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WALKS AVOIDING A QUADRANT AND THE REFLECTION PRINCIPLE

MIREILLE BOUSQUET-MÉLOU AND MICHAEL WALLNER

ABSTRACT. We continue the enumeration of plane lattice walks with small steps avoiding the negative quadrant, initiated by the first author in 2016. We solve in detail a new case, namely the king model where all eight nearest neighbour steps are allowed. The associated generating function is proved to be the sum of a simple, explicit D-finite series (related to the number of walks confined to the first quadrant), and an algebraic one. This was already the case for the two models solved by the first author in 2016. The principle of the approach is also the same, but challenging theoretical and computational difficulties arise as we now handle algebraic series of larger degree.

We expect a similar algebraicity phenomenon to hold for the seven *Weyl* step sets, which are those for which walks confined to the first quadrant can be counted using the reflection principle. With this paper, this is now proved for three of them. For the remaining four, we predict the D-finite part of the solution, and in three of the four cases, give evidence for the algebraicity of the remaining part.

1. INTRODUCTION

Over the last two decades, the enumeration of walks in the non-negative quadrant

$$\mathcal{Q} := \{(i, j) : i \geq 0 \text{ and } j \geq 0\}$$

has attracted a lot of attention and established its own scientific community with close to a hundred research papers; see, e.g., [11] and citing papers. One of its attractive features is the diversity of the used tools, such as algebra on formal power series [11, 35], bijective approaches [2, 17], computer algebra [6, 28], complex analysis [4, 38], probability theory [8, 18], and difference Galois theory [19]. Most of the attention has focused on walks *with small steps*, that is, taking their steps in a fixed subset \mathcal{S} of $\{-1, 0, 1\}^2 \setminus (0, 0)$. For each such step set \mathcal{S} (often called a *model* henceforth), one considers a trivariate generating function $Q(x, y; t)$ defined by

$$Q(x, y; t) = \sum_{n \geq 0} \sum_{i, j \in \mathcal{Q}} q_{i, j}(n) x^i y^j t^n, \quad (1)$$

where $q_{i, j}(n)$ is the number of quadrant walks with steps in \mathcal{S} , starting from $(0, 0)$, ending at (i, j) , and having in total n steps. For each \mathcal{S} , one now knows whether and where this series fits in the following classical hierarchy of series:

$$\text{rational} \subset \text{algebraic} \subset \text{D-finite} \subset \text{D-algebraic}.$$

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Recall that a series (say $Q(x, y; t)$ in our case) is *rational* if it is the ratio of two polynomials, *algebraic* if it satisfies a polynomial equation (with coefficients that are polynomials in the variables), *D-finite* if it satisfies three *linear* differential equations (one in each variable), again with polynomial coefficients, and finally *D-algebraic* if it satisfies three *polynomial* differential equations. It has been known since the 1980s [26] that the generating function of walks confined to a half-plane is algebraic. This explains why $Q(x, y; t)$ is algebraic in some cases, for instance when $\mathcal{S} = \{\rightarrow, \uparrow, \leftarrow\}$: indeed, confining walks to the first quadrant is then equivalent to confining them to the right half-plane $i \geq 0$. It was shown in [11] that exactly 79 (essentially distinct) quadrant problems with small steps are *not* equivalent to any half-plane problem. One central result in the classification of these 79 models is that $Q(x, y; t)$ is D-finite if and only if a certain group, which is easy to construct from the step set \mathcal{S} , is finite [6, 8, 11, 29, 33, 36].

Since any strictly convex closed cone can be deformed into the first quadrant, the enumeration of walks confined to \mathcal{Q} captures all such counting problems (provided we consider all possible step sets \mathcal{S} , not only small steps). Similarly, any non-convex closed cone in two dimensions can be deformed into the three-quadrant plane

$$\mathcal{C} := \{(i, j) : i \geq 0 \text{ or } j \geq 0\}, \quad (2)$$

and in 2016, the first author initiated the enumeration of lattice paths confined to \mathcal{C} [9]. Therein, the two most natural models of walks were studied: *simple walks* with steps in $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, and *diagonal walks* with steps in $\{\nearrow, \nwarrow, \swarrow, \searrow\}$. In both cases, the generating function

$$C(x, y; t) = \sum_{n \geq 0} \sum_{i, j \in \mathcal{C}} c_{i, j}(n) x^i y^j t^n \quad (3)$$

defined analogously to $Q(x, y; t)$ (see (1)) was proved to differ from the series

$$\frac{1}{3} \left(Q(x, y; t) - \frac{1}{x^2} Q\left(\frac{1}{x}, y; t\right) - \frac{1}{y^2} Q\left(x, \frac{1}{y}; t\right) \right) \quad (4)$$

by an *algebraic* one. In both cases, the underlying group is finite, hence $Q(x, y; t)$ is D-finite and $C(x, y; t)$ is D-finite as well.

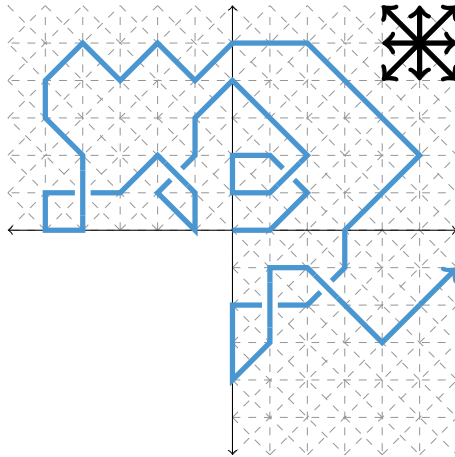


FIGURE 1. A king walk in the three-quadrant plane \mathcal{C} . The associated generating function is D-finite and transcendental (i.e., non-algebraic).

It became then natural to explore more three-quadrant problems, in particular to understand whether the D-finiteness of $C(x, y; t)$ was again related to the finiteness of the associated group – at least for the 74 three-quadrant problems that are not equivalent to a half-plane problem; see Section 2.2. Using an asymptotic argument, Mustapha quickly proved that the 51 three-quadrant problems associated with an infinite group have, like their quadrant counterparts, a non-D-finite solution [37]. Regarding exact solutions, Raschel and Trotignon obtained in [39] sophisticated integral expressions for eight step sets. Four of them have a finite group (namely $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, $\{\nearrow, \leftarrow, \downarrow\}$, $\{\rightarrow, \uparrow, \swarrow\}$, and $\{\rightarrow, \nearrow, \uparrow, \leftarrow, \swarrow, \downarrow\}$), and these expressions imply that they are D-finite (at least in x and y). In fact, the latter three are now known to be algebraic [14]. The other four have an infinite group and have been further studied by Dreyfus and Trotignon: one of them is D-algebraic, the other three are not [20]. Furthermore, the remarkable results of Budd [16] and Elvey Price [22] on the winding number of various families of plane walks provide explicit D-finite expressions for several generating functions of three-quadrant walks starting and ending close to the origin, in particular for step sets $\{\rightarrow, \nearrow, \leftarrow, \swarrow\}$ and $\{\rightarrow, \nwarrow, \leftarrow, \searrow\}$ in Budd’s paper, as well as $\{\uparrow, \leftarrow, \searrow\}$ and $\{\rightarrow, \uparrow, \nwarrow, \leftarrow, \downarrow, \searrow\}$ in Elvey Price’s¹.

Main results. In this paper we enrich the collection of completely solved cases with the *king walks*, in which all eight nearest neighbour steps $\rightarrow, \nearrow, \uparrow, \nwarrow, \leftarrow, \swarrow, \downarrow, \searrow$ are allowed; see Figure 1. This is again a finite group model, and the series $Q(x, y; t)$ is a well-understood D-finite series [11]. Here we determine $C(x, y; t)$, and show that the algebraicity phenomenon of [9] persists: the series $C(x, y; t)$ differs from the linear combination (4) by an algebraic series, this time of degree 216. For the simple and diagonal walks of [9] this algebraic series was of degree 72 “only”. The generating function $C_{i,j}$ of walks ending at a prescribed position (i, j) differs from a series of the form $\pm Q_{k,\ell}/3$ by an algebraic series of degree at most 24 (while this degree was bounded by 8 in the two models of [9]).

Moreover, we explain why we expect a similar property to hold (with variations on the linear combination (4) of series $Q(\cdot, \cdot)$) for the seven models of Table 1. These are precisely the models for which the quadrant problem can be solved using the reflection principle [27], and for this reason we call them *Weyl models*. We predict the relevant linear combination of series Q , and we give evidence of the algebraicity phenomenon for the three rightmost models of Table 1. However, we also expect the effective solution of these models to be challenging in computational terms, because the relevant algebraic series will most likely have very large degrees.

Outline of the paper. We begin in Section 2 with generalities on the enumeration of walks with small steps confined to the three-quadrant cone \mathcal{C} , and on the related functional equations. We describe the 74 (essentially distinct) models of interest – those that are not equivalent to a half-plane model – and define the group associated with a model. In Section 3 we state and justify our conjecture on the form of $C(x, y; t)$ in the seven Weyl cases. The next three sections are devoted to the solution of the king model in three quadrants: in Section 4 we state our results in details, and we prove them in Sections 5 and 6. In Section 7, we give combinatorial proofs, based on the reflection principle, of some identities obtained so far via

¹In both papers, when steps \nwarrow or \searrow are allowed, one includes in the enumeration walks using jumps from $(-1, 0)$ to $(0, -1)$ and vice-versa. Such jumps are forbidden in this paper, but we show in the last section that allowing them in king walks does not modify significantly the form of our results.

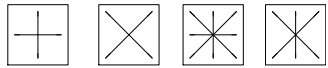


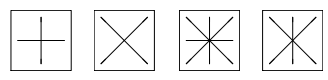
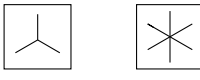

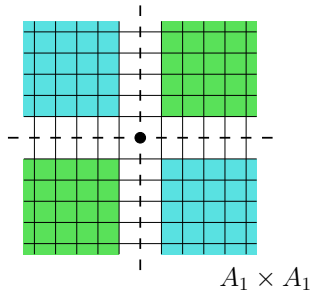
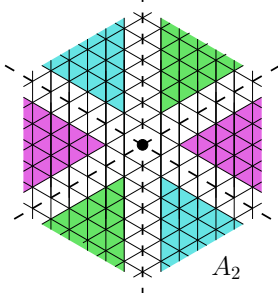
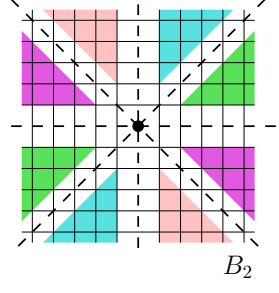
model \mathcal{S}	 simple diagonal king diabolo	 tandem double-tandem	 Gouyou-Beauchamps
group G	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y}), (x, \bar{y})$	$(x, y), (\bar{x}y, y),$ $(\bar{x}y, \bar{x}), (\bar{y}, \bar{x}),$ $(\bar{y}, x\bar{y}), (x, x\bar{y})$	$(x, y), (\bar{x}y, y),$ $(\bar{x}y, \bar{x}^2y), (\bar{x}, \bar{x}^2y),$ $(\bar{x}, \bar{y}), (x\bar{y}, \bar{y}),$ $(x\bar{y}, x^2\bar{y}), (x, x^2\bar{y})$
new steps			
Weyl chambers	 $A_1 \times A_1$	 A_2	 B_2

TABLE 1. The seven Weyl models, with their usual names and groups (defined in Section 2.3). The next rows show how to deform the steps and the plane so that walks in the first quadrant correspond to walks in a Weyl chamber, and walks avoiding the negative quadrant to walks avoiding a Weyl chamber.

functional equations, and generalize them to all seven Weyl models. In Section 8 we conclude with a few comments, in particular about what happens to the king model when one allows steps between $(0, -1)$ and $(-1, 0)$.

This paper is the full version of an extended abstract [13] that was published in the proceedings of the *Analysis of Algorithms* Conference in 2020.

2. ENUMERATION IN THE THREE-QUADRANT PLANE: BASIC TOOLS

Let us begin with some definitions and notation on formal power series. Let \mathbb{A} be a commutative ring and x an indeterminate. We denote by $\mathbb{A}[x]$ (resp. $\mathbb{A}[[x]]$) the ring of polynomials (resp. formal power series) in x with coefficients in \mathbb{A} . If \mathbb{A} is a field, then $\mathbb{A}(x)$ denotes the field of rational functions in x , and $\mathbb{A}((x))$ the field of Laurent series in x , that is, series of the form $\sum_{n \geq n_0} a_n x^n$, with $n_0 \in \mathbb{Z}$ and $a_n \in \mathbb{A}$. This notation is generalized to polynomials, fractions, and series in several indeterminates. The coefficient of x^n in a series $F(x)$ is denoted by $[x^n]F(x)$. We denote partial derivatives with indices: for instance, for a series F involving the indeterminate x , we write F_x for $\partial F / \partial x$.

We denote with bars the reciprocals of variables: that is, $\bar{x} = 1/x$, so that $\mathbb{A}[x, \bar{x}]$ is the ring of Laurent polynomials in x with coefficients in \mathbb{A} .

We will often handle series of $\mathbb{Q}(x)((t))$, and consider $\mathbb{Q}(x)$ as a subring of $\mathbb{Q}((x))$ (that is, we expand rational functions in x around $x = 0$). For $F(x; t) \in \mathbb{Q}((x))((t))$, of the form

$$F(x; t) = \sum_{n \geq n_0} t^n \sum_{i \geq i_0(n)} a(n, i) x^i,$$

the *non-negative part of F in x* is the following series in t and x :

$$[x^{\geq}]F(x; t) = \sum_{n \geq n_0} t^n \sum_{i \geq \max(0, i_0(n))} a(n, i) x^i.$$

We define analogously the *positive part* of F , denoted by $[x^>]F$. The *negative part* of F is $[x^<]F := F - [x^{\geq}]F$. Observe that, if $F(x; t) \in \mathbb{Q}(x)((t))$, this convention makes the roles of x and $1/x$ non-symmetric: the negative part of $F(x)$ is not always obtained by inverting x in the positive part of $F(\bar{x})$. For instance, if we take $F(x) = 1/(1+x)$, its negative part is 0, but $F(\bar{x}) = 1/(1+\bar{x}) = x/(1+x)$ has a non-trivial positive part. However, if $F(x; t) \in \mathbb{Q}[x, \bar{x}]((t))$, then the expected symmetry holds.

If \mathbb{A} is a field, a power series $F(x) \in \mathbb{A}[[x]]$ is *algebraic* (over $\mathbb{A}(x)$) if it satisfies a non-trivial polynomial equation $P(x, F(x)) = 0$ with coefficients in \mathbb{A} . Otherwise it is *transcendental*. It is *differentially finite* (or *D-finite*) if it satisfies a non-trivial linear differential equation with coefficients in $\mathbb{A}(x)$. For multivariate series, D-finiteness requires the existence of a differential equation *in each variable*. We refer to [30, 31] for general results on D-finite series.

We usually omit the dependency in t of our series, writing for instance $C(x, y)$ for $C(x, y; t)$. For a series $F(x, y) \in \mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$ and two integers i and j , we denote by $F_{i,j}$ the coefficient of $x^i y^j$ in $F(x, y)$. This is a series in $\mathbb{Q}[[t]]$. We also denote

$$F_{-,0}(\bar{x}) = \sum_{i < 0} F_{i,0} x^i \quad \text{and} \quad F_{0,-}(\bar{y}) = \sum_{j < 0} F_{0,j} y^j. \quad (5)$$

These two series lie in $\bar{x}\mathbb{Q}[\bar{x}][[t]]$ and $\bar{y}\mathbb{Q}[\bar{y}][[t]]$, respectively.

2.1. A FUNCTIONAL EQUATION

We fix a subset \mathcal{S} of $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ and we consider walks with steps in \mathcal{S} that start from $(0, 0)$ and remain in the cone \mathcal{C} defined by (2). By this, we mean that not only must every vertex of the walk lie in \mathcal{C} , but also every edge: a walk containing a step from $(-1, 0)$ to $(0, -1)$ (or vice versa) does not lie in \mathcal{C} . We often say for short that our walks *avoid the negative quadrant*. The *step polynomial* of \mathcal{S} is defined by

$$S(x, y) = \sum_{(i,j) \in \mathcal{S}} x^i y^j = \bar{y}H_-(x) + H_0(x) + yH_+(x) + \bar{x}V_-(y) + V_0(y) + xV_+(y), \quad (6)$$

for some Laurent polynomials H_-, H_0, H_+ and V_-, V_0, V_+ recording horizontal and vertical displacements, respectively. We denote by $C(x, y) \equiv C(x, y; t)$ the generating function (3) of walks confined to \mathcal{C} . In the expression (3), $c_{i,j}(n)$ is the number of walks of length n that go from $(0, 0)$ to (i, j) and are confined to \mathcal{C} .

Constructing walks confined to \mathcal{C} step by step gives the following functional equation:

$$C(x, y) = 1 + tS(x, y)C(x, y) - t\bar{y}H_-(x)C_{-,0}(\bar{x}) - t\bar{x}V_-(y)C_{0,-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0}\mathbb{1}_{(-1,-1) \in \mathcal{S}},$$

where we have used the notation (5). Note that the series $C_{-,0}(\bar{x})$ and $C_{0,-}(\bar{y})$ count walks ending on the horizontal and vertical boundaries of \mathcal{C} (but not at $(0, 0)$). On the right-hand

side, the term 1 accounts for the empty walk, the next term describes the extension of a walk in \mathcal{C} by one step of \mathcal{S} , and each of the other three terms corresponds to a “bad” move, either starting from the negative x -axis, or from the negative y -axis, or from $(0, 0)$. Equivalently,

$$K(x, y)C(x, y) = 1 - t\bar{y}H_-(x)C_{-,0}(\bar{x}) - t\bar{x}V_-(y)C_{0,-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0}\mathbb{1}_{(-1,-1)\in\mathcal{S}}, \quad (7)$$

where $K(x, y) := 1 - tS(x, y)$ is the *kernel* of the equation.

As recalled in the introduction, the enumeration of walks confined to the non-negative quadrant \mathcal{Q} has been studied intensively over the last 20 years. The associated generating function $Q(x, y) \equiv Q(x, y; t) \in \mathbb{Q}[x, y][[t]]$ defined in (1) satisfies a similarly looking equation [11, Lem. 4]:

$$K(x, y)Q(x, y) = 1 - t\bar{y}H_-(x)Q(x, 0) - t\bar{x}V_-(y)Q(0, y) + t\bar{x}\bar{y}Q(0, 0)\mathbb{1}_{(-1,-1)\in\mathcal{S}}. \quad (8)$$

Remark. In two recent references dealing with the winding number of plane lattice walks [16, 22], it seems more natural to count walks in which all vertices lie in \mathcal{C} , but not necessarily all edges: this means that there may be steps from $(-1, 0)$ to $(0, -1)$, and vice-versa. Counting these walks would add two terms to the right-hand side of (7), namely

$$t\bar{y}C_{-1,0}\mathbb{1}_{(1,-1)\in\mathcal{S}} + t\bar{x}C_{0,-1}\mathbb{1}_{(-1,1)\in\mathcal{S}}. \quad (9)$$

We discuss in the final section of the paper the enumeration of king walks in \mathcal{C} when these two steps are allowed. The results are qualitatively the same as when they are forbidden.

2.2. INTERESTING STEP SETS

As in the quadrant case [11], we can decrease the number of step sets that are worth being considered thanks to a few simple observations (*a priori*, there are 2^8 of them):

- Since the cone \mathcal{C} (as well as the quarter plane \mathcal{Q}) is x/y -symmetric, the counting problems defined by \mathcal{S} and by its mirror image $\bar{\mathcal{S}} := \{(j, i) : (i, j) \in \mathcal{S}\}$ are equivalent; the associated generating functions are related by $\bar{C}(x, y) = C(y, x)$.
- If all steps of \mathcal{S} are contained in the right half-plane $\{(i, j) : i \geq 0\}$, then *all* walks with steps in \mathcal{S} lie in \mathcal{C} , and the series $C(x, y) = 1/(1 - tS(x, y))$ is simply rational. The series $Q(x, y)$ is known to be algebraic in this case [1, 12, 21, 26].
- If all steps of \mathcal{S} are contained in the left half-plane $\{(i, j) : i \leq 0\}$, then confining a walk to \mathcal{C} is equivalent to confining it to the upper half-plane: the associated generating function is then algebraic, and so is $Q(x, y)$.
- If all steps of \mathcal{S} lie (weakly) above the first diagonal ($i = j$), then confining a walk to \mathcal{C} is again equivalent to confining it to the upper half-plane: the associated generating function is then algebraic, and so is $Q(x, y)$.
- If all steps of \mathcal{S} lie (weakly) above the second diagonal ($i + j = 0$), then all walks with steps in \mathcal{S} lie in \mathcal{C} , and $C(x, y) = 1/(1 - tS(x, y))$ is simply rational. In this case however, the series $Q(x, y)$ is not at all trivial [11, 36]. Such step sets are sometimes called *singular* in the framework of quadrant walks.
- Finally, if all steps of \mathcal{S} lie (weakly) below the second diagonal, then a walk confined to \mathcal{C} moves for a while along the second diagonal, and then either stops there or leaves it into the NW or SE quadrant using a South, South-West, or West step. It cannot leave the chosen quadrant anymore and behaves therein like a half-plane walk. By polishing this observation, one can prove that $C(x, y)$ is algebraic (while $Q(x, y) = 1$).

By combining these arguments, one finds that there are 74 essentially distinct models of walks avoiding the negative quadrant that are worth studying: the 79 models considered for quadrant walks (see [11, Tables 1–4]) except the 5 “singular” models for which all steps of \mathcal{S} lie weakly above the diagonal $i + j = 0$.

2.3. THE GROUP OF THE MODEL

One important tool in the systematic approach to quadrant walks is a certain group G of birational transformations associated with the step set \mathcal{S} . It was introduced in [11], and is an algebraic variant of a group introduced much earlier in the study of *random* walks in the quadrant [23, 25, 32].

