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# The roll-out of new mobile technologies as a timing game

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## ABSTRACT

When adopting a novel mobile technology, a mobile network operator faces the dilemma of determining which is the best time to install the next generation equipment onto the existing infrastructure. In a strategic context, the best possible time for deployment is the best response to competitors' actions, subject to normative and material constraints and to the customer's adoption curve. We formulate in this paper a finite timing game in discrete-time which captures the main features of the problem for a two players formulation. Under mild assumptions on the time scale at which operators decide on the installation, we provide a methodology to obtain a subgame-perfect equilibrium for the resulting extensive form game. Our numerical results compare the utility of each player with the social optimum and describe the trade off between fixed costs and installation strategies.

## KEYWORDS

mobile technology adoption, timing games, two-player extensive form games

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## 1 INTRODUCTION

In the telecommunication industry, the roll-out of a new mobile communication technology is a challenge faced periodically by mobile operators which are required to upgrade their infrastructure. In this paper, we consider the problem of determining the best strategy for the introduction of 5G on the market operated by telco operators [21]. The 4G mobile networks cannot support the ever-increasing traffic demand, therefore telco operators are forced to install the 5G technology on their sites to propose better Quality of Service to their customers. A key success element in this context is ensuring the most efficient timing to perform such installments on the network. Operators seek to delay the deployment costs, taking into account that infrastructure upgrades by other operators might let customers switch to such competitors. The optimal choice available to the operators must be thus defined in a competitive setting. In this work we introduce game-theoretical solution concepts to identify the optimal

technology deployment plans of two operators having to perform investments on a given set of sites. We formulate the problem in the form of a discrete-time timing game [1]. Therein, the actions sets available to each player are defined by logistical and normative constraints. Indeed, all operators must join the auction for the allocation of the 5G band spectrum and have thus to follow some constraints on its deployment set by the regulator [21]. In turn, the operators' utility is a function of the customers' technology adoption and the customers' distribution over the operator sites, the costs incurred for the technology upgrades and their promotion. We assume that the quality of service is a piece of public information, as it is constantly monitored by independent authorities (see [10] for France). A player, i.e., an operator is hence able to track the evolution of the competitors' 5G deployment. On the other hand, upon choosing a 5G service provider, we assume customers to bind to this operator until the end of the time horizon. To this aim, as a new technology is introduced, the operators use financial subsidies to accelerate its adoption e.g. using discounted offers or advertisement. We assume that subsidies are budgeted for the whole time horizon and simultaneously adopted by the operators at the beginning of the deployment phase.

*State of the art.* In the literature, several papers studied the investment optimization problem faced by operators aiming to mitigate installation costs of the 5G technology and yet satisfy increasing traffic demand. One approach considers the behaviour of the operator's customers [3, 8]; such models do not take into account the competition among operators. Another approach is to set a cooperative game which determines the social optimum and how investments should be allocated among the operators [4, 6, 15]. Some models do account for the fact that operators are in competition to serve potential customers [5, 7, 9], but they neglect the temporal dimension. In the economic theory, the introduction of a new technology belongs to a specific class of models, called *innovation timing games*. This category of games concerns two players selecting a specific time at which they act [13, 16, 24]. This standard scenario is concerned with the dynamics of an incumbent which defends from a possible entrant in a market [25]. Results for more than two players are derived in [2, 26] for specific contexts. In game theory, the class of games where players pick a time when they act are called *timing games* and can be either in continuous [18] or discrete [1] time. In the literature discrete timing games have been used for marketing decisions [12]. In this paper we introduce a two-player discrete timing game, in which two operators can decide when to start the roll-out. The model belongs to the class of sequential games [14], in which players act in turns one after another. In the literature results are only given for specific categories of timing games: stochastic games with one choice [22], Stackelberg games with random-ordered players [11] and games with small discrete time intervals [23].

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To the best of our knowledge, this is the first work that introduces a timing game model for the roll-out of the 5G technology. We factor in the model the customers' adoption dynamics, the operators' installation strategies on multiple sites and constraints related to logistics and regulation. The system is hence modeled with an extensive form game, which is solved with a tailored-made formulation based on a classic resolution method [27]. The paper is structured as follows. In Section 2 we introduce the model. Section 3 discusses the solution methods to determine the solution of the sequential game. Finally, in Section 4 we report for numerical results. A concluding section ends the paper.

