Method of asymptotic partial decomposition with discontinuous junctions

Marie-Claude Viallon, Grigory Panasenko

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Grigory Panasenko* and Marie-Claude Viallon †

Abstract

Method of asymptotic partial decomposition of a domain (MAPDD) proposed and justified earlier for thin domains (rod structures, tube structures consisting of a set of thin cylinders) generates some specific interface conditions between three-dimensional and one-dimensional parts. In the case of the heat equation these conditions ensure the continuity of the solution and continuity in average of the normal flux. However for some computational reasons the strong continuity condition for the solution may be undesirable. It may be preferable to replace it by more flexible condition of continuity in average over the interface cross section. In the present paper we introduce and justify this alternative junction condition allowing such discontinuity of the solution of both stationary and non-stationary problems. The closeness of solutions of these hybrid dimension problems to the solution of the fully three-dimensional setting is proved. At the discrete level, finite volume schemes are considered, an error estimate is established. The new version of the MAPDD is compared to the classical one via numerical tests. It shows better stability of the new scheme.

Keywords: method of asymptotic partial decomposition of the domain, heat equation, interface, asymptotic expansion, dimension reduction

AMS classification: 35B27, 35Q53, 35C20, 35J25, 65N12, 76M12

1 Introduction

For thin rod structures (domains depending on a small parameter ε, unions of "thin" cylinders having ratio of the diameter of a cross-section and the height of order of ε) we consider a new version of the method of asymptotic partial decomposition of the domain (MAPDD).

The MAPDD reduces the dimension in the main part of the domain keeping the full dimension for small parts of the domain only. At the junctions of parts of full and reduced dimension the special interface conditions are set. In the classical version proposed in [13] these conditions require the strong pointwise continuity for the solution and weak continuity (i.e. continuity of the mean value) for the fluxes. Althought the effectiveness of the classical version was confirmed by theoretical studies [17] and numerical experiments [1], [16] and [7] it turns out that for some computational reasons the strong continuity of the unknown function may be undesirable. Indeed, analysing the continuous and discontinuous versions of

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the Galerkin method, see for example [2], one can notice that the discontinuous version is more flexible with respect to the regularity of the mesh and it is more accurate when the exact solution changes fastly. That is why one can expect a similar comparison for the continuous and discontinuous versions of the MAPDD, confirmed numerically in [19]. In particular, applying the partial asymptotic dimension reduction we are interested to make the full dimension subdomains as small as possible. However, taking them too small we risk to loose the accuracy, because the solution may have a big gradient there. In this "shortest truncation" approach the discontinuous version seems to be more accurate than the continuous version. It is confirmed by the numerical experiments [19]. That's why in the present paper we discuss a new version of the MAPDD allowing the discontinuity of the solution at the interface of models of different dimensions. Namely this condition is the pointwise continuity of the flux and weak continuity of the unknown function. This condition is an inverse with respect to the classical version.

As model problems we consider in Section 2 the stationary diffusion equation and the nonstationary heat equation. In Section 3, we describe for each of them the reduced problem obtained by means of the new version of the MAPDD. Then we prove existence and uniqueness of the solution of these reduced problems, and a theorem on the error estimate is proved for both steady and nonsteady settings. Note that the proof of the existence and uniqueness of a solution of the MAPDD-reduced problem requires a generalization of the classical Poincaré - Friedrichs inequality for a Sobolev space which is wider than H^1. Namely the functions of this space may have discontinuities at several cross-sections of the cylinders belonging to the domain, however keeping the continuity of the mean value over the cross-section. For piecewise-H^1 functions it was proved in [6], but we need to know how its constant depends on the small parameter. So, the corresponding inequality is proved in the Appendix.

Let us recall briefly the definition of a rod structure (see [17], [16]). Let $O_1, O_2, \dots, O_N$ be N different points in $\mathbb{R}^s, s = 2, 3$, and $e_1, e_2, \dots, e_M$ be M closed segments each connecting two of these points (i.e. each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \dots, N\}, i_j \neq k_j$ and $\forall i \in \{1, ..., N\}, \exists j \in \{1, ..., M\}$ such that $O_i \in e_j$; $\forall j, k \in \{1, ..., M\}, e_j \cap e_k = \emptyset$ or is a part of the set $\{O_1, O_2, \dots, O_N\}$). The segments e_j are called edges of the graph. A point O_i is called node if it is the common end of at least two edges and O_i is called vertex if it is the end of the only one edge. Any two edges e_j and e_i can intersect only at the common node.

Let $\mathcal{B} = \bigcup_{j=1}^{N} e_j$ be the graph and assume that it is a connected set. For any $x = (x_1, ..., x_s)$ denote $x' = (x_2, ..., x_s)$.

Let e be some edge, $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in O_i and the axis $O_i x_1^{(e)}$ has the direction of the ray $[O_i O_j)$; the second one has the origin in O_j and the opposite direction, i.e. $O_j x_1^{(e)}$ is directed over the ray $[O_j O_i)$.

With every edge e_j we associate a bounded domain $\sigma_j \subset \mathbb{R}^{s-1}$ having Lipschitz boundary $\partial\sigma_j, j = 1, \dots, M$. For any edge $e_j = e$ and associated $\sigma_j = \sigma^{(e)}$ denote by $B_\varepsilon^{(e)}$ the cylinder

$$B_\varepsilon^{(e)} = \{x^{(e)} \in \mathbb{R}^s : x_1^{(e)} \in (0, |e|), \frac{x^{(e)\prime}}{\varepsilon} \in \sigma^{(e)}\},$$

where $x^{(e)\prime} = (x_2^{(e)}, \dots, x_s^{(e)})$, $|e|$ is the length of the edge e and $\varepsilon > 0$ is a small parameter. Notice that the edges e_j and Cartesian coordinates of nodes and vertices O_j, as well as domains σ_j, do not depend on ε.

Let $O_1, \dots, O_{N_1}$ be nodes and $O_{N_1+1}, \dots, O_N$ be vertices; $N > N_1$. Let $\omega^1, \dots, \omega^N$ be bounded independent of ε domains in $\mathbb{R}^s$ with Lipschitz boundaries $\partial\omega^j$; introduce the nodal domains $\omega_\varepsilon^j = \{x \in \mathbb{R}^s : \frac{x - O_j}{\varepsilon} \in \omega^j\}$.

By a rod structure we call the following domain

$$B_\varepsilon = \Big(\bigcup_{j=1}^{M} B_\varepsilon^{(e_j)} \Big) \bigcup \Big(\bigcup_{j=1}^{N} \omega_\varepsilon^j \Big).$$

Assume that it is a connected set and that its boundary $\partial B_\varepsilon \in C^2$. Denote $\gamma_\varepsilon^i = \partial B_\varepsilon \cap \partial \omega_\varepsilon^i$, $i = N_1 + 1, ..., N$. Without loss of generality assume that

$$\Big(B_\varepsilon^{(e_j)} \backslash \Big(\bigcup_{i=1}^{N_1} \omega_\varepsilon^i \Big) \Big) \cap \Big(B_\varepsilon^{(e_k)} \backslash \Big(\bigcup_{i=1}^{N_1} \omega_\varepsilon^i \Big) \Big) = \emptyset$$

for $j \neq k$. Denote $\gamma_\varepsilon = \bigcup_{i=N_1+1}^{N} \gamma_\varepsilon^i$.

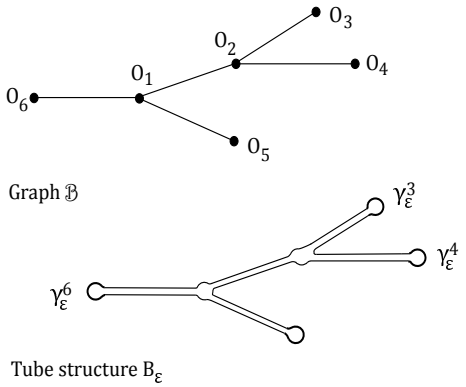

Figure 1: The graph and the tube structure.

Both model problems are set in B_ε. The reduced problems obtained by using the MAPDD are set in a geometrical multi-scale domain derived from B_ε, where parts of the cylinders are replaced by one-dimensional segments. Finite volume schemes for solving the reduced problems obtained from the new version of the MAPDD are considered in Section 4 in the case $s = 2$. An estimate on the discrete solution is proved in both cases. This estimate will be obtained by using a discrete Poincaré-Friedrichs inequality in the geometrical multi-scale domain, when the discrete average of the function from both sides of the cross sections is the same. This result is proved in the Appendix. An error estimate between the solution of the original problem and its approximation is proved in Section 5 for the diffusion equation and the heat equation. Then we provide in Section 6 numerical tests comparing this new version of the MAPDD and the classical one. Note that there is a vast literature on the asymptotic analysis of partial derivative equations in thin rod structures, see for example, [7],[10],[11],[12],[14],[17],[15].

2 Formulation of the steady diffusion equation and the heat equation in a rod structure

In order to introduce a new version of the MAPDD let us consider two model problems set in B_ε: stationary and non-stationary. The stationary problem reads:

$$\begin{aligned} -\Delta u_\varepsilon &= f(x), \quad x \in B_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n} &= 0, \quad x \in \partial B_\varepsilon \backslash \gamma_\varepsilon, \\ u_\varepsilon &= 0, \quad x \in \gamma_\varepsilon. \end{aligned} \tag{2.1}$$

The right hand side f is a function defined on B_ε such that $f(x) = f_j(x_1^{(e)})$, if $x \in B_\varepsilon^{(e_j)}$, $j = 1, ..., M$, where f_j are independent of ε $C^{J+4}-$smooth functions and they are constant with respect to x in some neighborhood of the nodes and vertices. The values of f in the domains ω_ε^i are equal to its value in the node or vertex O_i.

The variational formulation of problem (2.1) is : find $u_\varepsilon \in H^1_{\gamma,0}(B_\varepsilon)$ such that

$$\int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla v dx = \int_{B_\varepsilon} f v dx, \quad v \in H^1_{\gamma,0}(B_\varepsilon), \tag{2.2}$$

where

$$H^1_{\gamma,0}(B_\varepsilon) = \{v \in H^1(B_\varepsilon) | \ \ v|_{\gamma_\varepsilon} = 0\}.$$

The initial boundary value problem for the non-steady heat equation in the tube structure B_ε has the form

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f(x,t), \quad x \in B_\varepsilon, t \in (0,T), \\ \frac{\partial u_\varepsilon}{\partial n} = 0, \quad x \in \partial B_\varepsilon \backslash \gamma_\varepsilon, t \in (0,T), \\ u_\varepsilon = 0, \quad x \in \gamma_\varepsilon, t \in (0,T), \\ u_\varepsilon(x,0) = 0, x \in B_\varepsilon. \end{aligned} \tag{2.3}$$

The right hand side f is a function defined on $B_\varepsilon \times [0,T]$ such that $f(x,t) = f_j(x_1^{(e)}, t)$, if $x \in B_\varepsilon^{(e_j)}$, $j = 1, ..., M$, where f_j are independent of ε $C^{J+4}-$smooth functions and they are constant with respect to x in some neighborhood of the nodes and vertices. The values of f in the domains ω_ε^i are equal to its value in the node or vertex O_i. We assume that $f_j(., t) = 0$ for $t \leq t_0$, $t_0 > 0$.

The variational formulation of problem (2.3) is : find $u_\varepsilon \in H^1_{\gamma,0}(B_\varepsilon \times (0,T))$ such that for almost all $t \in (0,T)$,

$$\int_{B_\varepsilon} \Big(\frac{\partial u_\varepsilon}{\partial t} v + \nabla u_\varepsilon \cdot \nabla v \Big) dx = \int_{B_\varepsilon} f v dx, \quad v \in H^1_{\gamma,0}(B_\varepsilon), \tag{2.4}$$

$$u_\varepsilon|_{t=0} = 0, \tag{2.5}$$

where

$$H^1_{\gamma,0}(B_\varepsilon \times (0,T)) = \{v \in H^1(B_\varepsilon \times (0,T)) | \ \ v|_{\gamma_\varepsilon} = 0\}.$$

3 Asymptotic partial decomposition of the domain for the diffusion equation and for the heat equation

In this section we apply the method of partial asymptotic decomposition of the domain assuming that f_j are $C^{J+4}-$smooth functions, defined in Section 2.

Let us describe the algorithm of the MAPDD for the heat equation set in a tube structure B_ε. Let δ be a small positive number much greater than ε (it will be chosen of order $O(\varepsilon |ln\varepsilon|)$). For any edge $e = \overline{O_i O_j}$ connecting two points of the graph of the structure introduce two hyperplanes orthogonal to this edge and crossing it at the distance δ from its ends. Denote the cross-sections of the cylinder $B_\varepsilon^{(e)}$ containing e by these two hyperplanes respectively, $S_{i,j}$ (at the distance δ from O_i), and $S_{j,i}$ (at the distance δ from O_j), and denote the part of the cylinder $B_\varepsilon^{(e)}$ between these two cross-sections by $B_{ij}^{dec,\varepsilon}$. Denote $B_i^{\varepsilon,\delta}$ the connected truncated by the cross sections $S_{i,j}$, part of B_ε containing the vertex or the node O_i. Denote $e_{ij}^{dec,\delta}$ the part of the edge $\overline{O_i O_j}$ concluded between the cross-sections $S_{i,j}$ and $S_{j,i}$.