We assume from now on that \mathcal{S} contains positive and negative steps in the horizontal and vertical directions (otherwise the problem degenerates, as explained above). We define two birational transformations ϕ and ψ , acting on pairs (u, v) of coordinates (which will be, typically, rational functions of x and y):

$$\phi : (u, v) \mapsto \left(\bar{u} \frac{V_-(v)}{V_+(v)}, v \right) \quad \text{and} \quad \psi : (u, v) \mapsto \left(u, \bar{v} \frac{H_-(u)}{H_+(u)} \right),$$

where $H_-, H_+, V_-,$ and V_+ are defined by (6). Each transformation fixes one coordinate, and transforms the other *so as to leave the step polynomial $S(u, v)$, defined by (6), unchanged*. Note that ϕ and ψ are both involutions. The group G is the group generated by these two transformations. It is isomorphic to a dihedral group of order $2n$, with $n \in \mathbb{N} \cup \{\infty\}$. The *length* of $g \in G$, denoted $\ell(g)$, is the smallest ℓ such that g can be written as a product of ℓ generators ϕ and ψ . The *sign* of $g \in G$, denoted ε_g , is defined by $\varepsilon_g = (-1)^{\ell(g)}$. Note that for any $g \in G$, we have $S(g(x, y)) = S(x, y)$.

Among the 74 interesting models identified in the previous subsection, exactly 23 have a finite group; see [11]. For the remaining 51 models, an asymptotic argument implies that the series $C_{0,0}$ that counts walks ending at $(0, 0)$ is not D-finite, which implies that $C(x, y; t)$ is not D-finite; see [37]. Among the 23 models with a finite group,

- 16 have a vertical symmetry (say) and a group of order 4,
- 5 have a group of order 6, and
- 2 have a group of order 8.

These models and groups are listed in [11, Tables 1–3]. Another classification of these groups distinguishes the 7 + 4 models with a *monomial* group (meaning that for every $g \in G$, the pair $g(x, y)$ consists of two Laurent monomials in x and y), shown in Tables 1 and 3 of this paper, from the 12 non-monomial ones (Table 2).

\mathcal{S}							
G	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{1}{x+\bar{x}}),$ $(x, \bar{y} \frac{1}{x+\bar{x}})$	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{1}{x+1+\bar{x}}),$ $(x, \bar{y} \frac{1}{x+1+\bar{x}})$	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{x+\bar{x}}{x+1+\bar{x}}),$ $(x, \bar{y} \frac{x+\bar{x}}{x+1+\bar{x}})$	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} (x+1+\bar{x})),$ $(x, \bar{y} (x+1+\bar{x}))$	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{x+1+\bar{x}}{x+\bar{x}}),$ $(x, \bar{y} \frac{x+1+\bar{x}}{x+\bar{x}})$	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} (x+\bar{x})),$ $(x, \bar{y} (x+\bar{x}))$	

TABLE 2. The 12 models with a finite non-monomial group.

3. A CONJECTURAL FORM OF $C(x, y)$ FOR WEYL MODELS

It is shown in [11] that 19 of the 23 quadrant problems with a finite group can be solved in a uniform manner by considering the *orbit sum* of xy , where for any series $F(x, y)$ in $\mathbb{Q}(x, y)[[t]]$, we define the *orbit sum* of F by

$$\text{OS}(F(x, y)) := \sum_{g \in G} \varepsilon_g g(F(x, y)),$$

with $g(F(x, y)) := F(g(x, y))$. To explain the relevance of orbit sums observe that the quadrant equation (8), once multiplied by xy , reads

$$K(x, y)xyQ(x, y) = xy - R(x) - S(y),$$

where the series $R(x)$ does *not* involve y and the series $S(y)$ does not involve x . Recall moreover that for every $g \in G$ we have $K(g(x, y)) = 1 - tS(g(x, y)) = 1 - tS(x, y) = K(x, y)$. By linearity this gives

$$\begin{aligned} K(x, y) \sum_{g \in G} \varepsilon_g g(xyQ(x, y)) &= \text{OS}(K(x, y)xyQ(x, y)) \\ &= \text{OS}(xy) - \sum_{g \in G} \varepsilon_g g(R(x)) - \sum_{g \in G} \varepsilon_g g(S(y)) \\ &= \text{OS}(xy), \end{aligned} \tag{10}$$

since $\varepsilon_{g \circ \psi} = \varepsilon_{g \circ \phi} = -\varepsilon_g$ while $g(R(x)) = (g \circ \psi)(R(x))$ and $g(S(y)) = (g \circ \phi)(S(y))$. Analogously, for walks confined to \mathcal{C} , the form of the functional equation (7) implies that:

$$K(x, y) \sum_{g \in G} \varepsilon_g g(xyC(x, y)) = \text{OS}(xy). \tag{11}$$

Remark. If we decide to allow steps between $(-1, 0)$ and $(0, -1)$ in the walks that we count, thus adding the terms (9) to the right-hand side of the functional equation (7), the orbit sum of $xyC(x, y)$ is still $\text{OS}(xy)/K(x, y)$.

3.1. VANISHING ORBIT SUMS AND ALGEBRAICITY

As was first observed in [11], the orbit sum $\text{OS}(xy)$ is zero for exactly four of the 23 models with a finite group. The four models are shown in Table 3. For each of them, one has:

$$\sum_{g \in G} \varepsilon_g g(xyQ(x, y)) = 0 = \sum_{g \in G} \varepsilon_g g(xyC(x, y)).$$

That is, the orbit sums of $xyQ(x, y)$ and $xyC(x, y)$ vanish. It is known that these four models are precisely those for which $Q(x, y)$ is algebraic. One can derive the algebraicity from the fact that $\text{OS}(xy) = 0$ [3, 4, 7]. These derivations strongly suggest that, more generally, for any finite group model and any point $(a, b) \in \mathcal{Q}$ such that $\text{OS}(x^{a+1}y^{b+1}) = 0$, the generating function for walks in \mathcal{Q} starting from (a, b) is algebraic. This is proved in some cases beyond the case $(a, b) = (0, 0)$; see the discussion in [4, Sec. 7.2]. Note that the corresponding generating function $\tilde{Q}(x, y)$ is defined by

$$K(x, y)\tilde{Q}(x, y) = x^a y^b - t\bar{y}H_-(x)\tilde{Q}(x, 0) - t\bar{x}V_-(y)\tilde{Q}(0, y) + t\bar{x}\bar{y}\tilde{Q}(0, 0)\mathbb{1}_{(-1, -1) \in \mathcal{S}}.$$

By linearity, it is thus expected that for any polynomial $I(x, y) \in \mathbb{Q}[x, y]$ satisfying $\text{OS}(xyI) = 0$, the series $\tilde{Q}(x, y) \in \mathbb{Q}[x, y][[t]]$ defined by

$$K(x, y)\tilde{Q}(x, y) = I(x, y) - t\bar{y}H_-(x)\tilde{Q}(x, 0) - t\bar{x}V_-(y)\tilde{Q}(0, y) + t\bar{x}\bar{y}\tilde{Q}(0, 0)\mathbb{1}_{(-1, -1) \in \mathcal{S}}$$

is algebraic.

Could a similar algebraicity phenomenon hold for three-quadrant problems? That is, could it be that, for any finite group model, and any Laurent polynomial $I(x, y) \in \mathbb{Q}[x, \bar{x}, y, \bar{y}]$ having its support in \mathcal{C} and satisfying $\text{OS}(xyI) = 0$, the series $\tilde{C}(x, y)$ defined by

$$K(x, y)\tilde{C}(x, y) = I(x, y) - t\bar{y}H_-(x)\tilde{C}_{-,0}(\bar{x}) - t\bar{x}V_-(y)\tilde{C}_{0,-}(\bar{y}) - t\bar{x}\bar{y}\tilde{C}_{0,0}\mathbb{1}_{(-1, -1) \in \mathcal{S}} \quad (12)$$

is algebraic? (Again, if $I(x, y)$ is reduced to a single monomial $x^a y^b$, then \tilde{C} counts walks in \mathcal{C} starting from (a, b) .) Several results and guesses support this belief:

- (1) It was conjectured in [9] that for the four models of Table 3, for which $\text{OS}(xy) = 0$, the series $C(x, y)$ that counts walks in \mathcal{C} starting from $(0, 0)$ is algebraic. This was mostly based on a guessed polynomial equation satisfied by $C_{0,0}$. The algebraicity of $C(x, y)$ (and $C_{0,0}$) is now proved for the first three of these models [14], with explicit algebraic expressions. Moreover, Theorem 23 in [16], taken with $\alpha = 0$, $\beta_- = -\pi/2$, and $\beta_+ = 3\pi/4$, proves that $C_{0,0}$ is algebraic as well for the fourth model.









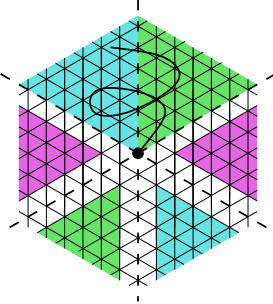
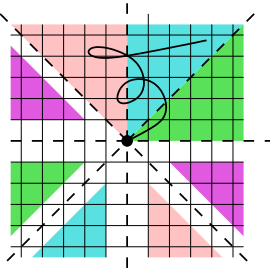
model				
	Kreweras	reverse Kreweras	double-Kreweras	Gessel
group		$(x, y), (\bar{x}\bar{y}, y),$ $(\bar{x}\bar{y}, x), (y, x),$ $(y, \bar{x}\bar{y}), (x, \bar{x}\bar{y})$		$(x, y), (\bar{x}\bar{y}, y),$ $(\bar{x}\bar{y}, x^2 y), (\bar{x}, x^2 y),$ $(\bar{x}, \bar{y}), (xy, \bar{y}),$ $(xy, \bar{x}^2 \bar{y}), (x, \bar{y} \bar{x}^2)$
new steps				
quadrant walks				

TABLE 3. The four models for which $\text{OS}(xy) = 0$, with their names and groups. After deformation, the first quadrant becomes a union of two or three Weyl chambers. The three-quadrant plane corresponds to the complement of this region.

- (2) For the simple square lattice model $\mathcal{S} = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, the orbit of (x, y) under the action of G is $\{(x, y), (\bar{x}, y), (x, \bar{y}), (\bar{x}, \bar{y})\}$. The orbit sum $\text{OS}(xy)$ is non-zero, and the series $C(x, y)$ is not algebraic. But it was proved that for walks starting at $(-1, 0)$, that is, for $I(x, y) = \bar{x}$, the generating function of walks confined to \mathcal{C} is algebraic [9, Thm. 6], and one observes that $\text{OS}(xyI) = \text{OS}(y) = 0$. Moreover, the generating function of walks in \mathcal{C} starting from $(-1, b)$ and ending at $(0, 0)$ is also algebraic [9, Cor. 2], and for $I(x, y) = \bar{x}y^b$ it also holds that $\text{OS}(xyI) = 0$.
- (3) Still for the square lattice model, the heart of the derivation of $C(x, y)$ in [9] is to prove that the series $\tilde{C}(x, y)$ defined by (12) with $I(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3$ is algebraic. Observe that $\text{OS}(xyI) = 0$.
- (4) Similar results hold for the diagonal model $\mathcal{S} = \{\nearrow, \nwarrow, \swarrow, \searrow\}$, which has the same group as the square lattice model. More precisely, for $I(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3$ the series $\tilde{C}(x, y)$ is algebraic [9, Thm. 4], and moreover walks confined to \mathcal{C} starting at $(-1, b)$ and ending at $(0, 0)$ have an algebraic generating function [9, Cor. 5].
- (5) Finally, it is conjectured by Raschel and Trotignon [39, p. 9] that for any finite group model, walks in \mathcal{C} starting from $(-1, b)$ (or $(b, -1)$) have an algebraic generating function. This series satisfies (12) with again $I(x, y) = \bar{x}y^b$, and thus $\text{OS}(xyI) = 0$.

We could add to this list more algebraicity results for walks with a fixed endpoint confined to certain cones; see [16, Thm. 23] and [22, Cor. 4]. In these two papers steps between $(-1, 0)$ and $(0, -1)$ are allowed (when they belong to \mathcal{C}), and thus the series under consideration obey slightly different functional equations. We also refer to [15] for more exotic series with zero orbit sums that are *not* algebraic.

3.2. VANISHING ORBIT SUMS FOR THREE-QUADRANT WALKS

As mentioned in item (3) above, the heart of the derivation of $C(x, y)$ for the simple square lattice is the solution of (12) with $I(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3$. Recall the associated series $xy\tilde{C}$ has orbit sum zero. What is then the connection between $\tilde{C}(x, y)$ and the series $C(x, y)$ we are interested in?

It is not hard to construct, for any finite group model, a series $\tilde{C}(x, y)$ related to $C(x, y)$ such that $xy\tilde{C}$ has orbit sum zero. In sight of the functional equations (10) and (11), an obvious choice is $\tilde{C}(x, y) := C(x, y) - Q(x, y)$. However, we would also like $\tilde{C}(x, y)$ to be characterized by a functional equation resembling (12), in which every unknown series is explicitly described as a sub-series of $\tilde{C}(x, y)$. But this is not the case for the above choice of \tilde{C} . Indeed, if for instance we take $\mathcal{S} = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, we have

$$K(x, y)Q(x, y) = 1 - t\bar{y}Q(x, 0) - t\bar{x}Q(0, y), \quad (13)$$

$$K(x, y)C(x, y) = 1 - t\bar{y}C_{-,0}(\bar{x}) - t\bar{x}C_{0,-}(\bar{y}), \quad (14)$$

and, if we take $\tilde{C}(x, y) := C(x, y) - Q(x, y)$, we obtain, by extracting terms of the forms $x^i y^0$ and $x^0 y^i$ with $i < 0$:

$$C_{-,0}(\bar{x}) = \tilde{C}_{-,0}(\bar{x}), \quad C_{0,-}(\bar{y}) = \tilde{C}_{0,-}(\bar{y}),$$

so that

$$K(x, y)\tilde{C}(x, y) = -t\bar{y}\tilde{C}_{-,0}(\bar{x}) - t\bar{x}\tilde{C}_{0,-}(\bar{y}) + t\bar{y}Q(x, 0) + t\bar{x}Q(0, y),$$

which does not look very encouraging. However, we can get more leeway as follows. Observe that for any model \mathcal{S} with a finite group G , any $h \in G$, and any series $F \in \mathbb{Q}(x, y)[[t]]$, we have $\text{OS}(h(F)) = \varepsilon_h \text{OS}(F)$. Therefore, for any $h \in G$, the series $\varepsilon_h h(xyQ(x, y))$ has the same orbit sum as $xyQ(x, y)$ and $xyC(x, y)$. Consequently, for any collection of real numbers $\lambda_h, h \in G$ such that $\sum_h \lambda_h = 1$, the series

$$\tilde{C}(x, y) := C(x, y) - \bar{x}\bar{y} \sum_{h \in G} \varepsilon_h \lambda_h h(xyQ(x, y)) \quad (15)$$

is such that $xy\tilde{C}$ has vanishing orbit sum. Can we choose the λ_h such that $\tilde{C}(x, y)$ is defined by an equation not involving Q ? Returning to the square lattice case, if we choose

$$\tilde{C}(x, y) := C(x, y) - \frac{\bar{x}\bar{y}}{3} (xyQ(x, y) - \bar{x}yQ(\bar{x}, y) - x\bar{y}Q(x, \bar{y})),$$

we obtain

$$C_{-,0}(\bar{x}) = \tilde{C}_{-,0}(\bar{x}) - \frac{\bar{x}^2}{3} Q(\bar{x}, 0), \quad C_{0,-}(\bar{y}) = \tilde{C}_{0,-}(\bar{y}) - \frac{\bar{y}^2}{3} Q(0, \bar{y}),$$

and the combination of (13) (written for (x, y) , (\bar{x}, y) , and (x, \bar{y})) and (14) results in the following simple equation:

$$K(x, y)\tilde{C}(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}\tilde{C}_{-,0}(\bar{x}) - t\bar{x}\tilde{C}_{0,-}(\bar{y}),$$

which involves no specialization of Q . The following proposition tells us that a similar choice exists for any of the Weyl models of Table 1. In fact, it is easy to check that this choice is always unique.

Proposition 3.1. *Let \mathcal{S} be one of the Weyl models shown in Table 1. Let $2d$ be the order of the associated group G . Then $d = 2, 3$ or 4 . Let $\omega := \phi\psi\phi \cdots$ (with d generators) be the only element of length $\ell(\omega) = d$ in G . Define*

$$A(x, y) = C(x, y) - \frac{\bar{x}\bar{y}}{2d-1} \sum_{h \in G \setminus \{\omega\}} \varepsilon_h h(xyQ(x, y)) \quad (16)$$

$$= C(x, y) - \frac{\bar{x}\bar{y}}{2d-1} \left(\frac{\text{OS}(xy)}{K(x, y)} - \varepsilon_\omega \omega(xyQ(x, y)) \right). \quad (17)$$

Then $xyA(x, y)$ has orbit sum zero, and is characterized by the following equation:

$$\begin{aligned} K(x, y)A(x, y) &= 1 - \frac{\bar{x}\bar{y}}{2d-1} \left(\text{OS}(xy) - (-1)^d \bar{x}\bar{y} \right) \\ &\quad - t\bar{y}H_-(x)A_{-,0}(\bar{x}) - t\bar{x}V_-(y)A_{0,-}(\bar{y}) - t\bar{x}\bar{y}A_{0,0}\mathbb{1}_{(-1,-1) \in \mathcal{S}}. \end{aligned} \quad (18)$$

Proof. First, the equivalence between the two expressions of $A(x, y)$ comes from (10). Then, the orbit sum of xyA vanishes because, as noticed above, xyC and each $\varepsilon_h h(xyQ(x, y))$ have the same orbit sum.

Now we want to write an equation for $A(x, y)$, using the defining equations of C and Q . Let us first express $C_{-,0}(\bar{x})$ in terms of A and Q , by extracting from (16) terms of the form

$x^i y^0$, with $i < 0$. By examination of the three possible groups, detailed in Table 1, we see that only one element h contributes, namely $\omega^- = \phi\psi \cdots$ with $d - 1$ generators. More explicitly,

$$C_{-,0}(\bar{x}) = A_{-,0}(\bar{x}) + \frac{(-1)^{d-1}}{2d-1} \begin{cases} \bar{x}^d Q(\bar{x}, 0) & \text{if } d = 2 \text{ or } 4, \\ \bar{x}^3 Q(0, \bar{x}) & \text{if } d = 3. \end{cases} \quad (19)$$

Analogously, when we extract from (16) terms of the form $x^0 y^j$, with $j < 0$, the only group element that contributes is $\omega^+ = \phi\psi \cdots$ with $(d + 1)$ generators, and

$$C_{0,-}(\bar{y}) = A_{0,-}(\bar{y}) + \frac{(-1)^{d+1}}{2d-1} \begin{cases} \bar{y}^m Q(0, \bar{y}) & \text{if } d = 2 \text{ or } 4, \\ \bar{y}^3 Q(\bar{y}, 0) & \text{if } d = 3, \end{cases} \quad (20)$$

where $m = 2$ for $d = 2$ and $m = 3$ for $d = 4$. Finally, the only element h that contributes to the coefficient of $x^0 y^0$ in $C(x, y)$ is the identity, and

$$C_{0,0} = A_{0,0} + \frac{1}{2d-1} Q(0, 0). \quad (21)$$

We now start from (17) to write an equation defining $A(x, y)$. By examining again the three possible groups we see that $\omega(x, y) = (\bar{u}, \bar{v})$ with

$$(u, v) = \begin{cases} (x, y) & \text{if } d = 2 \text{ or } 4, \\ (y, x) & \text{if } d = 3. \end{cases} \quad (22)$$

Let us finally denote $\delta = \mathbb{1}_{(-1,-1) \in \mathcal{S}}$. Then

$$\begin{aligned} K(x, y)A(x, y) &= K(x, y)C(x, y) - \frac{\bar{x}\bar{y}}{2d-1} \left(\text{OS}(xy) - (-1)^d K(x, y)\omega(xyQ(x, y)) \right) \\ &= 1 - t\bar{y}H_-(x)C_{-,0}(\bar{x}) - t\bar{x}V_-(y)C_{0,-}(\bar{y}) - t\delta\bar{x}\bar{y}C_{0,0} - \frac{\bar{x}\bar{y}}{2d-1} \text{OS}(xy) \\ &\quad + \frac{(-1)^d \bar{x}^2 \bar{y}^2}{2d-1} \left(1 - tvH_-(\bar{u})Q(\bar{u}, 0) - tuV_-(\bar{v})Q(0, \bar{v}) + t\delta uvQ(0, 0) \right). \end{aligned}$$

Here, we have used (7) to express $K(x, y)C(x, y)$, and (8) to express $K(x, y)\omega(Q(x, y)) = K(\bar{u}, \bar{v})Q(\bar{u}, \bar{v})$, with (u, v) given by (22).

We now express $C_{-,0}(\bar{x})$, $C_{0,-}(\bar{y})$, and $C_{0,0}$ thanks to (19–21), and examine separately the cases $d = 2, 4$ and $d = 3$.