## 2 SYSTEM MODEL

In this work, we focus on a two telecommunication operators framework where both operators act as rational players. Each player seeks the optimal strategy to maximize her own return. Each operator chooses the subsidy she offers from a discrete set  $S_i$ . Afterwards, they launch the deployment campaign for the new technology, i.e., they schedule when and where to deploy investments on their own sites. A site is an area where both operators can install the new technology. Since operators often build sites close the competitor's, we consider perfect overlap of the sites. Operators act on a discrete time horizon. We thus introduce the following parameters:

- $N = \{1, 2\}$ , set of players;
  - $S_i$ , set of possible subsidies for player  $i \in N$  chosen at time  $t = 0$ ;
  - $T = \{1, \dots, |T|\}$ , set of time-intervals over which operators act to install the new technology;
  - $\mathcal{A}$ , set of sites;
- The possible actions of a player are defined by the variables:
- $s_i \in S_i$ , subsidy chosen by player  $i$  at time 0;
  - $z_{i,a,t} \in \{0, 1\}$ , a binary variable indicating if the new technology is installed on site  $a$  by operator  $i$  at time  $t \geq 1$ . We call  $t_{i,a}$  the time at which operator  $i$  installs it on site  $a$ .

The operators' schedule is bounded by some constraints:

- *Logistic constraints*: the operator  $i$  can invest on a limited number  $Z_i$  of sites at each time  $t \geq 1$ . Thus for every player it holds  $\sum_{a \in \mathcal{A}} (z_{i,a,t+1} - z_{i,a,t}) \leq Z_i$ ;
- *Regulator constraints*: before every time  $t$  at least  $R(t)$  sites have to support the new technology. Thus for all players and for all  $t \geq 1$  it holds  $\sum_{a \in \mathcal{A}} z_{i,a,t} \geq R(t)$ .

The strategy  $(s_i, z_i) \in S_i \times (\mathcal{A} \times T)$  is a choice made by player  $i \in N$  on the subsidy  $s_i \in S$  and on the roll-out scheme  $z_i \in (\mathcal{A} \times T)$ . This choice is subject to the maximisation of the player's utility function  $u_i : S_1 \times S_2 \times (\mathcal{A} \times T)^2 \rightarrow \mathbb{R}$ . The utility functions depend on multiple parameters, which involve costs, characteristics of the market and adoption dynamics. We defer such analysis to Section 2.2. Players have conflicting interests, seeking to gain the largest share of the same market, while keeping the costs under control. Therefore an operator cannot compute her own solution independently from competitors. We thus identify an equilibrium of the game, i.e., a couple of strategies, one per player such that the players are satisfied with them if they are both played. We add a further assumption, i.e. that players can observe, at time  $t$ , the history of actions taken by the other player for  $t' < t$ , and react accordingly; the choice of the opponent's subsidy is observable by both players at time  $t = 0$ . A convenient model that can comply to such assumption is

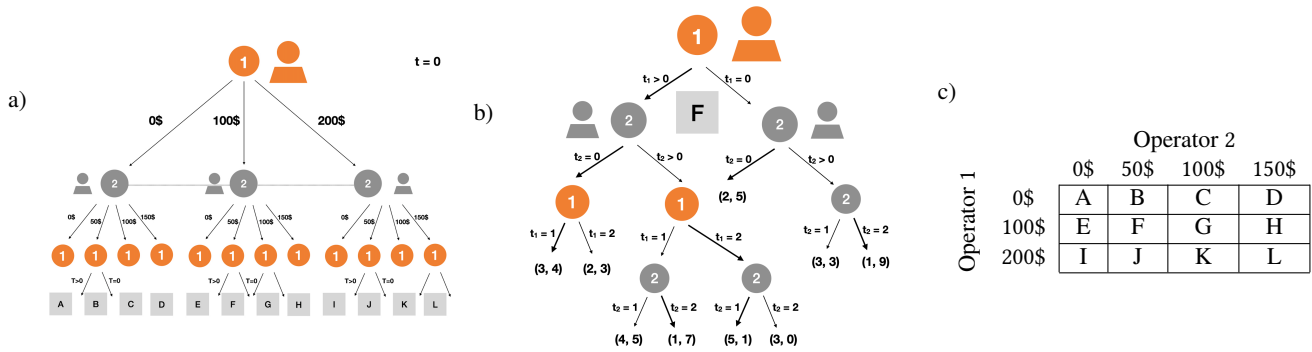
that of a *game in extensive form* [17], whose mathematical model is based on a *game tree*. In a game tree, arcs outgoing from a node represent the possible choices available to a player. The sequence of nodes and arcs represent the sequence of choices taken by the players. The leaves or terminal nodes reached after both players have performed  $|T|$  actions, have no outgoing arcs and represent the outcomes of the game. Such outcomes are assigned a pair of values  $(u_1, u_2)$ . Formally, an *outcome* is a pair of actions  $\{(s_i, z_i), i = 1, 2\}$  adopted by the two players. In our model, they are determined by the *utility function* introduced before evaluated at  $\{(s_i, z_i), i = 1, 2\}$ . The higher the value of  $u_i$ , the higher the value a player assigns to the combination of actions that leads to the final node. For our system the *game tree* is defined below:

*Definition 2.1 (Service providers (SP) game).* The service providers game  $\langle N, S_1, S_2, \mathcal{A}, T, u \rangle$  is an extensive form game with  $N = 2$  players  $\{1, 2\}$  competing over set of sites  $\mathcal{A}$  in which:

