi\sigma\|_{L_t^\infty \mathcal{H}^{j,l}}$. As before, we denote

$$\theta = \sigma/\varepsilon - 2(\mu+\lambda)\mathrm{div}^\varphi u.$$

By the equation of velocity,

$$\nabla^\varphi\theta = -\partial_t^\varphi u - f + \mu\Delta^\varphi v,$$

where

$$f = \frac{g_2 - \bar\rho}{\varepsilon}\varepsilon\partial_t^\varphi u + u\cdot\nabla^\varphi u, v = \mathbb{P}_t u.$$

We thus get that:

$$\begin{aligned} \varepsilon^{-\frac12}\|\nabla^\varphi\sigma\|_{L_t^\infty \mathcal{H}^{j,l}} &\lesssim \varepsilon^{\frac12}\|\nabla^\varphi \mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac12}\|(\mathbb{P}_t, \mathbb{Q}_t)\nabla^\varphi\theta\|_{L_t^\infty \mathcal{H}^{j,l}} \\ &\lesssim \varepsilon^{\frac12}\|\partial_t^\varphi\nabla^\varphi\Psi\|_{L_t^\infty \mathcal{H}^{j,l}} + \varepsilon^{\frac12}\|\nabla^\varphi \mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^{m-2}} \\ &\quad + \varepsilon^{\frac12}\big\|\big(\nabla^\varphi\pi, [\mathbb{Q}_t, \partial_t^\varphi]u, \nabla^\varphi q\big)\big\|_{L_t^\infty H_{co}^{m-2}} =: (11.5)_1 + (11.5)_2 + (11.5)_3. \end{aligned} \tag{11.5}$$

where we have used the defintion (9.7),(9.8). By the elliptic estimate (5.24),

$$
\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}\Psi\|_{L_t^\infty\mathcal{H}^{j,l}}\\
&\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac{3}{2}}}\\
&+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty\mathcal{H}^{m-2,0}}+\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi}u\|_{L_t^\infty\mathcal{H}^{j,l-1}}\mathbb{I}_{\{l\geq 1\}}\big).
\end{aligned}\tag{11.6}
$$

Next, by the elliptic estimate (5.13), we find:

$$
\begin{aligned}
\varepsilon^{\frac{1}{2}}\big\|\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\nabla^{\varphi}\Psi\big\|_{L_t^\infty H_{co}^{m-2}}&\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\big(\|\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^{m-2}}+|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}+|\partial_t h|_{L_t^\infty\tilde{H}^{m-2}}\big)\\
&\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Together with (11.6), this yields:

$$
\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}\partial_t^{\varphi}\nabla^{\varphi}\Psi\|_{L_t^\infty\mathcal{H}^{j,l}}\\
&\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac{3}{2}}}\\
&+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty\mathcal{H}^{m-2,0}}+\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi}u\|_{L_t^\infty\mathcal{H}^{j,l-1}}\mathbb{I}_{\{l\geq 1\}}\big).
\end{aligned}\tag{11.7}
$$

Let us control the terms $(11.5)_2, (11.5)_3$ appearing in (11.5):

- $(11.5)_2=\varepsilon^{\frac{1}{2}}\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^{m-2}}$. Thanks to the equation (9.3), we have:

$$
\begin{aligned}
&\varepsilon^{\frac{1}{2}}\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^{m-2}}\\
&\leq\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^\infty H_{co}^{m-1}}+\varepsilon^{\frac{3}{2}}\big\|\big(\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\sigma,\nabla^{\varphi}(\frac{g_2-1}{\varepsilon}\varepsilon\partial_t\sigma),\nabla^{\varphi}(g_2\underline{u}\cdot\nabla\sigma)\big)\big\|_{L_t^\infty H_{co}^{m-2}}\\
&\lesssim\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^\infty H_{co}^{m-1}}+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}\tag{11.8}
$$

- $(11.5)_3=\varepsilon^{\frac{1}{2}}\|(\nabla^{\varphi}q,\nabla^{\varphi}\pi,[\mathbb{Q}_t,\partial_t^{\varphi}]u)\|_{L_t^\infty H_{co}^{m-2}}$. By (9.10), (9.13), (9.15), we have that:

$$
\varepsilon^{\frac{1}{2}}\|(\nabla^{\varphi}q,\nabla^{\varphi}\pi,[\mathbb{Q}_t,\partial_t^{\varphi}]u)\|_{L_t^\infty H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^\infty H_{co}^{m-1}}+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\tag{11.9}
$$

Inserting (11.7)-(11.9) into (11.5), we achieve that:

$$
\begin{aligned}
\|\nabla^{\varphi}\sigma\|_{L_t^\infty\mathcal{H}^{j,l}}&\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac{3}{2}}}\\
&+(\|\mathrm{div}^{\varphi}u\|_{L_t^\infty\mathcal{H}^{j+1,l-1}}+\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty\mathcal{H}^{m-2,0}}+\|\varepsilon^{\frac{1}{2}}\nabla(\sigma,u)\|_{L_t^\infty H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big).
\end{aligned}\tag{11.10}
$$

Together with (11.4) and induction arguments, this yields (11.2). □

Remark 11.2. *By the estimates* (5.27) (5.26) *and* (9.1), (11.1), *we find*

$$
\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}\Psi\|_{L_t^\infty H_{co}^{m-2}}+\|\varepsilon^{\frac{1}{2}}\partial_t(\nabla^{\varphi})^2\Psi\|_{L_t^\infty H_{co}^{m-3}\cap L_t^2 H_{co}^{m-2}}\\
&\lesssim\Lambda\big(\frac{1}{c_0},|h|^2_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}+Y_m^2(0)\big)Y_m^2(0)+(\varepsilon+T)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

which further, together with (9.46)*, yields that:*

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H_{co}^{m-2}}+\|\varepsilon^{\frac{1}{2}}\partial_t\nabla u\|_{L_t^2 H_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0},|h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}+Y_m^2(0)\big)Y_m^2(0)+(\varepsilon+T)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}\tag{11.11}$$

11.2. **Uniform control of the gradient of the velocity-II.** In this subsection, we aim to control the $L_t^\infty H_{co}^{m-4}$ norm of $(\nabla u,\varepsilon^{\frac{1}{2}}\partial_t\nabla u)$ More precisely, the following lemma will be proved.

Lemma 11.3. *Under the assumption* (2.2)*, for any* $0<t\le T$*, we have the following estimate:*

$$\begin{aligned}&\|\nabla u\|^2_{L_t^\infty H_{co}^{m-4}}+\|\varepsilon^{\frac{1}{2}}\partial_t\nabla u\|^2_{L_t^\infty H_{co}^{m-4}}\\ &\lesssim \Lambda\bigg(\frac{1}{c_0},|h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}+Y_m^2(0)\bigg)Y_m^2(0)+(T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}\tag{11.12}$$

Proof. By the identities (4.8) and

$$\begin{aligned}|\mathbf{N}|\Pi(\partial_z^\varphi u)&=\Pi(\partial_{\mathbf{n}}^\varphi u-\mathbf{n}_1\partial_1 u-\mathbf{n}_2\partial_2 u)\\ &=\omega\times\mathbf{n}+\Pi\big((\nabla^\varphi u)^t\cdot\mathbf{n}-\mathbf{n}_1\partial_1 u-\mathbf{n}_2\partial_2 u\big)\\ &=\omega\times\mathbf{n}+\Pi(\partial_1 u\cdot\mathbf{n},\partial_2 u\cdot\mathbf{n},0)^t-\Pi(\mathbf{n}_1\partial_1 u+\mathbf{n}_2\partial_2 u),\end{aligned}$$

we have that:

$$\|\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-4}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|u\|_{L_t^\infty H_{co}^{m-3}}+\|(\omega\times\mathbf{n},\mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-4}},$$

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-4}}\lesssim{}&\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H_{co}^{m-3}}+\|\varepsilon^{\frac{1}{2}}\partial_t(\omega\times\mathbf{n},\mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-4}}\\ &+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, (11.12) can be derived from the estimate (11.11), Lemma 7.4 for $\mathrm{div}^\varphi u$, Lemma 9.3 for v, Lemma 6.1 for h as well as the next lemma for $\omega\times\mathbf{n}$. □

Lemma 11.4. *Suppose that Assumption* (2.2) *is true, then the following estimate holds:*

$$\begin{aligned}\|\omega\times\boldsymbol{n}\|^2_{L_t^\infty H_{co}^{m-4}}+\|\varepsilon^{\frac{1}{2}}\partial_t(\omega\times\boldsymbol{n})\|^2_{L_t^\infty H_{co}^{m-4}}\lesssim{}&Y_m^2(0)+(T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\ &+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|(v,\varepsilon^{\frac{1}{2}}\partial_t v,\varepsilon^{\frac{1}{2}}\nabla u)\|^2_{L_t^\infty H_{co}^{m-2}}.\end{aligned}\tag{11.13}$$

Proof. As explained in the introduction, although $\omega\times\mathbf{n}$ satisfies a transport-diffusion equation without singular terms, one cannot control it by direct energy estimates due to the lack of information of the trace of $\omega\times\mathbf{n}$ on the boundary. Since

$$(\omega\times\mathbf{n})|_{z=0}=2\Pi(\partial_1 u\cdot\mathbf{n},\partial_2 u\cdot\mathbf{n},0)^t|_{z=0}.$$

A natural attempt in order to do energy estimates is to introduce the modified vorticity: $\tilde{\omega}=\omega\times\mathbf{n}-\Pi(\partial_1 u\cdot\mathbf{n},\partial_2 u\cdot\mathbf{n},0)^t$. Nevertheless, if taking this way, we are confronted with the original difficulty due to the presence of a singular term in the equation of $\omega\times\mathbf{n}$. However, since the singular term appears only in the equation of the compressible part of the velocity, it is still useful to introduce the following quantity:

$$\omega_{\mathbf{n}}=\omega\times\mathbf{n}-2\Pi(\partial_1 v\cdot\mathbf{n},\partial_2 v\cdot\mathbf{n},0)^t.\tag{11.14}$$

where v is the incompressible part of the velocity. As will be seen later, the main advantage to work on $\omega_{\mathbf{n}}$ rather than $\omega \times \mathbf{n}$ is that up to remainders, one can reduce the estimate of $\omega_{\mathbf{n}}$ to that of the compressible part of the velocity and one can extract some extra power of T in the estimates, which is essential to establish the local existence on a uniform time interval.

Since away from the boundary, the conormal space is equivalent to the standard Sobolev space, it suffices to estimate $\omega_{\mathbf{n}}$ near the boundary. In the following, we shall focus on its control near the surface, the case near the bottom is similar (and is even simpler, one can refer to [47] for details). To overcome the difficulty resulting from the nontrivial boundary condition, the general strategy to get a uniform estimate for $\omega_{\mathbf{n}}$ is to split its system into two systems, one carries on the nonlinear terms and the initial data but with trivial Dirichlet boundary condition, while the other one is just a free heat equation with zero initial data and nontrivial Dirichlet boundary condition. The first system can be treated by direct energy estimates because of the homogeneous Dirichlet boundary condition. The analysis of the second system relies on the explicit formulae for the heat equation in the half-space.

To use the explicit formulae of the heat equation in the half-space, it is convenient to use a coordinate system in which the Laplacian has a good form. We thus use the following normal geodesic coordinates:

$$\begin{aligned} \tilde{\Phi}_t: \quad \mathcal{S}_\kappa = & \mathbb{R}^2 \times [-\kappa, 0] \longrightarrow \Omega_t \\ & (y,z) \to \begin{pmatrix} y \\ h(t,y) \end{pmatrix} + z\mathbf{n}^{b,1}(y) \end{aligned} \tag{11.15}$$

where $\mathbf{n}^{b,1} = \frac{\mathbf{N}^{b,1}}{|\mathbf{N}^{b,1}|} = (-\partial_1 h, -\partial_2 h, 1)/\sqrt{1+|\nabla h|^2}$ denotes the outward normal vector. Straightforward computations show that:

$$\mathrm{D}\tilde{\Phi}_t = \begin{pmatrix} 1 & 0 & \mathbf{n}_1^{b,1} \\ 0 & 1 & \mathbf{n}_2^{b,1} \\ \partial_1 h & \partial_2 h & \mathbf{n}_3^{b,1} \end{pmatrix} + z \begin{pmatrix} \partial_1 \mathbf{n}_1^b & \partial_2 \mathbf{n}_1^b & 0 \\ \partial_1 \mathbf{n}_2^b & \partial_2 \mathbf{n}_2^b & 0 \\ \partial_1 \mathbf{n}_3^b & \partial_2 \mathbf{n}_3^b & 0 \end{pmatrix}$$

Therefore, as long as $|h|_{2,\infty,T} < +\infty$, and κ small enough, we have that: $\det(D\tilde{\Phi}_t) > 0$ on $[0,T] \times \mathcal{S}_\kappa$, hence $\tilde{\Phi}_t$ is a diffeomorphism between $\mathcal{S}_\kappa$ and $\tilde{\Phi}_t(\mathcal{S}_\kappa)$. The Riemann metric induced by the pullback of the Euclidean metric in Ω_t through $\tilde{\Phi}_t^{-1}$ has the block structure:

$$g(y,z) = \begin{pmatrix} \tilde{g}(y,z) & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tilde{g}$ is a matrix that depends on the gradient of $\tilde{\Phi}_t$. Therefore, the Laplacian in this metric takes the form:

$$\Delta_g f = \partial_z^2 f + \frac{1}{2}\partial_z(\ln|g|)\partial_z f + \Delta_{\tilde{g}} f, \tag{11.16}$$

where

$$\Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1\leq i,j\leq 2} \partial_{y^i}(\tilde{g}^{ij}|\tilde{g}|^{\frac{1}{2}}\partial_{y^j} f) \quad |\tilde{g}| = \det \tilde{g}.$$

We take a cut off function $\chi = \chi_0(\frac{z}{C(\kappa)})$, where $\chi_0(s): \mathbb{R}_- \to \mathbb{R}$ is a smooth function supported on $[-\frac{3}{4}, 0]$ and equal to 1 on the interval $[-\frac{1}{2}, 0]$, $C(\kappa)$ is chosen such that $\Phi_t(\mathbb{R}^2 \times [-C_\kappa, 0]) \subset \tilde{\Phi}_t(\mathcal{S}_\kappa)$, the following task is to estimate $\chi\omega_{\mathbf{n}}$. Let us begin with the derivation of the equations satisfied by $\chi\omega_{\mathbf{n}}$. First of all, by taking the curl of $(1.16)_2$, we find that $\omega = \mathrm{curl}^\varphi u$ solves:

$$(\bar{\rho}\partial_t^\varphi - \mu\Delta^\varphi)\omega = G^\omega \tag{11.17}$$

with

$$G^\omega = -u\cdot\nabla^\varphi\omega + \omega\cdot\nabla^\varphi u - \omega\mathrm{div}^\varphi u - \frac{\nabla g_2}{\varepsilon}\times((\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)u) + \frac{\bar\rho-g_2}{\varepsilon}((\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)\omega).$$

Hence $\chi\omega\times\mathbf{n}$ is governed by:

$$(\bar\rho\partial_t^\varphi-\mu\Delta^\varphi)(\chi\omega\times\mathbf{n}) = G_\chi^\omega$$

with

$$G_\chi^\omega = \chi G^\omega\times\mathbf{n} - \mu\Delta^\varphi(\chi\mathbf{n})\omega - 2\mu\nabla^\varphi\omega\times\nabla^\varphi(\chi\mathbf{n}) + \bar\rho\omega\times\partial_t^\varphi(\chi\mathbf{n}). \tag{11.18}$$

By (9.6), v satisfies the equation:

$$\bar\rho\partial_t^\varphi v - \mu\Delta^\varphi v = -(f+\nabla^\varphi q + \bar\rho[\mathbb{P}_t,\partial_t^\varphi]u) - \nabla^\varphi\pi =: H,$$

which gives:

$$(\bar\rho\partial_t^\varphi-\mu\Delta^\varphi)(\partial_j v\cdot\mathbf{N}) = L_j$$

with

$$L_j = [\partial_j H + \partial_j(\frac{\partial_t\varphi}{\partial_z\varphi})\partial_z v - \mu[\partial_j,\Delta^\varphi]v]\cdot\mathbf{N} + \bar\rho\partial_j v\cdot\partial_t^\varphi\mathbf{N} - 2\mu\nabla^\varphi\partial_j v\cdot\nabla^\varphi\mathbf{N} - \mu\Delta^\varphi\mathbf{N}\cdot\partial_j v.$$

Denote $\varsigma = 2(\partial_1 v\cdot\mathbf{N},\partial_2 v\cdot\mathbf{N},0)^t, L=(L_1,L_2,0)^t$. Therefore, by recalling the definition of projection $\Pi = \mathrm{Id}_3-\mathbf{n}\otimes\mathbf{n}$, it holds that:

$$(\bar\rho\partial_t^\varphi-\mu\Delta^\varphi)(\chi\Pi\varsigma) = G_\chi^\varsigma$$

where

$$G_\chi^\varsigma = 2\chi\Pi L + \bar\rho\chi[\partial_t,\Pi]\varsigma - \bar\rho\chi\frac{\partial_t\varphi}{\partial_z\varphi}[\partial_z,\Pi]\varsigma + \mu\chi[\Pi,\Delta^\varphi]\varsigma + \bar\rho[\partial_t^\varphi,\chi]\Pi\varsigma + \mu[\chi,\Delta^\varphi]\Pi\varsigma. \tag{11.19}$$