If $d = 2$ or 4 , we have

$$\begin{aligned} K(x, y)A(x, y) &= 1 - t\bar{y}H_-(x)A_{-,0}(\bar{x}) - t\bar{x}V_-(y)A_{0,-}(\bar{y}) - t\delta\bar{x}\bar{y}A_{0,0} - \frac{\bar{x}\bar{y}(\text{OS}(xy) - \bar{x}\bar{y})}{2d-1} \\ &\quad - t\delta \frac{\bar{x}\bar{y}}{2d-1} (Q(0, 0) - \bar{x}\bar{y}xyQ(0, 0)) \\ &\quad + \frac{t\bar{y}}{2d-1} Q(\bar{x}, 0) \left(\bar{x}^d H_-(x) - \bar{x}^2 H_-(\bar{x}) \right) \\ &\quad + \frac{t\bar{x}}{2d-1} Q(0, \bar{y}) \left(\bar{y}^m V_-(y) - \bar{y}^2 V_-(\bar{y}) \right), \end{aligned}$$

and the announced equation follows by observing that for each of the 5 models under consideration (shown in the first and third columns of Table 1),

$$\bar{x}^{d-2} H_-(x) = H_-(\bar{x}) \quad \text{and} \quad \bar{y}^{m-2} V_-(y) = V_-(\bar{y}).$$

For the two models such that $d = 3$ (central column in Table 1), we have $\delta = 0$, and

$$\begin{aligned} K(x, y)A(x, y) &= 1 - t\bar{y}H_-(x)A_{-,0}(\bar{x}) - t\bar{x}V_-(y)A_{0,-}(\bar{y}) - \frac{\bar{x}\bar{y}(\text{OS}(xy) + \bar{x}\bar{y})}{2d-1} \\ &\quad - \frac{t\bar{y}}{2d-1}Q(0, \bar{x})\left(\bar{x}^d H_-(x) - \bar{x}^2 V_-(\bar{x})\right) \\ &\quad - \frac{t\bar{x}}{2d-1}Q(\bar{y}, 0)\left(\bar{y}^3 V_-(y) - \bar{y}^2 H_-(\bar{y})\right), \end{aligned}$$

and now the announced equation follows from the fact that

$$\bar{x}H_-(x) = V_-(\bar{x}). \quad \square$$

Remark. Let us explain why we only consider the seven Weyl models in Proposition 3.1.

- For the four models of Table 3, we have seen that $\text{OS}(xy) = 0$. Hence xyC has orbit sum zero, and we can simply take $A(x, y) = C(x, y)$.
- The remaining 12 models (Table 2) have a vertical symmetry, and a group of order 4. The orbit of (x, y) consists of the pairs (x, y) , $(\bar{x}, y) = \phi(x, y)$, and two pairs $(x, \bar{y}r(x)) = \psi(x, y)$ and $(\bar{x}, \bar{y}r(x)) = \phi\psi(x, y)$ where $r(x) = r(\bar{x})$ is a rational function in x , *not reduced to a monomial*. Let us consider a series $\tilde{C}(x, y)$ of the form (15). If at least one of the coefficients λ_ψ or $\lambda_{\phi\psi}$ is non-zero, then it is not clear how to express $C_{0,-}(\bar{y})$ in terms of Q and A , as we did in (20). If these two coefficients are taken to be 0, then

$$\tilde{C}(x, y) = C(x, y) - \lambda Q(x, y) + (1 - \lambda)\bar{x}^2 Q(\bar{x}, y),$$

and the reader can check that the equation for \tilde{C} still involves specializations of Q , for any choice of λ .

Let us thus return to the seven Weyl models. The reason for the construction of $A(x, y)$ (and of the notation change $\tilde{C} \rightarrow A$) is the following, partially proved, conjecture.

Conjecture 3.2. *For any of the seven Weyl models of Table 1, the series $A(x, y)$ defined in Proposition 3.1 is algebraic. In particular, $C(x, y)$ is D-finite.*

For any of the four models of Table 3, the series $C(x, y)$ is algebraic (as $Q(x, y)$ itself).

Let us recall that the conjecture is proved for the first two models of Table 1 in [9], and in this paper for the third (king steps). The second part of the conjecture is proved for the first three models of Table 3 in [14] (D-finiteness was established earlier in [39]). This leaves us with five models for which the conjecture is open: four Weyl models, and the (conjecturally algebraic) Gessel model. Based on the solved cases, we believe that algebraicity should hold in a strong sense, and in particular, that each series $A_{i,j}$ (for Weyl models) or $C_{i,j}$ (for the models of Table 3) should be algebraic. For the four remaining Weyl models, we have tried to guess (using the `gfun` package [40] in MAPLE) a polynomial equation for the series $A_{-1,0}$, which coincides with the generating function $C_{-1,0}$ of walks ending at $(-1, 0)$. We could not guess anything for the diabolo model (using the counting sequence up to length $n = 4000$), but we discovered equations of degree 24 for each of the next three. The degree of $C_{-1,0}$ is 4 (resp. 8, 24) for the three solved Weyl models. For Gessel's model, $C_{0,0}$ was conjectured to be algebraic of degree 24 in [9], and algebraicity was proved since then in [16, Thm. 23] (taken with $\alpha = 0$, $\beta_- = -\pi/2$, and $\beta_+ = 3\pi/4$).

4. THE KING WALKS: STATEMENT OF THE RESULTS

We now fix the step set to be $\mathcal{S} = \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. We still denote by $Q(x, y)$ the generating function of walks confined to the first quadrant, and by $C(x, y)$ the generating function of walks avoiding the negative quadrant. The orbit of (x, y) under the action of G is $\{(x, y), (\bar{x}, y), (x, \bar{y}), (\bar{x}, \bar{y})\}$. Recall from [11] that $Q(x, y)$ can be expressed in terms of a simple rational function:

$$xyQ(x, y) = [x^>y^>] \frac{\text{OS}(xy)}{K(x, y)} = [x^>y^>] \frac{(x - \bar{x})(y - \bar{y})}{1 - t(x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y})}.$$

From Proposition 3.1 we get an expression of $C(x, y)$ of the form

$$C(x, y) = A(x, y) + \frac{1}{3}(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y})),$$

where, as announced, $A(x, y)$ is algebraic. More precisely, we write

$$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x), \quad (23)$$

where $P(x, y)$ and $M(x, y)$ belong to $\mathbb{Q}[x, y][[t]]$, and prove that P and M are algebraic.

Theorem 4.1 (The GF of king walks). *The generating function of king walks starting from $(0, 0)$, confined to \mathcal{C} , and ending in the first quadrant (resp. at a negative abscissa) is*

$$\frac{1}{3}Q(x, y) + P(x, y), \quad \left(\text{resp. } -\frac{1}{3}\bar{x}^2 Q(\bar{x}, y) + \bar{x}M(\bar{x}, y) \right), \quad (24)$$

where $P(x, y)$ and $M(x, y)$ are algebraic of degree 216 over $\mathbb{Q}(x, y, t)$. The generating function of walks ending at a negative ordinate follows using the x/y -symmetry of the step set.

The series P can be expressed in terms of M by:

$$P(x, y) = \bar{x}(M(x, y) - M(0, y)) + \bar{y}(M(y, x) - M(0, x)), \quad (25)$$

and M is defined by the following equation:

$$\begin{aligned} K(x, y)(2M(x, y) - M(0, y)) &= \frac{2x}{3} - 2t\bar{y}(x + 1 + \bar{x})M(x, 0) + t\bar{y}(y + 1 + \bar{y})M(y, 0) \\ &+ t(x - \bar{x})(y + 1 + \bar{y})M(0, y) - t(1 + \bar{y}^2 - 2\bar{x}\bar{y})M_{0,0} - t\bar{y}M_{1,0}, \end{aligned} \quad (26)$$

where $K(x, y) = 1 - t(x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y})$. The specializations $M(x, 0)$ and $M(0, y)$ are algebraic of degree 72 over $\mathbb{Q}(x, t)$ and $\mathbb{Q}(y, t)$, respectively, and $M_{0,0}$ and $M_{1,0}$ have degree 24 over $\mathbb{Q}(t)$.

We give in the following two subsections a complete algebraic description of all the series needed to reconstruct $P(x, y)$ and $M(x, y)$ from (25) and (26), namely, the univariate series $M_{0,0}$ and $M_{1,0}$ (Section 4.1) and the bivariate series $M(x, 0)$ and $M(0, y)$ (Section 4.2).

A combinatorial proof of (25) is given in Section 7, together with a generalization to other starting points and other Weyl models.

4.1. UNIVARIATE SERIES

We define in three steps an extension of $\mathbb{Q}(t)$ of degree 24, schematized by

$$\mathbb{Q}(t) \xrightarrow{4} \mathbb{Q}(t, u) \xrightarrow{3} \mathbb{Q}(t, v) \xrightarrow{2} \mathbb{Q}(t, w),$$

where $u, v, w \in \mathbb{Q}[[t]]$ and the numbers give the degrees of the successive extensions. First, let $u = t + t^2 + \mathcal{O}(t^3)$ be the only power series in t satisfying the quartic equation

$$(1 - 3u)^3(1 + u)t^2 + (1 + 18u^2 - 27u^4)t - u = 0. \quad (27)$$

Equivalently,

$$\frac{u}{(1 + u)(1 - 3u)^3} = \frac{t(1 + t)}{1 - 8t}. \quad (28)$$

Second, let $v = t + 3t^2 + \mathcal{O}(t^3)$ be the only series with constant term zero satisfying the cubic equation

$$(1 + 3v - v^3)u - v(v^2 + v + 1) = 0. \quad (29)$$

Clearly, it holds that $u \in \mathbb{Q}(v)$, and hence we have $\mathbb{Q}(t, u, v) = \mathbb{Q}(t, v)$. The minimal equation of v over $\mathbb{Q}(t)$ can be written as follows:

$$\frac{v(v^2 + v + 1)(v^3 - 3v - 1)^3}{(v^2 + 4v + 1)(4v^3 + 3v^2 - 1)^3} = \frac{t(1 + t)}{1 - 8t}. \quad (30)$$

Third, define

$$w = \sqrt{1 + 4v - 4v^3 - 4v^4} = 1 + 2t + 4t^2 + \mathcal{O}(t^3). \quad (31)$$

One can check that w has degree 24 over $\mathbb{Q}(t)$. Hence the extension $\mathbb{Q}(t, w)$ contains v .

We can now make the series $M_{0,0}$ and $M_{1,0}$ occurring in (26) explicit. Note that by (24), the series $M_{0,0}$ coincides with the series $C_{-1,0}$ that counts walks in \mathcal{C} ending at $(-1, 0)$. It is

$$M_{0,0} = C_{-1,0} = \frac{1}{2t} \left(\frac{w(1 + 2v)}{1 + 4v - 2v^3} - 1 \right) = t + 2t^2 + 17t^3 + 80t^4 + 536t^5 + \mathcal{O}(t^6). \quad (32)$$

Analogously, we have

$$C_{0,0} = \frac{1}{3}Q_{0,0} + P_{0,0} \quad \text{and} \quad C_{-2,0} = -\frac{1}{3}Q_{0,0} + M_{1,0},$$

where $P_{0,0} = 2M_{1,0}$ (by (25)) and

$$M_{1,0} = \frac{1}{6t^2} \left(1 + 2t + \frac{(1 - 2t)(1 + 2v)(16v^6 + 24v^5 + 7v^4 - 24v^3 - 30v^2 - 10v - 1)}{w(v^4 + 8v^3 + 6v^2 + 2v + 1)(1 + 4v - 2v^3)} \right). \quad (33)$$

More generally, we have the following counterpart of [9, Cor. 2 and Cor. 5].

Proposition 4.2 (Walks ending at a prescribed position). *Let w be the above defined series in t . For $j \geq 0$, the series $C_{-1,j}$ belongs to $\mathbb{Q}(t, w)$, and is thus algebraic. More generally, for $i \geq 1$ and $j \geq 0$, the series $C_{-i,j}$ is D-finite of the form*

$$-\frac{1}{3}Q_{i-2,j} + \text{Rat}(t, w)$$

for some rational function Rat . It is transcendental as soon as $i \geq 2$.

For $i \geq 0$ and $j \geq 0$, the series $C_{i,j}$ is D-finite and transcendental of the form

$$\frac{1}{3}Q_{i,j} + \text{Rat}(t, w).$$

Another series of interest is $C(1, 1)$, which counts all walks in \mathcal{C} , regardless of their endpoint. It reads

$$C(1, 1) = A(1, 1) - \frac{1}{3}Q(1, 1),$$

and we prove that $A(1, 1)$ has degree 24 over $\mathbb{Q}(t)$. A rational expression for $A(1, 1)$ in terms of v and w is given in Proposition 6.4. However, $Q(1, 1)$ is transcendental [5] hence $C(1, 1)$ is transcendental too.

We also obtain detailed asymptotic results, which refine general results of Denisov–Wachtel [18] and Mustapha [37] (who only obtained estimates up to a multiplicative constant).

Corollary 4.3. *The number $c(n)$ of n -step king walks confined to \mathcal{C} and ending anywhere, and the number $c_{0,0}(n)$ of such walks in \mathcal{C} ending at the origin satisfy for $n \rightarrow \infty$:*

$$c(n) = \left(\frac{2^{32}K}{3^7}\right)^{1/6} \frac{1}{\Gamma(2/3)} \frac{8^n}{n^{1/3}} - \frac{8}{9\pi} \frac{8^n}{n} + \mathcal{O}\left(\frac{8^n}{n^{4/3}}\right),$$

$$c_{0,0}(n) = \left(\frac{2^{29}K}{3^7}\right)^{1/3} \frac{\Gamma(2/3)}{\pi} \frac{8^n}{n^{5/3}} - \left(\frac{2^{62}L}{3^{31}}\right)^{1/6} \frac{1}{\Gamma(2/3)n^{7/3}} + \mathcal{O}\left(\frac{8^n}{n^{8/3}}\right),$$

where K and L are the unique real roots of

$$101^6 K^3 - 601275603 K^2 + 92811 K - 1$$

and

$$101^{18} L^3 - 342130847546623941461342020714770 L^2$$

$$+ 25258724190403343220341683641 L - 5078^6.$$

4.2. BIVARIATE SERIES

It remains to describe the series $M(x, 0)$ and $M(0, x)$ involved in (26). Both are cubic over $\mathbb{Q}(t, w, x)$, and we express them explicitly in terms of a parametrizing series U_1 that satisfies a reasonably compact cubic equation over $\mathbb{Q}(t, v, x)$. Details are given in Proposition 6.3. We also refer to Figure 2 on page 31 for the structure of all series involved in the paper.

In Table 4, we compare the degrees of several relevant algebraic series in the king’s model and the simple and diagonal models solved in [9]². This gives a hint of the technical difficulties that arise in the solution of the king’s model.

Series	$M(x, y)$	$M(x, 0)$	$M(0, y)$	$M(1, 1)$	$M(0, 1)$	$M_{0,0}$	$M(1, 0), M_{1,0},$ $A(1, 1), A_{0,0}$
Simple/Diag.	72	24	12	16	8	4/–	8
King	216	72	72	48	48	24	24

TABLE 4. A comparison of the degrees of various algebraic series for the simple (or diagonal) model [9] and for the king’s model (this paper). The series $M_{0,0}$ is zero for the diagonal model.

²The details on the series $M(1, 0)$, $M(0, 1)$, $M(1, 1)$, and $A(1, 1)$ are not stated in [9], but they can be found in the MAPLE sessions accompanying this paper on the author’s webpage.

5. THE KING WALKS: AN EQUATION WITH ONLY ONE CATALYTIC VARIABLE

Our starting point is the functional equation (7), specialized to

$$\begin{aligned} S(x, y) &= (x + 1 + \bar{x})(y + 1 + \bar{y}) - 1 \\ &= x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}. \end{aligned}$$

We use the x/y -symmetry of $S(x, y)$, which induces a bijection between walks ending on the negative x - and y -axis, and implies that

$$C_{-,0}(\bar{x}) = C_{0,-}(\bar{x}) =: C_-(\bar{x}).$$

This gives

$$K(x, y)C(x, y) = 1 - t\bar{y}(x + 1 + \bar{x})C_-(\bar{x}) - t\bar{x}(y + 1 + \bar{y})C_-(\bar{y}) - t\bar{x}\bar{y}C_{0,0}, \quad (34)$$

where as usual, the kernel is $K(x, y) = 1 - tS(x, y)$. Multiplying by xy gives

$$xyK(x, y)C(x, y) = xy - t(x^2 + x + 1)C_-(\bar{x}) - t(y^2 + y + 1)C_-(\bar{y}) - tC_{0,0}.$$

As observed before, the generating function $Q(x, y)$ of quadrant walks satisfies similarly:

$$xyK(x, y)Q(x, y) = xy - t(x^2 + x + 1)Q(x, 0) - t(y^2 + y + 1)Q(0, y) + tQ_{0,0}.$$

The subsequent solution follows the same steps as for the simple walk and the diagonal walk in [9]. But in practise, the king model turns out to be much heavier, and raises serious computational difficulties. In what follows, we focus on the points of the derivation that differ from [9]. We have performed all computations with the computer algebra system MAPLE. The corresponding sessions are available on the authors' webpages.

5.1. A SERIES $A(x, y)$ WITH ORBIT SUM ZERO

As discussed in Section 3, and summarized in Proposition 3.1, it makes sense to introduce a new series $A(x, y)$ defined by

$$C(x, y) = A(x, y) + \frac{1}{3} (Q(x, y) - \bar{x}^2Q(\bar{x}, y) - \bar{y}^2Q(x, \bar{y})). \quad (35)$$

Note that any monomial $x^i y^j t^n$ that occurs in $A(x, y)$ is such that $(i, j) \in \mathcal{C}$. Then $xyA(x, y)$ has orbit sum zero, meaning that

$$xyA(x, y) - \bar{x}yA(\bar{x}, y) + \bar{x}\bar{y}A(\bar{x}, \bar{y}) - x\bar{y}A(x, \bar{y}) = 0. \quad (36)$$

Moreover, $A(x, y)$ is defined by the functional equation (18), which reads:

$$K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}(x + 1 + \bar{x})A_-(\bar{x}) - t\bar{x}(y + 1 + \bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}. \quad (37)$$

We now focus on the determination of $A(x, y)$, which should be algebraic according to our final Theorem 4.1. The next step is to split the series $A(x, y)$ into three parts, which involve polynomials in x and y instead of Laurent polynomials.

5.2. REDUCTION TO A QUADRANT-LIKE PROBLEM FOR $M(x, y)$

We now separate in $A(x, y)$ the contributions of the three quadrants, again using the x/y -symmetry of the step set:

$$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x), \quad (38)$$

where $P(x, y)$ and $M(x, y)$ lie in $\mathbb{Q}[x, y][[t]]$. Note that this identity defines P and M uniquely in terms of A . Replacing A by this expression, and extracting the positive part in x and y from the orbit equation (36) relates the series P and M by

$$xyP(x, y) = y(M(x, y) - M(0, y)) + x(M(y, x) - M(0, x)), \quad (39)$$

which is Equation (25) in Theorem 4.1, and also the same as [9, Eq. (22)]. For a combinatorial proof of this equation see Section 7.

We could now follow the lines of proof of [9, Sec. 2.3] to obtain the functional equation (26) for $M(x, y)$. However, we prefer to describe a slightly different – and more combinatorial – way to derive this equation. Clearly, $A(x, y)$ counts walks confined to \mathcal{C} , starting either from $(0, 0)$, $(-2, 0)$, or $(0, -2)$, with a weight $2/3$ in the first case and $1/3$ in each of the other two cases. In sight of the splitting (38) of $A(x, y)$, the series $P(x, y)$ counts such walks ending in the first quadrant, and $\bar{x}M(\bar{x}, y)$ those ending at a negative abscissa. By combining these two observations and constructing these walks step by step, we can write directly a pair of equations for P and M :

$$\begin{aligned} K(x, y)P(x, y) &= \frac{2}{3} - t\bar{y}(x+1+\bar{x})P(x, 0) - t\bar{x}(y+1+\bar{y})P(0, y) + t\bar{x}\bar{y}P_{0,0} \\ &\quad + t(x+1+\bar{x})M(0, x) - t\bar{x}M_{0,0} + t(y+1+\bar{y})M(0, y) - t\bar{y}M_{0,0}, \\ K(x, y)\bar{x}M(\bar{x}, y) &= \frac{1}{3}\bar{x}^2 - t\bar{y}(x+1+\bar{x})\bar{x}M(\bar{x}, 0) - t(y+1+\bar{y})M(0, y) + t\bar{y}M_{0,0} \\ &\quad + t\bar{x}(y+1+\bar{y})P(0, y) - t\bar{x}\bar{y}P_{0,0}. \end{aligned} \quad (40)$$

In the first equation for instance, the term $t(y+1+\bar{y})M(0, y) - t\bar{y}M_{0,0}$ counts walks that come from the NW quadrant and enter the non-negative quadrant through the y -axis. We will in fact ignore the first equation and replace it by the link (39) between P and M . Extracting the coefficient of x^1 in (39) gives

$$P(0, y) = M_x(0, y) + \bar{y}(M(y, 0) - M_{0,0}).$$

Extracting now the coefficient of y^0 gives

$$P_{0,0} = 2M_x(0, 0) = 2M_{1,0}.$$

We plug these two identities into (40): upon replacing x by \bar{x} and then dividing by x , we find:

$$\begin{aligned} K(x, y)M(x, y) &= \frac{1}{3}x - t\bar{y}(x+1+\bar{x})M(x, 0) - t\bar{x}(y+1+\bar{y})M(0, y) + t\bar{x}\bar{y}M_{0,0} \\ &\quad + t(y+1+\bar{y})(M_x(0, y) + \bar{y}(M(y, 0) - M_{0,0})) - 2t\bar{y}M_{1,0}. \end{aligned} \quad (41)$$

This is not yet (26), as there is one more series involved here, namely $M_x(0, y)$. However, by extracting the coefficient of x^0 in the above equation, we find one more relation:

$$t\bar{y}(y+1+\bar{y})M(y, 0) + (ty+t\bar{y}-1)M(0, y) + 2t(y+1+\bar{y})M_x(0, y) - t\bar{y}(\bar{y}+2+y)M_{0,0} - 3t\bar{y}M_{1,0} = 0.$$

Combined with (41), this now gives (26).

