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A Variational Marginalized Particle Filter for Jump Markov Nonlinear Systems with Unknown Transition Probabilities [☆]

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Abstract

This paper studies a new variational marginalized particle filter for jointly estimating the state and the system mode parameters of jump Markov nonlinear systems. Contrary to the Markovian assumption usually considered to model the evolution of the system modes, we introduce conjugate prior distributions for the system mode parameters. The joint posterior distribution of the state and system mode parameters is then marginalized with respect to the mode variables. The remaining state vector is sampled using a sequential Monte Carlo algorithm, and the mode parameters are sampled using variational Bayesian inference. In order to obtain analytical solutions for the different variational distributions, we use an extended factorized approximation simplifying the variational distributions. A comprehensive simulation study is conducted to compare the performance of the proposed approach with the state-of-the-art for a modified nonlinear benchmark model and maneuvering target tracking scenarios.

Keywords: Jump Markov nonlinear systems, marginalized particle filter, variational inference, extended factorized approximation.

1. Introduction

Jump Markov systems have been widely investigated in the literature, especially for state-space models that are conditionally linear Gaussian models, i.e., jump Markov linear systems (JMLSs), where a finite-state Markov chain enables switches between different modes with an appropriate transition probability matrix (TPM). Many algorithms have been proposed to solve this state estimation problem, mainly based on Gaussian mixture approximations, such as the interacting multiple model algorithm [1] or the generalized pseudo-Bayes algorithm [2]. In addition, efficient sequential Monte Carlo algorithms have been defined for JMLSs with a known TPM [3]. However, transition probabilities among different system modes are often unknown in practice [4]. For instance, the target tracking problem can be formulated as a jump Markov system where a finite number of possible maneuver models correspond to different target behavior modes and the model switching is usually modeled by a homogeneous Markov chain [5]. The transition probabilities of this Markov chain are treated by engineers as “design parameters” due to the absence of prior knowledge about the mean sojourn time for each mode. Other applications where the different modes are defined by unknown transition probabilities include the vertical take-off/landing of a helicopter [6], the mobile terminal positioning in wireless networks [7] and the internet-based networked control system [8]. In general, using an inaccurate TPM leads to performance degradation for the algorithms used for JMLS state estimation, mainly because the weights of mode-matched filters depend on the TPM [9].

On the one hand, a wide range of techniques has been investigated to design state estimators that are robust to uncertain TPs, such as the robust \mathcal{H}_∞ filter [10, 11] and the fuzzy $\mathcal{L}_2 - \mathcal{L}_\infty$ filtering [12, 13]. On the other

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hand, different approaches can be found in the literature for estimating the TPM jointly with the JMLS state. The proposed solutions consider that the transition probabilities are either deterministic quantities or random variables. When transition probabilities are treated as deterministic quantities, they can be estimated by a statistical approach based on the maximum likelihood (ML) principle, such as recursive Kullback-Leibler algorithms minimizing the Kullback-Leibler divergence between the likelihood function given the TPM and the true likelihood [14, 15]. An expectation-maximization (EM) algorithm was also studied in [16] for computing the ML estimator of the TPM, by calculating the ratio between statistics associated with a Markov chain jump between two different modes and statistics associated with the individual modes. A batch ML estimation of the TPM via convex optimisation was also investigated in [17]. When transition probabilities are treated as random variables, their posterior distribution can be derived by using the Bayesian rule. This strategy was proposed in [18] where Dirichlet priors were assigned to the TPM and the marginal posterior of the TPM was determined using Bayesian inference. Jilkov also proposed in [19] to approximate the posterior distribution of the TPM based on the minimum mean square error criterion.

The methods mentioned previously are not applicable to nonlinear state-space models, such as jump Markov nonlinear systems (JMNLs), which arise in various applications including target tracking [20, 21] and localization [22, 23]. The state-of-the-art about JMNLs estimation reduces to the Rao-Blackwellized particle filter (RBPF) [7] and an online EM approach embedded within the RBPF [24], which were proposed for estimating jointly the state and the TPM of a JMNLs. This paper studies a new variational marginalized particle filtering (MPF) for jointly estimating the state and the system mode parameters of JMNLs. The principle of the variational marginalized particle filtering for state estimation in a nonlinear state-space system is to marginalize out the unknown noise and state parameters of the state-space system, to approximate the posterior distribution of the state using an appropriate particle filter and to calculate the distribution of the noise parameters conditionally on each state particle using variational Bayesian inference. The MPF was studied in [25] for mixed linear/non-linear systems. It was combined with a variational Bayes approximation in [26] to address cases where the additive observation noise has a Student- t distribution, leading to a robust variational MPF. When the state and measurement equation depend on different modes, it is possible to consider a jump Markov nonlinear system whose state and measurement equations depend on an unknown mode variable [7]. The transitions between different modes at consecutive time instants are generally governed by a Markov model for simplicity. The main innovations of this paper are summarized below

- Contrary to the Markovian assumption usually considered to model the evolution of the system modes, we introduce conjugate prior distributions for the system mode parameters, which allows non Markovian transition between the different system modes to be considered.
- The joint posterior distribution of the state and system mode parameters is derived using a variational Bayes MPF
- Since analytical solutions for the different variational distributions cannot be obtained in closed form due to difficulties in calculating a multivariate log-inverse-beta function, we propose to use the extended factorized approximation introduced in [27] to obtain closed-form expressions for these variational distributions.

The paper is organized as follows: Section II formulates the joint state and mode parameter estimation problem for JMNLs and introduces the conjugate prior distributions for the system mode parameters. Section III studies the proposed new variational marginalized particle filter, allowing the state of JMNLs and the corresponding time-varying parameters to be estimated sequentially. The performance of the resulting filter is evaluated in Section IV using a modified nonlinear benchmark model and realistic target tracking scenarios. Conclusions are finally reported in Section V.

Nomenclature

Acronyms

EM expectation-maximization

JMLS jump Markov linear system
 JMNLS jump Markov nonlinear systems
 KL Kullback-Leibler
 MAP maximum a posteriori
 ML maximum likelihood
 MPF marginalized particle filtering
 pdf probability density function
 RBPF Rao-blackwellized particle filter
 RMSE root mean square error
 TPM transition probability matrix

Mathematical Notations

α concentration parameter vector of Dirichlet distribution
 θ mode parameter vector containing r , \mathbf{u} and α
 \mathbf{u} probability vector for system mode
 \mathbf{x} state vector
 \mathbf{y} measurement vector
 $(\cdot)^T$ transpose of a vector
 $E[\cdot]$ expectation
 ω weight of particle
 a shape parameter of gamma distribution
 b rate parameter of gamma distribution
 K number of system modes
 k, t, i integer indexes
 N number of particles
 N_m number of Monte Carlo runs
 r indicator variable for system mode

Other Symbols

\bar{x} expected quantity
 \hat{x} estimator quantity
 \tilde{x} approximate quantity
 x^-, x^+ quantity computed a priori and a posteriori

