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1

The effective equation method

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Summary. In this chapter we present a general method of constructing the effective equation which describes the behaviour of small-amplitude solutions for a nonlinear PDE in finite volume, provided that the linear part of the equation is a hamiltonian system with a pure imaginary discrete spectrum. The effective equation is obtained by retaining only the resonant terms of the nonlinearity (which may be hamiltonian, or may be not); the assertion that it describes the limiting behaviour of small-amplitude solutions is a rigorous mathematical theorem. In particular, the method applies to the three– and four–wave systems. We demonstrate that different possible types of energy transport are covered by this method, depending on whether the set of resonances splits into finite clusters (this happens, e.g. in case of the Charney-Hasegawa-Mima equation), or is connected (this happens, e.g. in the case of the NLS equation if the space-dimension is at least two). For equations of the first type the energy transition to high frequencies does not hold, while for equations of the second type it may take place. In the case of the NLS equation we use next some heuristic approximation from the arsenal of wave turbulence to show that under the iterated limit "the volume goes to infinity", taken after the limit "the amplitude of oscillations goes to zero", the energy spectrum of solutions for the effective equation is described by a Zakharov-type kinetic equation. Evoking the Zakharov ansatz we show that stationary in time and homogeneous in space solutions for the latter equation have a power law form. Our method applies to various weakly nonlinear wave systems, appearing in plasma, meteorology and oceanography.

1.1 Introduction

It is well known that solutions of linear evolution PDEs in finite volume are superpositions of normal modes of oscillations (in most cases of interest these are the Fourier modes). When a nonlinearity is added as a perturbation, different modes start to interact and the solutions of the equation can be approximated by suitable power series expansions, provided that the nonlinearity is sufficiently small (or, in other words, the PDE is *weakly* nonlinear). In such cases, the equation can be written as

$$u_t + L(u) = \varepsilon N(u)\,, \tag{1.1}$$

where L is the linear operator, N denotes the nonlinearity and ε is a small parameter, $0 < \varepsilon \ll 1$. The equation may contain a stochastic force, and in that case it reads

$$u_t + L(u) = \varepsilon N(u) + \sqrt{\varepsilon}\,\langle \text{random force}\rangle \tag{1.2}$$

(the scaling of the random force by the factor $\sqrt{\varepsilon}$ is the most natural, see below). We will show that the limiting, as $\varepsilon \to 0$, exchange of energy between the modes may be described by replacing the original system with a suitable *effective equation*. This result may be regarded as a PDE-version of the Bogolyubov averaging principle (see [1]) which implies a similar property for distribution of energy between the oscillating modes for small-amplitude oscillations in finite-dimensional nonlinear systems.

The mentioned above convergence that holds as $\varepsilon \to 0$ and various properties of the corresponding effective equations have been rigorously established (see [2, 3, 4, 5, 6, 7] and the discussion in [7]). The treatments of the deterministic and stochastic equations are similar, but the results, obtained in the stochastic case, are significantly stronger: while the deterministic effective equation controls the dynamics only on time intervals of order ε^{-1}, in the presence of stochastic forcing the corresponding effective equation also approximates the stationary measure for the original equation, thus controlling the asymptotical in time behaviour of solutions when $\varepsilon \ll 1$. Moreover, in the absence of forcing we only get information concerning the exchange of energy between the modes, whereas in the stochastically forced case, the stationary measure for the effective equation controls both the energies and the phases of the normal modes of solutions.

Below we explain how to construct the effective equations for eq. (1.1) and eq. (1.2) from the resonant terms of the nonlinearities. We will discuss two examples: the nonlinear Schrödinger equation and the Charney-Hasegawa-Mima equation on the β plane. These two equations display completely different types of energy exchange between modes, and we will explain why this happens.

1.2 How to construct the effective equation

We consider hamiltonian PDEs, whose linear parts have imaginary pure point spectra and are diagonal in Fourier modes. Written in terms of the complex Fourier coefficients $v = \{v_{\mathbf{k}}\}$ (also called waves), the equations which we study read

$$\frac{d}{dt} v_{\mathbf{k}} = i\omega_{\mathbf{k}} v_{\mathbf{k}} + P_{\mathbf{k}}(v) \ , \qquad \mathbf{k} \in \mathbb{Z}^d \ . \tag{1.3}$$

Here $\omega_{\mathbf{k}}$ are real numbers and $P(v) = (P_{\mathbf{k}}(v), \mathbf{k} \in \mathbb{Z}^d)$ is a polynomial nonlinearity in v of certain order q, of the form

$$P_{\mathbf{k}}(v) = \sum_{p \le q} \sum_{\mathbf{k}_1 \ldots \mathbf{k}_p \mathbf{k}_{p+1} \ldots \mathbf{k}_q} c_{\mathbf{k}_1 \ldots \mathbf{k}_q \mathbf{k}} v_{\mathbf{k}_1} \cdots v_{\mathbf{k}_p} v^*_{\mathbf{k}_p+1} \cdots v^*_{\mathbf{k}_q} \delta^{1\ldots p}_{p+1\ldots q\,\mathbf{k}} \ ,$$

where $c_{\mathbf{k}_1 \ldots \mathbf{k}_q \mathbf{k}}$ are some complex coefficients, v^* is the complex conjugate of v and

$$\delta^{1\ldots p}_{p+1\ldots q\mathbf{k}} = \begin{cases} 1 \text{ if } \mathbf{k}_1 + \ldots + \mathbf{k}_p = \mathbf{k}_{p+1} + \ldots + \mathbf{k}_q + \mathbf{k} \\ 0 \qquad\qquad\qquad\qquad \text{else} \end{cases} \ . \tag{1.4}$$

We always assume that "the nonlinearity does not pump energy in the system":

$$\mathrm{Re} \sum_{\mathbf{k}} P_{\mathbf{k}}(v) \bar{v}_{\mathbf{k}} \le 0 \tag{1.5}$$

(in most case of interest the l.h.s. vanishes).

The quantities $I_{\mathbf{k}} := |v_{\mathbf{k}}|^2/2$, $E_{\mathbf{k}} = \omega_{\mathbf{k}} I_{\mathbf{k}}$ and $\varphi_{\mathbf{k}} = \mathrm{Arg}(v_{\mathbf{k}})$ are called, respectively, the wave action, wave energy and wave phase. The relation between ω and $\mathbf{k}$, i.e. the function $\mathbf{k} \to \omega_{\mathbf{k}}$, is called the *dispersion relation*, or *dispersion function*.

The *weakly nonlinear regime* corresponds to solutions of small amplitude ε. We will study it in the presence of damping and, possibly, a random force, whose magnitude is controlled by another parameter, call it ν. So, instead of (1.3), we will consider

$$\frac{d}{dt}v_{\mathbf{k}} = i\omega_{\mathbf{k}}v_{\mathbf{k}} + \varepsilon^q P_{\mathbf{k}}(v) - \nu\gamma_{\mathbf{k}}v_{\mathbf{k}} + \mu\sqrt{\nu}\, b_{\mathbf{k}}\dot{\boldsymbol{\beta}}_{\mathbf{k}} \ , \qquad \mathbf{k}\in\mathbb{Z}^d \ , \tag{1.6}$$

where $\gamma_{\mathbf{k}} \geq \gamma_* > 0$ controls the damping term, $b_{\mathbf{k}} > 0$ controls the forcing and the parameter $\mu \in \{0,1\}$ is introduced to consider at the same time the forced and non-forced cases. The $\dot{\boldsymbol{\beta}}_{\mathbf{k}}$'s are complex white noises, independent from each other.[3] The factors ν and $\sqrt{\nu}$ in front of the damping and the dissipation are so chosen that, in the limit of $\nu \to 0$, the solutions stay of order one, uniformly in $t > 0$.

Note that, while ε controls the size of the solutions, $1/\nu$ is the time-scale on which the forcing acts significantly, as it is the time needed for the standard deviations of the processes $\sqrt{\nu}\boldsymbol{\beta}_{\mathbf{k}}$ to become of order one. If $\mu = 0$ and the system (1.6) is deterministic, still its time-scale is $1/\nu$ since, as we explain below, its solutions with initial data of order one stay of order one for $t \lesssim \nu^{-1}$, while for much bigger values of time they are very small since in view of (1.5) their ℓ_2-norms satisfy

$$|v(t)|^2_{\ell_2} \leq |v(0)|^2_{\ell_2} e^{-t\nu\gamma_*} \ .$$

We will consider the regime

$$\nu = \varepsilon^q \tag{1.7}$$

(where q is the degree of P), and study solutions of the equation with given initial conditions on the time-scale $1/\nu$, examining them under the limit $\nu \to 0$. Passing to the slow time $\tau = \nu t$ (so that time $t \sim 1/\nu$ corresponds to τ of order 1), eq. (1.6) becomes

$$\dot{v}_{\mathbf{k}} = i\nu^{-1}\omega_{\mathbf{k}}v_{\mathbf{k}} + P_{\mathbf{k}}(v) - \gamma_{\mathbf{k}}v_{\mathbf{k}} + \mu b_{\mathbf{k}}\dot{\boldsymbol{\beta}}_{\mathbf{k}} \ , \qquad \mathbf{k}\in\mathbb{Z}^d \ , \tag{1.8}$$

where the upper dot stands for $\frac{d}{d\tau}$.