Define some spaces. Let H be a subspace of functions from $L^2(B_\varepsilon)$ such that for each subdomain $B_{ij}^{dec,\varepsilon}$ or $B_i^{\varepsilon,\delta}$ they belong to $H^1(B_{ij}^{dec,\varepsilon})$ and $H^1(B_i^{\varepsilon,\delta})$ respectively, and at each cross section $S_{i,j}$ the

average of functions from both sides of it is the same. Let $H_{\gamma,0}$ be a subspace of H such that its functions vanish at γ_ε. So these spaces are Sobolev spaces of functions having discontinuities at $S_{i,j}$ preserving their average at the cross section. Let us define an inner product of two functions (f,g) in such spaces as the sum of inner products in $H^1(B_{ij}^{dec,\varepsilon})$ and $H^1(B_i^{\varepsilon,\delta})$ for all $B_{ij}^{dec,\varepsilon}$ and $B_i^{\varepsilon,\delta}$, so that $\|\cdot\|_H = \sqrt{\sum_{ij}\|\cdot\|^2_{H^1(B_{ij}^{dec,\varepsilon})} + \sum_i \|\cdot\|^2_{H^1(B_i^{\varepsilon,\delta})}}$

Let $H^1(B_\varepsilon,\delta)$ and $H^1_{\gamma,0}(B_\varepsilon,\delta)$ be subspaces of spaces H and $H_{\gamma,0}$ respectively such that their functions have vanishing transversal derivatives $\nabla'_{x^{(e)}}$ on every truncated cylinder $B_{ij}^{dec,\varepsilon}$.

The method of asymptotic partial domain decomposition (MAPDD) replaces the stationary diffusion problem by the following reduced problem (called partially decomposed) : find $u_\varepsilon \in H^1_{\gamma,0}(B_\varepsilon,\delta)$ such that

$$\int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla v dx = \int_{B_\varepsilon} f v dx, \quad v \in H^1_{\gamma,0}(B_\varepsilon,\delta). \tag{3.6}$$

Denote $\|\cdot\|_{H_T} = \sqrt{\sum_{ij}\|\cdot\|^2_{H^1(B_{ij}^{dec,\varepsilon}\times(0,T))} + \sum_i \|\cdot\|^2_{H^1(B_i^{\varepsilon,\delta}\times(0,T))}}$

Let $H^1_{\gamma,0}(B_\varepsilon\times(0,T),\delta)$ be a space of functions from $L^2(B_\varepsilon\times(0,T))$ such that for any $B_{ij}^{dec,\varepsilon}$ or $B_i^{\varepsilon,\delta}$ they belong to $H^1(B_{ij}^{dec,\varepsilon}\times(0,T))$ and $H^1(B_i^{\varepsilon,\delta}\times(0,T))$ respectively, at each cross section $S_{i,j}$ the average of functions from both sides of it is the same, functions have vanishing transversal derivatives $\nabla'_{x^{(e)}}$ on every truncated cylinder $B_{ij}^{dec,\varepsilon}$ and the functions vanish at the part γ_ε of the boundary. Let us define an inner product of two functions (f,g) in such spaces as the sum of inner products in $H^1(B_{ij}^{dec,\varepsilon}\times(0,T))$ and $H^1(B_i^{\varepsilon,\delta}\times(0,T))$ for all $B_{ij}^{dec,\varepsilon}$ and $B_i^{\varepsilon,\delta}$.

In the case of the heat equation the MAPDD replaces the problem (2.3) by its setting on $H^1_{\gamma,0}(B_\varepsilon\times(0,T),\delta)$: find $u_{\varepsilon,\delta,dec}\in H^1_{\gamma,0}(B_\varepsilon\times(0,T),\delta)$ such that for almost all $t\in(0,T)$,

$$\int_{B_\varepsilon}\Big(\frac{\partial u_{\varepsilon,\delta,dec}}{\partial t}v + \nabla u_{\varepsilon,\delta,dec}\cdot\nabla v\Big)dx = \int_{B_\varepsilon} f v dx, \quad v\in H^1_{\gamma,0}(B_\varepsilon,\delta), \tag{3.7}$$

and satisfying

$$u_{\varepsilon,\delta,dec}|_{t=0} = 0, \tag{3.8}$$

which implies:

$$\int_{B_\varepsilon\times(0,T)}\Big(\frac{\partial u_{\varepsilon,\delta,dec}}{\partial t}v + \nabla u_{\varepsilon,\delta,dec}\cdot\nabla v\Big)dxdt = \int_{B_\varepsilon\times(0,T)} f v dxdt, \quad v\in H^1_{\gamma,0}(B_\varepsilon\times(0,T),\delta), \tag{3.9}$$

$$u_{\varepsilon,\delta,dec}|_{t=0} = 0. \tag{3.10}$$

Theorem 3.1. *There exists a unique solution of both partially decomposed problems. The estimates hold: for the diffusion problem*

$$\|u_{\varepsilon,\delta,dec}\|_{H^1(B_\varepsilon,\delta)} \le C_{PF}\|f\|_{L^2(B_\varepsilon)} \tag{3.11}$$

and for the heat equation

$$\|u_{\varepsilon,\delta,dec}\|_{H^1(B_\varepsilon\times(0,T),\delta)} \le C_{PF}\|f\|_{L^2(B_\varepsilon\times(0,T))} \tag{3.12}$$

where the constant C_{PF} is independent of ε and δ.

The proof of this theorem repeats the proof of the existence and uniqueness of a solution to the diffusion equation (Lax-Milgram argument, see the Poincaré-Friedrichs inequality in the Appendix), and respectively to the heat equation where the Galerkin base is constructed in the space $H^1_{\gamma,0}(B_\varepsilon,\delta)$ instead of $H^1_{\gamma,0}(B_\varepsilon)$. Such base exists due to separability of spaces $H^1(B_{ij}^{dec,\varepsilon}\times(0,T))$ and $H^1(B_i^{\varepsilon,\delta}\times(0,T))$.

However, as far as we know, the Poincaré-Friedrichs inequality in the space $H^1_{\gamma,0}(B_\varepsilon,\delta)$ is not known. That is why its proof is given in the Appendix. Let us give a sketch of the proof for the non-stationary problem. Let $\phi_1,...,\phi_m,...$ be an orthogonal with respect to the inner product $\int_{B_\varepsilon}\Big(\nabla u\cdot\nabla v\Big)dx$ base of $H^1_{\gamma,0}(B_\varepsilon,\delta)$. Consider the span H_N of N first functions of the base and consider the projection of problem (2.3) on this subspace. Its solution u^N is saught in the form of a sum $\sum_{l=1}^N c_l(t)\phi_l$ with $c_l\in H^1(0,T)$, so that for the unknown functions c_l satisfy the system of ordinary differential equations with homogeneous initial conditions. Multiplying its equations by c_l and adding them, we get an estimate for u^N in the V^2 norm ($\|u\|_{V^2}=\sup_{t\in[0,T]}\|u(.,t)\|_{L_2(B_\varepsilon)}+\|\nabla u\|_{L_2(B_\varepsilon\times(0,T))}$). Multiplying then the equations by $\frac{dc_l}{dt}$ and adding them, we get an estimate for u^N in the $H^1(B_\varepsilon\times(0,T))$ norm. Then we apply the standard argument of the weak compactness of a ball in the Hilbert space and find that a weak limit of some subsequence is a solution of (3.7). The uniqueness follows from the identity (3.7) written for $v=u_{\varepsilon,\delta,dec}$. The estimates for u^N hold still for the weak limit $u_{\varepsilon,\delta,dec}$.

Remark 3.2. *Estimate* (3.12) *holds in the case if the right hand side is any function of* $L^2(B_\varepsilon\times(0,T))$ *free of the above regularity restrictions (and so it can depend on all components of* x*).*

Theorem 3.3. *There exists a constant* C_0 *independent of* ε *such that for any given integer* J *and for* δ *satisfying the inequality*

$$\delta\geq C_0J\varepsilon|ln(\varepsilon)| \tag{3.13}$$

the following estimates hold for the difference $u_\varepsilon-u_{\varepsilon,\delta,dec}$*:*

for the steady diffusion equation:

$$\|u_\varepsilon-u_{\varepsilon,\delta,dec}\|_H\leq C\varepsilon^J, \tag{3.14}$$

for the non-stationary heat equation

$$\|u_\varepsilon-u_{\varepsilon,\delta,dec}\|_{H_T}\leq C\varepsilon^J, \tag{3.15}$$

where constant C *is independent of* ε*.*

Notice that the integration by parts in the variational formulation (3.7) gives the differential version of the partially decomposed problem. Namely, denoting $\hat{u}$ the restriction of u on the part $e_{ij}^{dec,\delta}$ of the edge e we have

for the diffusion equation

$$\begin{gathered}-\Delta u_{\varepsilon,\delta,dec}=f(x),\quad x\in B_i^{\varepsilon,\delta},\ i=1,...,N,t\in(0,T),\\ -\frac{\partial^2\hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)2}}=\hat{f}(x_1^{(e)}),\quad x\in e_{ij}^{dec,\delta},\ \forall e;\\ \frac{\partial u_{\varepsilon,\delta,dec}}{\partial n}=0,\quad x\in(\partial B_i^{\varepsilon,\delta}\cap\partial B_\varepsilon)\backslash\gamma_\varepsilon,\ i=1,...,N,\\ u_{\varepsilon,\delta,dec}=0,\quad x\in\gamma_\varepsilon;\end{gathered} \tag{3.16}$$

for the heat equation :

$$\begin{gathered}\frac{\partial u_{\varepsilon,\delta,dec}}{\partial t}-\Delta u_{\varepsilon,\delta,dec}=f(x,t),\quad x\in B_i^{\varepsilon,\delta},\ i=1,...,N,t\in(0,T),\\ \frac{\partial\hat{u}_{\varepsilon,\delta,dec}}{\partial t}-\frac{\partial^2\hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)2}}=\hat{f}(x_1^{(e)},t),\quad x\in e_{ij}^{dec,\delta},\ \forall e;t\in(0,T),\\ \frac{\partial u_{\varepsilon,\delta,dec}}{\partial n}=0,\quad x\in(\partial B_i^{\varepsilon,\delta}\cap\partial B_\varepsilon)\backslash\gamma_\varepsilon,\ i=1,...,N,t\in(0,T),\\ u_{\varepsilon,\delta,dec}=0,\quad x\in\gamma_\varepsilon,t\in(0,T),\\ u_{\varepsilon,\delta,dec}(x,0)=0,x\in B_\varepsilon.\end{gathered} \tag{3.17}$$

with the junction condition at the sections $S_{i,j}$ corresponding to the value $x_1^{(e)} = \delta$ for the local variable:

$$\frac{1}{|S_{i,j}|}\int_{S_{i,j}} u_{\varepsilon,\delta,dec}|_{x_1^{(e)}=\delta} dx^{(e)\prime} = \hat{u}_{\varepsilon,\delta,dec}|_{x_1^{(e)}=\delta},$$
$$\frac{\partial u_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta} = \frac{\partial \hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta}. \tag{3.18}$$

Here, $\hat{f}(x_1^{(e)}(x)) = f(x)$, for $x \in e_{ij}^{dec,\delta}$ (respectively, $\hat{f}(x_1^{(e)}(x),t) = f(x,t)$, for $x \in e_{ij}^{dec,\delta}$ in the non-stationary case), where $x_1^{(e)}(x)$ is the relation between the local variable $x_1^{(e)}$ and the global variable$x$.

It means that we keep the n-dimensional in space setting $(3.16)_1$ for the diffusion equation ($(3.17)_1$ for the heat equation) within small pieces $B_i^{\varepsilon,\delta}$, $i = 1, ..., N$, (their diameters are of order $\varepsilon|ln(\varepsilon)|$), reduce the dimension to one and consider the one-dimensional diffusion or heat equation $(3.16)_2$ or $(3.17)_2$ on the pieces $e_{ij}^{dec,\delta}$ of edges e and add the junction conditions (3.18) between the n-dimensional and one dimensional parts. This reduction allows us to reduce the mesh $\frac{1}{\varepsilon|ln(\varepsilon)|}$ times and to keep an exponential precision of the computations.

Estimates (3.14), (3.15) justifiy the method of asymptotic partial decomposition of domain for the heat equation. The proof of the theorem is similar to [16]. Namely, we use the asymptotic expansion constructed there and check directly that it satisfies relations (3.18) exactly and relations (3.16) or (3.17) with residuals of order $O(\varepsilon^J)$ in corresponding L^2-norms (for more details see [16]).

Let $f_j \in C^\infty$ be a compact support function vanishing in some independent of ε neighborhood of the nodes. There exists a constant C_1 independent of ε such that if δ satisfies the following inequality $\delta \geq C_1\varepsilon|\ln\varepsilon|$ then

$$\|\nabla u_{\varepsilon,\delta,dec}\|_{L^\infty(B_\varepsilon\times(0,T))} \leq C, \quad \|\nabla^2 u_{\varepsilon,\delta,dec}\|_{L^\infty(B_\varepsilon\times(0,T))} \leq C\varepsilon^{-1}, \tag{3.19}$$

where constant C is independent of ε. The proof of these estimates is similar to the proof of Theorem 6.3 in [16]. Namely, we consider the difference between $u_{\varepsilon,\delta,dec}$ and the asymptotic expansion $u_{a\varepsilon}^J$ of order J constructed in [16]. This asymptotic expansion is obtained from the expansion $u_{a\varepsilon}^J$ by multiplication of the boundary layer correctors in the standard expansion by smooth cut-off functions vanishing at the distance δ from the nodes. [1] It can be proved exactly as in [16] that both solutions are $J+5$ times differentiable in time and one obtains the similar to (6.3) in [16] estimates for the time derivatives of their difference $w = u_{\varepsilon,\delta,dec} - u_{a\varepsilon}^J$. Out of some neighborhoods of cross sections $S_{i,j}$ one can prove the regularity and estimates via the ADN theory as it is done in [16]. As in [16] in the neighborhood of the cross-sections $S_{i,j}$ the regularity and estimates for w should be proved via its extension through $S_{i,j}$. However, this extension should be constructed in a different way (in [16] this extension was odd with respect to $S_{i,j}$).

