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KDV HAMILTONIAN AS A FUNCTION OF ACTIONS

EVGENY L. KOROTYAEV AND SERGEI KUKSIN

Abstract. We prove that the non-linear part of the Hamiltonian of the KdV equation on the circle, written as a function of the actions, defines a continuous convex function on the ℓ^2 space and derive for it lower and upper bounds in terms of some functions of the ℓ^2-norm. The proof is based on a new representation of the Hamiltonian in terms of the quasimomentum, obtained via the conformal mapping theory.

1. Introduction and main results

We consider the Korteweg de Vries (KdV) equation under zero mean-value periodic boundary conditions:

$$\begin{aligned} &q_t = -q_{xxx} + 6qq_x, \qquad\qquad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \\ &\int_0^1 q(x,t)\,dx = 0. \end{aligned} \tag{1.1}$$

For any $\alpha \in \mathbb{R}$ denote by $\mathscr{H}_\alpha$ the Sobolev space of real-valued 1-periodic functions with zero mean-value. In particular, we have

$$\mathscr{H}_\alpha = \mathscr{H}_\alpha(\mathbb{T}) = \left\{ q \in L^2(\mathbb{T}) : \ q^{(\alpha)} \in L^2(\mathbb{T}), \quad \int_0^1 q(x)\,dx = 0 \right\}, \qquad \alpha \geqslant 0.$$

We provide the spaces $\mathscr{H}_\alpha$ with the trigonometric base $\{e_{\pm 1}, e_{\pm 2}, e_{\pm 3}, ...\}$, where

$$e_j(x) = \sqrt{2}\cos 2\pi jx, \qquad e_{-j}(x) = -\sqrt{2}\sin 2\pi jx, \qquad j \geqslant 1.$$

We also introduce real spaces ℓ^p_α of sequences $f = (f_n)_1^\infty$, equipped with the norms

$$\|f\|^p_{p,\alpha} = \sum_{n\geqslant 1} (2\pi n)^{2\alpha} |f_n|^p, \qquad p \geqslant 1, \ \alpha \in \mathbb{R}, \tag{1.2}$$

and positive octants

$$\ell^p_{\alpha,+} = \left\{ f = (f_n)_1^\infty \in \ell^p_\alpha : f_n \geqslant 0, \ \forall\, n \geqslant 1 \right\}.$$

In the case $\alpha = 0$ we write $\ell^p = \ell^p_0, \ell^p_+ = \ell^p_{0,+}$ and $\|\cdot\|_p = \|\cdot\|_{p,0}$.

The operator $\frac{\partial}{\partial x}$ defines linear isomorphisms $\frac{\partial}{\partial x} : \mathscr{H}_\alpha \to \mathscr{H}_{\alpha-1}$. Denoting by $(\frac{\partial}{\partial x})^{-1}$ the inverse operator, we provide the spaces $\mathscr{H}_\alpha, \alpha \geqslant 0$, with a symplectic structure by means of the 2-form ω_2:

$$\omega_2(q_1, q_2) = -\langle (\partial/\partial x)^{-1} q_1, q_2 \rangle,$$

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where $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2(0,1)$. Then in any space $\mathscr{H}_\alpha, \alpha \geqslant 1$, the KdV equation (1.1) may be written as a Hamiltonian system with the Hamiltonian H_2, given by

$$H_2(q) = \frac{1}{2}\int_0^1 (q'(x)^2 + 2q^3(x))\,dx.$$

That is, as the system

$$q_t = \frac{\partial}{\partial x}\frac{\partial}{\partial q}H_2(q), \tag{1.3}$$

e.g. see [Ku, KaP, HK] (note that H_2 is an analytic function on any space $\mathscr{H}_\alpha,\ \alpha \geqslant 1$).

It is well known after the celebrated work of Novikov, Lax, Its and Matveev that the system (1.3) is integrable. It was shown by Kappeler and collaborators in a series of publications, starting with [Ka], that it admits global Birkhoff coordinates. Namely, for any $\alpha \in \mathbb{R}$ denote by $\mathfrak{h}_\alpha$ the Hilbert space, formed by real sequences $b = (b_n, b_{-n})_1^\infty$, equipped with the norm

$$\|b\|^2_{\mathfrak{h}_\alpha} = \sum_{n\geqslant 1}(2\pi n)^{2\alpha}(b_n^2 + b_{-n}^2).$$

We provide the spaces $\mathfrak{h}_\alpha$ with the usual symplectic form

$$\Omega_2 = \sum_{n\geqslant 1} db_n \wedge db_{-n},$$

and define the actions $I = (I_n)_1^\infty$ and the angles $\phi = (\phi_n)_1^\infty$ by

$$I_n = \frac{1}{2}(b_n^2 + b_{-n}^2), \qquad \phi_n = \arctan\frac{b_n}{b_{-n}}. \tag{1.4}$$

This is another set of symplectic coordinates on $\mathfrak{h}_\alpha$, since $\Omega_2 = dI \wedge d\phi$, at least formally. Then

1) There exists an analytic symplectomorphism $\Psi : \mathscr{H}_0 \to \mathfrak{h}_{\frac{1}{2}}$ which defines analytic diffeomorphisms $\Psi : \mathscr{H}_\alpha \to \mathfrak{h}_{\alpha+\frac{1}{2}}, \alpha \geqslant -1$, such that $d\Psi(0) = \Phi$, where

$$\Phi\Big(\sum_{j\geqslant 1}\Big(u_j e_j(x) + u_{-j}e_{-j}(x)\Big)\Big) = b, \qquad b_j = |2\pi j|^{\frac{1}{2}}u_j, \quad \forall\ j. \tag{1.5}$$

2) The transformed Hamiltonian $H_2(\Psi^{-1}(b))$ (which is an anlytic function on the space $\mathfrak{h}_{\frac{3}{2}}$) depends solely on the actions I, i.e. $K(I(b)) = H_2(\Psi^{-1}(b))$, where $K(I)$ is an analytic function on the octant $\ell^1_{\frac{3}{2},+}$. A curve $q(\cdot,t) \in C^1(\mathbb{R},\mathscr{H}_0)$ is a solution of (1.1) if and only if $b(t) = \Psi(q(\cdot,t))$ satisfies the following system of equations

$$\frac{\partial b_n}{\partial t} = -b_{-n}\frac{\partial K}{\partial I_n}, \qquad \frac{\partial b_{-n}}{\partial t} = b_n\frac{\partial K}{\partial I_n}, \qquad n \geqslant 1, \tag{1.6}$$

where $I = (I(b))$.

For 1)-2) with $\alpha \geqslant 0$ see [KaP] and with $\alpha = -1$ see [KaT]. See [KuP] for the important quasilinearity property of the transformation Ψ.

Note that

$$\|b\|_{h_\alpha} = 2|I|_{1,\alpha}.$$

Thus if $I \in \ell^p_+$ for some $p < \infty$, then $I \in \ell^1_{-\frac{1}{2},+}$ and the corresponding potential $q \in \mathscr{H}_{-1}$.

By 2), in the action-angle variables (I, ϕ) the KdV equation takes the form

$$I_t = 0, \qquad \phi_t = \frac{\partial}{\partial I} K(I). \tag{1.7}$$

This reduction of KdV is due to McKean-Trubowitz [MT1] and was found before the Birkhoff form (1.6). The action maps $\psi \mapsto I_j$, $j \geqslant 1$, are given by explicit formulas due to Arnold and are defined in a unique way. So the Hamiltonian $K(I)$ also is uniquelly defined, see [FM]. But the symplectic angles are defined only up to rotations $\phi \mapsto \phi + (\partial/\partial I)g(I)$, where g is any smooth function. So the transformation Ψ is not unique.

The Birkhoff coordinates b and the actions-angles (I, ϕ) make an effective tool to study properties of the KdV equation, see [KaT], and of its perturbations, see [Ku1, HK]. For both these goals it is important to understand properties of the Hamiltonian $K(I)$ which defines the dynamics (1.6) and (1.7). But the only information about the function $K(I)$ which follows from 1)-2) is that it is analytic on the spaces $\ell^1_{p,+}$, $p \geqslant 3/2$.[1]

Denote by P_j moments of the actions I, given by

$$P_j = \sum_{n \geqslant 1} (2\pi n)^j I_n, \qquad\qquad j \in \mathbb{Z}. \tag{1.8}$$

Note that

$$P_1 = \frac{1}{2}\|q\|^2, \qquad \text{if} \quad I = I(b),\ b = \Psi(q), \tag{1.9}$$

– this is the Parseval identity for the transformation Ψ, see [MT2, K5]. Due to (1.5), the linear part $dK(0)(I)$ of $K(I) = H_2(\Psi^{-1}(b))$ at the origin equals

$$\frac{1}{2}\int_0^1 \left(\frac{\partial}{\partial x}\left(\Phi^{-1} b\right)\right)^2 dx = P_3.$$

Therefore

$$K(I) = P_3(I) + O(I^2).$$

The cubic part $\int_0^1 q^3(x)\,dx$ of the Hamiltonian $H_2(q)$ is more regular than its quadratic part $\frac{1}{2}\int_0^1 q'(x)^2\,dx$. Thus it is natural to assume that the term P_3 is a singular part of $K(I)$ and to study smoothness of the more regular quadratic part V, given by

$$H_2(q) = K(I) = P_3(I) - V(I). \tag{1.10}$$

Here the minus-sign is convenient, since below we will see that $V \geqslant 0$. For any $N \geqslant 1$ denote by $\widetilde{\ell}^N \subset \ell^2$ the N-dimensional subspace

$$\widetilde{\ell}^N = \{I = (I_n)_1^\infty,\ I_n = 0 \quad \forall\ n > N\}, \tag{1.11}$$

and set $\widetilde{\ell}^\infty = \cup \widetilde{\ell}^N$. Clearly V is analytic on each octant $\widetilde{\ell}^N_+$ (i.e., it analytically extends to a neighbourhood of $\widetilde{\ell}^N_+$ in $\widetilde{\ell}^N$). So V is Gato-analytic on $\widetilde{\ell}^\infty_+$. That is, it is analytic on each interval $\{(a + tc) \in \widetilde{\ell}^\infty_+ | t \in \mathbb{R}\}$, where $a, c \in \widetilde{\ell}^\infty_+$. It is known that

$$\frac{\partial^2 V(0)}{\partial I_i \partial I_j} = 6\delta_{i,j} \qquad \forall \quad i, j \geqslant 1, \tag{1.12}$$

[1]About the behaviour of $K(I)$ of the finite-dimensional subspaces $\tilde{\ell}^N$, defined below in (1.11), we know more. See in [Ku] and below in Introduction.

see [BoKu] and [KaP, Ku, HK]. So $d^2V(0)(I) = 6\|I\|_2^2$. This suggests that the Hilbert space ℓ^2 rather than the Banach space $\ell^1_{\frac{3}{2}}$ (which is contained in ℓ^2) is a distinguished phase-space for the Hamiltonian $K(I)$. This guess is justified by the following theorem which is the main result of our work.