We claim that, in the limit when ν (or, equivalently, ε) goes to zero, the distribution of the energies $E_{\mathbf{k}}$ on times τ of order one is described by an *effective equation* whose nonlinearity is constituted by resonant terms of the nonlinearity (see (1.13) below).

It is easier to understand the role of resonances and the form of the effective equation by passing to the interaction representation (cf. [1, 8, 9]), i.e., by performing the time-dependent change of variables from $v_{\mathbf{k}}$ to

$$a_{\mathbf{k}} = e^{-i\nu^{-1}\omega_{\mathbf{k}}\tau}v_{\mathbf{k}} \ , \tag{1.9}$$

which transforms (1.8) to

$$\dot{a}_{\mathbf{k}} = \mathbf{R}_{\mathbf{k}}(a,\nu^{-1}\tau) - \gamma_{\mathbf{k}}a_{\mathbf{k}} + \mu b_{\mathbf{k}}e^{-i\nu^{-1}\omega_{\mathbf{k}}\tau}\dot{\boldsymbol{\beta}}_{\mathbf{k}} \ , \qquad \mathbf{k}\in\mathbb{Z}^d \ , \tag{1.10}$$

where $\mathbf{R}_{\mathbf{k}}$ denotes the nonlinearity, written in the a-variables. That is

$$\begin{aligned}\mathbf{R}_{\mathbf{k}}(a,\nu^{-1}\tau) = &\sum_{p\leq q}\sum_{\mathbf{k}_1\ldots\mathbf{k}_p\mathbf{k}_{p+1}\ldots\mathbf{k}_q} c_{\mathbf{k}_1\ldots\mathbf{k}_q\mathbf{k}}v_{\mathbf{k}_1}\cdots v_{\mathbf{k}_p}v^*_{\mathbf{k}_p+1}\cdots v^*_{\mathbf{k}_q}\delta^{1\ldots p}_{p+1\ldots q\,\mathbf{k}} \\ &\times\exp\left(i\nu^{-1}\tau\left(\omega_{\mathbf{k}_1}+\ldots+\omega_{\mathbf{k}_p}-\omega_{\mathbf{k}_{p+1}}-\ldots-\omega_{\mathbf{k}_q}-\omega_{\mathbf{k}}\right)\right) \ .\end{aligned}$$

Noting that the collection of the processes $\{e^{-\nu^{-1}\omega_{\mathbf{k}}\tau}\dot{\beta}_{\mathbf{k}}\}$ is another set of standard independent complex white noises, we re-write eq. (1.10) as

[3] That is, $\dot{\boldsymbol{\beta}}_{\mathbf{k}} = (d/dt)\boldsymbol{\beta}_{\mathbf{k}}$, $\boldsymbol{\beta}_{\mathbf{k}} = \boldsymbol{\beta}^+_{\mathbf{k}} + i\boldsymbol{\beta}\mathbf{k}^-$, where $\boldsymbol{\beta}^\pm_{\mathbf{k}}$ are standard independent real Wiener processes.

$$\dot{a}_{\mathbf{k}} = \mathbf{R}_{\mathbf{k}}(a, \nu^{-1}\tau) - \gamma_{\mathbf{k}} a_{\mathbf{k}} + \mu b_{\mathbf{k}} \dot{\boldsymbol{\beta}}_{\mathbf{k}} \ , \qquad \mathbf{k} \in \mathbb{Z}^d \ . \tag{1.11}$$

In the sum defining $\mathbf{R}_{\mathbf{k}}$, the terms for which the *resonance conditions*

$$\begin{cases} \omega_{\mathbf{k}_1} + \ldots + \omega_{\mathbf{k}_p} = \omega_{\mathbf{k}_{p+1}} + \ldots + \omega_{\mathbf{k}_q} + \omega_{\mathbf{k}} \\ \mathbf{k}_1 + \cdots + \mathbf{k}_p = \mathbf{k}_{p+1} + \cdots \mathbf{k}_q + \mathbf{k} \end{cases} \tag{1.12}$$

are satisfied (called the *resonant terms*) under the limit $\nu \to 0$ behave completely differently from others terms (called the *nonresonant terms*). Namely, the nonresonant terms oscillate faster and faster, whereas the resonant terms do not. We will say that a set of $\mathbb{Z}^d$-vectors $\{\mathbf{k}_1, \ldots, \mathbf{k}_q, \mathbf{k}\}$ forms a *resonance* if relations (1.12) are satisfied, if $c_{\mathbf{k}_1 \ldots \mathbf{k}_q \mathbf{k}} \neq 0$, and the set $\{\mathbf{k}_1, \ldots, \mathbf{k}_p\}$ does not equal the set $\{\mathbf{k}_{p+1}, \ldots, \mathbf{k}_q, \mathbf{k}\}$. The collection of all resonances is called the *resonant set*.

In the spirit of the finite-dimensional averaging, following the Bogolyubov averaging principle (see [1], the behaviour of solutions of (1.11) under the limit $\nu \to 0$ is obtained by replacing the nonlinearity $\mathbf{R}_{\mathbf{k}}$ with its time average, i.e. with

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{R}_{\mathbf{k}}(a, t) dt \ .$$

Since for any real number λ we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{i\lambda t} dt = \begin{cases} 1 \text{ if } \lambda = 0 \\ 0 \text{ if } \lambda \neq 0 \end{cases} \ ,$$

then only the resonant terms survive in the averaged nonlinearity. We write their sum as

$$R_{\mathbf{k}}(a) = \sum_{p \leq q} \sum_{\mathbf{k}_1 \ldots \mathbf{k}_p \mathbf{k}_{p+1} \ldots \mathbf{k}_q} c_{\mathbf{k}_1 \ldots \mathbf{k}_q \mathbf{k}} v_{\mathbf{k}_1} \cdots v_{\mathbf{k}_p} v^*_{\mathbf{k}_{p+1}} \cdots v^*_{\mathbf{k}_q} \delta^{1 \ldots p}_{p+1 \ldots q\, \mathbf{k}} \delta(\omega^{1 \ldots p}_{p+1 \ldots q\, \mathbf{k}}) \ ,$$

where

$$\delta(\omega^{1 \ldots p}_{p+1 \ldots q \mathbf{k}}) = \begin{cases} 1 \text{ if } \omega_{\mathbf{k}_1} + \ldots + \omega_{\mathbf{k}_p} = \omega_{\mathbf{k}_{p+1}} + \ldots + \omega_{\mathbf{k}_q} + \omega_{\mathbf{k}} \\ 0 \qquad\qquad\qquad\qquad \text{else} \end{cases} \ .$$

This suggests to take for the effective equation the following system:

$$\dot{\tilde{a}}_{\mathbf{k}} = R_{\mathbf{k}}(\tilde{a}) - \gamma_{\mathbf{k}} \tilde{a}_{\mathbf{k}} + \mu b_{\mathbf{k}} \dot{\boldsymbol{\beta}}_{\mathbf{k}} \ , \qquad \mathbf{k} \in \mathbb{Z}^d \ . \tag{1.13}$$

Indeed, it is proved in [7] (also see [2, 3, 4]) that, if the original equation (1.6) is well posed on time intervals $t \lesssim 1/\nu$, then eq. (1.13) describes the limiting behaviour of the variables $a_{\mathbf{k}}$ (and, as well, the distribution of energy since $|v_{\mathbf{k}}| = |a_{\mathbf{k}}|$) in the time-scale $t \sim 1/\nu$, for any sufficiently regular initial data. This holds both in the presence and in the absence of the random forcing (i.e., both for $\mu = 0$ and $\mu = 1$). Moreover, in the forced case we also control the limiting behaviour of the stationary solutions for eq. (1.6). So if the equation (1.6) and the effective equation both are mixing, then we control the behaviour of all solutions for (1.6) under the iterated limit $\lim_{\varepsilon \to 0} \lim_{t \to \infty}$. Remarkably, in this case the effective equation describes not only the limiting behaviour of the actions, but also that of the angles. I.e., it completely controls the limiting distribution of solutions. So if $f(v)$ is a functional on the space of sequences $v = (v_{\mathbf{k}})$, satisfying some mild restriction on its growth as the norm of v goes to infinity, and $v^\varepsilon(t)$ is any solution for (1.6), then

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \mathbf{E} f(v^\varepsilon(t)) \to \int f(v)\, \mu(dv) \, ,$$

where μ is the unique stationary measure for the effective equation (1.13) and $\mathbf{E}$ signifies the expectation. See in [3, 4, 7].