2. Problem Formulation

In this paper, we consider the following discrete-time JMNLs related to a hidden state vector $\mathbf{x}_t \in \mathbb{R}^{n_x}$ and the measurement vector $\mathbf{y}_t \in \mathbb{R}^{n_y}$

$$\mathbf{x}_t \sim f(\mathbf{x}_t | \mathbf{x}_{t-1}, r_t) \quad (1)$$

$$\mathbf{y}_t \sim g(\mathbf{y}_t | \mathbf{x}_t, r_t) \quad (2)$$

where $t = 1, \dots, T$ denotes the t th time instant, $r_t \in \{1, \dots, K\}$ is a discrete variable indicating the system mode at time t , K denotes the number of modes, and $f(\mathbf{x}_t | \mathbf{x}_{t-1}, r_t)$ and $g(\mathbf{y}_t | \mathbf{x}_t, r_t)$ are the conditional probability density functions (pdfs) of the state and measurement vectors associated with the r_t th system mode. In order to implement the state estimation of JMNLs, the statistical structure of the indicator variable r_t is generally assumed to be a first-order finite-state homogeneous Markov chain with a fixed TPM. This paper relaxes this assumption and considers a more general categorical distribution $\text{Cat}(\cdot)$ [28] denoted as

$$r_t \sim \text{Cat}(r_t | \mathbf{u}_t) \quad (3)$$

with

$$\text{Cat}(r_t | \mathbf{u}_t) = \prod_{k=1}^K u_{k,t}^{\mathbb{I}(r_t=k)}, \quad (4)$$

where $\mathbb{I}(r_t = k)$ is an indicator for the k th mode (equal to 1 when $r_t = k$ and 0 otherwise), $\mathbf{u}_t = (u_{1,t}, \dots, u_{K,t})^\top$ is a vector containing the probabilities of the different modes at the t th time instant, i.e.,

$$\Pr(r_t = k) = u_{k,t}, \quad k = 1, \dots, K \quad (5)$$

with $u_{k,t} > 0$ and $\sum_{k=1}^K u_{k,t} = 1$. In addition, the probability vector $\mathbf{u}_t = (u_{1,t}, \dots, u_{K,t})^\top$ is assigned a Dirichlet distribution $\text{Dir}(\cdot)$,

$$\mathbf{u}_t \sim \text{Dir}(\mathbf{u}_t | \boldsymbol{\alpha}_t) \quad (6)$$

with

$$\text{Dir}(\mathbf{u}_t | \boldsymbol{\alpha}_t) = \frac{\Gamma(\sum_{k=1}^K \alpha_{k,t})}{\prod_{k=1}^K \Gamma(\alpha_{k,t})} \prod_{k=1}^K u_{k,t}^{\alpha_{k,t}-1}, \quad (7)$$

where $\boldsymbol{\alpha}_t = (\alpha_{1,t}, \dots, \alpha_{K,t})^\top$ is a concentration parameter vector and $\Gamma(\alpha) \triangleq \int_0^\infty u^{\alpha-1} \exp(-u) du$ is the gamma function. Note that the Dirichlet distribution can reduce to the uniform distribution on the simplex $\{u_{k,t} > 0, \sum_{k=1}^K u_{k,t} = 1\}$ or can be more informative depending on the value of the hyperparameter vector $\boldsymbol{\alpha}_t$. This paper assumes that the K variables $\alpha_{k,t}$ are mutually independent and distributed according to gamma distributions

$$q(\alpha_{k,t}) = \gamma(\alpha_{k,t} | a_{k,t}, b_{k,t}), \quad (8)$$

where $k = 1, \dots, K$, $\gamma(\alpha_{k,t} | a_{k,t}, b_{k,t})$ denotes the gamma distribution with shape parameter $a_{k,t}$ and rate parameter (inverse scale parameter) $b_{k,t}$, whose pdf is defined as:

$$\gamma(\alpha | a, b) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} \exp(-b\alpha) \mathbb{I}_{\mathbb{R}^+}(\alpha), \quad (9)$$

where $\mathbb{I}_{\mathbb{R}^+}$ is the indicator function on \mathbb{R}^+ . Note that conjugate distributions have been assigned a priori to r_t , \mathbf{u}_t and $\boldsymbol{\alpha}_t$, which will simplify the analysis. As a consequence, the joint prior of $\{r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t\}$ is defined as

$$q(r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t) = \text{Cat}(r_t | \mathbf{u}_t) \text{Dir}(\mathbf{u}_t | \boldsymbol{\alpha}_t) \prod_{k=1}^K \gamma(\alpha_{k,t} | a_{k,t}, b_{k,t}). \quad (10)$$

When the system is possibly switching between different modes, the parameters $r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t$ are time-varying. Thus, the state estimation problem for JMNLs considered in this work is defined as estimating jointly the state and time-varying parameters of a state-space model. The Bayesian strategy adopted in this work is based on the joint posterior distribution of all unknown variables given the measurements $\mathbf{y}_{1:t} = \{\mathbf{y}_1, \dots, \mathbf{y}_t\}$, denoted as $p(\mathbf{x}_{1:t}, \boldsymbol{\theta}_t | \mathbf{y}_{1:t})$ with $\mathbf{x}_{1:t} = \{\mathbf{x}_1, \dots, \mathbf{x}_t\}$ and $\boldsymbol{\theta}_t = (r_t, \mathbf{u}_t^\top, \boldsymbol{\alpha}_t^\top)^\top$, which is studied in the next section.

3. A Variational Bayes Marginalized Particle Filter for the JMNLs

Since the joint posterior distribution $p(\mathbf{x}_{1:t}, \boldsymbol{\theta}_t | \mathbf{y}_{1:t})$ of JMNLs has a complex expression and cannot be calculated in closed form, we propose to study a particle filter approximating this posterior by using sequential importance sampling. Following the concept of the MPF [25], the joint posterior distribution of the state and system mode variables can be factorized as follows

$$p(\mathbf{x}_{1:t}, \boldsymbol{\theta}_t | \mathbf{y}_{1:t}) = p(\boldsymbol{\theta}_t | \mathbf{x}_{1:t}, \mathbf{y}_{1:t}) p(\mathbf{x}_{1:t} | \mathbf{y}_{1:t}), \quad (11)$$

where the mode vector $\boldsymbol{\theta}_t$ has been marginalized out in the second term of the right hand side. This paper proposes to approximate $p(\mathbf{x}_{1:t} | \mathbf{y}_{1:t})$ by using an empirical density following the principle of particle filters

$$p(\mathbf{x}_{1:t} | \mathbf{y}_{1:t}) \approx \sum_{i=1}^N \omega_t^i \delta(\mathbf{x}_{1:t} - \mathbf{x}_{1:t}^i), \quad (12)$$

where N is the number of particles, $\delta(\cdot)$ is the Dirac delta function, $\mathbf{x}_{1:t}^i$ is the i th particle path and ω_t^i is the corresponding weight at the t th time instant, which can be updated as follows

$$\omega_t^i \propto \frac{p(\mathbf{y}_t, \mathbf{x}_t^i | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t-1})}{q(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t})} \omega_{t-1}^i \quad (13)$$

with

$$\mathbf{x}_t^i \sim q(\mathbf{x}_t | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t}), \quad (14)$$

where $i = 1, \dots, N$ and $q(\mathbf{x}_t | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t})$ is the optimal importance distribution introduced in [29]. Replacing (12) in (11) leads to

$$p(\mathbf{x}_{1:t}, \boldsymbol{\theta}_t | \mathbf{y}_{1:t}) \approx \sum_{i=1}^N \omega_t^i p(\boldsymbol{\theta}_t | \mathbf{x}_{1:t}^i, \mathbf{y}_{1:t}) \delta(\mathbf{x}_{1:t} - \mathbf{x}_{1:t}^i). \quad (15)$$

