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Refining aggregation operations in finite ordinal scales

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Abstract

We discuss how to aggregate multiple criteria evaluations belonging to qualitative, linearly ordered scales. Qualitative aggregation operations such as min or max can be refined by discrimin and leximin orderings, in agreement with the Pareto ordering of vector evaluations. Further refinements of discrimin orderings, as well as the generalization of discrimin and leximin to functions other than min or max, are presented. Lastly, it is pointed out that a generalized leximin is not sufficient for describing any aggregation structure. However, the definition of aggregation structures in qualitative scales amounts, in practice, to the specification of a small number of positionings of aggregation results.

Keywords: Aggregation, Qualitative scale, Decision

1 Introduction

The fuzzy set framework provides a panoply of aggregation connectives for combining multiplecriteria evaluations, under a commensurability assumption: the merits of actions or items according to each criterion must be graded on the same scale, but it is usually on a numerical scale, namely [0,1]. In this paper, we investigate the possibility of not using numbers for modeling preference so as to avoid arbitrariness, improve robustness, and be more compatible with logical representations. Then the question is: can we solve multiple-criteria decision problems with a finite ordinal scale under a commensurability assumption? This might be of interest in information systems where we have to handle qualitative descriptions of preferences of users making queries.

Let be $X = \{\alpha_1 < \alpha_2 < ... < \alpha_p\}$ be a finite ordinal scale. Basic aggregation operations on ordinal scales are minimum, maximum, order-statistics (i.e. the kth smallest element in a multiset), and

the alpha-median $med(x, y, \alpha)$. Many wellbehaved aggregation operations on finite ordinal scales, such as associative smooth operations (Fodor, 2000), seem to be constant on significant subsets of their domains, which make these aggregations not so attractive in practice.

Besides, minimum or maximum, by retaining only the smallest or the largest evaluations, are not always sufficiently discriminating for rankordering possible choices in practice. Dubois, Fargier and Prade (1996) have proposed to use two refinements of the strict min (resp. max) ordering, in agreement with the strict Pareto the dominance, namely discrimin (resp. discrimax) and the leximin (resp. leximax) ordering. Let \bar{x} and $\bar{y} \in X^n$ be two eva-luation vectors. Let $\mathscr{D}(\bar{x}, \bar{y}) = \{i, x_i \neq y_i\}$ be the discriminating set of components for vectors \overline{x} and ÿ.

Then define $\bar{x} >_{\text{discrimin}} \bar{y}$

 $\Leftrightarrow \min\{\mathbf{x}_i, i \in \mathcal{D}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\} > \min\{\mathbf{y}_i, i \in \mathcal{D}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\}.$

The leximin ordering is similarly defined once the two vectors have been increasingly reordered. The extension of discrimin and leximin orderings to the comparison of vectors of different lengths has been discussed by Dubois and Fortemps (1999).

In the following we more particularly study further refinements of discrimin on the one hand, and generalizations of discrimin and leximin to monotone functions on the other hand, following ideas first suggested in Prade (2001) and Dubois (2001) respectively. Then we discuss how more general qualitative aggregation structures can be specified through the positioning of a few tuples.

2 Orderings between discrimin and leximin

Classical discrimin is based on the elimination of identical singletons at the same places in the comparison process of two vectors. Similarly, we can work with 2 elements subsets which are identical *and* pertain to the same pair of criteria.

However, simplifications can take place only one time. Thus, if the vectors are of the form $\overline{x} = (u, v, u, s)$ and $\overline{y} = (v, u, v, t)$ (with min $(u, v) \le \min(s, t)$ in order to have the two vectors minequivalent), we may either delete components of ranks 1 and 2, or of ranks 2 and 3, leading in both cases to compare (u, s) and (v, t), and to consider the first vector as smaller as soon as $u < \min(v, s, t)$.