We thus finally find that:

$$(\bar\rho\partial_t^\varphi-\mu\Delta^\varphi)(\chi\omega_n) = G_\chi^\varsigma + G_\chi^\omega. \tag{11.20}$$

For the sake of notational simplicity, we denote $\zeta = \chi\omega_{\mathbf{n}}, G_\chi^\zeta = G_\chi^\varsigma + G_\chi^\omega$. Consider

$$\tilde\zeta(t,x) = \zeta(t,\Phi_t^{-1}\circ\tilde\Phi_t(x)),$$

then $\tilde\zeta:[0,T]\times\mathcal{S}_\kappa\to\mathbb{R}$ solves the system:

$$\begin{cases} (\bar\rho\partial_t-\mu\Delta_g)\tilde\zeta = \widetilde{G_\chi^\zeta} + \bar\rho(\mathrm{D}\tilde\Phi_t)^{-1}\partial_t\tilde\Phi_t\cdot\nabla\tilde\zeta, \\ \tilde\zeta|_{t=0} = \zeta(\Phi_0^{-1}\circ\tilde\Phi_0), \\ \tilde\zeta|_{z=0} = -2\Pi(\partial_1\nabla^\varphi\Psi\cdot\mathbf{n},\partial_2\nabla^\varphi\Psi\cdot\mathbf{n},0)^t|_{z=0}. \end{cases}$$

where Δ_g is defined in (11.16). Since $\tilde\zeta$ vanishes in the vicinity of $\{z=-\kappa\}$, we can extend it by zero to the whole lower half space $\mathbb{R}^3_-$. Denote

$$\|f\|_{L_t^pH_{co}^k(\mathbb{R}^3_-)} = \sum_{|\alpha|\le k}\|Z^\alpha f\|_{L_t^pL^2(\mathbb{R}^3_-)}.$$

By Proposition 11.5, we have:

$$\|\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathcal{S})} \lesssim \|\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} \lesssim \Lambda(\frac{1}{c_0},|h|_{m-2,\infty,t})\|\tilde\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)},$$

$$\begin{aligned}&\|\varepsilon^{\frac12}\partial_t\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathcal{S})} \lesssim \|\varepsilon^{\frac12}\partial_t\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} \\ &\lesssim \Lambda(\frac{1}{c_0},|h|_{m-2,\infty,t})(\|\varepsilon^{\frac12}\partial_t\tilde\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} + \varepsilon^{\frac12}\|\tilde\zeta\|_{L_t^\infty H_{co}^{m-3}(\mathbb{R}^3_-)}) + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),\end{aligned}$$

$$\varepsilon^{\frac12}\|\tilde\zeta\|_{L_t^\infty H_{co}^{m-3}(\mathbb{R}^3_-)}) \lesssim \Lambda(\frac{1}{c_0},|h|_{m-2,\infty,t})\|\varepsilon^{\frac12}\zeta\|_{L_t^\infty H_{co}^{m-3}(\mathcal{S})}$$

$$\lesssim \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^\infty H_{co}^{m-3}(\mathcal{S})} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Therefore, (11.13) follows from the estimate:

$$\|(\tilde{\zeta}, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta})\|_{L_t^\infty H_{co}^{m-4}(\mathcal{S})} \lesssim Y_m^2(0) + (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \tag{11.21}$$

which is the consequence of Lemma 11.7 and Lemma 11.8. □

Proposition 11.5. *Suppose that $\mathcal{T}_t : \mathbb{R}_-^3 \to \mathbb{R}_-^3$ is a C^{m-3} diffeomorphism with $\mathcal{T}_t(y,0) = y, \forall y \in \mathbb{R}^2$. For any function $f(t,\cdot)$ which supported on $\mathcal{S}_\kappa$, and for $p = 2, +\infty$, it holds that*

$$\|\!|f(s,\mathcal{T}_s\cdot)|\!\|_{k,\infty,t} \lesssim \Lambda(\|\!|(\mathcal{T}, \partial_z\mathcal{T})|\!\|_{k,\infty,t})\|\!|f|\!\|_{k,\infty,t}, \tag{11.22}$$

$$\|f(s,\mathcal{T}_s\cdot)\|_{L_t^pH_{co}^k(\mathbb{R}_-^3)} \lesssim \Lambda(\|\!|(\mathcal{T}, \partial_z\mathcal{T})|\!\|_{k,\infty,t})\|f\|_{L_t^pH_{co}^k(\mathbb{R}_-^3)}, \tag{11.23}$$

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\partial_s[f(s,\mathcal{T}_s\cdot)]\|_{L_t^pH_{co}^k(\mathbb{R}_-^3)} &\lesssim \Lambda(\|\!|(\mathcal{T}, \partial_z\mathcal{T})|\!\|_{k,\infty,t})\|\varepsilon^{\frac{1}{2}}(\partial_t, \mathcal{Z})f\|_{L_t^pH_{co}^k(\mathbb{R}_-^3)}\\ &+ \varepsilon^{\frac{1}{2}}\Lambda(\|\!|\partial_t(\mathcal{T}, \partial_z\mathcal{T})|\!\|_{k-1,\infty,t})\|f\|_{L_t^pH_{co}^k(\mathbb{R}_-^3)} + \|\!|\mathcal{Z}\tilde{\zeta}|\!\|_{0,\infty,t}\Lambda(\|\partial_t\partial_z\mathcal{T}\|_{L_t^\infty H_{co}^k})\end{aligned} \tag{11.24}$$

where we denote $\mathcal{Z} = (\partial_{y_1}, \partial_{y_2}, Z_3)$ the spatial conormal derivatives.

Remark 11.6. *Since $\Phi_t^{-1}\circ\tilde{\Phi}_t = \Phi_t^{-1}(t, y_1 + z\boldsymbol{n}_1^{b,1}, y_2 + z\boldsymbol{n}_2^{b,1}, h + z\boldsymbol{n}_3^{b,1})$, and $|D\Phi_t^{-1}| \lesssim |h|_{1,\infty,t}$, we have that:*

$$\|\!|(\Phi_t^{-1}\circ\tilde{\Phi}_t, \partial_z(\Phi_t^{-1}\circ\tilde{\Phi}_t))|\!\|_{k,\infty,t} \lesssim \Lambda(\frac{1}{c_0}, |h|_{k+1,\infty,t}).$$

Proof. The proof of this lemma just follows from the Leibniz rule, we thus omit the proof. □

As explained before, to show (11.21), we write $\tilde{\zeta} = \tilde{\zeta}_1 + \tilde{\zeta}_2$, where $\tilde{\zeta}_1, \tilde{\zeta}_2$ satisfy the following two systems:

$$\begin{cases} (\bar{\rho}\partial_t - \mu\partial_z^2)\tilde{\zeta}_1 = 0, \quad (t,x) \in [0,T]\times\mathbb{R}_-^3,\\ \tilde{\zeta}_1|_{t=0} = 0, \quad \tilde{\zeta}_1|_{z=0} = \tilde{\zeta}|_{z=0} = -2\Pi(\partial_1\nabla^\varphi\Psi\cdot\mathbf{n}, \partial_2\nabla^\varphi\Psi\cdot\mathbf{n}, 0)^t|_{z=0}.\end{cases} \tag{11.25}$$

$$\begin{cases} (\bar{\rho}\partial_t - \mu\Delta_g)\tilde{\zeta}_2 = \widetilde{G_\chi^\zeta} + \bar{\rho}\partial_t\tilde{\Phi}_t(\mathrm{D}\tilde{\Phi}_t)^{-1}\nabla\tilde{\zeta} + \frac{1}{2}\mu\partial_z(\ln|g|)\partial_z\tilde{\zeta}_1 - \mu\Delta_{\tilde{g}}\tilde{\zeta}_1,\\ \tilde{\zeta}_2|_{t=0} = \tilde{\zeta}|_{t=0}, \quad \tilde{\zeta}_2|_{z=0} = 0.\end{cases} \tag{11.26}$$

Lemma 11.7. *Under the assumption (2.2), it holds that, for any $0 < t \le T$,*

$$\|(\tilde{\zeta}_1, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1)\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}_-^3)} + \|(\tilde{\zeta}_1, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1)\|_{L_t^2H_{co}^{m-3}(\mathbb{R}_-^3)} \lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \tag{11.27}$$

$$\|(Id, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y, Z_3)\tilde{\zeta}_1\|_{L^\infty([0,T]\times\mathbb{R}_-^3)} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.28}$$

Proof. We present the estimates for $\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1$ appearing in the inequality (11.27), the estimates for $\tilde{\zeta}_1$ is similar and easier. Let $\gamma = (\gamma', \gamma_3)$ a multi-index such that $|\gamma| \le m-4$, $Z^\gamma = Z_{tan}^{\gamma'}Z_3^{\gamma_3}$ where $Z_{tan}^{\gamma'} = Z_0^{\gamma_0}Z_1^{\gamma_1}Z_2^{\gamma_2}$. Taking $Z_{tan}^{\gamma'}$ on the equation of (11.25), we get:

$$\begin{cases} (\bar{\rho}\partial_t - \mu\partial_z^2)(Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}_1) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}_-^3,\\ Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}_1|_{t=0} = 0, \quad Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}_1|_{z=0} = Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}|_{z=0}.\end{cases}$$

By the explicit formulae of the heat equation on the half-line, we have that:

$$\varepsilon^{\frac{1}{2}} Z^{\gamma}\partial_t\tilde{\zeta}_1(t,y,z) = 2\tilde{\mu}\varepsilon^{\frac{1}{2}}\int_0^t \frac{1}{(4\pi\tilde{\mu}(t-s))^{\frac{1}{2}}} Z_3^{\gamma_3}\partial_z\big(e^{-\frac{z^2}{4\tilde{\mu}(t-s)}}\big) Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}|_{z=0}(s,y)\,\mathrm{d}s \tag{11.29}$$

where $\tilde{\mu}=\mu/\bar{\rho}$. To continue, we need the following estimate whose proof is elementary and is left for the reader: for any $l\geq 0$

$$\|Z_3^l\partial_z(e^{-\frac{z^2}{4\tilde{\mu}(t-s)}})\|_{L_z^2(0,\infty)} \lesssim (t-s)^{-\frac{1}{4}}. \tag{11.30}$$

Now, taking the $L_z^2L_y^2$ norm of (11.29) and applying (11.30), we find that for any $0<t\leq T$,

$$\varepsilon^{\frac{1}{2}}\|Z^{\gamma}\partial_t\tilde{\zeta}_1\|_{L_t^\infty L^2(\mathbb{R}_+^3)} \lesssim T^{\frac{1}{4}}|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}|_{z=0}|_{L_t^\infty\tilde{H}^{m-4}}.$$

By the trace inequality (3.17) and the estimate (5.27), we get that:

$$\begin{aligned}
|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1|_{z=0}|_{L_t^\infty\tilde{H}^{m-4}} &\lesssim |(\nabla^\varphi\Psi,\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi\Psi)|_{L_t^\infty\tilde{H}^{m-3}}\Lambda\big(\frac{1}{c_0},|(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{m-3,\infty,t}\big)\\
&\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t(\nabla^\varphi\Psi,\nabla\nabla^\varphi\Psi),(\nabla^\varphi\Psi,\nabla\nabla^\varphi\Psi)\|_{L_t^\infty H_{co}^{m-3}(\mathcal{S})}\Lambda\big(\frac{1}{c_0},|(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{m-3,\infty,t}\big)\\
&\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}$$

Combined the previous two inequalities, one finds:

$$\|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}_-^3)} \lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

Similarly, by employing Young's inequality and the estimate (5.25), we obtain that:

$$\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1\|_{L_t^2 H_{co}^{m-3}(\mathbb{R}_-^3)} \lesssim T^{\frac{1}{4}}|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}|_{z=0}|_{L_t^2\tilde{H}^{m-3}}\\
&\lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},|(h,\varepsilon\partial_t h|_{m-2,\infty,t}\big)\cdot\|\varepsilon^{\frac{1}{2}}\partial_t(\nabla^\varphi\Psi,\nabla\nabla^\varphi\Psi),\varepsilon^{-\frac{1}{2}}(\nabla^\varphi\Psi,\nabla\nabla^\varphi\Psi)\|_{L_t^2H_{co}^{m-2}(\mathcal{S})}\\
&\lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}$$

The above inequality then leads to (11.27). We now show the $L_{t,x}^\infty$ estimate (11.28). It results from (11.29) that: for any $t>0, z>0, j=1,2, Z^0=\mathrm{Id}, Z^1=(\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)$,

$$\begin{aligned}
\|Z_{tan}^j\tilde{\zeta}_1(t,\cdot,z)\|_{L_y^\infty} &\leq \big|Z_{tan}^j\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty}\int_0^t\sqrt{2\pi^{-1}\tilde{\mu}^2}z^{-2}\big(\frac{z^2}{2\tilde{\mu}(t-s)}\big)^{\frac{3}{2}}e^{-\frac{z^2}{4\tilde{\mu}(t-s)}}\mathrm{d}s\\
&\leq C(\tilde{\mu})\big|Z_{tan}^j\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty} \lesssim \Lambda(\varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\Psi\|_{2,\infty,t}+|h|_{2,\infty,t}+|\varepsilon^{\frac{1}{2}}\partial_t h|_{1,\infty,t})
\end{aligned} \tag{11.31}$$

where $C(\tilde{\mu})$ is a constant that depends only on $\tilde{\mu}$. In the same fashion, we have

$$\begin{aligned}
\|Z_3\tilde{\zeta}_1(t,\cdot,z)\|_{L_y^\infty} &\leq \big(\sqrt{2\pi^{-1}\tilde{\mu}^2}\phi(z)z^{-1}\int_0^t z^{-2}P\big(\frac{z}{\sqrt{2\tilde{\mu}s}}\big)\mathrm{d}s\big)\big|\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty}\\
&\leq C(\tilde{\mu})\big|\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty} \lesssim \Lambda(\|\nabla^\varphi\Psi\|_{1,\infty,t}+|h|_{1,\infty,t}),
\end{aligned} \tag{11.32}$$

where $P(z)=|(1-z^2)|z^3e^{-z^2}$. Note that $\phi(z)z^{-1}=(1+z)/(2-z)^2$ is uniformly bounded for $z>0$. The proof of (11.28) is now finished. $\square$

Lemma 11.8. *Suppose that (2.2) holds, for any $0<t\leq T$, we have the following estimates:*

$$\|\tilde{\zeta}_2\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}_-^3)}^2 + \|(\nabla\tilde{\zeta}_2,\varepsilon^{\frac{1}{2}}\partial_t\nabla\tilde{\zeta}_2)\|_{L_t^2H_{co}^{m-4}(\mathbb{R}_-^3)}^2 \lesssim Y_m^2(0)+T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big), \tag{11.33}$$

$$(11.34)\quad \|\varepsilon^{\frac12}\partial_t\tilde{\zeta}_2\|^2_{L^\infty_t H^{m-4}_{co}(\mathbb{R}^3_-)} + \|\varepsilon^{\frac12}\partial_t\nabla\tilde{\zeta}_2\|^2_{L^2_t H^{m-4}_{co}(\mathbb{R}^3_-)} \lesssim \Lambda\big(Y_m^2(0)+\widetilde{\mathcal{E}}^2_{m,t}\big)Y_m^2(0) + T^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

Proof. Again, we only give the details for the estimate of $\varepsilon^{\frac12}\partial_t\tilde{\zeta}_2$, the one of $\tilde{\zeta}_2$ is similar and slightly easier to deal with. Let β be a multi-index such that $|\beta| = k \le m-4$. Since

$$\Delta_{\tilde{g}} = \partial_i(\tilde{g}^{ij}\partial_j\cdot) - \partial_i(|\tilde{g}|^{-\frac12})\tilde{g}^{ij}|\tilde{g}|^{\frac12}\partial_i f,$$

to avoid losing derivatives on the surface, it is convenient to rewrite the system (11.26) as:

$$(11.35)\qquad \begin{cases} \big(\bar{\rho}\partial_t - \mu\partial_z^2 - \mu\partial_i(\tilde{g}^{ij}\partial_j\cdot)\big)\tilde{\zeta}_2 = F_\chi^{\tilde{\zeta}}, \\ \tilde{\zeta}_2|_{t=0} = \tilde{\zeta}|_{t=0}, \quad \tilde{\zeta}_2|_{z=0} = 0, \end{cases}$$

where

$$F_\chi^{\tilde{\zeta}} = \widetilde{G_\chi^{\zeta}} - \bar{\rho}\partial_t\tilde{\Phi}_t(\mathrm{D}\tilde{\Phi}_t)^{-1}\nabla\tilde{\zeta} + \frac12\mu\partial_z(\ln|g|)\partial_z\tilde{\zeta} + \mu\partial_i(\ln|g|)\tilde{g^{ij}}\partial_j\tilde{\zeta} + \mu\partial_i(\tilde{g}^{ij}\partial_j\tilde{\zeta}_1).$$