- at the root vertex both players choose at the same time  $t = 0$  and independently the subsidies  $s_1 \in S_1$  and  $s_2 \in S_2$ ;
- the players act in sequence at every round  $t \geq 1$ , starting from player 1. At every step they can decide on which sites  $A_{1t} \subseteq \mathcal{A}$  and  $A_{2t} \subseteq \mathcal{A}$  install the new technology, given the constraints;
- after  $|T|$  rounds the game ends and the actions chosen at each round are evaluated by the utility functions  $u_i$  hereafter defined;
- The utility function of a player  $i$  is  $u_i : S_1 \times S_2 \times (\mathcal{A} \times T)^2 \rightarrow \mathbb{R}$ ;

*Example.* Figures 1a) and 1b) show a representation of the game with the following properties:

- $|\mathcal{A}| = 1$ , a single site.  $A_{1t}, A_{2t} \subseteq \{\emptyset, \{1\}\}$ , it is possible to install the new technology on such site; no constraints are considered.
  - $T = \{1, 2\}$ , horizon of two time-intervals.
  - $S_1 = \{0\$, 100\$, 200\}$  set of possible subsidies for player 1.
  - $S_2 = \{0\$, 50\$, 100\$, 150\}$  set of possible subsidies for player 2.
- The game tree is represented in Figure 1a) and b). In Figure 1a), a line among the nodes where the second player moves indicates that the two players act at the same time; this is a standard representation of simultaneous actions in extensive form games. The solution of the game can be determined in two steps. First, at  $t = 0$  the operators pick a subsidy at the same time. The second operator does not know what the first operator has played, and vice versa. Then, in the second part (all subtrees rooted at  $t = 1$ ), they get to know what the other operator has chosen and decide one after another if installing on a site or not at each time step, starting by player 1. If the resulting subgame in extensive rooted at  $t = 1$  has a unique solution (or *equilibrium*), we can represent the first part with a matrix game whose entries  $(s_1, s_2)$  are the utilities corresponding to the said equilibrium (Figure 1c). Clearly, different choices of subsidies can lead to different timings of the investment. Formally, the equilibrium of the game is the Nash Equilibrium for the corresponding the matrix game [19]. Let denote  $\Gamma(s_1, s_2)$  the part of the game which starts from  $t = 1$ , given that the first operator has chosen  $s_1 \in S_1$  and the second operator  $s_2 \in S_2$ . Later in the section we show that for every  $\Gamma(s_1, s_2)$  there is a unique outcome  $\sigma(s_1, s_2) \in (\mathcal{A} \times T)^2$  which leads to a solution. Matrix  $M$  maps a couple of choices for the subsidies to the utility of such outcome  $(s_1, s_2) \rightarrow \sigma(s_1, s_2) \rightarrow M(s_1, s_2) = u(s_1, s_2, \sigma(s_1, s_2))$ . In the example of Figure 1, we have a  $|S_1| \times |S_2| = 3 \times 4$  matrix, whose entries correspond to a game tree of the type depicted in Figure 1b).



**Figure 1: a) Subsidies are picked first. b) After the subsidies are decided, players choose sequentially to add or not the new technology on the site at each round  $t \in T = \{1, 2, \dots, |T|\}$ . c) Matrix representation of the first part of the game.**

*Definition 2.2.* Given a SP game  $\langle N, S_i, \mathcal{A}, T, u \rangle$  and its correspondent matrix  $M : (s_1, s_2) \mapsto u(s_1, s_2, \sigma(s_1, s_2))$ , with  $(s_1, s_2)$  chosen at time  $t = 0$  and  $\sigma(s_1, s_2) \in (\mathcal{A} \times T)^2$  the optimal installation times chosen at times  $t \geq 1$ , we say  $(\bar{s}_1, \bar{s}_2) \in S_1 \times S_2$  is an equilibrium if for all  $s_1 \in S_1$  and  $s_2 \in S_2$  we have:

$$M_1(\bar{s}_1, \bar{s}_2) \geq M_1(s_1, \bar{s}_2), \quad M_2(\bar{s}_1, \bar{s}_2) \geq M_2(\bar{s}_1, s_2).$$

## 2.1 Sequential form of SP game

In order to populate matrix  $M$  we have to compute for every combination of subsidies  $(s_1, s_2) \in S_1 \times S_2$  the investment strategy that leads to the outcome  $\sigma(s_1, s_2) \in (\mathcal{A} \times T)^2$  that corresponds to an equilibrium for the game. We consider the part of the game  $\Gamma(s_1, s_2)$  in which players act in sequence, which is represented in its *sequential form*. A *sequence* of moves of player  $i$  is a collection of actions from the root to the leaf (irrespective of the other player's moves). The sequential form represents the collection of all the sequences of actions that lead from the vertex to the leaves of the tree.