Since the system mode parameters $\boldsymbol{\theta}_t = (r_t, \mathbf{u}_t^\top, \boldsymbol{\alpha}_t^\top)^\top$ is dependent of the state \mathbf{x}_t at the t th time instant, (15) can be simplified as follows

$$p(\mathbf{x}_{1:t}, \boldsymbol{\theta}_t | \mathbf{y}_{1:t}) \approx \sum_{i=1}^N \omega_t^i p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t}) \delta(\mathbf{x}_{1:t} - \mathbf{x}_{1:t}^i), \quad (16)$$

where the conditional distribution $p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})$ at the t th time instant is the distribution of the mode variables conditionally on the i th particle \mathbf{x}_t^i . In this paper, we propose to approximate the distribution $p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})$ using variational Bayesian inference. As a consequence, samples $\{\mathbf{x}_{1:t}^i\}_{i=1}^N$ distributed according to $p(\mathbf{x}_{1:t} | \mathbf{y}_{1:t})$ are generated using a sequential Monte Carlo method, and then the posterior distribution of $\boldsymbol{\theta}_t$ conditionally on \mathbf{x}_t^i is calculated according to variational Bayesian inference and replaced in (16), allowing the joint distribution of the state and system mode variables to be determined. These two steps are detailed in the following sections.

3.1. Generating samples $\mathbf{x}_{0:t}$ using a particle filter

In order to update the weight ω_t^i associated with the i th particle at the t th time, the conditional distribution $p(\mathbf{y}_t, \mathbf{x}_t | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t-1})$ appearing in the numerator of (13) is the mixture distribution of the pdfs $p(\mathbf{y}_t, \mathbf{x}_t, r_t^i | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t-1})$ where $r_t \in \{1, \dots, K\}$, i.e.,

$$\begin{aligned} p(\mathbf{y}_t, \mathbf{x}_t | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t-1}) &= \sum_{k=1}^K p(\mathbf{y}_t, \mathbf{x}_t, r_t^k | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t-1}) \\ &= \sum_{k=1}^K g(\mathbf{y}_t | \mathbf{x}_t, r_t^k) f(\mathbf{x}_t | \mathbf{x}_{t-1}^i, r_t^k) p(r_t^k | \mathbf{y}_{1:t-1}), \end{aligned} \quad (17)$$

where $i = 1, \dots, N$ and $p(r_t^i | \mathbf{y}_{1:t-1})$ is the predictive distribution of the indicator variable r_t for the i th particle at the t th time instant. The predictive distribution is defined as follows

$$p(r_t^i | \mathbf{y}_{1:t-1}) = \prod_{k=1}^K (u_{k,t}^{i,-})^{\mathbb{I}(r_t^i=k)}, \quad (18)$$

where $\mathbf{u}_t^{i,-} = (u_{1,t}^{i,-}, \dots, u_{K,t}^{i,-})^\top$ contains the predicted mode probabilities for the i th particle and the superscripts “-” means that the quantity are computed a priori. Note that the distribution of $\mathbf{u}_t^{i,-}$ is then a Dirichlet distribution according to (6)

$$\mathbf{u}_t^{i,-} \sim \text{Dir}(\mathbf{u}_t | \boldsymbol{\alpha}_t^{i,-}), \quad (19)$$

where $\boldsymbol{\alpha}_t^{i,-} = (\alpha_{1,t}^{i,-}, \dots, \alpha_{K,t}^{i,-})^\top$ is the so-called predicted concentration parameter vector for the i th particle. In order to maintain some conjugacy between the different considered distributions, we propose to decompose the parameters $\alpha_{k,t}^{i,-}$ as follows

$$\alpha_{k,t}^{i,-} = \frac{a_{k,t}^{i,-}}{b_{k,t}^{i,-}}, \quad k = 1, \dots, K \quad (20)$$

where $a_{k,t}^{i,-}$ and $b_{k,t}^{i,-}$ are the predicted hyperparameters of a gamma distribution associated with the i th particle, i.e.,

$$a_{k,t}^{i,-} = \rho a_{k,t-1}^{i,+}, \quad b_{k,t}^{i,-} = \rho b_{k,t-1}^{i,+}, \quad (21)$$

where $\rho \in (0, 1)$ is a forgetting factor (this strategy was used successfully in [30] for tracking the parameters of a randomly drifting stochastic resonator) and the superscripts “+” mean that the quantity are computed a posteriori. The a priori quantities $\{u_{k,t}^{i,-}, \alpha_{k,t}^{i,-}, a_{k,t}^{i,-}, b_{k,t}^{i,-}\}_{k=1}^K$ are then used as initial values for variational Bayesian iterations (detailed in the Section 3.2), leading to the a posteriori quantities $\{u_{k,t}^{i,+}, \alpha_{k,t}^{i,+}, a_{k,t}^{i,+}, b_{k,t}^{i,+}\}_{k=1}^K$ defined as hyperparameters of the variational distributions associated with the system mode parameters. As a consequence, $p(r_t^i | \mathbf{y}_{1:t})$ is also a categorical distribution denoted as

$$p(r_t^i | \mathbf{y}_{1:t}) = \prod_{k=1}^K (u_{k,t}^{i,+})^{\mathbb{I}(r_t^i=k)}. \quad (22)$$

Replacing (18) in (17) leads to

$$p(\mathbf{y}_t, \mathbf{x}_t | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t-1}) = \prod_{k=1}^K [g(\mathbf{y}_t | \mathbf{x}_t^i, r_t^i = k) f(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i, r_t^i = k) u_{k,t}^{i,-}]^{\mathbb{I}(r_t^i=k)}. \quad (23)$$

The generation of particles distributed according to the marginalized state posterior requires to define an appropriate importance distribution. In this work, the bootstrap proposal [31] is chosen as the important distribution defined in (14), i.e.,

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t}) = p(\mathbf{x}_t | \mathbf{x}_{t-1}^i, \mathbf{y}_{1:t-1}) = \prod_{k=1}^K [f(\mathbf{x}_t | \mathbf{x}_{t-1}^i, r_t^i = k) u_{k,t}^{i,-}]^{\mathbb{I}(r_t^i=k)}. \quad (24)$$

According to (23) and (24), (13) can be rewritten as follows

$$\omega_t^i \propto \frac{\prod_{k=1}^K [g(\mathbf{y}_t | \mathbf{x}_t^i, r_t^i = k) f(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i, r_t^i = k) u_{k,t}^{i,-}]^{\mathbb{I}(r_t^i=k)}}{\prod_{k=1}^K [f(\mathbf{x}_t | \mathbf{x}_{t-1}^i, r_t^i = k) u_{k,t}^{i,-}]^{\mathbb{I}(r_t^i=k)}} \omega_{t-1}^i \quad (25)$$

where

$$\mathbf{x}_t^i \sim \prod_{k=1}^K [f(\mathbf{x}_t | \mathbf{x}_{t-1}^i, r_t^i = k) u_{k,t}^{i,-}]^{\mathbb{I}(r_t^i=k)}, \quad i = 1, \dots, N. \quad (26)$$