Theorem 1.1. *The function $V : \widetilde{\ell}^\infty_+ \to \mathbb{R}$ extends to a non-negative continuous function on the ℓ^2-octant ℓ^2_+, such that $V(I) = 0$ for some $I \in \ell^2_+$ iff $I = 0$. Moreover,*

$$0 \leqslant V(I) \leqslant 8P_1P_{-1}, \qquad \forall\ I \in \ell^1_{1,+}, \tag{1.13}$$

and

$$\frac{\pi}{10}\frac{\|I\|_2^2}{(1+P_{-1}^{\frac{1}{2}})} \leqslant V \leqslant \Big(4^{\frac{11}{2}}(1+P_{-1}^{\frac{1}{2}})^{\frac{1}{2}}P_{-1}^2 + 6\pi e^{\sqrt{P_{-1}}}\|I\|_2\Big)\|I\|_2, \qquad \forall\ I \in \ell^2. \tag{1.14}$$

Let X be a Banach space which contains $\tilde{\ell}^\infty$ as a dense subsets. We say that the function $V(I)$ *agrees with the norm* $\|I\|_X$ if V extends to a continuous function on X_+ (= the closure of $\widetilde{\ell}^\infty_+$ in X) and

$$F_1(\|I\|_X) \leqslant V(I) \leqslant F_2(\|I\|_X), \qquad \forall\ I \in X_+,$$

where F_1, F_2 are monotonous continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $F_j(0) = 0$ and $F_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2$. It is easy to see that there exists at most one Banach space X as above (i.e., if X' is another space, then $X = X'$ and the two norms are equivalent).

Estimates (1.14) imply that the function $V(I)$ agrees with the norm $\|I\|_2$. So ℓ^2 is the natural phase space for the non-linear part V of the Hamiltonian $K(I)$. In Section 3 we shoe that ℓ^2-sequences I correspond to potentials $q \in \mathscr{H}_{-1}$ and in general these potentials do not belong to $\mathscr{H}_{1/2}$ (see there Remark 2).

A proof of the theorem is based on a new identity (see Theorem 4.2), representing $V(I)$ in terms of the quasimomentum of the Hill operator with a potential q. It uses properties of the conformal mapping, associated with this quasimomentum, developed in [K1] - [K5].

Remarks. 1) (1.13) improves the known estimate $|H_2(q)| \leqslant 4^5P_3(1+P_3^{\frac{4}{3}})$ from [K3].

2) We claim that the function V is real analytic on ℓ^2_+. This will be proven elsewhere.

3) The complete Hamiltonian $K(I)$ is analytic on the space $\ell^1_{3/2,+}$. Our results show that the function $K(I) - dK(0)(I) = -V(I)$ is smoother and continuously extends to a larger space ℓ^2_+. A natural question is if the function

$$K(I) - dK(0)(I) - \tfrac{1}{2}d^2K(0)(I,I) = K(I) - P_3 + 3\|I\|_2^2$$

is even smoother and continuously extends to a larger space, etc. We do not know the answer.

4) By Theorem 1.1, $V(I)$ admits a quadratic upper bound in terms of P_1. The estimate (1.14) implies the exponential upper bound for V in terms of $\|I\|_2$. The bottle neck of our proof which yields the unpleasant exponential factor in (1.14) is the Bernstein inequality, used in Section 3 to prove Lemma 3.1. We conjective that, in fact, $V(I)$ is bounded by a polynomials of $\|I\|_2$.

Consider the restriction of the function $V(I)$ to $\widetilde{\ell}^N_+$ with any $N \geqslant 1$. It is known that the corresponding Hessian is non-degenerate:

$$\det\Big\{\frac{\partial^2 V(I)}{\partial I_i \partial I_j}\Big\}_{1\leqslant i,j\leqslant N} \neq 0, \qquad \forall\ I \in \ell^+_N. \tag{1.15}$$

This result was proven in [Kri] with serious omissions, fixed in [BiKu] (see also Appendix 3.6 in [Ku] and [HK]). Since V is analytic on $\widetilde{\ell}^N_+$, then (1.12) and (1.15) yield that the Hessian of $V|_{\widetilde{\ell}^N_+}$ is a positive $N\times N$ matrix. Thus V is convex on ℓ^+_N. Since $\widetilde{\ell}^\infty=\cup\widetilde{\ell}^N$ is dense in ℓ^2, where V is continuous, we get

Corollary 1.2. *The function $V(I)$ is convex on ℓ^2_+.*

Remark 5). By (1.12) and remark 2, the function V is strictly convex in some vicinity of the origin in ℓ^2_+ (note that ℓ^2_+ is the only phase-space where V is strictly convex). We conjecture that it is strictly convex everywhere in ℓ^2_+.

In difference with $V(I)$, the total Hamiltonian $K(I)$ is not continuous on ℓ^2_+ since its linear part $P_3(I)$ is there an unbounded linear functional. But $P_3(I)$ contributes to equations (1.6) the linear rotations

$$\frac{\partial b_n}{\partial t}=-(2\pi n)^3 b_{-n},\qquad \frac{\partial b_{-n}}{\partial t}=(2\pi n)^3 b_n,\qquad n\geqslant 1.$$

So the properties of (1.6) essentially are determined by the component $-V(I)$ of the Hamiltonian $K(I)$. We also note that since $P_3(I)$ is a bounded linear functional on the space $\ell^1_{3/2,+}\subset\ell^2$, then the complete Hamiltonian $K(I)=P_3-V$ is concave on $\ell^1_{3/2,+}$. The flow of the KdV equation in the action-angle variables (1.7) is

$$(I,\phi)\to\Big(I,\phi(t)=\phi+tK'(I)\Big),\quad t\in\mathbb{R},\qquad K'(I)=\frac{\partial K(I)}{\partial I}.$$

Since the function K is concave and analytic on $\ell^1_{3/2,+}$, then the flow-maps are twisting:

$$\langle\phi(t;I_{(2)},\phi_{(1)})-\phi(t;I_{(1)},\phi_{(1)}),I_{(2)}-I_{(1)}\rangle=t\langle K'(I_{(2)})-K'(I_{(1)}),I_{(2)}-I_{(1)}\rangle\leqslant 0\qquad\forall\, t\geqslant 0.$$

If the assertion of Remark 5 above holds true, then L.H.S. is $\leqslant -Ct\|I_{(2)}-I_{(1)}\|_2^2$, where the positive constant C depends on $I_{(2)},I_{(1)}$.

In the finite-dimensional case convexity (and strict convexity) of an integrable Hamiltonian significantly simplifies the study of long time behavior of actions of solutions for perturbed equations. Similar, we are certain that results of this work will help to study perturbations of the KdV equation (1.1), especially, those which are Hamiltonian. It is important that as a phase space our results suggest the Hilbert space ℓ^2, rather than a weighted ℓ^1-space.

2. Momentum, quasimomentum and KdV equation

2.1. Spectrum of the Hill operator. We consider the Hill operator T acting in $L^2(\mathbb{R})$ and given by

$$T=-\frac{d^2}{dx^2}+q_0+q(x),$$

where a 1-periodic potential q (with zero mean-value) belongs to the Sobolev space $\mathscr{H}_\alpha,\alpha\geqslant -1$ and q_0 is a constant (so the potential q_0+q may be a distribution). Below we recall the results from [K2] on the Hill operator with potentials $q\in\mathscr{H}_{-1}$. The spectrum of T is absolutely continuous and consists of intervals (*spectral bands*) $\mathfrak{S}_n$, separated by *gaps* γ_n and is given by

$$\mathfrak{S}_n=[\lambda^+_{n-1},\lambda^-_n],\qquad \gamma_n=(\lambda^-_n,\lambda^+_n),\qquad \text{where}\qquad \lambda^-_{n-1}\leqslant\lambda^-_n\leqslant\lambda^+_n,\quad n\geqslant 1.$$

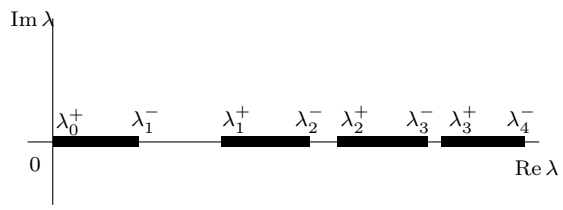

FIGURE 1. The spectral domain $\mathbb{C} \setminus \cup \mathfrak{S}_n$ and the bands $\mathfrak{S}_n = [\lambda_{n-1}^+, \lambda_n^-], n \geqslant 1$

We choose the constant $q_0 = q_0(q)$ in such a way that $\lambda_0^+ = 0$. Note that a gap-length $|\gamma_n| \geqslant 0$ may be zero. If the n-th gap degenerates, that is $\gamma_n = \emptyset$, then the corresponding spectral bands $\mathfrak{S}_n$ and $\mathfrak{S}_{n+1}$ merge. The sequence $0 = \lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \dots$ form the *energy spectrum* of T and is the spectrum of the equation $-y'' + (q_0 + q)y = \lambda y$ with the 2-periodic boundary conditions, i.e. $y(x+2) = y(x), x \in \mathbb{R}$. Here the equality means that $\lambda_n^- = \lambda_n^+$ is a double eigenvalue. The eigenfunctions, corresponding to $\lambda_n^\pm$, have period 1 when n is even, and they are antiperiodic, i.e., $y(x+1) = -y(x), \quad x \in \mathbb{R}$, when n is odd.