Note that we have used the summation convention for $i,j = 1,2$. Applying Z^β on the equation (11.35), we get that:

$$\varepsilon^{\frac12}\big(\bar{\rho}\partial_t - \mu\partial_z^2 - \mu\partial_i(\tilde{g}^{ij}\partial_j)\big)(Z^\beta\partial_t\tilde{\zeta}_2) = Z^\beta\varepsilon^{\frac12}\partial_t F_\chi^{\tilde{\zeta}} + \mu[Z^\beta\varepsilon^{\frac12}\partial_t,\partial_z^2]\tilde{\zeta} + \mu\partial_i[Z^\beta\varepsilon^{\frac12}\partial_t,\tilde{g}^{ij}]\tilde{\zeta},$$

from which we get the energy inequality:

$$\begin{aligned} &\bar{\rho}\varepsilon\|Z^\beta\partial_t\tilde{\zeta}_2(t)\|^2_{L^2(\mathbb{R}^3_-)} + \mu\varepsilon\|\partial_z Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L^2_tL^2(\mathbb{R}^3_-)} + \mu\int_0^t\int_{\mathbb{R}^3_-}\tilde{g}_{ij}\partial_i Z^\beta\partial_t\tilde{\zeta}_2\cdot\partial_j Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s \\ (11.36)\qquad &\le \bar{\rho}\varepsilon\|Z^\beta\partial_t\tilde{\zeta}(0)\|^2_{L^2(\mathbb{R}^3_-)} + \mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta\partial_t,\partial_z^2]\tilde{\zeta}_2\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big| \\ &+ \mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta,\tilde{g}^{ij}]\partial_j\tilde{\zeta}_2\partial_i Z^\beta\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big| + \varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}Z^\beta F_\chi^{\tilde{\zeta}}\cdot Z^\beta\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big|. \end{aligned}$$

As long as κ is chosen small enough, the matrix $\begin{pmatrix}\tilde{g}_{11} & \tilde{g}_{12}\\ \tilde{g}_{21} & \tilde{g}_{22}\end{pmatrix}$ is positive definite, so that the last two terms in the first line of (11.36) control $C_\kappa\|\varepsilon^{\frac12}\nabla Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L^2_tL^2(\mathbb{R}^3_-)}$. In the sequel, to lighten the notation load and without much ambiguity, we shall denote

$$\|\tilde{f}\|_{L^p_tH^k_{co}} = \|\tilde{f}\|_{L^p_tH^k_{co}(\mathbb{R}^3_-)},\ \|f\|_{L^p_tH^k_{co}} = \|f\|_{L^p_tH^k_{co}(\mathcal{S})}, \qquad p = 2,+\infty.$$

We begin now to estimate the last three terms of the right hand side of (11.36). At first, we have up to some smooth functions depending on ϕ,

$$[Z^\beta,\partial_z^2] = \sum_{|\tilde{\beta}|\le|\beta|-1} *_{\beta,\tilde{\beta}}\partial_z^2 Z^{\tilde{\beta}} + \sum_{|\gamma|\le|\beta|-1} *_{\beta,\gamma}\partial_z Z^\gamma,$$

Therefore, thanks to integration by parts and Young's inequality, we write:

(11.37)

$$\mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta,\partial_z^2]\partial_t\tilde{\zeta}\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big| \le \delta\varepsilon\|\partial_z Z^\beta\partial_t\tilde{\zeta}\|^2_{L^2_tL^2(\mathbb{R}^3_-)} + C_\delta\varepsilon(\|\partial_z\partial_t\tilde{\zeta}_2\|^2_{L^2_tH^{k-1}_{co}} + \|\partial_t\tilde{\zeta}_2\|^2_{L^2_tH^{m-5}_{co}}).$$

Similarly, by Young's inequality, we have:

$$
\begin{aligned}
(11.38)\qquad & \mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta\partial_t,\tilde{g}^{ij}]\partial_j\tilde{\zeta}_2\cdot\partial_i Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big|\leq\delta\varepsilon\|\nabla Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L^2_tL^2(\mathbb{R}^3_-)}\\
&+C_\delta\Lambda\big(\frac{1}{c_0},|(h,\varepsilon^{\frac12}\partial_t h)|_{m-2,\infty,t}+|\partial_t h|_{2,\infty,t}\big)\big(\|(\tilde{\zeta}_2,\varepsilon^{\frac12}\partial_t\tilde{\zeta}_2)\|^2_{L^2_tH^{m-4}_{co}}+\varepsilon\|\tilde{\zeta}_2\|^2_{L^2_tH^{m-3}_{co}}\big).
\end{aligned}
$$

We are now in position to control the last term in (11.36). We split it into several terms:

$$
\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t F^{\tilde{\zeta}}_\chi\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s=:\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4.
$$

with

$$
\mathcal{J}_1=\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t\widetilde{G^{\zeta}_\chi}\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s,\quad \mathcal{J}_2=\bar{\rho}\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t\big((\mathrm{D}\tilde{\Phi}_s)^{-1}\partial_s\tilde{\Phi}_s\cdot\nabla\tilde{\zeta}\big)\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s,
$$

$$
\mathcal{J}_3=\mu\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t\partial_i(\tilde{g}^{ij}\partial_j\tilde{\zeta}_1)\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s,\quad \mathcal{J}_4=\frac12\mu\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t\big(\partial_z(\ln|g|)\partial_z\tilde{\zeta}\big)\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s,
$$

$$
\mathcal{J}_5=\frac12\mu\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t\big(\partial_i(\ln|g|)\tilde{g}^{ij}\partial_j\tilde{\zeta}\big)\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s.
$$

To estimate $\mathcal{J}_2$, let us split it into two terms $\mathcal{J}_2=\mathcal{J}_{21}+\mathcal{J}_{22}$:

$$
\begin{aligned}
\mathcal{J}_{21}&=\bar{\rho}\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t\Big(\mathrm{div}\big((\mathrm{D}\tilde{\Phi}_s)^{-1}\partial_s\tilde{\Phi}_s\big)\tilde{\zeta}\Big)Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s,\\
\mathcal{J}_{22}&=\bar{\rho}\varepsilon\int_0^t\int_{\mathbb{R}^3_-}Z^\beta\partial_t\partial_l\Big(\big((\mathrm{D}\tilde{\Phi}_s)^{-1}\partial_s\tilde{\Phi}_s\big)_l\tilde{\zeta}\Big)Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

We emphasize that since there is no gain of the regularity of $\tilde{\Phi}$ from that of h (roughly speaking, one needs $k+1$ derivatives of h to control k derivatives of $\tilde{\Phi}$), careful attention needs to be paid to the regularity of the surface in the following computations. To estimate $\mathcal{J}_{21}$, in order not to lose regularity on the surface, we consider two cases. If Z^β contains at least one spatial conormal derivative, we integrate by parts in space, and then use Young's inequality to get:

$$
\begin{aligned}
\mathcal{J}_{21}\leq\delta\varepsilon\|\nabla Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L^2_tL^2(\mathbb{R}^3_-)}&+\big(\|(\tilde{\zeta},\varepsilon^{\frac12}\partial_t\tilde{\zeta})\|^2_{L^2_tH^{m-4}_{co}}+|\varepsilon^{\frac12}\partial_t^2h|^2_{L^2_t\tilde{H}^{m-3}}\big)\cdot\\
&\Lambda\big(\|\tilde{\zeta}\|_{1,\infty,t}+|(h,\varepsilon^{\frac12}\partial_t h)|_{m-2,\infty,t}+|\partial_t h|_{m-3,\infty,t}+|\varepsilon^{\frac12}\partial_t^2 h|_{m-5,\infty,t}\big).
\end{aligned}
$$

Moreover, we have by Proposition 11.5 and estimate (11.27) that for $l=3,4$

$$
\begin{aligned}
(11.39)\qquad &\|(\tilde{\zeta}_2,\varepsilon^{\frac12}\partial_t\tilde{\zeta}_2)\|_{L^2_tH^{m-l}_{co}}\leq\|(\tilde{\zeta},\varepsilon^{\frac12}\partial_t\tilde{\zeta}),(\tilde{\zeta}_1,\varepsilon^{\frac12}\partial_t\tilde{\zeta}_1)\|_{L^2_tH^{m-l}_{co}}\\
&\lesssim\|(\nabla u,\varepsilon^{\frac12}\partial_t\nabla u)\|_{L^2_tH^{m-l}_{co}}\Lambda\big(\frac{1}{c_0},|(h,\varepsilon^{\frac12}\partial_t h)|_{m-l+1,\infty,t}\big)+T^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\
&\lesssim\begin{cases} T^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big) & \text{if } l=4,\\ \lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big) & \text{if } l=3,\end{cases}
\end{aligned}
$$

and by (11.28) that:

$$
\begin{aligned}
(11.40)\qquad &\|(\mathrm{Id},\varepsilon^{\frac12}\partial_t,\partial_y,Z_3)\tilde{\zeta}\|_{L^\infty([0,T]\times\mathbb{R}^3_-)}\\
&\lesssim\|(\mathrm{Id},\varepsilon^{\frac12}\partial_t,\partial_y,Z_3)\zeta\|_{0,\infty,t}\Lambda\big(|\varepsilon^{\frac12}\partial_t h|_{2,\infty,t}+|h|_{3,\infty,t}\big)\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).
\end{aligned}
$$

Therefore, by combining (6.2), we obtain that in this case,

$$\mathcal{J}_{21} \le \delta\varepsilon \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.41}$$

If $Z^\beta = (\varepsilon\partial_t)^k$, $(k \le m-4)$, thanks to (6.2), (11.27), (11.39), (11.40), we can control $\mathcal{J}_{21}$ as:

$$\begin{aligned} \mathcal{J}_{21} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t \tilde{\zeta}_2\|_{L^2_t H^{m-4}_{co}} \Lambda\big(\frac{1}{c_0}, \|\!|(\tilde{\zeta}, \varepsilon\partial_t\tilde{\zeta})\|\!|_{0,\infty,t} + \mathcal{G}_{\infty,t}(h)\big) \cdot \\ &\qquad (|(\varepsilon^{\frac{1}{2}}\partial_t^2 h, \varepsilon^{\frac{3}{2}}\partial_t^3 h)|_{L^2_t \tilde{H}^{m-3}} + \|(\tilde{\zeta}_2, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2)\|_{L^2_t H^{m-4}_{co}}) \\ &\lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \end{aligned} \tag{11.42}$$

where

$$\mathcal{G}_{\infty,t}(h) \colon = |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t} + |\partial_t h|_{m-3,\infty,t} + |(\varepsilon^{\frac{1}{2}}\partial_t^2 h, \varepsilon^{\frac{3}{2}}\partial_t^3 h)|_{m-5,\infty,t}.$$

Note that by (6.1)-(6.2), and the Sobolev embedding $H^{\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$,

$$\mathcal{G}_{\infty,t}(h) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Collecting (11.41) and (11.42), we finally get that

$$\mathcal{J}_{21} \le \delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.43}$$

For $\mathcal{J}_{22}$, we write $Z^\beta \partial_l = [Z^\beta, \partial_l] + \partial_l Z^\beta$, we integrate by parts for the second term and follow similar arguments as in the estimate of $\mathcal{J}_{21}$ to get that:

$$\mathcal{J}_{22} \le \delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Combined with (11.43), this yields:

$$\mathcal{J}_{2} \le 2\delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.44}$$

For $\mathcal{J}_3$, we integrate by parts again and use the Cauchy-Schwarz inequality to get:

$$\begin{aligned} \mathcal{J}_3 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t(\tilde{g}^{ij}\partial_j\tilde{\zeta}_1)\|_{L^2_t H^{m-4}_{co}} \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L^2_t H^{m-3}_{co}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big) \|(\tilde{\zeta}_1, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1)\|_{L^2_t H^{m-3}_{co}} \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L^2_t H^{m-3}_{co}}. \end{aligned}$$

By estimates (11.27), (11.39), we find that:

$$\mathcal{J}_3 \lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.45}$$

We begin now to estimate $\mathcal{J}_4$. By writing

$$\partial_z(\ln|g|)\partial_z\tilde{\zeta} = -\partial_z^2(\ln|g|)\tilde{\zeta} + \partial_z\big(\partial_z(\ln|g|)\tilde{\zeta}\big),$$

we can follow the similar computations as in the estimates of $\mathcal{J}_2$ to obtain (it is indeed easier in the sense that $\partial_z^2(\ln|g|), \partial_z(\ln|g|)$ involve only two derivatives of h thanks to Remark 11.6)

$$\begin{aligned} \mathcal{J}_4 &\le \delta \|\varepsilon^{\frac{1}{2}}\partial_t\nabla Z^\beta\tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big) \|(\tilde{\zeta}, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta})\|^2_{L^2_t H^{m-4}_{co}} \\ &\le \delta \|\varepsilon^{\frac{1}{2}}\partial_t\nabla Z^\beta\tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \end{aligned} \tag{11.46}$$

We proceed to estimate $\mathcal{J}_5$. If $Z^\beta = (\varepsilon\partial_t)^k$, we control it by inequalities (6.2), (11.27), (11.39):

$$\begin{aligned}\mathcal{J}_5 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2H_{co}^{m-4}}\Lambda\big(\frac{1}{c_0}, \|\varepsilon^{\frac{1}{2}}\tilde{\zeta}\|_{1,\infty,t} + |(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big)\big(\|(\tilde{\zeta},\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta})\|_{L_t^2H_{co}^{m-3}} + |\varepsilon\partial_t^2 h|_{L_t^2\tilde{H}^{m-2}}\big)\\ &\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

If Z^β contains at least one spatial conormal derivative, we integrate by parts in space and control it in a similar way as $\mathcal{J}_3$:

$$\begin{aligned}\mathcal{J}_5 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t(\partial_i(\ln|g|)\tilde{g}^{ij}\partial_j\tilde{\zeta}))\|_{L_t^2H_{co}^{m-5}}\|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2H_{co}^{m-3}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big)\|(\tilde{\zeta},\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta})\|_{L_t^2H_{co}^{m-4}}\|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2H_{co}^{m-3}}\\ &\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

To summarize, we get that:

$$\mathcal{J}_5 \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{11.47}$$

We are now left to control the term $\mathcal{J}_1$. After checking every term of G_χ^ω and G_χ^ς defined in (11.18) and (11.19), we find that the problematic terms that may lead to a loss of derivatives are the following:

$$G_{\chi,1}^\omega = (u\cdot\nabla^\varphi\omega)\times\chi\mathbf{N},\quad G_{\chi,2}^\omega = \nabla^\varphi\omega\times\nabla^\varphi(\chi\mathbf{n}),\quad G_{\chi,1}^\varsigma = \chi\Pi([\partial_1,\Delta^\varphi]v\cdot\mathbf{N},[\partial_2,\Delta^\varphi]v\cdot\mathbf{N},0)^t.$$

All the other terms can be controlled directly through the Cauchy-Schwarz inequality, the estimate (11.39) and Proposition 11.9:

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3_-}\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\big(\widetilde{G_\chi^\zeta} - \widetilde{G_{\chi,1}^\omega} - \widetilde{G_{\chi,2}^\omega} - \widetilde{G_{\chi,1}^\varsigma}\big)\cdot\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\\ &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2H_{co}^{m-4}}\|\varepsilon^{\frac{1}{2}}\partial_t\big(\widetilde{G_\chi^\zeta} - \widetilde{G_{\chi,1}^\omega} - \widetilde{G_{\chi,2}^\omega} - \widetilde{G_{\chi,1}^\varsigma}\big)\|_{L_t^2H_{co}^{m-4}}\\ &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2H_{co}^{m-4}}\|\varepsilon^{\frac{1}{2}}\partial_t(G_\chi^\zeta - G_{\chi,1}^\omega - G_{\chi,2}^\omega - G_{\chi,1}^\varsigma)\|_{L_t^2H_{co}^{m-4}(\mathcal{S})}\\ &\lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Note that by Proposition 11.9,

$$\|G_\chi^\zeta - G_{\chi,1}^\omega - G_{\chi,2}^\omega - G_{\chi,1}^\varsigma\|_{L_t^2H_{co}^{m-3}(\mathcal{S})} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

It remains to control the remaining three terms. We shall explain the estimates of the term involving $G_{\chi,1}^\omega$. Let us first rewrite:

$$u\cdot\nabla^\varphi\omega = u_1\partial_{y_1}\omega + u_2\partial_{y_2}\omega + (u\cdot\mathbf{N})\cdot\partial_z\omega = R_1 - R_2.$$

where

$$R_1 = \partial_{y_1}(u_1\omega) + \partial_{y_2}(u_2\omega) + \partial_z\big((\frac{u\cdot\mathbf{N}}{\partial_z\varphi})\omega\big),\quad R_2 = \partial_{y_1}u_1\cdot\omega + \partial_{y_2}u_2\cdot\omega + \partial_z\big(\frac{u\cdot\mathbf{N}}{\partial_z\varphi}\big)\cdot\omega$$

Since

$$\partial_z\big(\frac{u\cdot\mathbf{N}}{\partial_z\varphi}\big) = \partial_z^\varphi u\cdot\mathbf{N} + u\cdot\partial_z\big(\frac{\mathbf{N}}{\partial_z\varphi}\big) = \mathrm{div}^\varphi u - \partial_{y_1}u_1 - \partial_{y_2}u_2 + u\cdot\partial_z\big(\frac{\mathbf{N}}{\partial_z\varphi}\big),$$

there is no term like $\partial_z u \cdot \partial_z u$ appearing in R_2, we thus can show by using similar arguments as in the proof of Proposition 11.9 that:

$$\|\varepsilon^{\frac{1}{2}}\partial_t R_2\|_{L_t^2 H_{co}^{m-4}} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),$$

which further yields:

$$\int_0^t \int_{\mathbb{R}^3_-} \varepsilon^{\frac{1}{2}} Z^\beta \partial_t \widetilde{R_2} \cdot \varepsilon^{\frac{1}{2}} Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s \lesssim T^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.48}$$