3.2. Calculating $p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})$ based on an extended factorized approximation

Considering that it is difficult to calculate the posterior distribution $p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})$ conditionally on the i th particle \mathbf{x}_t^i in closed form, we propose to approximate this posterior by the variational Bayesian inference. According to the mean-field theory in the variational Bayesian inference [32], the joint pdf of system mode variables $q(\boldsymbol{\theta}_t)$ can be factorized into single-variable factors, i.e., $q(\boldsymbol{\theta}_t) = q(r_t)q(\mathbf{u}_t)q(\boldsymbol{\alpha}_t)$. Note that the state \mathbf{x}_t carries information about r_t and thus serves as an extra measurement in this work. According to the factorized approximation [33], the logarithm of the marginal likelihood $\ln p(\mathbf{y}_t, \mathbf{x}_t^i | \boldsymbol{\theta}_t, \mathbf{y}_{1:t-1})$ can be defined by using the following identity

$$\ln p(\mathbf{y}_t, \mathbf{x}_t^i | \boldsymbol{\theta}_t, \mathbf{y}_{1:t-1}) = \mathcal{L} + \text{KL}[q(\boldsymbol{\theta}_t), p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})] \quad (27)$$

with

$$\begin{aligned} \mathcal{L} &= \mathbb{E}_{\boldsymbol{\theta}_t} \left[\ln \frac{p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})}{q(\boldsymbol{\theta}_t)} \right] \\ &= \int \int \int q(r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t) \ln \frac{p(\mathbf{y}_t, \mathbf{x}_t^i, r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t | \mathbf{y}_{1:t-1})}{q(r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t)} d\boldsymbol{\alpha}_t d\mathbf{u}_t dr_t \end{aligned} \quad (28)$$

and

$$\begin{aligned} \text{KL}[q(\boldsymbol{\theta}_t), p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})] &= \mathbb{E}_{\boldsymbol{\theta}_t} \left[\ln \frac{q(\boldsymbol{\theta}_t)}{p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})} \right] \\ &= \int \int \int q(r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t) \ln \frac{q(r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t)}{p(r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})} d\boldsymbol{\alpha}_t d\mathbf{u}_t dr_t, \end{aligned} \quad (29)$$

where \mathcal{L} is a variational objective function used in variational inference, $\text{KL}[q(\boldsymbol{\theta}_t), p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})]$ is the Kullback-Leibler divergence between the true posterior and its approximation. Accordingly, the joint pdf $p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})$ in (28) can be expressed according to (7), (8) and (23)

$$\begin{aligned} p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1}) &= p(\mathbf{y}_t | \mathbf{x}_t^i, r_t) p(\mathbf{x}_t^i | \mathbf{y}_{1:t-1}, r_t) q(r_t, \mathbf{u}_t, \boldsymbol{\alpha}_t) \\ &= \frac{\Gamma(\sum_{k=1}^K \alpha_{k,t})}{\prod_{k=1}^K \Gamma(\alpha_{k,t})} \prod_{k=1}^K \left\{ u_{k,t}^{\alpha_{k,t}-1} \gamma(\alpha_{k,t} | a_{k,t}, b_{k,t}) [u_{k,t} g(\mathbf{y}_t | \mathbf{x}_t^i, r_t^i = k) f(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i, r_t^i = k)]^{\mathbb{I}(r_t=k)} \right\}. \end{aligned} \quad (30)$$

Since the KL divergence is non-negative, minimizing the KL divergence can be achieved by maximizing the variational objective function \mathcal{L} . Accordingly, maximizing \mathcal{L} can be achieved by computing expectations of $\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})$ with respect to $q(r_t)$, $q(\mathbf{u}_t)$ and $q(\boldsymbol{\alpha}_t)$ in turn, i.e.,

$$\ln q(r_t) = \mathbb{E}_{\boldsymbol{\theta}_t \setminus r_t} [\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})], \quad (31)$$

$$\ln q(\mathbf{u}_t) = \mathbb{E}_{\boldsymbol{\theta}_t \setminus \mathbf{u}_t} [\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})], \quad (32)$$

$$\ln q(\boldsymbol{\alpha}_t) = \mathbb{E}_{\boldsymbol{\theta}_t \setminus \boldsymbol{\alpha}_t} [\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})], \quad (33)$$

where $\mathbb{E}_{\boldsymbol{\theta}_t \setminus \mathcal{X}}$ denotes the expectation with respect to the variational distributions of all variables in $\boldsymbol{\theta}_t$ except those contained in \mathcal{X} . However, analytical solutions for these variational distributions cannot be obtained in closed form due to the presence of the multivariate log-inverse-beta function defined by

$$\text{MLIB}(\boldsymbol{\alpha}_t) = \ln \frac{\Gamma(\sum_{k=1}^K \alpha_{k,t})}{\prod_{k=1}^K \Gamma(\alpha_{k,t})}. \quad (34)$$

In this work, the extended factorized approximation proposed in [27] is considered to derive analytical solutions for the variational distributions. According to the extended factorized approximation, a lower-bound $\tilde{\mathcal{L}}$ for the variational objective function \mathcal{L} can be defined as follows

$$\mathcal{L} \geq \tilde{\mathcal{L}} \quad (35)$$

with

$$\tilde{\mathcal{L}} = \mathbb{E}_{\theta_t} \left[\ln \frac{\tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})}{q(\boldsymbol{\theta}_t)} \right] \quad (36)$$

where $\mathbb{E}_{\theta_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})]$ satisfies the following inequality

$$\mathbb{E}_{\theta_t} [\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] \geq \mathbb{E}_{\theta_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})]. \quad (37)$$

Accordingly, maximizing this lower bound $\tilde{\mathcal{L}}$ is asymptotically equivalent to maximizing the variational objective function \mathcal{L} in (27) [27]. Appendix A explains how the lower-bound $\tilde{\mathcal{L}}$ and $\mathbb{E}_{\theta_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})]$ can be defined. Considering that maximizing $\tilde{\mathcal{L}}$ can be achieved by computing expectations of $\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})$ with respect to $q(r_t)$, $q(\mathbf{u}_t)$ and $q(\boldsymbol{\alpha}_t)$ in turn, the variational distributions can be approximated as follows

$$\begin{aligned} \ln q(r_t = k) &\approx \mathbb{E}_{\theta_t \setminus r_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] \\ &\propto \left\{ \mathbb{E} [\ln u_{k,t}] + \ln g(\mathbf{y}_t | \mathbf{x}_t^i, r_t^i) + \ln f(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i, r_t^i) \right\} \mathbb{I}(r_t = k), \end{aligned} \quad (38)$$

$$\begin{aligned} \ln q(\mathbf{u}_t) &\approx \mathbb{E}_{\theta_t \setminus \mathbf{u}_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] \\ &\propto \sum_{k=1}^K \left\{ \mathbb{E} [\mathbb{I}(r_t = k)] + \mathbb{E} [\alpha_{k,t}] - 1 \right\} \ln u_{k,t}, \end{aligned} \quad (39)$$