*Definition 2.3 (sequential game form).* Given a SP game  $\langle N, S_i, \mathcal{A}, T, u \rangle$  and a couple of subsidies chosen by the players  $(s_1, s_2) \in S_1 \times S_2$ , we consider  $\Gamma(s_1, s_2)$  the sequential game defined by all the sequences that lead a vertex generated by  $(s_1, s_2)$  to a leaf:

$$\Gamma = \{(\{z_{nt}\}, u(\{z_{nt}\})) : z_{nt} \in \text{Ant}, u : (A \times T)^2 \in \mathbb{R}\}.$$

In  $z_{nt}$  the index  $n$  represents the player acting at turn  $t$ . From now on, we omit  $(s_1, s_2)$  and identify the sequential game with the letter  $\Gamma$ , as we assume that a couple  $(s_1, s_2) \in S_1 \times S_2$  has already been chosen. We introduce the notion of subgame, i.e. the part of the tree that goes from a node to the leaves. In the sequential form, a sequence of actions  $a \in \Gamma$  leads from the vertex to a leaf. A node can be identified with a subsequence of actions  $b \subset a$ , i.e. those actions that lead from a vertex to the node itself. A subgame is defined as the set of subsequences  $a'$  that lead from a node to a leaf.

*Definition 2.4 (subgame).* Given a sequence  $a \in \Gamma = \langle \{a\}, u \rangle$  and a subsequence  $b \subset a$ , we consider the set of sequences  $\{a' : \exists a \in \Gamma, a = \{b, a'\}\}$  which provide utility  $u'(a') = u(\{b, a'\})$ . Then  $\Gamma'(b) = \langle \{a'\}, u' \rangle$  is a subgame of the game  $\Gamma$ .

We would like to identify a solution of the game under the assumption that every player at time  $t$  can observe the actions taken

by the other player at time  $t' < t$ . A standard requirement is for the solution to be an equilibrium also for every subgame, i.e., a *subgame perfect equilibrium* (SPE) [14]. In a game of perfect information, players can forecast future actions, starting from the bottom of the tree, since they both know that they choose the actions that maximise their utility. In recursive manner, in any parent subgame a player can identify the actions played subsequently and also identify the action that leads to the best outcome for her. Such algorithm that finds recursively the best action is called *backward induction algorithm* [14]. The solution identified by the algorithm corresponds to a SPE. A more efficient method to find the SPE is introduced in Section 3. Every finite perfect information extensive form game has a pure strategy SPE, but it is not unique if we need to arbitrate on equal payoffs for same player. However, if we assume that in every game  $\Gamma(s_1, s_2)$  no player has the same payoffs at any two terminal nodes, the uniqueness of the SPE in pure strategies is granted from a classic result.

**THEOREM 2.5** ([17]). *If no player has the same payoffs at any two terminal nodes, a finite extensive form game has a unique SPE equilibrium in pure strategies and it is generated by the backward induction algorithm.*

This assumption is assumed verified in the rest of the paper. The choice of the order of the players can be arbitrary. Indeed, in reality the players act simultaneously, so there is no natural order to be followed. Moreover, the solution does not change significantly if the order of the players is inverted. Operators play in turns with very small difference of time between one and another turn. Postponing an action of one interval does not change the outcome significantly. Both players can quickly adjust her strategy accordingly. This intuition is formalised in Theorem 2.6.

**THEOREM 2.6.** *Given two games  $\Gamma = \{z_{nt}, n = \{1, 2\}\}$  and  $\Gamma' = \{z'_{n't}, n' = \{2, 1\}\}$  in which players act in inverted order, if there is for every  $t \in T$  and for every subsequence  $z_{nt} \in \Gamma$  an action  $\bar{z}_{it}$  that can allow a player  $i \in N$  to postpone her action without changing significantly her utility (within an interval  $\epsilon > 0$ ):*

$$|u_i(\dots, z_{it}, \dots, z_{i,t+1}, \dots) - u_i(\dots, \bar{z}_{it}, \dots, z_{i,t}, \dots)| < \epsilon,$$

*then given two SPE  $\sigma \in \Gamma$ ,  $\sigma' \in \Gamma'$  we have  $|u_i(\sigma) - u_i(\sigma')| < \epsilon \forall i$ .*

## 2.2 Subscriber dynamics and operator utility

So far we have not discussed the operators' utility function  $\mathbf{u}$ . As discussed next, it highly depends on the dynamics. We consider first the dynamics for a single-site scenario, and then the case with more than one site.

*One-site model.* For the one-site model  $|\mathcal{A}| = 1$  and per site indexes are dropped for notation's sake. In order to characterize the utility function for the one-site model we fix some assumptions: 1) following [3], every customer on the site decides to switch to the new technology at a given time  $t$  and stick to the choice till the end of the horizon; 2) the quantity of customers switching at every time is known a priori by both operators according to a given dynamics; 3) every customer has a preference over the two operators which they subscribe for; if their preferred operator does not offer the technology at time  $t$  they subscribe to the other operator if it offers it, otherwise they wait for one of them to offer it; 4) an operator can admit only a limited amount of customers per time interval; 5) once a customer subscribes for an operator, they are bound to them for all the time  $t \in T$ .