Next, by the change of variable, we have:

$$\widetilde{R_1} = (\mathrm{D}(\Phi_t \circ \widetilde{\Phi_t}^{-1})^{-1})_{jl} \partial_l \big[\widetilde{I_j(u)\omega}\big], \qquad \text{where } I(u) = (u_1, u_2, \frac{u \cdot \mathbf{N}}{\partial_z \varphi}).$$

Therefore, using a similar strategy as the one employed in the estimate of $\mathcal{J}_2$, we find that:

$$\int_0^t \int_{\mathbb{R}^3_-} \varepsilon^{\frac{1}{2}} \partial_t Z^\beta \widetilde{R_1} \cdot \varepsilon^{\frac{1}{2}} Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s \leq \delta \|\varepsilon^{\frac{1}{2}} \nabla Z^\beta \partial_t \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),$$

which, together with (11.48), leads to:

$$\int_0^t \int_{\mathbb{R}^3_-} \varepsilon^{\frac{1}{2}} Z^\beta \partial_t \widetilde{G^\omega_{\chi,1}} \varepsilon^{\frac{1}{2}} Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s \leq \delta \|\varepsilon^{\frac{1}{2}} \nabla Z^\beta \partial_t \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Following similar arguments, one can also show that:

$$\int_0^t \int_{\mathbb{R}^3_-} \varepsilon^{\frac{1}{2}} \partial_t Z^\beta (\widetilde{G^\omega_{\chi,2}} + \widetilde{G^\varsigma_{\chi,1}}) \cdot \varepsilon^{\frac{1}{2}} Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s \leq \delta \|\nabla Z^\beta \varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

To summarize, we have obtained that:

$$\mathcal{J}_1 \leq 2\delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.49}$$

Gathering (11.44)-(11.47),(11.49) and using (11.39), we obtain:

$$\Big|\int_0^t \int_{\mathbb{R}^3_-} Z^\beta F^{\tilde{\zeta}}_\chi \cdot Z^\beta \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s\Big| \leq 10\delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.50}$$

Inserting (11.37), (11.38) and (11.50) into (11.36), we get by choosing δ small enough that for any $0 \leq k \leq m-4$,

$$\|\varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_2\|^2_{L_t^\infty H^k_{co}} + \|\varepsilon^{\frac{1}{2}} \partial_t \nabla \tilde{\zeta}_2\|^2_{L_t^2 H^k_{co}} \lesssim Y_m^2(0) + \|\varepsilon^{\frac{1}{2}} \partial_t \nabla \tilde{\zeta}_2\|^2_{L_t^2 H^{k-1}_{co}} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.51}$$

Note that in the above, we use the convention that $\|\cdot\|_{L_t^2 H^l_{co}} = 0$ if $l < 0$. Moreover, we can show by repeating the procedure to prove (11.51) that:

$$\|\tilde{\zeta}_2\|^2_{L_t^\infty H^k_{co}} + \|\nabla \tilde{\zeta}_2\|^2_{L_t^2 H^k_{co}} \lesssim Y_m^2(0) + \|\nabla \tilde{\zeta}_2\|^2_{L_t^2 H^{k-1}_{co}} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.52}$$

The estimate (11.33) then stems from (11.52) and an induction on $k \in [0, m-4]$, the estimate (11.34) can also be derived from (11.33) and induction arguments. $\square$

In the following, we show an estimate needed to control $\mathcal{J}_1$ in the above lemma.

Proposition 11.9. *Assume that* (2.2) *holds, then for any* $0 < t \leq T$,

$$\|(\mathrm{Id}\,, \varepsilon^{\frac{1}{2}} \partial_t)(G^\zeta_\chi - G^\omega_{\chi,1} - G^\omega_{\chi,2} - G^\varsigma_{\chi,1})\|_{L_t^2 H^{m-4}_{co}(\mathcal{S})} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Proof. One can show this estimate by bounding each term appearing in $G^{\zeta}_{\chi} - G^{\omega}_{\chi,1} - G^{\omega}_{\chi,2} - G^{\varsigma}_{\chi,1}$. We will give the details for one term, namely $\omega \cdot \nabla^{\varphi} u$, which is the most difficult one, the other terms can be controlled easily. Let us write

$$\omega \cdot \nabla^{\varphi} u = \omega_1 \partial_{y_1} u + \omega_2 \partial_{y_2} u + (\omega \cdot \mathbf{N}) \partial^{\varphi}_z u.$$

Furthermore, we have:

$$\begin{aligned} \omega \cdot \mathbf{N} &= \mathrm{div}^{\varphi}(u \times \mathbf{N}) + u \cdot (\nabla^{\varphi} \times \mathbf{N}) \\ &= -(u \times \mathbf{N}) \cdot \partial^{\varphi}_z \mathbf{N} - \partial_{y_1}(u \times \mathbf{N})_1 - \partial_{y_2}(u \times \mathbf{N})_2 + u \cdot (\nabla^{\varphi} \times \mathbf{N}). \end{aligned}$$

We thus see that $\omega \cdot \nabla^{\varphi} u = \partial_z u \cdot F_1(\partial_y u, \nabla^{\varphi}\mathbf{N}, u, \mathbf{N}, \frac{1}{\partial_z \varphi}) + F_2(\partial_y u, \partial_y u)$ where F_1, F_2 are some polynomials with degree 4. Let us control $\varepsilon^{\frac{1}{2}} \partial_t (\partial_z u \partial_y u)$ for example, the other ones can be bounded in a similar way (note that we do not lose regularity on the surface the terms involving $\mathbf{N}$). By counting the derivatives hitting on each term, one finds that:

$$\begin{aligned} &\|(\varepsilon^{\frac{1}{2}} \partial_t \partial_z u \cdot \partial_y u, \partial_z u \cdot \varepsilon^{\frac{1}{2}} \partial_t \partial_y u)\|_{L^2_t H^{m-4}_{co}} \\ &\lesssim \|\!|\varepsilon^{\frac{1}{2}} \partial_t \partial_z u\|\!|_{0,\infty,t} \|u\|_{L^2_t H^{m-3}_{co}} + \|\varepsilon^{\frac{1}{2}} \partial_t \partial_z u\|_{L^2_t H^{m-4}_{co}} \|\!|u\|\!|_{m-4,\infty,t} \\ &\quad + \|\!|\nabla u\|\!|_{1,\infty,t} \|\varepsilon^{\frac{1}{2}} \partial_t u\|_{L^2_t H^{m-3}_{co}} + \|\!|\varepsilon^{\frac{1}{2}} \partial_t u\|\!|_{m-5,\infty,t} \|\nabla u\|_{L^\infty_t H^{m-4}_{co}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \end{aligned}$$

□

11.3. **Estimate of the second order normal derivatives of the velocity.** To finish the a-priori estimates for the energy norms, we are left to estimate $\nabla^2 u$ in a non-uniform way which is the object of the following lemma.

Lemma 11.10. *Assume that* (2.2) *holds for some* $T > 0$, *then for any* $0 < t \le T$, *the following estimate holds,*

$$\|\varepsilon^{\frac{1}{2}} \nabla^2 u\|^2_{L^\infty_t H^{m-2}_{co} \cap L^2_t \mathcal{H}^{m-1}_{co}} \lesssim \Lambda\bigg(\frac{1}{c_0}, |h|^2_{L^\infty_t \tilde{H}^{m-\frac{1}{2}}} + Y^2_m(0)\bigg) Y^2_m(0) + (T+\varepsilon)^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{11.53}$$

Proof. We will prove the following two inequalities:

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \|\nabla^2 u\|_{L^\infty_t H^{m-2}_{co}} &\lesssim (T+\varepsilon)^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \|\varepsilon^{\frac{1}{2}} \nabla u\|_{L^\infty_t H^{m-1}_{co}} \\ &\quad + \varepsilon^{\frac{1}{2}} \|\partial_t u\|_{L^\infty_t H^{m-2}_{co}} + \varepsilon^{-\frac{1}{2}} \|\nabla^{\varphi} \sigma\|_{L^\infty_t H^{m-2}_{co}}, \end{aligned} \tag{11.54}$$

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \|\nabla^2 u\|_{L^2_t \mathcal{H}^{m-1}} &\lesssim (T+\varepsilon)^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \\ &\quad + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \|\varepsilon^{\frac{1}{2}} \nabla u\|_{L^2_t H^m_{co}} + \varepsilon^{-\frac{1}{2}} \|\nabla^{\varphi} \sigma\|_{L^2_t \mathcal{H}^{m-1}} \end{aligned} \tag{11.55}$$

where $\mathcal{N}_{m,T}$ is defined in (1.31). These two estimates, together with (7.1), (7.19), (9.1), (10.1), (11.1), yield (11.53). To prove (11.54) and (11.55), it suffices to control $\varepsilon^{\frac{1}{2}} \partial^2_z u$. Let us rewrite the equations $(1.16)_2$ as

$$\varepsilon^{\frac{1}{2}} \Delta^{\varphi} u = \varepsilon^{\frac{1}{2}} g_2 (\partial_t + \underline{u} \cdot \nabla) u + \varepsilon^{-\frac{1}{2}} \nabla^{\varphi} \sigma - \varepsilon^{\frac{1}{2}} \nabla^{\varphi} \mathrm{div}^{\varphi} u. \tag{11.56}$$

In view of (11.56), (8.29), we have by the product estimate (3.8) and the definition of $\mathcal{E}_{m,t}$ that:

$$\|\varepsilon^{\frac{1}{2}} \partial^2_z u\|_{L^\infty_t H^{m-2}_{co}} \lesssim \|\varepsilon^{\frac{1}{2}} \partial_t u\|_{L^\infty_t H^{m-2}_{co}} + \|\varepsilon^{-\frac{1}{2}} \nabla \sigma\|_{L^\infty_t H^{m-2}_{co}}$$

$$+\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\big(\varepsilon^{\frac12}\|(\sigma,u,\nabla\sigma,\nabla u)\|_{L_t^\infty H_{co}^{m-1}}+\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)|\varepsilon^{\frac12}h|_{L_t^\infty\tilde{H}^{m-\frac12}}\big)$$
$$\lesssim \|\varepsilon^{\frac12}\partial_t u\|_{L_t^\infty H_{co}^{m-2}}+\|\varepsilon^{-\frac12}\nabla\sigma\|_{L_t^\infty H_{co}^{m-2}}+\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\nabla u\|_{L_t^\infty H_{co}^{m-1}}$$
$$+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

We thus finish the proof of (11.54). The inequality (11.55) can be shown in a similar way, we thus omit the proof. □

12. Control of the $L^\infty_{t,x}$ norm

In this section, we prove Proposition 2.3, the a-priori estimate for $\mathcal{A}_{m,T}$:

$$\begin{aligned}\mathcal{A}_{m,T}(\sigma,u)=|h|_{m-2,\infty,t}+\|\nabla u\|_{1,\infty,T}+\|\varepsilon^{-\frac12}(\nabla^\varphi\sigma,\mathrm{div}^\varphi u)\|_{m-5,\infty,T}+\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{m-5,\infty,T}\\ +\|(\mathrm{Id},\varepsilon\partial_t)(\sigma,u)\|_{m-4,\infty,T}+\|\varepsilon^{\frac12}\nabla u\|_{m-3,\infty,T}+\|\varepsilon^{\frac12}(\sigma,u)\|_{m-2,\infty,T}.\end{aligned}\tag{12.1}$$

Remark 12.1. *By the identity* (12) *and the equation* $(1.16)_2$ *for* u*, we have that:*

$$\varepsilon^{\frac12}\|\partial_z^2 u\|_{m-5,\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).\tag{12.2}$$

Remark 12.2. *As* $[\frac{m}{2}]\leq m-4$ *if* $m\geq 7$*, we thus have:*

$$\|\varepsilon^{-\frac12}(\nabla^\varphi\sigma,\mathrm{div}^\varphi u)\|_{[\frac m2]-1,\infty,T}+\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{[\frac m2]-1,\infty,T}+\varepsilon^{\frac12}\|\partial_z^2u\|_{[\frac m2]-1,\infty,t}$$
$$+\|(Id,\varepsilon\partial_t)(\sigma,u)\|_{[\frac m2],\infty,T}+\|\varepsilon^{\frac12}\nabla u\|_{[\frac m2]+1,\infty,T}+\|\varepsilon^{\frac12}u\|_{[\frac m2]+2,\infty,T}\lesssim\mathcal{A}_{m,T}.$$

The other terms appearing in $\mathcal{A}_{m,T}$ can be obtained by the Sobolev embedding (3.16).

Proof of Proposition 2.3. By the Sobolev embedding $H^{\frac32}(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$, we have directly that:

$$|h|_{m-2,\infty,T}\lesssim|h|_{L_T^\infty\tilde{H}^{m-\frac12}}\lesssim\tilde{\mathcal{E}}_{m,T}.\tag{12.3}$$

Furthermore, thanks to the Sobolev embedding (3.16), the last four terms in (12.1) can be controlled by the ones appearing in $\mathcal{E}_{m,T}$. Indeed,

$$\varepsilon^{\frac12}\|\partial_t(\sigma,u)\|_{m-5,\infty,T}\lesssim\sup_{0\leq s\leq T}\big(\|\varepsilon^{\frac12}\partial_t u(s)\|_{H_{co}^{m-3}}+\|\varepsilon^{\frac12}\partial_t\nabla u(s)\|_{H_{co}^{m-4}}\big)\lesssim\tilde{\mathcal{E}}_{m,T},\tag{12.4}$$

$$\varepsilon^{\frac12}\|\nabla u(s)\|_{m-3,\infty,T}\lesssim\sup_{0\leq s\leq T}\big(\|\varepsilon^{\frac12}\nabla u(s)\|_{H_{co}^{m-1}}+\|\varepsilon^{\frac12}\nabla^2u(s)\|_{H_{co}^{m-2}}\big)\lesssim\tilde{\mathcal{E}}_{m,T},$$

$$\|(\sigma,u)\|_{m-4,\infty,T}\lesssim\sup_{0\leq s\leq T}\big(\|(\sigma,u)(s)\|_{H_{co}^{m-1}}+\|\nabla(\sigma,u)(s)\|_{H_{co}^{m-4}}\big)\lesssim\tilde{\mathcal{E}}_{m,T}.\tag{12.5}$$

$$\|\varepsilon\partial_t(\sigma,u)\|_{m-4,\infty,T}\lesssim\sup_{0\leq s\leq T}\big(\|\varepsilon^{\frac12}\partial_t(\sigma,u)(s)\|_{H_{co}^{m-2}}+\varepsilon^{\frac12}\|\varepsilon\partial_t\nabla(\sigma,u)(s)\|_{H_{co}^{m-3}}\big)\lesssim\tilde{\mathcal{E}}_{m,T}.$$

$$\varepsilon^{\frac12}\|(\sigma,u)\|_{m-2,\infty,T}\lesssim\sup_{0\leq s\leq T}\big(\|(\sigma,u)(s)\|_{H_{co}^{m}}+\|\varepsilon^{\frac12}\nabla(\sigma,u)(s)\|_{H_{co}^{m-1}}\big)\lesssim\tilde{\mathcal{E}}_{m,T}.\tag{12.6}$$

For the third term in (12.1), we can use the equation for σ to get that:

$$\begin{aligned}\varepsilon^{\frac12}\|\mathrm{div}^\varphi u\|_{m-5,\infty,T}&\lesssim\|\varepsilon^{\frac12}\partial_t\sigma\|_{m-5,\infty,T}+\varepsilon^{\frac12}\big(\|(u,\varepsilon\partial_t\sigma,\nabla\sigma\|_{m-5,\infty,T}+|h|_{m-4,\infty,T}\big)^2\\ &\lesssim\tilde{\mathcal{E}}_{m,T}+\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big).\end{aligned}\tag{12.7}$$

Moreover, in view of (12.3), (12.5) and identity $\Pi\nabla^\varphi = \Pi(\partial_1, \partial_2, 0)^t$,

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\sigma\|_{m-5,\infty,T} &\lesssim \varepsilon^{-\frac{1}{2}}\|\partial_y\sigma\|_{m-5,\infty,T}(1+|h|_{m-4,\infty,T})^2 + \varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\sigma\cdot\mathbf{n}\|_{m-5,\infty,T}|h|_{m-4,\infty,T}\\ &\lesssim \tilde{\mathcal{E}}_{m,T} + \tilde{\mathcal{E}}_{m,T}^3 + \|\nabla^\varphi\sigma\cdot\mathbf{n}\|^2_{[\frac{m}{2}]-1,\infty,T}.\end{aligned}$$

Indeed, we have used the Sobolev embedding (3.16) to get that:

$$\varepsilon^{-\frac{1}{2}}\|\partial_y\sigma\|_{m-5,\infty,T} \lesssim \varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\sigma\|_{L_t^\infty H_{co}^{m-3}} \lesssim \tilde{\mathcal{E}}_{m,T}.$$

Therefore, it remains to control $\varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\sigma\cdot\mathbf{n}\|_{[\frac{m}{2}]-1,\infty,T}$, which is the aim of the following lemma. $\square$