$$\begin{aligned} \ln q(\boldsymbol{\alpha}_t) &\approx \mathbb{E}_{\theta_t \setminus \boldsymbol{\alpha}_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] \\ &\propto \sum_{k=1}^K \left\{ a_{k,t} + \left[\Psi \left(\sum_{k=1}^k \bar{\alpha}_{k,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right] \bar{\alpha}_{k,t} - 1 \right\} \ln \alpha_{k,t} - \sum_{k=1}^K (b_{k,t} - \mathbb{E} [\ln u_{k,t}]) \alpha_{k,t}, \end{aligned} \quad (40)$$

where the detailed derivations for (38)-(40) are defined in Appendix B. The expectations appearing in the above equations to be computed as follows

$$\mathbb{E} [\ln u_{k,t}] = \Psi(\alpha_{k,t}) - \Psi \left(\sum_{k=1}^K \alpha_k \right), \quad (41)$$

$$\mathbb{E} [\mathbb{I}(r_t = k)] = \frac{q(r_t = k)}{\sum_{k=1}^K q(r_t = k)}, \quad (42)$$

$$\mathbb{E} [\alpha_{k,t}] = \frac{a_{k,t}}{b_{k,t}}. \quad (43)$$

As a consequence, the a posteriori quantities $u_{k,t}^{i,+}$, $\alpha_{k,t}^{i,+}$, $a_{k,t}^{i,+}$, and $b_{k,t}^{i,+}$ defined as hyperparameters of the variational distributions associated with the system mode parameters conditionally on the i th particle \mathbf{x}_k^i can be updated as follows (for $k = 1, \dots, K$ and $i = 1, \dots, N$)

$$u_{k,t}^{i,+} = \exp \left[\mathbb{E} (\ln u_{k,t}^i) + \ln g(\mathbf{y}_t | \mathbf{x}_t^i, r_t^i = k) + \ln f(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i, r_t^i = k) \right], \quad (44)$$

$$\alpha_{k,t}^{i,+} = \mathbb{E} [\mathbb{I}(r_t^i = k)] + \mathbb{E} [\alpha_{k,t}^i], \quad (45)$$

$$a_{k,t}^{i,+} = a_{k,t}^{i,-} + \left[\Psi \left(\sum_{k=1}^k \bar{\alpha}_{k,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right] \bar{\alpha}_{k,t}, \quad (46)$$

$$b_{k,t}^{i,+} = b_{k,t}^{i,-} - \mathbb{E} [\ln u_{k,t}^i]. \quad (47)$$

Thus $p(\boldsymbol{\theta}_t | \mathbf{x}_t^i, \mathbf{y}_{1:t})$ can be approximated by calculating (44)-(47) iteratively until a stopping rule has been satisfied. Finally, the proposed variational marginalized particle filter for JMNLSSs is summarized in Alg. 1.

Algorithm 1 Proposed variational marginalized particle filter for JMNLSSs.

Inputs: $\left\{x_{t-1}^i, \left\{a_{k,t-1}^i, b_{k,t-1}^i\right\}_{k=1}^K\right\}_{i=1}^N$

Outputs: $\left\{x_t^i, r_t^i, u_t^i, \alpha_{k,t}^i, \left\{a_{k,t}^i, b_{k,t}^i\right\}_{k=1}^K\right\}_{i=1}^N$

- 1: **for** $i = 1, \dots, N$ **do**
- 2: Compute $\left\{a_{k,t}^{i,-}, b_{k,t}^{i,-}, \alpha_{k,t}^{i,-}\right\}_{k=1}^K$ according to (21) and (20), and then generate $\left\{u_{k,t}^{i,-}\right\}_{k=1}^K$ by using (19).
- 3: Generate x_t^i by using (26), and then compute ω_t^i according to (25).
- 4: Initialize $u_{k,t}^{i,+}(0) = u_{k,t}^{i,-}, \alpha_{k,t}^{i,+}(0) = \alpha_{k,t}^{i,-}, a_{k,t}^{i,+}(0) = a_{k,t}^{i,-}$ and $b_{k,t}^{i,+}(0) = b_{k,t}^{i,-}$ where $k = 1, \dots, K$.
- 5: **for** $\kappa = 1, \dots, \kappa_{\max}$ **do**
- 6: Compute $\left\{u_{k,t}^{i,+}(\kappa)\right\}_{k=1}^K$ according to (44).
- 7: Compute $\left\{\alpha_{k,t}^{i,+}(\kappa)\right\}_{k=1}^K$ according to (45).
- 8: Compute $\left\{a_{k,t}^{i,+}(r), b_{k,t}^{i,+}(\kappa)\right\}_{k=1}^K$ according to (46) and (47).
- 9: **if** the parameters change by less than 0.1 **then**
- 10: stop the iteration;
- 11: **else**
- 12: set $\kappa = \kappa + 1$.
- 13: **end if**
- 14: **end for**
- 15: **end for**
- 16: Normalize $\tilde{\omega}_t^i = \omega_t^i / \sum_{i=1}^N (\omega_t^i)$ and perform particle resampling.
- 17: Recursion: $t = t + 1$.

4. Experimental results

4.1. Illustrative Example

In order to evaluate the proposed variational marginalized particle filter, we consider as a benchmark model a JMNLSS having the same state equation as in [34, 35] with different observation equations corresponding to different system modes

$$x_t = 0.5x_{t-1} + 25 \frac{x_{t-1}}{1 + x_{t-1}^2} + 8 \cos(1.2t) + \omega_{r,t} \quad (48)$$

$$y_t = \begin{cases} x_t + v_{r,t} & v_{r,t} \sim \mathcal{U}([-10, 10]), & \text{if } r_t = 1 \\ \frac{x_t^2}{20} + v_{r,t} & v_{r,t} \sim \mathcal{N}(0, 1), & \text{if } r_t = 2 \\ \frac{(x_t - 10)^2}{20} + v_{r,t} & v_{r,t} \sim \mathcal{N}(3, 5), & \text{if } r_t = 3 \end{cases} \quad (49)$$

with

$$\omega_{r,t} \sim \begin{cases} \mathcal{N}(0, 1), & \text{if } r_t = 1 \\ \mathcal{N}(0, 10), & \text{if } r_t = 2 \\ \mathcal{N}(0, 5), & \text{if } r_t = 3 \end{cases}$$

where $t = 1, \dots, T$, the simulation time T is set to 200 s, $\mathcal{U}([a, b])$ and $\mathcal{N}(\mu, \sigma^2)$ are the uniform distribution on the interval $[a, b]$ and the Gaussian distribution with mean μ and variance σ^2 , respectively. We compare the proposed approach with the RBPF-based online EM approach studied in [24] and the RBPF [7]. The number of particles N was set to 100 for the three approaches. The forgetting factor used in our experiments was set to $\rho = 0.1$. The impact of ρ on the algorithm will be studied in Section 4.2 devoted to a target tracking problem. In order to evaluate the different methods, the maximum a posteriori (MAP) estimates of r_t and the average root mean square errors (ARMSEs) of x_t are defined as

$$\hat{r}_t = \underset{r_t = \{1, \dots, K\}}{\operatorname{argmax}} \frac{1}{N_m} \sum_{m=1}^{N_m} \hat{u}_{k,t}(m) \quad (50)$$