Operators choose a pair of subsidies  $(s_1, s_2) \in S_1 \times S_2$ ; such choices influence the potential market of customers willing to switch to the new technology, which varies in time. For every  $t \in T$ , such customers are identified by the parameter  $\{y_t\}_{t \in T}$ , which is subject to the condition that  $\sum_t y_t = 1$ , i.e. that all the customers eventually switch to the new technology. We pick an adoption curve in the form  $y_{adpt}(t) = 1 - e^{-\lambda t}$ , with  $\lambda = \lambda(s_1, s_2) > 0$ , which is often used in the literature [20]. We can thus define from it the parameter  $y_t := y_{adpt}(t+1) - y_{adpt}(t)$ . Preferences over the offers proposed by operators 1 and 2 are a function of the subsidies: we assume such preferences distributed with the same proportion  $p_1(s_1, s_2)$  and  $p_2(s_1, s_2) = 1 - p_1(s_1, s_2)$  for all  $t \in T$ . We recall that  $y_t$  is the potential market at time  $t$ . In order to capture these customers the operators have to install the new technology. We suppose that operators need  $\tau > 0$  intervals of time to fill it. This constraint is due to the fact that customers do not discover the technology all at once, but little by little.

Let  $\alpha_1(t)$  and  $\alpha_2(t)$  the number of customers acquired at time  $t \in T$  with the new technology by operators 1 and 2, respectively. Since the customers, once acquired, are kept until the end of the horizon the final market share is  $\alpha(|T|)$ . On the other hand, there exist some costs related to the installation of the new technology and to subsidies. We denote the installation costs  $c \in (0, 1)$ : they are discounted by a factor  $e^{-\gamma t}$  which accounts for the depreciation since installation time  $t$  and lower maintenance costs over the period. We suppose that the subsidies  $s_i$  are fixed costs, thus they do not depend on the quantity of customers acquired. The utilities for the players are thus  $u_i(s_1, s_2, t_1, t_2) = \alpha_i(|T|) - c \cdot e^{-\gamma t_i} - s_i$ , where  $c$  and  $\gamma$  are parameters known a priori.

Let us formalise the dynamics, i.e. how the functions  $\alpha_1(t)$  and  $\alpha_2(t)$  depend on the strategies chosen by the players among the states  $t \in T$ . We define a state at time  $t$  with the following parameter and variables: 1)  $y_t$ , the number of users who decide to switch to the new technology at time  $t$ ; 2)  $t_i$ , time at which operator  $i$  installs the technology; 3)  $d_t$ , the demand of customers who want to switch to the new technology at time  $t$  and who are not served before time

$t$ ; 4)  $d_{it}$ , the fraction of  $d_t$  who prefer the operator  $i \in N$ ; 5)  $r_{it}$ , customers that operator  $i \in N$  can accept at time  $t$ .

The demand at time  $t$  is updated with the new costumers. All the customers added at time  $t$  are  $y_t$ , thus those who prefer  $i$  are  $p_i \cdot y_t$ , i.e.  $\forall i \in N: d_{it} \leftarrow d_{i,t-1} + p_i \cdot y_t$ . If an operator  $i \in N$  has installed the technology  $t \geq t_i$ , then she can accept up to  $\frac{1}{\tau}$  customers per interval of time:  $r_{it} = \frac{1}{\tau} \mathbf{1}_{t \geq t_i}$ . Up on their demand and on their supply, the operators add their customers. We have that for all  $i, j \in N$  with  $i \neq j: \alpha_i(t+1) \leftarrow \alpha_i(t) + \min(d_{it}, r_{it}) + [\min(d_{jt} - r_{jt}, r_{it} - d_{it})]^+$ . Those costumers served at time  $t$  are removed from the demand at time  $t+1: d_{i,t+1} \leftarrow [d_{it} - r_{it} - [r_{jt} - d_{jt}]^+]^+$ .

*Multi-site model.* Let us consider a model for more than one site  $|\mathcal{A}| > 1$ . We set as hypothesis that every customer is served on a single site and thus introduce the parameter  $h: \mathcal{A} \rightarrow [0, 1]$ , that maps every site  $a \in \mathcal{A}$  to the percentage of users  $h(a)$  that are served on the site. Every parameter is given an index referring to the site. The potential market  $y_{a,t}$  for a site  $a \in \mathcal{A}$  is subject to the condition  $\sum_t y_{a,t} = h(a) \cdot y_t$ . The utility function for the multi-site case differs from the one-site by the fact that it sums the customers acquired on every site and the costs of the respective installations. For every player  $i \in N$  the utility function is  $u_i(s_1, s_2, t_1, t_2) = \sum_a \alpha_{i,a}(|T|) - c \cdot \sum_a e^{-\gamma t_{i,a}} - s_i$ .

