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An extension of the Ising-Curie-Weiss model of self-organized criticality with long range interactions

Nicolas Forien*

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Abstract

In [CG16], Cerf and Gorny constructed a model of self-organized criticality, by introducing an automatic control of the temperature parameter in the generalized Ising Curie-Weiss model. In this article, we build upon this model by replacing the mean-field interaction of [CG16] with a one-dimensional interaction with a certain range d_n which varies as a function of the number n of particles. In the Gaussian case, we show that, for a very long range of interaction ($2d_n \sim \lambda n$), the model exhibits the same behaviour as in the mean-field case, whereas in the case of a nearest neighbour interaction ($d_n = 1$), the behaviour highlighted by Cerf and Gorny breaks out.

Mathematics Subject Classification : 82B20 (Primary) 82B27, 60K35 (Secondary)

1 Introduction

1.1 Definition of the model

This article is devoted to the study in two different regimes of a one-dimensional spin model with long range interactions and with a self-adjusted temperature. This model is defined as follows. First, we choose a sequence of integers $(d_n)_{n \geq 3}$ such that $0 < 2d_n < n$ for every $n \geq 3$, and which will represent the interaction range. Then, we define the Hamiltonian

$$H_n : (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto -\frac{1}{2d_n} \sum_{i=1}^n \sum_{j=1}^{d_n} x_i x_{i+j} \quad (1)$$

with the convention $x_{n+k} = x_k$ for all $k \in \{1, \dots, n\}$. This Hamiltonian corresponds to a one-dimensional chain of spins, with periodic boundary conditions, where every spin interacts with its $2d_n$ nearest neighbours. The model we consider is then given by the following probability distribution on \mathbb{R}^n :

$$d\mu_n(x_1, \dots, x_n) = \frac{1}{Z_n} \exp\left(-\frac{H_n(x_1, \dots, x_n)}{T_n(x_1, \dots, x_n)}\right) \mathbb{1}_{\{T_n > 0\}} \prod_{i=1}^n d\nu(x_i), \quad (2)$$

where Z_n is the normalization constant, ν is the standard normal distribution, and T_n is the self-adjusted temperature given by

$$T_n : (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto \frac{x_1^2 + \dots + x_n^2}{n}.$$

This model is well defined because

$$\forall (x_1, \dots, x_n) \in \mathbb{R}^n \quad |H_n(x_1, \dots, x_n)| \leq \frac{1}{2d_n} \sum_{i=1}^n \sum_{j=1}^{d_n} \frac{x_i^2 + x_{i+j}^2}{2} = \frac{n}{2} T_n(x_1, \dots, x_n). \quad (3)$$

*Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France

1.2 Results

In this article, we study the model defined by (2) in two extreme cases: on the one hand, in the case of very long range interactions, with d_n proportional to n , and on the other hand, in the case of nearest neighbour interactions, which corresponds to $d_n = 1$. In a subsequent article [For21], we will study an intermediate regime of the interaction range, showing that there is a threshold phenomenon when the range d_n is of order $n^{3/4}$.

1.2.1 The long range case

The following result indicates the asymptotic behaviour when the range of interaction is proportional to the total number of particles:

Theorem 1. *If the interaction range d_n is chosen such that*

$$2d_n \stackrel{n \rightarrow \infty}{\sim} \lambda n \quad \text{with} \quad 0 < \lambda < 1, \quad (4)$$

then, under the law μ_n defined by (2), we have the convergence in distribution

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \frac{\sqrt{2}}{\Gamma(1/4)} \exp\left(-\frac{z^4}{4}\right) dz.$$

The particular case $2d_n = n - 1$, which corresponds to a mean-field interaction, was already studied in [Gor14] (in fact there is a small difference, with a factor $1 - 1/n$ between our Hamiltonian and the one studied by Gorny, but this does not change the behaviour of the model). Thus, the above theorem shows that, as long as the interaction range remains proportional to n , the behaviour is the same as in the mean-field case.

1.2.2 The case of a nearest neighbour interaction

When $d_n = 1$, the Hamiltonian takes the simpler form

$$H_n(x_1, \dots, x_n) = -\frac{1}{2} \sum_{j=1}^n x_j x_{j+1},$$

with the convention $x_{n+1} = x_1$. In this case, the interaction is limited to the nearest neighbours, and the behaviour is very different from the long range case. Indeed, the following theorem shows that the phenomenon observed in the mean-field and long range cases does not occur in the nearest neighbour case, where the fluctuations of S_n are Gaussian and of order \sqrt{n} :

Theorem 2. *When $d_n = 1$, under the law μ_n defined by (2), we have the convergence in distribution*

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \sqrt{2} + 1).$$

1.3 Remarks and perspectives

1.3.1 A Curie-Weiss model of self-organized criticality

The initial motivation which lead us to consider this model comes from the work of Cerf and Gorny [Gor14, CG16], who constructed a simple mean-field model of self-organized criticality. To recover their model, let

us look at our Hamiltonian (1) in the extreme case $2d_n = n - 1$, where our model becomes a mean-field model. In this case, we have

$$H_n(x_1, \dots, x_n) = -\frac{1}{n-1} \sum_{i<j} x_i x_j = -\frac{1}{2(n-1)} \left[\left(\sum_{j=1}^n x_j \right)^2 - \sum_{j=1}^n x_j^2 \right] = -\frac{S_n^2 - nT_n}{2(n-1)},$$

and thus our definition (2) becomes

$$d\mu_n(x_1, \dots, x_n) = \frac{1}{Z_n} \exp\left(\frac{S_n^2}{2(n-1)T_n}\right) \mathbb{1}_{\{T_n > 0\}} \prod_{i=1}^n d\nu(x_i).$$

In [Gor14], Gorny studied this model (with n instead of $n-1$ in the denominator, but this does not change much), and he proved theorem 1 in this mean-field setting. The idea of Gorny was to obtain a toy model showing self-organized criticality, constructed as a variant of the generalized Ising Curie-Weiss model.

The concept of self-organized criticality was coined in by the physicists Bak, Tang and Wiesenfeld in their seminal article [BTW87], to explain the widespread presence of fractal structures in nature. They observed that some physical systems present a “critical-like” behaviour, with fractal structures and power-law correlations, without the need to finely tune a parameter (e.g., the temperature) to a critical value. They called this phenomenon “self-organized criticality”. The important difference with ordinary phase transitions is that the critical regime, instead of being a very specific regime which only appears for a very precise value of the parameters of the system, becomes an attracting point, the system being “forced” to look critical. Several mathematical models of self-organized criticality have been studied, but these models are often quite complex and not easily tractable [For20].

To construct a simple toy model of self-organized criticality, Gorny started from the generalized Ising Curie-Weiss model and, following an idea explained by Sornette [Sor92], he replaced the temperature parameter with a function of the spins, in order to introduce a kind of feedback from the configuration onto the temperature parameter. Starting from a model of the form $\exp(-H_n(\sigma)/T)$, with a phase transition for a critical temperature T_c , the technique consists of replacing this temperature parameter T with a function $T_n(\sigma)$, which tends to concentrate around the critical value T_c when $n \rightarrow \infty$. In [Gor14], Gorny proved that this construction indeed leads to a model exhibiting a behaviour similar to the critical regime of the initial model, but without the need to finely tune a parameter to a critical value: this is why we talk of self-organized criticality.

This work was then extended to more general distributions of the spins in [CG16], but still in a mean-field setting. In fact, the model studied in the present article could also be defined with more general distributions than only the standard normal law ν , but we chose to restrict ourselves to Gaussian spins, as in [Gor14], to enable exact computations.

1.3.2 Different kinds of long range interactions

The first motivation to study this model was to try to extend the construction of Cerf and Gorny, which was in a mean-field setting, to define a more geometrical model. Thus, we studied the behaviour of the model in one of the simplest geometrical setting, namely a one-dimensional nearest neighbour interaction with periodic boundary conditions. But, as theorem 2 shows, it turns out that this model does not present the same critical-like asymptotic behaviour as observed for the mean-field model. At this point, a natural question arises: if the interesting behaviour (with the fluctuations of order $n^{3/4}$ and the limiting distribution of the form $\exp(-\lambda n^{3/4})$) is observed in the mean-field case, i.e., when each spin interacts with all the other spins, but not when each spin only interacts with its two nearest neighbours, then what about the intermediate cases between these two extreme situations?

This led us to consider a model with an intermediate interaction range. But there are many different ways to interpolate between a mean-field interaction and a nearest neighbour interaction. Generally

speaking, one can consider a Hamiltonian H_n of the form

$$H_n(x_1, \dots, x_n) = - \sum_{1 \leq i, j \leq n} J(i, j) x_i x_j,$$

where the coupling constants $J(i, j) = J(|i - j|)$ are decreasing functions of the distance separating the particles i and j (with periodic boundary conditions). The behaviour of the model then depends on the decay rate of this coupling function. In [ACCN88], the key example of a coupling proportional to $|i - j|^{-2}$ is studied, and it is proved that the resulting Ising model presents a phase transition, as well as models constructed with a slower-decaying coupling function. Otherwise, if $J(i, j) = o(|i - j|^{-2})$, then the obtained Ising model does not anymore present a phase transition, which shows that a coupling of order $|i - j|^{-2}$ plays a pivotal role for the appearance of a phase transition.

Another way to design intermediate models consists in drawing random couplings $J(i, j)$. In [BG93], Bovier and Gaynard constructed such a model by taking for the $J(i, j)$ independent Bernoulli variables of parameter p , which amounts to considering the Ising model on an Erdős-Rényi random graph. This model exhibits different regimes characterized by different fluctuations of the sum of spins, depending on how the parameter p varies with the number n of particles. These different regimes were studied by Kabluchko, Löwe and Schubert [KLS19, KLS20] who proved in particular that, for a critical temperature and a parameter p_n chosen such that $n^{-3/4} = o(p_n)$, the behaviour resembles that of the critical Ising Curie-Weiss model, i.e., the sum of the spins is of order $n^{3/4}$ with fluctuations of the form $C \exp(-\lambda s^4)$. When the parameter p_n becomes of order $n^{-3/4}$, still at the critical temperature, the limiting distribution changes, and a quadratic term appears besides the term in s^4 . If $p_n = o(n^{-3/4})$ then this quadratic term dominates, which results in Gaussian fluctuations of the sum of spins.

This approach was generalized by Deb and Mukherjee [DM20], who studied the fluctuations of an Ising model defined on a more general set of graphs. Under certain conditions of homogeneity and connectivity, they obtain the same fluctuations as in the mean-field model, when the mean degree d_n in the graph satisfies $(n \ln n)^{1/3} = o(d_n)$ in the supercritical regime, $\sqrt{n} = o(d_n)$ in the subcritical regime, or $\sqrt{n} \ln n = o(d_n)$ at the critical point.

In our case, we chose an interaction of each spin with its $2d_n$ nearest neighbours, where d_n is a parameter which evolves with n . This corresponds to a coupling function of the form

$$J(i, j) = \frac{1}{4d_n} \times \begin{cases} 1 & \text{if } j \in \{i - d_n, \dots, i - 1\} \cup \{i + 1, \dots, i + d_n\} + n\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us remind that our model is different from the aforementioned models, because the spins are not valued in $\{-1, +1\}$ but are real-valued, and because our self-adjusted temperature in fact induces an interaction between all the spins. Therefore, it is not *a priori* evident which scale of d_n is relevant to observe a change of behaviour. This question is (partially) solved in the forthcoming article [For21], where we prove that the limiting distribution changes when d_n is of order $n^{3/4}$.

1.4 Strategy of proof

1.4.1 Diagonalization of the interaction Hamiltonian

The starting point to study our model is to diagonalize the interaction Hamiltonian. The matrix of the quadratic form H_n given by (1) is a symmetric circulant matrix, which writes

$$\mathfrak{M}(H_n) = -\frac{1}{4d_n} \sum_{m=1}^{d_n} (J^m + J^{-m}),$$

where the matrix J is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} = \left(\mathbb{1}_{j=i+1 \pmod n} \right)_{1 \leq i, j \leq n}.$$

The matrix J is diagonalizable, with eigenvalues

$$\text{sp}(J) = \left\{ e^{2ij\pi/n} : j \in \{1, \dots, n\} \right\}.$$

Therefore, for every $m \geq 1$, we have

$$\text{sp}(J^m + J^{-m}) = \left\{ 2 \cos\left(\frac{2jm\pi}{n}\right) : j \in \{1, \dots, n\} \right\},$$

and thus the eigenvalues of our Hamiltonian H_n are

$$\text{sp}(H_n) = \left\{ -\frac{\alpha_1^n}{2}, \dots, -\frac{\alpha_n^n}{2} \right\},$$

where

$$\forall j \in \{1, \dots, n\} \quad \alpha_j^n = \frac{1}{d_n} \sum_{m=1}^{d_n} \cos\left(\frac{2jm\pi}{n}\right). \quad (5)$$

1.4.2 Change of variables

Let $P \in \mathcal{O}_n(\mathbb{R})$ be an orthonormal matrix such that the matrix of the quadratic form H_n writes tPDP , where D is the diagonal matrix with diagonal coefficients $-\alpha_1^n/2, \dots, -\alpha_n^n/2$. We define the change of variables

$$\varphi : (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto \left(\sum_{k=1}^n P_{j,k} x_k \right)_{1 \leq j \leq n}.$$

With this change of variables $(y_1, \dots, y_n) = \varphi(x_1, \dots, x_n)$, the Hamiltonian H_n and the self-adjusted temperature T_n become

$$H_n(x_1, \dots, x_n) = -\frac{1}{2} \sum_{j=1}^n \alpha_j^n y_j^2, \quad \text{and} \quad T_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{j=1}^n y_j^2. \quad (6)$$

To see what happens to the sum $S_n = x_1 + \dots + x_n$ of the spins, note that $\alpha_n^n = 1$ and that this eigenvalue $-\alpha_n^n/2 = -1/2$ of H_n corresponds to the eigenvector $(1/\sqrt{n}, \dots, 1/\sqrt{n})$, whence $P_{n,k} = 1/\sqrt{n}$ for all $k \in \{1, \dots, n\}$. Therefore, we have

$$(y_1, \dots, y_n) = \varphi(x_1, \dots, x_n) \quad \Rightarrow \quad S_n = x_1 + \dots + x_n = \sqrt{n} y_n. \quad (7)$$

Also, the change of variable being orthonormal, we have

$$\prod_{i=1}^n d\nu(x_i) = d\nu^{\otimes n}(x_1, \dots, x_n) = d\nu^{\otimes n} \varphi^{-1}(y_1, \dots, y_n) = d\nu^{\otimes n}(y_1, \dots, y_n) = \prod_{i=1}^n d\nu(y_i),$$

which allows us to forget the variables x_1, \dots, x_n and to work only with the new variables y_1, \dots, y_n to study the limiting behaviour of S_n , which is now $S_n = \sqrt{n} y_n$.

1.4.3 Competition between two terms

Using the fact that $\alpha_n^n = 1$, we may write

$$H_n(x_1, \dots, x_n) = -\frac{1}{2} \sum_{j=1}^n \alpha_j^n y_j^2 = -\frac{y_n^2}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \alpha_j^n y_j^2 = -\frac{S_n^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \alpha_j^n y_j^2. \quad (8)$$

The first term exactly corresponds to the Hamiltonian in the mean-field model of Cerf and Gorny, as mentioned in section 1.3.1. When $2d_n = n - 1$ (mean-field case), we have $\alpha_j^n = -1/(2d_n)$ for every $j \neq n$ so the second term almost disappears (in fact it contributes as $-S_n^2/(4nd_n)$, but this has no significant effect). The behaviour of our model therefore results from the competition between these two terms, and we will see that, in the regime $2d_n \sim \lambda n$ with $\lambda \in (0, 1)$, the first term dominates, inducing a behaviour similar to the mean-field case. Thus, the key point is to control these $(n - 1)$ first eigenvalues, and to show that they are small enough to prevent this second term from perturbing the mean-field-like behaviour. The computational estimates used to control these eigenvalues are gathered in section 2.1.

1.4.4 Fourier inversion and saddle-point method

Having chosen the initial distribution ν of the spins to be a standard normal distribution enables us to perform exact computations. In section 2.2, we use the Fourier transform to obtain the distribution of (T_n, H_n, S_n) in our model μ_n . This provides us with an explicit integral expression for this distribution, involving the eigenvalues α_j^n defined above.

We then implement the saddle-point method (see for example [Cop04]), which consists in a judicious change of integration contour in the complex plane in order to obtain an integrand with an appropriate limiting behaviour. This idea to use the saddle-point method to study a long range Ising model, after having diagonalized the interaction matrix, was already presented by Canning in a series of publications [Can92a, Can92b, Can93]. However, Canning only discusses the saddle-point method in cases where the interaction matrix has a finite rank, and he does not give rigorous bounds on the precision of the obtained approximation. In our computations, it turns out that details such as the domination hypothesis to apply the dominated convergence theorem are not that evident and in fact require a precise control of the asymptotic behaviour of the eigenvalues α_j^n .

1.4.5 Study of the model with nearest neighbour interactions

We start with the study of the model with nearest neighbour interactions, which corresponds to the case $d_n = 1$. Before implementing the saddle-point method, as described above, as a preliminary step, we study the behaviour of T_n, H_n and S_n with a large-deviation approach. This approach seems natural because our model μ_n is constructed with an exponential term $\exp(-H_n/T_n)$: thus, the rate function in our model will easily follow from the rate function under the i.i.d. distribution $\nu^{\otimes n}$.

This is done in section 3, where we prove the following large deviation principle for T_n, H_n and S_n , using the Gärtner-Ellis theorem:

Theorem 3. *Under the law μ_n , when $d_n = 1$, the vector*

$$\left(T_n, \frac{H_n}{n}, \frac{S_n}{n} \right)$$

satisfies a large deviation principle governed by the good rate function

$$J(x, y, z) = \begin{cases} \frac{1}{2} \left[C + x - 1 - \ln x - \ln \left(1 + \frac{2y}{x} \right) - \ln \left(1 - \frac{2y + z^2}{x - z^2} \right) + \frac{2y}{x} \right] & \text{if } 2x > x - 2y > 2z^2, \\ +\infty & \text{otherwise,} \end{cases}$$

where the constant C is given by

$$C = -\ln(\sqrt{2}-1) - \ln 2 - \sqrt{2} + 1.$$

This large deviation principle is a first step forward, but it is not enough to obtain the convergence in distribution of S_n/\sqrt{n} announced in theorem 2. To prove this theorem, we use the Fourier transform to obtain an exact integral expression for the distribution of S_n , and we implement the saddle-point method to study the asymptotic behaviour of this expression. In the regime $d_n = 1$, the eigenvalues of the interaction Hamiltonian have a simple expression: the formula (5) becomes

$$\forall j \in \{1, \dots, n\} \quad \alpha_j^n = \cos\left(\frac{2j\pi}{n}\right).$$

Hence, the sums involving the α_j^n can be written as Riemann sums, which yields an explicit integral expression for the limit. The implementation of the saddle-point method in this case $d_n = 1$ is detailed in section 4.

1.4.6 Study of the model with long range interactions

To obtain theorem 1, we also use the Fourier inversion and the saddle-point method, but the eigenvalues α_j^n have a more complex expression. Thus, the important ingredient is a precise control of the asymptotic behaviour of these eigenvalues in the regime $2d_n \sim \lambda n$. In section 2.1, we study these eigenvalues and we note that, in this regime,

$$\lim_{n \rightarrow \infty} \alpha_j^n = \text{sinc}(j\lambda\pi) = \frac{\sin(j\lambda\pi)}{j\lambda\pi}$$

for every fixed $j \geq 1$. This allows us in section 5 to derive the limit of the integrand in the saddle-point method, yielding the desired convergence in distribution.

In the intermediate regime when $d_n \rightarrow \infty$ with $d_n = o(n)$, which is studied in a subsequent article [For21], the strategy of the proof is similar, but the behaviour of the eigenvalues changes, since $\alpha_j^n \rightarrow 1$ for every fixed $j \geq 1$. This forces to adapt the method and to change the scaling of the different variables, and this is why we restrict here the presentation to the two extreme cases $d_n = 1$ and $d_n \sim \lambda n$, which already require some work.

2 Preliminary computations

2.1 Estimates on the eigenvalues in the long range regime

We start with some preliminary computations on the eigenvalues $-\alpha_j^n/2$ of the interaction Hamiltonian, which are given by the formula (5). For every $j \in \{1, \dots, n-1\}$, we can write

$$\begin{aligned} \alpha_j^n &= \frac{1}{d_n} \Re \sum_{m=1}^{d_n} e^{2ijm\pi/n} \\ &= \frac{1}{d_n} \Re \left(e^{2ij\pi/n} \frac{e^{2ij d_n \pi/n} - 1}{e^{2ij\pi/n} - 1} \right) \\ &= \frac{\cos(j(d_n+1)\pi/n) \sin(jd_n\pi/n)}{d_n \sin(j\pi/n)} \end{aligned} \tag{9}$$

$$= \frac{1}{2d_n} \left(\frac{\sin((2d_n+1)j\pi/n)}{\sin(j\pi/n)} - 1 \right). \tag{10}$$

We will also be interested in the extrema of the spectrum (not counting the particular eigenvalue $\alpha_n^n = 1$), thus we define

$$M_n = \max \left\{ \alpha_j^n : 1 \leq j \leq n-1 \right\} \quad \text{and} \quad m_n = \max \left\{ -\alpha_j^n : 1 \leq j \leq n-1 \right\}. \quad (11)$$

Note that the eigenvalues are symmetric, that is to say, for any $j \in \{1, \dots, n-1\}$, we have $\alpha_{n-j}^n = \alpha_j^n$. Besides, it follows from (9) that

$$\forall j \in \left\{ 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \quad |\alpha_j^n| \leq \frac{1}{d_n \sin(j\pi/n)} \leq \frac{n}{2d_n j}. \quad (12)$$

From this we can deduce that

$$\sum_{j=1}^{n-1} |\alpha_j^n| \leq 2 \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{n}{2d_n j} = O\left(\frac{n \ln n}{d_n}\right), \quad (13)$$

and, similarly,

$$\sum_{j=1}^{n-1} (\alpha_j^n)^2 = O\left(\frac{n^2}{d_n^2}\right). \quad (14)$$

This last upper bound is not sharp, as can be seen by computing the trace of the square of the matrix of the quadratic form H_n , but this bound will be enough for our needs (it will only be used in the proof of lemma 14). We now prove some bounds on these eigenvalues in the regime (4), i.e., $2d_n \sim \lambda n$ with $\lambda \in (0, 1)$.