5.3. CANCELLING THE KERNEL: AN EQUATION BETWEEN BIVARIATE SERIES

Next we will cancel the kernel K . As a polynomial in y , the kernel admits only one root that is a formal power series in t :

$$Y(x) = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2(x + 1 + \bar{x})^2}}{2t(x + 1 + \bar{x})} = (x + 1 + \bar{x})t + \mathcal{O}(t^2).$$

Note that $Y(x) = Y(\bar{x})$. We specialize (26) to the pairs $(x, Y(x))$, $(\bar{x}, Y(x))$, $(Y(x), x)$, and $(Y(x), \bar{x})$ (the left-hand side vanishes for each specialization since $K(x, y) = K(y, x)$, yet this symmetry is not part of the group of the model), and we eliminate $M(0, Y)$, $M(Y, 0)$, and $M(\bar{x}, 0)$ from the four resulting equations. We obtain:

$$(x + 1 + \bar{x}) \left(Y(x) - \frac{1}{Y(x)} \right) (xM(0, x) - 2\bar{x}M(0, \bar{x})) + 3(x + 1 + \bar{x})M(x, 0) - \frac{2\bar{x}Y(x)}{t} + 3M_{1,0} + (2Y(x) - x - \bar{x})M_{0,0} = 0. \quad (42)$$

We have now eliminated the trivariate series $M(x, y)$. We are left with three bivariate series, namely $M(0, x)$, $M(0, \bar{x})$, and $M(x, 0)$. In the next section we eliminate the term $M(x, 0)$, so as to end with two specializations of the series $M(0, x)$.

5.4. AN EQUATION BETWEEN $M(0, x)$ AND $M(0, \bar{x})$

Let us denote the discriminant occurring in $Y(x)$ by

$$\Delta(x) := (1 - t(x + \bar{x}))^2 - 4t^2(x + 1 + \bar{x})^2 = (1 - t(3(x + \bar{x}) + 2))(1 + t(x + \bar{x} + 2)), \quad (43)$$

and introduce the notation

$$\begin{aligned} R(x) &:= t^2M(x, 0) = \frac{xt^2}{3} + \left(1 + \frac{x^2}{3}\right)t^3 + \mathcal{O}(t^4), \\ S(x) &:= txM(0, x) = x(1 + x)t^2 + 2x(1 + x + x^2)t^3 + \mathcal{O}(t^4). \end{aligned} \quad (44)$$

Note that $t^2M_{0,0} = R_0 = tS_1$ and $t^2M_{1,0} = R_1$. Then (42) reads

$$\begin{aligned} \sqrt{\Delta(x)} \left(S(x) - 2S(\bar{x}) + \frac{xR_0 - t}{t(1 + x + x^2)} \right) = \\ 3(x + 1 + \bar{x})R(x) + 3R_1 + \frac{1 - t(x + \bar{x})}{t(1 + x + x^2)}(xR_0 - t) - (x + \bar{x})R_0 =: \hat{R}(x), \end{aligned} \quad (45)$$

where we defined $\hat{R}(x)$ as a shorthand for the right-hand side. Observe that introducing

$$\hat{S}(x) := S(x) - \frac{3R_0/t - 2x - \bar{x}}{3(x + 1 + \bar{x})}, \quad (46)$$

allows us to rewrite the above equation as

$$\sqrt{\Delta(x)} \left(\hat{S}(x) - 2\hat{S}(\bar{x}) \right) = \hat{R}(x). \quad (47)$$

Before we go into the details of the next steps, let us describe their principle. We consider both sides of (45) as power series in t whose coefficients are Laurent series in x . We square Equation (45) and extract the negative part in x , as defined at the beginning of Section 2. On the right-hand side, the terms involving $R(x)$ (mostly) disappear as this series

involves only non-negative powers of x . On the left-hand side, the terms involving only $S(x)$ mostly disappear as well. There remain terms involving only $S(\bar{x})$, as well as the negative part of $\Delta(x)S(x)S(\bar{x})$. In other words, the result is an expression for the negative part of $\Delta(x)S(x)S(\bar{x})$ in terms of $S(\bar{x})$ and univariate series. Using the symmetry of $\Delta(x)$ in x and \bar{x} , we will then express the *positive* part of $\Delta(x)S(x)S(\bar{x})$ in terms of $S(x)$ and univariate series. We will thus reconstruct an expression of $\Delta(x)S(x)S(\bar{x})$ that does not involve $R(x)$, as in [9, Sec. 2.5].

In order to make the above programme effective, we need the following lemma, which tells us how to extract the non-negative part of certain series as those that we meet when we square (45).

Lemma 5.1. *Let $\zeta = e^{2i\pi/3}$ and $\bar{\zeta} = e^{-2i\pi/3}$ be the two primitive cubic roots of unity. Let $F(x) \in \mathbb{C}[x]((t))$. Then,*

$$[x^{\geq}] \frac{F(\bar{x})}{1+x+x^2} = \frac{1}{1-\zeta} \frac{F(\zeta)}{1-\zeta x} + \frac{1}{1-\bar{\zeta}} \frac{F(\bar{\zeta})}{1-\bar{\zeta}x}$$

and

$$[x^{\geq}] \frac{F(\bar{x})}{(1+x+x^2)^2} = \frac{2}{3} \left(\frac{1}{1-\zeta} \frac{F(\zeta)}{1-\zeta x} + \frac{1}{1-\bar{\zeta}} \frac{F(\bar{\zeta})}{1-\bar{\zeta}x} \right) + \frac{1}{(1-\zeta)^2} \left(\frac{\zeta F'(\zeta)}{1-\zeta x} + \frac{F(\zeta)}{(1-\zeta x)^2} \right) + \frac{1}{(1-\bar{\zeta})^2} \left(\frac{\bar{\zeta} F'(\bar{\zeta})}{1-\bar{\zeta}x} + \frac{F(\bar{\zeta})}{(1-\bar{\zeta}x)^2} \right).$$

In fact, the first formula holds for $F(x) \in \bar{x}\mathbb{C}[x]((t))$, and the second for $F(x) \in \bar{x}^3\mathbb{C}[x]((t))$.

Proof. By linearity, it suffices to prove the lemma when $F(x) = x^k$, for $k \geq -1$ in the first part, $k \geq -3$ in the second part. A key ingredient are the following partial fraction expansions:

$$\frac{1}{1+x+x^2} = \frac{1}{(1-\zeta)(1-\zeta x)} + \frac{1}{(1-\bar{\zeta})(1-\bar{\zeta}x)},$$

$$\frac{1}{(1+x+x^2)^2} = \frac{2}{3} \left(\frac{1}{(1-\zeta)(1-\zeta x)} + \frac{1}{(1-\bar{\zeta})(1-\bar{\zeta}x)} \right) + \frac{1}{(1-\zeta)^2(1-\zeta x)^2} + \frac{1}{(1-\bar{\zeta})^2(1-\bar{\zeta}x)^2}.$$

Then we work out each piece separately, first focussing on the case $k \geq 0$. For instance,

$$[x^{\geq}] \frac{\bar{x}^k}{1-\zeta x} = \bar{x}^k \sum_{n \geq k} \zeta^n x^n = \frac{\zeta^k}{1-\zeta x}$$

and

$$[x^{\geq}] \frac{\bar{x}^k}{(1-\zeta x)^2} = \bar{x}^k \sum_{n \geq k} (n+1)\zeta^n x^n = \zeta^k \sum_{n \geq 0} (k+n+1)\zeta^n x^n = \frac{k\zeta^k}{1-\zeta x} + \frac{\zeta^k}{(1-\zeta x)^2}.$$

To complete the proof, we check that the first (resp. second) identity of the lemma holds as well if $F(x) = \bar{x}^\ell$ for $\ell = 1$ (resp. $\ell = 1, 2, 3$). \square

By expanding (45) at $x = \zeta$ and $x = \bar{\zeta}$, we derive the values of $S(x)$ at these two points, which will be useful in sight of the above lemma:

$$S(\zeta) = S(\bar{\zeta}) = -\frac{R_0 + 3R_1}{1+t} = -t^2 - 11t^4 - 30t^5 + \mathcal{O}(t^6). \quad (48)$$

Now, as already observed, the right-hand side $\hat{R}(x)$ of (45) is mostly positive in x , meaning that the valuation in x of the coefficient of t^n is bounded from below, uniformly in n . We now square both sides. The negative part of (the square of) the right-hand side is easily obtained by an expansion around $x = 0$, and found to be

$$(2R_0 + t)^2 \bar{x}^2 + 2\bar{x}(2R_0 + t)(2R_0 + 6R_1 - 1 - t).$$

In the square of the left-hand side of (45) some terms are also mostly positive – in fact all terms that do not involve $S(\bar{x})$. Their negative parts can be extracted as above by an expansion around $x = 0$. Some other terms, like $\Delta(x)S(\bar{x})^2$, are mostly negative, and we subtract their non-negative parts, obtained via an expansion at $x = \infty$ (which is legitimate due to their Laurent polynomial coefficients in x). And finally there are two tricky terms:

$$\Delta(x)S(x)S(\bar{x}) \quad \text{and} \quad \Delta(x)\frac{S(\bar{x})(xR_0/t - 1)}{1 + x + x^2},$$

which require some care. We leave the first term untouched, since what we want to determine is precisely its negative part. The numerator of the second term is a series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$. We expand it at infinity, using $S_0 = 0$ and $S_1 = R_0/t$, and obtain,

$$\Delta(x)S(\bar{x})(xR_0/t - 1) = -3R_0^2 x^2 + F(\bar{x}),$$

for a series $F(x) \in \bar{x}\mathbb{Q}[x]((t))$. Then we divide this by $(1 + x + x^2)$. The term $x^2/(1 + x + x^2)$ has no negative part, and we apply Lemma 5.1 to express the negative part of $F(\bar{x})/(1 + x + x^2)$. After having treated all terms, we reach an identity of the form

$$[x^{<}] (\Delta(x)S(\bar{x})S(x)) = \Delta(x)S(\bar{x})^2 - \Delta(x)\frac{S(\bar{x})(xR_0 - t)}{t(1 + x + x^2)} + \frac{\text{Pol}(R_0, R_1, t, x)}{tx^2(1 + x + x^2)}$$

for some polynomial Pol with rational coefficients. We can now replace x by \bar{x} to obtain an expression of the positive part of $\Delta(x)S(\bar{x})S(x)$ (which is a series in $\mathbb{Q}[x, \bar{x}][[t]]$). We finally denote by P_0 the coefficient of x^0 in $\Delta(x)S(x)S(\bar{x})$, and obtain an expression of $\Delta(x)S(x)S(\bar{x})$ in terms of $S(x)$, $S(\bar{x})$, P_0 , R_0 , and R_1 , which can be written as:

$$\begin{aligned} \Delta(x) \left(S(x)^2 + S(\bar{x})^2 - S(x)S(\bar{x}) + \frac{S(x)(xt - R_0) + S(\bar{x})(\bar{x}t - R_0)}{t(x + 1 + \bar{x})} \right) = \\ (R_0 + 3R_1) \left((2R_0 + t) \left(x + \bar{x} + \frac{1 + t}{t(x + 1 + \bar{x})} \right) - 1 - t \right) \\ - (1 + 4t)(x + \bar{x})R_0 + (t^2 + tR_0 + R_0^2)(x^2 + \bar{x}^2) - P_0. \end{aligned} \quad (49)$$

As in [9] the numerator of the right-hand side as a polynomial in x is not divisible by $\Delta(x)$, nor by any of its factors.

Observe that (49) can also be written in terms of the series \hat{S} defined by (46), and then takes the following form:

$$\Delta(x) \left(\hat{S}(x)^2 - \hat{S}(x)\hat{S}(\bar{x}) + \hat{S}(\bar{x})^2 \right) = \frac{\text{Pol}(P_0, R_0, R_1, t, x)}{x^4 t^2 (x + 1 + \bar{x})^2}, \quad (50)$$

where Pol is another polynomial with rational coefficients. This simpler form in terms of \hat{S} will guide us in the following final step, in which we eliminate $S(\bar{x})$ and obtain an equation in which $S(x)$ is the only bivariate series.

5.5. AN EQUATION FOR $M(0, x)$ ONLY

We would like to extract the positive part of (49), but we are stopped by the mixed term $S(x)S(\bar{x})$. However, from the structure visible in (50), we observe that a multiplication by $\hat{S}(x) + \hat{S}(\bar{x})$ eliminates this mixed term, leaving us with the following cubic equation in \hat{S} :

$$\Delta(x) \left(\hat{S}(x)^3 + \hat{S}(\bar{x})^3 \right) = \left(\hat{S}(x) + \hat{S}(\bar{x}) \right) \frac{\text{Pol}(P_0, R_0, R_1, t, x)}{x^4 t^2 (x + 1 + \bar{x})^2}.$$

We then rewrite this in terms of S rather than \hat{S} , and extract the non-negative part in x , using the same tools as in the previous subsection. We refer for full details to the accompanying MAPLE worksheet. The terms that are mostly positive or mostly negative in x do not raise any difficulties. The two tricky terms are those that involve $S(\bar{x})^2/(1+x+x^2)$ and $S(\bar{x})/(1+x+x^2)^2$. Their non-negative parts are extracted using Lemma 5.1. When processing the latter term, three additional univariate series occur, namely S_2 , $S'(\zeta)$, and $S'(\bar{\zeta})$. We find it more convenient to work with the real and imaginary parts of $\zeta S'(\zeta)$, and to define series B_1 and B_2 by

$$\begin{aligned} (1+t)^2 \zeta S'(\zeta) &= B_1 + i\sqrt{3}B_2, \\ (1+t)^2 \bar{\zeta} S'(\bar{\zeta}) &= B_1 - i\sqrt{3}B_2. \end{aligned} \tag{51}$$

We use several times $S_1 = R_0/t$ and the expressions (48) of $S(\zeta)$ and $S(\bar{\zeta})$. At the end we obtain a polynomial identity between $S(x)$, P_0 , R_0 , R_1 , S_2 , B_1 , B_2 , x , and t .

We can reduce to four the number of univariate series involved in this equation as follows. First, we expand the equation around $x = 0$ at first order: this gives an expression of P_0 in terms of the 5 other univariate series. We replace P_0 by this expression in the functional equation, and now expand at first order around $x = \zeta$: this gives

$$3t^2 S_2 = -3tR_0 - 3t^2 - 2B_1.$$

In the end we get a cubic equation in $S(x)$:

$$\text{Pol}(S(x), R_0, R_1, B_1, B_2, t, x) = 0, \tag{52}$$

for a polynomial $\text{Pol}(x_0, x_1, x_2, x_3, x_4, t, x)$ with rational coefficients. In the terminology of [10], this is an equation with only one *catalytic variable*, namely x , as opposed to the original functional equation for $M(x, y)$ that had two catalytic variables, x and y .

We can describe the above polynomial Pol in a reasonably compact form thanks to some of its properties: first, when we introduce the series $\hat{S}(x)$ defined by (46), there is no quadratic term (in $\hat{S}(x)$). Then, the coefficients of the resulting equation are (almost) symmetric in x and \bar{x} , and they become symmetric if we introduce the series $\hat{S}(x)/(x - \bar{x})$. Now we can write the equation in terms of a new variable $y := x + 1 + \bar{x}$. Then we observe one more property, namely that the coefficients are (almost) invariant when we replace y by $\bar{y}(1 + 1/t)$. We refer to our MAPLE worksheet for details. If we denote

$$z = t(x + 1 + \bar{x}) + \frac{1+t}{x+1+\bar{x}} \tag{53}$$

and

$$\tilde{S}(x) = \frac{x+1+\bar{x}}{x-\bar{x}} \hat{S}(x) = \frac{1}{x-\bar{x}} \left((x+1+\bar{x})S(x) - \frac{R_0}{t} + \frac{2x}{3} + \frac{\bar{x}}{3} \right),$$

then Equation (52) reads:

$$\begin{aligned}
0 &= 27t^2(2t+z+1)(10t-3z+1)\tilde{S}(x)^3 \\
&+ \left((216t^2-27z^2+54t)R_0^2+27t(6R_1t-6R_1z+6t^2+2zt-z^2+2B_2+t+z)R_0 \right. \\
&\quad \left. -9t^2(27R_1^2-27R_1t+9R_1z+5t^2-2zt+3B_1-3B_2-9R_1+6t-2z+1) \right)\tilde{S}(x) \\
&+ (72t^2-9z^2+18B_1+18t)R_0^2+9t(6R_1t-6R_1z+6t^2+2zt-z^2+2B_1+2B_2+t+z)R_0 \\
&\quad -t^2(81R_1^2-81R_1t+27R_1z-5t^2-10zt+3z^2-9B_1-9B_2-27R_1+6t-4z+2).
\end{aligned} \tag{54}$$

6. THE KING WALKS: ALGEBRAICITY

In [10], a general method to solve equations in one catalytic variable was developed, proving in particular that their solutions are systematically algebraic (provided the equation is *proper* in a certain natural sense). In Section 6.1 we first use the results of [10] to obtain a system of four polynomial equations relating the series R_0, R_1, B_1 , and B_2 . Combined with a few initial terms, this system characterizes these four series. Unfortunately, it turns out to be too big for us to obtain individual equations for each of the four series, be it by bare hand elimination or using Gröbner bases: we did obtain polynomial equations for R_0 and R_1 , of degree 24 in each case, but not for the other two series. Instead, as detailed in Section 6.2, we have resorted to a guess-and-check approach, consisting in *guessing* such equations (of degree 12 or 24, depending on the series), and then *checking* that they satisfy the system.

6.1. A POLYNOMIAL SYSTEM RELATING R_0, R_1, B_1 , AND B_2

We start from the cubic equation (52). The approach of [10] instructs us to consider the series X (in t , or in a fractional power of t), satisfying

$$\text{Pol}_{x_0}(S(X), R_0, R_1, B_1, B_2, t, X) = 0, \tag{55}$$

where Pol_{x_0} stands for the derivative of Pol with respect to its first variable. [The number of such series \$X\$ and their first terms](#) depend only on the first terms of the series $S(x), R_0, R_1, B_1$, and B_2 ; see [10, Thm. 2]. We find that 6 such series exist:

$$\begin{aligned}
X_1(t) &= i + 2t^2 + 4t^3 + (36 - 2i)t^4 + \mathcal{O}(t^5), \\
X_2(t) &= -i + 2t^2 + 4t^3 + (36 + 2i)t^4 + \mathcal{O}(t^5), \\
X_3(t) &= \sqrt{t} + t + \frac{3}{2}t^{3/2} + 3t^2 + \frac{51}{8}t^{5/2} + 14t^3 + \mathcal{O}(t^{7/2}), \\
X_4(t) &= -\sqrt{t} + t - \frac{3}{2}t^{3/2} + 3t^2 - \frac{51}{8}t^{5/2} + 14t^3 + \mathcal{O}(t^{7/2}), \\
X_5(t) &= i\sqrt{t} - it^{3/2} + 2it^{5/2} + t^3 - 4it^{7/2} + 2t^4 + \mathcal{O}(t^{9/2}), \\
X_6(t) &= -i\sqrt{t} + it^{3/2} - 2it^{5/2} + t^3 + 4it^{7/2} + 2t^4 + \mathcal{O}(t^{9/2}).
\end{aligned}$$

Note that the coefficients of X_1 and X_2 (resp. X_5 and X_6) are conjugates of one another. As discussed in [10], each of these series X also satisfies

$$\text{Pol}_x(S(X), R_0, R_1, B_1, B_2, t, X) = 0, \tag{56}$$

where Pol_x is the derivative with respect to the last variable of Pol , and of course

$$\text{Pol}(S(X), R_0, R_1, B_1, B_2, t, X) = 0. \quad (57)$$

Using this, we can easily identify two of the series X_i : indeed, eliminating B_1 and B_2 between the three equations (55), (56), and (57) gives a polynomial equation between $S(X)$, R_0 , R_1 , t , and X , which factors. Remarkably, its simplest non-trivial factor only involves t and X , and reads

$$X^2 - t(1 + X)^2(1 + X^2). \quad (58)$$

By looking at the first terms of the X_i 's and at the other factors, one concludes that the above equation holds for X_3 and X_4 , which are thus explicit. The other four series X_i satisfy another equation in $S(X)$, X , R_0 , R_1 , and t , which we will not use.

Let $D(x_1, \dots, x_4, t, x)$ be the discriminant of $\text{Pol}(x_0, \dots, x_4, t, x)$ with respect to x_0 . According to [10, Thm. 14], each X_i is a *double root* of $D(R_0, R_1, B_1, B_2, t, x)$, seen as a polynomial in x . Hence this polynomial, which involves four unknown series R_0, R_1, B_1, B_2 , has (at least) 6 double roots. This seems more information than we need. However, we shall see that there is some redundancy in the 6 series X_i , which comes from the properties of Pol that we used at the end of Section 5.5 to write it in a compact form.