Consider a part B_p of a nodal domain $B_i^{\varepsilon,\delta}$ which is a "short" cylinder between $S_{i,j}$ and a parallel cross-section S' at the distance $\delta/3$ from $S_{i,j}$. Both the asymptotic solution $u_{a\varepsilon}^J$ and the solution of the problem of hybrid dimension (3.7) $u_{\varepsilon,\delta,dec}$ satisfy the homogeneous heat equation within B_p with the homogeneous Neumann boundary condition at the lateral boundary. At $S_{i,j}$ both solutions depend on t only. So, the difference $w = u_{\varepsilon,\delta,dec} - u_{a\varepsilon}^J$ also has these properties, in particular, it is equal to $c(t)$ at $S_{i,j}$, where c is a smooth function of time vanishing in some neighborhood of $t = 0$ and the L^2 norm of its derivatives is estimated as $O(\varepsilon^J)$. Consider a solution to the following problem:

[1] There is a missprint in the definition of $u_{a\varepsilon}^J$ in [16]: the ratio $\frac{|\ln\varepsilon||x-O_i|}{e_{\min}}$ should be read as $\frac{|x-O_i|}{\delta}$.

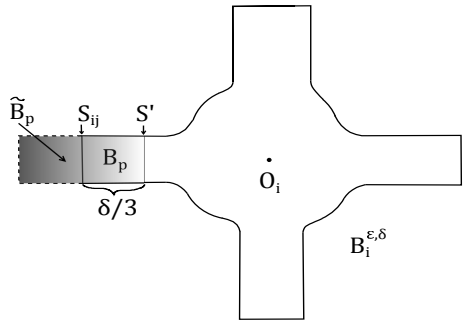

Figure 2: Extension through $S_{i,j}$.

$$\begin{aligned} \tfrac{\partial U}{\partial t} - \Delta U = 0, \quad & x \in B_p, \\ \frac{\partial U}{\partial n} = 0, \quad & x \in \partial B_p \backslash (S_{i,j} \cup S'), \\ \frac{\partial U}{\partial n} = c(t), \quad & x \in S_{i,j}, \\ \frac{\partial U}{\partial n} = -c(t), \quad & x \in S'. \end{aligned} \tag{3.20}$$

It depends only on t and londitudinal component of x and is a smooth function with a standard estimate for its derivatives. Let us subtract it from w and consider the difference $W = w - U$. This function W satisfies conditions

$$\begin{aligned} \frac{\partial W}{\partial t} - \Delta W = 0, \quad & x \in B_p, \\ \frac{\partial W}{\partial n} = 0, \quad & x \in \partial B_p \backslash S', \end{aligned}$$

and so, function W can be extended through $S_{i,j}$ as an even function with respect to the variable $x_1^{(e)} - \delta$. Let $\tilde{B}_p$ be the union of B_p and its symmetric (with respect to $S_{i,j}$) mapping and denote $\tilde{W}$ the even extension of W. It satisfies the homogeneous heat equation within $\tilde{B}_p$ and so it is a regular function. Applying now the ADN theory to this extension we complete the proof of the estimates (3.19).

4 Numerical scheme and error estimate

Finite volume schemes are used to solve the model problems. The method is known as a robust and cheap method for the discretization of conservation laws, and is well suited to approach a solution that is just continuous in average over the interface cross sections. The finite volume method is sometimes called a discontinuous finite element method, because the characteristic functions of the grid cells may be seen as shape functions. However, this choice of shape functions can not be used in the finite element framework (see [8]). The numerical schemes that are used in the paper are of order one. Discontinuous Galerkin high-order methods could also be used here (see [2]). Nevertheless, in this latter case, there are two values of the solution at the interface between each pair of neighboring cells (a value to the right of the interface and a value to the left), and the computational cost is higher. Moreover, this kind of scheme usually needs to be stabilized or requires a reconstruction of the solution.

From now on, we consider the steady diffusion equation and the non-stationary heat equation set in a two-dimensional ($s = 2$) rod structure. For every edge $e_j = e$, $\sigma^{(e)} = (-\frac{|\sigma^{(e)}|}{2}, \frac{|\sigma^{(e)}|}{2})$, and $|\sigma^{(e)}|$ is the width of the rectangle $B_\varepsilon^{(e)}$.

Let us consider the differential version of the partially decomposed problem (3.16) or (3.17) with the junction conditions (3.18). The numerical scheme used to approach $u_{\varepsilon,\delta,dec}$ is a finite volume method obtained by generalizing the scheme described in [19] for the heat equation. This is because the domain has only one node in [19]. The discretisation in time (where required) is unchanged, whereas some notations need to be adapted to describe the discretisation in space. In this paper, $\mathcal{T}_i$ stands for an admissible (see the definition in [8]) mesh of $B_i^{\varepsilon,\delta}$ (either $B_i^{\varepsilon,\delta}$ is assumed to be polygonal or, as in Section 5, the problem is set in a polygonal domain $B_{\varepsilon,poly}$ included in B_ε), $i = 1, ..., N$, and $\mathcal{E}_{int}^i = \{\sigma \in \mathcal{E}^i, \sigma \not\subset \partial B_i^{\varepsilon,\delta}\}$ where $\mathcal{E}^i$ is the family of edges of $\mathcal{T}_i$. Then, the mesh of $e_{lj}^{dec,\delta}$ consists of $N^{(e)}$ intervals $(x_{i-1/2}^e, x_{i+1/2}^e), i = 1, ..., N^{(e)}$, for all edges $e = \overline{O_l O_j}, l < j$, with increasing values in the direction of $[O_l, O_j)$, such that $x_{1/2}^e = \delta$ and $x_{N^{(e)}+1/2}^e = |e| - \delta$; and of $N^{(e)}$ associated points $x_i^e, i = 1, ..., N^{(e)}$, within. Let $h_i^e = x_{i+1/2}^e - x_{i-1/2}^e$ and $h_{i+1/2}^e = x_{i+1}^e - x_i^e$. A mesh of $B_{lj}^{dec,\varepsilon}$ contains the rectangular cells of height equal to $|S_{l,j}|$. The global mesh of B_ε is referred to as $\mathcal{T}$. We denote by h the size of the mesh and by k the time step. Let $N_k = \max\{n \in I\!N, nk < T\}$.

Let us begin by describing the scheme solving (3.17-3.18), that is the heat equation, in a rod structure. We have for $n = 0, ..., N_k$,

$$\begin{cases} h_i^e \dfrac{v_{e,i}^{n+1} - v_{e,i}^n}{k} + F_{i+1/2}^{e,n+1} - F_{i-1/2}^{e,n+1} = h_i^e f_i^{e,n},\ i = 1, ..., N^{(e)}, \forall e, \\ F_{i+1/2}^{e,n+1} = -\dfrac{v_{e,i+1}^{n+1} - v_{e,i}^{n+1}}{h_{i+1/2}^e},\ i = 0, \ldots, N^{(e)}, \forall e, \\ f_i^{e,n} = \dfrac{1}{h_i^e} \displaystyle\int_{x_{i-1/2}^e}^{x_{i+1/2}^e} \hat{f}(x_1^{(e)}, t_{n+1}) dx_1^{(e)},\ i = 1, ..., N^{(e)}, \forall e, \end{cases} \tag{4.21}$$

$$\begin{cases} |K| \dfrac{u_K^{n+1} - u_K^n}{k} + \displaystyle\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = |K| f_K^n,\ \forall K \in \mathcal{T}_i, i = 1, ..., N, \\ F_{K,\sigma}^{n+1} = -\dfrac{|\sigma|}{d_\sigma}(u_L^{n+1} - u_K^{n+1}),\ \forall \sigma \in \mathcal{E}_{int}^i\ , i = 1, ..., N,\ \text{if } \sigma = \sigma_{K/L}, \\ F_{K,\sigma}^{n+1} = \dfrac{|\sigma|}{d_\sigma} u_K^{n+1},\ \forall \sigma \subset \gamma_\varepsilon,\ \sigma \in \mathcal{E}_K, \\ F_{K,\sigma}^{n+1} = 0, \forall \sigma \subset (\partial B_i^{\varepsilon,\delta} \cap \partial B_\varepsilon) \backslash \gamma_\varepsilon, i = 1, ..., N, \\ f_K^n = \dfrac{1}{|K|} \displaystyle\int_K f(x, t_{n+1}) dx,\ \forall K \in \mathcal{T}_i, i = 1, ..., N, \end{cases} \tag{4.22}$$

The value of the solution is not assumed to be constant on the interfaces, and the approximate value of the solution on the edge σ if $\sigma \subset S_{i,j}$ at time t_n is denoted u_σ^n. Let us complete the definition of the approximated flux $F_{K,\sigma}^n$:

$$F_{K,\sigma}^n = -\frac{|\sigma|}{d_\sigma}(u_\sigma^n - u_K^n), \forall \sigma \subset S_{i,j}\ ,\ \sigma \in \mathcal{E}_K, \tag{4.23}$$

$$\frac{1}{|\sigma^{(e)}|\varepsilon} \sum_{\sigma \subset S_{i,j}, \sigma \in \mathcal{E}_K} |\sigma| u_\sigma^n = \begin{cases} v_{e,0}^n,\ \text{if } i < j, e = \overline{O_i O_j}, \\ v_{e,N^{(e)}+1}^n,\ \text{if } i > j, e = \overline{O_j O_i}, \end{cases} \tag{4.24}$$

$$\frac{u_\sigma^n - u_K^n}{d_\sigma} = \begin{cases} \dfrac{v_{e,1}^n - v_{e,0}^n}{h_{1/2}^e}, \forall \sigma \subset S_{i,j}\ ,\ \sigma \in \mathcal{E}_K,\ \text{if } i < j, e = \overline{O_i O_j}, \\ \dfrac{v_{e,N^{(e)}}^n - v_{e,N^{(e)}+1}^n}{h_{N^{(e)}+1/2}^e}, \forall \sigma \subset S_{i,j}\ ,\ \sigma \in \mathcal{E}_K,\ \text{if } i > j, e = \overline{O_j O_i}, \end{cases} \tag{4.25}$$

that gives for all edges e, $e = \overline{O_iO_j}, i < j$,

$$v^n_{e,0}=\left((\sum_{\sigma\subset S_{i,j}} |\sigma| d_\sigma) v^n_{e,1} + h^e_{1/2} \sum_{\sigma\subset S_{i,j}, \sigma\in\mathcal{E}_K} |\sigma| u^n_K \right)\left(\sum_{\sigma\subset S_{i,j}} |\sigma| d_\sigma + |\sigma^{(e)}| \varepsilon h^e_{1/2} \right)^{-1}$$
$$v^n_{e,N^{(e)}+1}=\left((\sum_{\sigma\subset S_{j,i}} |\sigma| d_\sigma) v^n_{e,N^{(e)}} + h^e_{N^{(e)}+1/2} \sum_{\sigma\subset S_{j,i}, \sigma\in\mathcal{E}_K} |\sigma| u^n_K \right)\left(\sum_{\sigma\subset S_{j,i}} |\sigma| d_\sigma + |\sigma^{(e)}| \varepsilon h^e_{N^{(e)}+1/2} \right)^{-1} \quad (4.26)$$

which are convex combinations of the approximated values of the solution on each side of the interface.

Let us add the initial condition

$$\begin{cases} u^0_K = 0, \forall K \in \mathcal{T}_i, i = 1, ..., N, \\ v^0_{e,i} = 0, i = 1, ..., N^{(e)}, \forall e. \end{cases}$$

The monolithic scheme that is used to solve (3.17-3.18) results from the equations (4.21-4.22-4.23-4.24-4.25), which is referred to as "dsa" (discontinuous selected alternative) as in [19].

The approximate solution is the piecewise constant function defined on $B_\varepsilon \times (0, (N_k + 1)k)$ by

$$u_{\varepsilon,\delta,dec,\mathcal{T},k}(x,t) = u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}(x), x \in B_\varepsilon, t \in [nk, (n+1)k), n \in \{0, ..., N_k\}$$

where $u^n_{\varepsilon,\delta,dec,\mathcal{T}}$ for $n \in \{0, ..., N_k + 1\}$ is given by

$$u^n_{\varepsilon,\delta,dec,\mathcal{T}}(x) = \begin{cases} u^n_K, x \in K, K \in \mathcal{T}_i, i = 1, ..., N, \\ v^n_{e,i}, x \in B^{dec,\varepsilon}_{lj}, x^{(e)}_1 \in (x^e_{i-1/2}, x^e_{i+1/2}), i = 1, ..., N^{(e)}, e = \overline{O_lO_j}. \end{cases} \quad (4.27)$$

The corresponding scheme to solve the diffusion equation that is (3.16-3.18) is similar to (4.21-4.22-4.23-4.24-4.25). The scheme is simply obtained by removing the terms related to the time derivative and by omitting the index n in the other terms.

The approximate solution is defined on B_ε by

$$u_{\varepsilon,\delta,dec,\mathcal{T}}(x) = \begin{cases} u_K, x \in K, K \in \mathcal{T}_i, i = 1, ..., N, \\ v_{e,i}, x \in B^{dec,\varepsilon}_{lj}, x^{(e)}_1 \in (x^e_{i-1/2}, x^e_{i+1/2}), i = 1, ..., N^{(e)}, e = \overline{O_lO_j}. \end{cases} \quad (4.28)$$

For the heat equation, the unknown of the scheme is U^n with
$(U^n)^T = \left(\{\{v^n_{e,i}, i = 1, ..., N^{(e)}\}, e \text{ edge}\}, \{\{u^n_K, K \in \mathcal{T}_i\}, i = 1, ..., N^{(e)}\}\right)$.
In the case $N_1 = 1$ and for a similar tube structure, it is proved in [19] that there is a unique solution (U^{n+1}) to equations (4.21-4.22-4.23-4.24-4.25), (U^n) being given. The generalization of this result below is straightforward.

For the diffusion equation, the unknown of the scheme is U, simply obtained by omitting the index n of U^n above.

Lemma 4.1. *There is a unique solution U to the linear system that corresponds to the monolithic scheme solving (3.16-3.18).*
There is a unique solution U^{n+1} to the linear system that corresponds to the monolithic scheme solving (3.17-3.18), U^n being given.