Lemma 1. *In this regime (4), for every fixed $j \geq 1$, we have*

$$\lim_{n \rightarrow \infty} \alpha_j^n = \text{sinc}(j\lambda\pi),$$

where sinc is the function $x \mapsto \sin x/x$. Furthermore, there exists $K = K_\lambda > 0$ such that

$$\forall n \geq 3 \quad \forall j \in \left\{ 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \quad |\alpha_j^n - \text{sinc}(j\lambda\pi)| \leq \frac{Kj}{n} + K \left| \frac{2d_n}{n} - \lambda \right|.$$

Proof. Let us write, with a O which is uniform over all $j \leq \lfloor n/2 \rfloor$,

$$\sin\left(\frac{j\pi}{n}\right) = \frac{j\pi}{n} \left[1 + O\left(\frac{j^2}{n^2}\right) \right].$$

Yet, the function $x \mapsto \sin x/x$ is bounded away from 0 on $[0, \pi/2]$ by a strictly positive constant, which allows us to take the inverse of this Taylor expansion:

$$\frac{1}{\sin(j\pi/n)} = \frac{n}{j\pi} \left[1 + O\left(\frac{j^2}{n^2}\right) \right].$$

In the meantime, also with a O which is uniform over all $j \leq \lfloor n/2 \rfloor$, we have

$$\sin\left(\frac{(2d_n+1)j\pi}{n}\right) = \sin\left(\frac{2d_n j\pi}{n}\right) + O\left(\frac{j}{n}\right).$$

Using the expression (10) of α_j^n , we deduce that

$$\begin{aligned}
\alpha_j^n &= \frac{\sin((2d_n + 1)j\pi/n)}{2d_n \sin(j\pi/n)} - \frac{1}{2d_n} \\
&= \frac{\sin(2d_n j\pi/n) + O(j/n)}{2d_n j\pi/n} \left[1 + O\left(\frac{j^2}{n^2}\right) \right] + O\left(\frac{1}{n}\right) \\
&= \operatorname{sinc}\left(\frac{2d_n j\pi}{n}\right) + O\left(\frac{j}{n}\right).
\end{aligned} \tag{15}$$

We carry on by writing, using the mean value theorem,

$$\operatorname{sinc}\left(\frac{2d_n j\pi}{n}\right) - \operatorname{sinc}(j\lambda\pi) = \left(\frac{2d_n}{n} - \lambda\right) j\pi \operatorname{sinc}'(\tau_{n,j}), \tag{16}$$

with

$$\tau_{n,j} \in \left[\frac{2d_n j\pi}{n}, j\lambda\pi\right] \cup \left[j\lambda\pi, \frac{2d_n j\pi}{n}\right].$$

Yet, for every $x > 0$, we have

$$|x \operatorname{sinc}'(x)| = \left| x \left(\frac{\cos x}{x} - \frac{\sin x}{x^2} \right) \right| = |\cos x - \operatorname{sinc} x| \leq 2.$$

Therefore, uniformly with respect to j and n , we have

$$|\operatorname{sinc}'(\tau_{n,j})| \leq \frac{2}{\tau_{n,j}} \leq \frac{2}{\tau_{n,1}} = O(1).$$

Plugging this into (16), we obtain

$$\operatorname{sinc}\left(\frac{2d_n j\pi}{n}\right) - \operatorname{sinc}(j\lambda\pi) = O\left(\frac{2d_n}{n} - \lambda\right). \tag{17}$$

Combining (15) and (17), we obtain

$$\alpha_j^n - \operatorname{sinc}(j\lambda\pi) = O\left(\frac{j}{n}\right) + O\left(\frac{2d_n}{n} - \lambda\right),$$

which holds uniformly for all the $j \leq [n/2]$, concluding the proof of the lemma. \square

We now deal with the extrema of the spectrum, as defined by (11).

Lemma 2. *In the regime (4), the smallest and largest eigenvalues satisfy:*

$$\lim_{n \rightarrow \infty} M_n = \operatorname{sinc}(\lambda\pi) \quad \text{and} \quad \lim_{n \rightarrow \infty} m_n = C_\lambda,$$

where C_λ is given by

$$C_\lambda = -\inf_{j \geq 1} \operatorname{sinc}(j\lambda\pi). \tag{18}$$

Proof. To begin with, note that, the eigenvalues being symmetric, we have

$$M_n = \max \{ \alpha_1^n, \dots, \alpha_{[n/2]}^n \} \quad \text{and} \quad m_n = -\min \{ \alpha_1^n, \dots, \alpha_{[n/2]}^n \}.$$

Since $\lambda \in (0, 1)$, we have $\sin(\lambda\pi) > 0$ and $C_\lambda > 0$, which allows us to define

$$j_0 = \left\lceil \frac{2\pi}{\sin(\lambda\pi)} \right\rceil \vee \left\lceil \frac{2}{\lambda C_\lambda} \right\rceil.$$

Let $n \geq 2j_0$. For any $j \in \{j_0, \dots, \lfloor n/2 \rfloor\}$, the upper bound (12) entails that

$$|\alpha_j^n| \leq \frac{n}{2d_n j} \leq \frac{n}{2d_n j_0}.$$

Yet, when $n \rightarrow \infty$, we have

$$\frac{n}{2d_n j_0} \xrightarrow{n \rightarrow \infty} \frac{1}{\lambda j_0}.$$

Therefore, for n large enough, we have

$$\frac{n}{2d_n j_0} \leq \frac{2}{\lambda j_0},$$

implying that

$$\max_{j_0 \leq j \leq \lfloor n/2 \rfloor} |\alpha_j^n| \leq \frac{2}{\lambda j_0}.$$

From this we can deduce, using the limit given by lemma 1, that

$$\begin{aligned} M_n &\leq \left(\max_{1 \leq j \leq j_0} \alpha_j^n \right) \vee \frac{2}{\lambda j_0} \leq \left(\max_{1 \leq j \leq j_0} \alpha_j^n \right) \vee \operatorname{sinc}(\lambda\pi) \\ &\xrightarrow{n \rightarrow \infty} \left(\max_{1 \leq j \leq j_0} \operatorname{sinc}(j\lambda\pi) \right) \vee \operatorname{sinc}(\lambda\pi) = \operatorname{sinc}(\lambda\pi). \end{aligned}$$

Yet, we also have

$$M_n \geq \alpha_1^n \xrightarrow{n \rightarrow \infty} \operatorname{sinc}(\lambda\pi),$$

and therefore, the squeeze theorem implies that $M_n \rightarrow \operatorname{sinc}(\lambda\pi)$. Similarly, we have

$$m_n \leq \left(-\min_{1 \leq j \leq j_0} \alpha_j^n \right) \vee \frac{2}{\lambda j_0} \leq \left(-\min_{1 \leq j \leq j_0} \alpha_j^n \right) \vee C_\lambda \xrightarrow{n \rightarrow \infty} \left(-\min_{1 \leq j \leq j_0} \operatorname{sinc}(j\lambda\pi) \right) \vee C_\lambda \leq C_\lambda.$$

Since $C_\lambda > 0$ and $\operatorname{sinc}(j\lambda\pi) \rightarrow 0$ when j tends to infinity, there exists $j_1 \geq 1$ such that $C_\lambda = -\operatorname{sinc}(j_1\lambda\pi)$, which allows us to write, for $n > j_1$,

$$m_n \geq -\alpha_{j_1}^n \xrightarrow{n \rightarrow \infty} -\operatorname{sinc}(j_1\lambda\pi) = C_\lambda,$$

and thus we find $m_n \rightarrow C_\lambda$. □

We now turn to an inequality which will be useful to prove integrability properties:

Lemma 3. *In the regime (4), there exists a constant $K = K_\lambda > 0$ such that*

$$\forall n \geq 3 \quad \forall a \in (0, n] \quad \forall \varepsilon \in \{-1, 1\} \quad \left| \left\{ j \in \{1, \dots, n-1\} : \varepsilon \alpha_j^n \geq \frac{1}{a} \right\} \right| \geq Ka - 2.$$

Proof. Let $n \geq 3$, $a \in (0, n]$ and $\varepsilon = (-1)^\eta$, with $\eta \in \{0, 1\}$. Let us define, for $k \geq 1$,

$$j_{n,k} = \left\lceil \frac{4nk + 2n\eta + n}{2(2d_n + 1)} - \frac{1}{2} \right\rceil.$$

Given that $4n > 2(2d_n + 1)$, the sequence $(j_{n,k})_{k \geq 1}$ is strictly increasing. Let us now prove that

$$\forall k \in \{1, \dots, K_n\} \quad \varepsilon \alpha_{j_{n,k}}^n \geq \frac{1}{a}, \quad (19)$$

where

$$K_n = \left\lfloor \frac{d_n a}{(n + 2d_n)\pi} \cos\left(\frac{(2d_n + 1)\pi}{2n}\right) - \frac{1}{4} - \frac{2d_n + 1}{4n} - \frac{\eta}{2} \right\rfloor.$$

Let $k \in \{1, \dots, K_n\}$. Recalling formula (10), we can write

$$\varepsilon \alpha_{j_{n,k}}^n = \frac{\varepsilon}{2d_n} \left(\frac{\sin((2d_n + 1)j\pi/n)}{\sin(j\pi/n)} - 1 \right) \geq \frac{\varepsilon \sin((2d_n + 1)j\pi/n)}{2d_n \sin(j\pi/n)} - \frac{1}{2d_n}. \quad (20)$$

We first bound the numerator, using that

$$(2k + \eta)\pi + \frac{\pi}{2} - \frac{(2d_n + 1)\pi}{2n} \leq \frac{(2d_n + 1)j_{n,k}\pi}{n} \leq (2k + \eta)\pi + \frac{\pi}{2} + \frac{(2d_n + 1)\pi}{2n},$$

which ensures that

$$\varepsilon \sin\left(\frac{(2d_n + 1)j_{n,k}\pi}{n}\right) \geq \sin\left(\frac{\pi}{2} - \frac{(2d_n + 1)\pi}{2n}\right) = \cos\left(\frac{(2d_n + 1)\pi}{2n}\right). \quad (21)$$

To deal with the denominator in (20), we write

$$\begin{aligned} j_{n,k} &\leq \frac{n(4K_n + 2\eta + 1)}{2(2d_n + 1)} + \frac{1}{2} \\ &\leq \frac{n}{2(2d_n + 1)} \left[\frac{4d_n a}{(n + 2d_n)\pi} \cos\left(\frac{(2d_n + 1)\pi}{2n}\right) - 1 - \frac{2d_n + 1}{n} - 2\eta + 2\eta + 1 \right] + \frac{1}{2} \\ &= \frac{2and_n}{(2d_n + 1)(n + 2d_n)\pi} \cos\left(\frac{(2d_n + 1)\pi}{2n}\right) \\ &\leq \frac{an}{(n + 2d_n)\pi} \cos\left(\frac{(2d_n + 1)\pi}{2n}\right). \end{aligned}$$

This implies that

$$2d_n \sin\left(\frac{j_{n,k}\pi}{n}\right) \leq 2d_n \frac{j_{n,k}\pi}{n} \leq \frac{2d_n a}{n + 2d_n} \cos\left(\frac{(2d_n + 1)\pi}{2n}\right). \quad (22)$$

Plugging (21) and (22) into (20), we obtain

$$\varepsilon \alpha_{j_{n,k}}^n \geq \frac{n + 2d_n}{2d_n a} - \frac{1}{2d_n} = \frac{1}{2d_n} \left(\frac{n}{a} - 1 \right) + \frac{1}{a} \geq \frac{1}{a}.$$

Thus, we have proved (19). Therefore, we have

$$\left| \left\{ j \in \{1, \dots, n-1\} : \varepsilon \alpha_j^n \geq \frac{1}{a} \right\} \right| \geq K_n \geq \frac{d_n a}{(n + 2d_n)\pi} \cos\left(\frac{(2d_n + 1)\pi}{2n}\right) - 2.$$

Now notice that

$$\lim_{n \rightarrow \infty} \frac{d_n}{(n + 2d_n)\pi} \cos\left(\frac{(2d_n + 1)\pi}{2n}\right) = \frac{\lambda}{2(1 + \lambda)\pi} \cos\left(\frac{\lambda\pi}{2}\right),$$

which implies that, for n larger than a certain rank n_0 ,

$$\frac{d_n}{(n+2d_n)\pi} \cos\left(\frac{(2d_n+1)\pi}{2n}\right) \geq \frac{\lambda}{4(1+\lambda)\pi} \cos\left(\frac{\lambda\pi}{2}\right).$$

The lemma follows by taking

$$K = \frac{\lambda}{4(1+\lambda)\pi} \cos\left(\frac{\lambda\pi}{2}\right) \wedge \frac{2}{n_0},$$

using the above argument if $n \geq n_0$, and noting that, if $n < n_0$ then $Ka - 2 \leq Kn_0 - 2 \leq 0$. \square

We now state another technical lemma, which will be useful in section 5.3:

Lemma 4. *The series*

$$\sum_{j \geq 1} \text{sinc}(j\lambda\pi)$$

converges and, when $j_0 \rightarrow \infty$, we have

$$\sum_{j=j_0}^{+\infty} \text{sinc}(j\lambda\pi) = O\left(\frac{1}{j_0}\right).$$

Proof. Take $1 \leq j_0 \leq j_1$. For every $n \geq 0$, writing

$$S_n = \sum_{j=0}^n \text{sinc}(j\lambda\pi) = \Im\left(\frac{e^{i(n+1)\lambda\pi} - 1}{e^{i\lambda\pi} - 1}\right) = \frac{\sin((n+1)\lambda\pi/2) \sin(n\lambda\pi/2)}{\sin(\lambda\pi/2)},$$

we see that the sequence $(S_n)_{n \in \mathbb{N}}$ is bounded. We now perform the Abel transformation:

$$\begin{aligned} \sum_{j=j_0}^{j_1} \text{sinc}(j\lambda\pi) &= \sum_{j=j_0}^{j_1} \frac{S_j - S_{j-1}}{j\lambda\pi} = \sum_{j=j_0}^{j_1} \frac{S_j}{j\lambda\pi} - \sum_{j=j_0-1}^{j_1-1} \frac{S_j}{(j+1)\lambda\pi} \\ &= \sum_{j=j_0}^{j_1} \frac{S_j}{j(j+1)\lambda\pi} - \frac{S_{j_0-1}}{j_0\lambda\pi} + \frac{S_{j_1}}{(j_1+1)\lambda\pi}. \end{aligned} \tag{23}$$

The boundedness of $(S_n)_{n \in \mathbb{N}}$ ensures that, when $j \rightarrow \infty$,

$$\frac{S_j}{j(j+1)\lambda\pi} = O\left(\frac{1}{j^2}\right)$$

implying that the series

$$\sum_{j \geq j_0} \frac{S_j}{j(j+1)\lambda\pi}$$

converges. Hence, letting j_1 tend to infinity in (23) yields

$$\sum_{j=j_0}^{+\infty} \text{sinc}(j\lambda\pi) = \sum_{j=j_0}^{+\infty} \frac{S_j}{j(j+1)\lambda\pi} - \frac{S_{j_0-1}}{j_0\lambda\pi} = O\left(\sum_{j=j_0}^{+\infty} \frac{1}{j(j+1)}\right) + O\left(\frac{1}{j_0}\right) = O\left(\frac{1}{j_0}\right),$$

which is the desired result. \square

2.2 Fourier inversion

We now perform a Fourier inversion to obtain an integral formula for the distribution of (T_n, H_n, S_n) in our model μ_n . The aim of this section is to prove the following result:

Lemma 5. *Assume that the parameter d_n is such that*

$$\lim_{n \rightarrow \infty} \frac{2d_n}{n} = \lambda \in [0, 1),$$

which contains the two regimes studied in this article. Then, for n large enough, under the law μ_n defined by (2), the vector (T_n, H_n, S_n) admits the following density with respect to the Lebesgue measure on \mathbb{R}^3 :

$$\begin{aligned} f_n : (x, y, z) \in \mathbb{R}^3 &\longmapsto \frac{1}{(2\pi)^{5/2} Z_n \sqrt{n}} \exp\left(-\frac{y}{x}\right) \mathbb{1}_\Omega\left(x, \frac{y}{n}, \frac{z}{n}\right) \\ &\times \int_{\mathbb{R}^2} du dv \exp\left[-iu\left(x - \frac{z^2}{n^2}\right) - iv\left(y + \frac{z^2}{2n}\right) - \frac{z^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln\left(1 - \frac{2iu}{n} + iv\alpha_j^n\right)\right], \end{aligned}$$

where the domain Ω is defined by

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : 2x > x - 2y > 2z^2 \right\}. \quad (24)$$

Proof. We start by computing the characteristic function of (T_n, H_n, S_n) under $\nu^{\otimes n}$, that is to say, when the variables y_j are i.i.d. standard normal variables. Using the expressions (6) and (7) obtained with our change of variable, this characteristic function writes

$$\begin{aligned} \phi_n : (u, v, w) \in \mathbb{R}^3 &\longmapsto \nu^{\otimes n}\left(e^{iuT_n + ivH_n + iwS_n}\right) \\ &= \int_{\mathbb{R}^n} \exp\left(\frac{iu}{n} \sum_{j=1}^n y_j^2 - \frac{iv}{2} \sum_{j=1}^n \alpha_j^n y_j^2 + iw\sqrt{n} y_n\right) \prod_{j=1}^n d\nu(y_j) \\ &= \int_{\mathbb{R}} \exp\left(\frac{iuy_n^2}{n} - \frac{iv\alpha_n^n y_n^2}{2} + iw\sqrt{n} y_n\right) d\nu(y_n) \prod_{j=1}^{n-1} \int_{\mathbb{R}} \exp\left(\frac{iuy_j^2}{n} - \frac{iv\alpha_j^n y_j^2}{2}\right) d\nu(y_j) \\ &= F\left(1 - \frac{2iu}{n} + iv\alpha_n^n, iw\sqrt{n}\right) \prod_{j=1}^{n-1} F\left(1 - \frac{2iu}{n} + iv\alpha_j^n, 0\right), \end{aligned}$$

where the function F is given by (29), which we recall here:

$$F : (a, b) \longmapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx e^{bx - ax^2/2}.$$

For $(a, b) \in \mathbb{C}^2$ with $\Re a > 0$, this integral is well defined, and it follows from lemma 4 in [Gor14] that

$$F(a, b) = \exp\left(\frac{b^2}{2a} - \frac{\ln(a)}{2}\right), \quad (25)$$

using the following determination of the logarithm:

$$\ln: \begin{cases} \mathbb{C} \setminus (-\infty, 0] \longrightarrow \mathbb{C} \\ z = x + iy \longmapsto \frac{1}{2} \ln(x^2 + y^2) + 2i \arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right). \end{cases}$$

Therefore, for every $(u, v, w) \in \mathbb{R}^3$, we obtain

$$\phi_n(u, v, w) = \exp \left[-\frac{nw^2}{2(1 - 2iu/n + iv)} - \frac{1}{2} \sum_{j=1}^n \ln \left(1 - \frac{2iu}{n} + iv \alpha_j^n \right) \right], \quad (26)$$

where we used the fact that $\alpha_n^n = 1$. Before using the Fourier inversion formula, we have to check that this function is integrable on \mathbb{R}^3 . Let us write, for $(u, v, w) \in \mathbb{R}^3$,

$$|\phi_n(u, v, w)| = \exp \left[-\frac{nw^2}{2(1 + (2u/n - v)^2)} - \frac{1}{4} \sum_{j=1}^n \ln \left(1 + \left(\frac{2u}{n} - v \alpha_j^n \right)^2 \right) \right].$$

If $\lambda > 0$, we use lemma 3, which ensures that

$$\forall n \geq 3 \quad \left| \left\{ j \in \{1, \dots, n-1\} : uv \alpha_j^n \leq 0 \quad \text{and} \quad |\alpha_j^n| \geq \frac{1}{n} \right\} \right| \geq Kn - 2,$$

where $K > 0$ is fixed. We can deduce that, for $n \geq n_I = 3 \vee 9/K$ and for every $(u, v, w) \in \mathbb{R}^3$,

$$\begin{aligned} |\phi_n(u, v, w)| &\leq \exp \left[-\frac{nw^2}{2(1 + 8u^2/n^2 + 2v^2)} - \frac{Kn - 2}{4} \ln \left(1 + \frac{4u^2}{n^2} + \frac{v^2}{n^2} \right) \right] \\ &\leq \exp \left[-\frac{nw^2}{2(1 + 8(u^2 + v^2)/n^2)} - \frac{7}{4} \ln \left(1 + \frac{u^2 + v^2}{n^2} \right) \right]. \end{aligned} \quad (27)$$

Now suppose that $\lambda = 0$, i.e. $d_n = o(n)$. If u and v have the same sign, then we only keep in the sum the terms of indexes $j \leq 7$. It follows from the formula (9) that, for any fixed $j \leq 7$, we have

$$\alpha_j^n = \frac{\cos(j(d_n + 1)\pi/n) \sin(jd_n\pi/n)}{d_n \sin(j\pi/n)} \xrightarrow{n \rightarrow \infty} 1,$$

since $d_n = o(n)$. Thus, for n large enough, we have $\alpha_j^n \geq 1/2$ for all $j \leq 7$ and therefore, u and v having the same sign,

$$\sum_{j=1}^n \ln \left(1 + \left(\frac{2u}{n} - v \alpha_j^n \right)^2 \right) \geq 7 \ln \left(1 + \frac{4u^2}{n} + \frac{v^2}{4} \right).$$

