



HAL
open science

An extension of the Ising-Curie-Weiss model of self-organized criticality with a threshold on the interaction range

Nicolas Forien

► **To cite this version:**

Nicolas Forien. An extension of the Ising-Curie-Weiss model of self-organized criticality with a threshold on the interaction range. 2021. hal-03382821v1

HAL Id: hal-03382821

<https://hal.science/hal-03382821v1>

Preprint submitted on 18 Oct 2021 (v1), last revised 11 Jan 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An extension of the Ising-Curie-Weiss model of self-organized criticality with a threshold on the interaction range

Nicolas Forien*

October 17, 2021

Abstract

In [CG16], Cerf and Gorny constructed a model of self-organized criticality, by introducing an automatic control of the temperature parameter in the generalized Ising Curie-Weiss model. We study an extension of this model where the mean-field interaction of [CG16] is replaced with a one-dimensional interaction with a certain range d_n which varies as a function of the number n of particles. In a previous article [For21], we studied the extreme cases of a very long range of interaction ($2d_n \sim \lambda n$), which behaves like the mean-field model, and of a nearest neighbour interaction ($d_n = 1$), which does not. In the present article, we study the intermediate regime and we show that the mean-field-like behaviour extends to interaction ranges $d_n \gg n^{3/4}$, and we prove that there is a threshold phenomenon with a different behaviour when the interaction range is of order $d_n \asymp n^{3/4}$.

Mathematics Subject Classification : 82B20 (Primary) 82B27, 60K35 (Secondary)

1 Introduction

1.1 Definition of the model

We study a one-dimensional chain of n spins, with periodic boundary conditions, where every spin interacts with its $2d_n$ nearest neighbours, with $0 < 2d_n < n$. Thus, we consider the Hamiltonian

$$H_n : (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto -\frac{1}{2d_n} \sum_{i=1}^n \sum_{j=1}^{d_n} x_i x_{i+j},$$

where we use the convention $x_{n+k} = x_k$ for all $k \in \{1, \dots, n\}$. Besides this Hamiltonian, the spins interact together through the following observable, which we think of as a “self-adjusted temperature parameter” (which is not really a temperature parameter because it is a function of the spins):

$$T_n : (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto \frac{x_1^2 + \dots + x_n^2}{n}.$$

Our model is then defined by the following probability distribution on \mathbb{R}^d :

$$d\mu_n(x_1, \dots, x_n) = \frac{1}{Z_n} \exp\left(-\frac{H_n(x_1, \dots, x_n)}{T_n(x_1, \dots, x_n)}\right) \mathbb{1}_{\{T_n > 0\}} \prod_{i=1}^n d\nu(x_i), \quad (1)$$

*Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France

where ν is the standard normal distribution, and Z_n is the normalization constant. In what follows, we will be interested in the behaviour, when n tends to infinity, of the “magnetization”

$$S_n(x_1, \dots, x_n) = x_1 + \dots + x_n,$$

when (x_1, \dots, x_n) is distributed according to μ_n .

1.2 Threshold phenomenon

In a previous work [For21], we studied this model in the particular case when the interaction range satisfies $2d_n \sim \lambda n$, with $\lambda \in (0, 1)$, and we proved, in this regime, the following convergence in distribution:

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \frac{\sqrt{2}}{\Gamma(1/4)} \exp\left(-\frac{z^4}{4}\right) dz.$$

This interesting behaviour was already observed by Cerf and Gorny [Gor14, CG16] in the mean-field case, which roughly corresponds to an interaction range $2d_n = n - 1$. In [For21], we also studied the other extreme case of a nearest neighbour interaction Hamiltonian, when $d_n = 1$, and we proved that, in this regime, we have the convergence in distribution

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \sqrt{2} + 1).$$

Therefore, the behaviour of the model depends on the interaction range d_n , but taking $2d_n \sim \lambda n$ and varying the constant λ is not enough to observe a change of behaviour. Thus, a natural idea is to consider an interaction range evolving like a power of n . In the present article, we prove that, when the interaction range is very large compared to $n^{3/4}$, the behaviour of S_n is the same as in the mean-field case:

Theorem 1. *If the interaction range d_n is such that $d_n/n^{3/4} \rightarrow \infty$ and $d_n/n \rightarrow 0$, then under the law μ_n defined by (1), we have the convergence in distribution*

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \frac{\sqrt{2}}{\Gamma(1/4)} \exp\left(-\frac{z^4}{4}\right) dz.$$

A natural question is now: is this exponent $3/4$ optimal? The following theorem provides a positive answer, showing that the limiting distribution changes when the interaction range d_n is of order $n^{3/4}$:

Theorem 2. *If the interaction range d_n is such that*

$$d_n \stackrel{n \rightarrow \infty}{\sim} \lambda n^{3/4} \quad \text{with} \quad \lambda > 0,$$

if $(Y_j)_{j \in \mathbb{Z}}$ is a family of i.i.d. standard normal variables, and if f is the density of the random variable

$$Z = \sqrt{2}Y_0 - \frac{3}{2\lambda^2\pi^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{Y_j^2}{j^2}, \quad (2)$$

then under the law μ_n defined by (1), we have the convergence in distribution

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \frac{f(z^2) dz}{\int_{\mathbb{R}} f(t^2) dt}.$$

This theorem indicates that there is a kind of phase transition phenomenon with respect to the interaction range. The obtained limiting law results from the competition between the two terms in (2). When $\lambda = \infty$, the second term disappears, leaving the distribution obtained in theorem 1.

1.3 Motivation: a Curie-Weiss model of self-organized criticality

The initial motivation to consider our model comes from the work of Cerf and Gorny [Gor14, CG16], who studied our model in the mean-field case, that is to say in the particular case $2d_n = n - 1$ (in fact, their Hamiltonian is equal to $1 - 1/n$ times our Hamiltonian, but this factor does not change the behaviour of the model). This mean-field model can be seen as a variant of the generalized Ising Curie-Weiss model [EN78, Ell06], which presents a phase transition for a critical value T_c of a temperature parameter. The idea behind the construction of the model of Cerf and Gorny is to replace this fixed temperature parameter T with a function $T_n(x_1, \dots, x_n)$ depending on the spins, which naturally concentrates around T_c when the number of spins tends to infinity. In [Gor14], Gorny proved theorem 1 but in the mean-field case $2d_n = n - 1$, and this result was extended to more general distributions of the spins in [CG16].

The interesting feature of this model is that it presents the same asymptotic behaviour as in the critical regime of the generalized Ising Curie-Weiss model, but without the need to finely tune a parameter like the temperature to a precise value. Hence, this toy model illustrates the phenomenon called “self-organized criticality”. This concept was coined by the physicists Bak, Tang and Wiesenfeld [BTW87] to explain the ubiquity of fractal structures in nature. The idea is that some physical systems naturally present a “critical-like” behaviour, with a self-similar geometry and power-law correlations, without the need to finely tune a parameter (e.g., the temperature) to a critical value.

1.4 Open questions

The intermediate regime In this article, we study the regime $d_n/n^{3/4} \rightarrow \infty$, and we prove that indeed something changes when d_n is of order $n^{3/4}$, but we leave open the question of the behaviour of the model in the regime $d_n = o(n^{3/4})$, which we have not studied yet. In fact, with our method, the study of each regime requires a careful control of the asymptotic behaviour of the eigenvalues and a well chosen change of variable to obtain a non-degenerate limit. So far, we have not managed to perform a similar computation for $d_n = o(n^{3/4})$, except in the much simpler case $d_n = 1$, treated in [For21].

More general distributions Our results only deal with the case of variables initially ν , that is, we restrict ourselves to the standard normal distribution ν . This restriction enables us to perform exact computations, as was done by Gorny in [Gor14], where he dealt with the Gaussian case of his model. However, the behaviour highlighted by Gorny for his mean-field model extends to a large variety of initial distributions satisfying an exponential moment condition [CG16], and therefore it may be possible to extend our result to a more general class of distributions for the spins.

Other interaction functions Instead of considering that each spin interacts with its $2d_n$ nearest neighbours, we could have constructed our model with an interaction function decreasing with the distance between two spins. For example, models with an coupling constant proportional to a power of the distance between two spins were extensively studied [ACCN88]. An other variant, called Kac-Ising models, consists of an interaction function (either with finite range or decreasing with the distance between spins) scaled by a factor γ , and one studies the limit of these models when $\gamma \rightarrow 0$ [KUH63, BZ97].

Random couplings An other way to build intermediate models between the nearest neighbour case and the mean-field case is to draw random coupling parameters between each pair of spins. This amounts to building a model on an Erdős-Rényi random graph, and it also leads to interesting threshold behaviours [KLS19, KLS20]. See section 1.4 of [For21] for a more detailed overview of this work.

1.5 Strategy of proof

The strategy to prove theorems 1 and 2 is similar to the strategy we followed in the previous article [For21]. However, the behaviour of the eigenvalues is quite different in the regimes we study here, which forces us to update most of the estimates, hence the presentation in two separate articles.

Change of variables to diagonalize the Hamiltonian. As was done in [For21], we let $P \in \mathcal{O}_n(\mathbb{R})$ be an orthonormal matrix such that the matrix of the quadratic form H_n writes tPDP , where D is the diagonal matrix with diagonal coefficients $-\alpha_1^n/2, \dots, -\alpha_n^n/2$, with

$$\forall j \in \{1, \dots, n\} \quad \alpha_j^n = \frac{1}{d_n} \sum_{m=1}^{d_n} \cos\left(\frac{2jm\pi}{n}\right).$$

Thus, with the change of variables

$$\varphi : (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto (y_1, \dots, y_n) = \left(\sum_{k=1}^n P_{j,k} x_k \right)_{1 \leq j \leq n},$$

we have

$$H_n(x_1, \dots, x_n) = -\frac{1}{2} \sum_{j=1}^n \alpha_j^n y_j^2, \quad T_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{j=1}^n y_j^2 \quad \text{and} \quad S_n = \sum_{j=1}^n x_j = \sqrt{n} y_n.$$

A competition between two terms. Given that $\alpha_n^n = 1$, we have

$$H_n(x_1, \dots, x_n) = -\frac{1}{2} \sum_{j=1}^n \alpha_j^n y_j^2 = -\frac{y_n^2}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \alpha_j^n y_j^2 = -\frac{S_n^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \alpha_j^n y_j^2. \quad (3)$$

In the mean-field case $2d_n = n - 1$ we have $\alpha_j^n = -1/(2d_n)$ for every $j \neq n$, so that the second term only contributes as $-S_n^2/(2n(n-1))$, leading to a mean-field interaction like in the model of Cerf and Gorny (see paragraph 1.3): the model behaves as if there was only the first term $-S_n^2/(2n)$. In fact, for a general interaction range d_n , there is a competition between the two terms in (3). As long as $d_n/n^{3/4} \rightarrow \infty$, what we will see is that the first term prevails, whence a behaviour similar to the mean-field case. The interesting change of behaviour intervenes when the second term becomes big enough to disturb the first term, and we will see that this happens when d_n is of order $n^{3/4}$.

Fourier inversion. Because we restricted our model to a Gaussian initial distribution of the spins, we are able to perform exact computations. In our previous article [For21], with a Fourier inversion, we obtained the following integral formula for the distribution (T_n, H_n, S_n) , which will also be useful in the present article:

Lemma 1 (lemma 5 of [For21]). *Assume that the parameter d_n is such that*

$$\lim_{n \rightarrow \infty} \frac{2d_n}{n} = \lambda \in [0, 1),$$

which is the case in the present article, since we assume that $d_n = o(n)$. Then, for n large enough, under the law μ_n defined by (1), the vector (T_n, H_n, S_n) admits the following density with respect to the Lebesgue

measure on \mathbb{R}^3 :

$$f_n : (x, y, z) \in \mathbb{R}^3 \mapsto \frac{1}{(2\pi)^{5/2} Z_n \sqrt{n}} \exp\left(-\frac{y}{x}\right) \mathbb{1}_\Omega\left(x, \frac{y}{n}, \frac{z}{n}\right) \\ \times \int_{\mathbb{R}^2} du dv \exp\left[-iu\left(x - \frac{z^2}{n^2}\right) - iv\left(y + \frac{z^2}{2n}\right) - \frac{z^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln\left(1 - \frac{2iu}{n} + iv \alpha_j^n\right)\right],$$

where Ω is the domain defined by

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : 2x > x - 2y > 2z^2 \right\}.$$

Saddle-point method. The remainder of this article consists in a careful implementation of the saddle-point method (see for example [Cop04]) to obtain the asymptotic behaviour of the above integral expression. This is done two times, once to prove theorem 1 in section 3, and a second time to prove theorem 2 in section 4. This method starts with the choice of a judicious change of variables, in order to obtain an integrand which has a non-degenerate limit when n tends to infinity. We then extend the integrand as a function with complex arguments, and we search for a saddle-point, that is to say, a critical point of the pointwise limit of the integrand in the complex plane. The core of the saddle-point method is then to move the integration contour, to make it pass through this saddle-point. The obtained integral then turns out to be well adapted to apply the dominated convergence lemma.

As often with the saddle-point method, a preliminary step is necessary to restrict the integration to a small domain close to the saddle-point, where everything goes fine. Without this restriction, one would not be able to verify the domination hypothesis, because of the behaviour of the integrand far from the saddle-point.

Control of the eigenvalues. At each step of the saddle-point method described above, the key ingredient is a precise control of the asymptotic behaviour of the eigenvalues α_j^n , which depends much on the regime of d_n we consider. Although the convergence result is the same in the two regimes $2d_n \sim \lambda n$ and $n^{3/4} \ll d_n \ll n$, the behaviour of the eigenvalues is quite different, and this is why we only deal here with the second regime, the second one being studied in [For21]. Indeed, in the regime $2d_n \sim \lambda n$, we have

$$\lim_{n \rightarrow \infty} \alpha_j^n = \text{sinc}(j\lambda\pi) = \frac{\sin(j\lambda\pi)}{j\lambda\pi}$$

for every fixed $j \geq 1$, whereas in the regime $n^{3/4} \ll d_n \ll n$, we have $\alpha_j^n \rightarrow 1$ for every fixed $j \geq 1$, which explains why it is necessary to consider a different scaling. Therefore, although the methods to study the different regimes are very similar, these slight differences in the behaviour of the eigenvalues induces differences in many technical points, which lead us to present these regimes separately, instead of trying to write a proof which would work for all the regimes, but which would end up maybe more confusing.

A Gaussian integration formula. As in [For21], we will also use the following formula, which follows from lemma 4 of [Gor14]:

$$\forall a \in \mathbb{R} \quad \forall b \in \mathbb{C} \quad \int_{\mathbb{R}} dx e^{bx - ax^2/2} = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right). \quad (4)$$

2 Asymptotics of the eigenvalues

Let us recall the expression of the eigenvalues α_j^n . We have $\alpha_n^n = 1$ and, for every $j \in \{1, \dots, n-1\}$,

$$\alpha_j^n = \frac{1}{d_n} \sum_{m=1}^{d_n} \cos\left(\frac{2jm\pi}{n}\right) = \frac{\cos(j(d_n+1)\pi/n) \sin(jd_n\pi/n)}{d_n \sin(j\pi/n)} \quad (5)$$

$$= \frac{1}{2d_n} \left(\frac{\sin((2d_n+1)j\pi/n)}{\sin(j\pi/n)} - 1 \right). \quad (6)$$

We will refer extensively to these different expressions for α_j^n throughout this section. We define the extrema

$$M_n = \max \left\{ \alpha_j^n : 1 \leq j \leq n-1 \right\} \quad \text{and} \quad m_n = \max \left\{ -\alpha_j^n : 1 \leq j \leq n-1 \right\}. \quad (7)$$

The eigenvalues are symmetric with respect to j , that is to say $\alpha_j^n = \alpha_{n-j}^n$ for every $j \in \{1, \dots, n-1\}$. It follows from (5) that

$$\forall j \in \left\{ 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \quad |\alpha_j^n| \leq \frac{1}{d_n \sin(j\pi/n)} \leq \frac{n}{2d_n j}, \quad (8)$$

which implies that

$$\sum_{j=1}^{n-1} |\alpha_j^n| \leq 2 \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{n}{2d_n j} = O\left(\frac{n \ln n}{d_n}\right). \quad (9)$$

As explained in section 1.5, we have $\alpha_j^n \rightarrow 1$ when $n \rightarrow \infty$, for every fixed $j \geq 1$. The following lemma controls the speed of this convergence, which will be a key point in our computations.