Lemma 12.3. *Suppose that* (2.2) *holds, then:*

$$\varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\sigma\cdot\boldsymbol{n}\|_{[\frac{m}{2}]-1,\infty,T} \lesssim Y_m^2(0) + \tilde{\mathcal{E}}_{m,T}^2 + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big). \tag{12.8}$$

Proof. By (8.21), we find that $\nabla^\varphi\sigma$ solves

$$\varepsilon^2 g_1(\partial_t + \underline{u}\cdot\nabla)\nabla^\varphi\sigma + \frac{1}{2\mu+\lambda}\nabla^\varphi\sigma = \mathcal{Q}_1 \tag{12.9}$$

where $\mathcal{Q}_1 = \mathcal{Q}_{11} + \mathcal{Q}_{12} + \mathcal{Q}_{13}$, with

$$\mathcal{Q}_{11} = -\varepsilon^2 g_1'\nabla^\varphi\sigma(\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)\sigma - \varepsilon^2 g_1\nabla^\varphi u\cdot\nabla^\varphi\sigma,$$

$$\mathcal{Q}_{12} = -\frac{\mu\varepsilon}{2\mu+\lambda}\operatorname{curl}^\varphi\omega, \quad \mathcal{Q}_{13} = -\frac{1}{2\mu+\lambda}g_2(\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)u.$$

Denote $R = \nabla^\varphi\sigma\cdot\mathbf{n}$, then by (8.21), R solves:

$$\varepsilon^2 g_1(\partial_t + \underline{u}\cdot\nabla)R + \frac{1}{2\mu+\lambda}R = \varepsilon^2 g_1\nabla^\varphi\sigma(\partial_t + \underline{u}\cdot)\mathbf{n} + \mathcal{Q}_1\cdot\mathbf{n} =: \mathcal{Q}_2 + \mathcal{Q}_1\cdot\mathbf{n}$$

For any multi-index with $|\beta| \le m-5$, denote $R^\beta = Z^\beta R$, then R^β satisfies:

$$\varepsilon^2 g_1(\partial_t + \underline{u}\cdot\nabla)R^\beta + \frac{1}{2\mu+\lambda}R^\beta = Z^\beta(\mathcal{Q}_2 + \mathcal{Q}_1\cdot\mathbf{n}) + \mathcal{C}^\beta_{R,1} + \mathcal{C}^\beta_{R,2} =: \mathcal{Q}^\beta,$$

where

$$\mathcal{C}^\beta_{R,1} = -\varepsilon^2[Z^\beta, g_1/\varepsilon]\varepsilon\partial_t R, \quad \mathcal{C}^\beta_{R,2} = -\varepsilon^2[Z^\beta, g_1\underline{u}\cdot\nabla]R.$$

Define $X_t(x) = X(t,x)$ the unique flow associated to $\underline{u}$:

$$\partial_t X(t,x) = \underline{u}(t, X(t,x)), \quad X(0,x) = x.$$

Note that since $\underline{u}\cdot\mathbf{n}|_{z=0} = 0$, and $u \in \mathrm{Lip}\,([0,T]\times\Omega)$, we have for each $t\in[0,T]$, $X_t : \mathcal{S}\to\mathcal{S}$ is a diffeomorphism. Denote $f^X = f(t, X(t,x))$, then $R^{\beta,X}$ solves the ODE:

$$\varepsilon^2(g_1\partial_t R^\beta)(t, X_t(x)) + \frac{1}{2\mu+\lambda}R^\beta(t, X_t(x)) = Q^\beta(t, X_t(x))$$

from which, we deduce that:

$$R^\beta(t, X_t(x)) = e^{-\int_0^t \frac{1}{\varepsilon^2 g_1(s, X_s(x))}\mathrm{d}s}R^\beta(0) + \int_0^t e^{-\int_\tau^t \frac{1}{\varepsilon^2 g_1(x, X_s(x))}\mathrm{d}s}\frac{1}{\varepsilon^2}Q^\beta(\tau, X_\tau(x))\mathrm{d}\tau.$$

By assumption (2.2), $c_0 \le g_1(t, X_t(x)) \le \frac{1}{c_0}$ for any $(t,x)\in[0,T]\times\mathcal{S}$. Therefore,

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|R^\beta\|_{0,\infty,T} &\lesssim \varepsilon^{-\frac{1}{2}}\sup_{(t,x)\in[0,T]\times\mathcal{S}}|R^\beta(t, X_t(x))|\\ &\lesssim \varepsilon^{-\frac{1}{2}}\|R^\beta(0)\|_{L^\infty(\mathcal{S})} + \varepsilon^{-\frac{1}{2}}\int_0^T e^{-c_0(t-s)/\varepsilon^2}\frac{1}{\varepsilon^2}\mathrm{d}s\|Q^\beta\|_{0,\infty,T} \qquad (12.10)\\ &\lesssim Y_m(0) + \varepsilon^{-\frac{1}{2}}\|Q^\beta\|_{0,\infty,T}.\end{aligned}$$

It thus suffices to control the term $\varepsilon^{-\frac{1}{2}}\|Q^\beta\|_{0,\infty,T}$. First of all, by the property (2.1), we get that:

$$\varepsilon^{-\frac{1}{2}}\|\mathcal{C}_{R,1}^{\beta}\|_{0,\infty,T} \lesssim \varepsilon^{\frac{3}{2}}\big(\|(\sigma,\nabla\sigma)\|_{m-5,\infty,T}+|h|_{m-4,\infty,T}\big)^2 \lesssim \varepsilon^{\frac{3}{2}}\mathcal{A}_{m,T}^2. \tag{12.11}$$

Next, by using that $\underline{u}\cdot\nabla = u_y\partial_y + \frac{U_z}{\phi}Z_3R$, we can control the second commutator term as:

$$\varepsilon^{\frac{3}{2}}\|\mathcal{C}_{R,2}^{\beta}\|_{0,\infty,T} \lesssim \varepsilon\big(\|(\sigma,u,\nabla\sigma,\varepsilon^{\frac{1}{2}}\nabla u)\|_{m-5,\infty,T}+|h|_{m+3,\infty,T}\big)^2 \lesssim \varepsilon\mathcal{A}_{m,T}^2. \tag{12.12}$$

Similarly, we can find some polynomial Λ, such that

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|Z^\beta(\mathcal{Q}_2+\mathcal{Q}_{11}\cdot\mathbf{n})\|_{0,\infty,T} &\lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\|(\sigma,u,\nabla\sigma,\varepsilon^{\frac{1}{2}}\nabla u)\|_{[\frac{m}{2}]-1,\infty,T}+|h|_{[\frac{m}{2}]+1,\infty,T}\big)\\ &\lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big).\end{aligned} \tag{12.13}$$

Moreover, in light of (12.3) and (12.5), we have

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|Z^\beta(\mathcal{Q}_{13}\cdot\mathbf{n})\|_{0,\infty,T} &\lesssim \|(\varepsilon^{\frac{1}{2}}\partial_t u\cdot\mathbf{n},\varepsilon^{\frac{1}{2}}\underline{u}\cdot\nabla u)\|_{m-5,\infty,T}+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\\ &\lesssim \tilde{\mathcal{E}}_{m,T}^2+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big).\end{aligned} \tag{12.14}$$

Finally, since

$$\begin{aligned}\mathrm{curl}^\varphi\,\omega\cdot\mathbf{n} &= \mathrm{div}^\varphi(\omega\times\mathbf{n})+\omega\cdot\mathrm{curl}^\varphi\,\mathbf{n}\\ &= -(\omega\times\mathbf{n})\cdot\partial_z^\varphi\mathbf{N}+\partial_1(\omega\times\mathbf{n})_1+\partial_2(\omega\times\mathbf{n})_2+\omega\cdot\mathrm{curl}^\varphi\,\mathbf{n}\end{aligned}$$

involves only tangential derivatives of $\nabla^\varphi u$, one has again by (12.3) and (12.5) that:

$$\varepsilon^{-\frac{1}{2}}\|Z^\beta(\mathcal{Q}_{12}\cdot\mathbf{n})\|_{0,\infty,T} \lesssim \big(\|\varepsilon^{\frac{1}{2}}\nabla u\|_{m-4,\infty,T}+|h|_{m-3,\infty,T}\big)^2 \lesssim \tilde{\mathcal{E}}_{m,T}^2. \tag{12.15}$$

Collecting (12.11)-(12.15), we find that:

$$\|\mathcal{Q}\|_{[\frac{m}{2}]-1,\infty,T} \lesssim \tilde{\mathcal{E}}_{m,T}^2+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big).$$

Inserting this inequality into (12.10), we eventually get (12.8). □

In the following Lemma, we obtain the $L_{t,x}^\infty$ estimates of ∇u, namely $\|\varepsilon^{\frac{1}{2}}\partial_t\nabla u\|_{0,\infty,t}$, $\|\nabla u\|_{1,\infty,t}$.

Lemma 12.4. *Assume that* (2.2) *holds, then we have that for any* $0<t\le T$,

$$\|\varepsilon^{\frac{1}{2}}\partial_t\nabla u\|_{0,\infty,t}+\|\nabla u\|_{1,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0},Y_m(0)\big)+\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\tilde{\mathcal{E}}_{m,T}+(T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{12.16}$$

Proof. In view of the identities (4.5) and

$$\begin{aligned}\Pi(\partial_z^\varphi u) &= \frac{1}{|\mathbf{N}|}\Pi\big(\omega\times\mathbf{N}+(\nabla^\varphi u)^t\cdot\mathbf{n}-\mathbf{n}_1\partial_1 u-\mathbf{n}_2\partial_2 u\big)\\ &= \frac{1}{|\mathbf{N}|}(\omega\times\mathbf{N})+\Pi\nabla^\varphi(u\cdot\mathbf{n})-\Pi\big((\nabla^\varphi\mathbf{n})^t u-\mathbf{n}_1\partial_1 u-\mathbf{n}_2\partial_2 u\big),\end{aligned}$$

one gets that:

$$\begin{aligned}&\|\nabla u\|_{1,\infty,t}+\varepsilon^{\frac{1}{2}}\|\partial_t\nabla u\|_{0,\infty,t} \lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\\ &+\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\big(\|u\|_{2,\infty,t}+\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{1,\infty,t}+\|\varepsilon^{-\frac{1}{2}}\mathrm{div}^\varphi u\|_{1,\infty,t}+\|\omega\|_{1,\infty,t}+\|\varepsilon^{\frac{1}{2}}\partial_t\omega\|_{0,\infty,t}\big).\end{aligned}$$

The inequality (12.16) then follows from (12.4), (12.5), (12.7) and the next lemma for the estimates of ω. □

Lemma 12.5. *Under the same assumption as in Lemma* (12.4),

$$\|\omega\|_{1,\infty,t} + \|\varepsilon^{\frac{1}{2}}\partial_t\omega\|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big) + \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\tilde{\mathcal{E}}_{m,T} + (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{12.17}$$

Proof. Away from the boundary, the conormal spaces are equivalent to the usual Sobolev space, the $L^\infty_{t,x}$ estimate for ω can be obtained directly from the usual Sobolev embedding. It thus suffices to establish the corresponding estimates near the boundaries. In what follows, we will detail their estimates near the upper boundary (which corresponds to the free surface), the one near the bottom being easier and has essentially been performed in [47]. As in the proof of Lemma 11.4, we will employ the normal geodesic coordinates (11.15) to take the benefit of the explicit formula for the heat equation on the half line. Taking the same cut off function $\chi = \chi_0(\frac{z}{C(\kappa)})$ introduced in Lemma 11.4 (which satisfies $\Phi_t(\operatorname{Supp}\chi) \Subset \tilde{\Phi}_t(\mathcal{S}_k)$), we use the equation (11.17) to obtain that:

$$(\bar{\rho}\partial_t - \mu\Delta^{\varphi})(\chi\omega) = \chi G^{\omega} - \mu\Delta^{\varphi}\chi\omega - \mu\partial_z\chi(\mathbf{N}\cdot\nabla^{\varphi})\omega =: G^{\chi,\omega}$$

where

$$G^{\omega} = -u\cdot\nabla^{\varphi}\omega + \omega\cdot\nabla^{\varphi}u - \omega\operatorname{div}^{\varphi}u - \frac{\nabla g_2}{\varepsilon}\times((\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)u) + \frac{\bar{\rho}-g_2}{\varepsilon}((\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)\omega).$$

For a function $f(t,\cdot)$ supported on $\mathbb{R}^2\times[-C(\kappa),0]$, we use the notation

$$\tilde{f}(t,x) = f(t, \Phi_t^{-1}\circ\tilde{\Phi}_t(x)).$$

By the change of variable, we find that $\widetilde{\chi\omega}$ satisfies the system:

$$\begin{aligned}(\bar{\rho}\partial_t - \mu\partial_z^2)\widetilde{\chi\omega} = \widetilde{F^{\chi,\omega}} &=: \widetilde{G^{\chi,\omega}} + \bar{\rho}(D\tilde{\Phi}_t)^{-1}\partial_t\tilde{\Phi}_t\cdot\nabla\widetilde{\chi\omega}\\ &\quad + \mu\big[\frac{1}{2}\partial_z(\ln|g|)\partial_z + \partial_i(\ln|g|)\tilde{g}^{ij}\partial_j + \partial_i(\tilde{g}^{ij}\partial_j\cdot)\big](\widetilde{\chi\omega})\end{aligned} \tag{12.18}$$

supplemented with the initial and the boundary conditions:

$$\widetilde{\chi\omega}|_{t=0} = \chi\omega|_{t=0}(\Phi_0^{-1}\circ\tilde{\Phi}_0), \qquad \widetilde{\chi\omega}|_{z=0} = \omega|_{z=0} =: \omega^{b,1}. \tag{12.19}$$

Let

$$E(t,z,z') = \tilde{\mu}\frac{1}{(4\pi\tilde{\mu}t)^{\frac{1}{2}}}\big(e^{-\frac{|z-z'|^2}{4\tilde{\mu}t}} - e^{-\frac{|z+z'|^2}{4\tilde{\mu}t}}\big), \qquad \tilde{\mu} = \bar{\rho}/\mu,$$

the solution to the system (12.18)-(12.19) can be expressed as:

$$\begin{aligned}\widetilde{\chi\omega}(t,y,z) = &-\int_0^t(\partial_{z'}E)(t-s,z,0)\omega^{b,1}(s,y)\,\mathrm{d}s + \int_{-\infty}^0 E(t,z,z')\widetilde{\chi\omega}|_{t=0}(y,z')\,\mathrm{d}z'\\ &+\int_0^t\int_{-\infty}^0 E(t-s,z,z')\widetilde{F^{\chi,\omega}}(s,y,z')\,\mathrm{d}z'\mathrm{d}s = (1)+(2)+(3).\end{aligned} \tag{12.20}$$

Control of the boundary term (1). As in the estimate of (11.31) and (11.32), we can bound the boundary term as:

$$\|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y,Z_3)(1)\|_{0,\infty,t} \le C(\mu)\|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)\omega^{b,1}\|_{L^\infty_{t,y}}$$

By the identities (4.4), (4.3), one sees that

$$\omega^{b,1} \approx F(u^{b,1}, \partial_y u^{b,1}, (\operatorname{div}^{\varphi}u)^{b,1}, \mathbf{n}^{b,1}, \nabla\mathbf{n}^{b,1}),$$

which, together with the previous inequality, yields that:

$$
\begin{aligned}
&\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y, Z_3)(1)\|_{0,\infty,t} \\
(12.21) \qquad &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{1,\infty,t} + \|\varepsilon^{-\frac{1}{2}}\mathrm{div}^{\varphi} u\|_{1,\infty,t} + \|u\|_{2,\infty,t}\big) + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\widetilde{\mathcal{E}}_{m,t} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

Control of the initial evolution (2). Since ∂_t, ∂_y commute with the operator $\bar{\rho}\partial_t - \mu\partial_z^2$, the following identity holds:

$$
(\varepsilon^{\frac{1}{2}}\partial_t, \partial_y)(2) = \int_{-\infty}^{0} E(t,z,z')(\varepsilon^{\frac{1}{2}}\partial_t, \partial_y)(\widetilde{\chi\omega})|_{t=0}(y,z')\,\mathrm{d}z',
$$

from which we derive that:

$$
\begin{aligned}
&\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y)(2)\|_{0,\infty,t} \lesssim \Big\| \int_{-\infty}^{0} |E(t,z,z')|\mathrm{d}z' \Big\|_{L_t^\infty L_z^\infty} \big\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y)\widetilde{\chi\omega})|_{t=0}\big\|_{L^\infty(\mathcal{S}_\kappa)} \\
(12.22) \qquad &\lesssim \Lambda\big(\frac{1}{c_0}, |h_0|_{2,\infty} + |\varepsilon^{\frac{1}{2}}\partial_t h|_{t=0}|_{1,\infty}\big)\big(\|(\varepsilon^{\frac{1}{2}}\partial_t\omega)|_{t=0}\|_{L^\infty(\mathcal{S})} + \|(\mathrm{Id}, \partial_y, Z_3)\omega_0\|_{L^\infty(\mathcal{S})}\big) \\
&\lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big).
\end{aligned}
$$