Otherwise, if u and v are of opposite signs, note that if j is such that

$$\left\lceil \frac{n}{2d_n + 1} \right\rceil \leq j \leq \left\lfloor \frac{n}{2d_n + 1} \right\rfloor + 6.$$

then we have

$$\pi \leq \frac{(2d_n + 1)j\pi}{n} \leq \pi + \frac{7(2d_n + 1)\pi}{n} \leq 2\pi$$

for n large enough, because $d_n = o(n)$. We then have, for these indexes j , using the expression (10),

$$\alpha_j^n = \frac{1}{2d_n} \left(\frac{\sin((2d_n + 1)j\pi/n)}{\sin(j\pi/n)} - 1 \right) \leq -\frac{1}{2d_n}.$$

From this we deduce that, when u and v are of opposite signs, for n large enough, we have

$$\sum_{j=1}^n \ln \left(1 + \left(\frac{2u}{n} - v \alpha_j^n \right)^2 \right) \geq 7 \ln \left(1 + \frac{4u^2}{n} + \frac{v^2}{4d_n^2} \right).$$

Thus, for n larger than a certain rank $n_I \in \mathbb{N}$, we have, for all $(u, v, w) \in \mathbb{R}^3$,

$$\begin{aligned} |\phi_n(u, v, w)| &\leq \exp \left[-\frac{nw^2}{2(1+8(u^2+v^2)/n^2)} - \frac{7}{4} \ln \left(1 + \frac{4u^2}{n^2} + \frac{v^2}{4d_n^2} \right) \right] \\ &\leq \exp \left[-\frac{nw^2}{2(1+8(u^2+v^2)/n^2)} - \frac{7}{4} \ln \left(1 + \frac{u^2+v^2}{n^2} \right) \right]. \end{aligned}$$

This upper bound, identical to (27), therefore holds whatever $\lambda \in [0, 1)$, which allows us to write, in $[0, +\infty[$:

$$\begin{aligned} \int_{\mathbb{R}^3} du dv dw |\phi_n(u, v, w)| &\leq \int_{\mathbb{R}^2} du dv \left(1 + \frac{u^2+v^2}{n^2} \right)^{-7/4} \int_{\mathbb{R}} dw \exp \left(-\frac{nw^2}{2(1+8(u^2+v^2)/n^2)} \right) \\ &= \sqrt{\frac{2\pi}{n}} \int_{\mathbb{R}^2} du dv \left(1 + \frac{u^2+v^2}{n^2} \right)^{-7/4} \sqrt{1 + \frac{8(u^2+v^2)}{n^2}} \\ &= \sqrt{\frac{2\pi}{n}} \int_0^{+\infty} \frac{2\pi r dr}{(1+r^2/n^2)^{7/4}} \sqrt{1 + \frac{8r^2}{n^2}} \\ &= n^{3/2} \sqrt{2\pi^3} \int_0^{+\infty} \frac{\sqrt{1+8\rho}}{(1+\rho)^{7/4}} d\rho, \end{aligned}$$

which is a converging integral, since

$$\frac{\sqrt{1+8\rho}}{(1+\rho)^{7/4}} = O\left(\frac{1}{\rho^{5/4}}\right) \quad \text{when } \rho \rightarrow +\infty.$$

The characteristic function of (T_n, H_n, S_n) is therefore integrable for $n \geq n_I$. Hence, we can apply the Fourier inversion formula to deduce that, for $n \geq n_I$, under $\nu^{\otimes n}$, the triplet (T_n, H_n, S_n) admits a density with respect to the Lebesgue measure:

$$(x, y, z) \mapsto \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} du dv dw \phi_n(u, v, w) e^{-iux-ivy-iwz}.$$

Given the definition (2) of our model μ_n , it follows that, under μ_n , the triplet (T_n, H_n, S_n) has a density

$$f_n : (x, y, z) \in \mathbb{R}^3 \mapsto \frac{1}{(2\pi)^3 Z_n} \exp\left(-\frac{y}{x}\right) \int_{\mathbb{R}^3} du dv dw \phi_n(u, v, w) e^{-iux-ivy-iwz}.$$

Replacing the characteristic function ϕ_n with its expression given by (26), it follows that, for $n \geq n_I$ and for any $(x, y, z) \in \mathbb{R}^3$,

$$\begin{aligned} f_n(x, y, z) &= \frac{1}{(2\pi)^3 Z_n} \exp\left(-\frac{y}{x}\right) \\ &\times \int_{\mathbb{R}^3} du dv dw \exp \left[-iux - ivy - iwz - \frac{nw^2}{2(1-2iu/n+iv)} - \frac{1}{2} \sum_{j=1}^n \ln \left(1 - \frac{2iu}{n} + iv \alpha_j^n \right) \right]. \end{aligned}$$

The integral over w is easily computed using the formula (25), yielding

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dw \exp \left(-iwz - \frac{nw^2}{2(1-2iu/n+iv)} \right) &= F \left(\frac{n}{1-2iu/n+iv}, -iz \right) \\ &= \frac{1}{\sqrt{n}} \exp \left(\frac{\ln(1-2iu/n+iv)}{2} - \frac{z^2}{2n} + \frac{iuz^2}{n^2} - \frac{ivz^2}{2n} \right). \end{aligned}$$

Therefore, we obtain

$$f_n(x, y, z) = \frac{1}{(2\pi)^{5/2} Z_n \sqrt{n}} \exp\left(-\frac{y}{x}\right) \times \int_{\mathbb{R}^2} du dv \exp\left[-iu\left(x - \frac{z^2}{n^2}\right) - iv\left(y + \frac{z^2}{2n}\right) - \frac{z^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln\left(1 - \frac{2iu}{n} + iv\alpha_j^n\right)\right].$$

To obtain the claimed formula, there remains to show that $f_n(x, y, z) = 0$ when $(x, y/n, z/n) \notin \Omega$. Recall that, as noted in (3), we always have $|2H_n/n| \leq T_n/2$. This inequality is even strict, $\nu^{\otimes n}$ -almost surely and therefore also μ_n -almost-surely. Similarly, with our change of variable, it follows from (8) that, $\nu^{\otimes n}$ -almost surely and thus also μ_n -almost-surely,

$$\left| \frac{2H_n}{n} + \frac{S_n^2}{n^2} \right| = \frac{1}{n} \left| \sum_{j=1}^{n-1} \alpha_j^n y_j^2 \right| < \frac{1}{n} \sum_{j=1}^{n-1} y_j^2 = T_n - \frac{S_n^2}{n^2}.$$

Therefore, we have

$$\forall n \geq 3 \quad \mu_n \left[\left(T_n, \frac{H_n}{n}, \frac{S_n}{n} \right) \in \Omega \right] = 1,$$

with

$$\Omega = \left\{ |2y| < x \right\} \cap \left\{ |2y + z^2| < x - z^2 \right\} = \left\{ 2x > x - 2y > 2z^2 \right\},$$

which concludes the proof of the lemma. \square

3 Large deviation principle for the nearest neighbour model

In this section, we study the case $d_n = 1$, where the interaction is limited to the nearest neighbours, and we prove the large deviation principle of theorem 3. A consequence of this theorem, which will be useful in section 4 to prove the convergence in distribution, is the following:

Theorem 4. *Consider an infinite triangular array of random variables $(X_j^n)_{1 \leq j \leq n}$ such that for all $n \geq 3$, the vector (X_1, \dots, X_n) is distributed according to μ_n defined by (2), with $d_n = 1$. Then, when $n \rightarrow \infty$, we have the convergences in probability:*

$$T_n \rightarrow 1, \quad \frac{H_n}{n} \rightarrow -\frac{\sqrt{2}-1}{2} \quad \text{and} \quad \frac{S_n}{n} \rightarrow 0.$$

3.1 Computation of the cumulant generating function

We wish to apply the Gärtner-Ellis theorem (theorem 2.3.6 in [DZ10]) to obtain a large deviation principle for $(T_n, H_n/n, S_n/n)$, first under the i.i.d. distribution $\nu^{\otimes n}$. Our approach is similar to the method suggested by Bercu, Gamboa and Rouault in [BGR97], to study quadratic forms defined in a similar way. We consider the renormalized cumulant generating function of $(T_n, H_n/n, S_n/n)$ under $\nu^{\otimes n}$, which is defined as

$$\Lambda_n : (\lambda, \mu, \nu) \in \mathbb{R}^3 \mapsto \frac{1}{n} \ln \nu^{\otimes n} \left[\exp \left(n\lambda T_n + n\mu \frac{H_n}{n} + n\nu \frac{S_n}{n} \right) \right].$$

Let us compute, in $[0, \infty]$,

$$\begin{aligned}
\Lambda_n(\lambda, \mu, \nu) &= \frac{1}{n} \ln \int_{\mathbb{R}^n} \exp \left(\lambda \sum_{j=1}^n y_j^2 - \frac{\mu}{2} \sum_{j=1}^n \alpha_j^n y_j^2 + \nu \sqrt{n} y_n \right) \prod_{j=1}^n d\nu(y_j) \\
&= \frac{1}{n} \sum_{j=1}^{n-1} \ln \int_{\mathbb{R}} \exp \left[\left(\lambda - \frac{\mu \alpha_j^n}{2} \right) y_j^2 \right] d\nu(y_j) + \frac{1}{n} \ln \int_{\mathbb{R}} \exp \left[\left(\lambda - \frac{\mu \alpha_n^n}{2} \right) y_n^2 + \nu \sqrt{n} y_n \right] d\nu(y_n) \\
&= \frac{1}{n} \sum_{j=1}^{n-1} \ln F(1 - 2\lambda + \mu \alpha_j^n, 0) + \frac{1}{n} \ln F(1 - 2\lambda + \mu \alpha_n^n, \nu \sqrt{n})
\end{aligned} \tag{28}$$

where the function $F : \mathbb{R}^2 \rightarrow [0, \infty]$ is given by

$$F(a, b) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx e^{bx - ax^2/2}. \tag{29}$$

If $a \leq 0$ then $F(a, b) = +\infty$, whereas for $a > 0$, we have

$$\begin{aligned}
F(a, b) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \exp \left[-\frac{a}{2} \left(x - \frac{b}{a} \right)^2 + \frac{b^2}{2a} \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dy \exp \left(-\frac{ay^2}{2} + \frac{b^2}{2a} \right) \\
&= \frac{1}{\sqrt{a}} \exp \left(\frac{b^2}{2a} \right).
\end{aligned} \tag{30}$$

Thus, if $2\lambda + |\mu| < 1$, which ensures that $1 - 2\lambda + \mu \alpha_j^n > 0$ for every $j \leq n$, the formula (30) yields

$$\begin{aligned}
\Lambda_n(\lambda, \mu, \nu) &= -\frac{1}{2n} \sum_{j=1}^{n-1} \ln(1 - 2\lambda + \mu \alpha_j^n) - \frac{1}{2n} \ln(1 - 2\lambda + \mu \alpha_n^n) + \frac{\nu^2}{2(1 - 2\lambda + \mu \alpha_n^n)} \\
&= -\frac{1}{2n} \sum_{j=1}^n \ln \left[1 - 2\lambda + \mu \cos \left(\frac{2j\pi}{n} \right) \right] + \frac{\nu^2}{2(1 - 2\lambda + \mu)}
\end{aligned} \tag{31}$$

$$\xrightarrow{n \rightarrow \infty} -\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(1 - 2\lambda + \mu \cos t) dt + \frac{\nu^2}{2(1 - 2\lambda + \mu)}, \tag{32}$$

since the function

$$t \mapsto \ln(1 - 2\lambda + \mu \cos t)$$

is continuous and 2π -periodic. We now detail several exact computations which will be useful in the sequel:

Lemma 6. For all $(a, b) \in \mathbb{R}^2$ with $|b| < a$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{a + b \cos t} = \frac{1}{\sqrt{a^2 - b^2}},$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos t}{a + b \cos t} dt = \frac{1}{b} \left(1 - \frac{1}{\sqrt{1 - b^2/a^2}} \right) \mathbb{1}_{b \neq 0},$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(a + b \cos t) dt = \ln a + 2 \ln \left[\cos \left(\frac{\arcsin(b/a)}{2} \right) \right].$$

Proof. Let $a > 0$ and let $b \in (-a, a)$. We start with the first integral. The change of variable $\theta = \tan(t/2)$ gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{a + b \cos t} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2 d\theta}{a(1 + \theta^2) + b(1 - \theta^2)} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2 d\theta}{(a + b) + (a - b)\theta^2}.$$

We carry on with the change of variable

$$\theta = u \sqrt{\frac{a + b}{a - b}},$$

which yields

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{a + b \cos t} = \frac{1}{2\pi\sqrt{a^2 - b^2}} \int_{\mathbb{R}} \frac{2 du}{1 + u^2} = \frac{1}{\sqrt{a^2 - b^2}}.$$

Thus, the first formula is proved. It implies that, for $b \neq 0$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos t}{a + b \cos t} dt = \frac{1}{2\pi b} \int_{-\pi}^{\pi} \left(1 - \frac{a}{a + b \cos t}\right) dt = \frac{1}{b} \left(1 - \frac{1}{\sqrt{1 - b^2/a^2}}\right).$$

To deal with the third integral, we define the function

$$f_a : b \in (-a, a) \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(a + b \cos t) dt.$$

This function is even and \mathcal{C}^1 , and for $0 < b < a$, we have

$$f'_a(b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos t}{a + b \cos t} dt = \frac{1}{b} \left(1 - \frac{1}{\sqrt{1 - b^2/a^2}}\right).$$

From this we deduce that, for $0 < b < a$,

$$f_a(b) = f_a(0) + \int_0^b f'_a(x) dx = \ln a + \int_0^b \frac{dx}{x} \left(1 - \frac{1}{\sqrt{1 - x^2/a^2}}\right) = \ln a + \int_0^{\frac{b}{a}} \frac{dt}{t} \left(1 - \frac{1}{\sqrt{1 - t^2}}\right).$$

With the change of variable $\theta = \arcsin t$, we obtain

$$\begin{aligned} f_a(b) - \ln a &= \int_0^{\arcsin(b/a)} \frac{(\cos \theta) d\theta}{\sin \theta} \left(1 - \frac{1}{\cos \theta}\right) = \int_0^{\arcsin(b/a)} \frac{\cos \theta - 1}{\sin \theta} d\theta \\ &= - \int_0^{\arcsin(b/a)} \frac{\sin(\theta/2)}{\cos(\theta/2)} d\theta = 2 \ln \left[\cos \left(\frac{\arcsin(b/a)}{2} \right) \right], \end{aligned}$$

which concludes the proof of the lemma. □

It follows from this lemma that, for $2\lambda + |\mu| < 1$,

$$\lim_{n \rightarrow \infty} \Lambda_n(\lambda, \mu, \nu) = \Lambda(\lambda, \mu, \nu),$$

with

$$\Lambda(\lambda, \mu, \nu) = -\frac{\ln(1 - 2\lambda)}{2} - \ln \cos \left[\frac{1}{2} \arcsin \left(\frac{\mu}{1 - 2\lambda} \right) \right] + \frac{\nu^2}{2(1 - 2\lambda + \mu)}. \quad (33)$$

If $2\lambda - \mu \geq 1$ then we have $1 - 2\lambda + \mu \leq 0$. whence for every $n \geq 1$,

$$F(1 - 2\lambda + \mu \alpha_n^n, \nu \sqrt{n}) = F(1 - 2\lambda + \mu, \nu \sqrt{n}) = +\infty.$$

Plugged into (28) this implies that $\Lambda_n(\lambda, \mu, \nu) = +\infty$. If $2\lambda + \mu > 1$, then we have

$$\lim_{n \rightarrow \infty} (1 - 2\lambda + \mu \alpha_{\lfloor n/2 \rfloor}^n) = 1 - 2\lambda - \mu < 0,$$

whence $\Lambda_n(\lambda, \mu, \nu) = +\infty$ for n large enough. In the last case, when $2\lambda - \mu < 1$ but $2\lambda + \mu = 1$, the behaviour depends on the parity of n . For n even, we have $\alpha_{n/2}^n = -1$ whence $\Lambda_n(\lambda, \mu, \nu) = +\infty$. In contrast, for n odd, we have $\alpha_j^n > -1$ for every $j \in \{1, \dots, n\}$, which leads to the same expression (31), namely

$$\Lambda_n(\lambda, \mu, \nu) = -\frac{\ln(1 - 2\lambda)}{2} - \frac{1}{2n} \sum_{j=1}^n \ln \left[1 + \cos \left(\frac{2j\pi}{n} \right) \right] + \frac{\nu^2}{4(1 - 2\lambda)}.$$

The sum is no longer the Riemann sum of a function which is continuous on a segment, because the function $f : t \mapsto \ln(1 + \cos t)$ is not defined for $t = \pi$. However, n being odd, we can write

$$-\frac{1}{2n} \sum_{j=1}^n \ln \left[1 + \cos \left(\frac{2j\pi}{n} \right) \right] = \frac{\ln 2}{2n} - \frac{1}{n} \sum_{j=0}^{\lfloor n/2 \rfloor} f_n \left(\frac{2j\pi}{n} \right),$$

where the function f_n is given by

$$f_n : t \in [0, \pi) \mapsto \ln \left[1 + \cos \left(\frac{2\pi}{n} \left\lfloor \frac{nt}{2\pi} \right\rfloor \right) \right].$$

We then have

$$\frac{1}{n} \sum_{j=0}^{\lfloor n/2 \rfloor} f_n \left(\frac{2j\pi}{n} \right) = \frac{1}{2\pi} \int_0^\pi f_n(t) dt + \frac{1}{2n} f_n \left(\frac{2\pi \lfloor n/2 \rfloor}{n} \right) = \frac{1}{2\pi} \int_0^\pi f_n(t) dt + o(1),$$

since

$$f_n \left(\frac{2\pi \lfloor n/2 \rfloor}{n} \right) = \ln \left[1 + \cos \left(\frac{2\pi \lfloor n/2 \rfloor}{n} \right) \right] = \ln \left[1 - \cos \left(\frac{\pi}{n} \right) \right] = O(\ln n).$$

Yet, the function $t \mapsto \ln(1 + \cos t)$ being decreasing on $[0, \pi)$, we have

$$\forall t \in [0, \pi) \quad \ln(1 + \cos t) \leq f_n(t) \leq \ln 2.$$

When $t \rightarrow \pi$ with $t < \pi$, we have $\ln(1 + \cos t) = O(\ln(\pi - t))$, so the function

$$t \mapsto (\ln 2) \vee (-\ln(1 + \cos t))$$

is integrable on $[0, \pi)$. Thereby, it follows from the dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \int_0^\pi f_n(t) dt = \int_0^\pi \lim_{n \rightarrow \infty} f_n(t) dt = \int_0^\pi \ln(1 + \cos t) dt,$$

whence

$$\lim_{n \rightarrow \infty} \Lambda_n(\lambda, \mu, \nu) = -\frac{\ln(1 - 2\lambda)}{2} - \frac{1}{2\pi} \int_0^\pi \ln(1 + \cos t) dt + \frac{\nu^2}{4(1 - 2\lambda)}.$$

Next, we write

$$\int_0^\pi \ln(1 + \cos t) dt = \int_0^\pi \lim_{b \xrightarrow{<} 1} \ln(1 + b \cos t) dt = \lim_{b \xrightarrow{<} 1} \int_0^\pi \ln(1 + b \cos t) dt,$$

using for example the monotone convergence theorem separately on $[0, \pi/2]$ and $[\pi/2, \pi]$. Using the last formula of lemma 6, this leads to

$$\frac{1}{2\pi} \int_0^\pi \ln(1 + \cos t) dt = \lim_{b \xrightarrow{<} 1} \ln \left[\cos \left(\frac{\arcsin b}{2} \right) \right] = \ln \cos \left[\frac{1}{2} \arcsin \left(\frac{\mu}{1 - 2\lambda} \right) \right].$$

Therefore, when n is odd and $2\lambda - \mu < 1$ but $2\lambda + \mu = 1$, we recover the same limit Λ whose expression is given by (33). To summarize, we obtain

$$\forall (\lambda, \mu, \nu) \in \mathbb{R}^3 \quad \lim_{n \rightarrow \infty} \Lambda_{2n}(\lambda, \mu, \nu) = \Lambda_p(\lambda, \mu, \nu) \quad \text{and} \quad \lim_{n \rightarrow \infty} \Lambda_{2n+1}(\lambda, \mu, \nu) = \Lambda_i(\lambda, \mu, \nu),$$

where the two limiting functions Λ_p and Λ_i are given by

$$\Lambda_p(\lambda, \mu, \nu) = \begin{cases} \Lambda(\lambda, \mu, \nu) & \text{if } 2\lambda + |\mu| < 1, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\Lambda_i(\lambda, \mu, \nu) = \begin{cases} \Lambda(\lambda, \mu, \nu) & \text{if } 2\lambda - \mu < 1 \quad \text{and} \quad 2\lambda + \mu \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

This small gap between even and odd indices is of no consequence, since the limiting functions Λ_i and Λ_p only differ on the set $\{2\lambda + \mu = 1\}$. Yet, as we will see, these points are not “exposing hyperplanes” (see [DZ10]), which ensures that both functions have the same Fenchel-Legendre transform. Thus, the Gärtner-Ellis theorem applied on the one hand to $(\Lambda_{2n})_n$ and on the other hand to $(\Lambda_{2n+1})_n$ will lead to two large deviation principles with the same rate function, which boil down to a single large deviation principle for $(\Lambda_n)_n$.