Lemma 2. *Assume that $d_n \rightarrow \infty$ with $d_n/n \rightarrow 0$. Then for every fixed $j \geq 1$ we have, when $n \rightarrow \infty$,*

$$1 - \alpha_j^n \underset{n \rightarrow \infty}{\sim} \frac{2d_n^2 j^2 \pi^2}{3n^2}. \quad (10)$$

If moreover $d_n/n^{2/3} \rightarrow \infty$, then there exists $K > 0$ such that, for $n \geq 3$ and $1 \leq j \leq \lfloor n/2 \rfloor$,

$$\left| \frac{1}{1 - \alpha_j^n} - \frac{3n^2}{2d_n^2 j^2 \pi^2} \right| \leq K \quad (11)$$

and

$$\left| \frac{\alpha_j^n}{1 - \alpha_j^n} - \frac{3n^2}{2d_n^2 j^2 \pi^2} \right| \leq K. \quad (12)$$

Proof. Let $(d_n)_n$ be such that $d_n \rightarrow \infty$ and $d_n/n \rightarrow 0$. Let $j \geq 1$ and let $n \geq 2j$. We can write, with a uniform O with respect to j and n ,

$$\sin\left(\frac{(2d_n+1)j\pi}{n}\right) = \frac{(2d_n+1)j\pi}{n} \left[1 - \frac{(2d_n+1)^2 j^2 \pi^2}{6n^2} + O\left(\frac{d_n^4 j^4}{n^4}\right) \right].$$

Besides, since $j\pi/n \leq \pi/2$, we have

$$\frac{1}{\sin(j\pi/n)} = \frac{n}{j\pi} \left[1 + O\left(\frac{j^2}{n^2}\right) \right].$$

Using our formula (6) for α_j^n , we obtain that, for $1 \leq j \leq \lfloor n/2 \rfloor$,

$$\begin{aligned}
\alpha_j^n &= \frac{\sin((2d_n + 1)j\pi/n)}{2d_n \sin(j\pi/n)} - \frac{1}{2d_n} \\
&= \frac{(2d_n + 1)j\pi/n}{2d_n j\pi/n} \left[1 - \frac{(2d_n + 1)^2 j^2 \pi^2}{6n^2} + O\left(\frac{d_n^4 j^4}{n^4}\right) \right] \times \left[1 + O\left(\frac{j^2}{n^2}\right) \right] - \frac{1}{2d_n} \\
&= \left(1 + \frac{1}{2d_n}\right) \left[1 - \frac{(2d_n + 1)^2 j^2 \pi^2}{6n^2} + O\left(\frac{j^2}{n^2}\right) + O\left(\frac{d_n^4 j^4}{n^4}\right) \right] - \frac{1}{2d_n} \\
&= 1 - \frac{(2d_n + 1)^2 j^2 \pi^2}{6n^2} + O\left(\frac{d_n j^2}{n^2}\right) + O\left(\frac{d_n^4 j^4}{n^4}\right) \\
&= 1 - \frac{2d_n^2 j^2 \pi^2}{3n^2} + O\left(\frac{d_n j^2}{n^2}\right) + O\left(\frac{d_n^4 j^4}{n^4}\right). \tag{13}
\end{aligned}$$

We can deduce that

$$1 - \alpha_j^n = \frac{2d_n^2 j^2 \pi^2}{3n^2} \left[1 + O\left(\frac{1}{d_n}\right) + O\left(\frac{d_n^2 j^2}{n^2}\right) \right],$$

with uniform O symbols for all $j \leq \lfloor n/2 \rfloor$. For a fixed j , since $d_n \rightarrow \infty$ and $d_n/n \rightarrow 0$, we obtain the asymptotic formula (10). We now assume that, in addition, we have $d_n/n^{2/3} \rightarrow \infty$. We then have

$$1 - \alpha_j^n = \frac{2d_n^2 j^2 \pi^2}{3n^2} \left[1 + O\left(\frac{d_n^2 j^2}{n^2}\right) \right]. \tag{14}$$

To obtain (11), we wish to take the inverse of this development. To this end, we distinguish between two cases, and we assume in the first place that $j \leq n/(2d_n + 1)$. In this case, we will show that the quantity between brackets in (14) is bounded away from 0. Let us go back to the formula (6) which reads

$$\alpha_j^n = \frac{\sin((2d_n + 1)j\pi/n)}{2d_n \sin(j\pi/n)} - \frac{1}{2d_n}. \tag{15}$$

A straightforward function study shows that, for every $x \geq 0$,

$$x \left(1 - \frac{x^2}{6}\right) \leq \sin x \leq x \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right). \tag{16}$$

From this we deduce that

$$\sin\left(\frac{j\pi}{n}\right) \geq \frac{j\pi}{n} \left(1 - \frac{j^2 \pi^2}{6n^2}\right).$$

Yet, for $j \leq n/(2d_n + 1)$, we have

$$\frac{j^2 \pi^2}{6n^2} \leq \frac{\pi^2}{6(2d_n + 1)^2} \leq \frac{1}{2}$$

for n large enough, because $d_n \rightarrow \infty$. Therefore, we get

$$\sin\left(\frac{j\pi}{n}\right)^{-1} \leq \frac{n}{j\pi} \left(1 - \frac{j^2 \pi^2}{6n^2}\right)^{-1} \leq \frac{n}{j\pi} \left(1 + \frac{j^2 \pi^2}{3n^2}\right), \tag{17}$$

using the convexity of the function $u \mapsto (1-u)^{-1}$ over $[0, 1/2]$. We now use the other inequality in (16) to write, for $j \leq n/(2d_n + 1)$,

$$\begin{aligned} \sin\left(\frac{(2d_n + 1)j\pi}{n}\right) &\leq \frac{(2d_n + 1)j\pi}{n} \left(1 - \frac{(2d_n + 1)^2 j^2 \pi^2}{6n^2} + \frac{(2d_n + 1)^4 j^4 \pi^4}{120n^4}\right) \\ &\leq \frac{(2d_n + 1)j\pi}{n} \left[1 - \left(\frac{\pi^2}{6} - \frac{\pi^4}{120}\right) \frac{(2d_n + 1)^2 j^2}{n^2}\right] \\ &\leq \frac{(2d_n + 1)j\pi}{n} \left(1 - \frac{(2d_n + 1)^2 j^2}{2n^2}\right). \end{aligned} \quad (18)$$

Using (17) and (18) in our formula (15) yields

$$\begin{aligned} \alpha_j^n &\leq \frac{2d_n + 1}{2d_n} \left(1 + \frac{j^2 \pi^2}{3n^2}\right) \left(1 - \frac{(2d_n + 1)^2 j^2}{2n^2}\right) - \frac{1}{d_n} \\ &\leq \left(1 + \frac{1}{2d_n}\right) \left(1 + \frac{j^2 \pi^2}{3n^2} - \frac{(2d_n + 1)^2 j^2}{2n^2}\right) - \frac{1}{d_n} \\ &= 1 - \left(1 + \frac{1}{2d_n}\right) \left(1 - \frac{2\pi^2}{3(2d_n + 1)^2}\right) \frac{(2d_n + 1)^2 j^2}{2n^2} \\ &= 1 - \left(1 + \frac{1}{2d_n}\right)^3 \left(1 - \frac{2\pi^2}{3(2d_n + 1)^2}\right) \frac{2d_n^2 j^2}{n^2}. \end{aligned}$$

Since $d_n \rightarrow \infty$, this implies that, for n large enough, we have

$$\forall j \leq \frac{n}{2d_n + 1} \quad 1 - \alpha_j^n \geq \frac{d_n^2 j^2}{n^2}.$$

This allows us to take the inverse of the development (14), yielding, with a uniform O symbol valid for all $j \leq n/(2d_n + 1)$,

$$\frac{1}{1 - \alpha_j^n} = \frac{3n^2}{2d_n^2 j^2 \pi^2} \left[1 + O\left(\frac{d_n^2 j^2}{n^2}\right)\right]. \quad (19)$$

Going back to our development of α_j^n in (13), we also have, for $j \leq n/(2d_n + 1)$,

$$\alpha_j^n = 1 + O\left(\frac{d_n^2 j^2}{n^2}\right) + O\left(\frac{d_n^4 j^4}{n^4}\right) = 1 + O\left(\frac{d_n^2 j^2}{n^2}\right).$$

Combining this with (19), we get

$$\frac{\alpha_j^n}{1 - \alpha_j^n} = \frac{3n^2}{2d_n^2 j^2 \pi^2} \left[1 + O\left(\frac{d_n^2 j^2}{n^2}\right)\right] = \frac{3n^2}{2d_n^2 j^2 \pi^2} + O(1), \quad (20)$$

with a uniform O valid for all $n \geq 3$ and $1 \leq j \leq n/(2d_n + 1)$. We now turn to the second case, and we take j such that

$$\frac{n}{2d_n + 1} < j \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad (21)$$

In this case, our formula (5) for α_j^n implies that

$$|\alpha_j^n| \leq \frac{1}{d_n \sin(j\pi/n)} \leq \frac{1}{d_n \sin(\pi/(2d_n + 1))} \stackrel{n \rightarrow \infty}{\sim} \frac{2}{\pi} < 1.$$

Therefore, uniformly for all j satisfying (21), we have

$$\frac{1}{1 - \alpha_j^n} = O(1) = \frac{3n^2}{2d_n^2 j^2 \pi^2} + O(1). \quad (22)$$

Since $|\alpha_j^n| \leq 1$ for all j , this implies that, uniformly for all j satisfying (21),

$$\frac{\alpha_j^n}{1 - \alpha_j^n} = O(1) = \frac{3n^2}{2d_n^2 j^2 \pi^2} + O(1). \quad (23)$$

In the end, combining our two cases (19) and (22), we obtain the existence of $K > 0$ such that (11) holds. Similarly, the upper bound (12) follows from (20) and (23). \square

Regarding the extreme eigenvalues, we prove the following result:

Lemma 3. *If $d_n \rightarrow \infty$ and $d_n/n \rightarrow 0$, then we have*

$$1 - M_n \stackrel{n \rightarrow \infty}{\sim} \frac{2\pi^2 d_n^2}{3n^2} \quad \text{and} \quad \liminf_{n \rightarrow \infty} m_n > 0.$$

Proof. Let us consider the function

$$u_n : t \in (0, \pi) \mapsto \frac{\sin t}{\sin(t/(2d_n + 1))}.$$

This function is differentiable and, for all $t \in (0, \pi)$, we have

$$u_n'(t) = \sin\left(\frac{t}{2d_n + 1}\right)^{-2} \left[\cos t \sin\left(\frac{t}{2d_n + 1}\right) - \frac{\sin t}{2d_n + 1} \cos\left(\frac{t}{2d_n + 1}\right) \right].$$

The sign of this derivative is governed by

$$u_n'(t) \leq 0 \quad \iff \quad \cot t \leq \frac{1}{2d_n + 1} \cot\left(\frac{t}{2d_n + 1}\right).$$

If $t \in [\pi/2, \pi)$, then we have $\cot t \leq 0$ whence $u_n'(t) \leq 0$. Otherwise, if $t \in (0, \pi/2)$, then we have

$$u_n'(t) \leq 0 \quad \iff \quad \tan t \geq (2d_n + 1) \tan\left(\frac{t}{2d_n + 1}\right),$$

and the latter inequality holds true, as a consequence of the convexity of the tangent function over $(0, \pi/2)$. Therefore, the function u_n is decreasing on the interval $(0, \pi)$. Yet, for $1 \leq j < n/(2d_n + 1)$, the formula (6) reads

$$\alpha_j^n = \frac{1}{2d_n} \left[u_n\left(\frac{(2d_n + 1)j\pi}{n}\right) - 1 \right].$$

The function u_n being decreasing, we can deduce that $\alpha_j^n \leq \alpha_1^n$ for all $j < n/(2d_n + 1)$. What's more, it stems from (5), the other formula for α_j^n , that for every integer j ,

$$\frac{n}{2d_n + 1} \leq j \leq n - \frac{n}{2d_n + 1} \quad \Rightarrow \quad |\alpha_j^n| \leq \left[d_n \sin\left(\frac{\pi}{2d_n + 1}\right) \right]^{-1} \stackrel{n \rightarrow \infty}{\sim} \frac{2}{\pi} < 1.$$

Hence, for n large enough, we have $M_n = \alpha_1^n$, and thus the equivalent of $1 - M_n$ follows from lemma 2. We now deal with the case of m_n , and we write

$$j(n) = \left\lfloor \frac{3n}{2(2d_n + 1)} \right\rfloor.$$

Going back to the formula (6) for α_j^n , we have

$$\alpha_{j(n)}^n = \frac{\sin((2d_n + 1)j(n)\pi/n)}{2d_n \sin(j(n)\pi/n)} - \frac{1}{2d_n} = \frac{\sin(3\pi/2) + o(1)}{2d_n(3\pi/(4d_n))(1 + o(1))} + o(1) \stackrel{n \rightarrow \infty}{\sim} \frac{\sin(3\pi/2)}{3\pi/2} = -\frac{2}{3\pi},$$

whence

$$\liminf_{n \rightarrow \infty} m_n \geq \frac{2}{3\pi} > 0,$$

which concludes the proof of this lemma \square

Next, we prove that the proportion of eigenvalues which are positive or negative stays away from 0 and 1:

Lemma 4. *If the interaction range d_n is such that $d_n \rightarrow \infty$ and $d_n/n \rightarrow 0$, then there exists $K > 0$ such that*

$$\forall n \geq 3 \quad \forall \varepsilon \in \{-1, 1\} \quad \left| \left\{ j \in \{1, \dots, n-1\} : \varepsilon \alpha_j^n \geq 0 \right\} \right| \geq Kn.$$

Proof. Let $n \geq 3$ and $\varepsilon = (-1)^\eta$, with $\eta \in \{0, 1\}$. Consider the set

$$E = \bigcup_{1 \leq k \leq d_n - 1} E_k \quad \text{where} \quad E_k = \mathbb{N} \cap \left[\frac{(2k + \eta)n}{2d_n}, \frac{(2k + \eta + 1)n}{2(d_n + 1)} \right].$$

We have $E \subset \{1, \dots, n-1\}$ and, if $j \in E_k$, then

$$k\pi \leq \frac{j d_n \pi}{n} \leq k\pi + \pi,$$

implying that

$$(-1)^k \sin\left(\frac{j d_n \pi}{n}\right) \geq 0.$$

Similarly, for every $j \in E_k$, we have

$$k\pi + \frac{\eta\pi}{2} \leq \frac{j(d_n + 1)\pi}{n} \leq k\pi + \frac{\eta\pi}{2} + \frac{\pi}{2},$$

whence

$$(-1)^{k+\eta} \cos\left(\frac{j(d_n + 1)\pi}{n}\right) \geq 0.$$

Using our formula (5) for α_j^n , we obtain that

$$\forall j \in E_k \quad \varepsilon \alpha_j^n = \frac{(-1)^{k+\eta} \cos(j(d_n + 1)\pi/n) (-1)^k \sin(j d_n \pi/n)}{d_n \sin(j\pi/n)} \geq 0.$$