To control $Z_3(2)$, we denote $E_\pm(t,z,z') = \tilde{\mu}\frac{1}{(4\pi\tilde{\mu}t)^{\frac{1}{2}}}e^{-\frac{|z\pm z'|^2}{4\tilde{\mu}t}}$. By writing $z = z - z' + z'$ or $z = z + z' - z'$, one can split $Z_3(2)$ into two terms:

$$
Z_3(2) = \int_{-\infty}^{0} \phi(z)\partial_z(E_- - E_+)(t,z,z')(\widetilde{\chi\omega})|_{t=0}\,\mathrm{d}z' = (Z_3(2))_1 + (Z_3(2))_2
$$

with

$$
\begin{aligned}
(Z_3(2))_1 &= \phi_1(z)\int_{-\infty}^{0} \big((z-z')\partial_z E_- - (z+z')\partial_z E_+\big)(t,z,z')(\widetilde{\chi\omega})|_{t=0}\,\mathrm{d}z', \\
(Z_3(2))_2 &= \phi_1(z)\int_{-\infty}^{0} E(t,z,z')\partial_{z'}(z'(\widetilde{\chi\omega})|_{t=0})\,\mathrm{d}z',
\end{aligned}
$$

where we use the notation $\phi(z) = \frac{z(1-z)}{(2-z)^2} = z\phi_1(z)$. By straightforward calculation, we obtain

$$
\big|\phi_1(z)\int_{-\infty}^{0} \big((z-z')\partial_z E_- - (z+z')\partial_z E_+\big)(t,z,z')\mathrm{d}z'\big| \leq C(\tilde{\mu})
$$

where $C(\tilde{\mu})$ is a constant depending only on $\tilde{\mu}$ (in particular, independent of z and t). The first term $(Z_3(2))_1$ can thus be bounded as:

$$
\|(Z_3(2))_1\|_{0,\infty,t} \lesssim \|(\widetilde{\chi\omega})|_{t=0}\|_{L^\infty(\mathcal{S}_\kappa)} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big).
$$

Next, by writing

$$
\partial_{z'}(z'(\widetilde{\chi\omega})|_{t=0}) = (\widetilde{\chi\omega})|_{t=0} + \frac{1}{\phi_1(z')}Z_3(\widetilde{\chi\omega})|_{t=0},
$$

and by observing that $\phi_1(z)$ has a uniform positive lower bound on $[-\kappa, 0]$, we control the second term as:

$$
\|(Z_3(2))_2\|_{0,\infty,t} \lesssim \|(\mathrm{Id}, Z_3)(\widetilde{\chi\omega})|_{t=0}\|_{L^\infty(\mathcal{S}_\kappa)} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big).
$$

To summarize, we have obtained that

$$\|Z_3(2)\|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big),$$

which, together with (12.22), yields that:

$$\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y, Z_3)(3)\|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big). \tag{12.23}$$

Control of the nonlinear term (3). We need to distinguish the terms appearing in $\widetilde{F^{\chi,\omega}}$ that involves one normal derivative of the vorticity and the others. Therefore, let us denote

$$\widetilde{F^{\chi,\omega}} = \bar{\rho}\chi\widetilde{u\cdot\nabla^{\varphi}}\omega + \bar{\rho}\partial_t\tilde{\Phi}_t\cdot\nabla(\widetilde{\chi\omega}) - \mu\partial_z\chi\widetilde{\mathbf{N}\cdot\nabla^{\varphi}}\omega + \frac{1}{2}\mu\partial_z(\ln|g|)\partial_z\widetilde{\chi\omega} + R, \tag{12.24}$$

where the remainder term R satisfies the estimate

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\partial_t R\|_{L_t^2H_{co}^2} + \|R\|_{L_t^2H_{co}^3} &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u,\nabla u)\|_{L_t^2H_{co}^4} + \|(\sigma,u,\nabla u)\|_{L_t^2H_{co}^5}\big)\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

By using the Sobolev embedding $H^2(\mathbb{R}^2)\hookrightarrow L_y^\infty(\mathbb{R}^2)$ we can deal with the term

$$\int_0^t\int_{-\infty}^0 E(t-s,z,z')R(s,y,z')\,\mathrm{d}z'\mathrm{d}s$$

as follows:

$$\begin{aligned}&\int_0^t\int_{-\infty}^0 E(t-s,z,z')(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)R(s,y,z')\,\mathrm{d}z'\mathrm{d}s\\ &\lesssim \Big(\int_0^t\int_{-\infty}^0 |E(t-s,z,z')|^2\mathrm{d}z'\mathrm{d}s\Big)^{\frac{1}{2}}\|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)R\|_{L_t^2L_{z'}^2L_y^\infty}\\ &\lesssim \Big(\int_0^t (t-s)^{-\frac{1}{2}}\mathrm{d}s\Big)^{\frac{1}{2}}\|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)R\|_{L_t^2H_{co}^2} \lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Moreover, as in the control of $Z_3(2)$, we have that:

$$\begin{aligned}&Z_3\int_0^t\int_{-\infty}^0 E(t-s,z,z')R(s,y,z')\,\mathrm{d}z'\mathrm{d}s \lesssim \|(\mathrm{Id},Z_3)R\|_{L_t^2H_{co}^2}\cdot\\ &\Big[\Big(\int_0^t\int_{-\infty}^0 |E|^2\mathrm{d}z'\mathrm{d}s\Big)^{\frac{1}{2}} + \Big(\int_0^t\int_{-\infty}^0 \big(|(z-z')\partial_z E_-|^2 + |(z+z')\partial_z E_+|^2\big)\mathrm{d}z'\mathrm{d}s\Big)^{\frac{1}{2}}\Big]\\ &\lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned} \tag{12.25}$$

We are left to treat the first four terms appearing in (12.24), for which we need to integrate by parts in order not to lose normal derivative. Let us explain the estimate for the term

$$\bar{\rho}\int_0^t\int_{-\infty}^0 E(t-s,z,z')\chi\widetilde{u\cdot\nabla^{\varphi}}\omega\,\mathrm{d}z'\mathrm{d}s$$

By straightforward calculation, we find that

$$\begin{aligned}\chi\widetilde{u\cdot\nabla^{\varphi}}\omega &= \widetilde{\chi u_k}(\mathrm{D}\tilde{\Phi})_{jk}\partial_j(\widetilde{\chi_1\omega}) + \widetilde{\chi u_k}(\mathrm{D}\tilde{\Phi})_{3k}\partial_z(\widetilde{\chi_1\omega})\\ &= \widetilde{\chi u_k}(\mathrm{D}\tilde{\Phi})_{jk}\partial_j(\widetilde{\chi_1\omega}) - \partial_z(\widetilde{\chi u_k}(\mathrm{D}\tilde{\Phi})_{3k})\widetilde{\chi_1\omega} + \partial_z(\widetilde{\chi u_k}(\mathrm{D}\tilde{\Phi})_{3k}\widetilde{\chi_1\omega})\end{aligned} \tag{12.26}$$

where χ_1 is a cut-off function supported on $[-C(\kappa),0]$ that satisfies $\chi_1\chi=\chi$. The Einstein summation convention is used for $j=1,2, k=1,2,3$. As the first two terms in the right hand side of the above identity does not involve normal derivatives of $(\widetilde{\chi_1\omega})$, we have by following the same procedure as in the estimate of R that:

$$\bar{\rho}\int_0^t\int_{-\infty}^0 E(t-s,z,z')\big(\widetilde{\chi u}_k(\mathrm{D}\tilde{\Phi})_{jk}\partial_j(\widetilde{\chi_1\omega})-\partial_z(\widetilde{\chi u}_k(\mathrm{D}\tilde{\Phi})_{3k})\widetilde{\chi_1\omega}\big)\,\mathrm{d}z'\mathrm{d}s\lesssim T^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

For the one whose integrand involves the last term of (12.26), we integrate by parts in z' to get that:

$$\begin{aligned}&\bar{\rho}\int_0^t\int_{-\infty}^0 E(t-s,z,z')\partial_{z'}(\widetilde{\chi u}_k(\mathrm{D}\tilde{\Phi})_{3k}\widetilde{\chi_1\omega})\mathrm{d}z'\mathrm{d}s\\ &\lesssim\int_0^t\|\partial_{z'}E(t-s,z,\cdot)\|_{L^2_{z'}}\mathrm{d}s\,\|\widetilde{\chi u}_k(\mathrm{D}\tilde{\Phi})_{3k}\widetilde{\chi_1\omega}\|_{L_t^\infty L^2_{z'}L^\infty_y}\\ &\lesssim T^{\frac14}\|\widetilde{\chi u}_k(\mathrm{D}\tilde{\Phi})_{3k}\widetilde{\chi_1\omega}\|_{L_t^\infty H^2_{co}}\lesssim T^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

In addition to the above two inequalities, we have also analogs of (12.25), that is to say:

$$\begin{aligned}&\bar{\rho}(\varepsilon^{\frac12}\partial_t,\partial_y,Z_3)\int_0^t\int_{-\infty}^0 E(t-s,z,z')\chi\widetilde{u\cdot\nabla^{\varphi}}\omega\,\mathrm{d}z'\mathrm{d}s\\ &\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\int_0^t\|E(t-s,z,\cdot),\partial_{z'}(E(t-s,z,\cdot),\,(z-\cdot)\partial_z E_-,\,(z+\cdot)\partial_z E_+)\|_{L^2_{z'}}\mathrm{d}s\\ &\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\int_0^t(t-s)^{-\frac34}\mathrm{d}s\lesssim T^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}\tag{12.27}$$

We have thus finished the estimate of the term $\int_0^t\int_{-\infty}^0 E(t-s,z,z')\chi\widetilde{u\cdot\nabla^{\varphi}}\omega\,\mathrm{d}z'\mathrm{d}s$. The other three terms in (12.24) can be dealt with in the same way. Consequently, we find that for any $t\in(0,T], z<0$,

$$\int_0^t\int_{-\infty}^0 E(t-s,z,z')\widetilde{F^{\chi,\omega}}\mathrm{d}z'\mathrm{d}s\lesssim T^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\tag{12.28}$$

Collecting (12.21), (12.23) and (12.28), we find that:

$$\|\!|(\mathrm{Id},\varepsilon^{\frac12}\partial_t,\partial_y,Z_3)(\widetilde{\chi\omega})\|\!|_{0,\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},Y_m(0)\big)+\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\tilde{\mathcal{E}}_{m,t}+(T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),$$

By the property (11.22), this leads to (12.17). □

13. Proof of Theorem 1.1.

This section is devoted to the proof of Theorem 1.1 which is based on the known local existence results (non-uniform with respect to ε) and the uniform estimates established in the previous sections. The local existence in the Sobolev-Slobodeskii space $H^{4,2}$ (see the definition (13.1)) is established in [57] [64] (see also [59] for the local existence in Hölder spaces). All these results deal with the case where the reference domain is a smooth bounded domain, nevertheless, by following the same arguments as in these papers, one can easily obtain a similar result when the reference domain is changed into a strip or half space. The following theorem corresponds to Theorem B of [57] or Theorem 6.2 in [64] in this framework.

Theorem 13.1. *Assume that the compatibility condition* (1.28) *holds up to order 2 and*

$$(\sigma_0^\varepsilon,u_0^\varepsilon)\in(H^3(\mathcal{S}))^4,\quad h_0^\varepsilon\in H^{\frac72}(\mathbb{R}^2),\quad 1+h_0^\varepsilon\geq 3c_0>0,$$

δ is chosen sufficiently small such that

$$\partial_z\varphi_0^\varepsilon(x)=1+\partial_z\eta_0^\varepsilon(1+z)+\eta_0^\varepsilon\geq 2c_0>0,\forall x\in\mathcal{S},$$

where η_0^ε is the extension of h_0^ε defined in (1.12). *Then for any $\varepsilon \in (0,1]$, we can find $T^\varepsilon > 0$ such that:*

$$(\sigma^\varepsilon, u^\varepsilon) \in C([0,T^\varepsilon], H^3(\mathcal{S})), \quad h^\varepsilon \in C([0,T^\varepsilon], H^{\frac{7}{2}}(\mathbb{R}^2)).$$

Moreover,

$$u^\varepsilon \in H^{4,2}([0,T^\varepsilon]\times\mathcal{S}) = \{u \big| \partial_t^j u \in L^2\big([0,T^\varepsilon], H^{4-2j}(\mathcal{S})\big), j=0,1,2\} \tag{13.1}$$

and (2.2) *holds.*

We shall combine this theorem with the uniform regularity estimates established in the previous sections. Set

$$T_*^\varepsilon = \sup\big\{T \big| (\sigma^\varepsilon, u^\varepsilon) \in C([0,T], H^3(S)), u^\varepsilon \in W^{4,2}([0,T^\varepsilon]\times\mathcal{S}) \text{ and (2.2) holds}\big\}.$$

Since the initial datum is assumed to belong to Y_m^ε, a space with higher regularity, by standard propagation of regularity arguments (for example based on applying finite difference instead of derivatives) and the computations presented in Section 6-Section 12, we can find the following uniform estimates of Theorem 2.1:

$$\mathcal{N}_{m,T}^\varepsilon \le P_5\big(\frac{1}{c_0}, Y_m^\varepsilon(0)\big) + (T+\varepsilon)^\vartheta P_6\big(\frac{1}{c_0}, Y_m^\varepsilon(0) + \mathcal{N}_{m,T}^\varepsilon\big). \tag{13.2}$$

where $0 < \vartheta < 1$ and P_5, P_6 are two increasing continuous functions that are independent of ε. By the fundamental theorem of calculus and Lemma 3.8, one finds for $0 \le t \le T$

$$\begin{aligned} \partial_z\varphi(t,x) &= \partial_z\varphi(0,x) + \int_0^t (\partial_t\eta + (1+z)\partial_t\partial_z\eta)(s,x)\,\mathrm{d}s \\ &\ge \partial_z\varphi(0,x) - C_1 T|\partial_t h(t)|_{L^\infty(\mathbb{R}^2)}, \end{aligned} \tag{13.3}$$

$$\|(\nabla\varphi, \nabla^2\varphi)(t)\|_{L^\infty(\mathcal{S})} \le \|(\nabla\varphi, \nabla^2\varphi)(0)\|_{L^\infty(\mathcal{S})} + C_2 T|h(t)|_{W^{2,\infty}(\mathbb{R}^2)}. \tag{13.4}$$

where C_1, C_2 are two constants independent of ε. Moreover, $\varepsilon\sigma^\varepsilon$ can be expanded by using the characteristic method:

$$\varepsilon\sigma^\varepsilon(t,x) = \varepsilon\sigma_0^\varepsilon(X^{-1}(t,x)) - \int_0^t (\mathrm{div}u^\varepsilon/g_1)(X(s,X^{-1}(t,x)))\mathrm{d}s \tag{13.5}$$

where $X(t,x)$ is the unique flow associated to $\underline{u}$. Let us define

$$\begin{aligned} T_*^\varepsilon &= \sup\{T \ge 0 \big| (\sigma^\varepsilon, u^\varepsilon) \in C([0,T], H^3), u^\varepsilon \in W^{4,2}([0,T]\times\mathcal{S})\}, \\ T_0^\varepsilon &= \sup\big\{0 \le T \le \min\{T_*^\varepsilon, 1\} \big| \mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \le 2P_5\big(1/c_0, M\big) \\ &\qquad\qquad (1.34) \text{ holds for all } (t,x) \in [0,T]\times\mathcal{S}\big\}. \end{aligned}$$

where M is chosen such that $M \ge \sup_{\varepsilon\in(0,1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon)$.

We now choose successively two constants $0 < \varepsilon_0 \le 1$ and $T_0 > 0$ (uniform in $\varepsilon \in (0, \varepsilon_0]$) which are small enough, such that:

$$(T_0 + \varepsilon_0)^\vartheta P_6\big(1/c_0, M + 2P_5(1/c_0, M)\big) < \frac{1}{2}P_5(1/c_0, M),$$

$$C_1 T_0 P_5(1/c_0, M)^2 \le c_0, \quad C_2 T_0 P_5(1/c_0, M) \le 1/(2c_0), \quad 2P_5(1/c_0, M)T_0/c_0 \le \bar{c}\bar{P}.$$

In order to prove Theorem 1.1, it suffices to show that $T_0^\varepsilon \ge T_0$ for every $0 < \varepsilon \le \varepsilon_0$. Suppose otherwise $T_0^\varepsilon < T_0$ for some $0 < \varepsilon \le \varepsilon_0$, then in view of inequalities (13.2)-(13.4) and the formula (13.5), we have by the definition of ε_0 and T_0 that:

$$\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \le \frac{3}{2}P_5(1/c_0, M) \qquad \forall T \le \tilde{T} = \min\{T_0, T_*^\varepsilon\}, \tag{13.6}$$

(13.7)
$$\partial_z\varphi^\varepsilon(t,x)\geq c_0,\quad |(\nabla\varphi^\varepsilon,\nabla^2\varphi^\varepsilon)(t,x)|\leq 1/c_0,\quad -2\bar{c}\bar{P}\leq\varepsilon\sigma^\varepsilon(t,x)\leq 2\bar{P}/\bar{c}\quad\forall(t,x)\in[0,\tilde{T}]\times\Omega.$$

We intend to prove that $\tilde{T}=T_0\leq T_*^\varepsilon$. This fact, combined with the definition of T_0^ε and the estimates (13.6), (13.7), yields $T_0^\varepsilon\geq T_0$, which is a contradiction with the assumption $T_0^\varepsilon<T_0$. To continue, we shall need the claim stated and proved below. Indeed, once the following claim holds, we have by (13.6) that $\|(\sigma^\varepsilon,u^\varepsilon)(T_0)\|_{H^3(\Omega)}<+\infty$. Using the local existence result stated in Theorem 13.1, we obtain that $T_*^\varepsilon>T_0=\tilde{T}$.