We can now introduce the definition of the (order) 2-discrimin. Let us build a set $\mathscr{D}_2(\bar{x},\bar{y})$ as any maximal set {(i, j), disjoint pairs such that ui = v_i and $u_i = v_i$ $\cup \{k, u_k = v_k\}$. Then the 2discrimin is just the minimum-based ordering once components corresponding to singletons and pairs in $\mathscr{D}_2(\bar{x},\bar{y})$ are deleted. Note that $\mathcal{D}_2(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is not always unique as shown by the above example. However this does not affect the result of the comparison of the vectors after the deletion of the components in $\mathscr{D}_2(\bar{x}, \bar{y})$ as it can be checked from the above formal example, since the minimum aggregation is not sensitive to the place of the terms. The 2-discrimin also includes the deletion of identical components as in the ordinary discrimin, since it would be strange to delete pairs of identical values in the comparison but not single identical values (which may blur the comparison).

The 2-discrimin refines the discrimin ordering. It deletes pairs of values which play a neutral role in the aggregation, but which may lead to ties if these values are not ignored.

Clearly this idea can be extended to 3 elementssets as well or more generally with k elementssets. However the definition of the 3-discrimin should be further refined by choosing the permutation, which lead to a discriminant situation if possible, as shown by the following example. Let

 $\mathbf{u} = (a, c, d, g, d, e, c) \mathbf{v} = (d, a, b, e, c, d, e)$

There exist two overlapping permutations :

(a, c, d) with (d, e, c) and (d, a, c) with (c, d, e).

where a < b < c < d < e < f < g is assumed. Ignoring the values involved in the first permutation, we obtain $\mathbf{v} <_{\min} \mathbf{u}$ since b < c, while using the second permutation, we get $\mathbf{v} =_{\min} \mathbf{u}$ since both vectors of remaining values lead to the same minimum a. Note that the discriminant permutation should involve the minimum value of the vectors, to be of interest. However they cannot exist two overlapping permutations which, after ignoring their components in the min-based comparison, would lead to opposite orderings (namely $\mathbf{u} > \mathbf{v}$ and $\mathbf{v} > \mathbf{u}$). This holds for the k-dicrimin as well. This can be seen by considering vectors of the form

$$\mathbf{u} = (t a b s x t y) and \mathbf{v} = (b s a t s x z)$$

where y and z are such that $\mathbf{u} =_{\min} \mathbf{v}$. Simplifying by the first four components leads to compare min(x, t, y) with min(s, x, z), while deleting (s, x, t) with (t, s, x) leads to the comparison of min(t, a, b, y) with min(b, s, a, z). Clearly the two comparisons cannot disagree with each other (i.e. cannot lead to $\mathbf{u} > \mathbf{v}$ and $\mathbf{v} > \mathbf{u}$ respectively). This remark could be further generalized by replacing some of the above vector components by subvectors. In case of several overlaps, this analysis can be iterated.

So the generalized procedure for applying kdiscrimin is to look for permutations of orders 1, 2, ..., k and to explore the different possibilities in case of overlapping permutations until a discriminating one is found. Note that 'kdiscrimin' amounts to a limited leximin on k-long subsequences. Thus 'k-discrimin' provides orderings in between discrimin and leximin. However the n-discrimin ordering may remain less discriminating than the leximin ordering.

The 'k-discrimin', defined as such, is not fully compatible with transitivity, since it can be established that $\bar{x} >_{k-discrimin} \bar{y}$ and $\bar{y} >_{k-discrimin} \bar{z}$ only entail $\bar{x} \ge_{k-discrimin} \bar{z}$. This is due to the possible overlapping of k elements-sets which are deleted in the comparison, as shown by the following example $\bar{x} = (a c e)$, $\bar{y} = (c a d)$, and z = (b d a) where $\bar{x} >_{2-discrimin} \bar{y}$, $\bar{y} >_{2-discrimin} > \bar{z}$ and neither $\bar{x} >_{2-discrimin} \bar{z}$ nor $\bar{z} >_{2-discrimin} \bar{x}$ with a < b < c < d < e. However $\bar{x} >_{leximin} \bar{z}$ anyway. This is particular for the k-discrimin (k ≥ 2), since the discrimin ordering $>_{discrimin} is$ indeed transitive.