Before proving the existence and uniqueness of the approximate solution in both cases and establishing an estimate of these solutions, let us generalize from [19] the definition of the discrete L^2 and H^1

norms for functions on B_ε such that the average from both sides of the cross sections $S_{i,j}$ is the same. We will then continue with the discrete L^2 and H^1 norms for functions on $B_\varepsilon \times (0,T)$.

Definition 4.2. *We define $X(\mathcal{T})$ the set of the functions from B_ε to $\mathbb{R}$ which are constant over each control volume of $\mathcal{T}$. Let $W \in X(\mathcal{T})$ such that,*

$$W(x) = \begin{cases} W_K, x \in K, K \in \mathcal{T}_i, i = 1, ..., N, \\ W_{e,i}, x \in B_{lj}^{dec,\delta}, x_1^{(e)} \in (x_{i-1/2}^e, x_{i+1/2}^e), i = 1, ..., N^{(e)}, e = \overline{O_l O_j}. \end{cases}$$

We define

$$\|W\|_{L^2(B_\varepsilon)} = \left(\sum_{i=1}^{N} \sum_{K \in \mathcal{T}_i} |K| W_K^2 + \sum_{j=1, e=e_j}^{M} |\sigma^{(e)}| \varepsilon \sum_{l=1}^{N^{(e)}} h_l^e W_{e,l}^2 \right)^{\frac{1}{2}}$$

and $| W |_{1,\mathcal{T}} =$

$$\left(\sum_{i=1}^{N} \sum_{\sigma \in \mathcal{E}_{int}^i, \sigma \subset \cup_j S_{i,j}, \sigma \subset \gamma_\varepsilon^i} |\sigma| d_\sigma \left(\frac{D_\sigma W}{d_\sigma} \right)^2 + \sum_{j=1, e=e_j}^{M} |\sigma^{(e)}| \varepsilon \sum_{l=0}^{N^{(e)}} \frac{(W_{e,l+1} - W_{e,l})^2}{h_{l+1/2}^e} \right)^{1/2}$$

with $\dfrac{D_\sigma W}{d_\sigma} =$

$$\begin{cases} \dfrac{| W_K - W_L |}{d_\sigma}, \sigma \in \mathcal{E}_{int}^i, i = 1, ..., N, \sigma = \sigma_{K|L}, \\ \dfrac{| W_K |}{d_\sigma}, \sigma \subset \gamma_\varepsilon, \ \sigma \in \mathcal{E}_K, \\ \dfrac{| W_{e,1} - W_{e,0} |}{h_{1/2}^e}, \sigma \subset S_{i,j}, i < j, e = \overline{O_i O_j}, \\ \dfrac{| W_{e,N^{(e)}} - W_{e,N^{(e)}+1} |}{h_{N^{(e)}+1/2}^e}, \sigma \subset S_{i,j}, j < i, e = \overline{O_j O_i}, \end{cases}$$

where $W_{e,0}$ and $W_{e,N^{(e)}+1}$ are defined according to a formula similar to (4.26), and γ_ε^i is assumed to be empty if $i \leq N_1$.
And finally, $\|W\|_{1,\mathcal{T}}^2 = \|W\|_{L^2(B_\varepsilon)}^2 + | W |_{1,\mathcal{T}}^2$, which is a discrete $H^1(B_\varepsilon)$ norm.

Remark 4.3. *At the discrete level, we have also the property that an inner product in H can be defined as the sum of inner products in $H^1(B_{ij}^{dec,\varepsilon})$ and $H^1(B_i^{\varepsilon,\delta})$ for all $B_{ij}^{dec,\varepsilon}$ and $B_i^{\varepsilon,\delta}$, so that* $|W|_H = \sqrt{\sum_{ij} |W|_{H^1(B_{ij}^{dec,\varepsilon})}^2 + \sum_i |W|_{H^1(B_i^{\varepsilon,\delta})}^2}$ *for $W \in X(\mathcal{T})$ with*

$$|W|_{H^1(B_i^{\varepsilon,\delta})}^2 = \sum_{\sigma \in \mathcal{E}_{int}^i, \sigma \subset \cup_j S_{i,j}, \sigma \subset \gamma_\varepsilon^i} |\sigma| d_\sigma \left(\frac{D_\sigma W}{d_\sigma} \right)^2$$

and $|W|_{H^1(B_{ij}^{dec,\varepsilon})}^2 = |\sigma^{(e)}| \varepsilon \sum_{l=0}^{N^{(e)}} \dfrac{(W_{e,l+1} - W_{e,l})^2}{h_{l+1/2}^e}$ *if* $e = \overline{O_i O_j}$,
with the same notations as above.

Definition 4.4. *We define $X(\mathcal{T}, k)$ the set of the functions from $B_\varepsilon \times (0, (N_k+1)k)$ to $\mathbb{R}$ which are constant over each control volume of $\mathcal{T}$ and over each time interval $[nk, (n+1)k), n = 0, ..., N_k$. Let $W \in X(\mathcal{T}, k)$ defined by,*

$$W(x,t) = W^n(x), t \in [nk, (n+1)k), n = 0, ..., N_k,$$

where

$$W^n(x) = \begin{cases} W_K^n, x \in K, K \in \mathcal{T}_i, i = 1, ..., N, \\ W_{e,i}^n, x \in B_{lj}^{dec,\delta}, x_1^{(e)} \in (x_{i-1/2}^e, x_{i+1/2}^e), i = 1, ..., N^{(e)}, e = \overline{O_l O_j}. \end{cases}$$

We define

$$\|W\|_{2,\mathcal{T},k}^2 = \|W\|_{L^2(B_\varepsilon \times (0,(N_k+1)k))}^2 = \sum_{n=0}^{N_k} k \|W^n\|_{L^2(B_\varepsilon)}^2$$

and

$| W |^2_{1,\mathcal{T},k} = \sum_{n=0}^{N_k} k \mid W^n \mid^2_{1,\mathcal{T}}$.

And finally, $\|W\|^2_{1,\mathcal{T},k} = \|W\|^2_{2,\mathcal{T},k} + | W |^2_{1,\mathcal{T},k}$, *which is a discrete* $H^1(B_\varepsilon \times (0,(N_k+1)k))$ *norm.*

Remark 4.5. *As long as* $N_k k < T \leq (N_k+1)k$, *we assume that the function* f *is defined a little beyond* T, *and for instance for* $t \in (0, 2T)$.

Theorem 4.6. *Under the assumptions of Theorem 3.3, the following estimates hold:*
for the diffusion equation

$$\|u_{\varepsilon,\delta,dec,\mathcal{T}}\|_{1,\mathcal{T}} \leq C \|f_{\mathcal{T}}\|_{L^2(B_\varepsilon)} \tag{4.29}$$

for the heat equation

$$\|u_{\varepsilon,\delta,dec,\mathcal{T},k}\|_{1,\mathcal{T},k} \leq C \|f_{\mathcal{T},k}\|_{2,\mathcal{T},k} \tag{4.30}$$

where the constant C *does not depend on* ε.

Proof. The proof is detailed only for the heat equation. The proof for the diffusion equation can be deduced easily in a similar way. Let $f_{\mathcal{T},k} \in X(\mathcal{T},k)$, with

$$f_{\mathcal{T},k}(x,t) = f^n_{\mathcal{T}}(x), t \in [nk,(n+1)k), n = 0, ..., N_k,$$

where

$$f^n_{\mathcal{T}}(x) = \begin{cases} f^n_K, x \in K, K \in \mathcal{T}_i, i = 1, ..., N, \\ f^{e,n}_i, x \in B^{dec,\delta}_{lj}, x^{(e)}_1 \in (x^e_{i-1/2}, x^e_{i+1/2}), i = 1, ..., N^{(e)}, e = \overline{O_l O_j}. \end{cases}$$

The proof is classical. We multiply $(4.21)_1$ by $v^{n+1}_{e,i}$, sum over i, multiply by $\mid \sigma^{(e)} \mid \varepsilon$, and sum over e. Then we multiply $(4.22)_1$ by u^{n+1}_K and sum over K. Reordering the summations and applying the Cauchy-Schwarz-Buniakowski inequality gives

$$\begin{aligned} &\frac{1}{k}\|u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}\|^2_{L^2(B_\varepsilon)} + |u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}|^2_{1,\mathcal{T}} \\ &\leq \frac{1}{k}\|u^n_{\varepsilon,\delta,dec,\mathcal{T}}\|_{L^2(B_\varepsilon)}\|u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}\|_{L^2(B_\varepsilon)} + \|f^n_{\mathcal{T}}\|_{L^2(B_\varepsilon)}\|u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}\|_{L^2(B_\varepsilon)}. \end{aligned}$$

At this stage, we need a Poincaré-Friedrichs inequality to be able to write that there exists a constant C such that
$\|u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}\|_{L^2(B_\varepsilon)} \leq C\, |u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}|_{1,\mathcal{T}}$.
This result is new in this context, and it is stated in Theorem A2, proved in Appendix 2. Then
$\frac{1}{k}\|u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}\|^2_{L^2(B_\varepsilon)} + |u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}|^2_{1,\mathcal{T}} \leq \frac{1}{k}\|u^n_{\varepsilon,\delta,dec,\mathcal{T}}\|^2_{L^2(B_\varepsilon)} + C\,\|f^n_{\mathcal{T}}\|^2_{L^2(B_\varepsilon)}$.
Finally, multiplying by k and summing over n from 0 to N_k gives
$\sum_{n=0}^{N_k} \|u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}\|^2_{L^2(B_\varepsilon)} + \sum_{n=0}^{N_k} k|u^{n+1}_{\varepsilon,\delta,dec,\mathcal{T}}|^2_{1,\mathcal{T}} \leq \sum_{n=0}^{N_k} \|u^n_{\varepsilon,\delta,dec,\mathcal{T}}\|^2_{L^2(B_\varepsilon)} + C\sum_{n=0}^{N_k} k\,\|f^n_{\mathcal{T}}\|^2_{L^2(B_\varepsilon)}$.
Almost all the terms of the first sum in the left hand side are simplified with those of the first sum in the right hand side. It follows that
$\|u^{N_k+1}_{\varepsilon,\delta,dec,\mathcal{T}}\|^2_{L^2(B_\varepsilon)} + |u_{\varepsilon,\delta,dec,\mathcal{T},k}|^2_{1,\mathcal{T},k} \leq \|u^0_{\varepsilon,\delta,dec,\mathcal{T}}\|^2_{L^2(B_\varepsilon)} + C\,\|f_{\mathcal{T},k}\|^2_{2,\mathcal{T},k}$.
Since $\|u^0_{\varepsilon,\delta,dec,\mathcal{T}}\|_{L^2(B_\varepsilon)} = 0$, using the Poincaré-Friedrichs inequality again allows us to complete the proof for the heat equation.

To adapt the proof to the case of the diffusion equation, all that remains is to give the definition of

$$f_{\mathcal{T}}(x) = \begin{cases} f_K, x \in K, K \in \mathcal{T}_i, i = 1, ..., N, \\ f^e_i, x \in B^{dec,\delta}_{lj}, x^{(e)}_1 \in (x^e_{i-1/2}, x^e_{i+1/2}), i = 1, ..., N^{(e)}, e = \overline{O_l O_j}. \end{cases}$$

□

Remark 4.7. *The proof of Lemma 4.1 can easily be deduced from the above by taking $f = 0$, or $u^n_{\varepsilon,\delta,dec,\mathcal{T}} = 0$ if necessary.*

5 Error estimate

As in [16], we can assume here that there exist N polygonal nodal domains $\omega^i_{\varepsilon,\text{poly}}, i = 1, ..., N$, included in ω^i_ε. Consider $B_{\varepsilon,\text{poly}} = \left(\cup_{j=1}^M B_\varepsilon^{(e_j)}\right) \cup \left(\cup_{i=1}^N \omega^i_{\varepsilon,\text{poly}}\right)$ as the polygonal domain where the problem is actually set. It is assumed that $\partial\omega^1_{\varepsilon,\text{poly}}$ and $\partial\omega^1_\varepsilon$ are so close that, at this part of the boundary, $\dfrac{\partial u_{\varepsilon,\delta,dec}}{\partial n}(x,t) = O(h)$.

Under this assumption, for the diffusion equation with continuous junctions, it is proved that

$$\|u_\varepsilon - u_{\varepsilon,\delta,dec,\mathcal{T}}\|_{L^2(B_{\varepsilon,\text{poly}})} = O\left(\frac{h}{\sqrt{\varepsilon}}\right) + O(\varepsilon^J).$$

Also, under this assumption, for the heat equation with continuous junctions, Theorem 8.1 [16] states that

$$\|u_\varepsilon - u_{\varepsilon,\delta,dec,\mathcal{T},k}\|_{L^2(B_{\varepsilon,\text{poly}}\times(0,T))} = O\left(\frac{h}{\sqrt{\varepsilon}} + k\right) + O(\varepsilon^J).$$

In the present case, the junction conditions are no longer continuous since they are given by (3.18), consequently the estimate must now depend on the jump of discontinuity of $u_{\varepsilon,\delta,dec}$ at the junctions. Fortunately, under the assumptions of Theorem 5.3, it is proved that this jump is small. Theoretically, for finite ε the convergence of the schemes towards the solution of the diffusion or the heat equation when the size of the mesh tends to zero can not be proved, but actually, since ε is a small parameter, the impact of the jump of discontinuity is limited.