3.2 Fenchel-Legendre transform

We now determine the Fenchel-Legendre transform of the function Λ_p , which is defined by

$$\Lambda_p^* : (x, y, z) \in \mathbb{R}^3 \mapsto \sup_{(\lambda, \mu, \nu) \in \mathbb{R}^3} \left(\lambda x + \mu y + \nu z - \Lambda_p(\lambda, \mu, \nu) \right), \quad (34)$$

and which takes its values in $(-\infty, +\infty]$. To compute this Fenchel-Legendre transform, we use the fact that the function Λ_p is differentiable on all the open set where it is finite. Let us recall the expression (32) of Λ_p , which is valid when $2\lambda + |\mu| < 1$:

$$\Lambda_p(\lambda, \mu, \nu) = -\frac{1}{4\pi} \int_{-\pi}^\pi \ln(1 - 2\lambda + \mu \cos t) dt + \frac{\nu^2}{2(1 - 2\lambda + \mu)}.$$

Using the exact computations of lemma 6, we can write, when $2\lambda + |\mu| < 1$,

$$\frac{\partial \Lambda_p}{\partial \lambda}(\lambda, \mu, \nu) = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{dt}{1 - 2\lambda + \mu \cos t} + \frac{2\nu^2}{2(1 - 2\lambda + \mu)^2} = \frac{1}{\sqrt{(1 - 2\lambda)^2 - \mu^2}} + \frac{\nu^2}{(1 - 2\lambda + \mu)^2}.$$

Similarly, we have

$$\begin{aligned}\frac{\partial \Lambda_p}{\partial \mu}(\lambda, \mu, \nu) &= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\cos t \, dt}{1 - 2\lambda + \mu \cos t} - \frac{\nu^2}{2(1 - 2\lambda + \mu)^2} \\ &= \frac{1}{2\mu} \left(\frac{1 - 2\lambda}{\sqrt{(1 - 2\lambda)^2 - \mu^2}} - 1 \right) \mathbb{1}_{\mu \neq 0} - \frac{\nu^2}{2(1 - 2\lambda + \mu)^2}\end{aligned}$$

and

$$\frac{\partial \Lambda_p}{\partial \nu}(\lambda, \mu, \nu) = \frac{\nu}{1 - 2\lambda + \mu}.$$

Let $(x, y, z) \in \mathbb{R}^3$. We search for the supremum of the application

$$(\lambda, \mu, \nu) \in \mathbb{R}^3 \mapsto \lambda x + \mu y + \nu z - \Lambda_p(\lambda, \mu, \nu). \quad (35)$$

The cumulant generating function Λ_n is convex for every integer n , thus the limiting function Λ_p is also convex. Therefore, the above function is concave, which implies that, if it admits a critical point then this point must be a global maximum. For this reason, we look for a critical point of the function (35), that is to say a solution of the system

$$\begin{cases} x = \frac{\partial \Lambda_p}{\partial \lambda} = \frac{1}{\sqrt{(1 - 2\lambda)^2 - \mu^2}} + \frac{\nu^2}{(1 - 2\lambda + \mu)^2} \\ y = \frac{\partial \Lambda_p}{\partial \mu} = \frac{1}{2\mu} \left(\frac{1 - 2\lambda}{\sqrt{(1 - 2\lambda)^2 - \mu^2}} - 1 \right) \mathbb{1}_{\mu \neq 0} - \frac{\nu^2}{2(1 - 2\lambda + \mu)^2} \\ z = \frac{\partial \Lambda_p}{\partial \nu} = \frac{\nu}{1 - 2\lambda + \mu}, \end{cases}$$

which is equivalent to the simplified system:

$$\begin{cases} x - z^2 = \frac{1}{\sqrt{(1 - 2\lambda)^2 - \mu^2}} \\ 2y + z^2 = \frac{1}{\mu} \left(\frac{1 - 2\lambda}{\sqrt{(1 - 2\lambda)^2 - \mu^2}} - 1 \right) \mathbb{1}_{\mu \neq 0} \\ z = \frac{\nu}{1 - 2\lambda + \mu}. \end{cases}$$

To solve this system, we will limit ourselves to $(x, y, z) \in \Omega$, where Ω is the open set defined by (24). Let $(x, y, z) \in \Omega$, and let us write $\tilde{x} = x - z^2$ and $\tilde{y} = 2y + z^2$. Thus, by definition of Ω , we have $|\tilde{y}| < \tilde{x}$. In this case, the above system admits a solution, which writes

$$\begin{cases} \lambda^*(x, y, z) = \frac{1}{2} - \frac{\tilde{x}^2 + \tilde{y}^2}{2\tilde{x}(\tilde{x}^2 - \tilde{y}^2)} \\ \mu^*(x, y, z) = \frac{2\tilde{y}}{\tilde{x}^2 - \tilde{y}^2} \\ \nu^*(x, y, z) = \frac{z(\tilde{x} + \tilde{y})}{\tilde{x}(\tilde{x} - \tilde{y})}. \end{cases}$$

We found a critical point of the function (35), which is therefore a global maximum, since this function is concave. It follows that

$$\forall (x, y, z) \in \Omega \quad \Lambda_p^*(x, y, z) = x\lambda^* + y\mu^* + z\nu^* - \Lambda_p(\lambda^*, \mu^*, \nu^*).$$

We are left with the computation of, on the one hand,

$$\begin{aligned}
x\lambda^* + y\mu^* + z\nu^* &= \frac{x}{2} - \frac{x(\tilde{x}^2 + \tilde{y}^2)}{2\tilde{x}(\tilde{x}^2 - \tilde{y}^2)} + \frac{2y\tilde{y}}{\tilde{x}^2 - \tilde{y}^2} + \frac{z^2(\tilde{x} + \tilde{y})}{\tilde{x}(\tilde{x} - \tilde{y})} \\
&= \frac{x}{2} - \frac{\tilde{x}^2 + \tilde{y}^2}{2(\tilde{x}^2 - \tilde{y}^2)} - \frac{z^2(\tilde{x}^2 + \tilde{y}^2)}{2\tilde{x}(\tilde{x}^2 - \tilde{y}^2)} + \frac{\tilde{y}^2}{\tilde{x}^2 - \tilde{y}^2} - \frac{z^2\tilde{y}}{\tilde{x}^2 - \tilde{y}^2} + \frac{z^2(\tilde{x} + \tilde{y})}{\tilde{x}(\tilde{x} - \tilde{y})} \\
&= \frac{x}{2} - \frac{1}{2} + \frac{z^2}{2\tilde{x}(\tilde{x}^2 - \tilde{y}^2)} (2(\tilde{x} + \tilde{y})^2 - 2\tilde{x}\tilde{y} - \tilde{x}^2 - \tilde{y}^2) \\
&= \frac{x-1}{2} + \frac{z^2(\tilde{x} + \tilde{y})}{2\tilde{x}(\tilde{x} - \tilde{y})}, \tag{36}
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\Lambda_p(\lambda^*, \mu^*, \nu^*) &= -\frac{\ln(1-2\lambda^*)}{2} - \ln \cos \left[\frac{1}{2} \arcsin \left(\frac{\mu^*}{1-2\lambda^*} \right) \right] + \frac{(\nu^*)^2}{2(1-2\lambda^* + \mu^*)} \\
&= -\frac{1}{2} \ln \left(\frac{\tilde{x}^2 + \tilde{y}^2}{\tilde{x}(\tilde{x}^2 - \tilde{y}^2)} \right) - \frac{1}{2} \ln \cos^2 \left[\frac{1}{2} \arcsin \left(\frac{2\tilde{x}\tilde{y}}{\tilde{x}^2 + \tilde{y}^2} \right) \right] + \frac{z^2(\tilde{x} + \tilde{y})}{2\tilde{x}(\tilde{x} - \tilde{y})}. \tag{37}
\end{aligned}$$

We then write

$$\begin{aligned}
\cos^2 \left[\frac{1}{2} \arcsin \left(\frac{2\tilde{x}\tilde{y}}{\tilde{x}^2 + \tilde{y}^2} \right) \right] &= \frac{1}{2} \left[1 + \cos \arcsin \left(\frac{2\tilde{x}\tilde{y}}{\tilde{x}^2 + \tilde{y}^2} \right) \right] \\
&= \frac{1}{2} \left(1 + \sqrt{1 - \frac{4\tilde{x}^2\tilde{y}^2}{(\tilde{x}^2 + \tilde{y}^2)^2}} \right) \\
&= \frac{1}{2} \left(1 + \frac{\tilde{x}^2 - \tilde{y}^2}{\tilde{x}^2 + \tilde{y}^2} \right) \\
&= \frac{\tilde{x}^2}{\tilde{x}^2 + \tilde{y}^2}.
\end{aligned}$$

Plugging this into (37) yields

$$\Lambda_p(\lambda^*, \mu^*, \nu^*) = -\frac{\ln \tilde{x}}{2} + \frac{\ln(\tilde{x}^2 - \tilde{y}^2)}{2} + \frac{z^2(\tilde{x} + \tilde{y})}{2\tilde{x}(\tilde{x} - \tilde{y})}. \tag{38}$$

Replacing (36) and (38) in the expression (34) of Λ_p^* , we obtain

$$\begin{aligned}
\forall (x, y, z) \in \Omega \quad \Lambda_p^*(x, y, z) &= x\lambda^* + y\mu^* + z\nu^* - \Lambda_p(\lambda^*, \mu^*, \nu^*) \\
&= \frac{x-1}{2} + \frac{\ln \tilde{x}}{2} - \frac{\ln(\tilde{x}^2 - \tilde{y}^2)}{2} \\
&= \frac{x-1}{2} + \frac{\ln \tilde{x}}{2} - \frac{\ln(\tilde{x} + \tilde{y})}{2} - \frac{\ln(\tilde{x} - \tilde{y})}{2} \\
&= \frac{x-1}{2} + \frac{\ln \tilde{x}}{2} - \frac{\ln(x+2y)}{2} - \frac{\ln \tilde{x}}{2} - \frac{1}{2} \ln \left(1 - \frac{\tilde{y}}{\tilde{x}} \right) \\
&= \frac{x-1}{2} - \frac{\ln(x+2y)}{2} - \frac{1}{2} \ln \left(1 - \frac{2y+z^2}{x-z^2} \right).
\end{aligned}$$

Note that this expression tends to $+\infty$ when $x + 2y \rightarrow 0$ or when $x - 2y - 2z^2 \rightarrow 0$, that is to say at the borders of the domain Ω . In addition, the Fenchel-Legendre transform of a convex function being convex, the function $\Lambda_p^* : \mathbb{R}^3 \rightarrow (-\infty, +\infty]$ is convex. Thus, for every $(x, y, z) \in \mathbb{R}^3 \setminus \Omega$, by considering the convex function

$$f : t \in [0, 1] \mapsto \Lambda_p^*(1 - t + tx, ty, tz),$$

which tends to $+\infty$ when t tends to

$$\tau = \sup \left\{ t \in [0, 1] : (1 - t + tx, ty, tz) \in \Omega \right\},$$

we can deduce that $\Lambda_p^*(x, y, z) = f(1) = +\infty$. This is not surprising since, as we have seen, the distribution of the triplet $(T_n, H_n, S_n)/n$ does not charge the set $\mathbb{R}^3 \setminus \Omega$. Lastly, our computation also holds for Λ_i , since the critical point $(\lambda^*, \mu^*, \nu^*)$ stays inside of the domain $\{2\lambda + |\mu| < 1\}$, on which the functions Λ_p and Λ_i are equal. Therefore, for all $(x, y, z) \in \mathbb{R}^3$, we have

$$\Lambda_p^*(x, y, z) = \Lambda_i^*(x, y, z) = \begin{cases} \frac{x - 1 - \ln(x + 2y) + \ln(x - z^2) - \ln(x - 2y - 2z^2)}{2} & \text{if } (x, y, z) \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

3.3 Large deviation principle

We now apply the Gärtner-Ellis theorem (theorem 2.3.6 in [DZ10], last point of the theorem), separately to the even and the odd indices. But, the two resulting rate functions being equal, we obtain that, under the i.i.d. law $\nu^{\otimes n}$, the triplet $(T_n, H_n/n, S_n/n)$ follows a large deviation principle with good rate function $\Lambda_p^* = \Lambda_i^*$.

To infer a large deviation principle in our model μ_n defined by (2), we use corollary B.8 of [Gor15], which is a consequence of Varadhan's lemma. Using this corollary, we obtain that, under μ_n , the triplet $(T_n, H_n/n, S_n/n)$ follows a large deviation principle with good rate function

$$J : (x, y, z) \in \mathbb{R}^3 \mapsto \begin{cases} I(x, y, z) - \inf_{\Omega} I & \text{if } (x, y, z) \in \Omega, \\ +\infty & \text{otherwise,} \end{cases} \quad (39)$$

where

$$I(x, y, z) = \Lambda_p^*(x, y, z) + \frac{y}{x} = \frac{x - 1 - \ln(x + 2y) + \ln(x - z^2) - \ln(x - 2y - 2z^2)}{2} + \frac{y}{x}.$$

We now search for the infimum of this function I over the domain Ω . For every $(x, y, z) \in \Omega$, we have

$$\frac{\partial I}{\partial z}(x, y, z) = \frac{2z}{x - 2y - 2z^2} - \frac{z}{x - z^2} = \frac{z(x + 2y)}{(x - z^2)(x - 2y - 2z^2)},$$

which has the same sign as z because, by the definition (24) of Ω , we have $x + 2y > 0$, $x - z^2 > 0$ and $x - 2y - 2z^2 > 0$. Thus, for any fixed x and y , the function $z \mapsto I(x, y, z)$ is minimal for $z = 0$, whence

$$\inf_{\Omega} I = \inf_{|2y| < x} I(x, y, 0).$$

For any $x > 0$ and $y \in (-x/2, x/2)$, we have

$$\frac{\partial I}{\partial y}(x, y, 0) = \frac{1}{x - 2y} - \frac{1}{x + 2y} + \frac{1}{x} = \frac{x^2 + 4xy - 4y^2}{x(x + 2y)(x - 2y)} = \frac{[2y + (\sqrt{2} - 1)x][(\sqrt{2} + 1)x - 2y]}{x(x + 2y)(x - 2y)}.$$

which has the sign of $2y + (\sqrt{2} - 1)x$ because $(\sqrt{2} + 1)x - 2y > x - 2y > 0$. Therefore, for every fixed $x > 0$, the function $(y, z) \mapsto I(x, y, z)$ is minimal for $y = (\sqrt{2} - 1)x$ and $z = 0$. There remains to compute, for $x > 0$,

$$\frac{\partial}{\partial x} I\left(x, -\frac{(\sqrt{2} - 1)x}{2}, 0\right) = \frac{\partial}{\partial x} \left(\frac{x - 1 - \ln x - \ln(2 - \sqrt{2}) - \ln \sqrt{2} - \frac{\sqrt{2} - 1}{2}}{2} \right) = \frac{1}{2} - \frac{1}{2x},$$

which implies that

$$\inf_{\Omega} I = I\left(1, -\frac{\sqrt{2} - 1}{2}, 0\right) = \frac{-\ln(2 - \sqrt{2}) - \ln \sqrt{2} - \frac{\sqrt{2} - 1}{2}}{2} = -\frac{\ln(\sqrt{2} - 1)}{2} - \frac{\ln 2}{2} - \frac{\sqrt{2} - 1}{2}.$$

Replacing this in (39), we obtain the large deviation principle announced in theorem 3. Furthermore, this point being the only critical point of J , the convergence in distribution claimed in theorem 4 follows from this large deviation principle. To go further, we can compute the Hessian of J at this minimum, which is

$$\text{Hess } J\left(1, -\frac{\sqrt{2} - 1}{2}, 0\right) = \frac{1}{2} \begin{pmatrix} 3 - \sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 8 + 4\sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2} - 2 \end{pmatrix}. \quad (40)$$

The last coefficient of this matrix leads to guess the result of theorem 2, that is to say

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{\sqrt{2} - 1}\right) = \mathcal{N}(0, \sqrt{2} + 1).$$

The proof of this convergence in distribution is given in section 4 below.

4 Convergence for the nearest neighbour model (proof of theorem 2)

In this section, we prove theorem 2, i.e., we study the limit in distribution of S_n/\sqrt{n} in the case $d_n = 1$. The method we implement here is maybe not optimal for this very particular case but, as we will see in the following section, it can be generalized to deal with other regimes of the interaction range. This part is therefore an occasion to present the approach in a setting where, in the end, no interesting unexpected behaviour arises.

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and bounded function. According to the convergence in probability given by theorem 4, we have

$$\forall \varepsilon > 0 \quad \lim_{n \rightarrow \infty} \mu_n \left(|T_n - 1| > \varepsilon \quad \text{or} \quad \left| \frac{H_n}{n} + \frac{\sqrt{2} - 1}{2} \right| > \varepsilon \right) = 0.$$

This restriction of the domain of T_n and H_n will help us to obtain an integral expression where the integrand can be bounded from above by an integrable function, enabling us to apply the dominated convergence theorem. Thus, we fix $\varepsilon = 2^{-25}$, which will be enough to obtain the upper bounds we will need, and we look at the limit when $n \rightarrow \infty$ of

$$E_n = \mu_n \left[g\left(\frac{S_n}{\sqrt{n}}\right) \mathbb{1}\left\{ |T_n - 1| \leq \varepsilon \right\} \mathbb{1}\left\{ \left| \frac{H_n}{n} + \frac{\sqrt{2} - 1}{2} \right| \leq \varepsilon \right\} \right] = \hat{\mu}_n \left[g\left(\frac{S_n}{\sqrt{n}}\right) \right] + o(1). \quad (41)$$

4.1 Right Riemann sum

We will need the following elementary result, which gives a bound on the approximation of an integral with a sum of rectangles:

Lemma 7. *Let $f : [0, 1] \rightarrow \mathbb{C}$ be a Lipschitz-continuous function with a Lipschitz constant $K > 0$. For every $n \geq 1$, we have*

$$\left| \frac{1}{n} \sum_{j=1}^n f\left(\frac{j}{n}\right) - \int_0^1 f(t) dt \right| \leq \frac{K}{2n}.$$

Proof. We write

$$\left| \frac{1}{n} \sum_{j=1}^n f\left(\frac{j}{n}\right) - \int_0^1 f(t) dt \right| \leq \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left| f\left(\frac{j}{n}\right) - f(t) \right| dt \leq \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} K \left| \frac{j}{n} - t \right| dt = \frac{K}{2n},$$

which is the desired inequality. \square

4.2 Exact integral expression

Plugging into (41) the density of (T_n, H_n, S_n) under μ_n which we have computed by Fourier inversion in lemma 5, we obtain, for all $n \geq n_I$,

$$E_n = \frac{1}{(2\pi)^{5/2} Z_n \sqrt{n}} \int_{1-\varepsilon}^{1+\varepsilon} dx \int_{(-\frac{\sqrt{2}-1}{2}-\varepsilon)n}^{(-\frac{\sqrt{2}-1}{2}+\varepsilon)n} dy \int_{-\sqrt{n^2 x/2-ny}}^{\sqrt{n^2 x/2-ny}} dz g\left(\frac{z}{\sqrt{n}}\right) \exp\left(-\frac{y}{x}\right) \\ \times \int_{\mathbb{R}^2} du dv \exp \left[-iu \left(x - \frac{z^2}{n^2}\right) - iv \left(y + \frac{z^2}{2n}\right) - \frac{z^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{n} + iv \alpha_j^n\right) \right]. \quad (42)$$

Thanks to the large deviation calculus of the previous section, we expect that $T_n - 1$ will be of order $1/\sqrt{n}$, and $H_n + n(\sqrt{2}-1)/2$ and S_n will both be of order \sqrt{n} . This leads us to proceed to the change of variables:

$$x = 1 + \frac{x'}{\sqrt{n}}, \quad y = -\frac{\sqrt{2}-1}{2}n + y'\sqrt{n}, \quad z = z'\sqrt{n}, \quad u = u'\sqrt{n}, \quad v = \frac{v'}{\sqrt{n}}.$$

We obtain

$$\forall n \geq n_I \quad E_n = \frac{1}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}^2} du dv e^{A_n(x,y,z,u,v)}, \quad (43)$$

where the domain \mathcal{D}_n is defined by

$$\mathcal{D}_n = \left\{ (x, y, z) \in \mathbb{R}^3 : |x| \leq \varepsilon\sqrt{n}, \quad |y| \leq \varepsilon\sqrt{n} \quad \text{and} \quad 2z^2 \leq n\sqrt{2} + x\sqrt{n} - 2y\sqrt{n} \right\}, \quad (44)$$

and where we let

$$A_n(x, y, z, u, v) = \frac{(\sqrt{2}-1)n - 2y\sqrt{n}}{2(1+x/\sqrt{n})} - iu\sqrt{n} - iux + \frac{iuz^2}{\sqrt{n}} + \frac{iv(\sqrt{2}-1)\sqrt{n}}{2} - ivy - \frac{ivz^2}{2\sqrt{n}} - \frac{z^2}{2} \\ - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{\sqrt{n}} + \frac{iv \alpha_j^n}{\sqrt{n}} \right).$$

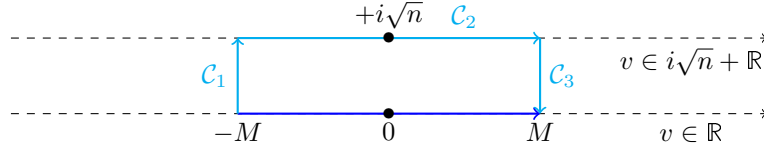


Figure 1: Cauchy's theorem allows us to replace the integral on the segment $[-M, M]$ by the integral along the three segments \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 .