In addition, for every $k \in \{1, \dots, \lfloor d_n/4 \rfloor\}$ we have

$$\begin{aligned} |E_k| &\geq \frac{(2k + \eta + 1)n}{2(d_n + 1)} - \frac{(2k + \eta)n}{2d_n} - 2 \\ &= \frac{n}{2(d_n + 1)} - \frac{\eta n}{2d_n(d_n + 1)} - \frac{kn}{d_n(d_n + 1)} - 2 \\ &\geq \frac{n}{4(d_n + 1)} - \frac{n}{2d_n(d_n + 1)} - 2. \end{aligned}$$

Therefore, we get

$$|E| \geq \left\lfloor \frac{d_n}{4} \right\rfloor \left(\frac{n}{4(d_n + 1)} - \frac{n}{2d_n(d_n + 1)} - 2 \right) \stackrel{n \rightarrow \infty}{\sim} \frac{n}{16},$$

which concludes the proof. \square

We now turn to an estimate which will be useful in section 4.6:

Lemma 5. *If the parameter d_n is such that $d_n/n^{2/3} \rightarrow \infty$, then we have*

$$\sum_{j=1}^{n-1} \ln(1 - \alpha_j^n) = O\left(\frac{n \ln n}{d_n}\right).$$

Proof. First, by symmetry of the eigenvalues, it is sufficient to show that

$$S_1 = O\left(\frac{n \ln n}{d_n}\right) \quad \text{and} \quad S_2 = O\left(\frac{n \ln n}{d_n}\right),$$

where

$$S_1 = \sum_{j=1}^{\lfloor n/d_n \rfloor} \ln(1 - \alpha_j^n) \quad \text{and} \quad S_2 = \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \ln(1 - \alpha_j^n).$$

For the first term, we write

$$|S_1| \leq \left\lfloor \frac{n}{d_n} \right\rfloor \left(|\ln(1 - M_n)| \vee (\ln 2) \right).$$

Yet, it follows from lemma 3 that

$$1 - M_n \stackrel{n \rightarrow \infty}{\sim} \frac{2\pi^2 d_n^2}{3n^2},$$

whence

$$S_1 = O\left(\frac{n \ln n}{d_n}\right). \tag{24}$$

To deal with the second term, we use the upper bound (8) whereby

$$\forall j \in \left\{ \left\lfloor \frac{n}{d_n} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \quad |\alpha_j^n| \leq \frac{n}{2d_{n,j}} \leq \frac{1}{2}.$$

The logarithm function being Lipschitz-continuous on $[1/2, 3/2]$, we deduce that

$$S_2 = O\left(\sum_{j=1}^{n-1} |\alpha_j^n|\right) = O\left(\frac{n \ln n}{d_n}\right), \tag{25}$$

where we have used the upper bound (9). Combining (24) and (25) yields the desired result. \square

3 Proof of theorem 1

This section is devoted to the proof of theorem 1. Hence, we assume that the interaction range d_n is such that $d_n/n^{3/4} \rightarrow \infty$ and $d_n/n \rightarrow 0$, we fix a bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ and we study the limit

$$\lim_{n \rightarrow \infty} \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right].$$

3.1 Exact integral expression

In our previous work [For21], we noticed that, under the law μ_n , the behaviour of T_n is simply given by (see lemma 11 of [For21]):

$$\sqrt{n}(T_n - 1) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 2).$$

Since $d_n/n^{3/4} \rightarrow \infty$, we can deduce that

$$\lim_{n \rightarrow \infty} \mu_n \left(|T_n - 1| > \frac{d_n}{n^{5/4}} \right) = 0. \quad (26)$$

Therefore, it is enough to study the limit of

$$E_n = \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \mathbb{1}_{|T_n - 1| \leq d_n/n^{5/4}} \right]. \quad (27)$$

With the density given by lemma 1, we can write, for n larger than a certain rank n_I ,

$$\begin{aligned} E_n &= \int_{1-d_n/n^{5/4}}^{1+d_n/n^{5/4}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz g \left(\frac{z}{n^{3/4}} \right) f_n(x, y, z) \\ &= \frac{1}{(2\pi)^{5/2} Z_n \sqrt{n}} \int_{1-d_n/n^{5/4}}^{1+d_n/n^{5/4}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz g \left(\frac{z}{n^{3/4}} \right) \exp \left(-\frac{y}{x} \right) \mathbb{1}_{2x > x - 2y/n > 2z^2/n^2} \\ &\quad \times \int_{\mathbb{R}^2} du dv \exp \left[-iu \left(x - \frac{z^2}{n^2} \right) - iv \left(y + \frac{z^2}{2n} \right) - \frac{z^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{n} + iv \alpha_j^n \right) \right]. \end{aligned}$$

We now proceed to the change of variables

$$x = 1 + \frac{x'}{\sqrt{n}}, \quad y = \frac{n^2 y'}{d_n^2} - \frac{z'^2 \sqrt{n}}{2}, \quad z = z' n^{3/4}, \quad u = u' \sqrt{n}, \quad v = \frac{d_n^2 v'}{n^2},$$

which yields, for $n \geq n_I$,

$$E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}^2} du dv e^{A_n(x, y, z, u, v)}, \quad (28)$$

where the integration domain \mathcal{D}_n is the following:

$$\mathcal{D}_n = \left\{ (x, y, z) \in \mathbb{R}^3 : |x| \leq \frac{d_n}{n^{3/4}} \quad \text{and} \quad 2\sqrt{n} + 2x > \sqrt{n} + x - \frac{2n^{3/2}}{d_n^2} y + z^2 > 2z^2 \right\}, \quad (29)$$

and where we let, for $(x, y, z) \in \mathcal{D}_n$ and $(u, v) \in \mathbb{R}^2$,

$$A_n = \frac{z^2 \sqrt{n} - 2yn^2/d_n^2}{2(1 + x/\sqrt{n})} - iu\sqrt{n} - iu(x - z^2) - ivy - \frac{z^2 \sqrt{n}}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{\sqrt{n}} + \frac{iv d_n^2 \alpha_j^n}{n^2} \right). \quad (30)$$

Our change of variable on y is guided by the assumption that, when writing the Hamiltonian as

$$H_n = -\frac{S_n^2}{2n} + R_n,$$

the first term will be of order \sqrt{n} and the second term will be of order n^2/d_n^2 , which is negligible compared to \sqrt{n} because $d_n/n^{3/4} \rightarrow \infty$. This is why the behaviour is the same as in the mean-field model, because the term $-S_n^2/(2n)$ prevails over the remainder R_n . This heuristics comes from the following computations, where we see that this is the right scaling to obtain a non-degenerate limit of the integrand when n tends to infinity. Let us now rewrite the expression (30) as

$$A_n = -\frac{xz^2}{2(1+x/\sqrt{n})} + \frac{yn^2/d_n^2}{1+x/\sqrt{n}} - iu\sqrt{n} - iu(x-z^2) - ivy - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{\sqrt{n}} + \frac{iv d_n^2 \alpha_j^n}{n^2} \right).$$

To obtain a non-degenerate limit, we must get rid of the diverging term yn^2/d_n^2 . To this end, we wish to move the integration contour of the variable v from \mathbb{R} to

$$v \in v_0 + \mathbb{R} \quad \text{where} \quad v_0 = \frac{in^2}{d_n^2}. \quad (31)$$

To justify this change of integration contour, we proceed as in section 4.2 of [For21]. We fix $n \geq n_I$ and $(x, y, z, u) \in \mathcal{D}_n \times \mathbb{R}$. The function $v \mapsto A_n(x, y, z, u, v)$ can be extended into a holomorphic function on the open set

$$\left\{ v \in \mathbb{C} : -\frac{n^2}{d_n^2 m_n} < \Im v < \frac{n^2}{d_n^2 M_n} \right\} \supset \left\{ v \in \mathbb{C} : 0 \leq \Im v \leq \frac{n^2}{d_n^2} \right\}.$$

As was done in [For21], it can be easily checked that the change of contour is legal, for n large enough. The formula (28) then becomes, for n large enough,

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv e^{F_n(x, y, z, u, v) + G_n(u, v)}, \quad (32)$$

where the constant C_n is given by

$$C_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \exp \left(-\frac{1}{2} \sum_{j=1}^{n-1} \ln(1 - \alpha_j^n) \right) = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \prod_{j=1}^{n-1} \frac{1}{\sqrt{1 - \alpha_j^n}}, \quad (33)$$

and the functions F_n and G_n are defined by

$$F_n(x, y, z, u, v) = -iu(x-z^2) - ivy - \frac{xz^2 - 2xy n^{3/2}/d_n^2}{2(1+x/\sqrt{n})}$$

and

$$G_n(u, v) = -iu\sqrt{n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{\sqrt{n}(1-\alpha_j^n)} + \frac{iv d_n^2 \alpha_j^n}{n^2(1-\alpha_j^n)} \right). \quad (34)$$

3.2 Pointwise limit of the integrand

We now study the pointwise convergence of these two functions F_n and G_n . Our assumption $d_n/n^{3/4} \rightarrow \infty$ implies that

$$\forall (x, y, z, u, v) \in \mathbb{R}^3 \times \mathbb{C}^2 \quad \lim_{n \rightarrow \infty} F_n(x, y, z, u, v) = F_\infty(x, y, z, u, v),$$

with

$$F_\infty(x, y, z, u, v) = -iu(x - z^2) - ivy - \frac{xz^2}{2}.$$

We now turn to the second function G_n . For the quantity $G_n(u, v)$ to be well defined, it is sufficient that

$$\forall j \in \{1, \dots, n-1\} \quad \Re \left(1 - \frac{2iu}{\sqrt{n}(1 - \alpha_j^n)} + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) > 0,$$

which is equivalent to

$$\forall j \in \{1, \dots, n-1\} \quad \frac{n^2}{d_n^2} + \frac{2n^{3/2}}{d_n^2} \Im u - \left(\Im v + \frac{n^2}{d_n^2} \right) \alpha_j^n > 0.$$

Rewriting this in terms of the extreme eigenvalues M_n and $-m_n$ defined by (7), this becomes

$$\frac{n^2}{d_n^2} + \frac{2n^{3/2}}{d_n^2} \Im u + \left(\Im v + \frac{n^2}{d_n^2} \right) m_n > 0 \quad \text{and} \quad \frac{n^2}{d_n^2} + \frac{2n^{3/2}}{d_n^2} \Im u - \left(\Im v + \frac{n^2}{d_n^2} \right) M_n > 0,$$

which amounts to

$$-\frac{n^2(1 + m_n)}{d_n^2 m_n} - \frac{2n^{3/2}}{d_n^2 m_n} \Im u < \Im v < \frac{n^2(1 - M_n)}{d_n^2 M_n} - \frac{2n^{3/2}}{d_n^2 M_n} \Im u. \quad (35)$$

Now recall the result of lemma 3 according to which

$$\lim_{n \rightarrow \infty} \frac{n^2(1 + m_n)}{d_n^2 m_n} = +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{n^2(1 - M_n)}{d_n^2 M_n} = \frac{2\pi^2}{3}.$$

What is more, for any fixed $u \in \mathbb{C}$, we have

$$\lim_{n \rightarrow \infty} \frac{2n^{3/2}}{d_n^2 m_n} \Im u = 0 = \lim_{n \rightarrow \infty} \frac{2n^{3/2}}{d_n^2 M_n} \Im u,$$

because $d_n/n^{3/4} \rightarrow \infty$. Therefore, for every $(u, v) \in \mathbb{C} \times V$, with

$$V = \left\{ v \in \mathbb{C} : \Im v < \frac{2\pi^2}{3} \right\},$$

the condition (35) is satisfied for n large enough. We can now state and prove the following result:

Lemma 6. *We have the pointwise convergence:*

$$\forall (u, v) \in \mathbb{C} \times V \quad \lim_{n \rightarrow \infty} G_n(u, v) = G_\infty(u, v),$$

where the function G_∞ is given by

$$G_\infty : (u, v) \in \mathbb{C} \times V \mapsto -u^2 - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3iv}{2\pi^2 j^2} \right).$$

Proof. Let us fix $(u, v) \in \mathbb{C} \times V$. We first study the limit of $G_n(0, v)$. Let us write

$$h = \frac{1}{2} \wedge \left(\frac{1}{2} - \frac{3\Im v}{4\pi^2} \right),$$

which is positive since $v \in V$. It stems from lemma 3 on the asymptotics of M_n and m_n that

$$\lim_{n \rightarrow \infty} \Re \left(1 + \frac{iv d_n^2 M_n}{n^2(1 - M_n)} \right) = 1 - \frac{3 \Im v}{2\pi^2} \geq 2h,$$

while

$$\lim_{n \rightarrow \infty} \Re \left(1 - \frac{iv d_n^2 m_n}{n^2(1 + m_n)} \right) = 1 \geq 2h.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \quad \forall j \in \{1, \dots, n-1\} \quad \Re \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) \geq h. \quad (36)$$

Let $n \geq n_0$. Using the symmetry relation $\alpha_{n-j}^n = \alpha_j^n$, we can write

$$G_n(0, v) - G_\infty(0, v) = \delta_n^1(v) + \delta_n^2(v) - \delta_n^3(v),$$

with

$$\begin{aligned} \delta_n^1(v) &= \sum_{j=1}^{\lfloor n/d_n \rfloor} \left[-\ln \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) + \ln \left(1 + \frac{3iv}{2\pi^2 j^2} \right) \right], \\ \delta_n^2(v) &= -\frac{1}{2} \sum_{j=\lfloor n/d_n \rfloor+1}^{n-\lfloor n/d_n \rfloor-1} \ln \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right), \\ \delta_n^3(v) &= -\sum_{j=\lfloor n/d_n \rfloor+1}^{+\infty} \ln \left(1 + \frac{3iv}{2\pi^2 j^2} \right). \end{aligned}$$

Control of the first term: We start with δ_n^1 . For every $j \leq \lfloor n/d_n \rfloor$, we have

$$\Re \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) \geq h \quad \text{and} \quad \Re \left(1 + \frac{3iv}{2\pi^2 j^2} \right) \geq 1 \wedge \left(1 - \frac{3 \Im v}{2\pi^2} \right) \geq h.$$

Thus, it follows from Taylor's theorem that

$$\left| -\ln \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) + \ln \left(1 + \frac{3iv}{2\pi^2 j^2} \right) \right| \leq \frac{1}{h} \left| \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} - \frac{3iv}{2\pi^2 j^2} \right|.$$

From this we deduce that

$$|\delta_n^1(v)| \leq \frac{d_n^2 |v|}{h n^2} \sum_{j=1}^{\lfloor n/d_n \rfloor} \left| \frac{\alpha_j^n}{1 - \alpha_j^n} - \frac{3n^2}{2d_n^2 \pi^2 j^2} \right|.$$

Following lemma 2, we obtain

$$|\delta_n^1(v)| = O \left(\frac{d_n^2}{n^2} \sum_{j=1}^{\lfloor n/d_n \rfloor} 1 \right) = O \left(\frac{d_n}{n} \right) = o(1).$$

Control of the second term: We now deal with δ_n^2 . If n is odd, using the symmetry of the eigenvalues, we can write

$$\delta_n^2(v) = - \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \ln \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right).$$

Otherwise, if n is even, there is a term counted twice in the above sum, whence

$$\delta_n^2(v) = \frac{1}{2} \ln \left(1 + \frac{iv d_n^2 \alpha_{n/2}^n}{n^2(1 - \alpha_{n/2}^n)} \right) - \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \ln \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right).$$

Yet, we have $\lim_{n \rightarrow \infty} \alpha_{n/2}^n = 0$, whence whatever the parity of n ,

$$\delta_n^2(v) = o(1) - \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \ln \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right).$$