## 3 SOLUTION METHODS

When the matrix game of Def. 2.2 has a unique Nash equilibrium in pure strategies, the optimal  $(\bar{s}_1, \bar{s}_2) \in S_1 \times S_2$  to be chosen at time  $t = 0$  can be easily identified by searching the matrix. In general, the solution of the game may be in mixed strategies.

However, the value of every  $M(s_1, s_2)$  can be computed only by analysing the SP game in *sequence form*, defined in Section 2.1, which represents times  $t \geq 1$ . Computing an equilibrium of an extensive game in sequence form can be performed with linear time complexity in the number of outcomes [27]. The solution method requires to list all the possible sequences for both players  $\Sigma_1$  and  $\Sigma_2$ . We consider  $x \in \{0, 1\}^{|\Sigma_1|}$  and  $y \in \{0, 1\}^{|\Sigma_2|}$ , vectors that define the probability that a sequence is played. The utilities of the players can be written in the form  $x^T A y$  and  $x^T B y$ . Indeed, given  $j = [0, \dots, 1] \in \Sigma_1$  and  $k = [0, \dots, 1] \in \Sigma_2$  two sequences that lead the players to play  $t_1$  and  $t_2$  we have that  $A_{jk} = u_1(t_1, t_2) > 0$  and  $B_{jk} = u_2(t_1, t_2) > 0$ . It is possible to add a translation factor such that the utilities are positive when an outcome is reached. We can then define the following bilevel problem. Matrices  $E$  and  $F$  link the sequences:

$$\begin{aligned} (\bar{x}, \bar{y}) &= \max_x \sum_j \sum_k A_{jk} \cdot x_j \cdot \bar{y}_k \\ \text{s.t.} \quad &\sum_l E_{lj} \cdot x_j = e_l \quad \forall l \in |E| \end{aligned} \quad (1)$$

$$x \in \{0, 1\}^{|\Sigma_1|} \quad (2)$$

$$\begin{aligned} \bar{y} &= \max_y \sum_j \sum_k B_{jk} \cdot x_j \cdot y_k \\ \text{s.t.} \quad &\sum_m F_{mk} \cdot y_k = f_m \quad \forall m \in |F| \end{aligned} \quad (3)$$

$$y \in \{0, 1\}^{|\Sigma_2|} \quad (4)$$

*Linear formulation.* We introduce the variable  $w_{jk} \in [0, 1]$  to linearize the constraint  $x_j \cdot y_k = w_{jk}$ :

$$w_{jk} \in [0, 1] \quad \forall j \in \Sigma_1, k \in \Sigma_2. \quad (5)$$

$$x_j \geq w_{jk} \quad \forall j \in \Sigma_1, k \in \Sigma_2, \quad (6)$$

$$y_k \geq w_{jk} \quad \forall j \in \Sigma_1, k \in \Sigma_2. \quad (7)$$

If we fix the value of  $x$ , i.e., the sequences chosen by the first player, the second player can reach a finite number of outcomes, i.e., there is a subset of indices  $k \in \Sigma_2$  such that  $\sum_j x_j B_{jk} > 0$ . We thus define  $v = \max_k (\sum_j x_j B_{jk})$  the maximum value achieved by the second player and  $k_{max} \in \Sigma_2$  the index for which such maximum is achieved:  $k' = \arg \max_k (\sum_j x_j B_{jk})$ . We define a binary variable  $y'_k \in \{0, 1\}$  which is equal 1 only for the index  $k \in \Sigma_2$  at which the maximum is achieved, i.e.  $y'_k = 1 \iff k = k_{max}$ . We thus add the following constraints:

$$y_k \geq y'_k \quad \forall k \in \Sigma_2, \quad (8)$$

$$y'_k \in \{0, 1\} \quad \forall k \in \Sigma_2, \quad (9)$$

$$\sum_{k \in \Sigma_2} y'_k = 1, \quad (10)$$

to select the index  $k \in \Sigma_2$  and make the second player choose it. We make  $v \in \mathbb{R}$  correspond to the maximum. Given  $M > 0$  a value large enough we have that:

$$\sum_{j \in \Sigma_1} B_{jk} x_j \leq v \quad \forall k \in \Sigma_2, \quad (11)$$

$$\sum_{j \in \Sigma_1} B_{jk} x_j \geq v - M(1 - y'_k) \quad \forall k \in \Sigma_2, \quad (12)$$

$$v \geq 0 \quad (13)$$

The linear formulation is thus:

$$\begin{aligned} & \max_x \quad \sum_j \sum_k A_{jk} \cdot w_{jk} \\ & s.t. \quad (1-13) \end{aligned} \quad (14)$$

*Valid constraints.* Let us suppose that we have  $B_{jk} > 1$  when an outcome is reached; utilities can be designed such that this is always true and the outcome of a game is invariant to the translation of utility. We want  $y'_k = 0$  when  $\sum_j B_{jk} x_j = 0$ , because such index cannot be selected, i.e.:

$$y'_k \leq \sum_{j \in \Sigma_1} B_{jk} x_j \quad \forall k \in \Sigma_2. \quad (15)$$