To get rid of the term $-2y\sqrt{n}$, we wish to move the integration contour of the variable v from \mathbb{R} to $v \in i\sqrt{n} + \mathbb{R}$. Had we made this displacement of the integration contour before the change of variables, it would have been equivalent to adding a term $+y$ in the exponential. We could have guessed from the beginning that the quotient $-y/x$ in the exponential would bring a term $-y$ to be compensated, since x concentrates around 1. This displacement of the integration contour could therefore have been avoided if, in the computation of lemma 5, we had expressed the characteristic function of (T_n, H_n, S_n) under the distribution

$$\frac{e^{-H_n}}{\mu_n(e^{-H_n})} d\nu^{\otimes n},$$

instead of the law $\nu^{\otimes n}$ (recall that ν is the standard normal distribution). Indeed, we would have obtained

$$\phi_n(u, v, w) = \frac{1}{\mu_n(e^{-H_n})} \nu^{\otimes n} \left(e^{iuT_n + i(v+i)H_n + iwS_n} \right),$$

which amounts to moving the integration contour from $v \in \mathbb{R}$ to $v \in i + \mathbb{R}$ in equation (42). After our change of variable $v = v'/\sqrt{n}$, this is equivalent to our displacement from $v \in \mathbb{R}$ to $v \in i\sqrt{n} + \mathbb{R}$. Let us now check that we have the right to move this integration contour in the complex plane. Let us fix $n \geq n_I$ and $(x, y, z, u) \in \mathcal{D}_n \times \mathbb{R}$. The function $v \mapsto A_n(x, y, z, u, v)$ is well defined and holomorphic on the open set

$$\left\{ v \in \mathbb{C} : -\frac{\sqrt{n}}{m_n} < \Im v < -\frac{\sqrt{n}}{M_n} \right\},$$

where m_n and M_n are the largest and smallest eigenvalues, as defined in (11). Since $M_n < 1$, the closed contour represented on figure 1 is contained in this open set, and thus Cauchy's theorem ensures that, for any $M > 0$, we have

$$\begin{aligned} & \int_{-M}^M dv e^{A_n(x, y, z, u, v)} \\ &= \int_{\mathcal{C}_1} dv e^{A_n(x, y, z, u, v)} + \int_{\mathcal{C}_2} dv e^{A_n(x, y, z, u, v)} + \int_{\mathcal{C}_3} dv e^{A_n(x, y, z, u, v)} \\ &= \int_0^1 i\sqrt{n} dt e^{A_n(x, y, z, u, -M+ti\sqrt{n})} + \int_{-M}^M dv e^{A_n(x, y, z, u, v+i\sqrt{n})} + \int_1^0 i\sqrt{n} dt e^{A_n(x, y, z, u, M+ti\sqrt{n})}. \end{aligned}$$

We then write, for $M > 0$ and $t \in [0, 1]$,

$$\begin{aligned}
& \Re A_n(x, y, z, u, \pm M + ti\sqrt{n}) \\
&= A_n(x, y, z, 0, 0) - t \frac{\sqrt{2}-1}{2} n + ty\sqrt{n} + \frac{tz^2}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left| 1 - \frac{2iu}{\sqrt{n}} \pm \frac{iM\alpha_j^n}{\sqrt{n}} - t\alpha_j^n \right| \\
&= A_n(x, y, z, 0, 0) - t \frac{\sqrt{2}-1}{2} n + ty\sqrt{n} + \frac{tz^2}{2} - \frac{1}{4} \sum_{j=1}^{n-1} \ln \left[(1 - t\alpha_j^n)^2 + \left(\frac{2u \mp M\alpha_j^n}{\sqrt{n}} \right)^2 \right] \\
&\leq A_n(x, y, z, 0, 0) - t \frac{\sqrt{2}-1}{2} n + ty\sqrt{n} + \frac{tz^2}{2} - \frac{n \ln(1 - M_n)}{2} - \frac{1}{2} \ln \left| \frac{2u \mp M_n M}{\sqrt{n}} \right| \\
&\xrightarrow{M \rightarrow \infty} -\infty,
\end{aligned}$$

uniformly for all $t \in [0, 1]$. It follows that

$$\lim_{M \rightarrow +\infty} \int_0^1 i\sqrt{n} dt e^{A_n(x, y, z, u, -M + ti\sqrt{n})} = \lim_{M \rightarrow +\infty} \int_1^0 i\sqrt{n} dt e^{A_n(x, y, z, u, M + ti\sqrt{n})} = 0,$$

whence

$$\int_{\mathbb{R}} dv e^{A_n(x, y, z, u, v)} = \int_{\mathbb{R}} dv e^{A_n(x, y, z, u, v + i\sqrt{n})}.$$

Thereby, our computation (43) becomes

$$\forall n \geq n_I \quad E_n = \frac{1}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv e^{B_n(x, y, z, u, v)}, \quad (45)$$

where

$$\begin{aligned}
B_n(x, y, z, u, v) &= -\frac{(\sqrt{2}-1)x\sqrt{n}}{2(1+x/\sqrt{n})} + \frac{xy}{1+x/\sqrt{n}} - iu\sqrt{n} - iux + \frac{iuz^2}{\sqrt{n}} + \frac{iv(\sqrt{2}-1)\sqrt{n}}{2} - ivy - \frac{ivz^2}{2\sqrt{n}} \\
&\quad - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \alpha_j^n - \frac{2iu}{\sqrt{n}} + \frac{iv\alpha_j^n}{\sqrt{n}} \right).
\end{aligned}$$

We now wish to change the order of integration between u and v , in order to be able to move the integration contour of the variable u . For every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, we have

$$\begin{aligned}
\Re B_n(x, y, z, u, v) &= B_n(x, y, z, 0, 0) - \frac{1}{4} \sum_{j=1}^{n-1} \ln \left(1 + \frac{(2u - v\alpha_j^n)^2}{n(1 - \alpha_j^n)^2} \right) \\
&\leq B_n(x, y, z, 0, 0) - \frac{1}{4} \sum_{j=1}^{n-1} \ln \left(1 + \frac{(2u - v\alpha_j^n)^2}{4n} \right),
\end{aligned}$$

because $|1 - \alpha_j^n| \leq 2$ for all $i \in \{1, \dots, n-1\}$. If u and v have opposite signs, in the sum we only keep the terms with indices $j \leq \lfloor n/6 \rfloor$, which are such that

$$\alpha_j^n = \cos\left(\frac{2j\pi}{n}\right) \geq \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

and we obtain

$$\Re B_n(x, y, z, u, v) \leq B_n(x, y, z, 0, 0) - \frac{1}{4} \left\lfloor \frac{n}{6} \right\rfloor \ln \left(1 + \frac{4u^2 + v^2/4}{4n} \right).$$

Otherwise, if u and v have the same sign, then we obtain the same lower bound by keeping only the terms with indices

$$\left\lfloor \frac{n}{2} \right\rfloor \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor.$$

It follows that, for all $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, we have

$$\left| e^{B_n(x, y, z, u, v)} \right| \leq e^{B_n(x, y, z, 0, 0)} \left(1 + \frac{16u^2 + v^2}{16n} \right)^{-\lfloor n/6 \rfloor / 4},$$

which is integrable over $(u, v) \in \mathbb{R}^2$ as soon as $n \geq 30$. Thus, Fubini's theorem allows us to swap the summations over u and v in (45), leading to

$$\forall n \geq 30 \vee n_I \quad E_n = \frac{1}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du e^{B_n(x, y, z, u, v)},$$

To compensate the exploding term $-(\sqrt{2}-1)x\sqrt{n}/2$ in the expression of B_n , we now wish to move the integration contour of the variable u from \mathbb{R} to $\mathbb{R} + i(\sqrt{2}-1)\sqrt{n}/2$. For any fixed $(x, y, z, v) \in \mathcal{D}_n \times \mathbb{R}$, the function $u \mapsto B_n(x, y, z, u, v)$ is well defined and holomorphic on the open set

$$\left\{ u \in \mathbb{C} : \Im u > -\frac{(1-M_n)\sqrt{n}}{2} \right\}.$$

This allows us to move the integration contour inside this open set using Cauchy's theorem, and with the same method as above, we can check that

$$\int_{\mathbb{R}} du \exp \left[B_n(x, y, z, u, v) \right] = \int_{\mathbb{R}} du \exp \left[B_n \left(x, y, z, u + \frac{i(\sqrt{2}-1)\sqrt{n}}{2}, v \right) \right],$$

which implies that, for n large enough,

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du e^{F_n(x, y, z, u, v) + G_n(u, v)}, \quad (46)$$

where the constant C_n is given by

$$C_n = \frac{1}{(2\pi)^{5/2} Z_n} \exp \left(\frac{(\sqrt{2}-1)n}{2} \right) \prod_{j=1}^{n-1} \frac{1}{\sqrt{\sqrt{2} - \alpha_j^n}}$$

and the functions F_n and G_n are given by

$$F_n(x, y, z, u, v) = \frac{(\sqrt{2}-1)x^2}{2(1+x/\sqrt{n})} + \frac{xy}{1+x/\sqrt{n}} - iux - ivy + \frac{iu z^2}{\sqrt{n}} - \frac{iv z^2}{2\sqrt{n}} - \frac{(\sqrt{2}-1)z^2}{2}$$

and

$$G_n(u, v) = -iu\sqrt{n} + \frac{iv(\sqrt{2}-1)\sqrt{n}}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu - iv\alpha_j^n}{\sqrt{n}(\sqrt{2} - \alpha_j^n)} \right).$$

4.3 Pointwise convergence of the integrand

We wish to study the convergence of $F_n + G_n$. In order to implement the saddle-point method, we study this function on a domain extended to complex values of the parameters u and v . For all $(x, y, z, u, v) \in \mathbb{R}^3 \times \mathbb{C}^2$, $F_n(x, y, z, u, v)$ is well defined for every $n > x^2$ and a simple Taylor expansion shows that

$$\lim_{n \rightarrow \infty} F_n(x, y, z, u, v) = F_\infty(x, y, z, u, v),$$

where

$$F_\infty(x, y, z, u, v) = \frac{(\sqrt{2}-1)x^2}{2} + xy - iux - ivy - \frac{(\sqrt{2}-1)z^2}{2}.$$

Concerning the function G_n , we will prove the following:

Lemma 8. *We have the pointwise convergence:*

$$\forall (u, v) \in \mathbb{C}^2 \quad \lim_{n \rightarrow \infty} G_n(u, v) = G_\infty(u, v),$$

where the limit G_∞ is given by

$$G_\infty(u, v) = -\sqrt{2}u^2 + uv - \frac{v^2}{4}.$$

Proof. Let us fix $(u, v) \in \mathbb{C}^2$. Let $n_0 \in \mathbb{N}$ be large enough such that

$$\frac{2|u| + |v|}{\sqrt{n_0}(\sqrt{2}-1)} \leq \frac{1}{2}.$$

Then for every $n \geq n_0$ and every $j \in \{1, \dots, n-1\}$, we have

$$\left| \frac{2iu - iv\alpha_j^n}{\sqrt{n}(\sqrt{2}-\alpha_j^n)} \right| \leq \frac{1}{2}. \quad (47)$$

Therefore, $G_n(u, v)$ is well defined for $n \geq n_0$. Applying Taylor's theorem to the function

$$f_n : t \in [0, 1] \mapsto G_n(tu, tv),$$

we obtain

$$\left| f_n(1) - f_n(0) - f'_n(0) - \frac{f''_n(0)}{2} \right| \leq \frac{1}{6} \sup_{t \in [0, 1]} |f_n^{(3)}(t)|. \quad (48)$$

First note that $f_n(1) = G_n(u, v)$ and $f_n(0) = G_n(0, 0) = 0$. Next, we look at

$$f'_n(0) = u \frac{\partial G_n}{\partial u}(0, 0) + v \frac{\partial G_n}{\partial v}(0, 0). \quad (49)$$

We have

$$\frac{\partial G_n}{\partial u}(0, 0) = -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{1}{\sqrt{2}-\alpha_j^n} = i\sqrt{n} \left[-1 + \frac{1}{n} \sum_{j=1}^{n-1} \left(\sqrt{2} - \cos\left(\frac{2j\pi}{n}\right) \right)^{-1} \right].$$

Using lemma 7 about the error of the right Riemann sum and the first integral formula from lemma 6, we obtain that, when $n \rightarrow \infty$,

$$\frac{\partial G_n}{\partial u}(0, 0) = i\sqrt{n} \left(\int_0^1 \frac{dt}{\sqrt{2}-\cos(2\pi t)} - 1 \right) + O\left(\frac{1}{\sqrt{n}}\right) = O\left(\frac{1}{\sqrt{n}}\right). \quad (50)$$

Similarly, we can write

$$\begin{aligned}\frac{\partial G_n}{\partial v}(0,0) &= \frac{i(\sqrt{2}-1)\sqrt{n}}{2} - \frac{i}{2\sqrt{n}} \sum_{j=1}^{n-1} \frac{\alpha_j^n}{\sqrt{2}-\alpha_j^n} = \frac{i\sqrt{n}}{2} \left((\sqrt{2}-1) - \frac{1}{n} \sum_{j=1}^{n-1} \frac{\cos(2j\pi/n)}{\sqrt{2}-\cos(2j\pi/n)} \right) \\ &= \frac{i\sqrt{n}}{2} \left((\sqrt{2}-1) - \int_0^1 \frac{\cos(2\pi t) dt}{\sqrt{2}-\cos(2\pi t)} \right) + O\left(\frac{1}{\sqrt{n}}\right) = O\left(\frac{1}{\sqrt{n}}\right),\end{aligned}\quad (51)$$

where we used again lemma 7, along with the second integral formula of lemma 6. Plugging (50) and (51) in the expression (49) of $f'_n(0)$, we obtain

$$f'_n(0) = O\left(\frac{1}{\sqrt{n}}\right). \quad (52)$$

We now turn to

$$f''_n(0) = u^2 \frac{\partial^2 G_n}{\partial u^2}(0,0) + 2uv \frac{\partial^2 G_n}{\partial u \partial v}(0,0) - v^2 \frac{\partial^2 G_n}{\partial v^2}(0,0). \quad (53)$$

Thus, we compute

$$\frac{\partial^2 G_n}{\partial u^2}(0,0) = -\frac{2}{n} \sum_{j=1}^{n-1} \frac{1}{(\sqrt{2}-\alpha_j^n)^2} = -2 \int_0^1 \frac{dt}{(\sqrt{2}-\cos(2\pi t))^2} + O\left(\frac{1}{n}\right). \quad (54)$$

With the change of variable $\tau = \tan(\pi t)$ followed by

$$\tau = s \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \frac{s}{\sqrt{2}+1},$$

we obtain

$$\begin{aligned}\int_0^1 \frac{dt}{(\sqrt{2}-\cos(2\pi t))^2} &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{(1+\tau^2) d\tau}{\left[(\sqrt{2}-1) + (\sqrt{2}+1)\tau^2\right]^2} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{(\sqrt{2}-1)^2(1+s^2)^2} \left(1 + \frac{\sqrt{2}-1}{\sqrt{2}+1} s^2\right) \frac{ds}{\sqrt{2}+1} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{(\sqrt{2}-1) + (\sqrt{2}-1)s^2}{(1+s^2)^2} ds \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sqrt{2} ds}{1+s^2} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{(1-s^2) ds}{(1+s^2)^2} \\ &= \sqrt{2} + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1-s^2}{1+s^2} \frac{2 ds}{1+s^2}.\end{aligned}$$

Carrying out the change of variable $s = \tan(\theta/2)$ now leads to

$$\int_0^1 \frac{dt}{(\sqrt{2}-\cos(2\pi t))^2} = \sqrt{2} + \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos \theta) d\theta = \sqrt{2}. \quad (55)$$

Thus, equation (54) becomes

$$\frac{\partial^2 G_n}{\partial u^2}(0,0) = -2\sqrt{2} + O\left(\frac{1}{n}\right). \quad (56)$$

Let us proceed by writing

$$\begin{aligned}
\frac{\partial^2 G_n}{\partial u \partial v}(0, 0) &= \frac{1}{n} \sum_{j=1}^{n-1} \frac{\alpha_j^n}{(\sqrt{2} - \alpha_j^n)^2} \\
&= \int_0^1 \frac{\cos(2\pi t) dt}{(\sqrt{2} - \cos(2\pi t))^2} + O\left(\frac{1}{n}\right) \\
&= \int_0^1 \frac{\sqrt{2} dt}{(\sqrt{2} - \cos(2\pi t))^2} + 2 \int_0^1 \frac{dt}{\sqrt{2} - \cos(2\pi t)} + O\left(\frac{1}{n}\right) \\
&= 1 + O\left(\frac{1}{n}\right), \tag{57}
\end{aligned}$$

using the first formula of lemma 6 and our computation (55). Similarly, we have

$$\begin{aligned}
\frac{\partial^2 G_n}{\partial v^2}(0, 0) &= -\frac{1}{2n} \sum_{j=1}^{n-1} \frac{(\alpha_j^n)^2}{(\sqrt{2} - \alpha_j^n)^2} \\
&= -\frac{1}{2} \int_0^1 \frac{\cos(2\pi t)^2 dt}{(\sqrt{2} - \cos(2\pi t))^2} + O\left(\frac{1}{n}\right) \\
&= -\frac{1}{2} \int_0^1 \frac{\sqrt{2} \cos(2\pi t) dt}{(\sqrt{2} - \cos(2\pi t))^2} + 2 \int_0^1 \frac{\cos(2\pi t) dt}{\sqrt{2} - \cos(2\pi t)} + O\left(\frac{1}{n}\right) \\
&= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2} - 1}{2} + O\left(\frac{1}{n}\right) \\
&= -\frac{1}{2} + O\left(\frac{1}{n}\right), \tag{58}
\end{aligned}$$

Replacing (56), (57) and (58) into (53), we get

$$f_n''(0) = -2\sqrt{2}u^2 + 2uv - \frac{v^2}{2} + O\left(\frac{1}{n}\right). \tag{59}$$

Lastly, for all $t \in [0, 1]$, we have

$$f_n^{(3)}(t) = \sum_{k=0}^3 \binom{3}{k} u^{3-k} v^k \frac{\partial^3 G_n}{\partial u^{3-k} \partial v^k}(tu, tv). \tag{60}$$

Let $k \in \{0, \dots, 3\}$, and let $t \in [0, 1]$. We have

$$\frac{\partial^3 G_n}{\partial u^{3-k} \partial v^k}(tu, tv) = -\frac{8i}{n^{3/2}} \sum_{j=1}^{n-1} \frac{(-\alpha_j^n/2)^k}{(\sqrt{2} - \alpha_j^n)^3} \left(1 - \frac{2itu - itv \alpha_j^n}{\sqrt{n}(\sqrt{2} - \alpha_j^n)}\right)^{-3}. \tag{61}$$

Using our upper bound (47) and the fact that $|\alpha_j^n| \leq 1$, we get, for all $n \geq n_0$,

$$\left| \frac{\partial^3 G_n}{\partial u^{3-k} \partial v^k}(tu, tv) \right| \leq \frac{8}{n^{3/2}} \sum_{j=1}^{n-1} \frac{1}{(\sqrt{2} - 1)^3} \left(1 - \frac{1}{2}\right)^{-3} \leq \frac{64}{\sqrt{n}(\sqrt{2} - 1)^3}.$$

Plugging this into (60) yields

$$\sup_{t \in [0, 1]} |f_n^{(3)}(t)| \leq \frac{64(|u| + |v|)^3}{\sqrt{n}(\sqrt{2} - 1)^3}. \tag{62}$$

Thus, combining (52), (59) and (62) with our Taylor expansion (48), we obtain that, for every fixed $(u, v) \in \mathbb{C}^2$,

$$|G_n(u, v) - G_\infty(u, v)| = O\left(\frac{1}{\sqrt{n}}\right),$$

which proves the claimed pointwise convergence. \square

4.4 Displacement of the integration contour

We now wish to implement the saddle-point method to obtain an approximation of the integral in equation (46). This method consists in changing the integration contour in the complex plane to make it pass through a saddle-point, that is to say a critical point of the function

$$(u, v) \mapsto F_\infty(x, y, z, u, v) + G_\infty(u, v),$$

which is holomorphic on \mathbb{C}^2 . This change of contour gives us an integral on which we may apply the dominated convergence theorem. Indeed, in the formula (46), it is manifest that the integrand is not an integrable function of $(x, y, z, u, v) \in \mathbb{R}^5$, hence there is no hope to apply the dominated convergence theorem if we do not change the integration contour. Thus, for every $(x, y, z) \in \mathbb{R}^3$, we search for a couple $(u, v) \in \mathbb{C}^2$ solution of the system

$$\begin{cases} 0 = \frac{\partial F_\infty}{\partial u}(x, y, z, u, v) + \frac{\partial G_\infty}{\partial u}(u, v) = -ix - 2\sqrt{2}u + v, \\ 0 = \frac{\partial F_\infty}{\partial v}(x, y, z, u, v) + \frac{\partial G_\infty}{\partial v}(u, v) = -iy + u - \frac{v}{2}. \end{cases}$$

This solution is given by

$$u^* = u^*(x, y) = -\frac{i(\sqrt{2}+1)(x+2y)}{2} \quad \text{and} \quad v^* = v^*(x, y) = -i(\sqrt{2}+1)(x+2\sqrt{2}y).$$

We start by moving the integration contour of the variable u , to integrate over $u \in u^* + \mathbb{R}$ instead of $u \in \mathbb{R}$. To do so, we fix $(x, y, z) \in \mathcal{D}_n$ and $v \in \mathbb{C}$. The function $u \mapsto F_n(x, y, z, u, v) + G_n(u, v)$ is defined and holomorphic on the open set

$$\left\{ u \in \mathbb{C} : \Im u > -\frac{(\sqrt{2}-1)\sqrt{n}}{2} \right\}.$$

It follows from the definition (44) of the domain \mathcal{D}_n that

$$|u^*| = \frac{(\sqrt{2}+1)|x+2y|}{2} \leq \frac{3}{2} \left(\frac{\sqrt{n}}{2^{25}} + \frac{\sqrt{n}}{2^{24}} \right) \leq \frac{\sqrt{n}}{2^{22}} < \frac{(\sqrt{2}-1)\sqrt{n}}{4}. \quad (63)$$

Thus, we may change the integration contour from $u \in \mathbb{R}$ to $u \in \mathbb{R} + u^*$, using the same method as in section 4.2 to obtain, for n large enough,

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du \exp\left(F_n(x, y, z, u^* + u, v) + G_n(u^* + u, v)\right).$$

It is then possible to check, as we did in section 4.2, that we have an integrable function of $(u, v) \in \mathbb{R}^2$, which allows us to swap the two last integrals using Fubini's theorem. Thus, for n large enough, we get

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv \exp\left(F_n(x, y, z, u^* + u, v) + G_n(u^* + u, v)\right).$$

For every fixed $(x, y, z, u) \in \mathcal{D}_n \times \mathbb{R}$, the function $v \mapsto G_n(u^* + u, v)$ is now holomorphic on the open set

$$\left\{ v \in \mathbb{C} : 2|\Im u^*| + |\Im v| < (\sqrt{2} - 1)\sqrt{n} \right\}. \quad (64)$$