Using again Taylor's theorem, we get

$$\delta_n^2(v) = o(1) + O \left(\frac{d_n^2}{n^2} \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{\alpha_j^n}{1 - \alpha_j^n} \right) + O \left(\frac{d_n^4}{n^4} \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{(\alpha_j^n)^2}{(1 - \alpha_j^n)^2} \right). \quad (37)$$

In the first sum, we write

$$\frac{\alpha_j^n}{1 - \alpha_j^n} = \alpha_j^n + \frac{(\alpha_j^n)^2}{1 - \alpha_j^n}.$$

Recalling the upper bound (8), we have

$$\forall j \in \left\{ \left\lfloor \frac{n}{d_n} \right\rfloor + 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \quad |\alpha_j^n| \leq \frac{n}{2d_n j} \leq \frac{1}{2}, \quad (38)$$

whence

$$\left| \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{\alpha_j^n}{1 - \alpha_j^n} \right| \leq \left| \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \alpha_j^n \right| + 2 \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} (\alpha_j^n)^2.$$

Thus, the equation (37) becomes

$$\delta_n^2(v) = o(1) + O \left(\frac{d_n^2}{n^2} \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \alpha_j^n \right) + O \left(\frac{d_n^2}{n^2} \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} (\alpha_j^n)^2 \right). \quad (39)$$

We now write, using the formula (6) for α_j^n ,

$$\left| \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \alpha_j^n \right| \leq \frac{n}{2d_n} + \frac{1}{2d_n} \left| \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{\sin((2d_n + 1)j\pi/n)}{\sin(j\pi/n)} \right|. \quad (40)$$

We then perform an Abel transform, letting, for $j \geq 0$,

$$\sigma_{n,j} = \sum_{k=0}^j \sin((2d_n + 1)k\pi/n).$$

We have

$$|\sigma_{n,j}| = \left| \frac{\sin((2d_n+1)j\pi/(2n)) \sin((2d_n+1)(j+1)\pi/(2n))}{\sin((2d_n+1)\pi/(2n))} \right| \leq \frac{1}{\sin((2d_n+1)\pi/(2n))} \leq \frac{n}{2d_n}.$$

This allows us to write

$$\begin{aligned} \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{\sin((2d_n+1)j\pi/n)}{\sin(j\pi/n)} &= \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{\sigma_{n,j} - \sigma_{n,j-1}}{\sin(j\pi/n)} \\ &= \sum_{j=\lfloor n/d_n \rfloor}^{\lfloor n/2 \rfloor - 1} \sigma_{n,j} \left(\frac{1}{\sin(j\pi/n)} - \frac{1}{\sin((j+1)\pi/n)} \right) - \frac{\sigma_{n,\lfloor n/d_n \rfloor}}{\sin(\lfloor n/d_n \rfloor \pi/n)} + \frac{\sigma_{n,\lfloor n/2 \rfloor}}{\sin(\lfloor n/2 \rfloor \pi/n)}. \end{aligned}$$

Therefore, we have

$$\left| \sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{\sin((2d_n+1)j\pi/n)}{\sin(j\pi/n)} \right| \leq \frac{n/d_n}{\sin(\lfloor n/d_n \rfloor \pi/n)} + \frac{n/d_n}{\sin(\lfloor n/2 \rfloor \pi/n)} = O(n).$$

Our upper bound (40) thus becomes

$$\sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} \alpha_j^n = O\left(\frac{n}{d_n}\right). \quad (41)$$

In addition, going back to the upper bound (38), we can write

$$\sum_{j=\lfloor n/d_n \rfloor + 1}^{\lfloor n/2 \rfloor} (\alpha_j^n)^2 \leq \sum_{j=\lfloor n/d_n \rfloor + 1}^{+\infty} \left(\frac{n}{2d_n j}\right)^2 \leq \frac{n^2}{4d_n^2} \int_{\lfloor n/d_n \rfloor}^{+\infty} \frac{dt}{t^2} \stackrel{n \rightarrow \infty}{\sim} \frac{n}{4d_n}. \quad (42)$$

Plugging (41) and (42) into (39) yields

$$\delta_n^2(v) = o(1) + O\left(\frac{d_n}{n}\right) = o(1).$$

Control of the third term: The series

$$\sum_{j \geq 1} \ln \left(1 - \frac{3iv}{\pi^2 j^2} \right)$$

being convergent, we know that $\delta_n^3(v)$ tends to 0 when n tends to infinity. Therefore, we have proved

$$\lim_{n \rightarrow \infty} G_n(0, v) = G_\infty(0, v).$$

Dependence on u : Let us fix $u \in \mathbb{C}$. Using again lemma 3 about the extreme eigenvalues, we have

$$\max_{1 \leq j \leq n-1} \left| \frac{2iu}{\sqrt{n}(1 - \alpha_j^n)} \right| = \frac{2|u|}{\sqrt{n}(1 - M_n)} \stackrel{n \rightarrow \infty}{\sim} \frac{3n^{3/2}|u|}{\pi^2 d_n^2} \stackrel{n \rightarrow \infty}{\rightarrow} 0,$$

since $d_n/n^{3/4} \rightarrow \infty$. This entails that, for n large enough,

$$\left| \frac{2iu}{\sqrt{n}(1-M_n)} \right| \leq \frac{h}{2}.$$

Combining this with (36), we obtain that, for n large enough,

$$\forall t \in [0, 1] \quad \forall j \in \{1, \dots, n-1\} \quad \Re \left(1 - \frac{2itu}{\sqrt{n}(1-\alpha_j^n)} + \frac{iv d_n^2 \alpha_j^n}{n^2(1-\alpha_j^n)} \right) \geq \frac{h}{2}. \quad (43)$$

As in our previous article [For21], using Taylor's theorem, we can write

$$\left| G_n(u, v) - G_n(0, v) - u \frac{\partial G_n}{\partial u}(0, v) - \frac{u^2}{2} \frac{\partial^2 G_n}{\partial u^2}(0, v) \right| \leq \frac{|u|^3}{6} \sup_{t \in [0, 1]} \left| \frac{\partial^3 G_n}{\partial u^3}(tu) \right|. \quad (44)$$

We now compute the successive partial derivatives of G_n with respect to u . We start by writing

$$\begin{aligned} \frac{\partial G_n}{\partial u}(0, v) &= -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{1}{1 - \alpha_j^n + iv d_n^2 \alpha_j^n / n^2} \\ &= -\frac{i}{\sqrt{n}} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left(\frac{1}{1 - \alpha_j^n + iv d_n^2 \alpha_j^n / n^2} - 1 \right) \\ &= -\frac{i}{\sqrt{n}} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{\alpha_j^n - iv d_n^2 \alpha_j^n / n^2}{1 - \alpha_j^n + iv d_n^2 \alpha_j^n / n^2} \\ &= -\frac{i}{\sqrt{n}} + \frac{i}{\sqrt{n}} \left(1 - \frac{iv d_n^2}{n^2} \right) \sum_{j=1}^{n-1} \frac{\alpha_j^n}{1 - \alpha_j^n} \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right)^{-1}. \end{aligned} \quad (45)$$

Yet, according to (43), we have

$$\left| 1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right| \geq \Re \left(1 + \frac{iv d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) \geq \frac{h}{2}.$$

Plugging this into (45), we obtain

$$\left| \frac{\partial G_n}{\partial u}(0, v) \right| \leq \frac{1}{\sqrt{n}} + \frac{2}{h\sqrt{n}} \left(1 + \frac{|v|d_n^2}{n^2} \right) \sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{1 - \alpha_j^n} \leq \frac{1}{\sqrt{n}} + \frac{4+4|v|}{h\sqrt{n}} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{|\alpha_j^n|}{1 - \alpha_j^n}.$$

Yet, we have proved in lemma 2 that

$$\frac{1}{1 - \alpha_j^n} = O(1) + O\left(\frac{n^2}{d_n^2 j^2}\right), \quad (46)$$

where the O is uniform over all $j \leq \lfloor n/2 \rfloor$. Combining this with the inequality (8) according to which

$$|\alpha_j^n| \leq 1 \wedge \frac{n}{2d_n j}, \quad (47)$$

we get

$$\frac{\alpha_j^n}{1 - \alpha_j^n} = O\left(\frac{n}{d_n j}\right) + O\left(\frac{n^2}{d_n^2 j^2}\right),$$

whence

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \frac{|\alpha_j^n|}{1 - \alpha_j^n} = O\left(\frac{n \ln n}{d_n}\right) + O\left(\frac{n^2}{d_n^2}\right). \quad (48)$$

Therefore, we get

$$\frac{\partial G_n}{\partial u}(0, v) = O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{\sqrt{n}(\ln n)}{d_n}\right) + O\left(\frac{n^{3/2}}{d_n^2}\right) = o(1),$$

since $d_n/n^{3/4} \rightarrow \infty$. Similarly, we have

$$\frac{\partial^2 G_n}{\partial u^2}(0, v) = -\frac{2}{n} \sum_{j=1}^{n-1} \frac{1}{(1 - \alpha_j^n + iv d_n^2 \alpha_j^n/n^2)^2} = -2 + \frac{2}{n} - \frac{2}{n} \sum_{j=1}^{n-1} \left(\frac{1}{(1 - \alpha_j^n + iv d_n^2 \alpha_j^n/n^2)^2} - 1 \right).$$

This implies that

$$\begin{aligned} \left| \frac{\partial^2 G_n}{\partial u^2}(0, v) + 2 \right| &\leq \frac{2}{n} + \frac{2}{n} \sum_{j=1}^{n-1} \frac{|2\alpha_j^n(1 - iv d_n^2/n^2) + (\alpha_j^n)^2(1 - iv d_n^2/n^2)^2|}{|1 - \alpha_j^n + iv d_n^2 \alpha_j^n/n^2|^2} \\ &\leq \frac{2}{n} + \frac{8}{nh^2} \left(1 + \frac{|v|d_n^2}{n^2}\right)^2 \left(2 \sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{(1 - \alpha_j^n)^2} + \sum_{j=1}^{n-1} \frac{|\alpha_j^n|^2}{(1 - \alpha_j^n)^2}\right) \\ &\leq \frac{2}{n} + \frac{48(1 + |v|)^2}{nh^2} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{|\alpha_j^n|}{(1 - \alpha_j^n)^2}. \end{aligned} \quad (49)$$

It follows from (46) and (47) that, for $j \leq \lfloor n/2 \rfloor$,

$$\frac{\alpha_j^n}{(1 - \alpha_j^n)^2} = O\left(\frac{n^4}{d_n^4 j^4}\right) + O\left(\frac{n}{d_n j}\right),$$

whence

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \frac{|\alpha_j^n|}{(1 - \alpha_j^n)^2} = O\left(\frac{n^4}{d_n^4}\right) + O\left(\frac{n \ln n}{d_n}\right).$$

Using this in our computation (49), we are left with

$$\frac{\partial^2 G_n}{\partial u^2}(0, v) = -2 + O\left(\frac{1}{n}\right) + O\left(\frac{n^3}{d_n^4}\right) + O\left(\frac{\ln n}{d_n}\right) = -2 + o(1).$$

Lastly, for every $t \in [0, 1]$, we have

$$\begin{aligned} \left| \frac{\partial^3 G_n}{\partial u^3}(tu) \right| &= \left| \frac{8}{n^{3/2}} \sum_{j=1}^{n-1} \frac{1}{(1 - \alpha_j^n - 2itu/\sqrt{n} + iv d_n^2 \alpha_j^n/n^2)^3} \right| \\ &\leq \frac{128}{n^{3/2} h^3} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{1}{(1 - \alpha_j^n)^3}, \end{aligned}$$

implying that

$$\frac{|u|^3}{6} \sup_{t \in [0, 1]} \left| \frac{\partial^3 G_n}{\partial u^3}(tu) \right| \leq \frac{64|u|^3}{3n^{3/2} h^3} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{1}{(1 - \alpha_j^n)^3}. \quad (50)$$

Yet, according to (46), we know that for $j \leq \lfloor n/2 \rfloor$,

$$\frac{1}{(1 - \alpha_j^n)^3} = O(1) + O\left(\frac{n^6}{d_n^{12} j^{12}}\right),$$

whence

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \frac{1}{(1 - \alpha_j^n)^3} = O(n) + O\left(\frac{n^6}{d_n^{12}}\right).$$

Inserting this into (50) yields

$$\frac{|u|^3}{6} \sup_{t \in [0,1]} \left| \frac{\partial^3 G_n}{\partial u^3}(tu) \right| = O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{n^{9/2}}{d_n^6}\right) = o(1),$$

where we have again used the fact that $d_n/n^{3/4} \rightarrow \infty$. Plugging all this in our Taylor formula (44) we get

$$\lim_{n \rightarrow \infty} G_n(u, v) = G_\infty(0, v) - u^2 = G_\infty(u, v),$$

concluding the proof of the lemma. \square

3.3 Moving the integration contour

We wish to use the saddle-point method to approximate the integral in our formula (32). Therefore, we look for a saddle-point of the function $F_\infty + G_\infty$, which is extended to complex values of u and v . For every $(x, y, z, u, v) \in \mathbb{R}^3 \times \mathbb{C} \times V$, we have

$$\frac{\partial F_\infty}{\partial u}(x, y, z, u, v) + \frac{\partial G_\infty}{\partial u}(u, v) = -i(x - z^2) - 2u.$$

Therefore, our saddle-point in u is

$$u^* = u^*(x, z) = -\frac{i(x - z^2)}{2}.$$

As for the second complex variable, the expression of the saddle-point being too complicated, we choose to set

$$v^* = v^*(y) = -i \operatorname{sg}(y) = \begin{cases} -i & \text{if } y \geq 0, \\ +i & \text{otherwise,} \end{cases}$$

which is not the saddle-point, but which will be enough to obtain our result. The only important point is that the term $-iyv^* = -|y|$ will allow us to obtain an integrable function of y , as we will see later. We now wish to move the integration contour from $(u, v) \in \mathbb{R}^2$ to $(u, v) \in (u^*, v^*) + \mathbb{R}^2$. As explained in (31), the displacement of the integration contour in v can be easily justified, taking this time $v_0 = v^*$. This gives

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv \exp\left(F_n(x, y, z, u, v^* + v) + G_n(u, v^* + v)\right). \quad (51)$$

In order to move the integration contour of the variable u , we now want to swap the two last integrals, over u and over v . To this end, we prove the following lemma, where we dominate the integrand by an integrable function, which will also be useful later when we will apply the dominated convergence theorem. This lemma is formulated with an additional parameter t , to avoid writing two very similar proofs (we now need the case $t = 0$, and we will later use the case $t = 1$).

Lemma 7. *There exists $K > 0$ such that, for n large enough, for every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$ and for all $t \in [0, 1]$,*

$$\Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) \leq M_K(u, v) + 2 \ln(202 + z^2) + 2 \ln 2,$$

where the function $M_K : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by

$$M_K(u, v) = -2 \ln(1 + Ku^2) - \ln\left(1 + K(|v| - |u|)_+^2\right). \quad (52)$$

where $a_+ = \max(a, 0)$.