In order to study the actions $I_n, n \geqslant 1$, we introduce the *quasimomentum function.* We can not introduce the standard fundamental solutions for the operator T, since the perturbation q is too singular if $\alpha < 0$. Instead we use another representation of T. Define a function $\rho(x)$ by

$$\rho(x) = e^{\int_0^x q_*(t)dt}, \qquad \text{where} \qquad q_* \in \mathscr{H}_0 \qquad q_*' = q.$$

Consider the unitary transformation $U : L^2(\mathbb{R}, \rho^2\, dx) \to L^2(\mathbb{R}, dx)$ given by the multiplication by ρ. Then T is unitarily equivalent to

$$T_1 y = U^{-1} T U y = -\frac{1}{\rho^2}(\rho^2 y')' + (q_0 - q_*^2)y = -y'' - 2q_* y' + (q_0 - q_*^2)y,$$

acting in $L^2(\mathbb{R}, \rho^2\, dx)$. Note that the norm in this space is equivalent to the original L^2- norm. This representation clearly is more convenient, since q_* and q_*^2 are regular functions. It is convenient to write the the spectral parameter λ as

$$\lambda = z^2.$$

Let $\varphi(x,z)$ and $\vartheta(x,z)$ be solutions of the equation

$$-y'' - 2q_* y' + (q_0 - q_*^2)y = z^2 y, \quad z \in \mathbb{C}, \tag{2.1}$$

satisfying $\varphi'(0,z) = \vartheta(0,z) = 1$ and $\varphi(0,z) = \vartheta'(0,z) = 0$. The *Lyapunov function* is defined by

$$\Delta(z) = \frac{1}{2}(\varphi'(1,z) + \vartheta(1,z)). \tag{2.2}$$

This function is entire and even, i.e., $\Delta(-z) = \Delta(z)$ for all $z \in \mathbb{C}$. It is known that $\Delta(\sqrt{\lambda_n^\pm}) = (-1)^n, n \geqslant 0$ and the function $\Delta'(z)$ has a unique zero z_n in each gap $[\sqrt{\lambda_n^-}, \sqrt{\lambda_n^+}] \subset \mathbb{R}_+$ (see e.g., [Kr], [K7]).

2.2. Momentum and quasimomentum. Consider a strongly increasing odd sequence $u_n, n \in \mathbb{Z}$, of real numbers, $u_n = -u_{-n}$, such that $u_n \to \pm\infty$ as $n \to \pm\infty$, and a non-negative sequence $h = (h_n)_1^\infty \in \ell_+^\infty$. We define the following domains

$$\mathcal{K}(h) = \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} \overline{\Gamma}_n, \qquad \mathcal{K}_+(h) = \mathbb{C}_+ \cap \mathcal{K}(h),$$

where

$$\Gamma_0 = \emptyset, \quad \Gamma_n = (u_n - ih_n, u_n + ih_n) = -\Gamma_{-n}, \qquad \text{and} \qquad \mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}.$$

We call $\mathcal{K}_+(h)$ the "*comb*" and denote its points by $k = u + iv$. Then there exists a unique conformal mapping $z = z(k)$:

$$z : \mathcal{K}_+(h) \to \mathbb{C}_+,$$

normalized by the condition $z(0) = 0$ and the asymptotics:

$$z(iv) = iv + o(v) \quad \text{as} \quad v \to +\infty, \quad \text{where} \quad z = x + iy, \quad k = u + iv. \tag{2.3}$$

We call $z(k)$ "*the comb mapping*". Define the inverse mapping

$$k = z^{-1} : \mathbb{C}_+ \to \mathcal{K}_+(h), \qquad k(z) = u(z) + iv(z). \tag{2.4}$$

This function is continuous in $\mathbb{C}_+$ up to the boundary, i.e., on the closure $\overline{\mathbb{C}}_+$. It is convenient to introduce "gaps" g_n, "bands" σ_n and the "spectrum" σ of the comb mapping by:

$$g_n = (z_n^-, z_n^+) = (z(u_n - 0), z(u_n + 0)), \qquad \sigma_n = [z_{n-1}^+, z_n^-], \qquad \sigma = \cup \sigma_n, \quad g_0 = \emptyset, \quad z_0^\pm = 0.$$

Note that the identities $\lambda_n^\pm = z_n^{\pm 2}$ yields

$$|\gamma_n| = z_n^{+2} - z_n^{-2} = |g_n|(z_n^+ + z_n^-), \quad \forall\, n \geqslant 1. \tag{2.5}$$

Define the momentum domain

$$\mathcal{Z} = \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} \overline{g}_n.$$

The function $k(z)$ may be continued from $\mathbb{C}_+$ to the domain $\mathcal{Z}$ by the symmetry the formula $k(z) = \overline{k}(\overline{z})$, $\operatorname{Im} z < 0$. Thus we obtain a conformal mapping $k : \mathcal{Z} \to \mathcal{K}(h)$, called *the quasimomentum mapping* (or shortly the quasimomentum), which generalizes the classical quasimomentum (see e.g. [RS]). A point $z \in \mathcal{Z}$ is called *momentum* and a point $k \in \mathcal{K}(h)$ is called *quasimomentum*. It is odd, i.e., $k(-z) = -k(z)$, since the domains $\mathcal{K}(h)$ and $\mathcal{Z}$ both are invariant under the inversion $z \to -z$.

If the spectrum of the comb mapping $k(z)$ has only finite number of open gaps, then $k(z)$ is called a *finite-gap quasimomentum*. Different properties of the finite-gap quasimomentum (and of more general conformal mappings) were studied by Hilbert one hundred years ago, see in [J].

The abstract quasimomentum, which we have just defined, is related to the spectral theory of the Hill operator T by the following construction invented in [MO]. Namely, let $\{z_n^\pm, n \in \mathbb{Z}\}$ be an odd sequence as above. For $n \geqslant 0$ denote $\lambda_n^\pm = (z_n^\pm)^2$. Then $\{\lambda_n^\pm, n \geqslant 0\}$ is the energy spectrum of the Hill operator T with a potential $q_0 + q$, where $q \in \mathscr{H}_\alpha, \alpha \geqslant 0$, if and only if the corresponding comb domain $\mathcal{K}(h)$ is such that $u_n = \pi n, n \in \mathbb{Z}$ and $h = (h_n)_1^\infty \in \ell^2_{\alpha+1}$. Moreover, in this case $\cos k(z) = \Delta(z)$ is the Lyapunov function for T.

In [K2] the construction was generalized for potentials from $\mathscr{H}_{-1}$, see below Theorem 2.1.

Despite the objects, treated by Theorem 1.1, are defined in terms of Hill operators with periodic potentials, for the proofs in Sections 3-4 below we need the quasimomentum mapping $k(z), k = u + iv, z = x + iy$, corresponding to general odd sequences $\{u_n\}$. Now we summarize their basic properties, refering for a proof to [KK2] [K1], [K2], [K4], [L1]-[L3], [MO], [M].

1) $v(z) \geqslant \operatorname{Im} z > 0$ *and* $v(z) = -v(\overline{z})$ *for all* $z \in \mathbb{C}_+$ *and*

$$k(-z) = -k(z), \qquad k(z) = \overline{k}(\overline{z}), \quad \text{all } z \in \mathcal{Z}, \tag{2.6}$$

2) $v(z) = 0$ *for all* $z \in \sigma_n = [z_{n-1}^+, z_n^-], n \in \mathbb{Z}$.