Again, it follows from the definition of the domain \mathcal{D}_n that

$$|v^*| = (\sqrt{2} + 1) \left| x + 2\sqrt{2}y \right| \leq \frac{3}{2} \left(\frac{\sqrt{n}}{2^{25}} + \frac{\sqrt{n}}{2^{23}} \right) \leq \frac{\sqrt{n}}{2^{21}} < \frac{(\sqrt{2} - 1)\sqrt{n}}{2}. \quad (65)$$

Combined with (63), this ensures that for all $(x, y, z) \in \mathcal{D}_n$, the saddle-point v^* belongs to the open set (64). This allows us to move the integral over v from $v \in \mathbb{R}$ to $v \in \mathbb{R} + v^*$, which yields, for n large enough,

$$E_n = C_n \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv \exp \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right). \quad (66)$$

Let us now examine more closely the limit function which we obtain after this change of the integration contour. It follows from the results of section 4.3 that, for all $(x, y, z, u, v) \in \mathbb{R}^5$, we have the pointwise convergence

$$\lim_{n \rightarrow \infty} F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) = F_\infty(x, y, z, u^* + u, v^* + v) + G_\infty(u^* + u, v^* + v).$$

Thus, we compute, on the one hand,

$$\begin{aligned} F_\infty(x, y, z, u^* + u, v^* + v) &= \frac{\sqrt{2} - 1}{2} x^2 + xy - i(u^* + u)x - i(v^* + v)y - \frac{\sqrt{2} - 1}{2} z^2 \\ &= \frac{\sqrt{2} - 1}{2} x^2 + xy - \frac{(\sqrt{2} + 1)(x + 2y)x}{2} - iux - (\sqrt{2} + 1)(x + 2\sqrt{2}y)y - ivy - \frac{\sqrt{2} - 1}{2} z^2 \\ &= -x^2 - (2\sqrt{2} + 1)xy - 2(\sqrt{2} + 2)y^2 - \frac{\sqrt{2} - 1}{2} z^2 - iux - ivy, \end{aligned}$$

and on the other hand,

$$\begin{aligned} G_\infty(u^* + u, v^* + v) &= -\sqrt{2}(u^* + u)^2 + (u^* + u)(v^* + v) - \frac{(v^* + v)^2}{4} \\ &= \frac{\sqrt{2}(\sqrt{2} + 1)^2(x + 2y)^2}{4} + iu\sqrt{2}(\sqrt{2} + 1)(x + 2y) - \sqrt{2}u^2 - \frac{(\sqrt{2} + 1)^2(x + 2y)(x + 2\sqrt{2}y)}{2} \\ &\quad - \frac{iv(\sqrt{2} + 1)(x + 2y)}{2} - iu(\sqrt{2} + 1)(x + 2\sqrt{2}y) + uv + \frac{(\sqrt{2} + 1)^2(x + 2\sqrt{2}y)^2}{4} \\ &\quad + \frac{iv(\sqrt{2} + 1)(x + 2\sqrt{2}y)}{2} - \frac{v^2}{4} \\ &= \frac{\sqrt{2} + 1}{4} x^2 + (\sqrt{2} + 1)xy + (\sqrt{2} + 2)y^2 + iux + ivy - \sqrt{2}u^2 + uv - \frac{v^2}{4}. \end{aligned}$$

Summing all this yields

$$F_\infty(x, y, z, u^* + u, v^* + v) + G_\infty(u^* + u, v^* + v) = Q(x, y, z, u, v),$$

where the quadratic form Q is given by

$$Q(x, y, z, u, v) = -\frac{3 - \sqrt{2}}{4} x^2 - \sqrt{2}xy - (\sqrt{2} + 2)y^2 - \frac{\sqrt{2} - 1}{2} z^2 - \sqrt{2}u^2 - 2uv - v^2. \quad (67)$$

There is no surprise here, since we recover the Hessian of the rate function that we computed in (40).

4.5 Domination

We plan to use the dominated convergence theorem to obtain the limit of the integral in (66). To this end, we prove the following upper bound:

Lemma 9. *There exists $K > 0$ such that, for n large enough, for all $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$,*

$$\Re \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right) \leq M(x, y, z, u, v),$$

where the function M is given by

$$M(x, y, z, u, v) = Q(x, y, z, 0, 0) + \frac{x^2}{8} + \frac{y^2}{8} + \frac{z^2}{2^{21}} - \frac{9}{8} \ln \left(1 + \frac{u^2 + v^2}{120} \right) + K. \quad (68)$$

Proof. Let $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$. We start by writing

$$\begin{aligned} \Re F_n(x, y, z, u^* + u, v^* + v) - F_\infty(x, y, z, u^*, v^*) &= F_n(x, y, z, u^*, v^*) - F_\infty(x, y, z, u^*, v^*) \\ &= -\frac{(\sqrt{2}-1)x^3}{2(x+\sqrt{n})} - \frac{x^2y}{x+\sqrt{n}} + \frac{iu^*z^2}{\sqrt{n}} - \frac{iv^*z^2}{2\sqrt{n}}. \end{aligned}$$

Yet, it follows from the definition (44) of the domain \mathcal{D}_n that $|x| \leq 2^{-22}\sqrt{n}$ and $|y| \leq 2^{-22}\sqrt{n}$, whence

$$-\frac{(\sqrt{2}-1)x^3}{2(x+\sqrt{n})} - \frac{x^2y}{x+\sqrt{n}} \leq \frac{|x|^3}{\sqrt{n}} + \frac{2x^2|y|}{\sqrt{n}} \leq \frac{x^2}{2^{23}}.$$

Using the upper bounds (63) and (65) on $|u^*|$ and $|v^*|$, we may write

$$\frac{iu^*z^2}{\sqrt{n}} - \frac{iv^*z^2}{2\sqrt{n}} \leq \frac{z^2}{2^{21}}.$$

Therefore, we obtain

$$\Re F_n(x, y, z, u^* + u, v^* + v) \leq F_\infty(x, y, z, u^*, v^*) + \frac{x^2}{2^{23}} + \frac{z^2}{2^{21}}. \quad (69)$$

Using again (63) and (65), we obtain that, for all $j \in \{1, \dots, n-1\}$,

$$\left| \frac{2iu^* - iv^* \alpha_j^n}{\sqrt{n}(\sqrt{2} - \alpha_j^n)} \right| \leq \frac{2|u^*| + |v^*|}{\sqrt{n}(\sqrt{2} - 1)} \leq \frac{1}{2^{18}} < \frac{1}{2}. \quad (70)$$

This allows us to write

$$\begin{aligned}
\Re G_n(u^* + u, v^* + v) - G_n(u^*, v^*) &= -\frac{1}{2} \sum_{j=1}^{n-1} \ln \left| 1 - \frac{2i(u^* + u) - i(v^* + v)\alpha_j^n}{\sqrt{n}(\sqrt{2} - \alpha_j^n)} \right| \\
&\quad + \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu^* - iv^*\alpha_j^n}{\sqrt{n}(\sqrt{2} - \alpha_j^n)} \right) \\
&= -\frac{1}{2} \sum_{j=1}^{n-1} \ln \left| 1 - \left(1 - \frac{2iu^* - iv^*\alpha_j^n}{\sqrt{n}(\sqrt{2} - \alpha_j^n)} \right)^{-1} \frac{2iu - iv\alpha_j^n}{\sqrt{n}(\sqrt{2} - \alpha_j^n)} \right| \\
&\leq -\frac{1}{4} \sum_{j=1}^{n-1} \ln \left[1 + \frac{4(2u - v\alpha_j^n)^2}{n(\sqrt{2} + 1)^2} \right] \\
&\leq -\frac{1}{4} \sum_{j=1}^{n-1} \ln \left(1 + \frac{(2u - v\alpha_j^n)^2}{n} \right).
\end{aligned}$$

Proceeding as in section 4.2, that is, keeping only the terms with $uv\alpha_j^n < 0$ and $|\alpha_j^n| \geq 1/2$, we get

$$\Re G_n(u^* + u, v^* + v) - G_n(u^*, v^*) \leq -\frac{1}{4} \left\lfloor \frac{n}{6} \right\rfloor \ln \left(1 + \frac{4u^2 + v^2/4}{n} \right) \leq -\frac{30}{4n} \left\lfloor \frac{n}{6} \right\rfloor \ln \left(1 + \frac{4u^2 + v^2/4}{30} \right),$$

as soon as $n \geq 30$, using the concavity of the logarithm function. For $n \geq 60$, we have

$$\frac{30}{4n} \left\lfloor \frac{n}{6} \right\rfloor \geq \frac{30}{4n} \left(\frac{n}{6} - 1 \right) \geq \frac{30}{4n} \left(\frac{n}{6} - \frac{n}{60} \right) = \frac{9}{8},$$

whence

$$\Re G_n(u^* + u, v^* + v) - G_n(u^*, v^*) \leq -\frac{9}{8} \ln \left(1 + \frac{u^2 + v^2}{120} \right), \quad (71)$$

Lastly, we deal with the term $G_n(u^*, v^*)$, using the same Taylor expansion as in the proof of lemma 8. Thus, we consider the (real-valued) function

$$f_n : t \in [0, 1] \mapsto G_n(tu^*, tv^*).$$

It follows from (63) and (65) that

$$u^* = O(\sqrt{n}) \quad \text{and} \quad v^* = O(\sqrt{n}),$$

with a O which is uniform over all the domain \mathcal{D}_n . Combining this with our computations (50) and (51) of the partial derivatives of G_n , we deduce that $f_n'(0) = O(1)$, uniformly over \mathcal{D}_n . Similarly, thanks to our estimates (56), (57) and (58) on the second partial derivatives of G_n , we know that

$$f_n''(0) = G_\infty(u^*, v^*) + O(1),$$

with again a uniform O for all $(x, y, z) \in \mathcal{D}_n$. Lastly, plugging the uniform bound (70) into our computation (61) of the third partial derivatives of G_n , we get

$$\sup_{t \in [0, 1]} |f_n^{(3)}(t)| \leq \frac{64(|u^*| + |v^*|)^3}{\sqrt{n}(\sqrt{2} - 1)^3}.$$

Using the bounds (63) and (65) on the saddle-points u^* and v^* , we can write

$$\sup_{t \in [0,1]} |f_n^{(3)}(t)| \leq \frac{3(|u^*| + |v^*|)^2}{2^{16}(\sqrt{2}-1)^3} \leq \frac{3(\sqrt{2}+1)^2(2|x|+4|y|)^2}{2^{16}(\sqrt{2}-1)^3} \leq \frac{3(\sqrt{2}+1)^5(x^2+y^2)}{2^{12}} \leq \frac{x^2+y^2}{16}.$$

Going back to our Taylor formula (48), we deduce that there exists $K > 0$ such that, for all $(x, y, z) \in \mathcal{D}_n$,

$$G_n(u^*, v^*) \leq G_\infty(u^*, v^*) + \frac{x^2+y^2}{16} + K. \quad (72)$$

Lastly, the addition of (69), (71) and (72) yields the desired result. \square

It remains to check the integrability hypothesis:

Lemma 10. *The function e^M , where M is defined by (68), is integrable on \mathbb{R}^5 .*

Proof. Let us write, in $[0, +\infty]$,

$$\begin{aligned} \int_{\mathbb{R}^5} dx dy dz du dv e^{M(x,y,z,u,v)} &= e^K \int_{\mathbb{R}^2} dx dy \exp\left(-\frac{5-2\sqrt{2}}{8}x^2 - \sqrt{2}xy - \frac{4\sqrt{2}+15}{8}y^2\right) \\ &\quad \times \int_{\mathbb{R}} dz \exp\left(-\frac{\sqrt{2}-1}{2}z^2 + \frac{z^2}{2^{21}}\right) \times \int_{\mathbb{R}^2} du dv \left(1 + \frac{u^2+v^2}{120}\right)^{-9/8}. \end{aligned}$$

We have

$$\begin{aligned} -\frac{5-2\sqrt{2}}{8}x^2 - \sqrt{2}xy - \frac{4\sqrt{2}+15}{8}y^2 &= -\frac{3-2\sqrt{2}}{8}x^2 - \frac{4\sqrt{2}-1}{8}y^2 - \frac{(x+2\sqrt{2}y)^2}{4} \\ &\leq -\frac{3-2\sqrt{2}}{8}x^2 - \frac{4\sqrt{2}-1}{8}y^2, \end{aligned}$$

which shows that

$$\int_{\mathbb{R}^2} dx dy \exp\left(-\frac{5-2\sqrt{2}}{8}x^2 - \sqrt{2}xy - \frac{4\sqrt{2}+15}{8}y^2\right) < +\infty.$$

Next, we can note that

$$-\frac{\sqrt{2}-1}{2} + \frac{1}{2^{21}} < 0,$$

whence

$$\int_{\mathbb{R}} dz \exp\left(-\frac{\sqrt{2}-1}{2}z^2 + \frac{z^2}{2^{21}}\right) < +\infty.$$

To conclude, we write

$$\int_{\mathbb{R}^2} du dv \left(1 + \frac{u^2+v^2}{120}\right)^{-9/8} = \int_0^{+\infty} 2\pi r dr \left(1 + \frac{r^2}{120}\right)^{-9/8} = 120\pi \int_0^{+\infty} \frac{d\rho}{(1+\rho)^{9/8}} < +\infty,$$

since $9/8 > 1$. Thus, we have proved that the function e^M is integrable on \mathbb{R}^5 . \square

4.6 Dominated convergence

We are now in a position to apply the dominated convergence theorem to the integral in the formula (66), which we recall here:

$$E_n = C_n \int_{\mathbb{R}^5} dx dy dz du dv g(z) \exp \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right) \mathbb{1}_{(x, y, z) \in \mathcal{D}_n}.$$

It stems from the definition (44) of the domain \mathcal{D}_n that

$$\liminf_{n \rightarrow \infty} \mathcal{D}_n = \bigcup_{n \geq 3} \bigcap_{k \geq n} \mathcal{D}_k = \mathbb{R}^3,$$

that is to say, for every $(x, y, z) \in \mathbb{R}^3$, we have $(x, y, z) \in \mathcal{D}_n$ for n large enough. Thus, for every $(x, y, z, u, v) \in \mathbb{R}^5$, following the results of section 4.3, we have the pointwise convergence

$$\lim_{n \rightarrow \infty} g(z) \exp \left(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) \right) \mathbb{1}_{(x, y, z) \in \mathcal{D}_n} = g(z) e^{Q(x, y, z, u, v)},$$

where Q is the quadratic form given (67). According to lemmas 9 and 10, the domination hypothesis is satisfied for n large enough. By virtue of the dominated convergence theorem, we can deduce that

$$E_n \stackrel{n \rightarrow \infty}{\sim} C_n \int_{\mathbb{R}^5} dx dy dz du dv g(z) \exp \left(Q(x, y, z, u, v) \right) = C_n D \int_{\mathbb{R}} dz g(x) \exp \left(-\frac{\sqrt{2}-1}{2} z^2 \right),$$

where

$$D = \int_{\mathbb{R}^4} dx dy du dv \exp \left(Q(x, y, 0, u, v) \right).$$

Yet, as written in (41), we have

$$\mu_n \left[g \left(\frac{S_n}{\sqrt{n}} \right) \right] = E_n + o(1).$$

Applying this with the constant function $g \equiv 1$ yields

$$1 \stackrel{n \rightarrow \infty}{\sim} E_n \stackrel{n \rightarrow \infty}{\sim} C_n D \int_{\mathbb{R}} dz \exp \left(-\frac{\sqrt{2}-1}{2} z^2 \right) = C_n D \sqrt{\frac{2\pi}{\sqrt{2}-1}}.$$

At the end of the day, we obtain that for every bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu_n \left[g \left(\frac{S_n}{\sqrt{n}} \right) \right] &= \sqrt{\frac{\sqrt{2}-1}{2\pi}} \int_{\mathbb{R}} dz g(x) \exp \left(-\frac{\sqrt{2}-1}{2} z^2 \right) \\ &= \frac{1}{\sqrt{2\pi(\sqrt{2}+1)}} \int_{\mathbb{R}} dz g(x) \exp \left(-\frac{z^2}{2(\sqrt{2}+1)} \right), \end{aligned}$$

which proves the convergence in distribution announced in theorem 2. □

5 Convergence for the long range model (proof of theorem 1)

We now turn to the proof of theorem 1. Thus, we assume that the interaction range d_n is such that

$$\lim_{n \rightarrow \infty} \frac{2d_n}{n} = \lambda \in (0, 1),$$

we consider a bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, and we study the limit when n tends to infinity of the quantity

$$\mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right].$$

5.1 Behaviour of the temperature

As we have seen in theorem 4, in the case $d_n = 1$ we have the convergence $T_n \rightarrow 1$ in probability. But in fact we can be much more precise: indeed, as shown by the lemma below, the temperature has the same behaviour in our model μ_n as in the case of independent Gaussian variables, whatever the interaction range d_n .

Lemma 11. *Under μ_n , we have the convergence in distribution*

$$\sqrt{n}(T_n - 1) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 2).$$

Proof. If Y_1, \dots, Y_n are i.i.d. standard Gaussian variables, then we will prove that the variables

$$T_n = \frac{Y_1^2 + \dots + Y_n^2}{n} \quad \text{and} \quad \left(\frac{Y_1^2}{T_n}, \dots, \frac{Y_n^2}{T_n} \right)$$

are independent. We take two bounded and continuous functions $g : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$, and we let

$$I_n = \mathbb{E} \left[g(T_n) h \left(\frac{Y_1^2}{T_n}, \dots, \frac{Y_n^2}{T_n} \right) \right].$$

Writing $T_n = (y_1^2 + \dots + y_n^2)/n$, we have

$$\begin{aligned} I_n &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} dy_1 \dots dy_n g(T_n) h \left(\frac{y_1^2}{T_n}, \dots, \frac{y_n^2}{T_n} \right) e^{-nT_n/2} \\ &= \left(\frac{2}{\pi} \right)^{n/2} \int_{(0, \infty)^n} dy_1 \dots dy_n g(T_n) h \left(\frac{y_1^2}{T_n}, \dots, \frac{y_n^2}{T_n} \right) e^{-nT_n/2}. \end{aligned}$$

We now perform the change of variable $z_i = y_i^2$ for every $i \in \{1, \dots, n\}$, which yields

$$\begin{aligned} I_n &= \frac{1}{(2\pi)^{n/2}} \int_{(0, \infty)^n} \frac{dz_1 \dots dz_n}{\sqrt{z_1} \dots \sqrt{z_n}} g \left(\frac{z_1 + \dots + z_n}{n} \right) h \left(\frac{nz_1}{z_1 + \dots + z_n}, \dots, \frac{nz_n}{z_1 + \dots + z_n} \right) \\ &\quad \times \exp \left(-\frac{z_1 + \dots + z_n}{2} \right). \end{aligned}$$

Letting $z_n = nt - z_1 - \dots - z_{n-1}$, we obtain

$$\begin{aligned} I_n &= \frac{n}{(2\pi)^{n/2}} \int_{(0, \infty)^{n-1}} \frac{dz_1 \dots dz_{n-1}}{\sqrt{z_1} \dots \sqrt{z_{n-1}}} \int_0^{+\infty} dt \frac{\mathbb{1}_{z_1 + \dots + z_{n-1} < nt}}{\sqrt{nt - z_1 - \dots - z_{n-1}}} g(t) \\ &\quad \times h \left(\frac{nz_1}{t}, \dots, \frac{nz_{n-1}}{t}, n - \frac{nz_1 + \dots + nz_{n-1}}{t} \right) e^{-nt/2}. \end{aligned}$$

We may now swap the two summation symbols, using Fubini's theorem, because the function

$$(z_1, \dots, z_{n-1}, t) \in (0, \infty)^n \mapsto \frac{e^{-nt/2} \mathbb{1}_{z_1 + \dots + z_{n-1} < nt}}{\sqrt{z_1} \dots \sqrt{z_{n-1}} \sqrt{nt - z_1 - \dots - z_{n-1}}}$$

is integrable. We then obtain

$$I_n = \frac{n}{(2\pi)^{n/2}} \int_0^{+\infty} dt g(t) e^{-nt/2} \int_{(0, \infty)^{n-1}} \frac{\mathbb{1}_{z_1 + \dots + z_{n-1} < nt} dz_1 \dots dz_{n-1}}{\sqrt{z_1} \dots \sqrt{z_{n-1}} \sqrt{mnt - z_1 - \dots - z_{n-1}}} \times h\left(\frac{z_1}{t}, \dots, \frac{z_{n-1}}{t}, n - \frac{z_1 + \dots + z_{n-1}}{t}\right).$$

Lastly, with the change of variable $z_i = tu_i$ for every $i = 1, \dots, n-1$, this becomes

$$I_n = \frac{n}{(2\pi)^{n/2}} \int_0^{+\infty} dt g(t) e^{-nt/2} t^{(n-2)/2} \times \int_{(0, \infty)^{n-1}} \frac{\mathbb{1}_{u_1 + \dots + u_{n-1} < n} du_1 \dots du_{n-1}}{\sqrt{u_1} \dots \sqrt{u_{n-1}} \sqrt{n - u_1 - \dots - u_{n-1}}} h(u_1, \dots, u_{n-1}, n - u_1 - \dots - u_{n-1}).$$

Applying this on the one hand to the constant function $g \equiv 1$, and on the other hand to $h \equiv 1$, we deduce that

$$I_n = \mathbb{E}[g(T_n)] \mathbb{E}\left[h\left(\frac{Y_1^2}{T_n}, \dots, \frac{Y_n^2}{T_n}\right)\right],$$

which proves that the variables T_n and

$$\left(\frac{Y_1^2}{T_n}, \dots, \frac{Y_n^2}{T_n}\right)$$

are independent. It follows that, under $\nu^{\otimes n}$, that is to say when the variables Y_j are i.i.d. standard Gaussian variables, the variable T_n is independent of

$$\exp\left(-\frac{H_n}{T_n}\right) = \exp\left(\frac{1}{2} \sum_{j=1}^n \alpha_j^n \frac{Y_j^2}{T_n}\right).$$