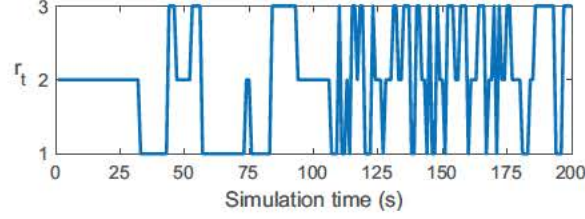


Figure 1: Typical time evolution of the indicator r_t for one realization of the JMNLs governed by the Markov process.

and

$$\text{ARMSE} = \sqrt{\frac{1}{TN_m} \sum_{t=1}^T \sum_{m=1}^{N_m} (\widehat{x}_t(m) - x_t)^2}, \quad (51)$$

where $\widehat{u}_{k,d}(m)$ and $\widehat{x}_t(m)$ are the m th estimates ($m = 1, \dots, N_m$) of the mode probability and state at time instant t . All algorithms were coded using MATLAB and run on a laptop with Intel i-5 and 8 GB RAM.

4.1.1. Case A

The JMNLs state and measurement equations are governed by a three-state mode sequence $r_t \in \{1, 2, 3\}$, which is a first-order three-state homogeneous Markov process with two 3×3 TPMs denoted as Π_1 and Π_2 associated with the time intervals $T_1 = [0, 100]$ and $T_2 = [101, 200]$ and defined as

$$\Pi_1 = \begin{pmatrix} 0.90 & 0.05 & 0.05 \\ 0.05 & 0.90 & 0.05 \\ 0.05 & 0.05 & 0.90 \end{pmatrix}, \Pi_2 = \begin{pmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{pmatrix}.$$

Fig. 1 displays typical changes of the indicator variable r_t over time for one realization of the JMNLs governed by the Markov process. The different methods have been evaluated using $N_m = 100$ MC simulations of the simulated JMNLs (one realization of r_t being displayed in Fig. 1). The TPM used in the RBPF is set to $\Pi_{\text{RBPF}} = \Pi_1$ (we assume here that the RBPF does not know that the TPM of the Markov chain governing r_t has changed at time $t = 101$).

Fig. 2 compares the MAP estimates of r_t for the RBPF [7], the RBPF-based online EM [24] and the proposed approach with the ground truth. The errors between the ground truth and the state estimates are indicated by the vertical red bars. The less red bars, the better the estimator. In addition, the corresponding error rates ε ¹ for the MAP estimates of r_t obtained with the three approaches for Case A are reported in Table 1. The error rate of the RBPF in the time interval T_2 is larger than that in the time interval T_1 . This is due to the fact that an inaccurate TPM provides inaccurate estimates of the mode probabilities obtained with the RBPF. The error rate of the RBPF-based online EM approach for the MAP estimates of r_t in the time interval T_1 is larger than that in the time interval T_2 . Since the TPM is estimated based on the ML principle in the RBPF-based online EM approach, less frequent switches between different modes in the time interval T_1 result in slow convergence of the TP estimates to the ground truth when implementing the ML estimator of the TPM. This slow convergence leads to more inaccurate estimates of the mode probabilities. Conversely, more frequent switches in the time interval T_2 facilitate the fast convergence of the TP estimates to the ground truth for the RBPF-based online EM approach, improving the estimation accuracy for the system mode parameters. Finally, the proposed approach provides the best results since the prior statistical properties of r_t is taken into account without relying on the Markovian assumption for r_t , leading to more robustness in the estimated TP.

4.1.2. Case B

The JMNLs state and measurement equations are governed by a three-state mode sequence $r_t \in \{1, 2, 3\}$, which has a categorical distribution, i.e., $r_t \sim \text{Cat}(r_t | \mathbf{u}_t)$ where $\mathbf{u}_t = (0.2, 0.5, 0.3)^T$. Accordingly, typical changes of the

¹ $\varepsilon = \frac{N_F}{N_T} \times 100\%$, where N_F and N_T denote the number of false MAP estimates and the total number of samples for r_t in the time interval T , respectively.

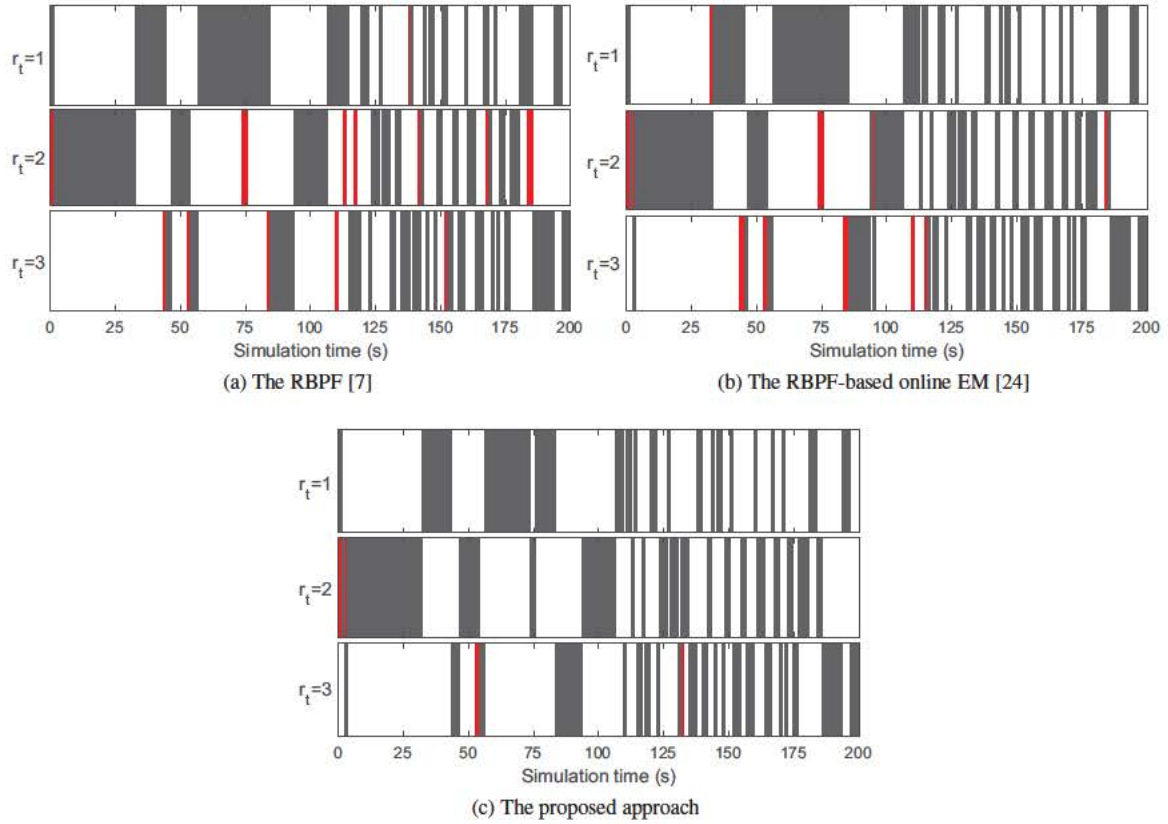


Figure 2: The MAP estimates of $r_t \in \{1, 2, 3\}$ (100 MC runs) when the JMNLs is governed by the Markov process. White part corresponds to a zero probability. Black and red parts correspond to probability one, where black denotes the estimates, and red denotes the ground truth when it does not coincide with the estimates (the less red, the better the estimate).