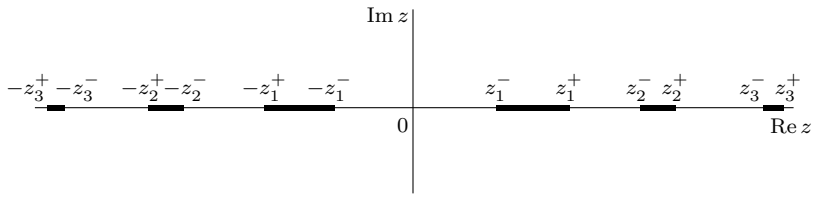

FIGURE 2. z-domain $\mathcal{Z} = \mathbb{C} \setminus \cup g_n$, where $z = \sqrt{\lambda}$ and momentum gaps $g_n = (z_n^-, z_n^+)$

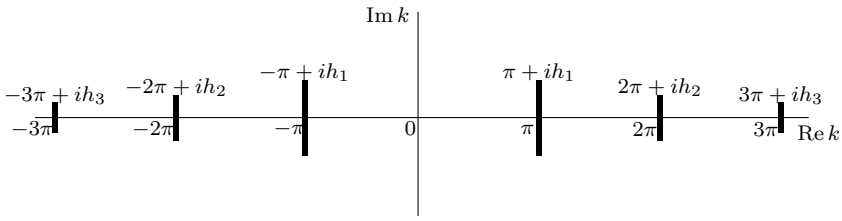

FIGURE 3. k-plane and cuts $\Gamma_n = (\pi n - ih_n, \pi n + ih_n), n \in \mathbb{Z}$

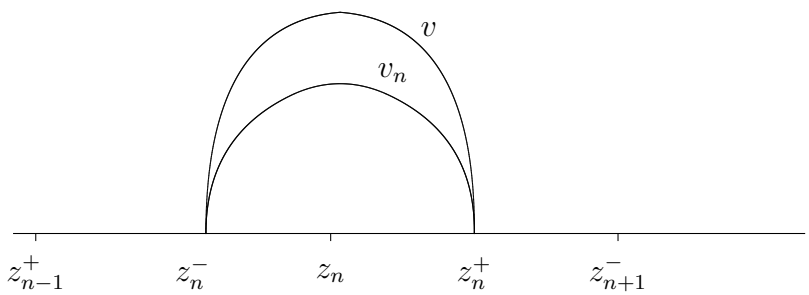

FIGURE 4. The graph of $v(z+i0)$, $z \in g_n \cup \sigma_n \cup \sigma_{n+1}$ and $|h_n| = v(z_n + i0) > 0$

3) If some $g_n \neq \emptyset, n \in \mathbb{Z}$, then

$$h_n \geqslant v(z+i0) = -v(z-i0) > 0, \qquad v''(z+i0) < 0 \qquad \forall\ z \in g_n, \tag{2.7}$$

see Fig. 4. The function $v(z+i0)|_{g_n} > 0$ attains its maximum at a point $z_n \in g_n$, where

$$h_n = v(z_n + i0), \quad v'(z_n) = 0. \tag{2.8}$$

Moreover,

$$v = 0 \qquad \text{on} \quad \mathbb{R} \setminus \cup_{n\in\mathbb{Z}} g_n, \tag{2.9}$$

$$v(z+i0) > v_n(z) = |(z - z_n^-)(z - z_n^+)|^{\frac{1}{2}} > 0, \qquad \forall\ z \in g_n, \tag{2.10}$$

$$|g_n| \leqslant 2h_n, \qquad |\sigma_n| \leqslant u_n - u_{n-1}, \qquad \forall\ n \in \mathbb{Z}. \tag{2.11}$$

4) $u'(z) > 0$ on each σ_n, and

$$u(z) = \pi n, \qquad \forall\ z \in g_n \neq \emptyset, n \in \mathbb{Z}. \tag{2.12}$$

5) The function $k(z)$ maps a horizontal cut (a gap) $\overline{g}_n$ onto the vertical cut $\overline{\Gamma}_n$ and maps a spectral band σ_n onto the segment $[\pi(n-1), \pi n]$, for all $n \in \mathbb{Z}$.
6) The following asymptotics hold true:

$$z_n^{\pm} = \pi n + o(1) \qquad \text{as} \quad n \to \infty. \tag{2.13}$$

7) If $h \in \ell^2$ and $\inf_{n\geqslant 1}(u_{n+1} - u_n) > 0$, then $v(z+i0), z \in \mathbb{R}$ belongs to $L^1(\mathbb{R})$ and the following identity holds true:

$$k(z) = z + \frac{1}{\pi}\int_{\bigcup_{n\in\mathbb{Z}} g_n} \frac{v(t)}{t-z}dt, \qquad \forall z \in \mathcal{Z}. \tag{2.14}$$

For additional properties of the comb mapping $z(k)$, see [K1]-[K6], [L1]-[L3].

2.3. Quasimomentum and the KdV Hamiltonian. Recall that we choose the constant $q_0 \geqslant 0$ in such a way that $\lambda_0^+ = 0$. If $q \in \mathscr{H}_0(\mathbb{T})$, then the quasimomentum $k(\cdot)$ has asymptotics

$$k(z) = z - \frac{Q_0}{z} - \frac{Q_2 + o(1)}{z^3} \qquad as \quad \operatorname{Im} z \to \infty, \tag{2.15}$$

see [K2]. If $q, q' \in \mathscr{H}_0$, then the asymptotics (2.15) may be improved:

$$k(z) = z - \frac{Q_0}{z} - \frac{Q_2}{z^3} - \frac{Q_4 + o(1)}{z^5} \quad as \quad z \to +i\infty, \tag{2.16}$$

and

$$k^2(z) = \lambda - S_{-1} - \frac{S_0}{\lambda} - \frac{S_1 + o(1)}{\lambda^2} \qquad \text{as} \quad \lambda = z^2, \quad z \to +i\infty, \tag{2.17}$$

where

$$Q_j = \frac{1}{\pi}\int_{\mathbb{R}} z^j v(z+i0)\,dz \geqslant 0, \quad j \geqslant 0, \qquad S_j = \frac{4}{\pi}\int_0^{\infty} z^{2j+1}u(z)v(z+i0)\,dz, \quad j \geqslant -1. \tag{2.18}$$

Note that $Q_{2j+1} = 0$ for all $j \geqslant 0$ by the symmetry. The involved quantities Q_j and S_j are defined by converging integrals (see [KK2], [K5]), and satisfy the following identities

$$q_0(q) = S_{-1} = 2Q_0 \qquad \text{if} \qquad q \in \mathscr{H}_{-1}, \tag{2.19}$$

$$H_1(q) = \int_0^1 q^2(x)\,dx = 2P_1 = 4S_0 = 8Q_2 - 4Q_0^2 \qquad \text{if} \qquad q \in \mathscr{H}_0, \tag{2.20}$$

$$H_2(q) = 8(S_1 - S_{-1}S_0), \qquad S_1 + 2Q_0Q_2 = 2Q_4 \qquad \text{if} \qquad q \in \mathscr{H}_1, \tag{2.21}$$

$$8Q_2 = \|q\|^2 + q_0^2, \qquad 2^4 Q_4 = H_2(q + q_0). \tag{2.22}$$

See [K2], [K5].

2.4. The KdV actions. The components I_n of the action vector $I = (I_n)_1^{\infty}$ (see (1.4)) may be calculated with the help of a general formula due to Arnold, which in the KdV-case takes the form

$$I_n = \frac{(-1)^{n+1}2}{\pi}\int_{g_n} \frac{z^2\Delta'(z)\,dz}{|\Delta^2(z) - 1|^{\frac{1}{2}}} \geqslant 0, \qquad n \geqslant 1, \tag{2.23}$$

see [FM]. These integrals may be re-written, using the quasimomentum. Indeed, since $\sin k(z) = \sqrt{1-\Delta^2(z)}$, then

$$I_n = -\frac{1}{\pi i}\int_{c_n} z^2 \frac{\Delta'(z)}{\sin k(z)}\, dz,$$

where c_n is a contour around g_n. It is convenient to introduce contours χ_n around Γ_n by

$$\chi_n = \left\{ k \in \mathcal{K}(h) : \text{dist } \ (k, \Gamma_n) = \frac{\pi}{4} \right\} \subset \mathcal{K}(h), \qquad n \geqslant 1, \tag{2.24}$$

and define the contours c_n as

$$c_n = z(\chi_n) \subset \mathcal{Z}, \qquad \forall\ n \geqslant 1.$$

The differentiation of $\Delta(z) = \cos k(z)$ gives $k'(z) = -\Delta'(z)/\sin k(z)$. This yields

$$I_n = \frac{1}{i\pi}\int_{c_n} z^2 k'(z)\, dz = -\frac{2}{i\pi}\int_{c_n} zk(z)\, dz = \frac{4}{\pi}\int_{g_n} zv(z+i0)\, dz \geqslant 0, \tag{2.25}$$

since on g_n the function $k = \pi n + iv$ and v satisfies (2.7). This representation for I_n is convenient and is crucial for our work. In particular, below in Lemma 3.1 we derive from (2.25) the following two-sided estimates:

$$\frac{2}{3\pi} h_n|\gamma_n| < I_n \leqslant \frac{2h_n|\gamma_n|}{\pi}, \qquad \text{if} \quad |\gamma_n| > 0.$$

Using (2.25) jointly with (2.12) and (2.9), we easily see that

$$P_3 = \sum_{n\geqslant 1} (2\pi n)^3 I_n = \frac{32}{\pi}\int_0^\infty zu^3(z)v(z+i0)\, dz. \tag{2.26}$$

Recall that $P_3(I)$ is the linear in I part of the Hamiltonian H_2, see (1.10).

2.5. Marchenko-Ostrovski construction for potentials $q \in \mathscr{H}_{-1}$. The Marchenko-Ostrovski construction, described in Section 2.1, defines the mapping $q \to h$, acting from $\mathscr{H}_{j-1}$ into $\ell^2_j, j = 0, 1, ...$ The results below are proven in [MO] for $j \geqslant 1$ and in [K2] for $j = 0$.