1.3 Structure of resonances

We intend to use the effective equations as a tool to investigate the energy transport in different physically relevant PDEs. We will show that the limiting, as $\varepsilon \to 0$, energy transport for any specific equation depends on the structure of the resonances (which, in turn, is determined by the form of the dispersion function $\omega_{\mathbf{k}}$).

Three possibilities can occur:

1) The resonant set is empty. Then if the degree q of the nonlinearity is even, the effective equation is linear. If q is odd, the equation may contain nonlinear integrable terms of the form $f(I)v_{\mathbf{k}}$. But these terms do not contribute to the dynamics of the wave actions. So in any case different modes do not exchange energy, and no energy transport to high frequencies occurs.

Now assume that the resonant set is not empty. We say that integer vectors $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{Z}^d$ are *equivalent* if there exist vectors $\mathbf{k}_3, \ldots, \mathbf{k}_q, \mathbf{k} \in \mathbb{Z}^d$, such that the relations (1.12) hold. This equivalence divides $\mathbb{Z}^d$ to clusters, formed by elements which can be joined by chains of equivalences (see [10, 11] for a discussion of the role of resonant clusters in weak turbulence).

The two remaining cases are:

2) All resonances are connected, so the whole $\mathbb{Z}^d$ is a single cluster. In this case, in the limit when the volume of the space-domain goes to infinity, under some additional assumptions a new type of kinetic equation can be derived, the energy transport takes place and power law stationary spectra, which depend only on the form of the dissipation, can be obtained.

3) All resonances are divided to non intersecting clusters. Now the energy transfer should be studied separately within each cluster. If sizes of the clusters are bounded, then no energy transport to high frequencies occurs.

See [3, 12] for the case 1), [4, 5] for the stochastic case 2) and [13] for the deterministic case, and see [6] for the stochastic case 3). See [7] for discussion and for theorems, applicable in all three cases, deterministic and stochastic.

Note that many examples of systems which fall to type 2) are given by equations (1.6) with completely resonant spectra $\{\omega_{\mathbf{k}}\}$, i.e. with spectra of the form $\omega_{\mathbf{k}} = \omega_* \Omega_{\mathbf{k}}$, where $\Omega_{\mathbf{k}}$ are integers. Averaging theorems for completely resonant deterministic equations (1.6) with $\nu = 0$ were discussed in [14, 15, 13]; also see [7].

Below we discuss examples for the case 2) when all resonances are connected (Section 1.4), and for the case 3) when the resonances make non intersecting finite clusters (Section 1.5). For more examples of systems of types 2) and 3) see [10].

1.3.1 The equations

Our first example is the cubic NLS equation on the d-dimensional torus of size L (see [4, 5]):

$$\partial_t u - i\Delta u = i\varepsilon^2 \delta |u|^2 u + \nu\langle \text{ dissipation}\rangle + \mu\sqrt{\nu}\langle \text{random forcing}\rangle\,, \quad u = u(t,\mathbf{x}) \in \mathbb{C}\,, \tag{1.14}$$

for $\mathbf{x} \in \mathbb{T}_L^d = \mathbb{R}^d/(2\pi L\mathbb{Z}^d)$, where the parameter $\delta = \delta(L)$ is introduced in order to control different scaling for solution as the size L of the torus varies.[4] The dissipation is a linear operator, diagonal in the exponential base of functions on $\mathbb{T}_L^d$

$$\{\phi_{\mathbf{k}}(x) = e^{iL^{-1}\mathbf{k}\cdot x},\ \mathbf{k} \in \mathbb{Z}^d\}\,. \tag{1.15}$$

[4] More exactly, (1.14) is the damped/driven cubic NLS equation. See [8, 9] for the non-perturbed NLS equations.

As before, by $v = \{v_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d\}$ we denote the the Fourier coefficients of $u(x)$:

$$u(x) = \sum_{\mathbf{k}\in\mathbb{Z}^d} v_{\mathbf{k}} \phi_{\mathbf{k}}(x)\,.$$

If $d = 1$, the resonance condition (1.12) takes the form

$$k_1^2 + k_2^2 = k_3^2 + k^2, \qquad k_1 + k_2 = k_3 + k\,.$$

All solutions for this system are such that $k_1 = k, k_2 = k_3$, or $k_1 = k_3, k_2 = k$. So in this case the resonant set is empty, and no energy cascade to high frequencies happens when $\varepsilon^2 = \nu \to 0$. This is well known.

Now consider a higher-dimensional NLS equation, write it in the Fourier variables $\{v_{\mathbf{k}},\ \mathbf{k} \in \mathbb{Z}^d\}$, and pass to the slow time $\tau = \nu t$. Then, if the forcing and the dissipation are chosen in accordance with the prescription of the previous section (cf. (1.6)), the equation reads

$$\dot{v}_{\mathbf{k}} = -i\nu^{-1}\omega_{\mathbf{k}}^N v_{\mathbf{k}} + i\delta \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\in\mathbb{Z}^d} v_{\mathbf{k}_1} v_{\mathbf{k}_2} v_{\mathbf{k}_3}^* \delta_{3\mathbf{k}}^{12} - \gamma_{\mathbf{k}} v_{\mathbf{k}} + \mu b_{\mathbf{k}} \dot{\boldsymbol{\beta}}_{\mathbf{k}}\,. \tag{1.16}$$

Here $-\gamma_{\mathbf{k}}$ are the eigenvalues of the dissipation operator. The eigenvalues of the operator $-\Delta$, call them $\omega_{\mathbf{k}}^N$, follow the dispersion relation

$$\omega_{\mathbf{k}}^N = |\mathbf{k}|^2 / L^2\ , \quad \mathbf{k} \in \mathbb{Z}^d\ . \tag{1.17}$$

Below we will see that if $d \geq 2$, then the whole $\mathbb{Z}^d$ forms a single cluster, so the equation fits the case 2).

An interesting example of the case 3) of isolated clusters is provided by the Charney–Hasegawa–Mima (CHM) equation on the β plane (see [6] and see [8, 9] for this equation with $\nu = 0$), which we write as

$$(-\Delta + F)\partial_t\psi - \varepsilon J(\psi, \Delta\psi) - \partial_x\psi = \nu\langle \text{ dissipation}\rangle + \mu\sqrt{\nu}\langle\text{random forcing}\rangle\,, \quad \psi = \psi(t,\mathbf{x}) \in \mathbb{R}\ .$$

Here the constant $F \geq 0$ is called the Froude number and $J(\psi, \Delta\psi)$ denotes the Jacobian determinant of the vector $(\psi, \Delta\psi)$. The space-domain is a strip of horizontal size ρ and vertical size one, under double periodic boundary conditions, i.e.,

$$\mathbf{x} = (x, y) \in \mathbb{T}^2_{\rho,1} = \mathbb{R}/(2\pi\rho\mathbb{Z}) \times S^1\ , \quad S^1 = \mathbb{R}/(2\pi\mathbb{Z})\ .$$

Again we pass to the Fourier modes[5] $\{v_{\mathbf{k}},\ \mathbf{k} = (m,n) \in \mathbb{Z}^2\}$ and to the slow time τ to re-write the equation as

$$\begin{aligned} \dot{v}_{\mathbf{k}} = -i\nu^{-1}\omega_{\mathbf{k}}^C v_{\mathbf{k}} + \frac{1}{\rho(m^2 + n^2\rho^2 + F\rho^2)} \sum_{\mathbf{k}_1,\mathbf{k}_2\in\mathbb{Z}^2} \left(m_1^2 + n_1^2\rho^2\right) \\ \times (m_1 n_2 - n_1 m_2)\, v_{\mathbf{k}_1} v_{\mathbf{k}_2} \delta_{\mathbf{k}}^{12} - \gamma_{\mathbf{k}} v_{\mathbf{k}} + \mu b_{\mathbf{k}} \dot{\boldsymbol{\beta}}_{\mathbf{k}}\ , \end{aligned} \tag{1.18}$$

where $\mathbf{k}_1 = (m_1, n_1)$, $\mathbf{k}_2 = (m_2, n_2)$ and the dispersion function has the form

$$\omega_{\mathbf{k}}^C = -\frac{m\rho}{m^2 + n^2\rho^2 + F\rho^2}\ , \quad \mathbf{k} = (m,n) \in \mathbb{Z}^2\ . \tag{1.19}$$

[5] Note that, due to the fact that the function ψ is real, $v_{\mathbf{k}} = v_{-\mathbf{k}}^*$.