3 Generalized discri-f and lexi-f

The discrimin and leximin ideas can be applied to functions other than min (or max). Given a finite, totally ordered set (X, \ge) with p elements top 1 and bottom 0, consider an aggregation function $f: X^n \rightarrow X$, which, by definition, is increasing in the wide sense and such that f(1, 1...1) = 1 and f(0, ...0) = 0. Clearly the discriminating power of qualitative aggregation operations is bound to be very weak since f classifies X^n into p ranked classes, and thus no aggregation function can be strictly increasing from X^n to X. A consequence is in general the existence for any aggregation function f of a maximizing set S such that some f-optimal solutions in S are not Pareto optimal. One way out of this difficulty may be to use functions from X^n to a bigger finite scale Y. However, this idea is not satisfactory from a practical point of view since the combinatorics of functions from

 X^{n} to Y become rapidly prohibitive as Y is bigger, and are thus much higher than those of functions

from X^n to X.

When f = min, the natural way to tackle the problem has been to introduce relations that naturally refine the min-ordering, and restore the Pareto optimality of the selected maximal solutions. Such relations are the discrimin ordering and the leximin ordering. The discrimination power of the latter is maximal, i.e. it is equal to that of the most discriminating symmetric aggregation operations.

We try here to generalize this refinement technique to more general families of aggregation operation. We restrict to the case of symmetric functions. Consider a family $\{f^p\}$ of symmetric functions $X^p \rightarrow X$. For any positive integer p, f^p is supposed to be

-) extensively preferentially consistent with
$$f^{p-1}$$
:
 $\forall u \in X, f^{p-1}(x_1 \dots x_{p-1}) \ge f^{p-1}(y_1 \dots y_{p-1}) \Longrightarrow$
 $f^p(x_1 \dots x_{p-1}, u) \ge f^p(y_1 \dots y_{p-1}, u)$

-) globally strictly monotone (assuming $p \ge n$):

if
$$x_i > y_i \forall i = 1, p$$
, then $f^p(\bar{x}) > f^p(\bar{y})$

where \bar{x} and $\bar{y} \in X^n$.

These conditions look natural in the scope of applications. The first condition is a weak form of preferential independence. They are satisfied by the minimum, the maximum (but not other order-statistics). By convention f^1 is the identity function on X. Call $\{f^1, \dots, f^n, \dots\}$ a qualitative aggregation structure. The notation f is used when the number of arguments is not emphasized. The generalization of discrimin and leximin to such aggregation structures is as follows:

$$\begin{split} \bar{x} &>_{discri-f} \bar{y} \Leftrightarrow \\ & f(\{x_i, i \in \mathcal{D}(\bar{x}, \bar{y})\}) > f(\{y_i, i \in \mathcal{D}(\bar{x}, \bar{y})\}). \\ \bar{x} &\geq_{lexi-f} \bar{y} \Leftrightarrow f(M(\bar{x}) - M(\bar{y})) \geq f(M(\bar{y}) - \\ & M(\bar{x})), \end{split}$$

where $M(\bar{x})$ denotes the multi-set induced by vector \overline{x} (the same grade may appear several times in \overline{x}), and where it can be easily checked that $M(\bar{x}) - M(\bar{y})$ and $M(\bar{y}) - M(\bar{x})$ have the same number of elements. It can be shown that mild conditions such as global under preferential monotonicity and extensive consistency, lexi-f and discri-f maximal solutions indeed Pareto-optimal, and that the are corresponding ordering of solutions is quite discriminant.

4 Specifying qualitative aggregation structures

Globally strictly monotone aggregation functions on X are easily proved to be idempotent on finite scales, since if $0 = x_1 < x_2 < ... < x_m = 1$ it follows that $f(x_i ... x_i) = x_i$. It rules out the Archimedean t-norms and conorm-like operations on finite sets (Fodor, 2000). Moreover, the only associative idempotent aggregation operations different from min and max, the α -medians ($f(x, y) = \text{median}(\alpha, x, y)$), are generally not globally strictly monotone, since they are constant on large subsets of X^n . The above extensions of leximin and discrimin orderings to these operations thus do not possess enough discrimination power; they can only be refined by directly adopting Paretoordering on the ranges where these aggregation functions are constant.