Theorem 2.1. *The mapping $q \to h$ acting from $\mathscr{H}_{j-1}$ into $\ell^2_j, j = 0, 1$ is a surjection. It satisfies the following estimates*

$$\|q\|_{-1} \leqslant 2\|h\|_2(1 + 4\|h\|_2), \qquad \|h\|_2 \leqslant 3\|q\|_{-1}(1 + 2\|q\|_{-1})^2, \quad \forall\ q \in \mathscr{H}_{-1}, \tag{2.27}$$

where $\|q\|_{-1} = \|q\|_{\mathscr{H}_{-1}}$. For each $h \in \ell^2$ there exists a function $q \in \mathscr{H}_{-1}$, and a unique conformal mapping $k(\cdot, h) : \mathcal{Z} \to K(h)$ defined in (2.4). Moreover,

$$\cos k(z, h) = \Delta(z, h), \qquad z \in \mathcal{Z}, \tag{2.28}$$

where $\Delta(z, h)$ is the Lyapunov function for q, and $k(z)$ satisfy

$$k(z, h) = z - \frac{Q_0 + o(1)}{z} \qquad \text{as} \qquad z \to i\infty, \tag{2.29}$$

$$k(z_n^{\pm}, h) = \pi n \pm i0, \qquad k(z_n \pm i0, h) = \pi n \pm ih_n, \quad n \geqslant 1. \tag{2.30}$$

In particular, the real numbers $z_n^{\pm 2}$, satisfying (2.30), form the energy spectrum of the operator T. Furthermore, if a sequence $h^\nu, \nu \geqslant 1$ converges strongly in ℓ^2 to h as $\nu \to \infty$, then $\Delta(z, h^\nu) \to \Delta(z, h)$ uniformly on bounded subsets of $\mathbb{C}$.

In order to prove our main result, Theorem 1.1, we use Theorem 2.1 to reformulate it as questions from the conformal mapping theory in terms of quasimomentum of the Hill operator. To proceed we need auxillary results from previous work of the first author:

Lemma 2.2. *Let $h \in \ell^2$ and let each $u_n = \pi n, n \geqslant 1$. Then following estimates hold true:*

$$\frac{\pi}{4} Q_0 \leqslant \|h\|_2^2 \leqslant \frac{\pi^2}{2}\left(1 + \frac{\sqrt{2}}{\pi} Q_0^{\frac{1}{2}}\right) Q_0, \tag{2.31}$$

$$\|\rho\|_2^2 \leqslant (16)^2 Q_0, \tag{2.32}$$

$$\frac{\|h\|_\infty^2}{2} \leqslant Q_0 \leqslant \frac{2}{\pi}\int_0^\infty \frac{zv(z)\,dz}{u(z)} = \sum_{j\geqslant 1} \frac{I_j}{2\pi j} = P_{-1}, \tag{2.33}$$

$$\|h\|_\infty \leqslant \frac{4}{\pi}\sum_{j=1}^\infty \frac{|\gamma_j|}{2\pi j}. \tag{2.34}$$

Proof. Estimates (2.31), (2.32) were proved in Theorem 2.1 from [K1]. The first estimate in (2.33) $\|h\|_\infty^2 \leqslant 2Q_0$ was established in [K6]. The second estimate in (2.33) $Q_0 \leqslant \frac{2}{\pi}\int_0^\infty zv(z)\,dz/u(z)$ was proved in [K5] (see p. 398). The identities (2.25), (2.9) and (2.12) imply

$$\frac{2}{\pi}\int_0^\infty \frac{zv(z)\,dz}{u(z)} = \sum_{j\geqslant 1} \frac{I_j}{2\pi j},$$

and we get (2.33). Finally, using (2.33), (2.8) and (2.12) we obtain

$$\|h\|_\infty^2 \leqslant \frac{4}{\pi}\int_0^\infty \frac{zv(z)\,dz}{u(z)} \leqslant \frac{4}{\pi}\sum_{j\geqslant 1} h_j \int_{g_j} \frac{z\,dz}{u(z)} = \frac{2}{\pi}\sum_{j\geqslant 1} h_j \frac{|\gamma_j|}{\pi j} \leqslant \frac{2\|h\|_\infty}{\pi}\sum_{j\geqslant 1}\frac{|\gamma_j|}{\pi j}, \tag{2.35}$$

which gives (2.34). ■

3. Local estimates

In this section we derive estimates for h_n, I_n and $|\gamma_n|$ with a fixed $n \geqslant 1$. We use the following constants

$$C_- = e^{\sqrt{P_{-1}}}, \qquad C_I = 1 + \sqrt{P_{-1}} \qquad C_0 = \operatorname{ch}\|h\|_\infty \leqslant e^{\sqrt{2P_{-1}}}, \tag{3.1}$$

where the inequality follows from lemma below.

Lemma 3.1. *Let $h \in \ell^2$ and let each $u_n = \pi n, n \geqslant 1$. Then for each $n \geqslant 1$ the following estimates hold true:*

$$\frac{2}{3\pi} h_n|\gamma_n| < \frac{2}{3\pi} h_n|g_n|(z_n + z_n^- + z_n^+) < I_n \leqslant \frac{2h_n|\gamma_n|}{\pi}, \qquad \text{if} \quad |\gamma_n| > 0, \tag{3.2}$$

$$z_n^\pm \leqslant \pi n + \sum_{j=1}^n |g_j|, \tag{3.3}$$

$$\pi n \leqslant 2z_n^\pm + \frac{\|\rho\|_2^2}{\pi}, \qquad \rho = (\rho_n)_1^\infty, \quad \rho_n = \pi - |\sigma_n|, \tag{3.4}$$

$$2n \leqslant C_0 z_n^\pm, \qquad h_n \leqslant \frac{\sqrt{C_0}}{2}|g_n|, \tag{3.5}$$

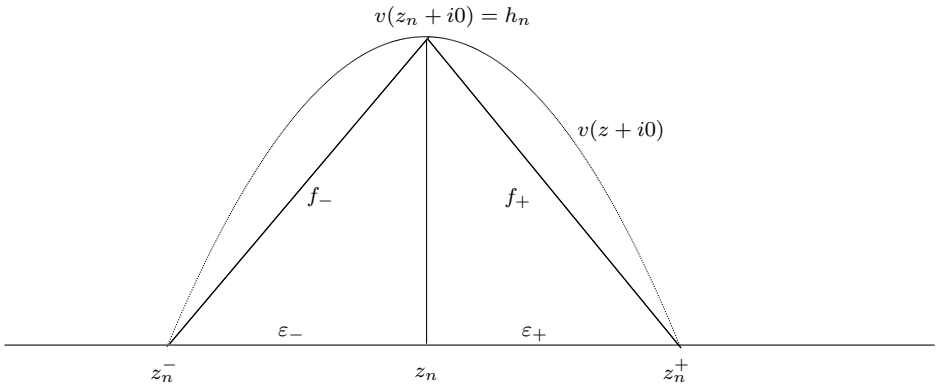

FIGURE 5. The graphs of $v(z+i0)$ and $f_\pm$

$$2\pi n h_n^2 \leqslant \sqrt{C_0}\frac{3\pi}{2} I_n + 2\frac{\|\rho\|^2}{\pi} h_n^2, \tag{3.6}$$

$$\frac{1}{C_I}\frac{|\gamma_n|}{4\pi n} \leqslant h_n \leqslant \frac{\pi C_0^{\frac{3}{2}}|\gamma_n|}{8\pi n}. \tag{3.7}$$

$$\frac{1}{3\pi C_I}\frac{|\gamma_n|^2}{(2\pi n)} \leqslant I_n \leqslant \frac{C_0^{\frac{3}{2}}}{2}\frac{|\gamma_n|^2}{(2\pi n)}, \tag{3.8}$$

$$\frac{8C_0^{-\frac{3}{2}}}{3\pi^2}(2\pi n)|h_n|^2 \leqslant I_n \leqslant 8nC_I h_n^2. \tag{3.9}$$

Proof. We show (3.2). Using (2.7), (2.8) and standard convexity arguments (see Fig. 5) we have

$$v(z_n^- + t + i0) \geqslant f_-(t) = t\frac{h_n}{\varepsilon_-}, \qquad t \in (0, \varepsilon_-), \quad \varepsilon_- = z_n - z_n^- > 0. \tag{3.10}$$

This yields

$$I_n^- = \frac{4}{\pi}\int_{z_n^-}^{z_n} zv(z)\,dz \geqslant \frac{4}{\pi}\int_0^{\varepsilon_-}(z_n^- + t)f_-(t)dt = \frac{4}{\pi}\frac{h_n}{\varepsilon_-}\left(z_n^-\frac{\varepsilon_-^2}{2} + \frac{\varepsilon_-^3}{3}\right) = \frac{2}{\pi}h_n\varepsilon_-\left(z_n - \frac{\varepsilon_-}{3}\right).$$

Let $f_+(t) = (\varepsilon_+ - t)\frac{h_n}{\varepsilon_+}, t \in (0, \varepsilon_+), \varepsilon_+ = z_n^+ - z_n > 0$. A similar argument gives

$$I_n^+ = \frac{4}{\pi}\int_{z_n}^{z_n^+} zv(z)\,dz \geqslant \frac{4}{\pi}\int_0^{\varepsilon_+}(z_n^+ + t - \varepsilon_+)f_+(t)dt = \frac{4}{\pi}\frac{h_n}{\varepsilon_+}\left(z_n^+\frac{\varepsilon_+^2}{2} - \frac{\varepsilon_+^3}{3}\right) = \frac{2}{\pi}h_n\varepsilon_+\left(z_n + \frac{\varepsilon_+}{3}\right).$$

Denoting $z_n^0 = \frac{1}{2}(z_n^+ + z_n^-)$, we obtain

$$I_n = I_n^- + I_n^+ > \frac{2}{\pi}h_n\left(\varepsilon_-(z_n - \frac{\varepsilon_-}{3}) + \varepsilon_+(z_n + \frac{\varepsilon_+}{3})\right) = \frac{2}{\pi}h_n\left(z_n|g_n| + \frac{\varepsilon_+^2 - \varepsilon_-^2}{3}\right)$$

$$= \frac{2}{\pi}h_n|g_n|\left(z_n + 2\frac{z_n^0 - z_n}{3}\right) = \frac{2}{3\pi}h_n|g_n|(z_n + 2z_n^0) \geqslant \frac{2}{3\pi}h_n|g_n|2z_n^0 = \frac{2}{3\pi}h_n|\gamma_n|.$$

This implies the first two estimates in (3.2). Using (2.7) for all $z \in g_n$, we get $I_n < \frac{4}{\pi}h_n\int_{g_n} z\,dz = \frac{2}{\pi}h_n|\gamma_n|$, which gives us the last estimate in (3.2).