The effective equations for eq. (1.16) and eq. (1.18) can be easily obtained on account of the general formula (1.13). Using it, for the NLS equation we get the effective equation

$$\dot{\tilde{a}}_{\mathbf{k}} = i\delta \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\in\mathbb{Z}^d} \tilde{a}_{\mathbf{k}_1}\tilde{a}_{\mathbf{k}_2}\tilde{a}^*_{\mathbf{k}_3}\delta^{12}_{3\mathbf{k}}\delta(\omega^{N^{12}}_{3\mathbf{k}}) - \gamma_{\mathbf{k}}\tilde{a}_{\mathbf{k}} + \mu b_{\mathbf{k}}\dot{\boldsymbol{\beta}}_{\mathbf{k}}\,, \quad \mathbf{k}\in\mathbb{Z}^d\,, \tag{1.20}$$

while for CHM the effective equation is the system

$$\begin{aligned}\dot{\tilde{a}}_{\mathbf{k}} = \frac{1}{\rho(m^2+n^2\rho^2+F\rho^2)} &\sum_{\mathbf{k}_1,\mathbf{k}_2\in\mathbb{Z}^2} \left(m_1^2+n_1^2\rho^2\right)(m_1n_2-n_1m_2) \\ &\times \tilde{a}_{\mathbf{k}_1}\tilde{a}_{\mathbf{k}_2}\delta^{12}_{\mathbf{k}}\delta(\omega^{C^{12}}_{\mathbf{k}}) - \gamma_{\mathbf{k}}\tilde{a}_{\mathbf{k}} + \mu b_{\mathbf{k}}\dot{\boldsymbol{\beta}}_{\mathbf{k}}\,, \quad \mathbf{k}\in\mathbb{Z}^2\,.\end{aligned} \tag{1.21}$$

It is clear that the behaviour of solutions for eqs. (1.20)-(1.21) is dictated by the structure of resonances since they determine the surviving terms of the nonlinearity. The geometric properties of the resonant set for the higher dimensional NLS equations are described in the following section, whereas the resonances for CHM are discussed in Section 1.3.3.

1.3.2 Structure of resonances for the NLS equation

In the case of $2d$ NLS equation each resonance is formed by four points of $\mathbb{Z}^2$ which have a simple geometrical characterization: they form the vertices of a rectangle. Indeed, if a quadruple $\{\mathbf{k},\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2\}$ satisfies (1.12) with $q=3$, then on account of the second relation we have $\mathbf{k}_1-\mathbf{k}=\mathbf{k}_3-\mathbf{k}_2$. So the polygonal $\{\mathbf{k},\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2\}$ is a parallelogram. Substituting $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3$ in the first relation and using (1.17) we get

$$2(\mathbf{k}_3\cdot\mathbf{k}_3+\mathbf{k}_1\cdot\mathbf{k}_2-\mathbf{k}_2\cdot\mathbf{k}_3-\mathbf{k}_1\cdot\mathbf{k}_3)=0 \quad\Rightarrow\quad (\mathbf{k}_3-\mathbf{k}_2)\cdot(\mathbf{k}_3-\mathbf{k}_1)=0\,.$$

I.e., $\mathbf{k}_3-\mathbf{k}_2$ is orthogonal to $\mathbf{k}_3-\mathbf{k}_1$. So $\{\mathbf{k},\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2\}$ is a rectangle in $\mathbb{Z}^2$.

It is easy to see that for any vectors $\mathbf{k},\mathbf{k}_1\in\mathbb{Z}^2$ there is an integer rectangle of the form $\{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\}$. So the equivalence, defined by the clusters of the 2d NLS equation makes $\mathbb{Z}^2$ a single cluster, and the equation falls in the case 2). A graphical illustration of some resonant quadruples and their connections in $\mathbb{Z}^2$ is displayed in fig. 1.1.

Similar all higher-dimensional NLS equations fall in case 2).

1.3.3 Structure of resonances for CHM

The structure of resonances for the CHM equation depends on the shape-factor ρ. Below we discuss it, supposing for simplicity that the Froude number F is kept fixed (see [6] for the general case). We start with explicitly rewriting for the present case the resonance condition (1.12) (recall that $\mathbf{k}_1=(m_1,n_1)$, $\mathbf{k}_2=(m_2,n_2)$ and $\mathbf{k}=(m,n)$):

$$\begin{aligned}&m_1+m_2=m\,, \qquad n_1+n_2=n\,,\\ &\frac{m_1}{m_1^2+n_1^2\rho^2+F\rho^2}+\frac{m_2}{m_2^2+n_2^2\rho^2+F\rho^2}=\frac{m}{m^2+n^2\rho^2+F\rho^2}\,.\end{aligned} \tag{1.22}$$

Solutions $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})$ to these equations can be divided to different classes, according to how many numbers among m_1,m_2 and m are non-zero:

(i) If all three are zero, then any triad $\mathbf{k}_1=(0,n_1)$, $\mathbf{k}_2=(0,n_2)$, $\mathbf{k}=(0,n_1+n_2)$ constitutes a solution. As $c_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}}$ vanish in this case (see (1.21)), such triads do not form a resonance.

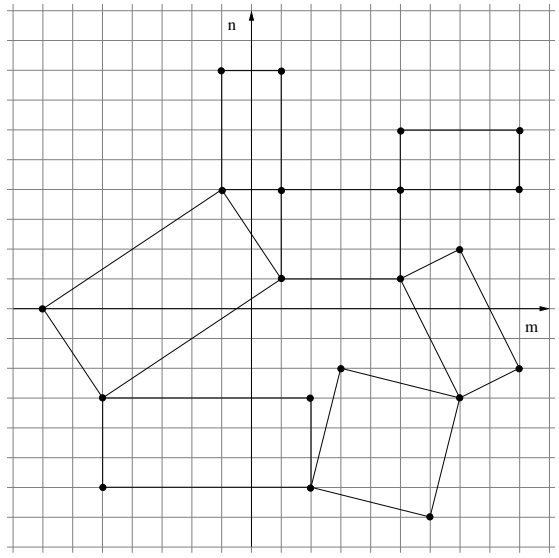

Fig. 1.1. Examples of connected resonant quadruples for the NLS equation in the $\mathbb{Z}^2$ lattice. Each point is the vertex of at least one rectangle.

(ii) If only one number is different from zero, then (1.22) admits no solution.

(iii) If only one among m_1, m_2 and m_3 vanishes, two subcases arise (as $\mathbf{k}_1$ and $\mathbf{k}_2$ play an exchangeable role):

a. if $m_1 = 0$ (which implies $m_2 = m$), then $n_2^2 = n^2$ and there are two solutions, one for $n_1 = 0$, $n_2 = n$, and another for $n_2 = -n_1/2 = -n$;

b. if $m = 0$ (which implies $m_1 = -m_2$), then $n_1^2 = n_2^2$ and again there are two solutions, one for $n = 0$, $n_1 = -n_2$, and another for $n_1 = n_2 = n/2$.

(iv) All three are different from zero. This is the only case when the solutions depend on ρ. Indeed, let us fix a triad $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k})$ and look for which values of ρ it constitutes a resonance. The second line of (1.22) may be re-written as

$$a_0(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) + a_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}, F)\rho^2 + a_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}, F)\rho^4 = 0\,, \tag{1.23}$$

where a_0, a_1 and a_2 are polynomials. In particular, $a_0 = m_1 m_2 m\,(m_2 m + m_1 m - m_1 m_2)$. In view of the inequality $(x^2 + y^2 + xy) > 0$, valid for nonvanishing x and y,

$$a_0 = m_1 m_2 m\left(m_1^2 + m_2^2 + m_1 m_2\right) \neq 0\ ,$$

where the use is made of the first line of (1.22). This shows that the second order polynomial in ρ^2 at the l.h.s. of (1.23) is nontrivial. So for any fixed triad $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k})$, where m_1, m_2 and m are nonzero, relation (1.23) holds for at most two nonnegative values of ρ.

The different types of resonances are represented in Fig. 1.2, where only the points above the horizontal axis are displayed (cf. footnote 5). There the resonances of type (iii) (which we will call standard resonances) are connected by solid lines: they form triangles symmetric with respect to the vertical axis $m = 0$, in which each point $(2m, 0)$ is connected with (m, n) and $(m, -n)$, for any n. The resonances of type (iv) (which we will call non-standard) are displayed as dashed lines: they constitute triangles in which none of the vertices lies on the vertical axis.