Claim. For all $\varepsilon\in(0,1]$, if $\mathcal{N}_{m,T}(\sigma^\varepsilon,u^\varepsilon)<+\infty$, then $(\sigma^\varepsilon,u^\varepsilon)\in C([0,T],H^3)$, $u^\varepsilon\in H^{4,2}([0,T]\times\mathcal{S})$.

Proof of claim. By the definition of $\mathcal{N}_{m,T}$, we derive that:

$$\varepsilon^{\frac{3}{2}}u^\varepsilon\in L^2([0,T],H^4),\quad \varepsilon^{\frac{3}{2}}\partial_t u^\varepsilon\in L^2([0,T],H^2),\quad \varepsilon^{\frac{3}{2}}\partial_t^2 u\in L^2([0,T],L^2)\quad \varepsilon^{\frac{1}{2}}\sigma^\varepsilon\in L^\infty([0,T],H^3),$$

which yields by interpolation that $\varepsilon^{\frac{3}{2}}u^\varepsilon\in C([0,T],H^3)\cap H^{4,2}([0,T]\times\mathcal{S})$. Moreover, carrying out direct energy estimates for σ^ε in $H^3(\Omega)$, one gets that:

$$|\partial_t R^\varepsilon(t)|\leq K^\varepsilon f^\varepsilon(t)\tag{13.8}$$

where $K^\varepsilon=\Lambda(1/c_0,\|(\sigma^\varepsilon,\nabla\sigma^\varepsilon,\nabla u^\varepsilon,\varepsilon^{\frac{1}{2}}\nabla^2u^\varepsilon)\|_{\infty,t})$ is uniformly bounded and

$$R^\varepsilon(t)=\|\varepsilon^{\frac{1}{2}}\sigma^\varepsilon(t)\|_{H^3}^2,\quad f^\varepsilon(t)=\|\varepsilon^{\frac{3}{2}}u^\varepsilon(t)\|_{H^4}^2+\|\varepsilon^{\frac{1}{2}}u^\varepsilon(t)\|_{H^3}^2+\|(\sigma^\varepsilon,\varepsilon^{-\frac{1}{2}}\nabla\sigma^\varepsilon)(t)\|_{H^2}^2\in L^1([0,T]).$$

Inequality (13.8) and the boundedness of $\|R^\varepsilon(\cdot)\|_{L^\infty([0,T])}$ leads to the fact that $R^\varepsilon(\cdot)\in C([0,T])$, which further yields that $\varepsilon^{\frac{1}{2}}\sigma^\varepsilon\in C([0,T],H^3)$. This ends the proof of the claim. Note that at this stage we do not require the norm $\|(\sigma^\varepsilon,u^\varepsilon)\|_{C([0,T],H^3)}$ to be bounded uniformly in ε. □

14. Convergence

This section aims to show Theorem 1.5. In the following, we denote $Q_{T_0}=[0,T_0]\times\mathcal{S}$, $\Gamma_{T_0}=[0,T_0]\times\mathbb{R}^2$.

First of all, for the surface, since $\partial_t h^\varepsilon$ is uniformly bounded in $L^\infty([0,T_0],H^{m-3/2}(\mathbb{R}^2))$, h^ε is uniformly bounded in $L^\infty([0,T_0],H^{m-1/2}(\mathbb{R}^2))$, one has that h^ε converges (say to h^0) in $C([0,T_0],H^s_{loc}(\mathbb{R}^2))$ for any $0\leq s<m-1/2$. Further, from the definition of φ^ε (1.11) and Lemma (3.8), we conclude also that $\varphi^\varepsilon\to\varphi_0$ in $C([0,T_0],H^s_{loc}(\mathcal{S})),0\leq s<m$ where φ^0 is defined in a similar way as (1.11) by replacing h^ε with h^0.

Next, since $(\varepsilon^{\frac{1}{2}}\partial_t\sigma^\varepsilon,\varepsilon^{\frac{1}{2}}\sigma^\varepsilon)$ is uniformly bounded in $L^\infty([0,T_0],H^1(\mathcal{S}))\times L^\infty([0,T_0],H^3(\mathcal{S}))$, we have that $\varepsilon^{\frac{1}{2}}\sigma^\varepsilon$ is uniformly bounded in $C^\gamma(Q_{T_0}),0<\gamma<\frac{1}{2}$. In view of the definition of $\sigma^\varepsilon:\sigma^\varepsilon=(P(\rho)-P(\bar{\rho}))/\varepsilon$, we have that $P(\rho^\varepsilon)\to P(\bar{\rho})$ in $C^\gamma(Q_{T_0})$, which, combined with the uniform boundedness of $\|\nabla P(\rho^\varepsilon)\|_{\infty,t}$, yields the convergence of ρ^ε to $\bar{\rho}$ in $C^\gamma(Q_{T_0})$.

Let us see the convergence of the velocity. We write $u^\varepsilon=\nabla^{\varphi^\varepsilon}\Psi^\varepsilon+v^\varepsilon$, where $\nabla^{\varphi^\varepsilon}\Psi^\varepsilon$ and v^ε denote the compressible and incompressible part of the velocity (see definitions (5.2), (5.3)). On the one hand, since $\varepsilon^{-\frac{1}{2}}\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon$, $\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon$ are both uniformly bounded in $L^\infty([0,T_0],H^1(\mathcal{S}))$, we get that $\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon\to 0$ in $C^\gamma([0,T_0],H^1(\mathcal{S})),0<\gamma<\frac{1}{2}$. By elliptic estimates (5.10), $\nabla^{\varphi^\varepsilon}\Psi^\varepsilon\to 0$ in $C^\gamma([0,T_0],H^2(\mathcal{S}))$. On the other hand, due to the uniform boundedness of $\partial_t v^\varepsilon$ in $L^2([0,T_0],H^{-1}(\mathcal{S}))$, and of v^ε in $L^\infty([0,T_0],H^1(\mathcal{S}))$, we obtain by Aubin-Lions lemma that up to extraction of subsequences, v^ε converges (say to u^0) in $C([0,T_0],L^2_{loc}(\mathcal{S}))$. Since we will prove that u^0 is the *unique* solution (in conormal spaces with additional regularity property), to the incompressible free-surface

Navier-Stokes equations this convergence holds indeed for the whole family. We thus proved that u^ε converges to u^0 in $C^\gamma([0,T_0],H^1(\mathcal{S}))+C([0,T_0],L^2_{loc}(\mathcal{S}))$.

To conclude, we have achieved that

$$\sigma^\varepsilon\to 0 \quad \rho^\varepsilon\to\bar\rho \quad \nabla^{\varphi^\varepsilon}\Psi^\varepsilon\to 0 \text{ in } \quad C^\gamma(Q_{T_0}) \quad v^\varepsilon\to u^0 \quad \text{in} \quad C\big([0,T_0],L^2_{loc}\big), \tag{14.1}$$

$$\varphi^\varepsilon\to\varphi^0 \text{ in } C\big([0,T_0],H^s_{loc}(\mathcal{S})\big) \qquad h^\varepsilon\to h^0 \quad \text{in} \quad C\big([0,T_0],H^s_{loc}(\mathbb{R}^2)\big), \quad 0\le s\le m-\frac{1}{2}. \tag{14.2}$$

We now show that there exists $\pi_0\in L^2([0,T_0],\mathcal{H}^{0,m-1})$ such that (u^0,π^0,h^0) is the (unique) solution to the incompressible free surface system (1.36). Let us rewrite the equations for the incompressible part of the velocity (see (9.6)) as follows:

$$\bar\rho(\partial_t^{\varphi^\varepsilon}v^\varepsilon+v^\varepsilon\cdot\nabla^{\varphi^\varepsilon}v^\varepsilon)-\mu\Delta^{\varphi^\varepsilon}v^\varepsilon+\nabla^{\varphi^\varepsilon}\tilde\pi^\varepsilon=F^\varepsilon. \tag{14.3}$$

where

$$\begin{aligned}\nabla^{\varphi^\varepsilon}\tilde\pi^\varepsilon&=\nabla^{\varphi^\varepsilon}(\pi^\varepsilon-q^\varepsilon)-[\partial_t^{\varphi^\varepsilon},\mathbb{P}_t]u^\varepsilon,\\ F^\varepsilon&=\varepsilon\frac{g_2-1}{\varepsilon}(\partial_t+\underline{u^\varepsilon}\cdot\nabla)u^\varepsilon-\bar\rho(v^\varepsilon\cdot(\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon+\nabla^{\varphi^\varepsilon}\Psi^\varepsilon\cdot\nabla^{\varphi^\varepsilon}u^\varepsilon).\end{aligned}$$

with $\nabla^{\varphi^\varepsilon}\pi^\varepsilon,\nabla^{\varphi^\varepsilon}q^\varepsilon$ defined in (9.7). Note that by the definition (5.2), (5.3) for $\mathbb{Q}_t,\mathbb{P}_t$, the commutator $-[\partial_t^{\varphi^\varepsilon},\mathbb{P}_t]u^\varepsilon$ can be expressed as a gradient:

$$-[\partial_t^{\varphi^\varepsilon},\mathbb{P}_t]u^\varepsilon=[\partial_t^{\varphi^\varepsilon},\mathbb{Q}_t]u^\varepsilon=\nabla^{\varphi^\varepsilon}(\partial_t^{\varphi^\varepsilon}\Psi^\varepsilon-\tilde\Psi^\varepsilon) \tag{14.4}$$

where we denote $\nabla^{\varphi^\varepsilon}\tilde\Psi^\varepsilon=\mathbb{Q}_t(\partial_t^{\varphi^\varepsilon}u^\varepsilon)$. By estimates established in (9.10), (9.14) and (9.15), we readily see that $\nabla\tilde\pi^\varepsilon$ is uniformly bounded in $L^2([0,T_0],\mathcal{H}^{0,m-2})$. Therefore, there exists $\pi^0\in L^2([0,T_0],\mathcal{H}^{0,m-1})$ such that $\nabla\tilde\pi^\varepsilon$ tends (up to subsequences) to $\nabla\pi^0$ in $L^2_w(Q_{T_0})$ and $\tilde\pi^\varepsilon$ converges to π_0 in $L^2_w([0,T_0],L^2_{loc}(\mathcal{S}))$. Next, by boundary conditions $(9.6)_2-(9.6)_3$ as well as the fact (14.4), we have that:

$$(2\mu S^{\varphi^\varepsilon}u^\varepsilon-\tilde\pi^\varepsilon\mathrm{Id})\mathbf{N}^\varepsilon=2\mu(\mathrm{div}^{\varphi^\varepsilon}u\mathrm{Id}-(\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon)\mathbf{N}^\varepsilon+(\frac{\partial_t h^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z\Psi^\varepsilon)\mathbf{N}^\varepsilon \quad \text{on } z=0, \tag{14.5}$$

$$v_3^\varepsilon=0,\ \mu\partial_z^{\varphi^\varepsilon}v_j^\varepsilon=au_j^\varepsilon \quad (j=1,2) \qquad \text{on } z=-1. \tag{14.6}$$

Let us now choose a smooth vector $\psi=(\psi_1,\psi_2,\psi_3)^t\in\big[C_c^\infty\big(\overline{Q_{T_0}}\big)\big]^3$ with condition $\psi_3|_{z=-1}=0$. Multiplying the equations (14.3) by ψ and integrating by parts in space and time, we find by using the boundary conditions (14.5), (14.6) that:

(14.7)

$$\begin{aligned}&\bar\rho\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(t,\cdot)\,\mathrm{d}\mathcal{V}_t^\varepsilon+2\mu\int_0^t\int_{\mathcal{S}}S^{\varphi^\varepsilon}v^\varepsilon\cdot\nabla^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s+\bar\rho\int_0^t\int_{\mathcal{S}}(v^\varepsilon\cdot\nabla^{\varphi^\varepsilon}v^\varepsilon)\cdot\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s\\ &=\bar\rho\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(0,\cdot)\,\mathrm{d}\mathcal{V}_0^\varepsilon+\int_0^t\int_{\mathcal{S}}F^\varepsilon\cdot\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s+\bar\rho\int_0^t\int_{\mathcal{S}}v^\varepsilon\cdot\partial_t^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s+\int_0^t\int_{\mathcal{S}}\tilde\pi^\varepsilon\mathrm{div}^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s\\ &\quad+a\int_0^t\int_{z=-1}(u_1^\varepsilon\cdot\psi_1+u_2^\varepsilon\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s+\int_0^t\int_{z=0}(v^\varepsilon\cdot\mathbf{N}^\varepsilon)(v^\varepsilon\cdot\psi)\,\mathrm{d}y\mathrm{d}s\\ &\quad+\int_0^t\int_{z=0}(2\mu\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon+\frac{\partial_t h^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z\Psi^\varepsilon)(\psi\cdot\mathbf{N}^\varepsilon)-(\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon\mathbf{N}^\varepsilon\cdot\psi\,\mathrm{d}y\mathrm{d}s\end{aligned}$$

where $\mathrm{d}\mathcal{V}_t^\varepsilon=\frac{1}{\partial_z\varphi^\varepsilon}(t,\cdot)\,\mathrm{d}y\mathrm{d}z$. Since $v^\varepsilon\to v^0$ in $C([0,T_0],L^2_{loc}(\mathcal{S}))$, $\partial_z\varphi^\varepsilon$ converges to $\partial_z\varphi^0$ in $C([0,T_0],C_{loc}(\mathcal{S}))$, we see that:

$$\bar\rho\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(t,\cdot)\,\mathrm{d}\mathcal{V}_t^\varepsilon\to\bar\rho\int_{\mathcal{S}}(u^0\cdot\psi)(t,\cdot)\,\mathrm{d}\mathcal{V}_t^0,\quad \bar\rho\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(0,\cdot)\,\mathrm{d}\mathcal{V}_0^\varepsilon\to\bar\rho\int_{\mathcal{S}}(u^0\cdot\psi)(0,\cdot)\,\mathrm{d}\mathcal{V}_0^0 \tag{14.8}$$

Let us now show the convergence of the last two terms in the left hand side of the above identity. Since

(14.9)

$$v^{\varepsilon} \to u^0 \text{ in } L^2([0,T_0],L^2_{loc}(\mathcal{S})),\ \nabla v^{\varepsilon} \rightharpoonup \nabla u^0 \text{ in } L^2(Q_{T_0}),\ v^{\varepsilon} \text{ uniformly bounded in } L^2([0,T_0],H^1(\mathcal{S}))$$

$$\varphi^{\varepsilon} \to \varphi^0 \text{ in } C([0,T_0],C^1_{loc}(\mathcal{S})), \qquad (\partial_z\varphi^{\varepsilon},\partial_z\varphi_0)(t,x) \geq c_0 > 0, \forall (t,x) \in Q_{T_0} \tag{14.10}$$

one gets that: $\mathcal{S}^{\varphi^{\varepsilon}} v^{\varepsilon} \rightharpoonup \mathcal{S}^{\varphi_0} v^0$, $\nabla^{\varphi^{\varepsilon}}\psi \to \nabla^{\varphi^0}\psi$ in $L^2(Q_{T_0})$, which leads to the fact:

$$\begin{aligned} &2\mu\int_0^t\int_{\mathcal{S}} S^{\varphi^{\varepsilon}} v^{\varepsilon}\cdot\nabla^{\varphi^{\varepsilon}}\psi\,\mathrm{d}\mathcal{V}^{\varepsilon}_s\mathrm{d}s + \bar{\rho}\int_0^t\int_{\mathcal{S}}(v^{\varepsilon}\cdot\nabla^{\varphi^{\varepsilon}}v^{\varepsilon})\cdot\psi\,\mathrm{d}\mathcal{V}^{\varepsilon}_s\mathrm{d}s \\ &\to 2\mu\int_0^t\int_{\mathcal{S}} S^{\varphi^0}u^0\cdot\nabla^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s + \bar{\rho}\int_0^t\int_{\mathcal{S}}(u^0\cdot\nabla^{\varphi^0}u^0)\cdot\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s \end{aligned} \tag{14.11}$$

It suffices to deal with the convergence of the the last four terms in the right hand side of (14.7). As $\nabla^{\varphi^{\varepsilon}}\psi^{\varepsilon} = \mathcal{O}(\varepsilon^{\frac{1}{2}})$ in $L^2_tH^1$ and $(u^{\varepsilon},\varepsilon^{\frac{1}{2}}\partial_t u^{\varepsilon})$ uniformly bounded in $L^2([0,T_0],H^1(\mathcal{S}))$, one readily see that $F^{\varepsilon} \to 0$ in $L^2(Q_{T_0})$, which gives that:

$$\int_0^t\int_{\mathcal{S}} F^{\varepsilon}\cdot\psi\,\mathrm{d}\mathcal{V}^{\varepsilon}_s\mathrm{d}s \to 0. \tag{14.12}$$