Table 1: The error rate ε for the MAP estimates of r_t obtained with different approaches for Case A.

Approaches	The error rate ε for Case A	
	$T_1 = [0, 100]$	$T_2 = [101, 200]$
RBPF	6%	9%
RBPF-based online EM	12%	3%
Proposed approach	4%	1%

indicator variable r_t over time for one realization of the JMNLs governed by the categorical distribution are depicted in Fig. 3 and $N_m = 100$ MC simulations on the realization of the simulated JMNLs (displayed in Fig. 3) have run for evaluating the different methods. The TPM used in the RBPF is set to $\Pi_{\text{RBPF}} = \Pi_1$ which is a typical value for high frequency mode switches [5]. Fig. 4 and Table 2 report the MAP estimates of r_t and the corresponding error rates for the RBPF, the RBPF-based online EM and the proposed approach, respectively. It is clear that the error rate associated with the proposed approach for the MAP estimates of r_t is lower than those obtained with the RBPF and the RBPF-based online EM, demonstrating that the proposed approach has a better applicability for the JMNLs with unknown TPs.

The ARMSEs of the state estimates obtained with the three approaches for Cases A and B are reported in Table 3. According to the results in Case A, the RBPF provides the best state estimation accuracy when the TP is perfectly known, i.e., on $T_1 = [0, 100]$. Conversely, the state estimation accuracy of the RBPF-based online EM and the proposed approach is worse than that obtained with the RBPF. Since the system mode parameters are unknown for

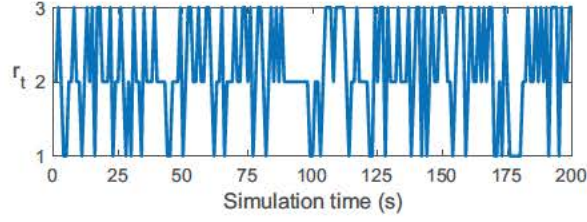


Figure 3: Typical time evolution of the indicator r_t for one realization of the JMNLs governed by the categorical distribution.

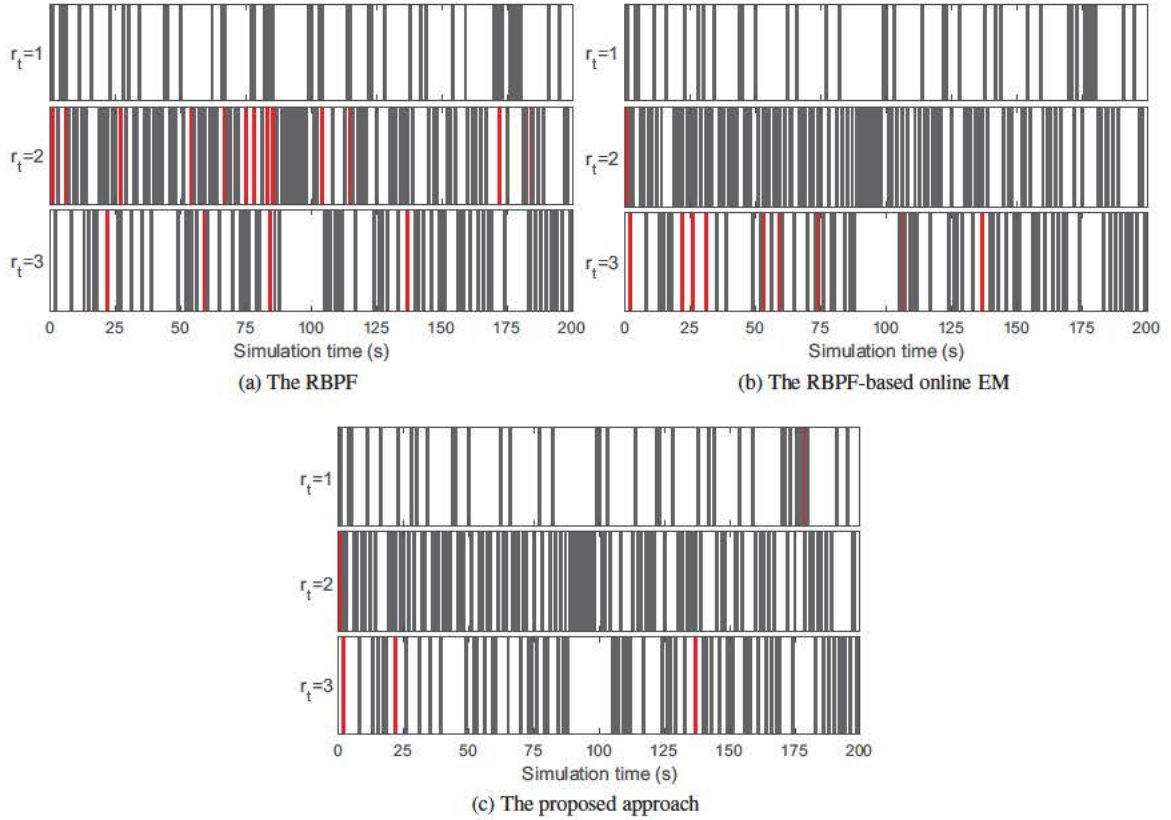


Figure 4: The MAP estimates of $r_t \in \{1, 2, 3\}$ (100 MC runs) when the JMNLs is governed by the categorical distribution. White part corresponds to a zero probability. Black and red parts correspond to probability one, where black denotes the estimates, and red denotes the ground truth when it does not coincide with the estimates (the less red, the better the estimate).

Table 2: The error rate ε for the MAP estimates of r_t obtained with different approaches for Case B.

Approaches	The error rate ε for Case B $T = [0, 200]$
RBPF	17%
RBPF-based online EM	11%
Proposed approach	5%

these approach at the beginning of T_1 , the system mode parameter estimates are inaccurate estimates, degrading the state estimation accuracies of the RBPF and RBPF-based online EM. In the time interval $T_2 = [101, 200]$ for Case A, the inaccurate knowledge about the TPM severely impairs the state estimation accuracy for the RBPF. Since

Table 3: ARMSEs of state estimates obtained with different approaches.

Approaches	Case A		Case B
	$T_1 = [0, 100]$	$T_2 = [101, 200]$	$T = [0, 200]$
RBPF	4.71	5.94	6.07
RBPF-based online EM	5.74	5.78	6.33
Proposed approach	5.46	5.49	5.76

Table 4: Execution times for different numbers of particles in Case A.

N	Execution times for 100 MC runs (s)		
	Proposed method with the stopping rule	Proposed method with $r_{\max} = 5$	RBPF-based online EM
100	340.27	565.58	248.29
200	687.31	996.58	473.95
400	1287.90	1858.55	950.10

the statistical properties of r_t can be estimated both using the RBPF-based online EM and the proposed variational approach, the state estimation accuracy of these two approaches is hardly impacted by the uncertainty regarding the TPM. According to the results in Case B, the state estimation accuracies of the RBPF and RBPF-based online EM are degraded when the system mode r_t does not satisfy the Markovian assumption. On the contrary, the proposed approach provides a better state estimation accuracy, demonstrating a good robustness to a bad knowledge about the TP of the JMNLs.