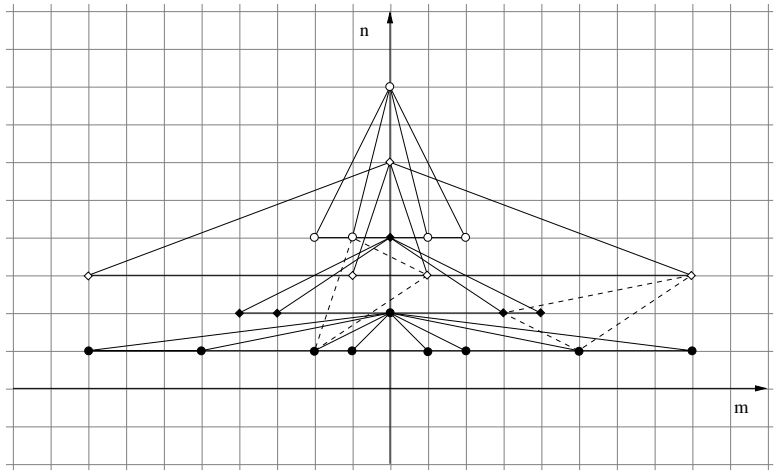

Fig. 1.2. Example of connected resonant triads for the CHM equation in the $\mathbb{Z}^2$ lattice. Points belonging to different clusters of standard resonances are marked with different symbols, solid lines connect standard resonances, dashed lines non-standard ones.

Since each non-standard resonance appears only for at most two values of ρ, then by removing (at most) a countable set of ρ's we kill all of them. Let us denote this removed set $\mathcal{Z} \subset \mathbb{R}_+$. The set $\mathbb{R}^+ \backslash \mathcal{Z}$ of remaining values of ρ, for which no non-standard resonance appear, can be regarded as "typical". Accordingly we will refer to $\rho \in \mathbb{R}^+ \backslash \mathcal{Z}$ as *typical* values of ρ (or as to the *typical case*). Below in Section 1.5 we show that in the typical case all resonances are divided to non-intersecting clusters of size at most 3, thus fitting the third option, considered in Section 1.3.

1.4 NLS: the power-law energy spectrum

Effective equation (1.20) for NLS (which, as we have seen, determines the energy spectrum) is not easy to handle since its completely connected resonance structure (see Section 1.3.2) makes impossible to split it to simpler subsystems (on contrary to the CHM equation, see Section 1.5). We present here a way to investigate the behaviour of solutions of (1.20) when the size L of the box goes to infinity, based on certain traditional for the wave turbulence heuristic approximation (see [9, 8, 16]), following our work [5]. This will lead us to a wave kinetic (WK) equation of the form, usually encountered in the wave turbulence. The treatment follows closely the paper [5], to which the reader can refer for further details.

1.4.1 The limit $L \to \infty$

From the point of view of mathematics, the limit when the size L of the torus $\mathbb{T}_L^d$ tends to infinity in equation (1.20) presents a serious problem. In particular, for what concerns the definition of a possible limiting stochastic equation. Instead of trying to resolve this difficulty, for any finite L we will study the expectations $\mathbf{E}(\tilde{I}_{\mathbf{k}})$ of the actions $\tilde{I}_{\mathbf{k}} = \frac{1}{2}|\tilde{a}_{\mathbf{k}}|^2$ of solutions for the corresponding equation (the function

$\mathbf{k} \mapsto \mathbf{E}(\tilde{I}_{\mathbf{k}})$ is called called *the wave-action spectrum*), and then pass to the limit as $L \to \infty$ only for these quantities. [6]

We fix L and, by making use of Ito's formula for $\tilde{I}_{\mathbf{k}}$, get from (1.20) that

$$\begin{aligned} \frac{d}{d\tau}\tilde{I}_{\mathbf{k}} = & \frac{i\delta}{2} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3 \in \mathbb{Z}^d} (\tilde{a}_{\mathbf{k}_1}\tilde{a}_{\mathbf{k}_2}\tilde{a}^*_{\mathbf{k}_3}\tilde{a}^*_{\mathbf{k}} - \tilde{a}^*_{\mathbf{k}_1}\tilde{a}^*_{\mathbf{k}_2}\tilde{a}_{\mathbf{k}_3}\tilde{a}_{\mathbf{k}})\, \delta^{12}_{3\mathbf{k}}\delta\left(\omega^{N\,12}_{\;\;3\mathbf{k}}\right) \\ & - \gamma_{\mathbf{k}}\tilde{I}_{\mathbf{k}} + \frac{\mu}{2} b_{\mathbf{k}} \left(\tilde{a}^*_{\mathbf{k}}\dot{\boldsymbol{\beta}}_{\mathbf{k}} + \tilde{a}_{\mathbf{k}}\dot{\boldsymbol{\beta}}^*_{\mathbf{k}}\right) + b^2_{\mathbf{k}} \ , \quad \mathbf{k} \in \mathbb{Z}^2 \, . \end{aligned} \tag{1.24}$$

Now we pass to the expected values, and define the moment $M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}(\tau)$ of $\tilde{a}(\tau)$ of order n_1+n_2 as

$$M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}(\tau) = \left\langle \tilde{a}_{\mathbf{k}_1}\cdots\tilde{a}_{\mathbf{k}_{n_1}}\tilde{a}^*_{\mathbf{k}_{n_1+1}}\cdots\tilde{a}^*_{\mathbf{k}_{n_1+n_2}}\right\rangle_\tau \ , \tag{1.25}$$

where $\langle\cdot\rangle_\tau$ stands for the expected value at time τ, i.e., $\langle f(\tilde{a})\rangle_\tau = \mathbf{E}\big(f(\tilde{a}(\tau))\big)$ for any measurable function $f(\tilde{a})$. Then from the system (1.24) we get

$$\dot{M}^{\mathbf{k}}_{\mathbf{k}} = -2\gamma_{\mathbf{k}}M^{\mathbf{k}}_{\mathbf{k}} + 2b^2_{\mathbf{k}} + 2\delta \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \operatorname{Im} M^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3}\delta^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3}\delta(\omega^{N\,\mathbf{k}_1\mathbf{k}_2}_{\;\;\mathbf{k}\mathbf{k}_3}) \ , \quad \mathbf{k} \in \mathbb{Z}^d \ . \tag{1.26}$$

This system is not closed since it involves the moments of order 4. By applying again Ito's formula, we can express the time derivative of moments of any order n_1+n_2 as a function of the moments of order n_1+n_2-2 and those of order n_1+n_2+2. The coupled system, containing the equations for all moments, is called the *chain of moments equation* (see [17]).[7] Systems of this kind are usually treated by approximating moments of high order by suitable functions of lower order moments in order to get a closed system of equations. We will show that if the quasi-stationary and quasi-Gaussian approximations (see below) are chosen to close the system of moment equations, then under the limit $L \to \infty$ we recover a modified version of the WK equation.

To study the sum in the r.h.s. of (1.26), we notice that if the Krönecker deltas are different from zero because $\mathbf{k}$ equals to one vector among $\mathbf{k}_1, \mathbf{k}_2$ and $\mathbf{k}_3$ is equal to another, then the moment is real and does not contribute to the sum. So we may assume that $\mathbf{k} \neq \mathbf{k}_1, \mathbf{k}_2$, $\mathbf{k}_3 \neq \mathbf{k}_1, \mathbf{k}_2$. In this case we calculate the fourth order moments in the r.h.s. of (1.26) through Ito's formula (see [5]) and get

$$\begin{aligned} \dot{M}^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3} = & -(\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3})M^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3} + i\delta \sum_{\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6} \Big(M^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4}_{\mathbf{k}_3\mathbf{k}_5\mathbf{k}_6}\delta^{\mathbf{k}\mathbf{k}_4}_{\mathbf{k}_5\mathbf{k}_6}\delta(\omega^{N\,\mathbf{k}\mathbf{k}_4}_{\;\;\mathbf{k}_5\mathbf{k}_6}) + \\ & M^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4}_{\mathbf{k}\mathbf{k}_5\mathbf{k}_6}\delta^{\mathbf{k}_3\mathbf{k}_4}_{\mathbf{k}_5\mathbf{k}_6}\delta(\omega^{N\,\mathbf{k}_3\mathbf{k}_4}_{\;\;\mathbf{k}_5\mathbf{k}_6}) - M^{\mathbf{k}_2\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}\delta^{\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}_1\mathbf{k}_4}\delta(\omega^{N\,\mathbf{k}_5\mathbf{k}_6}_{\;\;\mathbf{k}_1\mathbf{k}_4}) - M^{\mathbf{k}_1\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}\delta^{\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}_2\mathbf{k}_4}\delta(\omega^{N\,\mathbf{k}_5\mathbf{k}_6}_{\;\;\mathbf{k}_2\mathbf{k}_4}) \Big) \ . \end{aligned} \tag{1.27}$$

We make now the first approximation by neglecting the term containing the time derivative at the l.h.s. of (1.27). This can be justified, if τ is large enough, by the quasi-stationary approximation (cf. Section 2.1.3 in [8]). Namely, let us write equation (1.27) as

$$\left(\frac{d}{d\tau} + (\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3})\right) M^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3} = f \ .$$

Notice that since all $\gamma_{\mathbf{k}}$'s are positive, then the linear differential equation in the l.h.s. is exponentially stable. Assume that f as a function of τ is almost constant during time-intervals, sufficient for relaxation of the differential equation. Then

[6] In the case of the non-forced equation the expectations should be taken with respect to the distribution of the initial data, while for the forced equation – with respect to the distribution of the forcing (and, possibly, of the initial data).