We show (3.3). It is clear that

$$z_n^+ = \pi n + \sum_1^n\Big(|g_j| - (\pi - |\sigma_j|)\Big), \qquad \text{all } n \geqslant 1. \tag{3.11}$$

Since by (2.11), $\rho_j = \pi - |\sigma_j| \geqslant 0$ and $|g_n| = z_n^+ - z_n^-$, we get (3.3).

We show (3.4). Using identities (3.11) we obtain

$$\pi n \leqslant z_n^\pm + \sum_1^n \rho_j \leqslant z_n^\pm + n^{\frac{1}{2}}\|\rho\|_2 \leqslant z_n^\pm + \frac{\pi n}{2} + \frac{\|\rho\|_2^2}{2\pi},$$

which yields (3.4).

In order to prove (3.5) we use an argument from [MO]. This is a weak point in our proof, which gives the exponential factor in (3.5) and later in (1.14). The Taylor formula implies

$$2 = |\Delta(z_n^-) - \Delta(z_{n-1}^+)| \leqslant |\Delta'(\widetilde{z}_n)||\sigma_n|, \tag{3.12}$$

for some $\widetilde{z}_n \in \sigma_n = [z_{n-1}^+, z_n^-]$ and all $n \geqslant 1$. Using the Bernstein inequality for the bounded exponential type functions (see e.g., [S]) we obtain

$$\sup_{z\in\mathbb{R}}|\Delta'(z)| \leqslant \sup_{z\in\mathbb{R}}|\Delta(z)| = C_0 = \operatorname{ch}\|h\|_\infty \leqslant e^{\|h\|_\infty}. \tag{3.13}$$

Combining (3.12) and (3.13) we get $2n \leqslant C_0 z_n^\pm$ for all $n \geqslant 1$, which yields the first estimate in (3.4).

Let n be even and let $|z_n^- - z_n| \leqslant |g_n|/2$ (for other cases the proof is similar). The identity $\operatorname{ch} h_n = \Delta(z_n)$ (which follows from (2.30)) and the Taylor formula imply

$$\frac{h_n^2}{2} \leqslant \operatorname{ch} h_n - 1 = \Delta(z_n) - 1 = \frac{1}{2}\Delta''(\widetilde{z}_n^-)(z_n^- - z_n)^2 \tag{3.14}$$

for some $\widetilde{z}_n^- \in (z_n^-, z_n)$. Using again the Bernstein inequality, we obtain

$$\sup_{z\in\mathbb{R}}|\Delta''(z)| \leqslant \sup_{z\in\mathbb{R}}|\Delta(z)| = C_0. \tag{3.15}$$

Then combining (3.14) and (3.15) we get $h_n^2 \leqslant \frac{C_0}{4}|g_n|^2$, which gives the second estimate in (3.5).

We show (3.6). Using (3.4), (3.5), (3.2), (2.5) we obtain

$$\pi n h_n^2 \leqslant (z_n^- + z_n^+)h_n^2 + \frac{\|\rho\|_2^2}{\pi}h_n^2 \leqslant (z_n^- + z_n^+)\frac{\sqrt{C_0}}{2}|g_n|h_n + \frac{\|\rho\|_2^2}{\pi}h_n^2$$

$$= \frac{\sqrt{C_0}}{2}|\gamma_n|h_n + \frac{\|\rho\|_2^2}{\pi}h_n^2 \leqslant \frac{\sqrt{C_0}}{2}\frac{3\pi}{2}I_n + \frac{\|\rho\|_2^2}{\pi}h_n^2,$$

which yields (3.6).

We show (3.7). Using (2.5), (3.3), (2.33) and (2.11) we obtain

$$\frac{|\gamma_n|}{4\pi n} = \frac{(z_n^- + z_n^+)|g_n|}{4\pi n} \leqslant \left(1 + \frac{\|g\|_\infty}{\pi}\right)h_n \leqslant \left(1 + \frac{\sqrt{8P_{-1}}}{\pi}\right)h_n \leqslant C_I h_n.$$

Recalling that $C_0 = \operatorname{ch}\|h\|_\infty$ and using (3.5) and (2.5), we obtain

$$h_n \leqslant \frac{C_0^{\frac{1}{2}}|g_n|}{2} = \frac{C_0^{\frac{1}{2}}|\gamma_n|}{2(z_n^- + z_n^+)} \leqslant \frac{\pi C_0^{\frac{3}{2}}|\gamma_n|}{8\pi n},$$

and (3.7) is proven.

Estimates (3.7) and (3.2) imply the first estimate in (3.8):

$$\frac{|\gamma_n|^2}{(2\pi n)} \leqslant 2C_I h_n|\gamma_n| \leqslant 3\pi C_I I_n.$$

Combining the last estimate in (3.7) and (3.2) we obtain the second estimate in (3.8).

We show (3.9). Using (3.2) and (3.7) we obtain

$$I_n \leqslant \frac{2h_n|\gamma_n|}{\pi} \leqslant 8nC_I h_n^2,$$

which yields the second estimate in (3.9). Using (3.7) and (3.2), we obtain

$$2\pi n h_n^2 \leqslant \frac{\pi}{4}C_0^{\frac{3}{2}}h_n|\gamma_n| \leqslant \frac{3\pi^2}{8}C_0^{\frac{3}{2}}I_n,$$

and get the first. ■

For any $h \in \ell^\infty$ we define integrals V_n as

$$V_n = \frac{8}{\pi}\int_{g_n} zv^3(z)\,dz \geqslant 0, \qquad n \geqslant 1. \tag{3.16}$$

These quantities are important for our argument since, as we show below, $V = \sum_{n\geqslant 1}(4\pi n)V_n$ for $I \in \ell^2_+$.

Lemma 3.2. *Let $h \in \ell^\infty$ and let each $u_n = \pi n, n \geqslant 1$. Then for $n \geqslant 1$ the following relations hold true:*

$$\frac{1}{5}h_n^2 I_n \leqslant \frac{2}{5\pi}h_n^3|\gamma_n| \leqslant \frac{2}{5\pi}h_n^3|g_n|(3z_n + 2z_n^0) \leqslant V_n \leqslant 2h_n^2 I_n, \tag{3.17}$$

$$\frac{4}{\pi i}\int_{c_n} zk^4(z)\,dz = (4\pi n)V_n - (2\pi n)^3 I_n. \tag{3.18}$$

Proof. We show (3.17). Let $g_n \neq \emptyset$. Using (3.10) we get $v(z_n^- + t + i0) \geqslant f_-(t) = th_n/\varepsilon_-$, $t \in (0, \varepsilon_-)$, where $\varepsilon_- = z_n - z_n^- > 0$. Therefore

$$V_n^- := \frac{8}{\pi}\int_{z_n^-}^{z_n} zv^3(z)\,dz \geqslant \frac{8}{\pi}\int_0^{\varepsilon_-}(z_n^- + t)f_-^3(t)dt = \frac{8}{\pi}\frac{h_n^3}{\varepsilon_-^3}\left(z_n^-\frac{\varepsilon_-^4}{4} + \frac{\varepsilon_-^5}{5}\right)$$

$$= \frac{2}{\pi}h_n^3\varepsilon_-\left(z_n^- + \frac{4\varepsilon_-}{5}\right) = \frac{2}{\pi}h_n^3\varepsilon_-\left(z_n - \frac{\varepsilon_-}{5}\right).$$

Similar argument yields $v(z_n+t+i0) \geqslant f_+(t) = (\varepsilon_+ - t)h_n/\varepsilon_+$, $t \in (0,\varepsilon_+)$, where $\varepsilon_+ = z_n^+ - z_n > 0$. Thus

$$V_n^+ = \frac{8}{\pi}\int_{z_n}^{z_n^+} zv^3(z)\,dz \geqslant \frac{8}{\pi}\int_0^{\varepsilon_+}(z_n^+ + t - \varepsilon_+)f_+^3(t)dt = \frac{8}{\pi}\frac{h_n^3}{\varepsilon_+^3}\left(z_n^+\frac{\varepsilon_+^4}{4} - \frac{\varepsilon_+^5}{5}\right)$$
$$= \frac{2}{\pi}h_n^3\varepsilon_+\left(z_n^+ - \frac{4\varepsilon_+}{5}\right) = \frac{2}{\pi}h_n^3\varepsilon_+\left(z_n + \frac{\varepsilon_+}{5}\right).$$

Summing these relations we obtain

$$V_n = V_n^- + V_n^+ > \frac{2}{\pi}h_n^3\left(\varepsilon_-(z_n - \frac{\varepsilon_-}{5}) + \varepsilon_+(z_n + \frac{\varepsilon_+}{5})\right) = \frac{2}{\pi}h_n^3\left(z_n|g_n| + \frac{\varepsilon_+^2 - \varepsilon_-^2}{5}\right)$$
$$= \frac{2}{\pi}h_n^3|g_n|\left(z_n + 2\frac{z_n^0 - z_n}{5}\right) = \frac{2}{5\pi}h_n^3|g_n|(3z_n + 2z_n^0).$$

Using (2.7), we get $V_n \leqslant (8h_n^2/\pi)\int_{g_n} zv(z,h)\,dz = 2h_n^2 I_n$, which yields the last two estimates in (3.17). The second follows from (2.5) and the first follows from the last estimate in (3.2).