We first observe that D factors as

$$D(R_0, R_1, B_1, B_2, t, x) = 27x^2t^2(1 + x + x^2)^2\Delta(x)D_1(R_0, R_1, B_1, B_2, t, x),$$

where $\Delta(x)$ is defined by (43), and D_1 has degree 24 in x . It is easily checked that none of the X_i 's are roots of the prefactors, so they are double roots of D_1 . But we observe that $\bar{x}^{12}D_1$ is symmetric in x and \bar{x} . That is,

$$D_1(R_0, R_1, B_1, B_2, t, x) = x^{12}D_2(R_0, R_1, B_1, B_2, t, x + 1 + \bar{x}),$$

for some polynomial $D_2(x_1, \dots, x_4, t, y) \equiv D_2(y)$ of degree 12 in y . Since each X_i is a double root of D_1 , each series $Y_i := X_i + 1 + 1/X_i$, for $1 \leq i \leq 6$, is a double root of D_2 . The series Y_i , for $2 \leq i \leq 6$, are easily seen from their first terms to be distinct, but the first terms of Y_1 and Y_2 suspiciously agree: one suspects (and rightly so), that $X_2 = 1/X_1$, and carefully concludes that D_2 has (at least) 5 double roots in y . Moreover, since X_3 and X_4 satisfy (58), the corresponding series Y_3 and Y_4 are the roots of $1 + t = tY_i^2$, that is, $Y_{3,4} = \pm\sqrt{1 + 1/t}$. The other roots start as follows:

$$Y_2 = 1 + 4t^2 + 8t^3 + \mathcal{O}(t^4), \quad Y_{5,6} = \mp \frac{i}{\sqrt{t}} + 1 + t^2 \pm it^{5/2} + \mathcal{O}(t^3).$$

This is not yet the end of the story: indeed, D_2 appears to be almost symmetric in y and $1/y$. More precisely, we observe that

$$D_2(R_0, R_1, B_1, B_2, y) = y^6 D_3 \left(R_0, R_1, B_1, B_2, ty + \frac{t+1}{y} \right),$$

for some polynomial $D_3(R_0, R_1, B_1, B_2, t, z) \equiv D_3(z)$ of degree 6 in z . It follows that each series $Z_i := tY_i + (1 + t)/Y_i$, for $2 \leq i \leq 6$, is a root of $D_3(z)$, and even a double root, unless $tY_i^2 = 1 + t$, which precisely occurs for $i = 3, 4$. One finds $Z_{3,4} = \pm 2\sqrt{t(1 + t)}$,

$$Z_2 = 1 + 2t - 4t^2 + \mathcal{O}(t^3), \quad Z_{5,6} = 2t + 2t^3 + \mathcal{O}(t^4).$$

Since Z_5 and Z_6 seem indistinguishable, we conclude that $D_3(z)$ has (at least) two double roots Z_2 and Z_5 , and a factor $(z^2 - 4t(1+t))$ coming from the simple roots at Z_3 and Z_4 . We can thus write

$$D_3(z) = \sum_{i=0}^6 d_i z^i = (z^2 - 4t(1+t)) (\alpha z^2 + \beta z + \gamma)^2,$$

where the d_i are explicit in terms of $R_0, R_1, B_1,$ and B_2 . We can determine $\alpha, \beta,$ and γ in terms of the d_i by matching the three monomials of highest degree, and this gives:

$$D_3(z) = \sum_{i=0}^6 d_i z^i = \frac{(z^2 - 4t(1+t)) (8z^2 d_6^2 + 4z d_5 d_6 + 16t^2 d_6^2 + 16t d_6^2 + 4d_4 d_6 - d_5^2)^2}{64 d_6^3}.$$

Extracting from this identity the coefficients of z^0, \dots, z^3 gives four polynomial relations between the coefficients d_i , resulting in four polynomial relations between the four series $R_0, R_1, B_1,$ and B_2 . We give below the degrees and number of terms in each of them.

Degree in	R_0	R_1	B_1	B_2	t	Number of terms
Eq. 1	5	3	1	1	7	72
Eq. 2	6	4	2	2	7	132
Eq. 3	5	5	2	2	9	192
Eq. 4	6	6	3	3	10	276

TABLE 5. Properties of the four polynomial equations defining the four main unknown series $R_0, R_1, B_1,$ and B_2 .

We will now check that the solution of this system is unique if we add the conditions $R_0 = \mathcal{O}(t^3), R_1 = \mathcal{O}(t^2), B_1 = \mathcal{O}(t^2), B_2 = \mathcal{O}(t^2)$, which are directly deduced from the definitions of $R(x), B_1,$ and B_2 in (44) and (51). We write accordingly $R_0 = t^3 \tilde{R}_0, R_1 = t^2 \tilde{R}_1, B_1 = t^2 \tilde{B}_1, B_2 = t^2 \tilde{B}_2$ in the system, divide each equation by a power of t so that it becomes non-trivial at $t = 0$ (and, as it happens, linear in each series at this point). We finally form linear combinations of these four equations so that the system, evaluated at $t = 0$, is triangular. We refer again to our MAPLE sessions for details.

As explained at the beginning of this subsection, we have been able to derive directly from this system polynomial equations (of degree 24) for R_0 and R_1 by successive eliminations, but not for the other two series. At the end we resorted to a guess-and-check approach.

6.2. GUESS-AND-CHECK

The functional equation (37) defining $A(x, y)$ encodes a simple recurrence for the numbers $a_{i,j}(n)$ that count (weighted) walks of length n by the positions of their endpoints $(i, j) \in \mathcal{C}$:

$$a_{i,j}(n+1) = a_{i-1,j-1}(n) + a_{i-1,j}(n) + a_{i-1,j+1}(n) + a_{i,j-1}(n) \\ + a_{i,j+1}(n) + a_{i+1,j-1}(n) + a_{i+1,j}(n) + a_{i+1,j+1}(n),$$

with $a_{i,j}(n) = 0$ for $(i, j) \notin \mathcal{C}$ and initial conditions $a_{0,0}(0) = 2/3, a_{-2,0}(0) = a_{0,-2}(0) = 1/3,$ and $a_{i,j}(0) = 0$ otherwise. We implemented this recurrence in the programming language C using modular arithmetic and the Chinese remainder theorem to compute these numbers up

to $n = 2000$ (this effectively bounds i and j to 2000 as well, since $a_{i,j}(n) = 0$ if $i > n$ or $j > n$). For this purpose, we used approximately 100 primes of size $\approx 2^{64}$, and we actually computed $3A(x, y)$ rather than $A(x, y)$, as it has integer coefficients.

The series R_0, R_1, B_1 , and B_2 are related to $A(x, y)$ as follows. First, observe that by (44) it holds that $R_0 = t^2 A_{-1,0}$ and $R_1 = t^2 A_{-2,0}$. Second, for B_1 and B_2 defined in (51), we also start from (44), which implies that $S'(\zeta) = tM(0, \zeta) + t\zeta M_y(0, \zeta)$ where $M(0, y) = \sum_{j \geq 0} A_{-1,j} y^j$. In order to compute $M(0, \zeta)$ we used $\zeta^2 = -1 - \zeta$, with $\zeta = (-1 + i\sqrt{3})/2$, which implies that $6M(0, \zeta) = \alpha_1 + i\sqrt{3}\alpha_2$ with $\alpha_1, \alpha_2 \in \mathbb{Z}[[t]]$. Hence, the initial coefficients of the series α_1 and α_2 may be computed using modular arithmetic. The same holds for $\zeta M_y(0, \zeta)$, which then allows to reconstruct the coefficients of B_1 and B_2 . Then we were able to guess polynomial equations satisfied by R_0, R_1, B_1 , and B_2 using the `gfun` package in MAPLE [40]. We refer for full details to the accompanying MAPLE worksheet.

Of course, the equations obtained for R_0 and R_1 coincide with those that we derived from the system of the previous subsection. Details on the corresponding equations are shown in Table 6. We note that the degree 24 equation for B_2 is in fact a degree 12 equation for B_2^2 .

Generating function	Degree in GF	Degree in t	Number of terms
R_0	24	36	323
R_1	24	36	623
B_1	12	24	229
B_2	24	60	477

TABLE 6. Properties of the guessed polynomial equations for the four main unknown series R_0, R_1, B_1, B_2 .

We now have to check that the guessed series satisfy the system obtained in the previous subsection. This turns out to be much easier once the algebraic structure of these series is elucidated. We explain in Appendix A how this can be done. We believe that this can be of interest to readers handling algebraic series of large degree. After this step, one obtains expressions for R_0, R_1, B_1 , and B_2 in terms of the series v and w of Section 4.1. We have not tried a direct check of the system based on the four guessed equations of Table 6.

Proposition 6.1. *Let $u, v, w \in \mathbb{Q}[[t]]$ be the series defined in Section 4.1 by (27), (29), and (31), respectively. Then the four series that occur in the equation in one catalytic variable defining $S(x)$ are:*

$$R_0 = \frac{t}{2} \left(\frac{w(1+2v)}{1+4v-2v^3} - 1 \right),$$

$$R_1 = \frac{1}{6} \left(1 + 2t + \frac{(1-2t)(1+2v)(16v^6 + 24v^5 + 7v^4 - 24v^3 - 30v^2 - 10v - 1)}{w(v^4 + 8v^3 + 6v^2 + 2v + 1)(1+4v-2v^3)} \right),$$

$$B_1 = \frac{3v^2(1-8t)(1+4v+v^2)(v^2-1)(1+2v)}{2(1-3v^2-4v^3)^3(1+4v-2v^3)},$$

$$B_2 = \frac{(1+2v)(1-2t)}{2w(v^4 + 8v^3 + 6v^2 + 2v + 1)^2(2v^3 - 4v - 1)} (4tv^{12} + 68tv^{11} + 16(22t+1)v^{10})$$

$$\begin{aligned}
 &+12(67t+2)v^9 + 5(192t-5)v^8 + 8(61t-10)v^7 - (286t+41)v^6 - 2(394t-33)v^5 \\
 &- (738t-113)v^4 - 4(97t-17)v^3 - (126t-19)v^2 - 2(12t-1)v - 2t.
 \end{aligned}$$

Proof. It suffices to check that the four series above satisfy the initial conditions $R_0 = \mathcal{O}(t^3)$, $R_1 = \mathcal{O}(t^2)$, $B_1 = \mathcal{O}(t^2)$, $B_2 = \mathcal{O}(t^2)$, and the system of 4 polynomial equations established in Section 6.1, the properties of which are summarized in Table 5. The first point is straightforward. Then we take each equation of the system in turn, replace the four unknown series by the above expressions, take the numerator of the resulting equation (which is a polynomial in t , v , and w), and reduce it first modulo Equation (31) defining w over $\mathbb{Q}(v)$. In each case, we note that the remainder does not involve w , an encouraging sign. Then we reduce further modulo Equation (30) defining v over $\mathbb{Q}(t)$. In each case, we find zero, so that the system holds for the above values of R_0 , R_1 , B_1 , and B_2 . This completes the proof. \square

Note that this proves in particular the announced expressions (32) and (33) for the series $M_{0,0} = R_0/t^2$ and $M_{1,0} = R_1/t^2$; see (44). We claim that at this stage, we have proved the algebraicity of the series $P(x, y)$ and $M(x, y)$. Recall that by definition, walks in \mathcal{C} ending in the first quadrant (resp. at negative abscissa) have generating functions

$$\frac{1}{3}Q(x, y) + P(x, y), \quad \left(\text{resp. } -\frac{1}{3}\bar{x}^2Q(\bar{x}, y) + \bar{x}M(\bar{x}, y) \right).$$

Corollary 6.2. *The series $P(x, y)$ and $M(x, y)$ are algebraic over $\mathbb{Q}(t, x, y)$.*

Proof. We work our way backwards starting from the 4 univariate algebraic series of Proposition 6.1. Since $S(x) = txM(0, x)$ satisfies a cubic equation $\text{Pol}(S(x), R_0, R_1, B_1, B_2, t, x) = 0$, where the polynomial Pol has non-zero leading coefficient in its first variable, $S(x)$ and $M(0, x)$ are algebraic of degree at most 72. We will see that this bound is tight. It then follows from (45) that $R(x) = t^2M(x, 0)$ is algebraic as well. We now return to (26), which expresses $M(x, y)$: since $M_{0,0} = R_0/t^2$ and $M_{1,0} = R_1/t^2$, we conclude that $M(x, y)$ is algebraic. We finally use the relation (25) between $P(x, y)$ and $M(x, y)$ to conclude that $P(x, y)$ is algebraic. \square

In the next subsection, we determine the degree of all algebraic series of interest, and give closed form expressions for $S(x)$ and $R(x)$ in terms of the already defined series v and w , and a “simple” cubic extension of $\mathbb{Q}(t, v, x)$.

6.3. BACK TO $S(x)$ AND $R(x)$

In this subsection, we prove that $S(x)$ and $R(x)$ belong to the same cubic extension of $\mathbb{Q}(t, w, x)$, and describe this extension in (reasonably) compact terms. We give two descriptions of this extension by rational parametrizations (in fact, a third one hides in Appendix B). Remarkably, they define cubic extensions of $\mathbb{Q}(t, v, x)$ rather than $\mathbb{Q}(t, w, x)$. The first one is in terms of the variable $\tilde{y} := t(x + \bar{x} + 1)/(1 - 2t)$ and involves v but not t . The second one, however, involves the original variable x , and now t and v .

More precisely, let $U_1 \equiv U_1(x)$ be the unique series of the form $U_1 = xt^2 + \mathcal{O}(t^3)$ satisfying

$$\tilde{y} = \frac{\kappa}{U_1} \frac{N(U_1)}{N(r_1/U_1)}, \tag{59}$$

where

$$N(U) = U + v^2w^2 - \frac{v^4w^2(v^2-1)(v^2+v+1)}{U},$$

with $w^2 = 1 + 4v - 4v^3 - 4v^4$ as before, and

$$\begin{aligned}\kappa &= \frac{(v^3 - 3v - 1)^2 (v^2 + v + 1) v^2}{v^4 + 8v^3 + 6v^2 + 2v + 1}, \\ r_1 &= -v^3 w^2 (v^2 + v + 1) (v^3 - 3v - 1).\end{aligned}\tag{60}$$

Let $U_2 \equiv U_2(x)$ be the unique series $U_2 = \bar{x} + \mathcal{O}(t)$ that satisfies

$$x = U_2 \frac{M(U_2)}{M(1/U_2)},\tag{61}$$

where $M(U) = 1/U + \alpha + \beta U$, with

$$\alpha = v - \frac{(v^3 - 3v - 1)\beta}{v^2 + v + 1}$$

and

$$\beta = \frac{(v^2 + v + 1) \left((2v^5 + 15v^4 + 20v^3 + 16v^2 + 6v + 1)t + v(v^3 - 3v - 1) \right)}{t(v^4 + 8v^3 + 6v^2 + 2v + 1)(2v^3 + 3v^2 + 6v + 1)}.$$

The series U_1 and U_2 generate the same cubic extension of $\mathbb{Q}(t, v, x)$. In particular,

$$\frac{1}{U_1} = a \left(U_2 + \frac{1}{U_2} + \frac{v^2 + 4v + 1}{v^2 + v + 1} \right)\tag{62}$$

with

$$a = \frac{\beta t (v^4 + 8v^3 + 6v^2 + 2v + 1)}{v^3(1 - 2t)(v^2 - 1)(v^2 + v + 1)(v^3 - 3v - 1)}.$$

One can also express U_2 as an element of $\mathbb{Q}(t, v, x, U_1)$ by combining (61) and (62). Finally, one can check that U_1 and U_2 have degree $36 = 12 \times 3$ over $\mathbb{Q}(t, x)$. Therefore, we have $\mathbb{Q}(t, v, x, U_1) = \mathbb{Q}(t, x, U_1) = \mathbb{Q}(t, x, U_2)$.

Proposition 6.3. *Let v and w be the series of $\mathbb{Q}[[t]]$ defined by (29) and (31). Let $U_1(x)$ and $U_2(x)$ be defined above. The series $R(x) = t^2 M(x, 0)$ and $S(x) = txM(0, x)$ are algebraic of degree 72 over $\mathbb{Q}(x, t)$ and belong to $\mathbb{Q}(t, x, w, U_1) = \mathbb{Q}(t, x, w, U_2)$. More precisely, the series*

$$\tilde{S}(x) = \frac{1}{x - \bar{x}} \left((x + 1 + \bar{x})S(x) - \frac{R_0}{t} + \frac{2x}{3} + \frac{\bar{x}}{3} \right)\tag{63}$$

and

$$\hat{R}(x) = 3(x + 1 + \bar{x})R(x) + 3R_1 + \frac{1 - t\bar{x}(x + \bar{x})(x + 1)^2}{t(x + 1 + \bar{x})}(R_0 - t\bar{x}) + t(1 + \bar{x}^2)\tag{64}$$

belong respectively to $\mathbb{Q}(t, x, U_1(x))$ and $w\mathbb{Q}(t, x, U_1(x))$. In particular,

$$\tilde{S}(x) + \frac{1}{3} = -\frac{v^2 w^2 (1 + 2v) (v^2 + 4v + 1)^2}{2v^3 - 4v - 1} \frac{1}{D(U_1)D(r_1/U_1)}$$

where r_1 is given by (60) and

$$D(U) = (v + 1)U + vw^2(v^2 - 1) + (v - 1)\frac{r_1}{U}.$$

Recall that R_0 and R_1 lie in $\mathbb{Q}(t, w)$, and are given by Proposition 6.1.

Proof. We return to the cubic equation that defines $S(x)$, written in the form (54) in terms of z and $\tilde{S}(x)$ and we replace R_0, R_1, B_1, B_2 by their expressions in terms of t, v , and w . Then we observe that only even powers of w occur: hence, using the defining equation (31) of w , we obtain a cubic equation for $\tilde{S}(x)$ involving only t, v , and of course the variable z defined by (53). This equation has degree 2 in z . We lower the degree in t to 1 using the minimal polynomial (30) of v . Now the coefficient of z^2 does not involve t , the coefficient of z^1 is a multiple of $(1 - 2t)$, and the coefficient of z^0 is a multiple of $(1 - 8t)$. But observe that the minimal equation of v can also be written as

$$\frac{1 - 8t}{(1 - 2t)^2} = \frac{(v^2 + 4v + 1)(4v^3 + 3v^2 - 1)^3}{(4v^4 + 4v^3 - 4v - 1)(v^4 + 8v^3 + 6v^2 + 2v + 1)^2}.$$

This gives a cubic equation for $\tilde{S}(x)$, with coefficients in $\mathbb{Q}(v, \tilde{z})$ where

$$\tilde{z} = \frac{z}{1 - 2t} = \frac{yt}{1 - 2t} + \frac{1 + t}{y(1 - 2t)}, \quad (65)$$

where as before $y = x + 1 + \bar{x}$. It is remarkable that this equation does *not* involve t . Its genus (in \tilde{z} and \tilde{S}) is found to be zero and thus this equation admits a rational parametrization. We give one in Appendix B (see (78)), in terms of a series denoted by $U_0(x)$, for which we have

$$\tilde{S}(x) + \frac{1}{3} = -\frac{v^2(v^2 - 1)(2v + 1)(v^2 + 4v + 1)^2}{(2v^3 - 4v - 1)(w^2 U_0^2 + v^2(v^2 - 1)(2v + 1)(2v^3 + 3v^2 + 6v + 1))}.$$

But it may be better to parametrize our extensions in terms of x than \tilde{z} . Let us first get back to $y = x + 1 + \bar{x}$, or rather to $\tilde{y} = yt/(1 - 2t)$, and observe that \tilde{z} can be written as

$$\tilde{z} = \tilde{y} + \frac{t(1 + t)}{(1 - 2t)^2} \frac{1}{\tilde{y}} = \tilde{y} + \frac{q}{\tilde{y}} \quad (66)$$

where

$$q = -\frac{v(v^2 + v + 1)(v^3 - 3v - 1)^3}{w^2(v^4 + 8v^3 + 6v^2 + 2v + 1)^2}.$$

This means that $\tilde{S}(x)$ also satisfies a cubic equation with coefficients in $\mathbb{Q}(\tilde{y}, v)$, again not involving t . This equation is also found to have genus 0 (in \tilde{y} and \tilde{S}) and can be parametrized rationally by introducing the series U_1 defined by (59). Indeed, if, in the equation relating \tilde{y} and \tilde{S} , we replace \tilde{y} by its expression in terms of U_1 , the equation factors into a linear term in \tilde{S} , and a quadratic one. Provided we choose the correct determination of U_1 , given by $U_1 = xt^2 + \mathcal{O}(t^3)$, then the term that vanishes is the linear one, and this gives the expression of \tilde{S} stated in the proposition. Observe that replacing U_1 by r_1/U_1 in (59) replaces \tilde{y} by q/\tilde{y} (because $\kappa^2 = qr_1$), and thus leaves \tilde{z} unchanged; see (66). Analogously, the series $U_0(x)$ that parametrizes the equation in \tilde{z} and \tilde{S} (see Appendix B) is invariant by this transformation, and reads

$$U_0 = \frac{1 - v^2}{w^2} \left(U_1 + v^2 w^2 + \frac{r_1}{U_1} \right).$$

One can actually go even further, as the equation that relates the original variable x and the series U_1 (now with coefficients in $\mathbb{Q}(t, v)$) also has genus zero. It can be parametrized by introducing the series U_2 defined by (61). Indeed, if we replace, in the equation relating x

and U_1 , the variable x by its expression in terms of U_2 , we observe again a factorization, which leads to (62) once the correct determination of U_2 is chosen.

One readily checks that U_1 and U_2 (and U_0 as well) have degree 36 over $\mathbb{Q}(t, x)$.

Thus $\tilde{S}(x)$ belongs to $\mathbb{Q}(t, x, U_1) = \mathbb{Q}(t, x, U_2)$, while $S(x)$, which involves R_0 and hence w , belongs to $\mathbb{Q}(t, w, x, U_1) = \mathbb{Q}(t, w, x, U_2)$ and has degree at most 72. To prove that this bound is tight, one can eliminate w and v in the equation defining $S(x)$. It is enough to do it for $x = 2$, for instance, as we find that $S(2)$ has degree 72.