[7] Notice that, since the equation which we consider has a cubic nonlinearity, equations for moments of even order are decoupled from those for moments of odd order.

$$M^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3} \approx \frac{f}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}} \ .$$

We insert this in (1.26) and get

$$\begin{aligned}\dot{M}^{\mathbf{k}}_{\mathbf{k}} \approx & -2\gamma_{\mathbf{k}} M^{\mathbf{k}}_{\mathbf{k}} + 2b^2_{\mathbf{k}} + 2\delta^2 \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}} \delta^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3} \delta(\omega^{N\,\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3}) \\ & \times \mathrm{Re}\Bigg(\sum_{\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6} \Big(M^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4}_{\mathbf{k}_3\mathbf{k}_5\mathbf{k}_6} \delta^{\mathbf{k}\mathbf{k}_4}_{\mathbf{k}_5\mathbf{k}_6} \delta(\omega^{N\,\mathbf{k}\mathbf{k}_4}_{\mathbf{k}_5\mathbf{k}_6}) + M^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4}_{\mathbf{k}\mathbf{k}_5\mathbf{k}_6} \delta^{\mathbf{k}_3\mathbf{k}_4}_{\mathbf{k}_5\mathbf{k}_6} \delta(\omega^{N\,\mathbf{k}_3\mathbf{k}_4}_{\mathbf{k}_5\mathbf{k}_6}) \\ & - M^{\mathbf{k}_2\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4} \delta^{\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}_1\mathbf{k}_4} \delta(\omega^{N\,\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}_1\mathbf{k}_4}) - M^{\mathbf{k}_1\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4} \delta^{\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}_2\mathbf{k}_4} \delta(\omega^{N\,\mathbf{k}_5\mathbf{k}_6}_{\mathbf{k}_2\mathbf{k}_4}) \Big) \Bigg) \ . \end{aligned} \tag{1.28}$$

We then apply the second approximation, generally accepted in the WT (see [8, 18, 9, 16]) which enables us to transform the previous relation to a closed equation for the second order moments. This consists in the quasi–Gaussian approximation, i.e., in the assumption that the higher-order moments (1.25) can be approximated by polynomials of the second-order moments, as if the random variables $v_{\mathbf{k}}$ were independent complex Gaussian variables. In particular,

$$M^{\mathbf{l}_1\mathbf{l}_2\mathbf{l}_3}_{\mathbf{l}_4\mathbf{l}_5\mathbf{l}_6} \approx M^{\mathbf{l}_1}_{\mathbf{l}_1} M^{\mathbf{l}_2}_{\mathbf{l}_2} M^{\mathbf{l}_3}_{\mathbf{l}_3} \left(\delta^{\mathbf{l}_1}_{\mathbf{l}_4} (\delta^{\mathbf{l}_2}_{\mathbf{l}_5} \delta^{\mathbf{l}_3}_{\mathbf{l}_6} + \delta^{\mathbf{l}_2}_{\mathbf{l}_6} \delta^{\mathbf{l}_3}_{\mathbf{l}_5}) \ + \delta^{\mathbf{l}_1}_{\mathbf{l}_5} (\delta^{\mathbf{l}_2}_{\mathbf{l}_4} \delta^{\mathbf{l}_3}_{\mathbf{l}_6} + \delta^{\mathbf{l}_2}_{\mathbf{l}_6} \delta^{\mathbf{l}_3}_{\mathbf{l}_4}) + \delta^{\mathbf{l}_1}_{\mathbf{l}_6} (\delta^{\mathbf{l}_2}_{\mathbf{l}_4} \delta^{\mathbf{l}_3}_{\mathbf{l}_5} + \delta^{\mathbf{l}_2}_{\mathbf{l}_5} \delta^{\mathbf{l}_3}_{\mathbf{l}_4}) \right) \ . \tag{1.29}$$

At this point we pass in equation (1.28), closed using the relation (1.29), to the limit $L \to \infty$. This can be done by approximating sums with integrals if, instead of parametrising the modes by integer vectors $\mathbf{k} \in \mathbb{Z}^d$, we parametrise them by vectors $\tilde{\mathbf{k}} = \mathbf{k}/L$ from the shrunk lattice $\mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d$. Accordingly we define

$$\tilde{M}^{\tilde{\mathbf{k}}_1 \ldots \tilde{\mathbf{k}}_{n_1}}_{\tilde{\mathbf{k}}_{n_1+1} \ldots \tilde{\mathbf{k}}_{n_1+n_2}} := M^{\mathbf{k}_1 \ldots \mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1} \ldots \mathbf{k}_{n_1+n_2}} \ , \quad \tilde{\gamma}_{\tilde{\mathbf{k}}} := \gamma_{\mathbf{k}} \ , \quad \tilde{b}_{\tilde{\mathbf{k}}} := b_{\mathbf{k}} \ ,$$

and note that since the restriction, imposed by the Krönecker deltas, is homogeneous, then it does not change under this re-parametrisation. Abusing notation, we will drop the tildes in the rest of the Section, but will use the parametrisation by points of $\mathbb{Z}^d_L$.

We denote by $S_{\mathbf{k}}$ the sum, given by the second and third lines of (1.28), written in the new parametrisation, and note that it splits into a finite number of sums like

$$S^j_{\mathbf{k}} = \sum_{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6) \in \mathbb{Z}^{6d}_L \cap \Sigma^j_{\mathbf{k}}} F^j_{\mathbf{k}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6) \, .$$

Here $\Sigma^j_{\mathbf{k}}$ is a manifold in $\mathbb{R}^{6d}$, defined as

$$\begin{aligned}\Sigma^j_{\mathbf{k}} = \big\{ & (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\mathbf{x}_5,\mathbf{x}_6) \ : \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{k} + \mathbf{x}_3, |\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 = |\mathbf{k}|^2 + |\mathbf{x}_3|^2 \\ & \mathbf{x}^j + \mathbf{x}_4 = \mathbf{x}_5 + \mathbf{x}_6, |\mathbf{x}^j|^2 + |\mathbf{x}_4|^2 = |\mathbf{x}_5|^2 + |\mathbf{x}_6|^2, \mathbf{x}^j_1 = \mathbf{x}^j_2, \mathbf{x}^j_3 = \mathbf{x}^j_4, \mathbf{x}^j_5 = \mathbf{x}^j_6 \big\} \ , \end{aligned}$$

where $\mathbf{x}^j$ stands for one among the vectors $\mathbf{k}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, and $\{\mathbf{x}^j_1, \ldots, \mathbf{x}^j_6\}$ – for a permutation of the set $\{\mathbf{k}, \mathbf{x}_1, \ldots, \mathbf{x}_6\} \backslash \{\mathbf{x}^j\}$.[8] It is easy to see that since every F^j is a regular function, then when passing from the sums to integrals in the limit $L \to \infty$, each term $S^j_{\mathbf{k}}$ as a function of L becomes proportional to L^m, where m is the dimension of the manifold $\Sigma^j_{\mathbf{k}}$. A detailed analysis of all cases shows that the terms of the highest order in L in the integral correspond to terms of the form

[8] Note that the relations, defining $\Sigma^j_{\mathbf{k}}$ are not independent.

$$S^j_{\mathbf{k}} = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} F_{\mathbf{k}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3}\delta({\omega^N}^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3})$$

in the sum $S_{\mathbf{k}}$, where $\mathbf{k} := (\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in \mathbb{Z}^{3d}_L =: \mathcal{M}$. Denote

$$\Sigma_{\mathbf{k}} = \left\{\mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \in \mathbb{R}^{3d} : \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{k} + \mathbf{x}_3, |\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 = |\mathbf{k}|^2 + |\mathbf{x}_3|^2\right\} .$$

This is a manifold of dimension $3d - d - 1 = 2d - 1$, smooth outside the origin. The latter lies outside $\Sigma_{\mathbf{k}}$ if $\mathbf{k} \neq 0$, and is a singular point of $\Sigma_{\mathbf{k}}$ if $\mathbf{k} = 0$.

As shown in [5], in the limit $L \to \infty$ the sum $S_{\mathbf{k}}$ can be approximated by the integral

$$S_{\mathbf{k}} \approx L^{2d-1}\int_{\Sigma\setminus\{0\}} \frac{F_{\mathbf{k}}(\mathbf{x})}{\varphi_{\mathbf{k}}(\mathbf{x})} d\mathbf{x} , \tag{1.30}$$

where $\varphi_{\mathbf{k}}(\mathbf{x})$ is a certain function on $\Sigma_{\mathbf{k}}$, smooth outside zero, such that

$$\begin{aligned} &V_1 \le \varphi_{\mathbf{k}}(\mathbf{x}) \le V_1(3d)^{d-1/2} , \quad \varphi_{\mathbf{k}}(\mathbf{x}) := \varphi_{m\mathbf{k}}(m\mathbf{x}) , \\ &\varphi_{\mathbf{k}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \varphi_{\mathbf{k}}(\mathbf{x}_2,\mathbf{x}_1,\mathbf{x}_3) , \quad \varphi_{\mathbf{k}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \varphi_{\mathbf{x}_3}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{k}) , \end{aligned} \tag{1.31}$$

where V_1 is the volume of the 1-ball in $\mathbb{R}^{2d-1}$.