Theorem 5.1. *Under the assumptions of Theorem 3.3, the following estimates hold:*

for the diffusion equation

$$\|u_\varepsilon - u_{\varepsilon,\delta,dec,\mathcal{T}}\|_{L^2(B_{\varepsilon,poly})} = O\left(\frac{h}{\sqrt{\varepsilon}} + \sqrt{\frac{\varepsilon}{h}}[u_{\varepsilon,\delta,dec}]\right) + O(\varepsilon^J) \tag{5.31}$$

for the heat equation

$$\|u_\varepsilon - u_{\varepsilon,\delta,dec,\mathcal{T},k}\|_{L^2(B_{\varepsilon,poly}\times(0,T))} = O\left(\frac{h}{\sqrt{\varepsilon}} + \sqrt{\frac{\varepsilon}{h}}[u_{\varepsilon,\delta,dec}] + k\right) + O(\varepsilon^J) \tag{5.32}$$

where $[u_{\varepsilon,\delta,dec}]$ is the maximum of the jumps of discontinuity in space of $u_{\varepsilon,\delta,dec}$ at the junctions $S_{i,j}$.

The order of magnitude of $[u_{\varepsilon,\delta,dec}]$ is $O(\varepsilon^J)$.

Proof. The proof is given only for the heat equation. The proof for the diffusion equation can be deduced easily from the previous one.

Let $E^n_{\mathcal{T}} \in X(\mathcal{T}), n \in \{0, ..., N_k + 1\}$,

$$E^n_{\mathcal{T}}(x) = \begin{cases} E^n_K = u_{\varepsilon,\delta,dec}(x_K, t^n) - u^n_K, x \in K, K \in \mathcal{T}_i, i = 1, ..., N, \\ E^n_{e,i} = \hat{u}_{\varepsilon,\delta,dec}(x^e_i, t^n) - v^n_{e,i}, x \in B^{dec,\varepsilon}_{lj}, x^{(e)}_1 \in (x^e_{i-1/2}, x^e_{i+1/2}), i = 1, ..., N^{(e)}, e = \overline{O_l O_j}. \end{cases}$$

Let $E^n_{e,0}$ and $E^n_{e,N^{(e)}+1}$ be defined as $W^n_{e,0}$ and $W^n_{e,N^{(e)}+1}$ in Definition 4.2. The proof of the estimate (5.32) follows the proof of Theorem 8.1 [16]. The latter is produced in the case of one single node and for a simpler rod structure. However it may be easily transfered to the case of the general rod structure at hand. That is why we opt for the study of a junction. We intend to prove that $\|E^n_{\mathcal{T}}\|_{L^2(B_{\varepsilon,\text{poly}})} = O\left(\dfrac{h}{\sqrt{\varepsilon}} + \sqrt{\dfrac{\varepsilon}{h}}[u_{\varepsilon,\delta,dec}] + k\right)$ by analysing the new term $O\left(\sqrt{\dfrac{\varepsilon}{h}}[u_{\varepsilon,\delta,dec}]\right)$ coming from the consistency

error in space on the diffusion flux at the junctions. All other terms are already obtained and explained in [16].

Let us consider for instance the section $S_{i,j}, i<j$, at the distance δ from O_i on the edge $e=\overline{O_iO_j}$. Let us define

$$\overline{F}_{1/2}^{e,n}=-\frac{\partial \hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}(\delta,t^n),$$

and

$$\overline{F}_{K,\sigma}^{n}=-\int_\sigma \frac{\partial u_{\varepsilon,\delta,dec}}{\partial n}(.,t_n), \forall \sigma\in\mathcal{E}_K.$$

We have

$$\overline{F}_{K,\sigma}^{n}=|\sigma|\overline{F}_{1/2}^{e,n}, \forall \sigma\subset S_{i,j}, \sigma\in\mathcal{E}_K.$$

Let us define

$$F_{1/2}^{*,e,n}=-\frac{\hat{u}_{\varepsilon,\delta,dec}(x_1^e,t_n)-u_e^*(\delta,n)}{h_{1/2}^e}$$

with

$$u_e^*(\delta,n)=\left((\sum_{\sigma\subset S_{i,j}}|\sigma|d_\sigma)\hat{u}_{\varepsilon,\delta,dec}(x_1^e,t_n)+h_{1/2}^e\sum_{\sigma\subset S_{i,j},\sigma\in\mathcal{E}_K}|\sigma|u_{\varepsilon,\delta,dec}(x_K,t_n)\right)\left(\sum_{\sigma\subset S_{i,j}}|\sigma|d_\sigma+|\sigma^{(e)}|\varepsilon h_{1/2}^e\right)^{-1},$$

and

$$F_{K,\sigma}^{*,n}=|\sigma|F_{1/2}^{*,e,n}.$$

The left and right consistency errors (space variable) at the junction $S_{i,j}$ are $R_{1/2}^{e,n}=F_{1/2}^{*,e,n}-\overline{F}_{1/2}^{e,n}$ and $R_{K,\sigma}^{n}=\frac{1}{|\sigma|}(F_{K,\sigma}^{*,n}-\overline{F}_{K,\sigma}^{n})=R_{1/2}^{e,n}, \forall\sigma\subset S_{i,j}, \sigma\in\mathcal{E}_K$.

Let us define the following quantities :

$$\begin{cases} R_{1/2}^{*,e,n}=-\dfrac{\hat{u}_{\varepsilon,\delta,dec}(\delta,t_n)-u_e^*(\delta,n)}{h_{1/2}^e} \\ R_{1/2}^{\nabla,e,n}=-\dfrac{\hat{u}_{\varepsilon,\delta,dec}(x_1^e,t_n)-\hat{u}_{\varepsilon,\delta,dec}(\delta,t_n)}{h_{1/2}^e}+\dfrac{\partial \hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}(\delta,t_n) \end{cases}$$

such that

$$R_{1/2}^{e,n}=R_{1/2}^{*,e,n}+R_{1/2}^{\nabla,e,n}.$$

The estimate

$$R_{1/2}^{\nabla,e,n}=O\left(h\left\|\frac{\partial^2\hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)2}}\right\|_\infty\right) \tag{5.33}$$

holds but $R_{1/2}^{*,e,n}$ do not tend to zero when h tends to zero.

Let us consider the proof for the continuous junctions from there :

$$-|\sigma^{(e)}|\varepsilon R_{1/2}^{e,n+1}(E_{e,1}^{n+1}-E_{e,0}^{n+1})+\sum_{\sigma\subset S_{i,j},\sigma\in\mathcal{E}_K}|\sigma|R_{K,\sigma}^{n+1}(E_K^{n+1}-E_{e,0}^{n+1}).$$

This quantity can be found in [16], and can be grouped with the terms that are not related to the interface, to give at the end $O\left(\frac{h}{\sqrt{\varepsilon}}\right)$, with (3.19), because the term $R_{1/2}^{*,e,n}$ vanishes in the final overall estimate. Let us continue the proof assuming now (3.18). While the interior terms are unchanged from those described in [16], the quantity above reads instead

$$-R_{1/2}^{e,n+1}\frac{E_{e,1}^{n+1}-E_{e,0}^{n+1}}{h_{1/2}^e}\left(|\sigma^{(e)}|\varepsilon h_{1/2}^e+\sum_{\sigma\subset S_{i,j}}|\sigma|d_\sigma\right).$$

At this stage, the Cauchy-Schwarz-Buniakowski inequality is used. The part of the previous term: $\frac{E^{n+1}_{e,1}-E^{n+1}_{e,0}}{h^e_{1/2}}\left(|\sigma^{(e)}|\varepsilon h^e_{1/2}+\sum_{\sigma\subset S_{i,j}}|\sigma|d_\sigma\right)^{\frac{1}{2}}$ is used in $|E^{n+1}_{\mathcal{T}}|_{1,\mathcal{T}}$, the term $\left(|\sigma^{(e)}|\varepsilon h^e_{1/2}+\sum_{\sigma\subset S_{i,j}}|\sigma|d_\sigma\right)^{\frac{1}{2}}$ represents $O(\sqrt{\varepsilon h})$, and $R^{e,n+1}_{1/2}=O\left(\frac{[u_{\varepsilon,\delta,dec}]}{h}\right)$. It explains how the new term $O\left(\sqrt{\frac{\varepsilon}{h}}[u_{\varepsilon,\delta,dec}]\right)=O\left(\sqrt{\varepsilon h}\frac{[u_{\varepsilon,\delta,dec}]}{h}\right)$ appears in the final result. □

6 Numerical results

We consider model problems set in a domain with a single node O_1 ($N_1=1$) and two edges ($e_1=\overline{O_1O_2},\ e_2=\overline{O_1O_3},\ N=3$), without nodal domain at the vertices and with only one hyperplan crossing each edge at the distance δ from the node. The domain is illustrated in Figure 3. The problem corresponding to the diffusion equation has already been described in [15], and with the heat equation in [16] and [19]. Each time, the right hand side is such that the exact solution is continuous.

The numerical experiments compare the approximate solutions of the partially decomposed problems of hybrid dimension (generated by the MAPDD) obtained either directly (monolithic schemes described in Section 4) or by subdomain iteration (domain decomposition methods). Indeed domain decomposition methods are frequently used to approach the solution of geometrical multi-scale problems like the partially decomposed problems. Specifically, they are methods where subdomains do not overlap. In our case, the subdomains are delimited by interfaces at the junction between the 2D and the 1D problems. The domain decomposition methods that we use are generalizations of the Steklov-Poincaré problem. They reduce to interface systems with a small number of unknowns, which ensure that the transmission conditions are satisfied. Following for instance the methods proposed in [4, 5, 9], we obtain reduced systems of four equations and four unknowns (double trace formulation : two unknowns per interface, two interfaces). They are solved by using the matrix-free GMRES method (see [18]), and it takes four iterations to achieve convergence. When a direct solver is used to solve the reduced systems, the L^2 error between the solution of the monolithic scheme and the solution obtained from the reduced system, is in the order of 10^{-14}. For comparative purposes, the numerical experiments are carried out with two kinds of transmission conditions : pointwise continuity or continuity in average over the junctions, which correspond respectively to the classical and new version of the MAPDD. Questions are being asked about whether the milestone of 10^{-14} might be reached with the GMRES algorithm in either case, in order to see if the new transmission conditions allow more stability.

Regarding the domain decomposition strategy, the transmission conditions (3.18) are transformed into two equations on each interface. Another option is to enforce one of them through the discretization to obtain a reduced system with just one equation per interface, that ensures the other one. Such an option leads to reduced systems with just one unknown per interface (single-trace formulation). However, no differentiation can be made from the latter between the continuous and the alternative transmission conditions regarding the milestone of 10^{-14}. On the contrary, a significant difference can be seen by solving reduced systems with two equations per interface, at least in the case of the heat equation.

The methodology used to solve the diffusion equation is now described. In order to define the separated 1D and 2D problems, let us consider the equations below taken from (3.16):

$$\begin{cases} -\dfrac{\partial^2\hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)2}}(x)=\hat{f}(x_1^{(e)}),\quad x\in e_{1l}^{dec,\delta},\ l=j+1,\ j\in\{1,2\},\\ \hat{u}_{\varepsilon,\delta,dec}|_{x_1^{(e)}=|e|}=0,\ e=e_j,\quad j\in\{1,2\}, \end{cases} \tag{6.34}$$

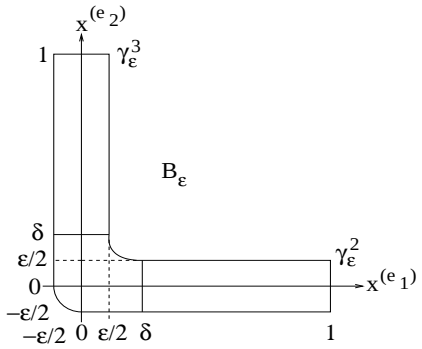

Figure 3: The domain B_ε.

$$\left\{\begin{array}{l} -\triangle u_{\varepsilon,\delta,dec}(x) = f(x), \quad x \in B_1^{\varepsilon,\delta}, \\ \dfrac{\partial u_{\varepsilon,\delta,dec}}{\partial n}(x) = 0, \quad x \in \partial B_1^{\varepsilon,\delta} \cap \partial B_\varepsilon. \end{array}\right. \tag{6.35}$$

The junction condition (3.18) may be written

$$\left\{\begin{array}{l} \hat{u}_{\varepsilon,\delta,dec}|_{x_1^{(e)}=\delta} = \alpha_j, \ e = e_j, \ j \in \{1,2\}, \\ \dfrac{\partial \hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta} = \beta_j, \ e = e_j, \ j \in \{1,2\}, \end{array}\right. \tag{6.36}$$

$$\left\{\begin{array}{l} \dfrac{1}{|S_{1,l}|}\displaystyle\int_{S_{1,l}} u_{\varepsilon,\delta,dec}|_{x_1^{(e)}=\delta} dx^{(e)\prime} = \alpha_j, \ e = e_j, \ l = j+1, \ j \in \{1,2\}, \\ \dfrac{\partial u_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta} = \beta_j, \ e = e_j, \ j \in \{1,2\}. \end{array}\right. \tag{6.37}$$

Boundary conditions must be added to (6.34) and (6.35) so that the problems are well posed. First, we define the operator $S_{1D,j}^{DN}$ from $I\!R$ to $I\!R$, $j \in \{1,2\}$, by

$$S_{1D,j}^{DN}(\alpha_j) = \frac{\partial \hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta}, \ e = e_j,$$

where $\hat{u}_{\varepsilon,\delta,dec}$ is solution of (6.34) with $(6.36)_1$.

It is underlined that (6.35) is well posed neither with the boundary conditions $(6.37)_1$ nor $(6.37)_2$. Then we cannot consider continuity in average of $u_{\varepsilon,\delta,dec}$ on both interfaces. Though, this makes it possible to test the alternative transmission condition at only one interface ($j = 1$), the solution being continuous on the other junction ($j = 2$). Then, the junction condition actually used is not (3.18) as such but rather (6.36) and (6.37) if (only) $j = 1$ to which is added

$$\left\{\begin{array}{c} \dfrac{1}{|S_{1,l}|}\displaystyle\int_{S_{1,l}} \frac{\partial u_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta} dx^{(e)\prime} = \beta_j, \ e = e_j, \ l = j+1, \ j \in \{1,2\}, \\ u_{\varepsilon,\delta,dec}|_{x_1^{(e)}=\delta} = \alpha_j, \ e = e_j, \ j \in \{1,2\}, \end{array}\right. \tag{6.38}$$

if (only) $j = 2$.