We now wish to determine the series $R(x) = t^2 M(x, 0)$, which is expressed in terms of $S(x)$ and $S(\bar{x})$ in (47). Equivalently,

$$\hat{R}(x) = \frac{(x - \bar{x})\sqrt{\Delta(x)}}{x + 1 + \bar{x}} \left(\tilde{S}(x) + 2\tilde{S}(\bar{x}) \right).$$

We could of course eliminate $\tilde{S}(x)$ and $\tilde{S}(\bar{x})$ to determine a polynomial equation satisfied by $\hat{R}(x)$ over $\mathbb{Q}(t, x, v)$, but there is an algebraic structure in the above equation, which will save us these calculations. Let us denote $\text{Pol}(s) = s^3 + ps + q$ the monic minimal polynomial of $\tilde{S}(x)$ over $\mathbb{Q}(\tilde{z}, v)$. One of its root is of course $s_1 = \tilde{S}(x)$, another one is $s_2 = \tilde{S}(\bar{x})$ (because \tilde{z} is invariant under $x \mapsto \bar{x}$) and the third one is $s_3 = -\tilde{S}(x) - \tilde{S}(\bar{x})$ (because there is no quadratic term in Pol). Hence $\tilde{S}(x) + 2\tilde{S}(\bar{x}) = s_2 - s_3$. It is not hard to see that, if we denote by $\delta(\tilde{z}) = -4p^3 - 27q^2$ the discriminant of $\text{Pol}(s)$, and choose its square root so that

$$\sqrt{\delta(\tilde{z})} = (s_1 - s_2)(s_1 - s_3)(s_2 - s_3),$$

then

$$\sqrt{\delta(\tilde{z})}(s_2 - s_3) = -6ps_1^2 + 9qs_1 - 4p^2.$$

Hence

$$\hat{R}(x) = \frac{(x - \bar{x})}{x + 1 + \bar{x}} \sqrt{\frac{\Delta(x)}{\delta(\tilde{z})}} \left(9q\tilde{S}(x) - 6p\tilde{S}(x)^2 - 4p^2 \right), \quad (67)$$

for some $p, q \in \mathbb{Q}(\tilde{z}, v)$. Hence the proof of the proposition will be complete if we prove that $\Delta(x)/(w^2\delta(\tilde{z}))$ is a square in $\mathbb{Q}(t, v, x)$. After several reductions, described in our MAPLE session, we obtain

$$\sqrt{\frac{\Delta(x)}{\delta(\tilde{z})}} = \frac{w\Delta(x)^2(v^4 + 8v^3 + 6v^2 + 2v + 1)^2(2v^3 - 4v - 1)^3(x - \bar{x})^3}{y^2(ty^2 - t - 1)(v^2 + 4v + 1)^2(v^2 - 1)(2v + 1)(1 - 2t)^2P(\tilde{z})} \quad (68)$$

where we denote as before $y = x + 1 + \bar{x}$ and

$$\begin{aligned} P(\tilde{z}) = & -w^2\tilde{z}^2 (v^4 + 8v^3 + 6v^2 + 2v + 1)^2 \\ & - (v - 1) (8v^7 + 16v^6 + 40v^5 + 72v^4 + 85v^3 + 53v^2 + 13v + 1) (v^4 + 8v^3 + 6v^2 + 2v + 1) \tilde{z} \\ & + 2v (2v^{11} + 2v^{10} + 12v^9 + 18v^8 + 23v^7 + 22v^6 + 5v^5 - 29v^4 - 57v^3 - 40v^2 - 11v - 1). \end{aligned}$$

From this point on, we can combine (67) and (68) with the various parametrizations (by U_0 , U_1 , or U_2) introduced above to write closed-form expressions for $\hat{R}(x)$. We give one in Appendix B in terms of U_0 ; see (80). The degree of $R(x)$ is clearly 72 at most. We determine it at $x = 2$ by elimination of U_0 , w , and v , and find it to be 72; hence the bound is tight. \square

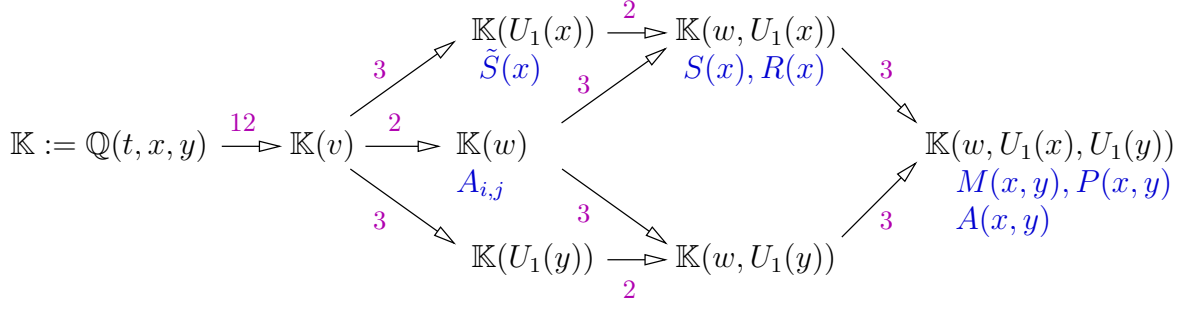


FIGURE 2. Structure of the various fields involved in the solution of king walks in \mathcal{C} . We have indicated the degrees, and where the main series lie.

End of the proof of Theorem 4.1. We have proved all statements of this theorem, except for the degrees of the trivariate generating functions $M(x, y)$, $P(x, y)$, and $A(x, y)$. It is clear from (26), (25), and (23) that they belong to $\mathbb{K}(w, U_1(x), U_1(y))$, where $\mathbb{K} = \mathbb{Q}(t, x, y)$ and hence that they have degree at most $72 \times 3 = 216$ over \mathbb{K} . We check (by specializing x, y , and even t to real values where all series converge, like $x = 3, y = 2$, and $t = 1/100$) that there is no unexpected degree reduction. \square

We get the final picture of the algebraic extensions shown in Figure 2.

6.4. SOME INTERESTING UNIVARIATE SERIES

In this subsection we examine various univariate series of interest, like those that are involved in the enumeration of all walks in \mathcal{C} , or of walks ending on the boundaries of \mathcal{C} . We also prove the results of Proposition 4.2 dealing with walks ending at a specific point, and the asymptotic results of Corollary 4.3.

Proposition 6.4. *The series $R(1) = t^2 M(1, 0)$ is algebraic of degree 24 over $\mathbb{Q}(t)$ and belongs to $\mathbb{Q}(t, w)$. More precisely,*

$$R(1) + \frac{t}{3} = -\frac{\text{num}}{3w(2v^3 - 4v - 1)(v^4 + 8v^3 + 6v^2 + 2v + 1)(2v^3 + 3v^2 + 6v + 1)},$$

where

$$\begin{aligned} \text{num} = & v(v+1)(2v^3 + 4v^2 + 5v + 1)(4v^6 + 3v^5 - 8v^4 - 6v^3 + 12v^2 + 11v + 2) \\ & + (96v^{10} + 272v^9 + 446v^8 + 384v^7 + 3v^6 - 464v^5 - 553v^4 - 298v^3 - 87v^2 - 14v - 1)t. \end{aligned}$$

The series $S(1) = tM(0, 1)$ is algebraic of degree 48 over $\mathbb{Q}(t)$ and belongs to a quadratic extension of $\mathbb{Q}(t, w)$. More precisely,

$$S(1) + \frac{1}{2} = w\tilde{w}, \tag{69}$$

where $\tilde{w} = 1/2 + \mathcal{O}(t)$ has degree 2 over $\mathbb{Q}(t, v)$, and satisfies (81) (in Appendix C).

The series $M(1, 1)$ and $P(1, 1)$ are algebraic of degree 48 and belong to $\mathbb{Q}(t, w, \tilde{w})$.

The series $A(1, 1)$ and $A_{0,0}$ are algebraic of degree 24 and belong to $\mathbb{Q}(t, w)$. More precisely,

$$A(1, 1) + \frac{1}{3t} = -\frac{w \times \text{num}'}{3t(1-2t)(4v^3 + 3v^2 - 1)^2(2v^3 - 4v - 1)(2v^3 + 3v^2 + 6v + 1)},$$

with

$$\begin{aligned} \text{num}' &= 2(4v^3 + 3v^2 + 4v + 1)(4v^3 + 3v^2 - 1)^2t \\ &\quad + (v + 1)(16v^9 + 72v^8 + 94v^7 + 86v^6 + 3v^5 + 61v^4 + 68v^3 + 24v^2 + 7v + 1), \end{aligned}$$

while

$$A_{0,0} = P_{0,0} = \frac{2R_1}{t^2},$$

where R_1 is given in Proposition 6.1.

Proof. We begin with the series $S(1)$: we set $x = 1$ in the cubic equation (52) satisfied by $S(x)$, and observe that the equation factors. The factor that vanishes is quadratic in $S(1)$. (The fact that $S(1)$ is quadratic can also be seen from (49).) Then we replace R_0, R_1, B_1, B_2 by their expressions from Proposition 6.1. We then reduce the degree of t and w in this equation by taking remainders (in t and w) modulo (30) and (31). The coefficient of w in this equation has a factor $(1 + 2S(1))$, which suggests to write (69). Now \tilde{w} is quadratic over $\mathbb{Q}(t, v)$, but is found not to belong to $\mathbb{Q}(t, w)$. Its minimal equation over $\mathbb{Q}(t, v)$ can be written as (81).

Now in order to determine $R(1)$, we set $x = 1$ in the square of (45), and perform similar reductions as for $S(1)$. For $M(1, 1)$, we use the defining equation of $M(x, y)$ (see (26)), of course at $x = y = 1$, and obtain

$$(1 - 8t)M(1, 1) = \frac{1}{3} - \frac{R_1 + 3R(1)}{2t} + (1 - 8t)\frac{S(1)}{2t},$$

from which the properties stated in the proposition easily follow. We then combine the above expression of $M(1, 1)$ with (25) to obtain

$$(1 - 8t)P(1, 1) = \frac{2}{3} - \frac{R_1 + 3R(1)}{t} - (1 - 8t)\frac{S(1)}{t}.$$

Since $A(x, y)$ is given by (38), we then find

$$(1 - 8t)A(1, 1) = \frac{4}{3} - 2\frac{R_1 + 3R(1)}{t}.$$

We observe that the series $S(1)$ is not involved in this expression, and therefore $A(1, 1)$ has degree 24 only. Finally, we obtain from (25) that $A_{0,0} = P_{0,0} = 2M_{1,0} = 2R_1/t^2$, which thus also has degree 24. \square

Let us now prove Proposition 4.2, which deals with walks ending at a specific point.

Proof of Proposition 4.2. According to (24) and (25), it suffices to prove that all series $M_{i,j}$ belong to $\mathbb{Q}(t, w)$.

Let us first prove this when $i = 0$ or $j = 0$, that is, for the coefficients of the series $S(x) = txM(0, x)$ and $R(x) = t^2M(x, 0)$. For $S(x)$, we write $S(x) = xT(x)$, and observe that the cubic equation (52) satisfied by $S(x)$, with coefficients in $\mathbb{Q}(t, x, R_0, R_1, B_1, B_2)$, reads

$$3t(R_0^2 + R_0t + t^2)(T(x) - R_0/t) = x\widetilde{\text{Pol}}(t, x, T(x), R_0, R_1, B_1, B_2),$$

for some polynomial $\widetilde{\text{Pol}}$. This implies that $T_0 = S_1 = R_0/t$, as we already know from the definitions of $R(x)$ and $S(x)$, and then, by induction on i , that the series S_i belong to $\mathbb{Q}(t, w)$ (because the series R_0, R_1, B_1, B_2 do). It then follows that the coefficients of $R(x)$ also belong to this field, using (64), (67), and (68).

We finally return to the equation (26) that defines $M(x, y)$. It reads $K(x, y)M(x, y) = F(t, x, y)$, where $F(t, x, y)$ is a Laurent series in x and y , having coefficients in $\mathbb{Q}(t, w)$ as we have just proved. We extract the coefficient of $x^i y^j$ in this equation, for $i, j \geq 0$, and thus obtain a linear expression $tM_{i+1, j+1}$ in terms of series $M_{k, \ell}$, where $k \leq i+1$ and $\ell \leq j+1$, one equality being strict, and series of $\mathbb{Q}(t, w)$. We then conclude by induction on $i+j$.

The fact that $C_{i, j}$ is transcendental (except for $i = -1$ or $j = -1$), follows from the fact that $Q_{i, j}$ is transcendental for $i, j \geq 0$, because its coefficients grow like $8^n n^{-3}$, which contradicts algebraicity. \square

We finally prove the asymptotic results of Corollary 4.3.

Proof of Corollary 4.3. We apply here the principles of the *singularity analysis* of algebraic series [24, Sec. VII.7]. The series u defined by (27) is found to have radius of convergence $1/8$, and a unique singularity of minimal modulus, at $t = 1/8$. Moreover, as t approaches $1/8$ from below, u has the following Puiseux expansion:

$$u = \frac{1}{3} - \frac{2}{9} 6^{1/3} (1-8t)^{1/3} + \frac{1}{27} 6^{2/3} (1-8t)^{2/3} + \frac{1}{27} (1-8t) + \mathcal{O}\left((1-8t)^{4/3}\right).$$

Then the series v defined in (29), seen as a series in u , has a radius of convergence larger than $u_c := 1/3$, and is thus analytic at u_c . At this point it attains the value $v_c \approx 0.455\dots$, which is the only real root of $4v^3 + 3v^2 - 1$. As t approaches $1/8$ from below, one finds

$$v = v_c - \frac{1}{3} v_c (1+2v_c) 6^{1/3} (1-8t)^{1/3} + \frac{(8v_c^2 + 11v_c + 2)}{54} (1-8t) + \mathcal{O}\left((1-8t)^{4/3}\right).$$

Finally, the series w , seen as a series in v , is analytic at v_c , where it is equal to $w_c := \sqrt{3v_c^2 + 12v_c + 3}/2$. As t approaches $1/8$ from below, one finds

$$w = w_c - \frac{2}{9} 6^{2/3} v_c w_c (1+2v_c) (1-8t)^{2/3} + \mathcal{O}\left((1-8t)\right).$$

More terms of the singular expansions of these three series are available in our MAPLE session. We plug these expansions in the expressions of $A(1, 1)$ and $A_{0,0}$ given in the previous proposition and obtain

$$A(1, 1) = -\frac{2^5 6^{1/3} w_c (28v_c^2 + 61v_c - 86)}{3^3 101 (1-8t)^{2/3}} + cst + \mathcal{O}\left((1-8t)^{1/3}\right), \quad (70)$$

$$\begin{aligned} A_{0,0} = cst - & \frac{2^9 6^{2/3} w_c (6716 v_c^2 + 2165 v_c - 1582) (1-8t)^{2/3}}{3^4 101^2} + cst (1-8t) \\ & + \frac{2^8 6^{1/3} w_c (344660 v_c^2 + 688535 v_c - 718546) (1-8t)^{4/3}}{3^5 101^3} + \mathcal{O}\left((1-8t)^{5/3}\right), \end{aligned}$$

where each symbol cst stands for a real constant that may vary from place to place, but has no implication on the asymptotic behaviour of the coefficients of our series. The series we are really interested in are

$$C(1, 1) = A(1, 1) - \frac{1}{3} Q(1, 1)$$

and

$$C_{0,0} = A_{0,0} + \frac{1}{3} Q_{0,0}.$$

Recall from [24, Thm. VI.1] that for $\alpha \notin \{-1, -2, \dots\}$, it holds that

$$[t^n](1-8t)^{-\alpha-1} = \frac{8^n n^\alpha}{\Gamma(\alpha+1)} + \mathcal{O}(8^n n^{\alpha-1}).$$

In particular, the n th coefficient in $A(1, 1)$ grows like $8^n n^{-1/3}$, while the estimate corresponding to the remainder is in $8^n n^{-4/3}$. Moreover, it is proved in [5, 34] that

$$[t^n]Q(1, 1) = \frac{8}{3\pi} \frac{8^n}{n} + \mathcal{O}\left(\frac{8^n}{n^2}\right),$$

so that $Q(1, 1)$ contributes to the second order term in the asymptotic behaviour of the number $c(n)$ of n -step walks in \mathcal{C} . We then compute the minimal polynomial over \mathbb{Q} of the constant occurring in the first term of (70), and put the two contributions together to obtain the first part of the corollary.

Now consider the series $C_{0,0}$. Since the coefficient of t^n in $Q_{0,0}$ grows like $8^n/n^3$ (see [5, 18]), the first two terms in the expansion of $c_{0,0}(n)$ come from the above expansion of $A_{0,0}$, and this yields the second part of the corollary. \square

7. COMBINATORIAL PROOFS OF SOME IDENTITIES ON SQUARE LATTICE WALKS

As already observed in [9, Sec. 7.1] for the simple and diagonal models, the first two equations of Theorem 4.1, combined with the x/y -symmetry of our step set, imply that for $i, j \geq 0$,

$$C_{i,j} = Q_{i,j} + C_{-i-2,j} + C_{i,-j-2}.$$

As suggested in [9], this can be proved using the reflection principle. This is what we do in this section. Further, we establish identities of this type for more general starting points and endpoints, and all Weyl models of Table 1. We begin in Section 7.1 with the four models having a group of order 4, and develop in Section 7.2 a general setting.

7.1. A GROUP OF ORDER 4: SIMPLE, DIAGONAL, KING, AND DIABOLO WALKS

As shown in Table 1 there are four step sets associated with the Weyl group $A_1 \times A_1$, of order 4. Mimicking the action of this group on \mathbb{R}^2 , we decompose the three-quarter plane \mathcal{C} into three disjoint parts:

$$\begin{aligned} \mathcal{Q} &= \{(i, j) : i \geq 0 \text{ and } j \geq 0\} && \text{(the first quadrant),} \\ \mathcal{L} &= \{(i, j) : i \leq -1 \text{ and } j \geq 0\} && \text{(the left quadrant),} \\ \mathcal{B} &= \{(i, j) : i \geq 0 \text{ and } j \leq -1\} && \text{(the bottom quadrant).} \end{aligned}$$

As before, let $C_{i,j}$ (resp. $Q_{i,j}$) be the number of walks confined to \mathcal{C} (resp. \mathcal{Q}) ending at (i, j) . More generally, for any starting point (a, b) we write $Q_{i,j}^{a,b}$ (resp. $C_{i,j}^{a,b}$) for the length generating function of walks confined to \mathcal{Q} (resp. \mathcal{C}), starting from (a, b) and ending at (i, j) . A step set \mathcal{S} is called *vertically symmetric* (or *v-symmetric*) if for all $(i, j) \in \mathcal{S}$ one has $(-i, j) \in \mathcal{S}$; it is called *horizontally symmetric* (or *h-symmetric*) if for all $(i, j) \in \mathcal{S}$ one has $(i, -j) \in \mathcal{S}$. The four models that we consider in this subsection are the only v- and h-symmetric models among all small step models.

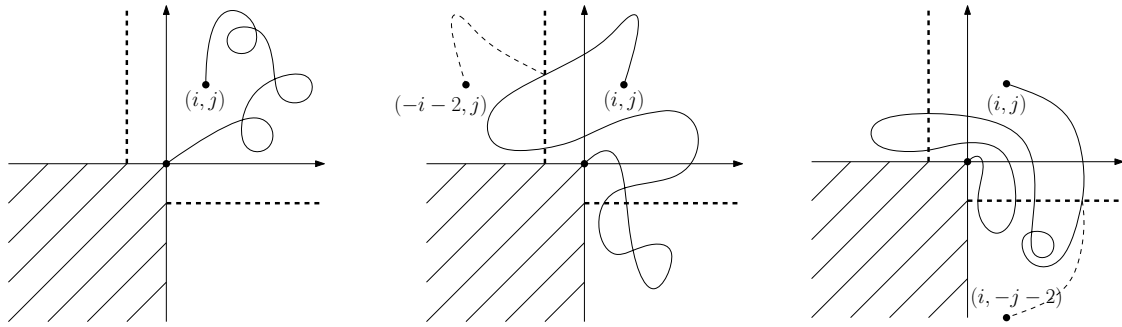


FIGURE 3. By the reflection principle, walks in the three-quarter plane \mathcal{C} from $(0,0)$ to (i,j) with $i, j \geq 0$ are in bijection with the union of three sets of walks: walks in \mathcal{C} ending at $(-i-2, j)$, walks in \mathcal{C} ending at $(i, -j-2)$, and walks staying completely in the first quadrant \mathcal{Q} , ending at (i, j) . For more such identities see Proposition 7.1.

Proposition 7.1. *Let \mathcal{S} be one of the four v - and h -symmetric small step models, and let (a, b) be a starting point in \mathcal{C} . For $(i, j) \in \mathcal{Q}$ we have*

$$C_{i,j}^{a,b} = C_{-i-2,j}^{a,b} + C_{i,-j-2}^{a,b} + \begin{cases} Q_{i,j}^{a,b} & \text{if } a, b \geq 0, \\ 0 & \text{if } a = -1 \text{ or } b = -1, \\ -Q_{i,j}^{-a-2,b} & \text{if } a < -1, \\ -Q_{i,j}^{a,-b-2} & \text{if } b < -1. \end{cases}$$

Furthermore, there exists an explicit bijection proving each of these identities.

Proof. The proof idea is to suitably reflect the walks along the lines $x = -1$ and $y = -1$ which directly results in bijections for the claimed identities. We fix an endpoint $(i, j) \in \mathcal{Q}$.

First, for a starting point $(a, b) \in \mathcal{Q}$ we partition the walks confined to \mathcal{C} into three classes as shown in Figure 3: a walk either always stays in the first quadrant and is therefore counted by $Q_{i,j}^{a,b}$, or it leaves the first quadrant. In the latter case it either touches the line $x = -1$ or $y = -1$. We cut the walk at the last point (k, ℓ) where this happens and reflect the second part of the walk, going from (k, ℓ) to (i, j) , along this line. As \mathcal{S} is v - and h -symmetric we get a walk in \mathcal{C} with steps in \mathcal{S} ending either at $(-i-2, j)$ or $(i, -j-2)$. The reverse bijection is analogous. One key point here is that a walk from (a, b) to $(-i-2, j)$ (say) will necessarily touch the line $x = -1$, and will touch it after any visit to the line $y = -1$.

Second, if the starting point (a, b) is on the line $x = -1$ or $y = -1$ then the same argument applies, with $Q_{i,j}^{a,b} = 0$ because no path can be entirely in the first quadrant \mathcal{Q} .

Third, if $a < -1$, the path starts left of the line $x = -1$, and thus cannot be contained in the first quadrant either. Moreover, a difficulty arises when defining the reverse construction: a walk starting from (a, b) and ending at $(-i-2, j)$ may not touch the line $x = -1$, and thus cannot be reflected along this line (there is no such problem with walks ending at $(i, -j-2)$). But these walks are in essence walks in a quadrant: reflecting them along the line $x = -1$ gives walks from $(-a-2, b)$ to (i, j) confined to the first quadrant \mathcal{Q} .