By substituting (1.29) in (1.28) and using (1.30) we get the limiting (as $L \to \infty$) equation in the form

$$\begin{aligned} \dot{M}^{\mathbf{k}}_{\mathbf{k}} \approx &- 2\gamma_{\mathbf{k}} M^{\mathbf{k}}_{\mathbf{k}} + 2b^2_{\mathbf{k}} + 4\delta^2 L^{2d-1}\int_{\mathbb{R}^{3d}\setminus\{0\}} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \frac{\varphi^{-1}_{\mathbf{k}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}}\delta^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3} \\ &\times \delta({\omega^N}^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3})\Big(M^{\mathbf{k}_1}_{\mathbf{k}_1}M^{\mathbf{k}_2}_{\mathbf{k}_2}M^{\mathbf{k}_3}_{\mathbf{k}_3} + M^{\mathbf{k}}_{\mathbf{k}}M^{\mathbf{k}_1}_{\mathbf{k}_1}M^{\mathbf{k}_2}_{\mathbf{k}_2} - M^{\mathbf{k}}_{\mathbf{k}}M^{\mathbf{k}_2}_{\mathbf{k}_2}M^{\mathbf{k}_3}_{\mathbf{k}_3} - M^{\mathbf{k}}_{\mathbf{k}}M^{\mathbf{k}_1}_{\mathbf{k}_1}M^{\mathbf{k}_3}_{\mathbf{k}_3}\Big) . \end{aligned}$$

Finally, we define

$$n_{\mathbf{k}} = L^d M^{\mathbf{k}}_{\mathbf{k}}/2 , \quad \mathfrak{b}_{\mathbf{k}} = L^{d/2} b_{\mathbf{k}} , \tag{1.32}$$

(so that $\sum_{\mathbf{k}} M^{\mathbf{k}}_{\mathbf{k}}/2 \to \int n_{\mathbf{k}}$ and $\sum_{\mathbf{k}} b^2_{\mathbf{k}} \to \int \mathfrak{b}^2_{\mathbf{k}}$ as L goes to infinity), choose

$$\delta(L) = \tilde{\varepsilon}^2 L^{1/2} = \frac{\varepsilon^{2q*}}{\nu}\tilde{\varepsilon}^2 L^{1/2} , \tag{1.33}$$

for some $\tilde{\varepsilon} > 0$, and get the kinetic equation

$$\begin{aligned} \dot{n}_{\mathbf{k}} = &-2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \mathfrak{b}^2_{\mathbf{k}} + 16\tilde{\varepsilon}^4 \int_{\mathbb{R}^{3d}\setminus\{0\}} d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3 \delta^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3}\delta({\omega^N}^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3}) \\ &\times \frac{\varphi^{-1}_{\mathbf{k}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}}\Big(n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3} + n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_2} - n_{\mathbf{k}}n_{\mathbf{k}_2}n_{\mathbf{k}_3} - n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_3}\Big) . \end{aligned} \tag{1.34}$$

We have thus shown that, with the proper scaling of δ and b given by (1.32)-(1.33)), the function $n_{\mathbf{k}}$ satisfies a kinetic equation, similar to the WK equation for NLS in the classical wave turbulence theory (see, for instance, formula (6.81) of [9], where $d = 2$). The differences are two: obviously in our case there are forcing and dissipation, absent in the traditional WK equations. More interesting is the nonvanishing denominator $\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}$ which regularises the integral since it growths to infinity with $\mathbf{k}$, and which modifies the spectra.

1.4.2 Power law spectra

Now, under some additional approximation and using the well known Zakharov argument (see [8, 9, 16]) we will get stationary solutions of eq. (1.34) with power law energy spectra (more properly, wave-action spectra) $\{n_{\mathbf{k}}\}$.

To do this we have to restrain our analysis to the inertial interval, i.e., to the spectral interval, where the damping and forcing are negligible. That is, we should consider eq. (1.34), supposing that the wave-vector $\mathbf{k}$ belongs to a sufficiently large spectral region, where the first two terms at the r.h.s. of (1.34) can be neglected, compared with the third. This happens, e.g., if there the solution $\{n_{\mathbf{k}}\}$ is of order one, while $\mathfrak{b}_{\mathbf{k}} \ll 1$ and $\gamma_{\mathbf{k}} \ll 1$ (i.e., the damping and the dissipation are small at that spectral region). In the inertial interval we end up with the equation

$$\begin{aligned} \dot{n}_{\mathbf{k}} \approx 16\,\tilde{\varepsilon}^4 \int_{\mathbb{R}^{3d}\setminus\{0\}} d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3 \delta^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3} \delta(\omega^{N^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3}}) \frac{\varphi_{\mathbf{k}}^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}} \\ \times \Big(n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3} \Big) . \end{aligned} \tag{1.35}$$

Notice that, while in the inertial interval we can simply approximate $\mathfrak{b}_{\mathbf{k}}$ with zero, this cannot be done to $\gamma_{\mathbf{k}}$ since these numbers appear in the denominator of the integral at the r.h.s. of (1.34) (their sum makes the denominator of the so-called *collision term*), and play an essential role in determining of the spectrum.

The previous equation has the form of the four-wave kinetic equation (see, for instance, formula 2.1.29 of [8]). It is well known (see [8, 9]) how to solve such equations for stationary spectra with the aid of the Zakharov transformations, provided that the terms

$$\mathcal{T}^{\mathbf{k},\mathbf{k}_3}_{\mathbf{k}_1,\mathbf{k}_2} = \frac{\varphi_{\mathbf{k}}^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}}$$

satisfy, for some $m \in \mathbb{R}$, the following conditions of symmetry and homogeneity:

$$\mathcal{T}^{\mathbf{k},\mathbf{k}_3}_{\mathbf{k}_1,\mathbf{k}_2} = \mathcal{T}^{\mathbf{k}_3,\mathbf{k}}_{\mathbf{k}_1,\mathbf{k}_2} = \mathcal{T}^{\mathbf{k},\mathbf{k}_3}_{\mathbf{k}_2,\mathbf{k}_1} = \mathcal{T}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathbf{k},\mathbf{k}_3} , \qquad \mathcal{T}^{\lambda\mathbf{k},\lambda\mathbf{k}_3}_{\lambda\mathbf{k}_1,\lambda\mathbf{k}_2} = \lambda^m \mathcal{T}^{\mathbf{k},\mathbf{k}_3}_{\mathbf{k}_1,\mathbf{k}_2} .$$

Since φ is a homogeneous function of degree 0 due to (1.31), the requirements above are met if on the inertial range the function $\gamma_{\mathbf{k}}$ can be approximated by a homogeneous function of the form $\gamma_{\mathbf{k}} = \varepsilon' |\mathbf{k}|^m$, where m is a real number and $\varepsilon' \ll 1$ is a small parameter to guaranty that the dissipation term indeed is negligible.

We abbreviate $|\mathbf{k}| = k$ and look for stationary isotropic, spectra behaving as power laws of k, i.e. $n_{\mathbf{k}} = n_k \propto k^\nu$ for some real ν, by searching ν such that the r.h.s. of (1.35) vanishes. The result (see [5]) is that, in addition to the equilibrium solutions $n_k = C$ and $n_k = C/k^2$, which correspond, respectively, to the equipartition of the wave action and of the quadratic energy (Rayleigh-Jeans distribution), two nontrivial power law stationary distributions appear. These are the solutions:

$$n_k \propto k^{-(m+3d-2)/3} , \qquad n_k \propto k^{-(m+3d)/3} . \tag{1.36}$$

If $m = 0$, they coincide with the well known in the wave turbulence power-law spectra for the free NLS equation $(1.14)_{\nu=0}$ (for $d \geq 2$, see [9]), but the dissipation modifies the power law of the decay if $m \neq 0$.

1.5 CHM: resonance clustering

Let us consider in more detail the effective equation (1.21) for the CHM equation for typical values of the shape-factor ρ. By the definition of a typical ρ (see Section 1.3.3), no resonances corresponding to

the case (iv) of Section 1.3.3 occur. We can then write the effective equation explicitly, following [6]. It will only involve resonances of type (iii).