Using (2.7), (2.12) and (2.25) we obtain

$$\frac{4}{\pi i}\int_{c_n} zk^4(z)\,dz = \frac{4}{\pi i}\int_{c_n} z(u^2 - v^2 + 2iuv)^2\,dz = -\frac{8}{\pi i}\int_{g_n} 2z(u^2 - v^2)(2iuv)\,dz$$
$$= \frac{32}{\pi}\int_{g_n} z(v^2 - u^2)uv\,dz = (4\pi n)V_n - (2\pi n)^3 I_n.$$

This proves (3.18). ■

Remark. 1) In particular, (3.8) yields that $I \in \ell^2_+$ iff $\gamma \in \ell^4_{-1}$.

2) Due to [K2], for any $N_0 > 1$ and $\varepsilon \in (0, \frac{1}{4})$ there exists a potential $q \in \mathscr{H}_{-1}$ such that $|\gamma_n| = n^\varepsilon$, for all $n > N_0$. Then $\gamma = (|\gamma_n|)_1^\infty \in \ell^4_{-1}$ and (3.8) gives that $I \in \ell^2_+$. It is clear that $q \notin L^2(\mathbb{T})$, since the gap length $|\gamma_n|$ is increasing.

Note that if $I \in \ell^2_+$, then (3.7) gives $\sum_{n\geqslant 1}|\gamma_n|^4/n^2 < \infty$, which yields $(|\gamma_n|)_1^\infty \in \ell^2_{-\frac{1}{2}}$. This and the standard relationship between the gap lengths and the Fourier coefficients of potential imply $q \in \mathscr{H}_{-\frac{1}{2}}$ (e.g., see [K2], where an analogy of this relation is established for $(|\gamma_n|)_1^\infty \in \ell^2_{-1}$ and for $(|\gamma_n|)_1^\infty \in \ell^2$).

3) Relations (3.8) show that asymptotically the actions I_n are equivalent to the weighted squared gap-length $|\gamma_n|^2/2\pi n$. It is known that the gap-length $|\gamma_n|$ is asymptotically equivalent to the module of the Fourier coefficients $|\hat{q}_n|$ of the potential q (see [MO], [M] for the equivalence and see [K4] for the corresponding estimates). The asymptotical equivalences $nI_n \sim |\gamma_n|^2 \sim |\hat{q}_n|^2$ and the corresponding estimates are important for the spectral theory of the Hill operator T and the theory of KdV.

4. Proof of Theorem 1.1

We remind that $V(I) = P_3 - H_2$ is the non-linear part of the KdV Hamiltonian, written as a function of actions, see (1.8) and (1.10). Identities (2.21) and (2.25) represent H_2 and I_j as integrals in terms of the quasimomentum $k = u + iv$. They allow to write V in a similar form. We start with an integral representation for V for the case of smooth potentials.

Lemma 4.1. *Let $q \in \mathscr{H}_1$. Then V is finite, nonnegative and satisfies*

$$V = \frac{32}{\pi} \int_0^\infty z u(z) v^3(z)\, dz. \tag{4.1}$$

Proof. Since $q \in \mathscr{H}_1$, then H_2 is finite and $I \in \ell^1_{3/2,+}$. So P_3 is finite, as well as V. To prove (4.1) we start with a finite-gap aproximation for the momentum spectrum $\sigma_M = \mathbb{R} \setminus \cup g_n$ of the potential $q_0 + q$, obtained by closing the gaps g_m with large m. Namely, we fix $r > 0$ and consider a new momentum spectrum $\sigma_M^r = \sigma_M \cup (-\infty, -r) \cup (r, \infty)$, where the new gaps g_n^r are given by

$$g_n^r = \begin{cases} g_n & if \quad g_n \subset (-r, r) \\ \emptyset & if \quad g_n \not\subset (-r, r) \end{cases}.$$

The variables corresponding to σ_M^r will be indicated the upper index r. Due to the general construction, presented in Section 2.2, for the finite-gap momentum spectrum σ_M^r there exists a unique conformal mapping

$$k^r : \mathbb{C} \setminus g^r \to \mathbb{C} \setminus \Gamma_n^r, \qquad \Gamma_n^r = (u_n^r + ih_n^r, u_n^r - ih_n^r), \quad h_n^r \geqslant 0,$$

satisfying the asymptotics $k^r(z) = z - O(1)z^{-1}$ as $|z| \to \infty$. By (2.14), each function $k^r(z) - z$ is analytic at ∞. The sequence of real numbers $u_n^r, n \in \mathbb{Z}$ is odd, strongly increasing and $u_n^r \to \pm\infty$ as $n \to \pm\infty$. In general, k^r is not a quasimomentum for some periodic potential, since not necessarily $u_n = \pi n$ for all n.

For each r we introduce Q_m^r, S_m^r, P_3^r and V^r by relations (2.18), (2.26) and (4.1) respectively, where $k = k^r, u = u^r$ and $v = v^r$. Since $v^r(x) = 0$ for large real x and $v^r(x + i0), u^r(x) \geqslant 0$ for real $x \geqslant 0$, then all these quantities are finite and non-negative. It is known (see [L1]-[L3]) that

$$v^r(x) \nearrow v(x), \quad |u^r(x)| \nearrow |u(x)| \qquad x \in \mathbb{R}, \quad \text{as} \quad r \to \infty,$$

and that k^r converges to k uniformly on compact sets from $\mathbb{C} \setminus \sigma_M$. From these convergence and Levy's theorem it follows that

$$Q_m^r \nearrow Q_m, \qquad S_m^r \nearrow S_m, \qquad P_m^r \nearrow P_m, \qquad V^r \nearrow V \quad \text{as} \quad r \to \infty, \tag{4.2}$$

for $m = -1, 0, 1, 2, ...$ (some limits may be infinite).

Assume that for each r sufficiently large we have proved that

$$8(S_1^r - S_{-1}^r S_0^r) = P_3^r - V^r. \tag{4.3}$$

Then sending $r \to \infty$ using (4.2) and evoking (2.21), we get that $H_2 = P_3 -$ (r.h.s. of (4.1)). Since $H_2 = P_3 - V$, we recover (4.1).

So it remains to show (4.3). Fix $r > 1$ large enough and consider the integral $\int_{|z|=t} z k^4(z)\, dz$. The function $z(k^r(z))^4$ is analytic in $\{|z| > r\}$. For any $m \geqslant 1$ we write its Tailor series at infinity, omitting the index r for brevity:

$$k(z) = z - \frac{Q_0}{z} - \frac{Q_2}{z^3} ... - \frac{Q_{2m}}{z^{2m+1}} + \frac{O(1)}{z^{2m+2}} \qquad \text{as} \qquad |z| \to \infty. \tag{4.4}$$

Due to (4.4) we get

$$zk^4 = z\left(z^2 - S_{-1} - \frac{S_0}{z^2} - \frac{S_1 + o(1)}{z^4}\right)^2 = z^5\left(1 - \frac{S_{-1}}{z^2} - \frac{S_0}{z^4} - \frac{S_1 + o(1)}{z^6}\right)^2$$

$$= z^5\Big(1 - 2\Big(\frac{S_{-1}}{z^2} + \frac{S_0}{z^4} + \frac{S_1 + o(1)}{z^6}\Big) + \Big(\frac{S_{-1}}{z^2} + \frac{S_0}{z^4}\Big)^2 + ..\Big) = z^5 + .. - 2\frac{S_1 - S_0S_{-1}}{z} + \frac{O(1)}{z^2}.$$

If $t > r$, then

$$\frac{1}{2\pi i}\int_{|z|=t} zk^4(z)\,dz = -2(S_1 - S_0S_{-1}), \tag{4.5}$$

since $v(z) = 0$ for real z such that $|z| > r$. Thus

$$\frac{1}{2\pi i}\int_{|z|=t} zk^4(z)\,dz = \frac{1}{2\pi i}\int_{|z|=t} z(u^2 - v^2 + 2iuv)^2\,dz = \frac{-1}{\pi i}\int_{\mathbb{R}} 2z(u^2 - v^2)(2iuv)dz$$

$$= -\frac{8}{\pi}\int_{\cup_{n\geqslant 1} g_n} z(u^2 - v^2)uv\,dz. \tag{4.6}$$

By (2.21), (4.5), (2.26) and (4.6) we get that

$$8(S_1 - S_0S_{-1}) = \frac{32}{\pi}\int_0^\infty z(u^2 - v^2)uv\,dz = \frac{32}{\pi}\sum_{n\geqslant 1}\int_{g_n} z(u^2 - v^2)uv\,dz = P_3 - V,$$

which yields (4.3). ■

Let $0 < a < \frac{1}{4}$. We note that since $|h\|_{2,a}^2 = \sum_{n\geqslant 1}(2\pi n)^{2a-1}(2\pi n h_n^2)$, then

$$\|h\|_{2,a}^2 \leqslant C_{2-4a}\|h\|_{4,1}, \qquad \textit{where} \quad C_t^2 = \sum_{n\geqslant 1}\frac{1}{(2\pi n)^t} < \infty \quad \text{if } t > 1. \tag{4.7}$$

Theorem 4.2. *A sequence $h = h(I)$ belongs to ℓ_1^4 if and only if $I \in \ell_+^2$. The series*

$$W = \sum_{n\geqslant 1}(4\pi n)V_n, \tag{4.8}$$

where $V_n = V_n(I) \geqslant 0$ is defined by (3.16), converges for $I \in \ell_+^2$ and defines there a finite non-negative function $W(I)$. Moreover,

i) this function equals to

$$\frac{32}{\pi}\int_0^\infty zu(z)v^3(z)\,dz,$$

ii) satisfies the following estimates

$$\frac{1}{5}\sum_{n\geqslant 1}(4\pi n)h_n^2 I_n \leqslant W \leqslant 2\sum_{n\geqslant 1}(4\pi n)h_n^2 I_n, \tag{4.9}$$

$$W \leqslant 4\|h\|_\infty^2 P_1; \tag{4.10}$$

iii) is continuous on ℓ_+^2;

iv) on the octant $\tilde{\ell}_{3/2,+}^\infty$ it coincides with $V(I)$.