If we add this class of constraints we can tighten the problems and relax the constraints  $y'_k \in \{0, 1\}$  to  $y'_k \in [0, 1]$ :

$$y'_k \in [0, 1] \quad \forall k \in \Sigma_2. \quad (16)$$

We have then that the constraint concerning the variable  $y_k$  become useless. We thus remove such variable and its constraints:

$$y'_k \geq w_{jk} \quad \forall j \in \Sigma_1, k \in \Sigma_2. \quad (17)$$

**PROPOSITION 3.1 (RELAXATION).** *The problem (1-14) and its relaxation (1-2,5-6,9-17) provide the same solution.*

According to [27], the number of sequences  $|\Sigma_1|, |\Sigma_2|$  and the number of constraints  $|E|$  are linear in the number of outcomes, which is  $O(T^{2|\mathcal{A}|})$ . Therefore  $|\Sigma_1|, |\Sigma_2|$  and  $|E|$  are  $O(T^{2|\mathcal{A}|})$ . The parameter  $A$  and  $B$  are matrices of dimension  $O(T^{2|\mathcal{A}|}) \times O(T^{2|\mathcal{A}|})$  with  $O(T^{2|\mathcal{A}|})$  non-zero values. The model has  $O(T^{4|\mathcal{A}|})$  variables,  $O(T^{2|\mathcal{A}|})$  binary constraints and  $O(T^{4|\mathcal{A}|})$  linear constraints.

*Stackelberg.* In Section 4 we compare the results of the aforementioned problem with the classic Stackelberg formulation. An operator chooses before the other at which time install the new technology. She announces the time chosen and then she is forced to stick to it. The problem is formalised hereafter:

$$\begin{aligned} (t_1, t_2) &= \max_{\tau_1} u_1(\tau_1, t_2) \\ s.t. \quad t_2 &= \max_{\tau_2} u_2(t_1, \tau_2) \end{aligned}$$

## 4 SIMULATION

First of all, we consider the sequential game. In order to understand how the performances evolve with increasing intervals of time  $T \in [7, 30]$ , we pick the following values for the parameters:  $\tau = \min(T, 15)$ ,  $\lambda = \gamma = \frac{-\ln(0.01)}{T}$ ,  $c = 1.0$ ,  $p_1 = 0.7$ ,  $p_2 = 0.3$ . Some of the values obtained are reported in Figure 3a). As expected, the time of resolution grows polynomially  $O(T^4)$  with the parameter  $T$ . If customers are acquired in a short interval of time ( $\tau = T$ ), the first player tends to anticipate the second player. On the other hand, when  $\tau < T$  the first player waits until the second player is forced to act. The results are very close to the social optimum, which is defined as the value of the greater sum of the two utilities. In the Stackelberg problem the first player announces her installation time and thus gives more information to the second player; indeed her utility is lower than in the sequential game.

*Subsidies.* We now consider the full game, in which operators have to choose their subsidies among  $S_1 = \{0.0, 0.1, 0.2, 0.3, 0.4\}$  and  $S_2 = \{0.0, 0.1, 0.2, 0.3\}$ . The chosen setting is  $T = 10$ ,  $\tau = 5$ ,  $\lambda = \gamma \cdot (1 + s_1, s_2)$ ,  $c = 1.0$ ,  $p_1 = \frac{0.7+s_1}{1+s_1+s_2}$ ,  $p_2 = \frac{0.3+s_2}{1+s_1+s_2}$ . For every scenario  $(s_1, s_2)$  we build a sequential game  $\Gamma(s_1, s_2)$ , which leads to an optimal choices of the installation times  $\sigma(s_1, s_2) \in T^2$ . The matrix  $\sigma$  maps every scenario  $(s_1, s_2) \in S_1 \times S_2$  to the optimal installation times  $(t_1, t_2) = \sigma(s_1, s_2) \in T^2$  (cf. Figure 2b), which are then evaluated  $M(s_1, s_2) = u(s_1, s_2, t_1, t_2)$  (cf. Figure 3b). Once the matrix  $M$  is filled, we can identify the solution of the game, i.e. a combination  $(s_1, s_2) \in S_1 \times S_2$  such that no player has an incentive to change its strategy unilaterally. A priori, a solution in pure strategies may not exist nor be unique. In this case we have a unique solution, which is  $(s_1, s_2) = (0.0, 0.1)$ . Our model thus forecasts that the first operator will have no subsidies ( $s_1 = 0.0$ ), while the second operator will choose to have  $s_2 = 0.1$ . Then, the first operator chooses to install the new technology at time  $t_1 = 7$ , after the second operator, who chooses to add at time  $t_2 = 6$ .