As a result, we are able to define the operator S_{2D}^{ND} from $I\!R^2$ to $I\!R^2$, by

$$S_{2D}^{ND}(\beta_1,\alpha_2) = \left(\frac{1}{|S_{1,2}|}\int_{S_{1,2}} u_{\varepsilon,\delta,dec}|_{x_1^{(e)}=\delta} dx^{(e)\prime} \ ; \ \frac{1}{|S_{1,3}|}\int_{S_{1,3}} \frac{\partial u_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta} dx^{(e)\prime} \right)$$

where $u_{\varepsilon,\delta,dec}$ is solution of (6.35) with $(6.37)_2$ with $j = 1$ and $(6.38)_2$ with $j = 2$.

Let $\alpha_j, j \in \{1,2\}$, denote the unknown (average) values and $\beta_j, j \in \{1,2\}$, the unknown (average) values of the normal derivative, then $\hat{u}_{\varepsilon,\delta,dec}$ and $u_{\varepsilon,\delta,dec}$ are the solution of (6.34-6.35-6.36) and (6.37) with $j = 1$ and (6.38) with $j = 2$ if (only)

$$\begin{cases} \beta_j - S_{1D,j}^{DN}(\alpha_j) = 0, j \in \{1,2\}, \\ (\alpha_1, \beta_2) - S_{2D}^{ND}(\beta_1, \alpha_2) = 0, \end{cases} \tag{6.39}$$

which is the reduced system that reflects $(6.36)_2$, $(6.37)_1$ with $j = 1$ and $(6.38)_1$ with $j = 2$ (see [19] where a similar approach is used for the heat equation).

The scheme above will be referred to as *NmDDN*. The capital letters "D" and "N" represent the interface conditions "Dirichlet" and "Neumann", writing firstly the input (on $S_{1,2}$) and the output (on $S_{1,2}$) values of the 2D problem, and secondly those of the 1D problems. The letter "m" in front of "D" stands for "mean value", that is to say the output value on $S_{1,2}$ of the 2D problem is the mean value of the solution (this name does not reflect the input/output values on $S_{1,3}$).

Other schemes can be defined by using in turn α_j and β_j as input values for the separated problems. The same naming is used for all schemes (for both the diffusion and heat equations). Each scheme is denominated by the four corresponding letters, preceded by "m" where appropriate. The scheme *NmDND* uses the operator $S_{1D,j}^{ND}$, which can easily be deduced from $S_{1D,j}^{DN}$, $j \in \{1,2\}$. In what follows, for the sake of simplicity, unlike the previous schemes, as long as the 2D problem is well posed, there is the same boundary condition on both interfaces (for both the diffusion and heat equations). For the purpose of comparison, results from the schemes *DmNND* and *DmNDN* (continuous junctions on both interfaces) are also reported. These schemes are defined by the means of the operator S_{2D}^{DN},

$$S_{2D}^{DN}(\alpha_1, \alpha_2) = \left(\frac{1}{|S_{1,2}|} \int_{S_{1,2}} \frac{\partial u_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta} dx^{(e)\prime} \; ; \; \frac{1}{|S_{1,3}|} \int_{S_{1,3}} \frac{\partial u_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}|_{x_1^{(e)}=\delta} dx^{(e)\prime} \right)$$

where $u_{\varepsilon,\delta,dec}$ is solution of (6.35) and $(6.38)_2$. The outcomes can be seen on Table 1.

-	*DmNND*	*DmNDN*	*NmDND*	*NmDDN*
preconditioning	continous	continuous	discontinuous	discontinuous
$\alpha-\beta$	$5.3\,10^{-15}$	$1.9\,10^{-14}$	$1.1\,10^{-14}$	$2.0\,10^{-14}$
$\alpha-\alpha$	$5.3\,10^{-15}$	$1.8\,10^{-14}$	$1.1\,10^{-14}$	$2.3\,10^{-14}$
$\beta-\alpha$	$4.3\,10^{-15}$	$1.9\,10^{-14}$	$1.1\,10^{-14}$	$2.4\,10^{-14}$
$\beta-\beta$	$5.2\,10^{-15}$	$1.9\,10^{-14}$	$1.1\,10^{-14}$	$2.4\,10^{-14}$

Table 1: Diffusion equation. The L^2 error between the solution of the monolithic scheme and the solution obtained by using the domain decomposition method.

Several diagonal preconditioners of the reduced systems are considered (even if we are not seeing the positive effect in the case of the diffusion equation), by selecting the coefficients of the variables equal to 1 when one acts as if the solution of the separated problems are nil. The variables in question can be read in the first column of Table 1 : they are the variables having the coefficient equal to 1 in the equations describing the 1D side and the 2D side respectively. The four versions of the schemes, whether continuity is enforced or is not on the junctions, give great accuracy. The GMRES algorithm gives results in the order of 10^{-14} in either case, as if a direct solver has been used to solve the interface system. Regarding the diffusion equation, the experiments do not allow us to distinguish between the two kinds of interface conditions with regard to stability properties.

Rather, in the case of the heat equation, the results differ significantly. We will not recall the domain decomposition strategy used for the heat equation (for details see [19]). Indeed, after performing an Euler's implicite time-discretization, it is very similar to that described previously. There is, however, an essential difference between the diffusion equation and the heat equation. The separated 2D problem for the heat equation is well posed even with alternative junctions on both interfaces. The schemes *NmDND* and *NmDDN* involve the solution of the 2D separated problem with the boundary conditions $(6.37)_2$. Some interesting outcomes are resumed in Table 2 below, a part of them comes from [19]. The schemes *DmNND* and *DmNDN* correspond to continuous junctions as above.

-	*DmNND*	*DmNDN*	*NmDND*	*NmDDN*
preconditioning	continous	continuous	discontinuous	discontinuous
$\alpha-\beta$	$4.8\,10^{-10}$	$9.2\,10^{-7}$	$1.0\,10^{-11}$	$4.0\,10^{-7}$
$\alpha-\alpha$	$2.4\,10^{-11}$	$1.7\,10^{-13}$	$2.1\,10^{-10}$	$2.5\,10^{-6}$
$\beta-\alpha$	0.017	$2.0\,10^{-7}$	$4.2\,10^{-6}$	$4.2\,10^{-6}$
$\beta-\beta$	$1.9\,10^{-5}$	$2.4\,10^{-9}$	$3.5\,10^{-14}$	$8.1\,10^{-14}$

Table 2: Heat equation. The L^2 error between the solution of the monolithic scheme and the solution obtained by using the domain decomposition method.

For the heat equation, the $\alpha-\alpha$ version of the scheme *DmNDN* gives accuracy of the order of 10^{-13}, it is an appropriate scheme for continuous junctions. The $\beta-\beta$ version of the schemes *NmDDN* and *NmDND* gives a greater accuracy in the case of alternative junctions. These latter schemes give a result in the order of 10^{-14} as if a direct solver has been used to solve the interface system. The GMRES algorithm can not achieve comparable accuracy for the schemes that correspond to continuous junctions. It can be concluded that the schemes that make use of alternative junctions have better properties of stability, even if we are only able to achieve this result by using a preconditioner.

7 Conclusion

We introduce a new version of the MAPDD allowing the discontinuity of the solution at the interface of models of different dimensions. At the junctions the pointwise continuity of the flux and continuity of the mean value of the unknown function are required. These conditions are inverse with respect to the classical version of the MAPDD : the pointwise continuity of the unknown function and continuity of the mean value of the flux. This new version of the MAPDD is applied to the stationary diffusion equation and to the non-stationary heat equation set in a rod structure with Neumann's condition at the lateral boundary. Thus, the original problem is reduced to a partially decomposed problem of hybrid dimension. We prove that the partially decomposed problem admits a unique solution. These problems are non-standard because their solutions are discontinuous at the interface. The proof of the theorem of existence of solution uses a specific Poincaré - Friedrichs inequality for functions having discontinuities at the junction interface but keeping the continuous mean value. A theorem on the error estimate is proved. Then the hybrid dimension problem is solved numerically by a special finite volume scheme. For this scheme the convergence result is proved and the error is estimated. This estimate depends on the small parameter and on the steps of the scheme. The jumps of discontinuity in space of the unknown function at the junctions are involved in the error estimate, but the order of magnitude of this term is small.

Numerical experiments compare the approximate solutions of the hybrid dimensional partially decomposed problem (generated by the MAPDD) obtained either directly or by subdomain iteration. In

the latter case, the interfaces are located at the junctions and the interface systems are solved using the GMRES method. The numerical results show that the L^2 error between the approximate solutions cannot be as small as 10^{-14} with the classical transmission conditions in the case of the heat equation, while this milestone is reached with the new transmission conditions.

This new version of the MAPDD allows the discontinuity of the solution at the junctions. Only the weak continuity of the unknown function is required. In the same way, specific Robin interface conditions in relation to the mean value of the unknown function and pointwise value of the flux could just as well be considered.

8 Appendix: The Poincaré and the Poincaré - Friedrichs inequalities for a finite rod structure with discontinuities conserving the average at the cross-section

A1 Appendix 1: The Poincaré and the Friedrichs inequalities for a finite rod structure with discontinuities conserving the average at the cross-section

The goal of this appendix is to prove the Poincaré - Friedrichs inequality for a function $u \in H^1_{\gamma,0}(B_\varepsilon)$. Below we consider domains called $L-$domains which are connected and bounded open sets in $\mathbb{R}^s$ or $\mathbb{R}^{s-1}$ with a Lipschitz boundary. We suppose also that ε is sufficiently small. For the reader's convenience let us recall several lemmata from [17].

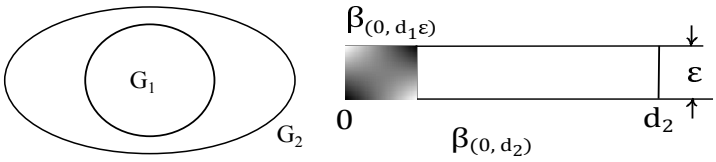

Figure 4: (a) Domains of lemma A1. $G_1 \subset G_2$. (b) Domains of lemma A3.

Lemma A1. *Let G_1, G_2 be $L-$domains, $G_1 \subset G_2$. Then for each function $u \in H^1(G_2)$ the estimate holds true*

$$(\{u\}_1 - \{u\}_2)^2 \leq c \int_{G_2} (\nabla u)^2 dx,$$

where c is a constant depending on G_1, G_2, and $\{u\}_i = \frac{1}{|G_i|} \int_{G_i} u\, dx$. Lemma is proved in [17] Chapter 4 Appendix A2.

Lemma A2. *Let $G_i^\varepsilon = \{x \in \mathbb{R}^s \mid \frac{x}{\varepsilon} \in G_i\}$, where G_1, G_2 are $L-$domains, $G_1 \subset G_2$. Then for each $u \in H^1(G_2^\varepsilon)$ the estimate holds*

$$(\{u\}_1^\varepsilon - \{u\}_2^\varepsilon)^2 \leq c\varepsilon^{2-s} \int_{G_2^\varepsilon} (\nabla u)^2 dx,$$

where c is independent of ε, but depends on G_1 and G_2,

$\{u\}_i^\varepsilon = \frac{1}{|G_i^\varepsilon|} \int_{G_i^\varepsilon} u\, dx, \quad i = 1, 2.$ Lemma is proved in [17].

Lemma A3. *Let β be an $(s-1)$ dimensional $L-$domain, $\beta_{(a,b)}$ be a cylinder $(a,b) \times \beta$; denote*

$$\beta^\varepsilon_{(a,b)} = \{x \in \mathbb{R}^s \mid (\frac{x_2}{\varepsilon}, ..., \frac{x_s}{\varepsilon}) \in \beta, \quad x_1 \in (a,b)\}.$$

Let d_1, d_2 be constants independent of ε. Then

$$(\{u\}^{\varepsilon}_{(0,d_1\varepsilon)} - \{u\}^{\varepsilon}_{(0,d_2)})^2 \le \frac{d_2}{|\beta|} \quad \varepsilon^{1-s} \int_{\beta^{\varepsilon}_{(0,d_2)}} (\frac{\partial u}{\partial x_1})^2 \quad dx,$$

where

$$u \in H^1(\beta_{(0,d_2)}), \quad \{u\}^{\varepsilon}_{(a,b)} = \frac{1}{|\beta^{\varepsilon}_{(a,b)}|} \int_{\beta^{\varepsilon}_{(a,b)}} u\, dx.$$

Lemma is proved in [17].

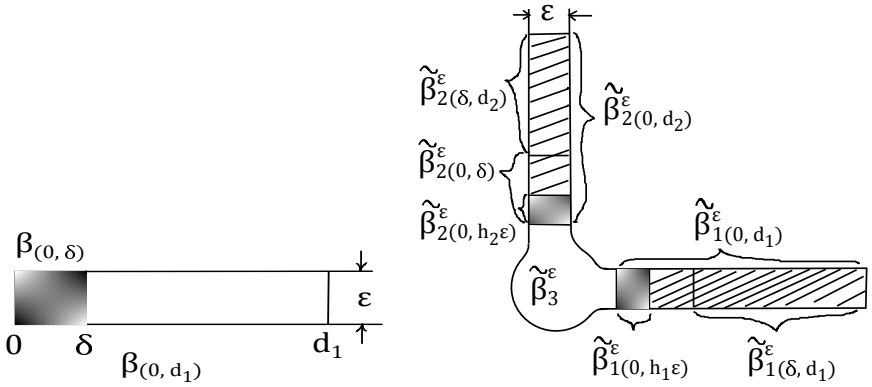

Figure 5: (a) Domains of lemma A4. (b) Domains of lemma A5.