Proof. Estimates (3.9) imply that $h \in \ell_1^4$ iff $I \in \ell_+^2$.

Due to the last inequality in (3.17),

$$W \leqslant \sum_{n\geqslant 1}(4\pi n)2h_n^2 I_n \leqslant 4\|I\|_2^{1/2}\|h\|_{4,1}^2,$$

So W is defined by a converging series and satisfies the second estimate in (4.9). Using the lower bound for V_n in (3.17) we recover the first estimate in (4.9). Estimate (4.10) follows from (4.9).

Since $u = \pi n$ on g_n and u vanishes outside $\cup g_n$, then $W(I)$ has the integral representation, required by i).

Let a sequence $I^s = (I^s_n)_1^\infty \to I$ strongly in ℓ^2 as $s \to \infty$. To prove iii) we need to show that

$$W(I^s) \to W(I) \qquad \text{as} \qquad s \to \infty. \tag{4.11}$$

Using (3.9) and (3.1) we have

$$\|h^s\|_{4,1}^4 \leqslant \frac{3\pi^2}{8} C_0^{\frac{3}{2}} \|I^s\|_2^2,$$

where $C_0 \leqslant \exp\sqrt{2P_{-1}(I^s)}$ and $P_{-1}(I) = \sum_{n\geqslant 1} I_n/(2\pi n) \leqslant C_2\|I\|_2$. Together with (4.7) this yields the estimates

$$\sup_{s\geqslant 1} \|h^s\|_{4,1} < \infty, \qquad \sup_{s\geqslant 1} \|h^s\|_{2,a} < \infty \ \text{ if } \ a < \frac{1}{4}. \tag{4.12}$$

We claim that

$$h^s \to h \qquad \text{weakly in} \quad \ell^2_a \quad \text{as} \quad s \to \infty, \tag{4.13}$$

for some $h \in \ell^2_a$. Indeed, assume that this is not the case. Then by (4.12) there are two different vectors $h', h'' \in \ell^2_a$ and two subsequence $\{s'_j\}$ and $\{s''_j\}$ such that

$$h^{s'_j} \to h', \quad h^{s''_j} \to h'' \qquad\qquad \text{weakly in} \quad \ell^2_a. \tag{4.14}$$

Then

$$h^{s'_j} \to h', \quad h^{s''_j} \to h'' \qquad\qquad \text{strongly in} \quad \ell^2_\nu,$$

for each $\nu < a$. Using Theorem 2.1 and the identity

$$k(z,h) = \int_0^z \frac{\Delta'(t,h)}{\sqrt{1-\Delta^2(t,h)}} dt, \qquad z \in \mathcal{Z},$$

which easily follows from (2.28), we deduce that the corresponding conformal mappings k converge to limits:

$$k(z, h^{s'_j}) \to k(z,h'), \qquad k(z, h^{s'_j}) \to k(z,h') \qquad as \ \ j \to \infty, \tag{4.15}$$

uniformly on bounded subsets in $\mathbb{C}$. These convergences and (2.25) imply that for each n the actions $I_n(h^{s'_j})$ and $I_n(h^{s''_j})$ converge to limits $I_n(h')$ and $I_n(h'')$, which must equal I_n. That is, h' and h'' belong to the same iso-spectral class. Since $h', h'' \in \ell^2$, then by Theorem 2.1 we have $h' = h''$. This proves (4.13).

Due to (3.18)

$$(4\pi n)V_n - (2\pi n)^3 I_n = \frac{4}{\pi i}\int_{c_n} zk^4(z)\, dz = -\frac{8}{\pi i}\int_{\chi_n} z^2(k,h)k^3 dk.$$

By this relation, (2.25) and (4.15) we have

$$V_n(I^{s_j}) \to V_n(I) \quad \text{as} \quad j \to \infty \quad \text{for each} \quad n = 1,2,3..... \tag{4.16}$$

For any N denote

$$W^{(N)}(I^s) = \sum_{n\geqslant N} (4\pi n) V_n(I^s).$$

Using (3.17) we get

$$W^{(N)}(I^s) \leqslant \sum_{n\geqslant N} (8\pi n)(h^s_n)^2 I^s_n = A_N + B_N,$$

where

$$A_N = \sum_{n\geqslant N}(8\pi n)(h_n^s)^2 I_n, \qquad B_N = \sum_{n\geqslant N}(8\pi n)(h_n^s)^2(I_n^s - I_n).$$

Since

$$A_N \leqslant 4\|h^s\|_{4,1}\bigg(\sum_{n>N} I_n^2\bigg)^{\frac{1}{2}}, \qquad B_N \leqslant 4\|h^s\|_{4,1}\|I^s - I\|_2,$$

then (4.12) and (4.16) yield the required convergence (4.11).

The last assertion follows from the integral representation for $W(I)$ and (4.1) since $q \in \mathscr{H}_1$ means that $I \in \ell^1_{3/2,+}$. ■

Proof of Theorem 1.1. Theorem 4.2 gives that the function $V : \widetilde{\ell}^\infty \to \mathbb{R}$ extends to a non-negative continuous function on the octant ℓ^2_+. Using estimates (4.10) and (2.33) we obtain $V \leqslant 4\|h\|_\infty^2 P_1 \leqslant 8P_{-1}P_1$, which yields (1.13). If $I = 0$, then (1.13) implies $V(I) = 0$. Finally, let $V = 0$ for some I. Since the terms V_n are non-negative, then each $V_n = 0$ and (3.17) implies that $I = 0$.

It remains to prove (1.14). Estimates (4.9) and (3.9) give

$$V \geqslant \frac{2}{5}\sum_{n\geqslant 1}(2\pi n)h_n^2 I_n \geqslant \frac{\pi}{10C_I}\|I\|_2^2, \tag{4.17}$$

which yields the first inequality in (1.14). Now we show the second. Using (4.9) and (3.6) we find that

$$V \leqslant \sum_{n\geqslant 1}(8\pi n h_n^2)I_n \leqslant \sum_{n\geqslant 1}\bigg(C_0^{\frac{1}{2}}6\pi I_n^2 + 8\frac{\|\rho\|_2^2}{\pi}h_n^2 I_n\bigg) \leqslant 6\pi C_0^{\frac{1}{2}}\|I\|_2^2 + 8\frac{\|\rho\|_2^2}{\pi}\|h\|_2\|h\|_\infty\|I\|_2.$$

Using (2.31), (2.32) and (2.33) we obtain

$$\|\rho\|_2^2\|h\|_2\|h\|_\infty \leqslant \pi 4^4\bigg(1 + Q_0^{\frac{1}{2}}\bigg)^{\frac{1}{2}}Q_0^2.$$

Combining these estimates we get that

$$V \leqslant 6\pi\sqrt{C_0}\|I\|_2^2 + 4^{\frac{11}{2}}(1 + Q_0^{\frac{1}{2}})^{\frac{1}{2}}Q_0^2\|I\|_2.$$

Together with (2.33) this yields the required estimate, since $C_0 \leqslant C_- = \exp\sqrt{2P_{-1}}$. ■

Finally, as a by-product of some relations, derived above in this work, we get two-sided algebraical bounds on the norm $\|q'\|$ in terms of $P_3 = \|I\|_{1,\frac{3}{2}}$ (see [K4] for two-sided algebraical estimates of $\|q^{(m)}\|$ in terms of $P_{m+\frac{1}{2}}$ for all $m \geqslant 0$).

Proposition 4.3. *The following estimates hold true:*

$$\|q'\|^2 \leqslant 4(P_3 + 2P_1^2), \tag{4.18}$$

$$P_3 \leqslant \frac{\|q'\|^2}{2} + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 + 2\pi\|q\|^3(1 + \|q\|^{\frac{1}{3}}). \tag{4.19}$$

Proof. Since $H_2 = P_3 - V$ and $\|q\|_\infty = \sup_{x\in[0,1]} |q(x)| \leqslant \frac{\|q'\|}{\sqrt{2}}$, then

$$\frac{\|q'\|^2}{2} = H_2(q) - \int_0^1 q^3(x)\,dx \leqslant P_3 + \|q\|_\infty \|q\|^2 \leqslant P_3 + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 \leqslant P_3 + \frac{\|q\|^4}{2} + \frac{\|q'\|^2}{4},$$

which together with (1.9) yields (4.18). Using (4.10) and relations $\|q\|_\infty \leqslant \frac{\|q'\|}{\sqrt{2}}$, $\|q\|^2 = 2P_1$ (see (1.9)), we obtain

$$P_3 = \frac{\|q'\|^2}{2} + \int_0^1 q^3(x)\,dx - V \leqslant \frac{\|q'\|^2}{2} + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 + 4\|h\|_\infty^2 P_1 = \frac{\|q'\|^2}{2} + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 + 2\|h\|_\infty^2\|q\|^2.$$

As $\|h\|_\infty^2 \leqslant \pi\|q\|(1 + \|q\|^{\frac{1}{3}})$ (see Theorem 2.3 in [K3]), then we get (4.19). ■

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