Fourth, for $b < -1$ the reasoning is analogous. \square

The above proposition implies in particular the three formulas given in [9, Sec. 7.1]: for $i, j \geq 0$ we have for any v- and h-symmetric step set and the three starting points $(0, 0)$, $(-1, 0)$, and $(-2, 0)$:

$$\begin{aligned} C_{i,j}^{0,0} &= C_{-i-2,j}^{0,0} + C_{i,-j-2}^{0,0} + Q_{i,j}^{0,0}, \\ C_{i,j}^{-1,0} &= C_{-i-2,j}^{-1,0} + C_{i,-j-2}^{-1,0}, \\ C_{i,j}^{-2,0} &= C_{-i-2,j}^{-2,0} + C_{i,-j-2}^{-2,0} - Q_{i,j}^{0,0}. \end{aligned}$$

Let us reformulate Proposition 7.1 in terms of trivariate (rather than univariate) generating functions. For $(a, b) \in \mathcal{C}$, let $C^{a,b}(x, y)$ denote the generating function of walks in \mathcal{C} that start from (a, b) :

$$C^{a,b}(x, y) = \sum_{(i,j) \in \mathcal{C}} C_{i,j}^{a,b} x^i y^j. \quad (71)$$

We also define (uniquely) series $P^{a,b}(x, y)$, $L^{a,b}(x, y)$, and $B^{a,b}(x, y)$ in $\mathbb{Q}[x, y][[t]]$ by

$$C^{a,b}(x, y) = P^{a,b}(x, y) + \bar{x}L^{a,b}(\bar{x}, y) + \bar{y}B^{a,b}(x, \bar{y}). \quad (72)$$

Then Proposition 7.1 can be reformulated as follows.

Proposition 7.2. *Let \mathcal{S} be one of the four v- and h-symmetric small step models. For $(a, b) \in \mathcal{C}$, the above defined series are related by*

$$\begin{aligned} P^{a,b}(x, y) &= \bar{x} \left(L^{a,b}(x, y) - L^{a,b}(0, y) \right) + \bar{y} \left(B^{a,b}(x, y) - B^{a,b}(x, 0) \right) \\ &+ \begin{cases} Q^{a,b}(x, y) & \text{if } a, b \geq 0, \\ 0 & \text{if } a = -1 \text{ or } b = -1, \\ -Q^{-a-2,b}(x, y) & \text{if } a < -1, \\ -Q^{a,-b-2}(x, y) & \text{if } b < -1. \end{cases} \end{aligned}$$

Proof. We multiply the identities of Proposition 7.1 by $x^i y^j$ and sum over all $i, j \geq 0$. \square

Now we will use these results to generalize Equation (25) to the four models under consideration. First, we define the generating function $A(x, y)$ as in Proposition 3.1, or equivalently by (35). It satisfies the following functional equation:

$$K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}H_-(x)A_{-,0}(\bar{x}) - t\bar{x}V_-(y)A_{0,-}(\bar{y}) - t\bar{x}\bar{y}A_{0,0}\mathbb{1}_{(-1,-1) \in \mathcal{S}}. \quad (73)$$

Hence $A(x, y) = \sum_{(i,j) \in \mathcal{C}} A_{i,j} x^i y^j$ can be interpreted as the generating function of walks starting from $(0, 0)$, $(-2, 0)$, or $(0, -2)$ with weights $2/3$, $1/3$, and $1/3$, respectively. In particular, if we now define the series $P(x, y), L(x, y), B(x, y) \in \mathbb{Q}[x, y][[t]]$ by

$$A(x, y) = P(x, y) + \bar{x}L(\bar{x}, y) + \bar{y}B(x, \bar{y}), \quad (74)$$

(observe that $B(x, \bar{y}) = L(\bar{y}, x)$ for an x/y -symmetric model), we have

$$P(x, y) = \frac{1}{3} (2P^{0,0}(x, y) + P^{-2,0}(x, y) + P^{0,-2}(x, y)), \quad (75)$$

and analogously for the series L and B . Then Proposition 7.2 implies the following generalization of Equation (25).

Corollary 7.3. *In the case of simple, diagonal, king, or diabolito walks, the power series $P(x, y)$, $L(x, y)$, and $B(x, y)$ defined in (74) obey the following identity*

$$P(x, y) = \bar{x}(L(x, y) - L(0, y)) + \bar{y}(B(x, y) - B(x, 0)).$$

Proof. Applying Proposition 7.2 for $(a, b) = (0, 0)$, $(-2, 0)$, and $(0, -2)$ to (75) makes all contributions of the series Q vanish and shows the claim. \square

Remark. Let us define the series $A(x, y)$ as in (73), but with weights w_0 , w_x , and w_y for walks starting from $(0, 0)$, $(-2, 0)$, and $(0, -2)$ respectively (rather than $2/3$, $1/3$, $1/3$). This only changes the initial term in (73), and Corollary 7.3 still holds, provided that $w_0 = w_x + w_y$.

In the next subsection, we give a higher level explanation of what happens here, more in the spirit of Gessel's and Zeilberger's proof of the reflection principle in [27], and thus obtain statements that are valid for all Weyl models.

7.2. A GENERAL RESULT FOR WEYL MODELS

We now consider one of the seven Weyl models \mathcal{S} of Table 1, with a group G of order $2d$, $d \in \{2, 3, 4\}$. Recall the definition of this group from Section 2.3, and the definition of the length $\ell(g)$ and sign ε_g of $g \in G$. This group acts on steps, seen as elements of the vector space \mathbb{Z}^2 : for $g \in G$, the corresponding element \dot{g} sends (i, j) to (k, ℓ) if $g(x^i y^j) = x^k y^\ell$ (recall that we have defined $g(F(x, y)) := F(g(x, y))$ for any rational function $F(x, y)$). By construction of G , the set of steps \mathcal{S} is invariant under this action of G . The group G also acts on points of the plane, that is, on the affine space \mathbb{Z}^2 , by $\dot{g}(a, b) = (c, d)$ where $x^c y^d = \bar{x}\bar{y} g(x^{a+1} y^{b+1})$ (the shift by xy is a bit unfortunate, and would be avoided by considering the positive quadrant $\{(a, b) : a > 0, b > 0\}$ rather than the non-negative quadrant \mathcal{Q}).

For $g \in G$, we denote $\mathcal{Q}_g = \dot{g}(\mathcal{Q})$. The $2d$ domains \mathcal{Q}_g , for $g \in G$, are disjoint; see Figure 4. For $(a, b) \in \mathcal{Q}$, the orbit of (a, b) under the affine action of G consists of $2d$ distinct points of the plane. In particular, the orbit sum $\text{OS}(x^{a+1} y^{b+1}) = \sum_g \varepsilon_g g(x^{a+1} y^{b+1})$ is non-zero. The union of the $2d$ domains \mathcal{Q}_g does not cover the whole plane. For the points (a, b) that are not in this union, the orbit of (a, b) under the affine action of G has cardinality less than $2d$, and in fact $\text{OS}(x^{a+1} y^{b+1}) = 0$. The complement of $\cup_g \mathcal{Q}_g$ is the union of d lines (also called *walls* to match the terminology of [27]), defined, for each $g \in G$ such that $\varepsilon_g = -1$ (i.e., $\ell(g)$ is odd), by $W_g = \{(a, b) \in \mathbb{Z}^2 : \dot{g}(a, b) = (a, b)\}$. The lines are dashed in our figures, and correspond to the reflection axes once the steps are straightened (as in Table 1). For instance, in all cases we have $W_\phi = \{(a, b) : a = -1\}$ and $W_\psi = \{(a, b) : b = -1\}$. Any two of the lines W_g intersect at the point $(-1, -1)$. An important property is that a walk that is not entirely contained in a domain \mathcal{Q}_g must touch one of these lines.

We adopt the same notation $C_{i,j}^{a,b}$ and $C^{a,b}(x, y)$ as in the previous subsection. The generalization of Proposition 7.1 reads as follows.

Proposition 7.4. *Let \mathcal{S} be one of the Weyl models of Table 1. Let $2d$ be the order of the associated group G . Let $\omega = \phi\psi\psi\cdots$ (with d generators) be the only element of length d in G . For any starting point $(a, b) \in \mathcal{C}$ and any endpoint $(i, j) \in \mathcal{Q}$, we have:*

$$\sum_{g \in G \setminus \{\omega\}} \varepsilon_g C_{\dot{g}(i,j)}^{a,b} = \begin{cases} 0 & \text{if } (a, b) \notin \cup_g \mathcal{Q}_g, \\ \varepsilon_h Q_{i,j}^{\dot{h}(a,b)} & \text{if } \dot{h}(a, b) \in \mathcal{Q} \text{ for } h \in G. \end{cases}$$

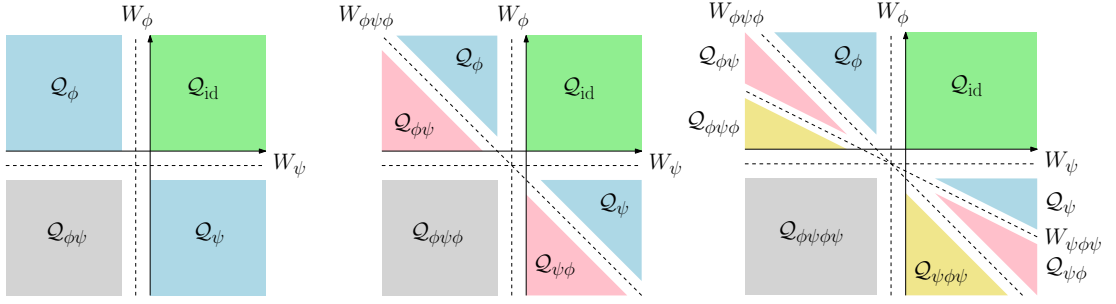


FIGURE 4. The $2d$ domains \mathcal{Q}_g for $g \in G$, where G has order $2d = 4, 6, 8$. They are separated by the d walls W_g , for $g \in G$ such that $\varepsilon_g = -1$.

Proof. Recall that for $(i, j) \in \mathcal{Q}$, the $2d$ endpoints $\dot{g}(i, j)$ are distinct. Hence the left-hand side of the above identity counts walks in \mathcal{C} , starting from (a, b) and ending at one of the $2d - 1$ points in the (affine) orbit of (i, j) that are not in the negative quadrant, with a sign that depends on the domain \mathcal{Q}_g where the walk ends. Observe that for each walk the parameters (a, b) , (i, j) , and g are uniquely determined. We will define a (partial) sign-reversing involution ι on these walks. The idea is sketched in Figure 5 for a group of order 6, that is, for tandem or double tandem walks.

Let w be such a walk. If it does not intersect any of the walls, then $\iota(w)$ is undefined. In this case, the starting point (a, b) of w must be in one (and exactly one) of the domains \mathcal{Q}_g , say in $\mathcal{Q}_{h^{-1}}$ (so that $\dot{h}(a, b) \in \mathcal{Q}$). Then the endpoint of w must be in $\mathcal{Q}_{h^{-1}}$ as well, and applying \dot{h} to the walk w (seen as a sequence of vertices) sends w to a walk joining $\dot{h}(a, b)$ to (i, j) in \mathcal{Q} . Hence the signed number of walks that do not intersect any wall is given by the right-hand side of the identity.

Now assume that w intersects one of the d walls, and write $w = (w_0, \dots, w_n)$ where the w_i 's are points of \mathcal{C} . Consider the largest m such that w_m is on one of the walls W_h . Note that the group element h is uniquely defined, because the walls only intersect at $(-1, -1)$, which is not in \mathcal{C} . Moreover, we have $m < n$ because the final point $\dot{g}(i, j)$ is not on a wall. More generally, all points $w_{m+1}, \dots, w_n = \dot{g}(i, j)$ lie in \mathcal{Q}_g . Now, form the walk $\iota(w) := (w_0, \dots, w_m = \dot{h}(w_m), \dot{h}(w_{m+1}), \dots, \dot{h}(w_n))$. Note that the points $\dot{h}(w_{m+1}), \dots, \dot{h}(w_n)$ lie in the domain \mathcal{Q}_{hg} . The new walk has still steps in \mathcal{S} , because \mathcal{S} is invariant under the (vectorial) action of G .

Let us prove that it lies in the three-quadrant cone \mathcal{C} . This holds obviously for the first m steps. If this were not true for the rest of the walk, then either the step $(w_m, \dot{h}(w_{m+1}))$ would be one of the two forbidden steps joining $(-1, 0)$ to $(0, -1)$ (but this is impossible because $\dot{h}(w_{m+1})$ is not on a wall), or all points $\dot{h}(w_{m+1}), \dots, \dot{h}(w_n)$ would be in the domain \mathcal{Q}_ω . But this is not possible either since $w_m = \dot{h}(w_m)$ would then have both coordinates negative.

Since $\iota(w)$ ends at $\dot{h} \circ \dot{g}(i, j)$, its sign is $-\varepsilon_g$ (because h has odd sign). Its last visit to a wall is clearly $w_m \in W_h$, so $\iota \circ \iota(w) = w$ and we have indeed constructed a sign reversing involution of walks that visit at least one wall. This concludes the proof. \square

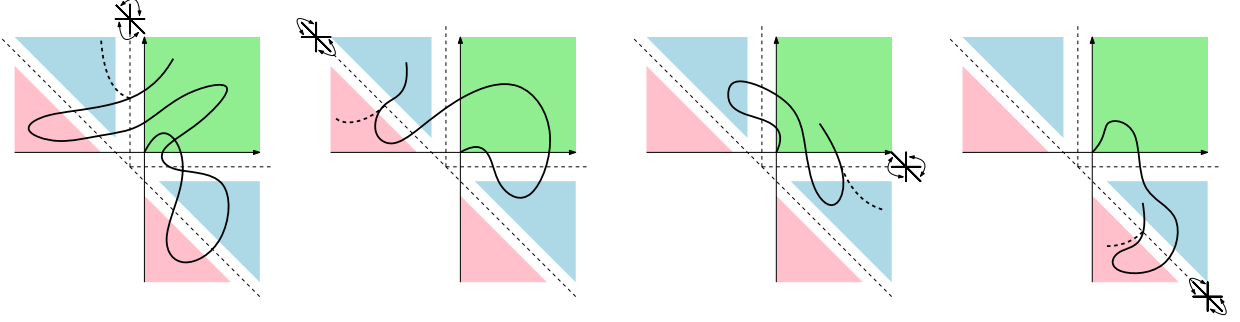


FIGURE 5. The involution of Proposition 7.4 for walks starting at $(0, 0)$ and a group of order 6. The above eight walks capture all possible values of the pair (g, h) , where \mathcal{Q}_g is the domain in which the walk ends and W_h the last visited wall. The figure also shows the action of h on the steps.

Let us now reformulate the above proposition in terms of trivariate generating functions, as in Proposition 7.2. Given $(a, b) \in \mathcal{C}$ and $g \in G$, the generating function of walks in \mathcal{C} starting from (a, b) and ending in \mathcal{Q}_g reads

$$\begin{aligned} \sum_{(k, \ell) \in \mathcal{Q}_g} C_{k, \ell}^{a, b} x^k y^\ell &= \sum_{(i, j) \in \mathcal{Q}} C_{\dot{g}(i, j)}^{a, b} \bar{x} \bar{y} g(x^{i+1} y^{j+1}) \\ &= \bar{x} \bar{y} g(xy) P_g^{a, b}(g(x, y)), \end{aligned}$$

where $P_g^{a, b}(x, y) = \sum_{(i, j) \in \mathcal{Q}} C_{\dot{g}(i, j)}^{a, b} x^i y^j$ is a series in $\mathbb{Q}[x, y][[t]]$. For instance, when G has order 4, it follows from (72) that

$$\begin{aligned} P_{\text{id}}^{a, b}(x, y) &= P^{a, b}(x, y), \\ P_{\phi}^{a, b}(x, y) &= \bar{x} \left(L^{a, b}(x, y) - L^{a, b}(0, y) \right), \\ P_{\psi}^{a, b}(x, y) &= \bar{y} \left(B^{a, b}(x, y) - B^{a, b}(x, 0) \right). \end{aligned}$$

The generalization of Proposition 7.2 reads as follows.

Proposition 7.5. *Let \mathcal{S} be one of the Weyl models of Table 1. Let $2d$ be the order of the associated group G . Let $\omega = \phi\psi\psi\cdots$ (with d generators) be the only element of length d in G . For any starting point $(a, b) \in \mathcal{C}$, we have:*

$$\sum_{g \in G \setminus \{\omega\}} \varepsilon_g P_g^{a, b}(x, y) = \begin{cases} 0 & \text{if } (a, b) \notin \bigcup_g \mathcal{Q}_g, \\ \varepsilon_h Q_{\dot{h}(a, b)}^{a, b}(x, y) & \text{if } \dot{h}(a, b) \in \mathcal{Q} \text{ for } h \in G. \end{cases}$$

Proof. Multiply the identity of Proposition 7.4 by $x^i y^j$ and sum over $i, j \geq 0$. \square

Our final result deals with the series $A(x, y)$ defined in Proposition 3.1. Observe that the orbit sum of xy can be written in terms of the affine orbit of $(0, 0)$:

$$\bar{x} \bar{y} \text{OS}(xy) = \sum_{\substack{h \in G \\ (c, d) = \dot{h}(0, 0)}} \varepsilon_h x^c y^d.$$

Hence, from the functional equation (18) satisfied by $A(x, y)$, we conclude that this series counts weighted walks in \mathcal{C} , and more precisely, that

$$A(x, y) = \frac{2d-2}{2d-1} C^{0,0}(x, y) - \frac{1}{2d-1} \sum_{h \in G \setminus \{\text{id}, \omega\}} \varepsilon_h C^{\dot{h}(0,0)}(x, y).$$

For $g \in G$, let us denote by $\bar{x}\bar{y}g(xy)P_g(g(x, y))$ the contribution in $A(x, y)$ of (weighted) walks ending in Q_g . As before, this notation is designed so that $P_g(x, y) \in \mathbb{Q}[x, y][[t]]$. Then

$$P_g(x, y) = \frac{2d-2}{2d-1} P_g^{0,0}(x, y) - \frac{1}{2d-1} \sum_{h \in G \setminus \{\text{id}, \omega\}} \varepsilon_h P_g^{\dot{h}(0,0)}(x, y). \quad (76)$$

We can now state the generalization of Corollary 7.3.

Corollary 7.6. *The above defined series $P_g(x, y)$ are related by*

$$\sum_{g \in G \setminus \{\omega\}} \varepsilon_g P_g(x, y) = 0.$$

Proof. We first use the identity (76), and then Proposition 7.5. This gives:

$$\begin{aligned} \sum_{g \in G \setminus \{\omega\}} \varepsilon_g P_g(x, y) &= \frac{2d-2}{2d-1} \sum_{g \in G \setminus \{\omega\}} \varepsilon_g P_g^{0,0}(x, y) - \frac{1}{2d-1} \sum_{h \in G \setminus \{\text{id}, \omega\}} \varepsilon_h \sum_{g \in G \setminus \{\omega\}} \varepsilon_g P_g^{\dot{h}(0,0)}(x, y) \\ &= \frac{2d-2}{2d-1} Q^{0,0}(x, y) - \frac{1}{2d-1} \sum_{h \in G \setminus \{\text{id}, \omega\}} \varepsilon_h \varepsilon_{h^{-1}} Q^{0,0}(x, y) \\ &= 0, \end{aligned}$$

since $\varepsilon_h \varepsilon_{h^{-1}} = 1$ and G has order $2d$. □

Remark. It is easy to see that the results of this section hold as well if we allow steps between the points $(0, -1)$ and $(-1, 0)$, as in [16, 22].

8. FINAL COMMENTS

The first question raised by this paper is whether all seven models of Table 1 actually obey the pattern described in Conjecture 3.2. Does $C(x, y)$ differ from the linear combination of series $Q(\cdot, \cdot)$ given in Proposition 3.1 by an algebraic series? This is now proved for three of these seven models.

In terms of techniques, one can of course try to extend the approach of this paper to the other four Weyl models. Another idea would be to try to use the technique, based on *invariants*, that has been used recently [14] to solve the first three (algebraic) models of Table 3. In fact, it is shown in [14] that this approach also works for the simple and diagonal models. Can it be adapted to the four unsolved Weyl cases? The only monomial model that would remain to fully prove Conjecture 3.2 would then be Gessel's model (number four in Table 3).

Next to these $7 + 4 = 11$ models, there remain 12 models with finite and non-monomial group, as shown in Table 2. The non-monomial group action when applied to power series, prevents the efficient extraction of (positive/negative) parts. For this reason the methods of this paper become even more complicated, and probably new approaches have to be developed.

Another question is whether the generating function for walks in other cones – possibly larger than 2π , as in [16, 22] – may satisfy a similar algebraicity phenomenon; that is, decompose into a simple D-finite series with the same orbit sum and an algebraic one.

We conclude with a sketch of the solution of the king model in which we allow moves from $(0, -1)$ to $(-1, 0)$ and back, as in [16, 22].

ALLOWING STEPS BETWEEN $(-1, 0)$ AND $(0, -1)$ IN KING WALKS

As already mentioned in this paper, in two recent references dealing with the winding number of plane lattice walks [16, 22], it seems more natural to count walks in which all vertices lie in \mathcal{C} , but not necessarily all edges: that is, one allows steps from $(-1, 0)$ to $(0, -1)$, and vice versa. It is natural to ask whether this choice leads to simpler series. This is why we have re-run our MAPLE sessions on this variant of the king model. The first steps of the derivation, until the determination of the series R_0 , R_1 , B_1 , and B_2 (as in Proposition 6.1) appear to be a bit simpler, but this stops being the case as soon as we return to the series $R(x)$ and $S(x)$. Let us give a few details.