Proof. Let $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, and let $t \in [0, 1]$. We start by writing

$$\Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) = -\frac{1}{4} \sum_{j=1}^{n-1} \ln \left(1 + \frac{(2u/\sqrt{n} - v d_n^2 \alpha_j^n/n^2)^2}{(1 - \alpha_j^n - 2itu^*/\sqrt{n} + iv^* d_n^2 \alpha_j^n/n^2)^2} \right).$$

As a first step, we isolate the sum over the first four eigenvalues:

$$S_1 = -\frac{1}{4} \sum_{j=1}^4 \ln \left(1 + \frac{(2u/\sqrt{n} - v d_n^2 \alpha_j^n/n^2)^2}{(1 - \alpha_j^n - 2itu^*/\sqrt{n} + iv^* d_n^2 \alpha_j^n/n^2)^2} \right).$$

It follows from lemma 2 that, for n large enough,

$$\max_{j \leq 4} (1 - \alpha_j^n) \leq \frac{200d_n^2}{n^2}.$$

This implies that, for $j \leq 4$,

$$1 - \alpha_j^n - \frac{2itu^*}{\sqrt{n}} + \frac{iv^* d_n^2 \alpha_j^n}{n^2} \leq \frac{200d_n^2}{n^2} - \frac{x - z^2}{\sqrt{n}} + \frac{d_n^2}{n^2} \leq \frac{d_n^2}{n^2} \left(201 + \frac{n^{3/4}}{d_n} + \frac{z^2 n^{3/2}}{d_n^2} \right),$$

where we have used that $|v^*| = 1$ and $|x| \leq d_n/n^{3/4}$. In addition, we have $n^{3/4}/d_n \leq 1$ for n large enough, whence

$$1 - \alpha_j^n - \frac{2iu^*}{\sqrt{n}} + \frac{iv^* d_n^2 \alpha_j^n}{n^2} \leq \frac{d_n^2}{n^2} (202 + z^2).$$

Therefore, we obtain

$$S_1 \leq -\frac{1}{4} \sum_{j=1}^4 \ln \left(1 + \frac{(2u/\sqrt{n} - v d_n^2 \alpha_j^n/n^2)^2}{(d_n^2/n^2)(202 + z^2)^2} \right) \leq -\frac{1}{4} \sum_{j=1}^4 \ln \left(1 + \frac{(v \alpha_j^n - 2u n^{3/2}/d_n^2)^2}{(202 + z^2)^2} \right).$$

Using again the fact that, for n large enough, $n^{3/4}/d_n \leq 1/2$ and $\alpha_j^n \geq 1/4$ for all $j \leq 4$, we get

$$\left| v \alpha_j^n - \frac{2u n^{3/2}}{d_n^2} \right| \geq \left(|v \alpha_j^n| - \left| \frac{2u n^{3/2}}{d_n^2} \right| \right)_+ \geq \frac{(|v| - |u|)_+}{2},$$

for n larger than a certain fixed rank which does not depend on (x, y, z, u, v) . Thus, for n large enough, we have

$$S_1 \leq -\ln \left(1 + \frac{(|v| - |u|)_+^2}{4(203 + z^2)^2} \right) \leq -\ln \left(1 + (|v| - |u|)_+^2 \right) + 2 \ln(202 + z^2) + 2 \ln 2. \quad (53)$$

We now study the terms coming from the other eigenvalues:

$$S_2 = -\frac{1}{4} \sum_{j=5}^{n-1} \ln \left(1 + \frac{(2u/\sqrt{n} - v d_n^2 \alpha_j^n / n^2)^2}{(1 - \alpha_j^n - 2itu^*/\sqrt{n} + iv^* d_n^2 \alpha_j^n / n^2)^2} \right).$$

According to the definition (29) of the domain \mathcal{D}_n , we have $x - z^2 \geq -\sqrt{n}$. This implies that, for every $j \in \{5, \dots, n-1\}$,

$$1 - \alpha_j^n - \frac{2itu^*}{\sqrt{n}} + \frac{iv^* d_n^2 \alpha_j^n}{n^2} \leq 2 - \frac{x - z^2}{\sqrt{n}} + \frac{d_n^2}{n^2} \leq 4,$$

allowing us to write

$$S_2 \leq -\frac{1}{4} \sum_{j=5}^{n-1} \ln \left[1 + \frac{1}{16} \left(\frac{2u}{\sqrt{n}} - \frac{v d_n^2 \alpha_j^n}{n^2} \right)^2 \right].$$

By lemma 4, there exists $K_0 > 0$ such that

$$|J_{n,u,v}| \geq K_0 n,$$

where

$$J_{n,u,v} = \left\{ j \in \{5, \dots, n-1\} : uv \alpha_j^n \leq 0 \right\}.$$

Therefore, we have

$$S_2 \leq -\frac{|J_{n,u,v}|}{4} \ln \left(1 + \frac{u^2}{4n} \right) \leq -\frac{K_0 n}{4} \ln \left(1 + \frac{u^2}{4n} \right).$$

As soon as $n \geq 8/K_0$, the concavity of the logarithm function ensures that

$$S_2 \leq -\frac{K_0}{4} \left\lceil \frac{8}{K_0} \right\rceil \ln \left(1 + \frac{u^2}{4 \lceil 8/K_0 \rceil} \right).$$

Adding this to (53), and letting

$$K = 1 \wedge \frac{1}{4 \lceil 8/K_0 \rceil},$$

we obtain the desired upper bound (the factor K in the second term of M_K is unnecessary, but it will appear in the next section, in the course of the proof of theorem 2, so we include it here so that we use the same function M_K for both cases). \square

We now prove check:

Lemma 8. *For every $K > 0$, the function e^{M_K} , where $M_K : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by (52), is integrable over \mathbb{R}^2 .*

Proof. Let $K > 0$. We can write, in $[0, +\infty]$,

$$\int_{\mathbb{R}^2} du dv e^{M_K(u,v)} = \int_{\mathbb{R}} \frac{du}{(1 + Ku^2)^2} \int_{\mathbb{R}} \frac{dv}{1 + K(|v| - |u|)_+^2}.$$

We then compute, for $u \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{dv}{1 + K(|v| - |u|)_+^2} = 2|u| + 2 \int_0^{+\infty} \frac{dv}{1 + Kv^2} = 2|u| + \frac{\pi}{\sqrt{K}},$$

so that we have

$$\int_{\mathbb{R}^2} du dv e^{M_0(u,v)} \leq \int_{\mathbb{R}} \frac{(2|u| + \pi/\sqrt{K})}{(1 + Ku^2)^2} du < +\infty,$$

since

$$\frac{(2|u| + \pi/\sqrt{K})}{(1 + Ku^2)^2} = O\left(\frac{1}{|u|^3}\right)$$

when $|u| \rightarrow \infty$. □

Let us now go back to our equation (51), where we wanted to invert the summations over u and over v . For every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, we have

$$\Re F_n(x, y, z, u, v^* + v) = F_n(x, y, z, 0, v^*).$$

Thus, it follows from lemma 7 applied with $t = 0$ that, for n larger than a certain rank which is the same for all $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, we have

$$\left| e^{F_n(x, y, z, u, v^* + v) + G_n(u, v^* + v)} \right| \leq (203 + z^2)^2 e^{F_n(x, y, z, 0, v^*) + G_n(0, v^*) + M_K(u, v)},$$

which is an integrable function of (u, v) by virtue of lemma 8. Therefore, Fubini's theorem allows us to swap the two last integrals in the formula (51), to obtain

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du \exp\left(F_n(x, y, z, u, v^* + v) + G_n(u, v^* + v)\right).$$

We now fix $(x, y, z, v) \in \mathcal{D}_n \times \mathbb{R}$ and n , and we wish to move the integration contour of the variable u . The function $u \mapsto F_n(x, y, z, u, v^* + v)$ is defined and holomorphic on all the complex plane. As for the function $u \mapsto G_n(u, v^* + v)$, it is holomorphic on the open set

$$\left\{ u \in \mathbb{C} \quad : \quad \forall j \in \{1, \dots, n-1\} \quad 1 + \frac{2\Im u}{\sqrt{n}(1 - \alpha_j^n)} + \frac{\text{sg}(y)d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} > 0 \right\}. \quad (54)$$

We wish to move the integration path of u from \mathbb{R} to $\mathbb{R} + u^*$, where u^* is our saddle-point given by

$$u^* = -\frac{i(x - z^2)}{2}.$$

Yet, by the definition (29) of the domain \mathcal{D}_n , we have $|x| \leq d_n/n^{3/4} \leq n^{1/4}$, whence

$$\forall t \in [0, 1] \quad 1 + \frac{2t\Im u^*}{\sqrt{n}(1 - \alpha_j^n)} + \frac{\text{sg}(y)d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \geq 1 - \frac{d_n}{n^{5/4}(1 - M_n)} - \frac{d_n^2}{n^2(1 - M_n)}. \quad (55)$$

According to lemma 3, we have

$$1 - M_n \underset{n \rightarrow \infty}{\sim} \frac{2\pi^2 d_n^2}{3n^2},$$

whence

$$1 - \frac{d_n}{n^{5/4}(1 - M_n)} - \frac{d_n^2}{n^2(1 - M_n)} = 1 - O\left(\frac{n^{3/4}}{d_n}\right) - \frac{3}{2\pi^2}(1 + o(1)) = 1 - \frac{3}{2\pi^2} + o(1). \quad (56)$$

This implies that, for n larger than a certain rank, which does not depend on $(x, y, z, v) \in \mathcal{D}_n \times \mathbb{R}$, we have

$$1 - \frac{d_n}{n^{5/4}(1 - M_n)} - \frac{d_n^2}{n^2(1 - M_n)} > 0,$$

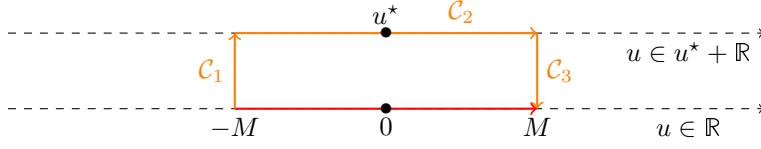


Figure 1: Cauchy's theorem allows us to replace the integral over $[-M, M]$ with the integral on the contours \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 .

and in this case, the contour represented on figure 1 is included in the open set given by (54). Thus, by virtue of Cauchy's theorem, we obtain that for $M > 0$,

$$\begin{aligned} \int_{-M}^M du e^{F_n(x,y,z,u,v^*+v)+G_n(u,v^*+v)} &= \int_0^1 u^* dt e^{F_n(x,y,z,-M+tu^*,v^*+v)+G_n(-M+tu^*,v^*+v)} \\ &+ \int_{-M}^M du e^{F_n(x,y,z,u^*+u,v^*+v)+G_n(u^*+u,v^*+v)} \\ &+ \int_1^0 u^* dt e^{F_n(x,y,z,M+tu^*,v^*+v)+G_n(M+tu^*,v^*+v)}. \end{aligned}$$

Using the domination result of lemma 7, we can write, for $M > 0$ and $t \in [0, 1]$,

$$\left| e^{F_n(x,y,z,\pm M+tu^*,v^*+v)+G_n(\pm M+tu^*,v^*+v)} \right| \leq (202+z^2)^2 e^{F_n(x,y,z,tu^*,v^*)+G_n(tu^*,v^*)+M_0(\pm M,v)}$$

$$\xrightarrow{M \rightarrow \infty} 0,$$

uniformly with respect to $t \in [0, 1]$, implying that

$$\lim_{M \rightarrow +\infty} \int_0^1 u^* dt e^{F_n(x,y,z,\pm M+tu^*,v^*+v)+G_n(\pm M+tu^*,v^*+v)} = 0,$$

whence

$$\int_{\mathbb{R}} du e^{F_n(x,y,z,u,v^*+v)+G_n(u,v^*+v)} = \int_{\mathbb{R}} du e^{F_n(x,y,z,u^*+u,v^*+v)+G_n(u^*+u,v^*+v)}.$$

At the end of the day, we obtain that, for n large enough,

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du \exp \left(F_n(x,y,z,u^*+u,v^*+v) + G_n(u^*+u,v^*+v) \right). \quad (57)$$

3.4 Domination

We now need to bound from above the modulus of the integrand in (57), uniformly with respect to n . This is done in the following lemma:

Lemma 9. *There exist $K, K_1 > 0$ such that, for n large enough,*

$$\forall (x,y,z,u,v) \in \mathcal{D}_n \times \mathbb{R}^2 \quad \Re \left(F_n(x,y,z,u^*+u,v^*+v) + G_n(u^*+u,v^*+v) \right) \leq M(x,y,z,u,v),$$

where the function $M : \mathbb{R}^5 \rightarrow \mathbb{R}$ is defined by

$$M(x,y,z,u,v) = -\frac{9}{70}(x-z^2)^2 - \frac{xz^2}{2} + 2|x-z^2| - \frac{|y|}{2} + z^2 + 2 \ln(202+z^2) + K_1 + M_K(u,v).$$

with M_K being the function given by (52).

Proof. Upper bound on the first term: We first deal with the real part of the function F_n . For every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, we have

$$\begin{aligned} \Re F_n(x, y, z, u^* + u, v^* + v) &= -i(x - z^2)u^* - iyv^* - \frac{xz^2}{2(1 + x/\sqrt{n})} + \frac{xy n^{3/2}}{d_n^2(1 + x/\sqrt{n})} \\ &= -\frac{(x - z^2)^2}{2} - |y| - \frac{xz^2}{2} + \frac{x^2 z^2}{2(x + \sqrt{n})} + \frac{xy n^2}{d_n^2(x + \sqrt{n})}. \end{aligned} \quad (58)$$

By definition of the domain \mathcal{D}_n , we have $|x| \leq d_n/n^{3/4} \leq n^{1/4}$, whence

$$\sqrt{n} + x \geq \sqrt{n} - n^{1/4} \geq \frac{\sqrt{n}}{2},$$

as soon as $n \geq 16$. We then have, on the one hand,

$$\frac{x^2 z^2}{2(x + \sqrt{n})} \leq \frac{(n^{1/4})^2 z^2}{\sqrt{n}} = z^2, \quad (59)$$

and on the other hand,

$$\frac{xy n^2}{d_n^2(x + \sqrt{n})} \leq \frac{2n^{3/2}}{d_n^2} |xy| \leq \frac{2n^{3/4}}{d_n} |y|.$$

Since we assumed that $d_n/n^{3/4} \rightarrow \infty$, we have $d_n \geq 4n^{3/4}$ for n large enough, whence

$$\frac{xy n^2}{d_n^2(x + \sqrt{n})} \leq \frac{|y|}{2}. \quad (60)$$

Therefore, plugging (59) and (60) into (58), we obtain that, for n large enough,

$$\forall (x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2 \quad \Re F_n(x, y, z, u^* + u, v^* + v) \leq -\frac{(x - z^2)^2}{2} - \frac{xz^2}{2} - \frac{|y|}{2} + z^2. \quad (61)$$

Upper bound on the second term: Let $(x, y, z) \in \mathcal{D}_n$. We now determine an upper bound for the term $G_n(u^*, v^*)$ (which is real, since u^* and v^* are pure imaginary). Recalling our computations (55) and (56), we have

$$\Re \left(1 - \frac{2itu^*}{\sqrt{n}(1 - \alpha_j^n)} + \frac{iv^* d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) \geq 1 - \frac{d_n}{n^{5/4}(1 - M_n)} - \frac{d_n^2}{n^2(1 - M_n)} \xrightarrow{n \rightarrow \infty} 1 - \frac{3}{2\pi^2} > \frac{9}{13},$$

uniformly for all $(x, y, z) \in \mathcal{D}_n$ and for all $t \in [0, 1]$. Consequently, for n large enough, we have

$$\forall (x, y, z) \in \mathcal{D}_n \quad \forall t \in [0, 1] \quad \Re \left(1 - \frac{2itu^*}{\sqrt{n}(1 - \alpha_j^n)} + \frac{iv^* d_n^2 \alpha_j^n}{n^2(1 - \alpha_j^n)} \right) \geq \frac{9}{13}, \quad (62)$$

which corresponds to the lower bound (43) that we used to establish the pointwise convergence, but it now holds uniformly. The choice of the constant $9/13$ is a bit arbitrary, but the important point is that it is strictly larger than $2/3$, in order for the quadratic form we will obtain in the end to be positive definite. By the way, note that the constant $3/(2\pi^2)$ which appears above is purely contingent, since it is a consequence

of our arbitrary choice to take $v^* = -i \operatorname{sg}(y)$. Had we needed a smaller constant there, we would have taken a smaller v^* . Taking the derivative of the expression (34) of G_n , we get, for all $t \in [0, 1]$,

$$\begin{aligned} \frac{\partial G_n}{\partial u}(tu^*, v^*) &= -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{1}{1 - \alpha_j^n - 2itu^*/\sqrt{n} + iv^* d_n^2 \alpha_j^n/n^2} \\ &= -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left(\frac{1}{1 - \alpha_j^n - 2itu^*/\sqrt{n} + iv^* d_n^2 \alpha_j^n/n^2} - 1 \right) \\ &= -\frac{i}{\sqrt{n}} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{\alpha_j^n + 2itu^*/\sqrt{n} - iv^* d_n^2 \alpha_j^n/n^2}{1 - \alpha_j^n - 2itu^*/\sqrt{n} + iv^* d_n^2 \alpha_j^n/n^2}. \end{aligned}$$