Let us consider the equations for the variables $\tilde{a}_{\mathbf{k}}$ with $\mathbf{k} = (m, n)$, separating the cases $m = 0$ and $m \neq 0$. When $m = 0$, the only terms which survive in the nonlinearity $R_{\mathbf{k}}(\tilde{a})$ are those where $\mathbf{k}_1$ and $\mathbf{k}_2$ satisfy the relation (iii-b) of Section 1.3.3, while for $m \neq 0$ only the terms falling in the case (iii-a) give contribution. For $m = 0$, the nonlinearity vanishes if n is odd, while if it is even, then

$$R_{\mathbf{k}}(\tilde{a}) = \frac{1}{\rho(m^2 + n^2\rho^2 + F\rho^2)} \sum_{m_1 \in \mathbb{Z}} \left(m_1^2 + \frac{n^2\rho^2}{4} \right) m_1 n \, \tilde{a}_{(m_1, n/2)} \, \tilde{a}_{(-m_1, n/2)} \ , \quad m = 0 \ ,$$

which in turn vanishes because the odd symmetry in m_1. On the other hand, if $m_1 \neq 0$, then $\mathbf{k}_1$ and $\mathbf{k}_2$ are completely determined by $\mathbf{k}$. So we get that

$$R_{\mathbf{k}}(\tilde{a}) = \left(\frac{2mn}{\rho(m^2 + n^2\rho^2 + F\rho^2)} \left(3n^2\rho^2 - m^2\right) \tilde{a}_{\bar{\mathbf{k}}} \, \tilde{a}_{(0,2n)} \right) , \tag{1.37}$$

where we denoted $\bar{\mathbf{k}} := (m, -n)$. Note that this formula applies for the both case $m = 0$ and $m \neq 0$.

Expression (1.37) entails the remarkable consequence that the hamiltonian part of the effective equation, i.e., the system in which forcing and dissipation are removed,

$$\frac{d}{d\tau} \tilde{a}_{\mathbf{k}} = R_{\mathbf{k}}(\tilde{a}) \ , \quad \mathbf{k} \in \mathbb{Z}^2 \ , \tag{1.38}$$

is integrable and decomposes to invariant subsystems of complex dimension at most three. Indeed, if m or n vanish, then $R_{\mathbf{k}} = 0$ and $\tilde{a}_{\mathbf{k}}(t) = \text{const}$. Now let $m, n \neq 0$. If $3\rho^2 n^2 = m^2$, then again the equation for $\tilde{a}_{\mathbf{k}}$ trivialises. Suppose that $3L^2 n^2 \neq m^2$ and denote

$$A_{\mathbf{k}} = \frac{2mn}{m^2 + n^2\rho^2 + F\rho^2} \left(3n^2\rho^2 - m^2\right) \in \mathbb{R} \, . \tag{1.39}$$

Then $A_{\bar{\mathbf{k}}} \equiv -A_{\mathbf{k}}$. Eq. (1.38) (with any fixed $\mathbf{k}$) belongs to the following invariant sub-system of (1.38):

$$\begin{aligned} &\frac{d}{d\tau} \tilde{a}_{\mathbf{k}} = A_{\mathbf{k}} \, \tilde{a}_{(0,2n)} \tilde{a}_{\bar{\mathbf{k}}}, \\ &\frac{d}{d\tau} \tilde{a}_{\bar{\mathbf{k}}} = -A_{\mathbf{k}} \, \tilde{a}^*_{(0,2n)} \tilde{a}_{\mathbf{k}}, \\ &\frac{d}{d\tau} \tilde{a}_{(0,2n)} = 0 \end{aligned} \tag{1.40}$$

(we recall that $\tilde{a}_{(0,2n)} = \tilde{a}^*_{(0,-2n)}$ by the reality condition, see footnote 5). This system is explicitly soluble: if $\tilde{a}_{(0,2n)}(0) \neq 0$, then

$$\begin{aligned} &\tilde{a}_{(0,2n)}(t) = \text{ Const} \, , \\ &\tilde{a}_{\mathbf{k}}(t) = \tilde{a}_{\mathbf{k}}(0) \cos(|A_{\mathbf{k}} \tilde{a}_{(0,2n)}| t) + \tilde{a}_{\bar{\mathbf{k}}}(0) \, \mathrm{sgn}(A_{\mathbf{k}} \tilde{a}_{(0,2n)}) \sin(|A_{\mathbf{k}} \tilde{a}_{(0,2n)}| t) \, , \end{aligned} \tag{1.41}$$

where for a complex number z we denote

$$\mathrm{sgn}(z) = z/|z| \text{ if } z \neq 0, \text{ and } \ \mathrm{sgn}(0) = 0 \, .$$

The formula for $\tilde{a}_{\bar{\mathbf{k}}}(t)$ is obtained from that for $\tilde{a}_{\mathbf{k}}(t)$ by swapping $\mathbf{k}$ with $\bar{\mathbf{k}}$ and replacing $\tilde{a}_{(0,2n)}$ by its complex conjugate. All these solutions are periodic, and it is easy to check that $|\tilde{a}_{\mathbf{k}}|^2 + |\tilde{a}_{\bar{\mathbf{k}}}|^2$ and $|\tilde{a}_{(0,2n)}|^2$ are integrals of motion for eq. (1.40).

We have established that there is no Hamiltonian exchange of energy between different modes, apart the coupled modes $\tilde{a}_{\mathbf{k}}$ and $\tilde{a}_{\bar{\mathbf{k}}}$. The situation does not change much when we switch in the

forcing and the dissipation since the effective equation (1.21), too, splits to invariant subsystems of complex dimension one (if $mn = 0$ or $3n^2\rho^2 = m^2$), or of dimension three (otherwise). These systems either are independent, or have *catalytic interaction* through the variables $\tilde{a}_{(0,2n)}$, which satisfy the Ornstein–Uhlenbeck equation

$$\frac{d}{d\tau}\tilde{a}_{(0,2n)} = -\gamma_{(0,2n)}v_{(0,2n)} + \mu b_{(0,2n)}\dot{\boldsymbol{\beta}}_{(0,2n)} ,$$

and are independent from other variables.

Being particularly interested in the exchange of energy, let us consider the equation for the actions $\tilde{I}$. Due to the conservation of $|\tilde{a}_{\mathbf{k}}|^2 + |\tilde{a}^2_{\bar{\mathbf{k}}}|^2$, Ito's formula gives (cf. (1.24))

$$\frac{d}{d\tau}\left(\tilde{I}_{\mathbf{k}} + \tilde{I}_{\bar{\mathbf{k}}}\right) = -\gamma_{\mathbf{k}}\tilde{I}_{\mathbf{k}} - \gamma_{\bar{\mathbf{k}}}\tilde{I}_{\bar{\mathbf{k}}} + \frac{\mu}{2}b_{\mathbf{k}}\left(\tilde{a}^*_{\mathbf{k}}\dot{\boldsymbol{\beta}}_{\mathbf{k}} + \tilde{a}_{\mathbf{k}}\dot{\boldsymbol{\beta}}^*_{\mathbf{k}}\right) + \frac{\mu}{2}b_{\bar{\mathbf{k}}}\left(\tilde{a}^*_{\bar{\mathbf{k}}}\dot{\boldsymbol{\beta}}_{\bar{\mathbf{k}}} + \tilde{a}_{\bar{\mathbf{k}}}\dot{\boldsymbol{\beta}}^*_{\bar{\mathbf{k}}}\right) + b^2_{\mathbf{k}} + b^2_{\bar{\mathbf{k}}} .$$

By taking the expected value, we see that the second order moments satisfy

$$\dot{M}^{\mathbf{k}}_{\mathbf{k}} + \dot{M}^{\bar{\mathbf{k}}}_{\bar{\mathbf{k}}} = -\gamma_{\mathbf{k}}M^{\mathbf{k}}_{\mathbf{k}} - \gamma_{\bar{\mathbf{k}}}M^{\bar{\mathbf{k}}}_{\bar{\mathbf{k}}} + 2(b^2_{\mathbf{k}} + b^2_{\bar{\mathbf{k}}}) \ , \dot{M}^{(0,n)}_{(0,n)} = -\gamma_{(0,n)}M^{(0,n)}_{(0,n)} + b^2_{(0,n)} \tag{1.42}$$

This equations should be compared with (1.26): they show that the amount of energy contained in a given cluster is not transferred to other clusters and depends only on the forcing and the dissipation, acting in its interior. Thus the energy cascades cannot occur for *typical* values of ρ.

1.6 Discussion

In this Chapter we presented a method to study a weakly nonlinear PDE by investigating properties of the corresponding *effective equation*, written in terms of the nonlinearity and the resonances in the spectrum of the linear part of the equation.

We have considered two examples of equations, where the structures of resonances are completely different. Namely, for the NLS equation all resonances are connected, and for it we have provided a way of getting power law stationary energy spectra (evoking a heuristic physical argument in addition to a rigorous mathematical theory), while for the Charney–Hasegawa–Mima equation the resonances form finite clusters, and for this equation we have shown (completely rigorously) that, in the *typical* case, no exchange of energy between different oscillating modes occurs.

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