First, the only changes in our basic functional equations are a term $t(\bar{x} + \bar{y})C_{-1,0}$ in the right-hand side of (34), and a term $t\bar{x}M_{0,0}$ in the right-hand side of (40). The new series $\hat{S}(x)$ is obtained from (46) by deleting the term in R_0 . We still denote $\tilde{S}(x) = (x + \bar{x} + 1)\hat{S}(x)/(x - \bar{x})$, and then the equation in one catalytic variable that we have to solve reads, with the same notation as in (54):

$$\begin{aligned} 0 = & 27(2t + z + 1)(10t - 3z + 1)\tilde{S}(x)^3 + ((54 - 243R_1 + 54t)R_0 - 243R_1^2 + 243R_1t \\ & - 81zR_1 - 45t^2 + 18zt - 27B_1 + 27B_2 + 81R_1 - 54t + 18z - 9)\tilde{S}(x) \\ & - 81R_0^2 + (81t + 27 - 27z - 162R_1)R_0 - 81R_1^2 + 81R_1t - 27zR_1 \\ & + 5t^2 + 10zt - 3z^2 + 9B_1 + 9B_2 + 27R_1 - 6t + 4z - 2 = 0. \end{aligned}$$

The system defining R_0 , R_1 , B_1 , and B_2 is also a bit more compact, and in fact we can derive polynomial equations for each individual series without having to guess them first. They are now all of degree 24 (while B_1 had degree 12 in the first setting), and are found to belong to $\mathbb{Q}(t, w)$. For instance, we now have

$$R_0 = \frac{v(1 - 2t)}{v^4 + 8v^3 + 6v^2 + 2v + 1} \left(1 + 2v + \frac{2v^3 - 4v - 1}{2w} \right).$$

Then we get back to $S(x)$, and that is where things become after all a bit more complicated than in the first setting. For instance, $\tilde{S}(x)$ has now degree 72 rather than 36. It can be written as $\tilde{S}_0(x) + w\tilde{S}_1(x)$, where both series $\tilde{S}_i(x)$, now of degree 36, belong to $\mathbb{Q}(v, U_0(x))$ and hence to $\mathbb{Q}(v, U_1(x))$, where U_0 and U_1 are series defined in Appendix B and Section 4.2, respectively. Finally, both series $R(x)$ and $S(x)$ are found to belong to $\mathbb{Q}(t, w, x, U_1(x))$ and have degree 72 over $\mathbb{Q}(t, x)$.

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We are extremely thankful to Mark van Hoeij, who helped us a lot in finding a simple description of our algebraic series of high degree. Our warm thanks also go to Bruno Salvy for his help with several MAPLE problems that we met.

**APPENDIX A. FROM LARGE POLYNOMIAL EQUATIONS TO SIMPLE
SUB-EXTENSIONS**

In this section we explain how to derive a “simple” expression for a series F , similar to those of Proposition 6.1, from a large polynomial equation satisfied by F , like the polynomial equations for the series R_0 , R_1 , B_1 , and B_2 that we have guessed in Section 6.2; see Table 6. More specifically, we describe how to find subextensions over $\mathbb{Q}(t)$ of the fields $\mathbb{Q}(t, R_0)$, \dots , $\mathbb{Q}(t, B_2)$, and “simple” series in these extensions. For this section, we have greatly benefited from the help of Mark van Hoeij (<https://www.math.fsu.edu/~hoeij/>). We also refer to the appendix that he wrote in [6]. The final picture is shown in Figure 6.

$$\begin{array}{ccccccc}
 & & \mathbb{Q}(t) & \xrightarrow{4} & \mathbb{Q}(t, u) & \xrightarrow{3} & \mathbb{Q}(t, v) & \xrightarrow{2} & \mathbb{Q}(t, w) \\
 & \textcircled{2} & & & & & & & \\
 & \nearrow & & & & & & & \\
 & & & & & & & & \\
 \mathbb{Q}(s) & \xrightarrow{\textcircled{4}} & \mathbb{Q}(u) & \xrightarrow{\textcircled{3}} & \mathbb{Q}(v) & \xrightarrow{2} & \mathbb{Q}(v, w) & & \\
 & & & & & & & &
 \end{array}$$

FIGURE 6. Algebraic structure of the fields $\mathbb{Q}(t, B_1) = \mathbb{Q}(t, v)$ and $\mathbb{Q}(t, R_1) = \mathbb{Q}(t, B_1) = \mathbb{Q}(t, B_2) = \mathbb{Q}(t, w)$. The numbers above the arrows give the degree of the extensions; circles mark rational parametrizations. We set $s = \frac{t(1+t)}{(1-8t)}$.

We begin with the simplest series, B_1 , of (conjectured) degree 12. We denote by $P(F)$ its guessed monic minimal polynomial with coefficients in $\mathbb{Q}(t)$.

A.1. FINDING SUB-EXTENSIONS

In principle, the `Subfields` command of MAPLE can determine all subextensions of $\mathbb{Q}(t, B_1)$ of a prescribed degree. But we were unable to use it successfully with the variable t . Instead, we used it for several specific values of t . For instance, for $t = 1$ the polynomial $P(F)$ is irreducible over \mathbb{Q} , and the command `evala(Subfields(subs(t=1,P(F)),d),d)`, with $d = 2, 3, 4, 6$, shows the existence of a subfield of degree 4 over \mathbb{Q} , generated by a number u that satisfies

$$199974741 u^4 - 76156920 u^3 - 34589883726 u^2 + 248642276448 u - 521380624943,$$

but of no subfield of degree 2, 3, or 6. By repeating this calculation with several fixed rational values of t , one conjectures that the extension $\mathbb{Q}(t, B_1)$ indeed possesses a subfield $\mathbb{K} = \mathbb{Q}(t, u)$ of degree 4 over $\mathbb{Q}(t)$. For each fixed t , MAPLE gives a generator u , but it is not canonical. How can we then construct u for a generic t ?

If $\mathbb{Q}(t, B_1)$ has indeed a subfield \mathbb{K} of degree 4 over $\mathbb{Q}(t)$, then $P(F)$ factors over \mathbb{K} into the form $P_3(F)P_9(F)$, where P_3 (resp. P_9) is a monic polynomial of degree 3 (resp. 9) with coefficients in \mathbb{K} . This factorization should be reflected in the factorization of $P(F)$ over $\mathbb{Q}(t, B_1)$, which should then be of the form

$$P(F) = (F - B_1)Q_2(F)Q_9(F),$$

where indices still indicate degrees. This time the polynomials Q_2 and Q_9 should have coefficients in $\mathbb{Q}(t, B_1)$, and we would then have

$$P_3(F) = (F - B_1)Q_2(F).$$

If we can compute this factorization using MAPLE (see below what to do otherwise), we thus obtain an expression of the minimal monic polynomial of B_1 over \mathbb{K} , namely $P_3(F)$, as a polynomial in F with explicit coefficients in $\mathbb{Q}(t, B_1)$. Now, let us write

$$P_3(F) = F^3 + p_2 F^2 + p_1 F + p_0.$$

By eliminating B_1 from the expressions of the p_i 's (using the equation $P(B_1) = 0$), we obtain the minimal monic polynomial of each p_i over $\mathbb{Q}(t)$, say $M_i(p_i) = 0$, where $M_i(p)$ has coefficients in $\mathbb{Q}(t)$. Since p_i must belong to \mathbb{K} , each $M_i(p)$ should be of degree at most 4 in p . Conversely, for each M_i of degree 4 (if any), we can take p_i as a generator of \mathbb{K} over $\mathbb{Q}(t)$.

If the command `factor(P(F), RootOf(P(B1), B1))` fails, as happened for us, we can perform this factorization for several rational values of t . The above procedure then gives the value of the minimal monic polynomial $M_i(p)$ at this specific value of t . Since the coefficients of M_i are rational functions in t , we then reconstruct the value of this polynomial for a generic t by rational interpolation. In practise, we were able to reconstruct the minimal polynomial $M_2(p)$, of degree 4 in p , from its values obtained for $t = 3, \dots, 30$ (we start at $t = 3$ because $P(F, t)$ is reducible for $t = 2$). At this stage, we can conjecture that $\mathbb{Q}(t, B_1)$ has a subextension of degree 4 generated by a root of $M_2(p)$, namely p_2 . We denote by $u_1 := p_2$ this first generator of \mathbb{K} . Note that we have not identified p_2 but just its minimal polynomial over $\mathbb{Q}(t)$.

A.2. FINDING "SIMPLE" GENERATORS

However, the polynomial $M_2(p)$ is still too big for our taste. In particular, its numerator, denoted $N_2(p, t)$, is a polynomial in p and t , of degree 12 in t . Using the `algebra` package, we find that $N_2(p, t)$ has genus 2, and the `is_hyperelliptic` command tells us that it is hyperelliptic. This implies that the equation $N_2(p, t) = 0$ can be written as $g^2 = \text{Pol}(f)$ where f and g are rational functions in p and t , and conversely, p and t can be expressed rationally in terms of f and g . The command `Weierstrassform` determines such a pair (f, g) .

Next, we compute the minimal polynomials of f and g over $\mathbb{Q}(t)$, in the hope that they are simpler than M_2 . This is indeed the case, and we finally take $u_2 := g$ as a new generator of \mathbb{K} . The coefficients of its minimal polynomial over $\mathbb{Q}(t)$ are found to have several common factors. This leads us to introduce a new generator u_3 , which only differs from u_2 by a factor of $\mathbb{Q}(t)$, and satisfies

$$9u_3^4 - 4 \frac{(112t^2 + 120t - 1)(16t^2 + 72t - 7)}{(4t + 1)^4} u_3^3 + 30u_3^2 - 12u_3 + 1 = 0.$$

Remarkably, this can be rewritten so that u_3 and t are separated:

$$\frac{(3u_3^2 + 6u_3 - 1)^2}{u_3^3} = 64 \frac{(16t^2 + 24t - 1)^2}{(4t + 1)^4}.$$

Hence, one of the square roots of u_3 , denoted u_4 , has also degree 4 (and thus generates the field \mathbb{K}) and satisfies

$$\frac{3u_4^4 + 6u_4^2 - 1}{u_4^3} = 8 \frac{16t^2 + 24t - 1}{(4t + 1)^2}$$

or equivalently,

$$\frac{(u_4 + 1)^3 (3u_4 - 1)}{16(u_4 - 1)^3 (3u_4 + 1)} = -\frac{t(1 + t)}{1 - 8t}.$$

Finally, with $u_5 := (u_4 - 1/3)/(u_4 + 1)$, we have reached

$$\frac{u_5}{(1 + u_5)(1 - 3u_5)^3} = \frac{t(1 + t)}{1 - 8t}, \quad (77)$$

where we recognize Equation (28) satisfied by the series u of Section 4.1. Note that this equation also shows the existence of a non-trivial subfield of $\mathbb{Q}(t)$ and $\mathbb{Q}(u)$, namely $\mathbb{Q}(s)$ with $s := \frac{t(1+t)}{1-8t}$, which we, however, have not used; see Figure 6.

A.3. PROVING THE GUESSED SUB-EXTENSION

At this stage, we suspect that the field $\mathbb{Q}(t, B_1)$ contains a field $\mathbb{K} = \mathbb{Q}(t, u_5)$, where u_5 is one of the roots of (77). In order to check this, and identify the correct root u_5 , we factor the (guessed) minimal polynomial of B_1 , denoted $P(F)$ above, using the command `factor(P(F), RootOf(Alg(u5), u5))`, where `Alg(u)` is the minimal polynomial of u_5 . Actually a bug in the version of MAPLE that we use forces us to have a monic polynomial instead of `Alg(u)`, which is why we consider in practise $u' = u_5/(27t(1+t))$ instead of u_5 . Then the factorization works, and tells us that $P(F)$ has indeed a factor $P_3(F)$ of degree 3 with coefficients in $\mathbb{Q}(t, u_5)$. By expanding $P_3(B_1)$ around $t = 0$ for each of the roots of (77), we see that u_5 must be the root u defined in Section 4.1 as the only solution of (77) that is a formal power series in t .

We have now proved that for the (guessed) series B_1 , the field $\mathbb{Q}(t, B_1)$ admits indeed $\mathbb{Q}(t, u)$ as a subextension of degree 4.

A.4. CONSTRUCTION OF THE SERIES v

We would now like to find in $\mathbb{Q}(t, B_1)$ a series v that is also cubic above $\mathbb{Q}(t, u)$ (like B_1), but satisfies a simpler equation, and, why not, an equation that does not involve t . To investigate this, we now look at B_1 as an algebraic element over $\mathbb{Q}(u)$. We construct its minimal monic polynomial $\tilde{P}(F)$ over $\mathbb{Q}(u)$, of degree 6 in F , by eliminating t between $P_3(F)$ and the minimal polynomial of u . We now repeat the procedure of Section A.1, but with \tilde{P} and u rather than P and t . The `Subfields` command, used for specific values of u , suggests that $\mathbb{Q}(t, B_1) = \mathbb{Q}(u, B_1)$ contains an extension of $\mathbb{Q}(u)$ of degree 2 (which is $\mathbb{Q}(t, u)$), and another of degree 3, say $\tilde{\mathbb{K}}$, above which B_1 should have degree 2. We then factor $\tilde{P}(F)$ over $\mathbb{Q}(B_1)$ for various values of u , and observe the following pattern:

$$\tilde{P}(F) = (F - B_1)\tilde{Q}_1(F)\tilde{Q}_2(F)\hat{Q}_2(F),$$

where indices indicate the degree. Thus, the minimal polynomial of B_1 over $\tilde{\mathbb{K}}$ should be

$$\tilde{P}_2(F) = (F - B_1)\tilde{Q}_1(F) = F^2 + \tilde{p}_1F + \tilde{p}_0.$$

We reconstruct it again by rational interpolation in u . Its two coefficients \tilde{p}_1 and \tilde{p}_0 are found indeed to have degree 3 over $\mathbb{Q}(u)$. In particular, \tilde{p}_1 has a cubic minimal polynomial $\tilde{M}_1(p)$, of degree 21 in u . By observing the repeated factors in the coefficients of $\tilde{M}_1(p)$, we introduce a series v_1 that differs of p_1 by a multiplicative factor, and satisfies

$$\begin{aligned} (3u^3 - 15u^2 + 9u + 21)v_1^3 + (9u^4 - 72u^3 + 126u^2 + 36u + 9)v_1^2 \\ - (18u^5 - 36u^4 - 99u^3 - 53u^2 - 7u - 1)v_1 + 3u^2(1+u)^4 = 0. \end{aligned}$$

The degree in u has reduced to 6.

Now in the field $\mathbb{Q}(u, v_1)$, we would like to find an even simpler generator than v_1 . The above curve is found to have genus 0, so we have a rational parametrization this time, which MAPLE can compute. Since this parametrization looks pretty big, one can first use the `NormalBasis` package³ of van Hoeij and Novocin [41], which gives a new generator v_2 satisfying an equation that is cubic in v_2 (of course) and in u , and then parametrize this simpler equation with the `parametrization` command. This is how we obtained Equation (29). We then check that $\tilde{P}(F)$ actually factors over $\mathbb{Q}(u, v) = \mathbb{Q}(v)$, with one factor $\tilde{P}_2(F)$ of degree 2, and that the root of (29) such that this factor of degree 2 vanishes is the one with constant term zero. Now we have proved the existence of a subfield $\mathbb{Q}(u, v) = \mathbb{Q}(v)$ in $\mathbb{Q}(u, B_1) = \mathbb{Q}(t, B_1)$.

A.5. EXPRESSION OF B_1

We return to the minimal polynomial of B_1 over $\mathbb{Q}(v)$, namely \tilde{P}_2 and factor it over $\mathbb{Q}(t, v)$ using `factor(P2(F), RootOf(alg(t, v), t))`, where $alg(t, v)$ is the minimal polynomial of v over $\mathbb{Q}(t)$, which has degree 12 in v but only 2 in t . This gives us the expression of B_1 in Proposition 6.1.

A.6. EXPRESSION OF R_0 AND CONSTRUCTION OF w

We return to the guessed minimal polynomial of R_0 over $\mathbb{Q}(t)$, which has degree 24. We use the minimal polynomial (30) of v , and the first terms of R_0 , to obtain the minimal polynomial of R_0 over $\mathbb{Q}(v)$, which has degree 4. This polynomial further factors over $\mathbb{Q}(t, v)$, and we obtain an equation of degree 1 in t , of the form

$$c_2(v)R_0^2 + tc_1(v)R_0 + c_0(t, v) = 0.$$

This suggests to look at the quadratic equation satisfied by R_0/t , which is found to have coefficients in $\mathbb{Q}(v)$. We solve it, which leads us to introduce the series w defined by (31), and we finally obtain the expression for R_0 stated in Proposition 6.1.

A.7. EXPRESSIONS OF R_1 AND B_2

We return to the guessed minimal polynomial of R_1 , of degree 24 over $\mathbb{Q}(t)$, and derive as above an equation of degree 4 over $\mathbb{Q}(v)$. This equation factors into four linear terms in $\mathbb{Q}(t, v, w)$, and this gives us the expression for R_1 stated in Proposition 6.1.

We apply the same steps to the minimal polynomial of B_2 . Recall that it has degree 12 in B_2^2 . As a result, the minimal polynomial of B_2 over $\mathbb{Q}(v)$ is found to be bi-quadratic.

APPENDIX B. ANOTHER PARAMETRIZATION FOR $S(x)$ AND $R(x)$

In Section 6.3 we gave two parametrizations for $S(x)$ and $R(x)$, in terms of series U_1 and U_2 . Here we give another one in terms of a series denoted U_0 . We have mentioned it in the proof of Proposition 6.3.

The series $\tilde{S}(x)$ defined by (63) satisfies over $\mathbb{Q}(\tilde{z}, v)$ (where \tilde{z} is defined by (65)) a cubic equation, which can be parametrized rationally by introducing the unique series U_0 such that $U_0 = \tilde{x}t + \mathcal{O}(t^2)$ and

$$\tilde{z} = \frac{\text{num}}{(v^4 + 8v^3 + 6v^2 + 2v + 1)\text{den}}, \quad (78)$$

³Source code available online: <https://www.math.fsu.edu/~hoeij/files/NormalBasis/>.

where

$$\begin{aligned} \text{num} &= w^2 U_0^3 + v w^2 (v^3 + 3v + 2) U_0^2 \\ &\quad - v(2v + 1)(4v^8 + 4v^7 + 14v^6 + 19v^5 + 7v^4 - 22v^3 - 32v^2 - 11v - 1) U_0 \\ &\quad - v^3 (v^2 - 1) (v^5 + v^4 + 6v^3 + 8v^2 + 11v + 3) (2v + 1)^2 \end{aligned}$$

and

$$\text{den} = w^2 U_0^2 + v w^2 (2v + 1) U_0 - v^2 (v^2 - 1) (v^2 + v + 1) (2v + 1)^2. \quad (79)$$

We have denoted, as usual, $w^2 = 1 + 4v - 4v^3 - 4v^4$. Then we have

$$\tilde{S}(x) + \frac{1}{3} = -\frac{v^2 (v^2 - 1) (2v + 1) (v^2 + 4v + 1)^2}{(2v^3 - 4v - 1) (w^2 U_0^2 + v^2 (v^2 - 1) (2v + 1) (2v^3 + 3v^2 + 6v + 1))}.$$

We can also express $\hat{R}(x)$ in terms of U_0 :

$$\hat{R}(x) = -\frac{y(1 - 2t)^2(1 + 2v)N_1(U_0)N_2(U_0)N_3(U_0)}{w(ty^2 - t - 1)(2v^3 - 4v - 1)(v^4 + 8v^3 + 6v^2 + 2v + 1)^2 \text{den}^2} \quad (80)$$

where $y = x + 1 + \bar{x}$, den is given by (79) and

$$\begin{aligned} N_1(U) &= w^2 U^2 + 2v(2v + 1)w^2 U - v(3v^5 + 3v^4 + 2v^3 + 3v + 1)(2v + 1)^2, \\ N_2(U) &= w^2 U^2 + 2v^2 w^2 (v^2 - 1)U - v^3(8v^3 + 12v^2 + 15v + 4)(v^2 - 1)^2, \\ N_3(U) &= w^2 U^2 - (v^2 + 4v + 1)^2 U + v^2(v^2 - 1)(2v + 1)(2v^3 + 3v^2 + 6v + 1). \end{aligned}$$

APPENDIX C. ANOTHER QUADRATIC EXTENSION OF $\mathbb{Q}(t, v)$

This extension is different from $\mathbb{Q}(t, w)$, and is involved in the description of the series $S(1)$ in Proposition 6.4. The series \tilde{w} defined by $S(1) + 1/2 = w\tilde{w}$ has degree 24 over $\mathbb{Q}(t)$, degree 2 over $\mathbb{Q}(t, v)$, and satisfies:

$$\tilde{w}^2 + \frac{(2v + 1)\tilde{w}}{3(2v^3 - 4v - 1)} + \frac{\text{num}''}{12w^2(1 - 2t)(2v^3 + 3v^2 + 6v + 1)^2(2v^3 - 4v - 1)^2(4v^3 + 3v^2 - 1)} \quad (81)$$

with

$$\begin{aligned} \text{num}'' &= 2(4v^3 + 3v^2 - 1)(64v^{12} + 576v^{11} + 2336v^{10} + 5136v^9 + 6896v^8 \\ &\quad + 4652v^7 - 832v^6 - 4756v^5 - 4495v^4 - 2300v^3 - 682v^2 - 108v - 7)t \\ &\quad - 128v^{15} - 608v^{14} - 1312v^{13} - 2624v^{12} - 4560v^{11} - 6808v^{10} - 5476v^9 \\ &\quad + 2088v^8 + 10500v^7 + 11309v^6 + 5096v^5 + 559v^4 - 220v^3 - 45v^2 + 4v + 1. \end{aligned}$$

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