Using the lower bound (62), this becomes

$$\begin{aligned} \left| \frac{\partial G_n}{\partial u}(tu^*, v^*) \right| &\leq \frac{1}{\sqrt{n}} + \frac{13}{9\sqrt{n}} \left(1 + \frac{|v^*| d_n^2}{n^2} \right) \sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{1 - \alpha_j^n} + \frac{26t|u^*|}{9n} \sum_{j=1}^{n-1} \frac{1}{1 - \alpha_j^n} \\ &\leq 1 + \frac{26}{9\sqrt{n}} \sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{1 - \alpha_j^n} + \frac{26t|u^*|}{9} \left(1 + \frac{1}{n} \sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{1 - \alpha_j^n} \right). \end{aligned} \quad (63)$$

Recall now the estimate (48) according to which

$$\sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{1 - \alpha_j^n} = O\left(\frac{n \ln n}{d_n}\right) + O\left(\frac{n^2}{d_n^2}\right) = o(\sqrt{n}).$$

Therefore, for n large enough, we have

$$\sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{1 - \alpha_j^n} \leq \sqrt{n} \leq \frac{n}{35},$$

and thus (63) becomes

$$\left| \frac{\partial G_n}{\partial u}(tu^*, v^*) \right| \leq 1 + \frac{26}{9} + \frac{26t|u^*|}{9} \times \frac{36}{35} \leq 4 + \frac{104t|u^*|}{35}.$$

Integrating this and writing

$$K_1 = \sup_n \max(|G_n(0, i)|, |G_n(0, -i)|),$$

which is finite because of the pointwise convergence of G_n , we deduce that, for n large enough, for every $(x, y, z) \in \mathcal{D}_n$,

$$\begin{aligned} |G_n(u^*, v^*)| &= \left| G_n(0, v^*) + u^* \int_0^1 \frac{\partial G_n}{\partial u}(tu^*, v^*) dt \right| \\ &\leq K_1 + 4|u^*| + \frac{52|u^*|^2}{35} = K_1 + 2|x - z^2| + \frac{13(x - z^2)^2}{35}. \end{aligned} \quad (64)$$

Combining the upper bounds (61) and (64) with the domination result of lemma 7, we obtain the claimed result. \square

There remains to check that our upper bound is integrable:

Lemma 10. *The function e^M (where M was defined in lemma 9) is integrable over \mathbb{R}^5 .*

Proof. We have in $[0, +\infty]$,

$$\int_{\mathbb{R}^5} dx dy dz du dv e^{M(x,y,z,u,v)} = e^{K_1} \int_{\mathbb{R}^3} dx dy dz e^{M(x,y,z,0,0)} \times \int_{\mathbb{R}^2} du dv e^{M_K(u,v)}.$$

We have already checked the integrability of the function e^{M_K} over \mathbb{R}^2 in lemma 8. Therefore, we are left with proving

$$\int_{\mathbb{R}^3} dx dy dz e^{M(x,y,z,0,0)} < +\infty.$$

To this end, we write, for $(x, y, z) \in \mathbb{R}^3$,

$$M(x, y, z, 0, 0) = -\frac{9}{70} \left[(x - z^2)^2 + \frac{35}{9} x z^2 \right] + 2|x - z^2| - \frac{|y|}{2} + z^2 + 2 \ln(202 + z^2)$$

Let us now remark that

$$\begin{aligned} (x - z^2)^2 + \frac{35}{9} x z^2 &= x^2 + \frac{17}{9} x z^2 + z^4 \\ &= \left(x + \frac{17}{18} z^2 \right)^2 + \left(1 - \frac{17^2}{18^2} \right) z^4 \\ &= \left(x + \frac{17}{18} z^2 \right)^2 + \frac{35}{324} z^4. \end{aligned}$$

so that we may rewrite

$$M(x, y, z, 0, 0) = -\frac{9}{70} \left(x + \frac{17}{18} z^2 \right)^2 - \frac{z^4}{72} + 2|x - z^2| - \frac{|y|}{2} + z^2 + 2 \ln(202 + z^2).$$

In the integral, we then perform the change of variable

$$x = \tau - \frac{17}{18} z^2,$$

and we get, in $[0, +\infty]$,

$$\begin{aligned} &\int_{\mathbb{R}^3} dx dy dz e^{M(x,y,z,0,0)} \\ &= \int_{\mathbb{R}^3} d\tau dy dz (202 + z^2)^2 \exp \left(-\frac{9\tau^2}{70} - \frac{z^4}{72} + 2 \left| \tau - \frac{35z^2}{18} \right| - \frac{|y|}{2} + z^2 \right) \\ &\leq \int_{\mathbb{R}} d\tau \exp \left(2|\tau| - \frac{9\tau^2}{70} \right) \times \int_{\mathbb{R}} dy e^{-|y|/2} \times \int_{\mathbb{R}} dz (202 + z^2)^2 \exp \left(\frac{44z^2}{9} - \frac{z^4}{72} \right) < +\infty. \end{aligned}$$

Thus, the function e^M is integrable over \mathbb{R}^5 . □

3.5 Dominated convergence

We are now in a position to apply the dominated convergence theorem and to conclude the proof of theorem 1. Let us recall our formula (57) :

$$E_n = C_n \int_{\mathbb{R}^3} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du e^{F_n(x,y,z,u^*+u,v^*+v)+G_n(u^*+u,v^*+v)} \mathbb{1}_{(x,y,z) \in \mathcal{D}_n}.$$

In section 3.2, we proved the pointwise convergence, for all $(x, y, z, u, v) \in \mathbb{R}^5$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \exp \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right) \mathbb{1}_{(x, y, z) \in \mathcal{D}_n} \\ = \exp \left(F_\infty(x, y, z, u^* + u, v^* + v) + G_\infty(u^* + u, v^* + v) \right), \end{aligned}$$

where

$$F_\infty(x, y, z, u^* + u, v^* + v) = -\frac{(x - z^2)^2}{2} - iu(x - z^2) - |y| - ivy - \frac{xz^2}{2}$$

and

$$G_\infty(u^* + u, v^* + v) = \frac{(x - z^2)^2}{4} + iu(x - z^2) - u^2 - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3i(v^* + v)}{2\pi^2 j^2} \right).$$

Adding these two lines, we obtain, for all $(x, y, z, u, v) \in \mathbb{R}^5$,

$$\lim_{n \rightarrow \infty} \exp \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right) \mathbb{1}_{(x, y, z) \in \mathcal{D}_n} = \exp \left(J(x, y, z, u, v) \right),$$

with

$$J(x, y, z, u, v) = -\frac{x^2}{4} - \frac{z^4}{4} - u^2 - |y| - ivy - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3i(v^* + v)}{2\pi^2 j^2} \right).$$

According to lemmas 9 and 10 and to the fact that the function g is bounded, the domination hypothesis is satisfied for n large enough. Hence, the dominated convergence theorem entails that

$$E_n \stackrel{n \rightarrow \infty}{\sim} C_n \int_{\mathbb{R}^5} dx dy dz du dv g(z) e^{J(x,y,z,u,v)} = 2\pi C_n D \int_{\mathbb{R}} dz g(z) e^{-z^4/4},$$

where

$$D = \int_{\mathbb{R}^2} dy dv \exp \left[-|y| - ivy - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3i(v^* + v)}{2\pi^2 j^2} \right) \right].$$

Thus, the variable z finds itself separated from the other variables, thus the complicated sum appearing in D plays no role in the limiting distribution of $S_n/n^{3/4}$. This is what will change in the following section, that is to say in the regime $d_n \sim \lambda n^{3/4}$, because then we will have a term linking the two variables y and z , which will not disappear when taking the limit. Going back to the present case, recall that, as we wrote in (26), we have

$$\mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = E_n + \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \mathbb{1}_{|T_n - 1| > d_n/n^{5/4}} \right] = E_n + o(1).$$

Therefore, we obtain

$$\mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] \stackrel{n \rightarrow \infty}{\sim} E_n \stackrel{n \rightarrow \infty}{\sim} 2\pi C_n D \int_{\mathbb{R}} dz g(z) e^{-z^4/4}.$$

Applying this to the constant function $g \equiv 1$ yields

$$1 \stackrel{n \rightarrow \infty}{\sim} 2\pi C_n D \int_{\mathbb{R}} dz e^{-z^4/4}.$$

Thus, for any bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, we obtain

$$\lim_{n \rightarrow \infty} \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = \frac{\int_{\mathbb{R}} dz g(z) e^{-z^4/4}}{\int_{\mathbb{R}} dz e^{-z^4/4}} = \frac{\sqrt{2}}{\Gamma(1/4)} \int_{\mathbb{R}} dz g(z) e^{-z^4/4},$$

which proves the convergence in distribution announced in theorem 1. \square

4 Proof of theorem 2

The goal of this section is to obtain theorem 2. Thus, we assume that the parameter d_n scales as

$$d_n \stackrel{n \rightarrow \infty}{\sim} \lambda n^{3/4} \quad \text{with} \quad \lambda > 0,$$

we fix a bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, and we want to compute

$$\lim_{n \rightarrow \infty} \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right].$$

We proceed using the same technique as for the proof of theorem 1 (see section 3), with however some changes here and there.

4.1 A bound on the Hamiltonian

We start by proving a lower bound on the interaction Hamiltonian, which will be useful later to obtain the domination by an integrable function.

Lemma 11. *For any positive sequence $(a_n)_{n \in \mathbb{N}}$ such that $d_n a_n / (n \ln n) \rightarrow \infty$, we have*

$$\mu_n \left(H_n > a_n \right) = O \left(\frac{e^{-a_n/4}}{Z_n} \right).$$

Proof. Let (a_n) be a sequence of positive integers satisfying $d_n a_n / (n \ln n) \rightarrow \infty$. We start by writing (where ν is the standard normal distribution)

$$\mu_n \left(H_n > a_n \right) = \frac{1}{Z_n} \nu^{\otimes n} \left[\exp \left(-\frac{H_n}{T_n} \right) \mathbb{1}_{H_n > a_n} \right] \leq \frac{1}{Z_n} \nu^{\otimes n} \left(H_n > a_n \right). \quad (65)$$

Next, according to Chebychev's inequality, we have

$$\begin{aligned} \nu^{\otimes n} \left(H_n > a_n \right) &\leq e^{-a_n/2} \nu^{\otimes n} \left(e^{H_n/2} \right) \\ &= e^{-a_n/2} \prod_{j=1}^n \int_{\mathbb{R}} e^{-\alpha_j^n y^2/4} \nu(dy) \\ &= \exp \left[-\frac{a_n}{2} - \frac{1}{2} \sum_{j=1}^n \ln \left(1 + \frac{\alpha_j^n}{2} \right) \right]. \end{aligned} \quad (66)$$

For every $j \in \{1, \dots, n\}$, we have $|\alpha_j^n| \leq 1$. The logarithm function being Lipschitz on the segment $[1/2, 3/2]$, we deduce that

$$\sum_{j=1}^n \ln \left(1 + \frac{\alpha_j^n}{2} \right) = O \left(\sum_{j=1}^n |\alpha_j^n| \right) = O \left(\frac{n \ln n}{d_n} \right),$$

following the upper bound (9). Since $d_n a_n / (n \ln n) \rightarrow \infty$, we have, for n large enough,

$$-\frac{1}{2} \sum_{j=1}^n \ln \left(1 + \frac{\alpha_j^n}{2} \right) \leq \frac{a_n}{4}.$$

Hence, for n large enough, our upper bound (66) becomes

$$\nu^{\otimes n} \left(H_n > a_n \right) \leq \exp \left(-\frac{a_n}{4} \right).$$

Plugging this into (65), we obtain the claimed inequality. \square

4.2 Exact integral expression

As explained at the beginning of section 3.1, we use the result of [For21] about the behaviour of T_n , according to which

$$\sqrt{n}(T_n - 1) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 2),$$

implying that

$$\lim_{n \rightarrow \infty} \mu_n \left(|T_n - 1| > n^{-1/4} \right) = 0.$$

Besides, lemma 11 ensures that

$$\mu_n \left(H_n > n^{3/4} \right) = O \left(\frac{e^{-n^{3/4}/4}}{Z_n} \right),$$

which will enable us to prove that this term tends to 0, once we will know the order of magnitude of Z_n (see section 4.6). Therefore, we leave this term apart for the moment, and we let

$$E_n = \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \mathbb{1}_{|T_n - 1| \leq n^{-1/4}} \mathbb{1}_{H_n \leq n^{3/4}} \right].$$

With the density given by lemma 1, we have, for $n \geq n_I$,

$$\begin{aligned} E_n &= \frac{1}{(2\pi)^{5/2} Z_n \sqrt{n}} \int_{1-n^{-1/4}}^{1+n^{-1/4}} dx \int_{-\infty}^{n^{3/4}} dy \int_{\mathbb{R}} dz g \left(\frac{z}{n^{3/4}} \right) \exp \left(-\frac{y}{x} \right) \mathbb{1}_{2x > x - 2y/n > 2z^2/n^2} \\ &\quad \times \int_{\mathbb{R}^2} du dv \exp \left[-iu \left(x - \frac{z^2}{n^2} \right) - iv \left(y + \frac{z^2}{2n} \right) - \frac{z^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{n} + iv \alpha_j^n \right) \right]. \end{aligned}$$

We now perform the change of variable

$$x = 1 + \frac{x'}{\sqrt{n}}, \quad y = y' \sqrt{n}, \quad z = z' n^{3/4}, \quad u = u' \sqrt{n}, \quad v = \frac{v'}{\sqrt{n}},$$

which yields

$$\forall n \geq n_I \quad E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}^2} du dv e^{A_n(x,y,z,u,v)},$$

where the domain \mathcal{D}_n is now defined by

$$\mathcal{D}_n = \left\{ (x, y, z) \in \mathbb{R}^3 : |x| \leq n^{1/4}, \quad y \leq n^{1/4} \quad \text{and} \quad 2\sqrt{n} + 2x > \sqrt{n} + x - 2y > 2z^2 \right\}, \quad (67)$$

and where the function A_n is given by

$$A_n(x, y, z, u, v) = -\frac{y\sqrt{n}}{1+x/\sqrt{n}} - iu\sqrt{n} - iu(x-z^2) - ivy - \frac{ivz^2}{2} - \frac{z^2\sqrt{n}}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{\sqrt{n}} + \frac{iv\alpha_j^n}{\sqrt{n}} \right).$$

We then move the integration contour as in section 3.1, but this time from $v \in \mathbb{R}$ to $v \in i\sqrt{n} + \mathbb{R}$. This change of contour can be justified just the same way as previously, and this gives us, for n large enough,

$$E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv e^{B_n(x,y,z,u,v)},$$

where

$$B_n(x, y, z, u, v) = \frac{xy}{1+x/\sqrt{n}} - iu\sqrt{n} - iu(x-z^2) - ivy - \frac{ivz^2}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \alpha_j^n - \frac{2iu}{\sqrt{n}} + \frac{iv\alpha_j^n}{\sqrt{n}} \right).$$