Lemma A4 *If $u \in H^1(\beta^{\varepsilon}_{(0,\delta)}) \cap H^1(\beta^{\varepsilon}_{(\delta,d_1)})$, where $\delta > 0$, $\delta < d_1$ and if the integral of the trace $\int_S u(\delta, x')dx'$ is the same from both sides of the cross section $S = \{x_1 = \delta, x'/\varepsilon \in \beta\}$ then*

$$(\{u\}^{\varepsilon}_{(0,\delta)} - \{u\}^{\varepsilon}_{(\delta,d_1)})^2 \le \frac{d_1}{|\beta|} \quad \varepsilon^{1-s} \int_{\beta^{\varepsilon}_{(0,\delta)} \cup \beta^{\varepsilon}_{(\delta,d_1)}} (\frac{\partial u}{\partial x_1})^2 \quad dx.$$

Here and below an intersection $H^1(A) \cap H^1(B)$ denotes the space of functions defined on $A \cup B$ with restrictions belonging to both spaces $H^1(A)$ and $H^1(B)$.

Proof

As usual, we prove first the inequality for C^1-smooth functions and then passing to the limit we generalize it for the Sobolev spaces.

Consider $t_1 \in (0, \delta)$ and $x_1 \in (\delta, d_1)$, applying the Newton -Leibnitz formula, we get:

$$u(x_1, x') - u(t_1, x') - u(\delta + 0, x') + u(\delta - 0, x') = \int_{t_1}^{\delta} \frac{\partial u}{\partial x_1}(\theta, x')d\theta + \int_{\delta}^{x_1} \frac{\partial u}{\partial x_1}(\theta, x')d\theta,$$

integrate it for $t_1 \in (0, \delta), x_1 \in (\delta, d_2)$ and $x' \in \beta_\varepsilon$, $\quad \beta_\varepsilon = \{x' : x'/\varepsilon \in \beta\}$. Taking into account that $\int_{\beta_\varepsilon}(-u(\delta + 0, x') + u(\delta - 0, x'))dx' = 0$, we obtain

$$\delta \int_{\beta^{\varepsilon}_{(\delta,d_1)}} u(x)dx - (d_1 - \delta) \int_{\beta^{\varepsilon}_{(0,\delta)}} u(x)dx = \int_{\beta_\varepsilon} \Big((d_1 - \delta) \int_0^{\delta} \int_{t_1}^{\delta} \frac{\partial u}{\partial x_1}(\theta, x')d\theta dt_1 + \delta \int_{\delta}^{d_1} \int_{\delta}^{x_1} \frac{\partial u}{\partial x_1}(\theta, x')d\theta dx_1\Big)dx',$$

so

$$\{u\}^{\varepsilon}_{(0,\delta)} - \{u\}^{\varepsilon}_{(\delta,d_1)} = \frac{1}{\delta(d_1 - \delta)|\beta|\varepsilon^{s-1}} \int_{\beta_\varepsilon} \Big((d_1 - \delta) \int_0^{\delta} \int_{t_1}^{\delta} \frac{\partial u}{\partial x_1}(\theta, x')d\theta dt_1 + \delta \int_{\delta}^{d_1} \int_{\delta}^{x_1} \frac{\partial u}{\partial x_1}(\theta, x')d\theta dx_1\Big)dx',$$

and so, applying the Cauchy-Schwarz-Buniakowski inequality, we get

$$|\{u\}^{\varepsilon}_{(0,d_1)} - \{u\}^{\varepsilon}_{(0,d_1\varepsilon)}| \le \frac{1}{\varepsilon^{s-1}|\beta|} ||\frac{\partial u}{\partial x_1}||_{L^1(\beta^{\varepsilon}_{(0,d_1)})} \le \frac{\sqrt{d_1}}{\sqrt{\varepsilon^{s-1}|\beta|}} ||\frac{\partial u}{\partial x_1}||_{L^2(\beta^{\varepsilon}_{(0,d_1)})}.$$

Lemma is proved. It generalizes Lemma3.

Lemma A5 *Let δ be a positive number greater than $h_i\varepsilon$, β_1 and β_2 be two $L-$domains in $I\!R^{s-1}$, β_3 be an $L-$domain in $I\!R^s$. Denote*

$$\beta_i^\varepsilon = \{x \in I\!R^{s-1} \mid \frac{x}{\varepsilon} \in \beta_i\} \quad (i = 1, 2), \quad \beta_3^\varepsilon = \{x \in I\!R^s \mid \frac{x}{\varepsilon} \in \beta_3\},$$

and let $\tilde{\beta}^\varepsilon_{i(a,b)}$ be a cylinder obtained from the cylinder $(a, b) \times \beta_i^\varepsilon$ by some rotations and translations. Let $\tilde{\beta}^\varepsilon_{i(0,d_i)} \cap \beta_3^\varepsilon = \tilde{\beta}^\varepsilon_{i(0,h_i\varepsilon)}$, where d_i, h_i do not depend on ε. Let u be function defined in B_ε such that $u \in H^1(B_U) \cap \cap_{i=1}^2 H^1(\tilde{\beta}^\varepsilon_{i(\delta,d_i)})$, where $B_U = (\bigcup_{i=1}^2 \quad \tilde{\beta}^\varepsilon_{i(0,\delta)}) \quad \bigcup \quad \beta_3^\varepsilon$ and the average of u at the cross sections of the cylinders $\tilde{\beta}^\varepsilon_{i(0,d_i)}$ corresponding to $x_1 = \delta$ (in local variables with respect to $(0, d_i) \times \beta_i^\varepsilon$) has the same value from both sides of the cross section.

Then

$$\frac{1}{|\tilde{\beta}^\varepsilon_{1(\delta,d_1)}|}(\int_{\tilde{\beta}^\varepsilon_{(\delta,d_1)}} u dx)^2, \frac{1}{|\tilde{\beta}^\varepsilon_{1(0,\delta)}|}(\int_{\tilde{\beta}^\varepsilon_{(0,\delta)}} u dx)^2, \quad \frac{1}{|\beta_3^\varepsilon|}(\int_{\beta_3^\varepsilon} u dx)^2 \le$$

$$\frac{c_1}{|\tilde{\beta}^\varepsilon_{(\delta,d_2)}|}(\int_{\tilde{\beta}^\varepsilon_{2(0,d_2)}} u dx)^2 \quad + \quad c_2\Big(\int_{B_U} (\nabla u)^2 dx + \sum_{i=1}^2 \int_{\tilde{\beta}^\varepsilon_{i(\delta,d_i)}} (\nabla u)^2 dx\Big),$$

where c_1, c_2 do not depend on ε and δ.

Proof.

Denote

$\frac{1}{|\tilde{\beta}^\varepsilon_{i(a,b)}|}(\int_{\tilde{\beta}^\varepsilon_{i(a,b)}} u dx)$ as $\{u\}_{i,\varepsilon}^{(a,b)}$, $i = 1, 2$, and $\{u\}_{3,\varepsilon} = \frac{1}{|\beta_3^\varepsilon|}\int_{\beta_3^\varepsilon} u dx$, then applying the Young's inequality, Lemma A4, and then Lemma A3 we get:

$$|\tilde{\beta}^\varepsilon_{1(\delta,d_1)}|(\{u\}_{1,\varepsilon}^{(\delta,d_1)})^2 \le 2d_1\varepsilon^{s-1}|\beta_1|((\{u\}_{1,\varepsilon}^{(0,\delta)})^2 \quad + \quad (\{u\}_{1,\varepsilon}^{(\delta,d_1)} \quad - \quad \{u\}_{1,\varepsilon}^{(0,\delta)})^2)$$

$$\le 2d_1\varepsilon^{s-1}|\beta_1|(\{u\}_{1,\varepsilon}^{(0,\delta)})^2 \quad + \quad 2d_1^2(\int_{\tilde{\beta}^\varepsilon_{1(0,\delta)}} (\nabla u)^2 dx + \int_{\tilde{\beta}^\varepsilon_{1(\delta,d_1)}} (\nabla u)^2 dx)$$

$$\le 4d_1\varepsilon^{s-1}|\beta_1|((\{u\}_{1,\varepsilon}^{(0,h_1\varepsilon)})^2 \quad + \quad (\{u\}_{1,\varepsilon}^{(0,\delta)} \quad - \quad \{u\}_{1,\varepsilon}^{(0,\varepsilon h_1)})^2) \quad + \quad 2d_1^2\Big(\int_{\tilde{\beta}^\varepsilon_{1(0,\delta)}} (\nabla u)^2 dx + \int_{\tilde{\beta}^\varepsilon_{1(\delta,d_1)}} (\nabla u)^2 dx\Big)$$

$$\le 4d_1\varepsilon^{s-1}|\beta_1|((\{u\}_{1,\varepsilon}^{(0,h_1\varepsilon)})^2 \quad + \quad 6d_1^2(\int_{\tilde{\beta}^\varepsilon_{1(0,\delta)}} (\nabla u)^2 dx + \int_{\tilde{\beta}^\varepsilon_{1(\delta,d_1)}} (\nabla u)^2 dx).$$

According to the Young's inequality and Lemma A2 we get:

$$4d_1\varepsilon^{s-1}|\beta_1|((\{u\}_{1,\varepsilon}^{(0,h_1\varepsilon)})^2 \quad \le 8d_1\varepsilon^{s-1}|\beta_1|\Big((\{u\}_{3,\varepsilon})^2 \quad + \quad (\{u\}_{1,\varepsilon}^{(0,h_1\varepsilon)} \quad - \quad \{u\}_{3,\varepsilon})^2\Big)$$

$$\le 8d_1\varepsilon^{s-1}|\beta_1|\Big((\{u\}_{3,\varepsilon})^2 \quad + \quad c\varepsilon^{2-s}\int_{\beta_3^\varepsilon} (\nabla u)^2 dx\Big).$$

Similar considerations for $\tilde{\beta}^\varepsilon_{2(0,h_2\varepsilon)}, \tilde{\beta}^\varepsilon_{2(0,\delta)}, \tilde{\beta}^\varepsilon_{2(\delta,d_2)}$ finalize the proof of the estimate for $\frac{1}{|\tilde{\beta}^\varepsilon_{1(\delta,d_1)}|}(\int_{\tilde{\beta}^\varepsilon_{(\delta,d_1)}} u dx)^2$.

Lemma A6. *Let G be an $L-$domain in $I\!R^s$, $G^\varepsilon \quad = \quad \{x \in I\!R^s | \frac{x}{\varepsilon} \in G\}$, and let u be a function, $u \in H^1(G^\varepsilon)$. Then the estimate takes place*

$$||u||^2_{L^2(G^\varepsilon)} \quad \le \quad \frac{1}{|G^\varepsilon|}(\int_{G^\varepsilon} u dx)^2 \quad + \quad \varepsilon^2 c||\nabla u||^2_{L^2(G^\varepsilon)}.$$

Here constant c depends on G. This assertion is a standard scaling of the classical Poincaré - Friedrichs inequality.

Lemma A7. *Let $u \in H^1(\beta^\varepsilon_{(0,d)})$ then*

$$||u||^2_{L^2(\beta^\varepsilon_{(0,d)})} \leq \frac{4}{|\beta^\varepsilon_{(0,d)}|}(\int_{\beta^\varepsilon_{(0,d)}} u(x)dx)^2 + 8d^2 \int_{\beta^\varepsilon_{(0,d)}} (\nabla u)^2 dx$$

for all sufficiently small ε.

Proof see [17].

Lemma A7 admits the following generalization:

Lemma A8. *Let the domain $\tilde{\beta}^\varepsilon_{(a,b)}$ be obtained from $\beta^\varepsilon_{(a,b)}$ by means of rotation and translation, then the estimate holds*

$$\|u\|^2_{L^2(\tilde{\beta}^\varepsilon_{(0,d)})} \quad \leq \quad \frac{4}{|\tilde{\beta}^\varepsilon_{(0,d)}|} \quad (\int_{\tilde{\beta}^\varepsilon_{(0,d)}} udx)^2 \quad + \quad c_3 \int_{\tilde{\beta}^\varepsilon_{(0,d)}} (\nabla u)^2 dx,$$

where c_3 is independent of d and ε, ε is sufficiently small.

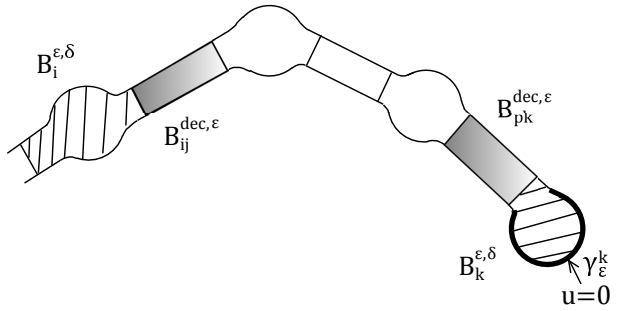

Figure 6: A chain of the proof of Theorem A1.

Theorem A1. (Poincaré-Friedrichs inequality for the rod structure) *Let u be a function of space $H^1_{\gamma,0}(B_\varepsilon, \delta)$, then*

$$\|u\|^2_{L^2(B_\varepsilon)} \leq C_{PF}(\sum_{i,j} \|\nabla u\|^2_{L^2(B^{dec,\varepsilon}_{ij})} + \sum_i \|\nabla u\|^2_{L^2(B^{\varepsilon,\delta}_i)}),$$

where C_{PF} is a constant independent of ε and δ.