*Multi-site.* We consider the case of  $|\mathcal{A}| = 2$  sites and the value of the subsidies fixed to  $s_1 = 0.0$  and  $s_2 = 0.0$ . We set  $h(\mathcal{A}) = (0.75, 0.25)$  and, for sake of clearness, we normalise the total number of customers to  $\sum_a \sum_t y_{a,t} = 2$ . Every site can thus admit at most respectively 1.5 and 0.5 unit of customers. We set  $T = [3, 7]$ ,  $\tau = T-1$  and  $Z_i = 1$  for all  $i \in N$ , i.e. the installment can be done on at most  $Z_i = 1$  site for every interval of time. We fix also  $R(T) = 2$ , i.e. that

T	$t(s)$	$u_1 [S]$	$u_2 [S]$	% opt	solution [S]
10	3.4	0.44 [0.44]	0.34 [0.34]	97.4	$t_1 : 4 [4] t_2 : 6 [6]$
15	16.8	0.44 [0.44]	0.34 [0.34]	97.6	$t_1 : 6 [6] t_2 : 9 [9]$
20	50.9	0.42 [0.39]	0.44 [0.36]	96.6	$t_1 : 12 [9] t_2 : 11 [9]$
25	85.8	0.44 [0.40]	0.46 [0.28]	98.2	$t_1 : 17 [9] t_2 : 16 [11]$
30	203	0.44 [0.38]	0.45 [0.21]	95.6	$t_1 : 22 [9] t_2 : 21 [12]$

$s_1$	$s_2 = 0.0$	$s_2 = 0.1$	$s_2 = 0.2$	$s_2 = 0.3$
$s_1 = 0.0$	(4, 4)	<b>(7, 6)</b>	(7, 6)	(7, 6)
$s_1 = 0.1$	(7, 6)	(7, 6)	(7, 6)	(7, 6)
$s_1 = 0.2$	(7, 6)	(4, 4)	(7, 6)	(7, 6)
$s_1 = 0.3$	(4, 4)	(7, 6)	(7, 6)	(7, 6)
$s_1 = 0.4$	(5, 5)	(4, 4)	(4, 4)	(7, 6)

Figure 2: a) Sequential game. The results of the Stackelberg formulation are shown in [S]. b) Solution of the game for  $T = 10$ .

T	$t(s)$	site 1	site 2	$u_1$	$u_2$	$t_{i,1}$	$t_{i,2}$
3	1.10	[1.00,0.46]	[0.03,0.47]	0.77	0.66	{1, 2}	{2, 1}
4	9.61	[1.00,0.33]	[0.16,0.34]	0.81	0.54	{1, 3}	{3, 2}
5	62.5	[1.00,0.50]	[0.25,0.25]	0.83	0.66	{1, 3}	{4, 4}
6	336	[0.80,0.60]	[0.29,0.31]	0.83	0.66	{2, 3}	{4, 4}
7	3500	[0.83,0.67]	[0.35,0.15]	0.84	0.64	{2, 3}	{4, 5}

$s_1$	$s_2 = 0.0$	$s_2 = 0.1$	$s_2 = 0.2$	$s_2 = 0.3$
$s_1 = 0.0$	(0.4, 0.2)	<b>(0.4, 0.3)</b>	(0.5, 0.2)	(0.4, 0.0)
$s_1 = 0.1$	(0.2, 0.4)	(0.2, 0.3)	(0.2, 0.1)	(0.2, 0.0)
$s_1 = 0.2$	(0.0, 0.4)	(0.0, 0.0)	(0.0, 0.1)	(0.0, 0.0)
$s_1 = 0.3$	(-0.2, 0.4)	(-0.2, 0.1)	(-0.2, 0.1)	(-0.2, 0.0)
$s_1 = 0.4$	(-0.4, 0.3)	(-0.4, 0.1)	(-0.4, -0.1)	(-0.4, 0.0)

Figure 3: a) Multi-site case: game solutions for increasing  $T$  and  $s_1 = s_2 = 0$ ; b) matrix  $M$  entries ( $T = 10$ ).

the installments have to be done on all the sites at the end of the time horizon. Results are displayed in Figure 3a). The first player always tries systematically to anticipate the investment on the first site. The second player shows two possible behaviors, depending on the parameters: either she chooses to anticipate the first player on the second site while delaying the investment on the first site ( $T = \{3, 4\}$ ) or she follows the investment of the first player on both sites ( $T = \{5, 6, 7\}$ ).

## 5 CONCLUSION

In this work we introduced a game theoretical model, the *Service providers (SP) game*, to determine the optimal timing of the roll-out of the 5G. The model is enough flexible to be used for any general investment timing problem. We applied it to the case of two operators having to perform investments on a set of sites. A model for analysing the dynamics of this specific problem is defined. We provided a solution concept for the game and a method to compute it. The simulations on the case with a single site identify different optimal strategies, that can change with the choice of the parameters. The complexity of the game grows exponentially with the number of sites. Partial results for a two-site case show that new strategic patterns can be identified when more than one site is considered. In the future we plan to study different methods to analyse larger games and categorize the optimal strategies.

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