Lastly, we perform the change of variable

$$v = v' + 2u,$$

which yields, for n large enough,

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv e^{F_n(x,y,z,u,v) + G_n(u,v)},$$

with the same constant C_n given by (33), and with

$$F_n(x, y, z, u, v) = -iu(x+2y) - ivy - \frac{ivz^2}{2} + \frac{xy}{1+x/\sqrt{n}}$$

and

$$G_n(u, v) = -iu\sqrt{n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{\sqrt{n}} + \frac{iv\alpha_j^n}{\sqrt{n}(1-\alpha_j^n)} \right). \quad (68)$$

4.3 Pointwise convergence of the integral

We easily see that

$$\forall (x, y, z, u, v) \in \mathbb{R}^3 \times \mathbb{C}^2 \quad \lim_{n \rightarrow \infty} F_n(x, y, z, u, v) = F_\infty(x, y, z, u, v),$$

where

$$F_\infty(x, y, z, u, v) = -iu(x+2y) - ivy - \frac{ivz^2}{2} + xy.$$

As in section 3.2, the term $G_n(u, v)$ is defined for n large enough, as long as $(u, v) \in \mathbb{C} \times V$, with now

$$V = \left\{ v \in \mathbb{C} : \Im v < \frac{\pi^2 \lambda^2}{6} \right\}.$$

The limit of G_n when $n \rightarrow \infty$ is then given by the following lemma:

Lemma 12. *We have the pointwise convergence*

$$\forall (u, v) \in \mathbb{C} \times V \quad \lim_{n \rightarrow \infty} G_n(u, v) = G_\infty(u, v),$$

where

$$G_\infty(u, v) = -u^2 - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3iv}{2\pi^2 \lambda^2 j^2} \right).$$

Proof. Let $(u, v) \in \mathbb{C} \times V$. We prove that $G_n(0, v) \rightarrow G_\infty(0, v)$ exactly as in the proof of lemma 6, except that there is an additional term involved in the upper bound of $\delta_n^1(v)$. Indeed, we write

$$\delta_n^1(v) = \sum_{j=1}^{\lfloor n/d_n \rfloor} \left[-\ln \left(1 + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right) + \ln \left(1 + \frac{3iv}{2\pi^2 \lambda^2 j^2} \right) \right],$$

and we have

$$\begin{aligned} |\delta_n^1(v)| &= O \left(\sum_{j=1}^{\lfloor n/d_n \rfloor} \left| \frac{\alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} - \frac{3}{2\pi^2 \lambda^2 j^2} \right| \right) \\ &= O \left(\sum_{j=1}^{\lfloor n/d_n \rfloor} \left| \frac{\alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} - \frac{3n^{3/2}}{2\pi^2 d_n^2 j^2} \right| + \sum_{j=1}^{\lfloor n/d_n \rfloor} \left| \frac{3n^{3/2}}{2\pi^2 d_n^2 j^2} - \frac{3}{2\pi^2 \lambda^2 j^2} \right| \right) \\ &= O \left(\sum_{j=1}^{\lfloor n/d_n \rfloor} \frac{1}{\sqrt{n}} \right) + O \left(\sum_{j=1}^{\lfloor n/d_n \rfloor} \frac{1}{j^2} \left| \frac{n^{3/2}}{d_n^2} - \frac{1}{\lambda^2} \right| \right) \\ &= O \left(\frac{\sqrt{n}}{d_n} \right) + o \left(\sum_{j=1}^{\lfloor n/d_n \rfloor} \frac{1}{j^2} \right) = o(1). \end{aligned}$$

Regarding the variations with respect to u , the fact that there is no more $(1 - \alpha_j^n)$ in the denominator of the term in u simplifies a bit the computation. As in the proof of lemma 6, we have at our disposal $h > 0$ such that, for n large enough,

$$\forall t \in [0, 1] \quad \forall j \in \{1, \dots, n-1\} \quad \Re \left(1 - \frac{2itu}{\sqrt{n}} + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right) \geq h. \quad (69)$$

We now have

$$\begin{aligned} \frac{\partial G_n}{\partial u}(0, v) &= -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left(1 + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^{-1} \\ &= -\frac{i}{\sqrt{n}} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left[\left(1 + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^{-1} - 1 \right] \\ &= -\frac{i}{\sqrt{n}} - \frac{v}{n} \sum_{j=1}^{n-1} \frac{\alpha_j^n}{1 - \alpha_j^n} \left(1 + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^{-1}, \end{aligned}$$

whence

$$\frac{\partial G_n}{\partial u}(0, v) = O \left(\frac{1}{\sqrt{n}} \right) + O \left(\frac{1}{n} \sum_{j=1}^{n-1} \frac{|\alpha_j^n|}{1 - M_n} \right) = o(1) + O \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} |\alpha_j^n| \right),$$

where we used the equivalent of $1 - M_n$ provided by lemma 3. Using now the upper bound (9), we get

$$\frac{\partial G_n}{\partial u}(0, v) = o(1) + O\left(\frac{\sqrt{n}(\ln n)}{d_n}\right) = o(1).$$

Similarly, we can write

$$\begin{aligned} \frac{\partial^2 G_n}{\partial u^2}(0, v) &= -\frac{2}{n} \sum_{j=1}^{n-1} \left(1 + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)}\right)^{-2} \\ &= -\frac{2(n-1)}{n} - \frac{2}{n} \sum_{j=1}^{n-1} \left[\left(1 + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)}\right)^{-2} - 1 \right] \\ &= -2 + \frac{2}{n} + \frac{2}{n} \sum_{j=1}^{n-1} \left[\frac{2iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} + \left(\frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)}\right)^2 \right] \left(1 + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)}\right)^{-2}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \frac{\partial^2 G_n}{\partial u^2}(0, v) &= -2 + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^{3/2}(1 - M_n)} \sum_{j=1}^{n-1} |\alpha_j^n|\right) + O\left(\frac{1}{n^2(1 - M_n)^2} \sum_{j=1}^{n-1} (\alpha_j^n)^2\right) \\ &= -2 + o(1) + O\left(\frac{1}{n} \sum_{j=1}^{n-1} |\alpha_j^n|\right) = -2 + o(1). \end{aligned}$$

Lastly, using (69) we obtain, for every $t \in [0, 1]$,

$$\left| \frac{\partial^3 G_n}{\partial u^3}(tu) \right| = \left| \frac{8}{n^{3/2}} \sum_{j=1}^{n-1} \left(1 - \frac{2itu}{\sqrt{n}} + \frac{iv \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)}\right)^{-3} \right| \leq \frac{8}{h^3 \sqrt{n}},$$

and thus we have proved

$$\lim_{n \rightarrow \infty} G_n(u, v) = G_\infty(0, v) - u^2 = G_\infty(u, v),$$

concluding the proof of the lemma. \square

4.4 Moving the integration contour

The saddle-point in u is now given by

$$u^* = -\frac{i(x + 2y)}{2}, \tag{70}$$

and, as an ersatz for the saddle-point in v , we define

$$v^* = -i, \tag{71}$$

which will give us a term $-iz^2 v^* = -z^2$, which will ensure the integrability with respect to the variable z . Moving the integration contour of (u, v) in the complex plane to make it pass through u^* and v^* can be justified exactly as in section 3.3. To swap the two integrals over u and v , we prove:

Lemma 13. *There exists $K > 0$ such that, for n large enough,*

$$\forall (x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2 \quad \forall t \in [0, 1] \quad \Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) \leq M_K(u, v),$$

where the function $M_K : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by equation (52).

Proof. Let $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, and let $t \in [0, 1]$. We start by writing

$$\begin{aligned} & \Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) \\ &= -\frac{1}{4} \sum_{j=1}^{n-1} \ln \left[1 + \left(\frac{2u}{\sqrt{n}} + \frac{v \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^2 \left(1 - \frac{2itu^*}{\sqrt{n}} + \frac{iv^* \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^{-2} \right]. \end{aligned} \quad (72)$$

Replacing u^* and v^* by their definitions given by (70) and (71) yields

$$1 - \frac{2itu^*}{\sqrt{n}} + \frac{iv^* \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} = 1 - \frac{t(x + 2y)}{\sqrt{n}} + \frac{\alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \leq 1 - \frac{t(x + 2y)}{\sqrt{n}} + \frac{1}{\sqrt{n}(1 - M_n)}.$$

Yet, by definition of the domain \mathcal{D}_n , we have $x + 2y > -\sqrt{n}$, whence

$$1 - \frac{2itu^*}{\sqrt{n}} + \frac{iv^* \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \leq 2 + \frac{1}{\sqrt{n}(1 - M_n)}.$$

According to lemma 3, we have

$$1 - M_n \stackrel{n \rightarrow \infty}{\sim} \frac{2\pi^2 d_n^2}{3n^2} \stackrel{n \rightarrow \infty}{\sim} \frac{2\pi^2 \lambda^2}{3\sqrt{n}},$$

whence

$$\frac{1}{\sqrt{n}(1 - M_n)} \stackrel{n \rightarrow \infty}{\sim} \frac{3}{2\pi^2 \lambda^2} < \frac{1}{\lambda^2}.$$

Thus, for n large enough, we have

$$\frac{1}{\sqrt{n}(1 - M_n)} \leq \frac{1}{\lambda^2}.$$

We then have, for all $(x, y, z) \in \mathcal{D}_n$ and for all $t \in [0, 1]$,

$$1 - \frac{2itu^*}{\sqrt{n}} + \frac{iv^* \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \leq 2 + \frac{1}{\lambda^2}.$$

Replacing this in (72), we obtain that, for n large enough, for every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$,

$$\Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) \leq -\frac{1}{4} \sum_{j=1}^{n-1} \ln \left[1 + \frac{1}{2 + \lambda^{-2}} \left(\frac{2u}{\sqrt{n}} - \frac{v \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^2 \right].$$

As in the proof of lemma 7, we first study

$$S_1 = -\frac{1}{4} \sum_{j=1}^4 \ln \left[1 + \frac{1}{2 + \lambda^{-2}} \left(\frac{2u}{\sqrt{n}} - \frac{v \alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^2 \right].$$

Lemma 2 teaches us that, for every $j \leq 4$,

$$\frac{\alpha_j^n}{\sqrt{n}(1-\alpha_j^n)} \underset{n \rightarrow \infty}{\sim} \frac{3}{2\pi^2 \lambda^2 j^2}.$$

Therefore, from a certain rank n_0 , we have

$$\forall j \in \{1, 2, 3, 4\} \quad \frac{\alpha_j^n}{\sqrt{n}(1-\alpha_j^n)} \geq \frac{3}{64\pi^2 \lambda^2} \geq \frac{1}{214\lambda^2}.$$

We deduce that for $n \geq n_0 \vee 214^2 \lambda^4$ and for $j \leq 4$,

$$\left| \frac{2u}{\sqrt{n}} - \frac{v \alpha_j^n}{\sqrt{n}(1-\alpha_j^n)} \right| \geq \left(\left| \frac{v \alpha_j^n}{\sqrt{n}(1-\alpha_j^n)} \right| - \left| \frac{2u}{\sqrt{n}} \right| \right)_+ \geq \frac{(|v| - |u|)_+}{214\lambda^2}.$$

Hence, for n large enough, we have for every $(u, v) \in \mathbb{R}^2$,

$$S_1 \leq -\ln \left(1 + \frac{1}{2 + \lambda^{-2}} \frac{(|v| - |u|)_+^2}{214^2 \lambda^4} \right).$$

The other term

$$S_2 = -\frac{1}{4} \sum_{j=5}^{n-1} \ln \left[1 + \left(\frac{u}{\sqrt{n}} + \frac{v \alpha_j^n}{\sqrt{n}(1-\alpha_j^n)} \right)^2 \right]$$

may be dealt with exactly as in the proof of lemma 7, yielding the desired inequality. \square

Therefore, the function $e^{M\kappa}$ being integrable according to lemma 8, we can swap the summations over u and over v . We then note that the function $u \mapsto G_n(u, v^* + v)$ is defined and holomorphic on the open set

$$\left\{ u \in \mathbb{C} : 1 + \frac{2\Im u}{\sqrt{n}} - \frac{m_n}{\sqrt{n}(1+m_n)} > 0 \right\} \supset \left\{ u \in \mathbb{C} : 1 + \frac{2\Im u}{\sqrt{n}} - \frac{1}{2\sqrt{n}} > 0 \right\}.$$

Yet, it follows from the definition (67) of the domain \mathcal{D}_n that

$$|x| \leq n^{1/4} \quad \text{and} \quad y \leq n^{1/4},$$

whence

$$\Im u^* = -\frac{x+2y}{2} \geq -\frac{3}{2}n^{1/4}.$$

Therefore, for all $(x, y, z) \in \mathcal{D}_n$, we have

$$1 + \frac{2\Im u}{\sqrt{n}} - \frac{1}{2\sqrt{n}} \geq 1 - \frac{2}{n^{1/4}} - \frac{1}{2\sqrt{n}} > 0,$$

which allows us to move the integration contour of u from \mathbb{R} to $\mathbb{R} + u^*$, as we did in section 3.3. We then obtain the same formula as in (57), namely

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du \exp \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right). \quad (73)$$

4.5 Domination

We now search for an upper bound on the modulus of the integrand in (73), uniformly with respect to n . Namely, we prove the following result:

Lemma 14. *There exists $K, K_1 > 0$ such that, for n large enough,*

$$\forall (x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2 \quad \Re \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right) \leq M(x, y, z, u, v),$$

where the function $M : \mathbb{R}^5 \rightarrow \mathbb{R}$ is defined by

$$M(x, y, z, u, v) = -\frac{x^2 + 2xy + 8y^2}{6} + |x| + 5|y| - \frac{z^2}{2} + K_1 + M_K(u, v). \quad (74)$$

with M_K given by (52).