Therefore, for every bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \mu_n[g(T_n)] &= \frac{1}{Z_n} \nu^{\otimes n} \left[\exp\left(-\frac{H_n}{T_n}\right) g(T_n) \right] \\ &= \frac{1}{Z_n} \nu^{\otimes n} \left[\exp\left(-\frac{H_n}{T_n}\right) \right] \nu^{\otimes n} [g(T_n)] = \nu^{\otimes n} [g(T_n)]. \end{aligned}$$

Thus, the variable T_n has the same distribution in our model μ_n as if the variables Y_j were i.i.d. standard Gaussian variables, so the central limit theorem ensures that

$$\sqrt{n}(T_n - 1) = \frac{Y_1^2 + \dots + Y_n^2 - n}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \nu(Y_1^4) - \nu(Y_1^2)^2 = 3 - 1 = 2,$$

whence the claimed convergence result. \square

5.2 Exact integral expression

According to lemma 11, we have

$$\sqrt{n}(T_n - 1) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 2). \quad (73)$$

This implies that

$$\lim_{n \rightarrow \infty} \mu_n \left(|T_n - 1| > n^{-3/8} \right) = 0.$$

Thus, we may study the limit when $n \rightarrow \infty$ of

$$E_n = \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \mathbb{1}_{|T_n - 1| \leq n^{-3/8}} \right].$$

As in section 4.2, this limitation is important to enable us to obtain an integral expression satisfying the domination hypothesis. The Fourier inversion formula given by lemma 5 enables us to write, for every $n \geq n_I$,

$$\begin{aligned} E_n &= \int_{1-n^{-3/8}}^{1+n^{-3/8}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz g \left(\frac{z}{n^{3/4}} \right) f_n(x, y, z) \\ &= \frac{1}{(2\pi)^{5/2} Z_n \sqrt{n}} \int_{1-n^{-3/8}}^{1+n^{-3/8}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz g \left(\frac{z}{n^{3/4}} \right) \exp \left(-\frac{y}{x} \right) \mathbb{1}_{2nx > nx + y > 2z^2/n} \\ &\quad \times \int_{\mathbb{R}^2} du dv \exp \left[-iu \left(x - \frac{z^2}{n^2} \right) - iv \left(y + \frac{z^2}{2n} \right) - \frac{z^2}{2n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{n} + iv \alpha_j^n \right) \right]. \end{aligned}$$

The behaviour (73) of T_n and the expected order of magnitude of S_n lead us to make the change of variable

$$x = 1 + \frac{x'}{\sqrt{n}}, \quad z = z' n^{3/4}, \quad u = u' \sqrt{n},$$

To choose the change of variable for y and v , recall that

$$H_n = -\frac{1}{2} \sum_{j=1}^{n-1} \alpha_j^n Y_j^2 - \frac{S_n^2}{2n}, \quad (74)$$

with $\alpha_j^n \rightarrow \text{sinc}(j\lambda\pi)$ for every fixed j , when $n \rightarrow \infty$. Thus, we opt for the change of variable

$$y = y' - \frac{z'^2 \sqrt{n}}{2}, \quad v = v',$$

which amounts to betting that the first term in (74) will be of order $O(1)$, while the term $-S_n^2/(2n)$ will be of order \sqrt{n} . Yet, the term $-S_n^2/(2n)$ precisely corresponds to the Hamiltonian of the model described in paragraph 1.3.1, which explains that, provided that we obtain a non-degenerate limit at the end of our computation, we expect to recover the behaviour of the mean-field model. With these changes of variables, we obtain

$$E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}^2} du dv e^{F_n(x,y,z,u,v) + G_n(u,v)}, \quad (75)$$

where the set \mathcal{D}_n is defined by

$$\mathcal{D}_n = \left\{ (x, y, z) \in \mathbb{R}^3 : |x| \leq n^{1/8} \quad \text{and} \quad 2\sqrt{n} + 2x > \sqrt{n} + x + \frac{y}{\sqrt{n}} - \frac{z^2}{2} > 2z^2 \right\}, \quad (76)$$

and the functions F_n and G_n are given by

$$F_n(x, y, z, u, v) = -iu(x - z^2) - ivy - \frac{xz^2}{2(1 + x/\sqrt{n})} - \frac{y}{1 + x/\sqrt{n}}$$

and

$$G_n(u, v) = -iu\sqrt{n} - \frac{1}{2} \sum_{j=1}^{n-1} \ln \left(1 - \frac{2iu}{\sqrt{n}} + iv \alpha_j^n \right). \quad (77)$$

5.3 Pointwise convergence of the integrand

First, we have the convergence:

$$\forall (x, y, z, u, v) \in \mathbb{R}^3 \times \mathbb{C}^2 \quad \lim_{n \rightarrow \infty} F_n(x, y, z, u, v) = F_\infty(x, y, z, u, v), \quad (78)$$

where

$$F_\infty(x, y, z, u, v) = -iu(x - z^2) - ivy - \frac{xz^2}{2} - y.$$

We now study the pointwise limit of the function G_n . Recall that we study this function on a domain extended to complex values of u and v . For $G_n(u, v)$ to be well defined, we require the complex variable v to satisfy the condition

$$\forall j \in \{1, \dots, n-1\} \quad \Re \left(1 - \frac{2iu}{\sqrt{n}} + iv \alpha_j^n \right) > 0.$$

According to lemma 2 about the largest and smallest eigenvalues, the above condition is satisfied for n large enough if

$$\Re(1 + iv \operatorname{sinc}(\lambda\pi)) > 0 \quad \text{and} \quad \Re(1 - iv C_\lambda) > 0,$$

where C_λ is the constant defined by (18). This leads us to consider the domain

$$\begin{aligned} V &= \left\{ v \in \mathbb{C} : \forall j \in \mathbb{N} \setminus \{0\} \quad \Re(1 + iv \operatorname{sinc}(j\lambda\pi)) > 0 \right\} \\ &= \left\{ v \in \mathbb{C} : -\frac{1}{C_\lambda} < \Im v < \frac{1}{\operatorname{sinc}(\lambda\pi)} \right\}. \end{aligned}$$

Thus, for all $(u, v) \in \mathbb{C} \times V$, the function $G_n(u, v)$ is well defined for n large enough, so it makes sense to study its pointwise limit. We now prove the following result:

Lemma 12. *We have the pointwise convergence:*

$$\forall (u, v) \in \mathbb{C} \times V \quad \lim_{n \rightarrow \infty} G_n(u, v) = G_\infty(u, v),$$

where the function G_∞ is given by

$$G_\infty : (u, v) \in \mathbb{C} \times V \mapsto -u^2 + \frac{iv}{2\lambda} - \sum_{j=1}^{+\infty} \ln(1 + iv \operatorname{sinc}(j\lambda\pi)).$$

Proof. We fix $(u, v) \in \mathbb{C} \times V$. When $j \rightarrow \infty$, we have

$$\ln(1 + iv \operatorname{sinc}(j\lambda\pi)) = iv \operatorname{sinc}(j\lambda\pi) + O\left(\frac{1}{j^2}\right).$$

Yet, lemma 4 ensures that the series $\sum \operatorname{sinc}(j\lambda\pi)$ converges, therefore the series in the definition of G_∞ is also convergent. As a first step, we study the limit of $G_n(0, v)$. We define

$$h = \frac{1}{2} \inf_{j \geq 1} \Re(1 + iv \operatorname{sinc}(j\lambda\pi)) = \frac{(1 - \operatorname{sinc}(\lambda\pi)\Im v) \wedge (1 + C_\lambda \Im v)}{2},$$

which is positive because $v \in V$. It follows from lemma 2 on the smallest and largest eigenvalues that

$$\lim_{n \rightarrow \infty} \min_{1 \leq j \leq n-1} \Re(1 + iv \alpha_j^n) = 2h,$$

whence, from a certain rank n_0 ,

$$\forall j \in \{1, \dots, n-1\} \quad \Re(1 + iv \alpha_j^n) \geq h.$$

Let $n \geq n_0$. We now define

$$u_n = \left\lfloor \frac{n^{1/4}}{2} \wedge \left(\frac{2d_n}{n} - \lambda \right)^{-1/2} \right\rfloor.$$

This sequence (u_n) tends to infinity because $2d_n \sim \lambda n$. Using the symmetry relation $\alpha_{n-j}^n = \alpha_j^n$, we can write

$$G_n(0, v) - G_\infty(0, v) = \delta_n^1(v) + \delta_n^2(v) - \delta_n^3(v),$$

with

$$\begin{aligned} \delta_n^1(v) &= \sum_{j=1}^{u_n} \left[-\ln(1 + iv \alpha_j^n) + \ln(1 + iv \operatorname{sinc}(j\lambda\pi)) \right], \\ \delta_n^2(v) &= -\frac{iv}{2\lambda} - \frac{1}{2} \sum_{j=u_n+1}^{n-u_n-1} \ln(1 + iv \alpha_j^n), \\ \delta_n^3(v) &= -\sum_{j=u_n+1}^{+\infty} \ln(1 + iv \operatorname{sinc}(j\lambda\pi)). \end{aligned}$$

Control of the first term: Let us start with δ_n^1 . For all $z, z' \in \mathbb{C}$, it follows from the mean value theorem that

$$\Re z \geq h \quad \text{and} \quad \Re z' \geq h \quad \Rightarrow \quad |\ln z' - \ln z| \leq |z' - z| \sup_{t \in [0,1]} \frac{1}{|(1-t)z + tz'|} \leq \frac{|z' - z|}{h}.$$

Therefore, for every $j \in \{1, \dots, u_n\}$, we have

$$\left| \ln(1 + iv \alpha_j^n) - \ln(1 + iv \operatorname{sinc}(j\lambda\pi)) \right| \leq \frac{|v|}{h} |\alpha_j^n - \operatorname{sinc}(j\lambda\pi)|.$$

We can deduce that

$$|\delta_n^1(v)| \leq \frac{|v|}{h} \sum_{j=1}^{u_n} |\alpha_j^n - \operatorname{sinc}(j\lambda\pi)|.$$

Using the upper bound of lemma 1, this implies

$$\begin{aligned} |\delta_n^1(v)| &\leq \frac{K|v|}{h} \sum_{j=1}^{u_n} \left(\frac{j}{n} + \left| \frac{2d_n}{n} - \lambda \right| \right) = O\left(\frac{u_n^2}{n}\right) + O\left(u_n \left| \frac{2d_n}{n} - \lambda \right| \right) \\ &= O\left(\frac{1}{\sqrt{n}}\right) + O\left(\left| \frac{2d_n}{n} - \lambda \right|^{1/2}\right) = o(1). \end{aligned}$$

Control of the second term: We now deal with δ_n^2 . Let us rewrite this term as

$$\delta_n^2(v) = -\sum_{j=u_n+1}^{n-u_n-1} \left(\ln(1 + iv \alpha_j^n) - iv \alpha_j^n \right) - iv \left(\frac{1}{\lambda} + \sum_{j=u_n+1}^{n-u_n-1} \alpha_j^n \right).$$

Taylor's theorem ensures that for all $z \in \mathbb{C}$, we have

$$\Re z \geq h \quad \text{and} \quad 1 \geq h \quad \Rightarrow \quad |\ln z - (z - 1)| \leq \frac{|z - 1|^2}{2} \sup_{t \in [0,1]} \frac{1}{|1 + t(z - 1)|^2} \leq \frac{|z - 1|^2}{2h^2}.$$

This entails that

$$\left| \sum_{j=u_n+1}^{n-u_n-1} \left(\ln(1 + iv\alpha_j^n) - iv\alpha_j^n \right) \right| \leq \frac{|v|^2}{2h^2} \sum_{j=u_n+1}^{n-u_n-1} |\alpha_j^n|^2,$$

implying that

$$|\delta_n^2(v)| \leq \frac{|v|^2}{2h^2} \sum_{j=u_n+1}^{n-u_n-1} |\alpha_j^n|^2 + |v| \left| \frac{1}{\lambda} + \sum_{j=u_n+1}^{n-u_n-1} \alpha_j^n \right|. \quad (79)$$

Using the formula (10) for α_j^n , we obtain

$$\begin{aligned} \frac{1}{\lambda} + \sum_{j=u_n+1}^{n-u_n-1} \alpha_j^n &= \frac{1}{\lambda} - \frac{n - 2u_n - 1}{2d_n} + \sum_{j=u_n+1}^{n-u_n-1} \frac{\sin((2d_n + 1)j\pi/n)}{2d_n \sin(j\pi/n)} \\ &= o(1) + \frac{1}{2d_n} \sum_{j=u_n+1}^{n-u_n-1} \frac{\sin((2d_n + 1)j\pi/n)}{\sin(j\pi/n)}, \end{aligned}$$

where we used the fact that $n/(2d_n) \rightarrow 1/\lambda$ and $u_n = o(n)$ whence $u_n = o(d_n)$. Using the symmetry property of the eigenvalues, this becomes

$$\frac{1}{\lambda} + \sum_{j=u_n+1}^{n-u_n-1} \alpha_j^n = o(1) + \frac{1}{d_n} \sum_{j=u_n+1}^{\lfloor n/2 \rfloor} \frac{\sin((2d_n + 1)j\pi/n)}{\sin(j\pi/n)}, \quad (80)$$

where the term $j = \lfloor n/2 \rfloor$ was counted two times if n is even, but it tends to 0 anyway. We then perform an Abel transform, letting, for $j \geq 0$,

$$\sigma_{n,j} = \sum_{k=0}^j \sin((2d_n + 1)k\pi/n).$$

We have

$$\begin{aligned} |\sigma_{n,j}| &= \left| \frac{\sin((2d_n + 1)j\pi/(2n)) \sin((2d_n + 1)(j + 1)\pi/(2n))}{\sin((2d_n + 1)\pi/(2n))} \right| \\ &\leq \frac{1}{\sin((2d_n + 1)\pi/(2n))} \leq \frac{n}{2d_n + 1} \xrightarrow{n \rightarrow \infty} \frac{1}{\lambda}, \end{aligned}$$

hence there exists a constant $M > 0$ such that $|\sigma_{n,j}| \leq M$ for all $n \geq 1$ and all $j \geq 0$. We then write

$$\begin{aligned} \sum_{j=u_n+1}^{\lfloor n/2 \rfloor} \frac{\sin((2d_n + 1)j\pi/n)}{\sin(j\pi/n)} &= \sum_{j=u_n+1}^{\lfloor n/2 \rfloor} \frac{\sigma_{n,j} - \sigma_{n,j-1}}{\sin(j\pi/n)} \\ &= \sum_{j=u_n}^{\lfloor n/2 \rfloor - 1} \sigma_{n,j} \left(\frac{1}{\sin(j\pi/n)} - \frac{1}{\sin((j + 1)\pi/n)} \right) - \frac{\sigma_{n,u_n}}{\sin(u_n\pi/n)} + \frac{\sigma_{n,\lfloor n/2 \rfloor}}{\sin(\lfloor n/2 \rfloor \pi/n)}. \end{aligned}$$

Therefore, we get

$$\left| \sum_{j=u_n+1}^{\lfloor n/2 \rfloor} \frac{\sin((2d_n+1)j\pi/n)}{\sin(j\pi/n)} \right| \leq \frac{2M}{\sin(u_n\pi/n)} + \frac{2M}{\sin(\lfloor n/2 \rfloor \pi/n)} \leq \frac{2Mn}{u_n}.$$

Going back to (80), we obtain

$$\frac{1}{\lambda} + \sum_{j=u_n+1}^{n-u_n-1} \alpha_j^n = o(1) + O\left(\frac{1}{u_n}\right) = o(1). \quad (81)$$

Eventually, using the upper bound (12) on α_j^n , we have

$$\sum_{j=u_n+1}^{n-u_n-1} |\alpha_j^n|^2 \leq \frac{n^2}{2d_n^2} \sum_{j=u_n+1}^{\lfloor n/2 \rfloor} \frac{1}{j^2} = O\left(\frac{1}{u_n}\right) = o(1). \quad (82)$$

Plugging (81) and (82) into (79) yields

$$\lim_{n \rightarrow \infty} \delta_n^2(v) = 0.$$

Control of the third term: The convergence of the series

$$\sum_{j \geq 0} \ln(1 + iv \operatorname{sinc}(j\lambda\pi))$$

implies that $\delta_n^3(v)$ tends to 0, since $u_n \rightarrow \infty$ when n tends to infinity. Thus, we have checked that

$$\lim_{n \rightarrow \infty} G_n(0, v) = G_\infty(0, v).$$

Dependence on u : Let $n'_0 = n_0 \vee 16|u|/h^2$, and let $n \geq n'_0$. We have

$$\left| \frac{2iu}{\sqrt{n}} \right| \leq \frac{h}{2},$$

which ensures that

$$\forall j \in \{1, \dots, n-1\} \quad \Re \left(1 - \frac{2iu}{\sqrt{n}} + iv \alpha_j^n \right) \wedge \Re \left(1 - \frac{2iu}{\sqrt{n}} + iv \operatorname{sinc}(j\lambda\pi) \right) \geq \frac{h}{2}. \quad (83)$$

It follows from Taylor's theorem that

$$\left| G_n(u, v) - G_n(0, v) - u \frac{\partial G_n}{\partial u}(0, v) - \frac{u^2}{2} \frac{\partial^2 G_n}{\partial u^2}(0, v) \right| \leq \frac{|u|^3}{6} \sup_{t \in [0,1]} \left| \frac{\partial^3 G_n}{\partial u^3}(tu) \right|. \quad (84)$$

Thus, we study the successive derivatives of G_n with respect to u . First, we write

$$\frac{\partial G_n}{\partial u}(0, v) = -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{1}{1 + iv \alpha_j^n} = -\frac{i}{\sqrt{n}} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left(\frac{1}{1 + iv \alpha_j^n} - 1 \right).$$

This implies that

$$\left| \frac{\partial G_n}{\partial u}(0, v) \right| \leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \left| \frac{iv \alpha_j^n}{1 + iv \alpha_j^n} \right| \leq \frac{1}{\sqrt{n}} + \frac{2|v|}{h\sqrt{n}} \sum_{j=1}^{n-1} |\alpha_j^n|.$$

Using our upper bound (13), this becomes

$$\frac{\partial G_n}{\partial u}(0, v) = O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{\ln n}{\sqrt{n}}\right) = o(1).$$

Similarly, we have

$$\frac{\partial^2 G_n}{\partial u^2}(0, v) = -\frac{2}{n} \sum_{j=1}^{n-1} \frac{1}{(1 + iv \alpha_j^n)^2} = -2 + \frac{2}{n} - \frac{2}{n} \sum_{j=1}^{n-1} \left(\frac{1}{(1 + iv \alpha_j^n)^2} - 1 \right).$$

From this we can deduce that

$$\begin{aligned} \left| \frac{\partial^2 G_n}{\partial u^2}(0, v) + 2 \right| &\leq \frac{2}{n} + \frac{2}{n} \sum_{j=1}^{n-1} \frac{|2iv \alpha_j^n - v^2 (\alpha_j^n)^2|}{|1 + iv \alpha_j^n|^2} \leq \frac{2}{n} + \frac{16|v|}{nh} \sum_{j=1}^{n-1} |\alpha_j^n| + \frac{8|v|^2}{nh^2} \sum_{j=1}^{n-1} |\alpha_j^n|^2 \\ &= O\left(\frac{\ln n}{n}\right) + O\left(\frac{1}{n}\right) = o(1). \end{aligned}$$

Lastly, we have

$$\sup_{0 \leq t \leq 1} \left| \frac{\partial^3 G_n}{\partial u^3}(tu) \right| = \sup_{0 \leq t \leq 1} \left| \frac{8}{n^{3/2}} \sum_{j=1}^{n-1} \frac{1}{(1 - 2iut/\sqrt{n} + iv \alpha_j^n)^3} \right| \leq \frac{64}{h^3 \sqrt{n}} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, the inequality (84) becomes

$$G_n(u, v) = G_\infty(0, v) + o(1) - u^2 + o(1) = G_\infty(u, v) + o(1),$$

concluding the proof of the lemma. \square

5.4 Displacement of the integration contour

We wish to implement the saddle-point method to approximate the integral in (75). To this end, we take $(x, y, z, u, v) \in \mathbb{R}^3 \times \mathbb{C} \times V$, and we look for a zero of the partial derivative:

$$\frac{\partial F_\infty}{\partial u}(x, y, z, u, v) + \frac{\partial G_\infty}{\partial u}(u, v) = -i(x - z^2) - 2u.$$

Thus, the saddle-point in u is

$$u^* = u^*(x, z) = -\frac{i(x - z^2)}{2}.$$

As for the second complex variable, we choose to consider

$$v^* = v^*(y) = -\frac{i \operatorname{sg}(y)}{2} \left(1 + \frac{1}{\operatorname{sinc}(\gamma\pi)} \right),$$

which is not the saddle-point, but which will be more convenient for our computations. We now check that we have the right to move the integration contour of the variables u and v to pass through u^* and v^* ,

as we did in section 4.4. We start with the variable v : let us fix $n \geq n_I$, $(x, y, z) \in \mathcal{D}_n$ and $u \in \mathbb{R}$. The function $v \mapsto F_n(x, y, z, u, v) + G_n(u, v)$ is defined and holomorphic on the open set

$$\left\{ v \in \mathbb{C} : -\frac{1}{m_n} < \Im v < \frac{1}{M_n} \right\}. \quad (85)$$

Yet, according to lemma 2, we have

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} = \frac{1}{\text{sinc}(\lambda\pi)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{m_n} = \frac{1}{C_\lambda} \geq \frac{1}{\text{sinc}(\lambda\pi)}. \quad (86)$$

Since $\lambda > 0$, we have $\text{sinc}(\lambda\pi) < 1$, whence

$$\frac{1}{\text{sinc}(\lambda\pi)} > \frac{1}{2} \left(1 + \frac{1}{\text{sinc}(\lambda\pi)} \right) = |\Im v^*|. \quad (87)$$