Proof Let a cylinder $B^{dec,\varepsilon}_{ij}$ or a domain $B^{\varepsilon,\delta}_j$ be connected with the part of the boundary $\gamma^k_\varepsilon \subset B^{\varepsilon,\delta}_k$ by a chain of cylinders $B^{dec,\varepsilon}_{\alpha r}$ and domains $B^{\varepsilon,\delta}_q$. Using the same scheme as in Theorem 4.A2.1 Appendix 4.A2 [17], we apply successfully Lemma A5 to the connected triplets $B^{dec,\varepsilon}_{\alpha r} \cup B^{\varepsilon,\delta}_r \cup B^{dec,\varepsilon}_{rm}$ of this chain and we get the estimate

$$\|u\|^2_{L^2(B^{dec,\varepsilon}_{ij})}, \|u\|^2_{L^2(B^{\varepsilon,\delta}_i)} \leq C_1(\sum_{i,j} \|\nabla u\|^2_{L^2(B^{dec,\varepsilon}_{ij})} + \sum_i \|\nabla u\|^2_{L^2(B^{\varepsilon,\delta}_i)} + \frac{1}{|B^{dec,\varepsilon}_{pk}|}(\int_{B^{dec,\varepsilon}_{pk}} udx)^2).$$

Using Lemmata A2-A4 we majorate the last term as

$$c\Big(\frac{1}{|\omega^k_\varepsilon|}\Big(\int_{\omega^k_\varepsilon} u(x)dx\Big)^2 + \|\nabla u\|^2_{L^2(B^{dec,\varepsilon}_{pk} \cup B^{\varepsilon,\delta}_k)}\Big),$$

where c is a constant independent of small parameters.

Applying now the Cauchy-Schwarz-Bunyakowskii inequality and then the standard Poincaré- Friedrichs inequality

$$\frac{1}{|\omega^k_\varepsilon|}\Big(\int_{\omega^k_\varepsilon} u(x)dx\Big)^2 \leq \|u\|^2_{L^2(\omega^k_\varepsilon)} \leq c\|\nabla u\|^2_{L^2(\omega^k_\varepsilon)}$$

we get the assertion of the theorem.

Theorem is proved.

Remark A1. (Poincaré inequality for the rod structure) *Let u be a function from $L^2(B_\varepsilon)$ belonging to $H^1(B_{ij}^{dec,\varepsilon}) \cap H^1(B_i^{\varepsilon,\delta})$, then*

$$\|u\|^2_{L^2(B_\varepsilon)} \leq \frac{1}{|B_\varepsilon|}(\int_{B_\varepsilon} udx)^2 + C_P(\sum_{i,j} \|\nabla u\|^2_{L^2(B_{ij}^{dec,\varepsilon})} + \sum_i \|\nabla u\|^2_{L^2(B_i^{\varepsilon,\delta})}),$$

where C_P is a constant independent of ε and δ.

The Proof follows the scheme of the proof of Theorem 4.A2.1 of Appendix 4.A2 [17] using Lemma A5.

A2 Appendix 2: The discrete mean Poincaré and the Poincaré-Friedrichs inequalities for a finite rod structure with discontinuities conserving the average at the cross-section

The continuity of the mean value of the unknown function at the interfaces plays a key role in the proof. First, a Poincaré-Wirtinger type inequality (mean Poincaré) is established. We rely in particular on [8],[21] and [3]. Further references can be found in [20] where a Poincaré inequality is proved for the classical version of the MAPDD whith continuity at the cross-sections. Then the Poincaré - Friedrichs inequality follows in the functional space we are interested in, where functions vanish at γ_ε.

The proof follows the scheme of the proof of the lemma 10.2 [8]. However the justification of the MAPDD needs the estimates with an explicite expression of the dependence of the constants on ε.

Lemma A9. *Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^s, s = 2$, which is a thin domain. Let ω be an open convex subset of Ω. Then, there exists $C \geq 0$, such that for all admissible meshes $\mathcal{T}$, and for any piecewise constant function u defined on the control volumes of $\mathcal{T}$, the following inequality holds*

$$\|u - m_\omega\|^2_{L^2(\omega)} \leq C \mid u \mid^2_{1,\mathcal{T}}$$

with m_ω the mean value of u on ω, and $\mid u \mid^2_{1,\mathcal{T}} = \sum_{\sigma \in \mathcal{E}_{int}} |\sigma| d_\sigma \left(\frac{D_\sigma u}{d_\sigma}\right)^2$, $D_\sigma u = \mid u_K - u_L \mid, \sigma \in \mathcal{E}_{int}$, the set of interior edges, $\sigma = \sigma_{K|L}$. The constant C does not depend on the thickness of the domain.

Proof It is proved in the lemma 10.2 [8] that

$$\|u - m_\omega\|^2_{L^2(\omega)} \leq \frac{\mathrm{diam}(\Omega)}{|\omega|} \mid u \mid^2_{1,\mathcal{T}} \left(\int_{\omega'} |z| dz\right)$$

with $\omega' = \{z \in \mathbb{R}^2 : \exists x \in \omega, \exists y \in \omega, z = y - x\}$. Then

$$\|u - m_\omega\|^2_{L^2(\omega)} \leq (\mathrm{diam}(\Omega))^2 \frac{|\omega'|}{|\omega|} \mid u \mid^2_{1,\mathcal{T}} \leq C \mid u \mid^2_{1,\mathcal{T}}$$

Indeed, if ω is a thin domain, then this is also the case for ω'. Consequently, $|\omega'|$ and $|\omega|$ are in the same order of magnitude, and the constant C only depends on $\mathrm{diam}(\Omega)$. Lemma A9 is proved.

Lemma A10. *Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^2$. Let ω be an open convex subset of Ω. Let $I \subset \partial\omega, |I| > 0$. Then, there exists $C \geq 0$, such that for all admissible meshes $\mathcal{T}$, and for any piecewise constant function u defined on the control volumes of $\mathcal{T}$, the following inequality holds*

$$\|u - m_I\|^2_{L^2(\omega)} \leq C \mid u \mid^2_{1,\mathcal{T}}$$

with m_I the mean value of u on I, and $| u |^2_{1,\mathcal{T}} = \sum_{\sigma\in\mathcal{E}_{int},\sigma\subset I} |\sigma| d_\sigma \left(\frac{D_\sigma u}{d_\sigma}\right)^2$, $D_\sigma u = | u_K - u_L |, \sigma \in \mathcal{E}_{int}$, *the set of interior edges, $\sigma = \sigma_{K|L}$. Let $u(x) = u_\sigma, x \in \sigma, \sigma \subset I$, then $D_\sigma u = | u_K - u_\sigma |, \sigma \subset I, \sigma \in \mathcal{E}_K$.*

If Ω is a thin domain, the constant C does not depend on the thickness of the domain, C only depends on the diameter of Ω.

Theorem A2. (Poincaré-Friedrichs inequality for the rod structure) *Let u be a piecewise constant function from $X(\mathcal{T})$ such that u vanishes at γ_ε, and at each cross section $S_{i,j}$ the discrete average of u from both sides of it is the same, then*

$$\|u\|^2_{L^2(B_\varepsilon)} \leq C \mid u \mid^2_{1,\mathcal{T}},$$

where C is a constant independent of ε, and $| \, . \, |_{1,\mathcal{T}}$ is defined in Definition 4.2.

The result is true with any arbitrary (admissible) mesh of $B^{dec,\varepsilon}_{ij}$ (modulo an adaptation of Definition 4.2 if the mesh is no more rectangular).

Proof

The proof follows the scheme of the proof of lemma 10.2 [8]. The proof requires the domain to be polygonal. As in [16], we can assume that there exists a polygonal domain $B_{\varepsilon,poly}$ included in B_ε, and prove the discrete inequality above in $B_{\varepsilon,poly}$ instead of B_ε. Here, for the sake of simplicity, B_ε is assumed to be polygonal, just as each $B^{\varepsilon,\delta}_i$.
Since $B^{\varepsilon,\delta}_i$ is polygonal, there exist n_i disjoint convex polygonal sets, denoted by $\Omega^p_i, p = 1, ..., n_i$, such that $\overline{B^{\varepsilon,\delta}_i} = \bigcup_{p=1}^{n_i} \overline{\Omega^p_i}$, and $B_\varepsilon = \Big(\bigcup_{i=1}^{N}\bigcup_{p=1}^{n_i} \Omega^p_i\Big) \bigcup \Big(\bigcup_{ij} B^{dec,\varepsilon}_{ij}\Big)$ is the union of disjoint convex polygonal sets. We assume there are two constants a and b, independent of ε, such that $\forall i, p,\ a\varepsilon \leq |\Omega^p_i| \leq b\varepsilon$.

Let $u \in X(\mathcal{T})$. Let $m_\varepsilon, m^p_i, m_{ij}$ be the mean value of u over $B_\varepsilon, \Omega^p_i, B^{dec,\varepsilon}_{ij}$ respectively.
First,

$$\begin{aligned}
\|u\|^2_{L^2(B_\varepsilon)} &\leq \|u - m_\varepsilon\|^2_{L^2(B_\varepsilon)} + |B_\varepsilon| \, m^2_\varepsilon \\
&\leq \sum_{i=1}^{N}\sum_{p=1}^{n_i} \|u - m_\varepsilon\|^2_{L^2(\Omega^p_i)} + \sum_{ij} \|u - m_\varepsilon\|^2_{L^2(B^{dec,\varepsilon}_{ij})} + |B_\varepsilon| \, m^2_\varepsilon
\end{aligned}$$

We have

$$\begin{aligned}
\|u - m_\varepsilon\|^2_{L^2(\Omega^p_i)} &\leq \|u - m^p_i\|^2_{L^2(\Omega^p_i)} \\
&\quad + |\Omega^p_i| \left(\sum_{l=1}^{N}\sum_{q=1}^{n_l} \frac{|\Omega^q_l|}{|B_\varepsilon|}(m^p_i - m^q_l) + \sum_{lj} \frac{|B^{dec,\varepsilon}_{lj}|}{|B_\varepsilon|}(m^p_i - m_{lj})\right)^2 \\
\|u - m_\varepsilon\|^2_{L^2(B^{dec,\varepsilon}_{ij})} &\leq \|u - m_{ij}\|^2_{L^2(B^{dec,\delta}_{ij})} \\
&\quad + |B^{dec,\delta}_{ij}| \left(\sum_{l=1}^{N}\sum_{q=1}^{n_l} \frac{|\Omega^q_l|}{|B_\varepsilon|}(m_{ij} - m^q_l) + \sum_{lr} \frac{|B^{dec,\varepsilon}_{lr}|}{|B_\varepsilon|}(m_{ij} - m_{lr})\right)^2
\end{aligned}$$

In order to obtain a bound of the quantities $m^p_i - m^q_l$, $m^p_i - m_{lj}$ and $m_{ij} - m_{lr}$, a chain must be constructed for connecting all the sets Ω^p_i and $B^{dec,\varepsilon}_{lj}$, in pairs by using the mean value of u on the intersection. Let us illustrate the process to connect for instance Ω^p_i and $B^{dec,\varepsilon}_{ij}$ when $\Omega^p_i \cap B^{dec,\varepsilon}_{ij} = I$, with $I = S_{i,j}$. Let m_I be the mean value of u on I, we deduce from Lemmas A9 and A10 that there exist constants C_i and C_{ij} such that

$$|m^p_i - m_{ij}| \leq |m^p_i - m_I| + |m_I - m_{ij}| \leq \left(\frac{C_i}{\sqrt{|\Omega^p_i|}} + \frac{C_{ij}}{\sqrt{|B^{dec,\varepsilon}_{ij}|}}\right) \mid u \mid_{1,\mathcal{T}}.$$

Since the domain is thin, each $|\Omega_i^p|$ and each $|B_{ij}^{dec,\varepsilon}|$ are in the same order of magnitude that is $O(\varepsilon)$. We can conclude that
$|m_i^p - m_l^q|$, $|m_i^p - m_{lj}|$, $|m_{ij} - m_{lr}| \leq D \mid u \mid_{1,\mathcal{T}}$ with $D = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$.
At this stage of the proof, it can be said that

$$\|u\|^2_{L^2(B_\varepsilon)} \leq |B_\varepsilon|\, m_\varepsilon^2 + C \mid u \mid^2_{1,\mathcal{T}},$$

where C is a constant independent of ε. Since we have

$$m_\varepsilon = \sum_{l=1}^{N} \sum_{q=1}^{n_l} \frac{|\Omega_l^q|}{|B_\varepsilon|} m_l^q + \sum_{lj} \frac{|B_{lj}^{dec,\varepsilon}|}{|B_\varepsilon|} m_{lj}$$

therefore

$$|B_\varepsilon|\, m_\varepsilon^2 \leq \sum_{l=1}^{N} \sum_{q=1}^{n_l} |\Omega_l^q| (m_l^q)^2 + \sum_{lj} |B_{lj}^{dec,\varepsilon}| (m_{lj})^2$$

by applying the Cauchy-Schwarz-Bunyakowskii inequality.

As in the proof of Theorem A1, let Ω_k^r be a part of $B_k^{\varepsilon,\delta}$ which has a part of its boundary that is common with γ_ε^k. Since $u|_{\gamma_\varepsilon^k} = 0$, the standard Poincaré- Friedrichs inequality gives

$$|\Omega_k^r| (m_k^r)^2 \leq \|u\|^2_{L^2(\Omega_k^r)} \leq C \mid u \mid^2_{1,\mathcal{T}},$$

where C is a constant independent of ε. It is then easy to bound each $|\Omega_l^q| (m_l^q)^2$ and $|B_{lj}^{dec,\varepsilon}| (m_{lj})^2$, by connecting it to Ω_k^r by a chain of domains Ω_i^p and cylinders, knowing that, as stated above, $|m_l^q - m_k^r|$, $|m_{lj} - m_k^r| \leq D \mid u \mid_{1,\mathcal{T}}$ with $D = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$. We obtain

$$|B_\varepsilon|\, m_\varepsilon^2 \leq C_1 \mid u \mid^2_{1,\mathcal{T}} + C_2 |\Omega_k^r| (m_k^r)^2 \leq C_3 \mid u \mid^2_{1,\mathcal{T}},$$

where C_1, C_2, C_3 are constants independent of ε. The theorem is proved.

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