Proof. Upper bound on the first term: For every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, we have

$$\begin{aligned} \Re F_n(x, y, z, u^* + u, v^* + v) &= -i(x + 2y)u^* - iyv^* - \frac{iz^2v^*}{2} + xy - \frac{x^2y}{x + \sqrt{n}} \\ &= -\frac{(x + 2y)^2}{2} - y - \frac{z^2}{2} + xy - \frac{x^2y}{x + \sqrt{n}}. \end{aligned}$$

By definition of the domain \mathcal{D}_n , we have $|x| \leq n^{1/4}$, whence, for n large enough

$$\frac{x^2y}{x + \sqrt{n}} \leq \frac{\sqrt{n}}{\sqrt{n} - n^{1/4}}|y| \leq 2|y|.$$

Hence, for n large enough, we have for every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$,

$$\Re F_n(x, y, z, u^* + u, v^* + v) \leq -\frac{x^2}{2} - xy - 2y^2 + 3|y| - \frac{z^2}{2}. \quad (75)$$

Upper bound on the second term: Let $(x, y, z) \in \mathcal{D}_n$. We proceed as in the proof of lemma 9. Going back to the expression (68) of G_n , we have, for every $t \in [0, 1]$,

$$\begin{aligned} \frac{\partial G_n}{\partial u}(tu^*, v^*) &= -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left(1 - \frac{2itu^*}{\sqrt{n}} + \frac{\alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^{-1} \\ &= -\frac{i}{\sqrt{n}} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left(\frac{2itu^*}{\sqrt{n}} - \frac{\alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right) \left(1 - \frac{2itu^*}{\sqrt{n}} + \frac{\alpha_j^n}{\sqrt{n}(1 - \alpha_j^n)} \right)^{-1}. \end{aligned}$$

We deduce that

$$\left| \frac{\partial G_n}{\partial u}(tu^*, v^*) \right| \leq \frac{1}{\sqrt{n}} + \left(2t|u^*| + \frac{1}{n(1 - M_n)} \sum_{j=1}^{n-1} |\alpha_j^n| \right) \left(1 - \frac{t(x + 2y)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \right)^{-1}. \quad (76)$$

Yet, by the definition (67) of the domain \mathcal{D}_n , we know that

$$|x| \leq n^{1/4} \quad \text{and} \quad y \leq n^{1/4},$$

whence

$$1 - \frac{t(x+2y)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \geq 1 - \frac{3}{n^{1/4}} - \frac{1}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 1.$$

Thus, for n large enough, for every $(x, y, z) \in \mathcal{D}_n$ and every $t \in [0, 1]$, we have

$$1 - \frac{t(x+2y)}{\sqrt{n}} - \frac{1}{\sqrt{n}} \geq \frac{3}{4},$$

and therefore, inequality (76) becomes

$$\left| \frac{\partial G_n}{\partial u}(tu^*, v^*) \right| \leq \frac{1}{\sqrt{n}} + \frac{8t|u^*|}{3} + \frac{4}{3n(1-M_n)} \sum_{j=1}^{n-1} |\alpha_j^n|.$$

Yet, it follows from lemma 3 and from the upper bound (9) that

$$\frac{4}{3n(1-M_n)} \sum_{j=1}^{n-1} |\alpha_j^n| = O\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} |\alpha_j^n|\right) = O\left(\frac{\sqrt{n}(\ln n)}{d_n}\right) = o(1).$$

Thus, for n large enough, for every $(x, y, z) \in \mathcal{D}_n$ and every $t \in [0, 1]$, we have

$$\left| \frac{\partial G_n}{\partial u}(tu^*, v^*) \right| \leq 2 + \frac{4t|u^*|}{3}.$$

Integrating this, we get

$$|G_n(u^*, v^*)| \leq |G_n(0, -i)| + 2|u^*| + \frac{2|u^*|^2}{3} = |G_n(0, -i)| + |x+2y| + \frac{(x+2y)^2}{6}. \quad (77)$$

The pointwise convergence of G_n ensures that the sequence $G_n(0, -i)$ converges, and is therefore bounded from above by a certain $K_1 \in \mathbb{R}$. Therefore, combining (75), (77) and the result of lemma 13, we obtain the desired inequality. \square

We now check that the resulting function is indeed integrable:

Lemma 15. *The function e^M , where M is defined by (74), is integrable over \mathbb{R}^5 .*

Proof. We have, in $[0, +\infty]$,

$$\int_{\mathbb{R}^5} dx dy dz du dv e^{M(x,y,z,u,v)} = \int_{\mathbb{R}^2} dx dy e^{M(x,y,0,0,0)} \times \int_{\mathbb{R}} dz e^{-z^2/2} \times \int_{\mathbb{R}^2} du dv e^{M_\kappa(u,v)}.$$

We then write

$$\begin{aligned} M(x, y, 0, 0, 0) &= -\frac{x^2 + 2xy + 8y^2}{6} + |x| + 5|y| = -\frac{(x+y)^2 + 7y^2}{6} + |x| + 5|y| \\ &\leq -\frac{(x+y)^2 + 7y^2}{6} + |x+y| + 6|y|, \end{aligned}$$

which leads to

$$\int_{\mathbb{R}^2} dx dy e^{M(x,y,0,0,0)} \leq \int_{\mathbb{R}} da e^{|a|-a^2/6} \times \int_{\mathbb{R}} dy e^{6|y|-7y^2/6} < +\infty.$$

The function e^{M_κ} being integrable over \mathbb{R}^2 according to lemma 8, we obtain a finite integral. \square

4.6 Dominated convergence

We are now in a position to apply the dominated convergence theorem. Following the results of section 4.3, for all $(x, y, z) \in \mathbb{R}^3$, we have the pointwise convergence

$$\lim_{n \rightarrow \infty} F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) = J(x, y, z, u, v),$$

with

$$J(x, y, z, u, v) = -\frac{x^2}{4} - y^2 - y - \frac{z^2}{2} - u^2 - ivy - \frac{ivz^2}{2} - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3i(v-i)}{2\pi^2 \lambda^2 j^2} \right).$$

The domination hypothesis is satisfied for n large enough, thanks to lemmas 14 and 15. Thus, by virtue of the dominated convergence theorem, we have

$$E_n \stackrel{n \rightarrow \infty}{\sim} C_n \int_{\mathbb{R}^5} dx dy dz du dv g(z) e^{J(x, y, z, u, v)} = 2\pi C_n \int_{\mathbb{R}^3} dy dz dv g(z) e^{J(0, y, z, 0, v)}.$$

In the case of the constant function $g \equiv 1$, going back to the definition (27) of E_n , we get

$$E_n \leq \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = 1,$$

whence $C_n = O(1)$. Recalling the definition (33) of C_n , we deduce the following upper bound on the normalization constant:

$$\frac{1}{Z_n} = O \left(n^{1/4} \prod_{j=1}^{n-1} \frac{1}{\sqrt{1 - \alpha_j^n}} \right). \quad (78)$$

Yet, according to lemma 5, there exists $K > 0$ such that, for n large enough,

$$\left| \sum_{j=1}^{n-1} \ln(1 - \alpha_j^n) \right| \leq K \frac{n \ln n}{d_n}.$$

Consequently, our estimation (78) becomes

$$\frac{1}{Z_n} = O \left[n^{1/4} \exp \left(K \frac{n \ln n}{d_n} \right) \right]. \quad (79)$$

Recall now that, as explained at the beginning of section 4.2, for every bounded and continuous function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, we have

$$\mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = E_n + o(1) + O \left(\frac{e^{-n^{3/4}/4}}{Z_n} \right).$$

In view of (79) and given the fact that $d_n \sim \lambda n^{3/4}$, this becomes

$$\mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = E_n + o(1) + O \left[n^{1/4} \exp \left(K \frac{n \ln n}{d_n} - \frac{n^{3/4}}{8} \right) \right] = E_n + o(1).$$

In the case of the constant function $g \equiv 1$, we obtain

$$1 \stackrel{n \rightarrow \infty}{\sim} E_n \stackrel{n \rightarrow \infty}{\sim} 2\pi C_n \int_{\mathbb{R}^3} dy dz dv e^{J(0, y, z, 0, v)}.$$

Therefore, we have

$$\lim_{n \rightarrow \infty} C_n = \left(2\pi \int_{\mathbb{R}^3} dy dz dv e^{J(0,y,z,0,v)} \right)^{-1}.$$

Thus, for every bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\lim_{n \rightarrow \infty} \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = \frac{I(g)}{I(1)}, \quad (80)$$

where

$$I(g) = \int_{\mathbb{R}^3} dy dz dv g(z) e^{J(0,y,z,0,v)}. \quad (81)$$

4.7 Identification of the limit law

We can write

$$J(0, y, z, 0, v) = -y^2 + f_{(y+z^2/2)}(v-i),$$

where, for $\theta \in \mathbb{R}$, we write

$$f_\theta : v \mapsto -i\theta v - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3iv}{2\pi^2 \lambda^2 j^2} \right),$$

which is defined and holomorphic on the open set

$$V = \left\{ v \in \mathbb{C} : \Im v < \frac{2\pi^2 \lambda^2}{3} \right\}.$$

With this notation, equation (81) becomes

$$I(g) = \int_{\mathbb{R}} dz g(z) \int_{\mathbb{R}} dy e^{-y^2} \int_{\mathbb{R}} dv e^{f_{(y+z^2/2)}(v-i)}.$$

We are now going to move the integration contour of the variable v to get rid of the term $-i$, which amounts to cancelling our last displacement of the contour in v (which was nevertheless useful to obtain an expression satisfying the domination hypothesis). Let $\theta \in \mathbb{R}$. The function e^{f_θ} being holomorphic on the open set V , Cauchy's theorem ensures that, for every $M > 0$,

$$\int_{-M}^M dv e^{f_\theta(v)} = \int_0^1 (-i) dt e^{f_\theta(-M-it)} + \int_{-M}^M dv e^{f_\theta(v-i)} + \int_1^0 (-i) dt e^{f_\theta(M-it)}.$$

Yet, for every $v \in V$ we have

$$\left| e^{f_\theta(v)} \right| = \exp \left[\theta \Im v - \frac{1}{2} \sum_{j=1}^{+\infty} \ln \left(1 + \frac{9|v|^2}{4\pi^4 \lambda^2 j^4} \right) \right] \leq e^{\theta \Im v} \left(1 + \frac{9|v|^2}{4\pi^4 \lambda^4} \right)^{-1/2},$$

whence

$$\left| \int_0^1 (-i) dt e^{f_\theta(\pm M-it)} \right| \leq e^{|\theta|} \left(1 + \frac{9M^2}{4\pi^4 \lambda^4} \right)^{-1/2} \xrightarrow{M \rightarrow +\infty} 0.$$

Therefore, taking the limit $M \rightarrow \infty$ we obtain

$$\int_{\mathbb{R}} dv e^{f_\theta(v)} = \int_{\mathbb{R}} dv e^{f_\theta(v-i)},$$

whence

$$I(g) = \int_{\mathbb{R}} dz g(z) \int_{\mathbb{R}} dy e^{-y^2} \int_{\mathbb{R}} dv e^{f_{(y+z^2/2)}(v)}.$$

We now fix $z \in \mathbb{R}$, and we wish to swap the integral over y and the integral over v . For every $(y, v) \in \mathbb{R}^2$, we have

$$\left| e^{-y^2 + f_{(y+z^2/2)}(v)} \right| = e^{-y^2} \prod_{j=1}^{+\infty} \left(1 + \frac{9v^2}{4\pi^4 \lambda^4 j^4} \right)^{-1/2} \leq e^{-y^2} \left(1 + \frac{9v^2}{4\pi^4 \lambda^4} \right)^{-1/2} \left(1 + \frac{9v^2}{64\pi^4 \lambda^4} \right)^{-1/2},$$

which is an integrable function (y, v) over \mathbb{R}^2 . Thus, we may swap the two summations, to obtain

$$\begin{aligned} I(g) &= \int_{\mathbb{R}} dz g(z) \int_{\mathbb{R}} dv \exp \left[-\frac{ivz^2}{2} - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3iv}{2\pi^2 \lambda^2 j^2} \right) \right] \int_{\mathbb{R}} dy e^{-ivy - y^2} \\ &= \sqrt{\pi} \int_{\mathbb{R}} dz g(z) \int_{\mathbb{R}} dv \exp \left[-\frac{ivz^2}{2} - \frac{v^2}{4} - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3iv}{2\pi^2 \lambda^2 j^2} \right) \right], \end{aligned} \quad (82)$$

where we used formula (4) to compute the integral over y . To identify the result announced in theorem 2, we consider a family $(Y_j)_{j \in \mathbb{Z}}$ of i.i.d. standard normal variables, and we define the random variable

$$Z = \sqrt{2}Y_0 - \frac{3}{2\pi^2 \lambda^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{Y_j^2}{j^2}.$$

We have, when $j \rightarrow +\infty$,

$$\mathbb{P} \left(|Y_j| \leq \sqrt[4]{j} \right) = \frac{2}{\sqrt{2\pi}} \int_{\sqrt[4]{j}}^{+\infty} dy e^{-y^2/2} \leq \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} dt e^{-\sqrt{j}/2 - t^2/2} = e^{-\sqrt{j}/2},$$

whence

$$\sum_{j \in \mathbb{Z}} \mathbb{P} \left(|Y_j| \leq \sqrt[4]{j} \right) \leq \sum_{j \in \mathbb{Z}} e^{-\sqrt{|j|}/2} < +\infty.$$

Thus, Borel-Cantelli's lemma ensures that

$$\mathbb{P}(|Z| < \infty) \geq \mathbb{P} \left(\bigcup_{k \geq 1} \bigcap_{|j| \geq k} \left\{ |Y_j| \leq |j|^{1/4} \right\} \right) = 1.$$

Therefore, our random variable Z is almost surely finite, and we can compute its Fourier transform ϕ_Z . For every $v \in \mathbb{R}$, we have

$$\phi_Z(v) = \mathbb{E} [e^{ivZ}] = \mathbb{E} [e^{iv\sqrt{2}Y_0}] \prod_{j \in \mathbb{Z} \setminus \{0\}} \mathbb{E} \left[\exp \left(-\frac{3ivY_j^2}{2\pi^2 \lambda^2 j^2} \right) \right].$$

Using again our Gaussian integration formula (4), we get

$$\phi_Z(v) = \exp \left[-v^2 - \sum_{j=1}^{+\infty} \ln \left(1 + \frac{3iv}{\pi^2 \lambda^2 j^2} \right) \right].$$

This function ϕ_Z is integrable, since

$$\forall v \in \mathbb{R} \quad |\phi_Z(v)| \leq e^{-v^2}.$$

Therefore, the variable Z admits a density f given by the inverse Fourier transform of ϕ_Z , that is to say

$$\forall z \in \mathbb{R} \quad f(z) = \int_{\mathbb{R}} dv e^{-ivz} \phi_Z(v).$$

Thus, the equation (82) now reads

$$\begin{aligned} I(g) &= \sqrt{\pi} \int_{\mathbb{R}} dz g(z) \int_{\mathbb{R}} dv e^{-ivz^2/2} \phi_Z(v/2) \\ &= \sqrt{\pi} \int_{\mathbb{R}} dz g(z) \int_{\mathbb{R}} dv e^{-ivz^2} \phi_Z(v) \\ &= \sqrt{\pi} \int_{\mathbb{R}} dz g(z) f(z^2). \end{aligned}$$

Replacing this in (80), we obtain

$$\lim_{n \rightarrow \infty} \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = \frac{\int_{\mathbb{R}} dz g(z) f(z^2)}{\int_{\mathbb{R}} dz f(z^2)},$$

which concludes the proof of the convergence in distribution announced in theorem 2. □

References

- [ACCN88] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman. Discontinuity of the magnetization in one-dimensional $1/|x - y|^2$ Ising and Potts models. *J. Statist. Phys.*, 50(1-2):1–40, 1988.
- [BTW87] Per Bak, Chao Tang, and Kurt Wiesenfeld. Self-organized criticality: An explanation of the $1/f$ noise. *Physical review letters*, 59(4):381, 1987.
- [BZ97] Anton Bovier and Miloš Zahradník. The low-temperature phase of Kac-Ising models. *J. Statist. Phys.*, 87(1-2):311–332, 1997.
- [CG16] Raphaël Cerf and Matthias Gorny. A Curie-Weiss model of self-organized criticality. *Ann. Probab.*, 44(1):444–478, 2016.
- [Cop04] E. T. Copson. *Asymptotic expansions*, volume 55 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004. Reprint of the 1965 original.
- [Ell06] Richard S. Ellis. *Entropy, large deviations, and statistical mechanics*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1985 original.
- [EN78] Richard S. Ellis and Charles M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete*, 44(2):117–139, 1978.
- [For21] Nicolas Forien. An extension of the Ising Curie-Weiss model of self-organized criticality with long range interactions. *preprint available on arXiv*, 2021.
- [Gor14] M. Gorny. A Curie-Weiss model of self-organized criticality: the Gaussian case. *Markov Process. Related Fields*, 20(3):563–576, 2014.

- [KLS19] Zakhar Kabluchko, Matthias Löwe, and Kristina Schubert. Fluctuations of the magnetization for Ising models on dense Erdős-Rényi random graphs. *J. Stat. Phys.*, 177(1):78–94, 2019.
- [KLS20] Zakhar Kabluchko, Matthias Löwe, and Kristina Schubert. Fluctuations of the magnetization for Ising models on Erdős-Rényi random graphs—the regimes of small p and the critical temperature. *J. Phys. A*, 53(35):355004, 37, 2020.
- [KUH63] M. Kac, G. E. Uhlenbeck, and P. C. Hemmer. On the van der Waals theory of the vapor-liquid equilibrium. I. Discussion of a one-dimensional model. *J. Mathematical Phys.*, 4:216–228, 1963.