The simplest non trivial example of finite totally ordered scale is $X = \{\alpha_1 < \alpha_2 < \alpha_3\}$, that we shall write it as $\{1, 2, 3\}$ for short. Adopting the lexi-f² ordering for X does not leave many degrees of freedom: one must indicate the relative position of f²(1, 3) and f²(2, 2). If f²(1, 3) > f²(2, 2), this is the leximax ordering. If f²(1, 3) < f²(2, 2), this is the leximin ordering. If f²(1, 3) = f²(2, 2), this is a kind of ordinal average (which is less discriminant). With three arguments, the ordering of 3-tuples is fixed by further positioning (2, 2, 2) with respect to (1, 1, 3) and (1, 3, 3) (note that (1, 3, 3) >_{lexi-f} (1, 1, 3) in any case). The lexi-f positions of (1, 2, 2) with respect to (1, 1, 3), (1, 2, 3) with respect to (2, 2, 2), and (2, 2, 3) with respect to (1, 3, 3), are enforced by the position of (2, 2) with respect to (1, 3). If f²(1, 3) > f²(2, 2), then $f^{3}(1, 3, 3) \ge f^{3}(2, 2, 2)$ and only the relative position of (2, 2, 2) and (1, 1, 3) is left open. If $f^{2}(2, 2) > f^{2}(1, 3)$, then $f^{3}(2, 2, 2) \ge f^{3}(1, 1, 3)$ and only the relative position of (2, 2, 2) and (1, 3, 3) is left open.

The generation of complete preorderings of tuples of elements from a finite ordered scale in agreement with Pareto-dominance and symmetry has been considered in Moura-Pires and Prade (2000) in the scope of fuzzy constraint satisfaction problems. A natural question is whether any such complete preorderings of X^n can be obtained as a lexi-f ordering for some qualitative aggregation structure $\{f^1 \dots f^n\}$, and more generally, can be generated by a small number of extra constraints on the relative positioning of a few tuples.

Unfortunately the answer for lexi-f ordering is negative. A counterexample is obtained using a 4-element scale $X = \{0, 1, 2, 3\}$. Then f^2 is characterized by the relative positionings of (0, 2)w.r.t. (1, 1), (1, 3) w.r.t. (2, 2), and (0, 3) w.r.t. (1, 1)and (2, 2). However, using $f: X^2 \rightarrow X$, we have $f^2(0, 3) \in \{f^2(0, 0), f^2(1, 1), f^2(2, 2), f^2(3, 3)\}$. Neither the discri-f nor the lexi-f extension, nor even supplemented by Pareto-dominance itself, can generate the complete preorders such that (1, 1) < (0, 3) < (2, 2). There are 12 total orderings which are Pareto-compatible and respect symmetry in this example, and only 8 of them can be generated as a lexi-f ordering via an aggregation structure $X^n \rightarrow X$. Generating the other total orderings requires a function $X^2 \rightarrow Y$ where Y has 5 levels.

More generally the problem is how to efficiently describe Pareto-compatible rankings using a small number of conditions on the relative positioning of a few tuples. This includes the open question: how to characterize the descriptive power of lexi-f. An example of property which may simplify the study of qualitative aggregation structures is the following regularity : let $(i, j) \in \mathbb{N}^2$ (0.1.2)

 $X^2 = \{0, 1, 2, ..., n\},\$

if $f^{2}(i, j) > f^{2}(i+1, j-1)$, then $f^{2}(i+1, j+1) > f^{2}(i+2, j)$.

The combinatorics of such regular aggregations functions look moderate and deserve further exploration. Another issue is the expansion of a qualitative aggregation structure from f^k to f^{k+1} .

5 Concluding remarks

Numerical scales such as [0, 1] are not always meaningful in practice when users are only able to discriminate between a small number of satisfaction levels. Qualitative aggregation of fuzzy sets with membership grades belonging to such qualitative scales have been discussed. This can be relevant for modelling users' preferences and comparing solutions of constraint satisfaction problems.

The above study has exhibited some intrinsic limitations of the otherwise appealing finite setting for criteria aggregation using a single finite scale, whereby concise representations and functions having good algebraic properties turn out to lack expressivity, even under natural lexicographic-like extensions.

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