Next, since $\partial_t\varphi^{\varepsilon} \to \partial_t\varphi^0$ in $L^2_w([0,T_0],L^2(\mathcal{S}))$, we have by combining (14.10) that $\partial_t^{\varphi^{\varepsilon}}\psi \rightharpoonup \partial_t^{\varphi^0}\psi$ in $L^2(Q_{T_0})$ This, together with (14.9) gives that:

$$\bar{\rho}\int_0^t\int_{\mathcal{S}} v^{\varepsilon}\cdot\partial_t^{\varphi^{\varepsilon}}\psi\,\mathrm{d}\mathcal{V}^{\varepsilon}_s\mathrm{d}s \to \bar{\rho}\int_0^t\int_{\mathcal{S}} u^0\cdot\partial_t^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s. \tag{14.13}$$

As for (14.11), we have also that:

$$\int_0^t\int_{\mathcal{S}}\tilde{\pi}^{\varepsilon}\mathrm{div}^{\varphi^{\varepsilon}}\psi\,\mathrm{d}\mathcal{V}^{\varepsilon}_s\mathrm{d}s \to \int_0^t\int_{\mathcal{S}}\pi^0\mathrm{div}^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s. \tag{14.14}$$

To proceed, we prove that $(u^{\varepsilon})^{b,j},(v^{\varepsilon})^{b,j}$ both convergent to $(u^0)^{b,j}$ in $L^2_{loc}([0,T_0]\times\mathbb{R}^2)$ where $j=1,2$. Indeed, by the trace inequality and the fact (14.9), one has for any $K\subset\mathbb{R}^2$ compact,

$$|(v^{\varepsilon})^{b,j}-(u^0)^{b,j}|_{L^2([0,T_0]\times K)} \lesssim \|v^{\varepsilon}-u^0|^{\frac{1}{2}}_{L^2([0,T_0],L^2(\tilde{K}\times[-1,0]))}\|v^{\varepsilon}-u^0|^{\frac{1}{2}}_{L^2([0,T_0],H^1(\mathcal{S})}\to 0.$$

where $\tilde{K}\subset\mathbb{R}^2$ is a compact set such that $K\Subset\tilde{K}$. The same argument applies also for u^{ε}. Therefore, one deduces that:

$$\begin{aligned} &a\int_0^t\int_{z=-1}(u^{\varepsilon}_1\cdot\psi_1+u^{\varepsilon}_2\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s + \int_0^t\int_{z=0}(v^{\varepsilon}\cdot\mathbf{N}^{\varepsilon})(v^{\varepsilon}\cdot\psi)\,\mathrm{d}y\mathrm{d}s \\ &\to a\int_0^t\int_{z=-1}(u^0_1\cdot\psi_1+u^0_2\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s + \int_0^t\int_{z=0}(u^0\cdot\mathbf{N}^0)(u^0\cdot\psi)\,\mathrm{d}y\mathrm{d}s \end{aligned} \tag{14.15}$$

Finally, by the trace inequality $\mathrm{div}^{\varphi^{\varepsilon}}u^{\varepsilon},\nabla^{\varphi^{\varepsilon}}\Psi^{\varepsilon},(\nabla^{\varphi^{\varepsilon}})^2\Psi^{\varepsilon} = \mathcal{O}(\varepsilon^{\frac{1}{2}})$ in $L^2([0,T_0],L^2(\mathbb{R}^2))$, which yields that:

$$\int_0^t\int_{z=0}(2\mu\mathrm{div}^{\varphi^{\varepsilon}}u^{\varepsilon}+\frac{\partial_t h^{\varepsilon}}{\partial_z\varphi^{\varepsilon}}\partial_z\Psi^{\varepsilon})(\psi\cdot\mathbf{N}^{\varepsilon})-(\nabla^{\varphi^{\varepsilon}})^2\Psi^{\varepsilon}\mathbf{N}^{\varepsilon}\cdot\psi\,\mathrm{d}y\mathrm{d}s\to 0. \tag{14.16}$$

Plugging (14.8) and (14.11)-(14.16) into (14.7), we find that (u^0,π^0,h^0) satisfies (1.40). Finally, it is direct to see that u^0 has the additional regularity (1.35). In particular, u^0 is Lipschitz continuous, which is sufficient to verify the uniqueness. For the reader's convenience, we will sketch the proof in the following subsection.

14.1. **Uniqueness of limit system.** Suppose that there are two solutions $(h^1, u^1, \nabla\pi^1)$ and $(h^2, u^2, \nabla\pi^2)$ to the system (1.36)-(1.39) on the time interval $[0, T_0]$ with the same initial data (φ^1, φ^2 are defined through (1.11) and (1.12) associated to h^1, h^2). Let $h = h^1 - h^2, u = u^1 - u^2, \pi = \pi^1 - \pi^2$. We prove that $h = 0, u = 0$. By direct calculation, we find that (h, u) solves the following system:

$$\partial_t h + (u^1)^{b,1} \cdot \nabla_y h + u^{b,1} \cdot \nabla_y h^2 + u_3^{b,1} = 0 \tag{14.17}$$

$$(\partial_t + \underline{u}^1 \cdot \nabla) u + \nabla^{\varphi^1} \pi - \mu \Delta^{\varphi^1} u = F \tag{14.18}$$

where

$$F = -(\underline{u}^1 - \underline{u}^2) \cdot \nabla u^2 + (\nabla^{\varphi^2} - \nabla^{\varphi^1})\pi^2 + \mu(\Delta^{\varphi^1} - \Delta^{\varphi^2})u^2, \quad \underline{u}^i = (u_1^i, u_2^i, \frac{u^i \cdot \mathbf{N}^i - \partial_t \varphi^i}{\partial_z \varphi^i}), i = 1, 2, \tag{14.19}$$

and with boundary conditions:

$$(S^{\varphi^1} u - \pi \mathrm{Id}_3)\mathbf{n}^1 = [(S^{\varphi^2} - S^{\varphi^1})u^2]\mathbf{n}^1 + (-S^{\varphi^2} u^2 + \pi^2 \mathrm{Id}_3)(\mathbf{n}^1 - \mathbf{n}^2) \quad \text{on } \{z = 0\}, \tag{14.20}$$

$$\mu \partial_z u_j = a u_j \, (j = 1, 2) \quad u_3 = 0 \quad \text{on } \{z = -1\}. \tag{14.21}$$

Define

$$E(t) = |h(t)|^2_{H^{\frac{3}{2}}(\mathbb{R}^2)} + \|(u, \partial_y u)(t)\|^2_{L^2(\mathcal{S})}.$$

It suffices to prove that

$$E(t) + \int_0^t \|\nabla(u, \partial_y u)(s)\|^2_{L^2(\mathcal{S})} \mathrm{d}s \leq \Lambda(R) \int_0^t E(s) \mathrm{d}s, \, \forall t \in [0, T_0]. \tag{14.22}$$

where

$$R = \sum_{i=1}^2 \big(\|(u^i, \nabla u^i, \partial_y \nabla u^i)\|_{0,\infty,t} + \|(\pi^i, \nabla \pi^i)\|_{0,\infty,t} + |h^i|_{L_t^\infty H^4} \big).$$

Direct energy estimates on h lead to:

$$\begin{aligned} |h(t)|^2_{H^{\frac{3}{2}}(\mathbb{R}^2)} &\lesssim \Lambda(R) \big(|h|^2_{L_t^2 H^{\frac{3}{2}}(\mathbb{R}^2)} + |h|_{L_t^2 H^{\frac{3}{2}}(\mathbb{R}^2)} |u^{b,1}|_{L_t^2 H^{\frac{3}{2}}(\mathbb{R}^2)} \big) \\ &\leq \frac{1}{2} \int_0^t \|\nabla(u, \partial_y u)(s)\|^2_{L^2(\mathcal{S})} \, \mathrm{d}s + \Lambda(R) \int_0^t E(s) \, \mathrm{d}s. \end{aligned} \tag{14.23}$$

Thanks to Lemma 3.12 and boundary condition (14.21), we can obtain the energy equality:

$$\begin{aligned} &\frac{1}{2} \int_{\mathcal{S}} |u(t)|^2 \, \mathrm{d}\mathcal{V}_t^1 + 2\mu \int_0^t \int_{\mathcal{S}} |S^{\varphi^1} u|^2 \, \mathrm{d}\mathcal{V}_s^1 \mathrm{d}s + a \int_0^t \int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}y \mathrm{d}s \\ &= \int_0^t \int_{\mathcal{S}} \pi \, \mathrm{div}^{\varphi^1} u \, \mathrm{d}\mathcal{V}_s^1 \mathrm{d}s + \int_0^t \int_{\mathcal{S}} F \cdot u \, \mathrm{d}\mathcal{V}_s^1 \mathrm{d}s + 2\mu \int_0^t (S^{\varphi^1} u - \pi \mathrm{Id}_3)\mathbf{n}^1 \cdot u \, \mathrm{d}y \mathrm{d}s, \end{aligned}$$

where $\mathrm{d}\mathcal{V}_t^1 = \partial_z \varphi^1(t, \cdot) \, \mathrm{d}x$. In light of the definition (14.19) for F, boundary condition (14.20) as well as the identity:

$$\mathrm{div}^{\varphi^1} u = (\mathrm{div}^{\varphi^2} - \mathrm{div}^{\varphi^1}) u^2,$$

we can obtain, after lengthy but direct computations, that:

$$\int_{\mathcal{S}} |u(t)|^2 \mathrm{d}\mathcal{V}^1 + \int_0^t \int_{\mathcal{S}} |\nabla u|^2 \mathrm{d}\mathcal{V}^1 \mathrm{d}s \leq \Lambda(R) \big(\int_0^t E(s) \, \mathrm{d}s + \|\pi\|_{L_t^2 L^2(\mathcal{S})} |h|_{L_t^2 H^{\frac{1}{2}}} \big).$$

Following similar arguments, one can also show that:

$$\int_{\mathcal{S}} |\partial_y u(t)|^2 \mathrm{d}\mathcal{V}^1 + \int_0^t \int_{\mathcal{S}} |\nabla \partial_y u|^2 \mathrm{d}\mathcal{V}^1 \mathrm{d}s \leq \Lambda(R) \big(\int_0^t E(s) \, \mathrm{d}s + \|\pi\|_{L_t^2 H^1(\mathcal{S})} |h|_{L_t^2 H^{\frac{3}{2}}} \big).$$

By the elliptic estimates performed in Section 5, we can find that:

$$\|\pi\|_{L_t^2 H^1(\mathcal{S})} \lesssim \Lambda(R)\big(|h|_{L_t^2 H^{\frac{3}{2}}(\mathbb{R}^2)} + \|(u, \partial_y u)\|_{L_t^2 H^1(\mathcal{S})}\big).$$

Combining the previous three inequalities and using Young's inequality, we have:

$$\|(u, \partial_y u)(t)\|_{L^2(\mathcal{S})}^2 + \int_0^t \|\nabla(u, \partial_y u)(s)\|_{L^2(\mathcal{S})}^2 \mathrm{d}s \leq \Lambda(R) \int_0^t E(s)\, \mathrm{d}s.$$

Together with (14.23), this yields (14.22).

15. Remarks for other reference domains.

In this section, we shall explain how to extend the uniform estimates results established in sections 5-12 to the case when the reference domain is a channel with infinite depth or a bounded domain. We will only explain the former case since the latter can be dealt with by using the similar covering as in [48] and by working in local coordinates based on the former case.

Assume now that Ω_t^ε is given by:

$$\Omega_t^\varepsilon = \{x = (y, z) |\, y \in \mathbb{R}^2, z < h^\varepsilon(t, y)\}.$$

The first step is still to use the so-called harmonic extension transformation to reduce the problem to a fixed domain. Consider the map

$$\begin{aligned} \Phi_t^\varepsilon : \mathbb{R}_-^3 &\to \Omega_t^\varepsilon \\ (y, z) &\to \Phi^\varepsilon(t, y, z) = (y, \varphi^\varepsilon(t, y, z))^t \end{aligned} \tag{15.1}$$

where

$$\varphi^\varepsilon(t, y, z) = Az + \eta^\varepsilon(t, x) \tag{15.2}$$

Here η is given by (1.12) and A is a constant which is chosen sufficiently large such that $\partial_z \varphi^\varepsilon > 0$. We introduce the conormal vector fields

$$Z_0 = \varepsilon \partial_t, \quad Z_1 = \partial_{y_1}, \quad Z_2 = \partial_{y_2}, \quad Z_3 = \phi(z)\partial_z.$$

where the weight function $\phi(z) = z/(1-z)$. We can define conormal spaces analogous to those in Section 1.2 by using these vector fields. Furthermore, we can use the quantity $\mathcal{N}_{m,T}^\varepsilon$ defined in (1.31) (with the conormal norms being changed accordingly in the current definition). The projections $\mathbb{Q}_t, \mathbb{P}_t$ that send a vector field in $(L^2(\mathbb{R}_-^3\, \mathrm{d}\mathcal{V}_t))^3, (\mathrm{d}\mathcal{V}_t = \partial_z \varphi\, \mathrm{d}y \mathrm{d}z)$ to its compressible part and incompressible part are defined as: $\mathbb{P}_t = \mathrm{Id} - \mathbb{Q}_t$ and

$$\begin{aligned} \mathbb{Q}_t : \quad L^2(\mathbb{R}_-^3\, \mathrm{d}\mathcal{V}_t)^3 &\to L^2(\mathbb{R}_-^3\, \mathrm{d}\mathcal{V}_t)^3 \\ f &\to \mathbb{Q}_t f = \nabla^{\varphi^\varepsilon} \varrho \end{aligned} \tag{15.3}$$

where ϱ satisfies the elliptic equation with trivial Dirichlet boundary condition:

$$\begin{cases} -\Delta^{\varphi^\varepsilon} \varrho = -\mathrm{div}^{\varphi^\varepsilon} f \quad \text{in } \mathbb{R}_-^3 \\ \varrho|_{z=0} = 0 \end{cases} \tag{15.4}$$

Denote further $v^\varepsilon = \mathbb{P}_t u^\varepsilon, \nabla^{\varphi^\varepsilon} \Psi^\varepsilon = \mathbb{Q}_t u^\varepsilon$.

Following the similar (and even easier since there is no lower boundary) computations done in Section 5-12, we can prove uniform estimates analogous to those of Theorem 2.1, we thus do not detail them. We comment that one crucial point that we have used in the computations is that $\|\nabla \Psi^\varepsilon\|_{0,\infty,t}$ can be controlled by the $L_t^\infty H_{co}^1$ norm of $\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon$ (rather than u^ε) which has a size of $\varepsilon^{\frac{1}{2}}$. This is achieved by Sobolev embedding and elliptic estimate similar to (5.10). In the current situation, due to the lack of suitable Poincaré inequality, only $\|\nabla^2\Psi\|_{L_t^\infty H_{co}^1}$ (but not $\|\nabla\Psi^\varepsilon\|_{L_t^\infty H_{co}^2}$) can be controlled by $\|\mathrm{div}^\varphi u^\varepsilon\|_{L_t^\infty H_{co}^1}$. Nevertheless, in the current situation, one has the following Sobolev embedding:

$$\|f\|_{L^\infty(\mathbb{R}_-^3)} \lesssim \|\nabla f\|_{H^1_{tan}(\mathbb{R}_-^3)}$$

which leads to:

$$\|\nabla\Psi^\varepsilon\|_{0,\infty,t} \lesssim \|\nabla^2\Psi^\varepsilon\|_{L_t^\infty H_{co}^1} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^1}.$$

16. Appendix

We give a short proof of (3.4). The proof of $|fg|_{H^s(\mathbb{R}^2)} \lesssim |f|_{H^s}|g|_{W^{1,\infty}}, (0 \leq s \leq 1)$ can be found in Theorem 15.2 of [46]. The case for $-1 < s < 0$ is derived by duality. We thus focus on the proof of inequality: $|fg|_{H^s(\mathbb{R}^2)} \lesssim |f|_{H^s}|g|_{H^{1^+}}, (-1 < s \leq 1)$. We shall use Bony's decomposition:

$$fg = T_g f + \tilde{T}_f g = \sum_{j\geq 0} S_{j-1}g\Delta_j f + \sum_{k\geq -1} S_{k+2}f\Delta_k g.$$

One can refer to [p.61, [6]] for the definition of nonhomogeneous dyadic block Δ_k and nonhomogeneous low-frequency cut-off operator S_k. For any $s \in \mathbb{R}$, one can control $T_g f$ as:

$$|T_g f|_{H^s(\mathbb{R}^2)} \lesssim |g|_{L^\infty}|f|_{H^s} \lesssim |g|_{H^{1^+}}|f|_{H^s}.$$

As for $\tilde{T}_f g$, if $s < 0$, we control it with the aid of Bernstein inequality:

$$\begin{aligned}\big(2^{js}|\Delta_j\tilde{T}_f g|_{L^2}\big)_{l^2} &\lesssim \bigg(2^{j(s+1)}|\Delta_j\big(\sum_k S_{k+2}f\Delta_k g\big)|_{L^1}\bigg)_{l_j^2} \\ &\lesssim \big(2^{js}\sum_{k\leq j+5}|\Delta_k g|_{L^2}\big)_{l_j^2}\sup_k(2^{ks}|S_{k+2}f|_{L^2}) \lesssim |g|_{H^1}|f|_{H^s},\end{aligned}$$

and if $s > 0$,

$$|\tilde{T}_f g| \lesssim \sup_k\big(2^{k(s-1-\kappa)}|S_{k+2}f|_{L^\infty}\big)|g|_{H^{1+\kappa}} \lesssim |f|_{H^s}|g|_{H^{1+\kappa}},$$

where $\kappa > 0$ is a number that can be arbitrarily close to 0. The proof is now complete.

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