Therefore, (86) and (87) ensure that, for n large enough, the point v^* belongs to the open set (85). This allows us to move the integration contour using Cauchy's theorem, to obtain, for all $M > 0$,

$$\begin{aligned} \int_{-M}^M dv e^{F_n(x,y,z,u,v)+G_n(u,v)} &= \int_0^1 v^* dt e^{F_n(x,y,z,u,-M+tv^*)+G_n(u,-M+tv^*)} \\ &\quad + \int_{-M}^M dv e^{F_n(x,y,z,u,v^*+v)+G_n(u,v^*+v)} \\ &\quad + \int_1^0 v^* dt e^{F_n(x,y,z,u,M+tv^*)+G_n(u,M+tv^*)}. \end{aligned}$$

As in section 4.4, we have

$$\lim_{M \rightarrow \infty} \int_0^1 v^* dt e^{F_n(x,y,z,u,\pm M+tv^*)+G_n(u,\pm M+tv^*)} = 0,$$

thus, letting M tend to infinity, we obtain

$$\int_{\mathbb{R}} dv e^{F_n(x,y,z,u,v)+G_n(u,v)} = \int_{\mathbb{R}} dv e^{F_n(x,y,z,u,v^*+v)+G_n(u,v^*+v)}.$$

Therefore, our formula (75) becomes

$$\forall n \geq n_I \quad E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} du \int_{\mathbb{R}} dv e^{F_n(x,y,z,u,v^*+v)+G_n(u,v^*+v)}. \quad (88)$$

Next, we wish to swap the integrations on the two variables u and v , in order to be able to move the integration contour in u . To do so, we need to check that, for every fixed $(x, y, z) \in \mathcal{D}_n$, the function

$$(u, v) \in \mathbb{R}^2 \mapsto e^{F_n(x,y,z,u,v^*+v)+G_n(u,v^*+v)}$$

is integrable. Since

$$\left| e^{F_n(x,y,z,u,v^*+v)} \right| = e^{\Re F_n(x,y,z,u,v^*+v)} = e^{F_n(x,y,z,0,v^*)},$$

we only need to bound $|e^{G_n(u,v^*+v)}|$ by an integrable function. Given that later on, we will also need to bound $|e^{G_n(u^*+u,v^*+v)}|$ and the computations are very similar, the following lemma is formulated with a parameter t which allows us to treat both cases at the same time:

Lemma 13. *There exists $K > 0$ such that, for n large enough, for all $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$ and for all $t \in [0, 1]$,*

$$\Re G_n(tu^* + u, v^* + v) \leq G_n(tu^*, v^*) - \ln(1 + Ku^2) - \ln(1 + Kv^2).$$

Proof. Let $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, and let $t \in [0, 1]$. Note that, u^* and v^* being pure imaginary, we have $G_n(tu^*, v^*) \in \mathbb{R}$. Thus, we start by writing

$$\Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) = -\frac{1}{4} \sum_{j=1}^{n-1} \ln \left(1 + \frac{(2u/\sqrt{n} - v\alpha_j^n)^2}{(1 - 2itu^*/\sqrt{n} - iv^*\alpha_j^n)^2} \right).$$

It follows from the definition (76) of the domain \mathcal{D}_n that $z^2 - x \leq \sqrt{n}$, implying that for all $j \in \{1, \dots, n-1\}$,

$$1 - \frac{2itu^*}{\sqrt{n}} + iv^*\alpha_j^n = 1 + \frac{t(z^2 - x)}{\sqrt{n}} + \frac{\text{sg}(y)}{2} \left(1 + \frac{1}{\text{sinc}(\lambda\pi)} \right) \alpha_j^n \leq B_\lambda,$$

with

$$B_\lambda = \frac{5}{2} + \frac{1}{2 \text{sinc}(\lambda\pi)}.$$

Therefore, we have

$$\Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) \leq -\frac{1}{4} \sum_{j=1}^{n-1} \ln \left[1 + \frac{1}{B_\lambda^2} \left(\frac{2u}{\sqrt{n}} - v\alpha_j^n \right)^2 \right]. \quad (89)$$

According to lemma 3, there exists a constant $K_0 > 0$ such that, for n large enough,

$$|I_{n,u,v}| \geq K_0 \frac{6}{K_0} - 2 = 4,$$

where

$$I_{n,u,v} = \left\{ j \in \{1, \dots, n-1\} : uv\alpha_j^n \leq 0 \text{ and } |\alpha_j^n| \geq \frac{K_0}{6} \right\}.$$

Applying again lemma 3 but this time with $a = n$, we obtain that, for $n \geq 12/K_0$,

$$|J_{n,u,v}| \geq K_0 n - 6 \geq \frac{K_0 n}{2},$$

with

$$J_{n,u,v} = \left\{ j \in \{1, \dots, n-1\} \setminus I_{n,u,v} : uv\alpha_j^n \leq 0 \right\}.$$

Using this, the inequality (89) becomes

$$\begin{aligned} \Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) &\leq -\frac{1}{4} \sum_{j \in I_{n,u,v}} \ln \left(1 + \frac{K_0^2 v^2}{6B_\lambda^2} \right) - \frac{1}{4} \sum_{j \in J_{n,u,v}} \ln \left(1 + \frac{4u^2}{B_\lambda^2 n} \right) \\ &\leq -\ln \left(1 + \frac{K_0^2 v^2}{6B_\lambda^2} \right) - \frac{K_0 n}{8} \ln \left(1 + \frac{4u^2}{B_\lambda^2 n} \right). \end{aligned}$$

If moreover $n \geq 8/K_0$, then the concavity of the logarithm function implies that

$$\ln \left(1 + \frac{4u^2}{B_\lambda^2 n} \right) \geq \frac{8}{K_0 n} \ln \left(1 + \frac{4K_0 u^2}{8B_\lambda^2} \right),$$

whence

$$\Re G_n(tu^* + u, v^* + v) - G_n(tu^*, v^*) \leq -\ln\left(1 + \frac{K_0^2 v^2}{6B_\lambda^2}\right) - \ln\left(1 + \frac{K_0 u^2}{2B_\lambda^2}\right).$$

Letting

$$K = \frac{K_0^2}{6B_\lambda^2} \wedge \frac{K_0}{2B_\lambda^2},$$

we obtain the upper bound announced in the lemma. \square

The function

$$(u, v) \in \mathbb{R}^2 \mapsto \frac{1}{(1 + Ku^2)(1 + Kv^2)}$$

being integrable, Fubini's theorem allows us to swap the integral over u and the integral over v in the equation (88) to obtain, for n large enough,

$$E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du e^{F_n(x, y, z, u, v^* + v) + G_n(u, v^* + v)}.$$

We now fix $(x, y, z) \in \mathcal{D}_n$ and $v \in \mathbb{R}$ and we want to move the integration contour of the variable u . The function $u \mapsto F_n(x, y, z, u, v^* + v) + G_n(u, v^* + v)$ is defined and holomorphic on the open set

$$\left\{ u \in \mathbb{C} : \forall j \in \{1, \dots, n-1\} \quad \Re\left(1 - \frac{2iu}{\sqrt{n}} + iv^* \alpha_j^n\right) > 0 \right\}.$$

Yet, for all $j \in \{1, \dots, n-1\}$, we have

$$\begin{aligned} 1 - \frac{2iu^*}{\sqrt{n}} + iv^* \alpha_j^n &= 1 + \frac{z^2 - x}{\sqrt{n}} + \frac{\text{sg}(y) \alpha_j^n}{2} \left(1 + \frac{1}{\text{sinc}(\lambda\pi)}\right) \\ &\geq 1 - \frac{x}{\sqrt{n}} - \frac{M_n}{2} \left(1 + \frac{1}{\text{sinc}(\lambda\pi)}\right) \\ &\geq 1 - \frac{1}{n^{3/8}} - \frac{M_n}{2} - \frac{M_n}{2 \text{sinc}(\lambda\pi)} \xrightarrow{n \rightarrow \infty} \frac{1 - \text{sinc}(\lambda\pi)}{2} > 0, \end{aligned}$$

where we have used the fact that $|x| \leq n^{1/8}$ and $M_n \rightarrow \text{sinc}(\lambda\pi)$, as proved in lemma 2. Therefore, for n large enough, we have for all $(x, y, z) \in \mathcal{D}_n$,

$$1 - \frac{2iu^*}{\sqrt{n}} + iv^* \alpha_j^n \geq \frac{1 - \text{sinc}(\lambda\pi)}{4} > 0, \quad (90)$$

so that we can change the integration contour from $u \in \mathbb{R}$ to $u \in u^* + \mathbb{R}$. This change of contour can be justified exactly as the previous changes of contour, and we obtain, for n large enough,

$$E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathcal{D}_n} dx dy dz g(z) \int_{\mathbb{R}} dv \int_{\mathbb{R}} du e^{F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v)}. \quad (91)$$

5.5 Domination

We now search for an upper bound on the integrand in (91). We prove the following result:

Lemma 14. *There exist constants $K_1, K_2 > 0$ such that, for n large enough,*

$$\forall (x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2 \quad \Re(F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v)) \leq M(x, y, z, u, v),$$

where the function $M : \mathbb{R}^5 \rightarrow \mathbb{R}$ is defined by

$$M(x, y, z, u, v) = K_1 + |x| - \frac{3x^2}{16} - \frac{xz^2}{8} - \frac{3z^4}{16} - \frac{|y|}{8} \left(\frac{1}{\text{sinc}(\lambda\pi)} - 1 \right) + 2z^2 - \ln(1 + K_2 u^2) - \ln(1 + K_2 v^2). \quad (92)$$

Proof. Upper bound on the first term: We first deal with $\Re F_n$. For every $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$, we have

$$\begin{aligned} \Re F_n(x, y, z, u^* + u, v^* + v) &= -i(x - z^2)u^* - iyv^* - \frac{xz^2}{2(1 + x/\sqrt{n})} - \frac{y}{1 + x/\sqrt{n}} \\ &= -\frac{(x - z^2)^2}{2} - \frac{|y|}{2} \left(1 + \frac{1}{\text{sinc}(\lambda\pi)} \right) - \frac{xz^2}{2} + \frac{x^2 z^2}{2(x + \sqrt{n})} - y + \frac{xy}{\sqrt{n} + x} \\ &\leq -\frac{x^2}{2} + \frac{xz^2}{2} - \frac{z^4}{2} - \frac{|y|}{2} \left(\frac{1}{\text{sinc}(\lambda\pi)} - 1 \right) + \frac{x^2 z^2}{2(x + \sqrt{n})} + \frac{|xy|}{\sqrt{n} + x} \end{aligned}$$

Yet, with our definition (76) of the set \mathcal{D}_n , we have $|x| \leq n^{1/4}$, whence

$$\sqrt{n} + x \geq \sqrt{n} - n^{1/4} \geq \frac{\sqrt{n}}{2},$$

as soon as $n \geq 16$. Thus we have, on the one hand,

$$\frac{x^2 z^2}{2(x + \sqrt{n})} \leq \frac{z^2}{n^{1/4}} \leq z^2,$$

and on the other hand,

$$\frac{|xy|}{x + \sqrt{n}} \leq \frac{2|y|}{n^{3/8}} \leq \frac{|y|}{4} \left(\frac{1}{\text{sinc}(\lambda\pi)} - 1 \right),$$

for n large enough. Combining all this, we obtain that for n large enough, we have for all $(x, y, z, u, v) \in \mathcal{D}_n \times \mathbb{R}^2$,

$$\Re F_n(x, y, z, u^* + u, v^* + v) \leq -\frac{x^2}{2} + \frac{xz^2}{2} - \frac{z^4}{2} - \frac{|y|}{8} \left(\frac{1}{\text{sinc}(\lambda\pi)} - 1 \right) + z^2. \quad (93)$$

Upper bound on the second term: We now turn to the term $G_n(u^* + u, v^* + v)$. Following lemma 13 applied with $t = 1$, we have

$$\Re G_n(u^* + u, v^* + v) \leq G_n(u^*, v^*) - \ln(1 + Ku^2) - \ln(1 + Kv^2).$$

Therefore, we only need to bound the (real) term $G_n(u^*, v^*)$. With our lower bound (90), we know that for n large enough,

$$\forall (x, y, z) \in \mathcal{D}_n \quad \forall t \in [0, 1] \quad 1 - \frac{2itu^*}{\sqrt{n}} + iv^* \alpha_j^n \geq \frac{1 - \text{sinc}(\lambda\pi)}{4}, \quad (94)$$

which corresponds to the upper bound (83) that we used to establish the pointwise convergence, but now holds uniformly over all the domain \mathcal{D}_n . Going back to the expression (77) of the function G_n , we have, for all $t \in [0, 1]$,

$$\begin{aligned}
\frac{\partial G_n}{\partial u}(tu^*, v^*) &= -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{1}{1 - 2itu^*/\sqrt{n} + iv^* \alpha_j^n} \\
&= -i\sqrt{n} + \frac{i}{\sqrt{n}} \sum_{j=1}^{n-1} \left(\frac{1}{1 - 2itu^*/\sqrt{n} + iv^* \alpha_j^n} - 1 \right) \\
&= -\frac{i}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{2tu^*/\sqrt{n} - v^* \alpha_j^n}{1 - 2itu^*/\sqrt{n} + iv^* \alpha_j^n} \\
&= -\frac{i}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{2tu^*/\sqrt{n} - v^* \alpha_j^n}{1 - 2itu^*/\sqrt{n}} \\
&\quad + \frac{v^*}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{2tu^* \alpha_j^n / \sqrt{n} - v^* (\alpha_j^n)^2}{(1 - 2itu^*/\sqrt{n})(1 - 2itu^*/\sqrt{n} + iv^* \alpha_j^n)}.
\end{aligned}$$

Using (94) and the fact that

$$1 - \frac{2itu^*}{\sqrt{n}} = 1 + \frac{t(z^2 - x)}{\sqrt{n}} \geq 1 - \frac{1}{n^{3/8}} \geq \frac{1}{2},$$

we deduce that

$$\begin{aligned}
\left| \frac{\partial G_n}{\partial u}(tu^*, v^*) \right| &\leq \frac{1}{\sqrt{n}} + \frac{2t|u^*|}{1 - n^{-3/8}} + \frac{2|v^*|}{\sqrt{n}} \sum_{j=1}^{n-1} |\alpha_j^n| \\
&\quad + \frac{16t|u^*||v^*|}{(1 - \text{sinc}(\lambda\pi))n} \sum_{j=1}^{n-1} |\alpha_j^n| + \frac{8|v^*|^2}{(1 - \text{sinc}(\lambda\pi))\sqrt{n}} \sum_{j=1}^{n-1} (\alpha_j^n)^2.
\end{aligned}$$

According to the upper bounds (13) and (14), in our regime $2d_n \sim \lambda n$ we have

$$\sum_{j=1}^{n-1} |\alpha_j^n| = O(\ln n) \quad \text{and} \quad \sum_{j=1}^{n-1} (\alpha_j^n)^2 = O(1).$$

What's more, we have $v^* = O(1)$ uniformly over all the domain \mathcal{D}_n (since v^* can only take two different values). Therefore, we can write, uniformly for all $(x, y, z) \in \mathcal{D}_n$,

$$\frac{2|v^*|}{\sqrt{n}} \sum_{j=1}^{n-1} |\alpha_j^n| = O\left(\frac{\ln n}{\sqrt{n}}\right) = o(1),$$

along with

$$\frac{16|v^*|}{(1 - \text{sinc}(\lambda\pi))n} \sum_{j=1}^{n-1} |\alpha_j^n| = O\left(\frac{\ln n}{n}\right) = o(1),$$

and

$$\frac{8|v^*|^2}{(1 - \text{sinc}(\lambda\pi))\sqrt{n}} \sum_{j=1}^{n-1} (\alpha_j^n)^2 = O\left(\frac{1}{\sqrt{n}}\right) = o(1).$$

Therefore, for n large enough, we have for all $(x, y, z) \in \mathcal{D}_n$

$$\frac{2}{1 - n^{-3/8}} + \frac{16|v^*|}{(1 - \text{sinc}(\lambda\pi))n} \sum_{j=1}^{n-1} |\alpha_j^n| \leq \frac{5}{2},$$

and

$$\frac{1}{\sqrt{n}} + \frac{2|v^*|}{\sqrt{n}} \sum_{j=1}^{n-1} |\alpha_j^n| + \frac{8|v^*|^2}{(1 - \text{sinc}(\lambda\pi))\sqrt{n}} \sum_{j=1}^{n-1} (\alpha_j^n)^2 \leq 2,$$

which implies that for all $t \in [0, 1]$,

$$\left| \frac{\partial G_n}{\partial u}(tu^*, v^*) \right| \leq 2 + \frac{5t|u^*|}{2}.$$

Integrating this, we deduce that, for n large enough, for all $(x, y, z) \in \mathcal{D}_n$, we have

$$|G_n(u^*, v^*)| = \left| G_n(0, v^*) + u^* \int_0^1 \frac{\partial G_n}{\partial u}(tu^*, v^*) dt \right| \leq |G_n(0, v^*)| + 2|u^*| + \frac{5|u^*|^2}{4}.$$

Since v^* can only take two different values, the term $G_n(0, v^*)$ is bounded from above by a constant $K_1 > 0$, uniformly for all $(x, y, z) \in \mathcal{D}_n$. Thus, we obtain that for n large enough, for all $(x, y, z) \in \mathcal{D}_n$,

$$|G_n(u^*, v^*)| \leq K_1 + |x - z^2| + \frac{5(x - z^2)^2}{16} \leq K_1 + |x| + z^2 + \frac{5(x - z^2)^2}{16}.$$

Combining this with (93) and with the result of lemma 13 (applied this time with $t = 1$), we obtain the announced domination. \square

Now there remains to check:

Lemma 15. *The function e^M , where M is given by (92), is integrable on \mathbb{R}^5 .*

Proof. Let us write, in $[0, +\infty]$,

$$\begin{aligned} \int_{\mathbb{R}^5} e^M &= e^{K_1} \int_{\mathbb{R}^2} dx dz \exp\left(|x| - \frac{3x^2}{16} - \frac{xz^2}{8} - \frac{3z^4}{16} + 2z^2\right) \times \int_{\mathbb{R}} dy \exp\left[-\frac{|y|}{8} \left(\frac{1}{\text{sinc}(\lambda\pi)} - 1\right)\right] \\ &\quad \times \left(\int_{\mathbb{R}} \frac{du}{1 + K_2 u^2}\right)^2 \\ &= e^{K_1} \int_{\mathbb{R}^2} dx dz \exp\left[|x| - \frac{1}{16} \left(3x + \frac{z^2}{3}\right)^2 - \frac{z^4}{18} + 2z^2\right] \times \frac{16 \text{sinc}(\lambda\pi)}{1 - \text{sinc}(\lambda\pi)} \times \frac{\pi^2}{K_2}, \end{aligned}$$

which shows that the integral is indeed finite. \square

5.6 Dominated convergence

We are now in a position to apply the dominated convergence theorem. Let us recall that

$$E_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathbb{R}^5} g(z) e^{F_n(x, y, z, u^*+u, v^*+v) + G_n(u^*+u, v^*+v)} \mathbb{1}_{(x, y, z) \in \mathcal{D}_n}.$$

It follows from equation (78) and lemma 12 that for all $(x, y, z, u, v) \in \mathbb{R}^5$, we have the pointwise convergence

$$\lim_{n \rightarrow \infty} F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) = F_\infty(x, y, z, u^* + u, v^* + v) + G_\infty(u^* + u, v^* + v),$$

where

$$F_\infty(x, y, z, u^* + u, v^* + v) = -\frac{(x - z^2)^2}{2} - iu(x - z^2) - i(v^* + v)y - \frac{xz^2}{2} - y$$

and

$$G_\infty(u^* + u, v^* + v) = \frac{(x - z^2)^2}{4} + iu(x - z^2) - u^2 + \frac{i(v^* + v)}{2\lambda} - \sum_{j=1}^{+\infty} \ln(1 + i(v^* + v)\text{sinc}(j\lambda\pi)).$$

Summing these, we obtain

$$\lim_{n \rightarrow \infty} F_n(x, y, z, u^* + u, v^* + v) + G_n(u^* + u, v^* + v) = J(x, y, z, u, v),$$

with

$$J(x, y, z, u, v) = -\frac{x^2}{4} - \frac{z^4}{4} - u^2 - y - i\left(y - \frac{1}{2\lambda}\right)(v^* + v) - \sum_{j=1}^{+\infty} \ln(1 - 2i(v^* + v)\text{sinc}(j\lambda\pi)).$$

Note that, in this function J , the only term where the variable z (which is the variable we are interested in, since it corresponds to S_n) appears is $-z^4/4$. This is the reason why we obtain the same behaviour as in the mean-field case. In the subsequent article [For21], we will study a regime where there is another term combining z and v , therefore leading to a different limit theorem. According to lemmas 14 and 15, the domination hypothesis is satisfied for n large enough. Thus, the dominated convergence theorem entails that

$$E_n \stackrel{n \rightarrow \infty}{\sim} \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathbb{R}^5} dx dy dz du dv g(z) e^{J(x, y, z, u, v)} = C_n \int_{\mathbb{R}} dz g(z) e^{-z^4/4},$$

where

$$C_n = \frac{n^{1/4}}{(2\pi)^{5/2} Z_n} \int_{\mathbb{R}^4} dx dy du dv e^{J(x, y, 0, u, v)}.$$

Yet, going back to our computations at the beginning of section 5.2, we know that

$$\mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = E_n + o(1).$$

In the case of the constant function $g = 1$, this yields

$$1 = \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] \stackrel{n \rightarrow \infty}{\sim} E_n \stackrel{n \rightarrow \infty}{\sim} C_n \int_{\mathbb{R}} dz g(z) e^{-z^4/4} = \frac{C_n \Gamma(1/4)}{\sqrt{2}}.$$

Turning back to the general case of a bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, we obtain

$$\lim_{n \rightarrow \infty} \mu_n \left[g \left(\frac{S_n}{n^{3/4}} \right) \right] = \frac{\sqrt{2}}{\Gamma(1/4)} \int_{\mathbb{R}} dz g(z) e^{-z^4/4},$$

which concludes the proof of the convergence in distribution announced in theorem 1. \square

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