When the number of variational Bayesian inference iterations is maximum, the computational complexities of the RBPF-based online EM and the proposed approach are $\mathcal{O}(NT)$ and $\mathcal{O}(\kappa_{\max}NT)$, respectively. Table 4 shows the execution times for 100 MC runs by using different numbers of particles for the RBPF-based online EM and the proposed approach in Case A (similar results should be obtained in Case B). In addition, the proposed approach was evaluated with the stopping rule defined as Line 10 of Alg. 1. and the fixed iterations $\kappa_{\max} = 5$ ², respectively. The computations of the variational Bayesian iterations, that are embedded in the RBPF for updating the posterior pdf of r_t , lead to a higher computational cost. However, it is interesting to note that this cost can be efficiently reduced by introducing the stopping rule of Line 10 of Alg. 1.

4.2. Application to Target Tracking

This section shows the interest of the proposed variational marginalized particle filter for a generic air traffic control problem [36, 37]. In order to track an aircraft moving within the $x - y$ plane, the motion of the aircraft is modeled by using a coordinated-turn of the form [21]

$$\mathbf{x}_t = \begin{pmatrix} 1 & \frac{\sin \Omega_t \Delta T}{\Omega_t} & 0 & -\frac{1 - \cos \Omega_t \Delta T}{\Omega_t} \\ 0 & \cos \Omega_t \Delta T & 0 & -\sin \Omega_t \Delta T \\ 0 & \frac{1 - \cos \Omega_t \Delta T}{\Omega_t} & 1 & \frac{\sin \Omega_t \Delta T}{\Omega_t} \\ 0 & \sin \Omega_t \Delta T & 0 & \cos \Omega_t \Delta T \end{pmatrix} \mathbf{x}_{t-1} + \begin{pmatrix} \frac{\Delta T^2}{2} & 0 \\ \Delta T & 0 \\ 0 & \frac{\Delta T^2}{2} \\ 0 & \Delta T \end{pmatrix} \mathbf{v}_{t-1} \quad (52)$$

where $\mathbf{x}_t = (x_t, \dot{x}_t, y_t, \dot{y}_t)^T$ is a vector containing the position and velocity of the target in the $x - y$ plane, Ω_t is the turn rate (this model simplifies to the nearly constant velocity model when $\Omega_t = 0$), ΔT denotes the sampling period, the process noise \mathbf{v}_t is modeled as a zero-mean Gaussian white noise with covariance $\mathbf{Q}_t = \eta_t^2 \mathbf{I}_{2 \times 2}$, where η_t denotes the standard deviation and $\mathbf{I}_{2 \times 2}$ is the 2×2 identity matrix. Noisy nonlinear observations of the target in the form of

²According to the simulation results, the parameters in the variational Bayesian iterations change by less than 0.1 for $\kappa_{\max} \leq 5$.

Table 5: The coordinated-turn model parameters for the different system modes.

Parameters	System mode r_t				
	#1	#2	#3	#4	#5
Ω ($^\circ/\text{s}$)	-6.0	-3.0	0.0	3.0	6.0
η (m/s^2)	1.0	0.5	0.1	0.5	1.0

range, Doppler speed and bearing measurements are acquired by a fixed observer positioned at $\mathbf{s}_t = (s_x, s_y)^\text{T}$, i.e.,

$$\mathbf{y}_t = \begin{pmatrix} \frac{\sqrt{(x_t - s_x)^2 + (y_t - s_y)^2}}{(x_t \dot{x}_t + y_t \dot{y}_t)} \\ \arctan\left(\frac{y_t - s_y}{x_t - s_x}\right) \end{pmatrix} + \boldsymbol{\epsilon}_t \quad (53)$$

where the observation noise $\boldsymbol{\epsilon}_t$ is modeled as a zero-mean Gaussian white noise with covariance matrix \mathbf{R}_t . A target trajectory was simulated according to the target maneuver models in (52) with 200 time steps with sampling period $\Delta T = 1$ s, where the turn rate Ω_t takes its values in the set $\{0, 3, 0, -6, 0, 6, 0, -3, 0\}$ $^\circ/\text{s}$ with changes occurring at times $\{30, 60, 75, 90, 105, 135, 150, 180\}$, respectively. The corresponding standard deviation η_t of the process noise belongs to the set $\{0.1, 0.5, 0.1, 1.0, 0.1, 1.0, 0.1, 0.5, 0.1\}$ m/s^2 and the covariance of the measurement noise was set to $\mathbf{R}_t = \text{diag}[(25 \text{ m})^2, (2 \text{ m}/\text{s})^2, (0.2^\circ)^2]$, where $\text{diag}[\lambda_1, \dots, \lambda_N]$ denotes a diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_N$. The initial state of the target was chosen as $\mathbf{x}_0 = (30 \text{ km}, 120 \text{ m}/\text{s}, 30 \text{ km}, 0 \text{ m}/\text{s}, 0^\circ/\text{s})^\text{T}$ and the observer position was fixed to $\mathbf{s}_t = (0 \text{ km}, 0 \text{ km})^\text{T}$. We considered five system modes (i.e., $K = 5$ and $r_t \in \{1, \dots, 5\}$) defining the target motion in the $x-y$ plane. The corresponding parameters in the coordinated-turn model are reported in Table 5. Note that $\Omega = \pm 3^\circ/\text{s}$ and $\pm 6^\circ/\text{s}$ correspond to standard and emergency situation turn rates for an aircraft [38], and the process noise with standard deviation $0.1 \text{ m}/\text{s}^2$ is used to model the nearly constant velocity model, whereas the maneuver is defined by higher standard deviations [36].

In order to evaluate the proposed algorithm, we compared its accuracy for position estimation with that obtained using the RBPF-based online EM approach and the RBPF, where the TPM used in the RBPF is defined as follows

$$\mathbf{\Pi} = \begin{pmatrix} 0.96 & 0.01 & 0.01 & 0.01 & 0.01 \\ 0.01 & 0.96 & 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.96 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0.96 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.96 \end{pmatrix}.$$

The number of particle N for the three approaches was set to 2000 and $N_m = 100$ Monte Carlo simulations were considered for each filter. The forgetting factor ρ used in the proposed approach was adjusted as in the illustrative example.

The first results show the means of $\widehat{\boldsymbol{\alpha}}_t = (\widehat{\alpha}_{1,t}, \dots, \widehat{\alpha}_{5,t})^\text{T}$ computed using 100 Monte Carlo simulations, which are depicted in Fig. 5. The estimators $\widehat{\alpha}_{k,t}$ ($k = 1, \dots, 5$) gradually increase when the system stays in the same mode and tend to a small value otherwise, leading to an accurate posterior distributions for the mode probability \mathbf{u}_t . In order to evaluate the effect of different forgetting factors on the mode probability estimate, the means of $\widehat{\mathbf{u}}_t = (\widehat{u}_{1,t}, \dots, \widehat{u}_{5,t})^\text{T}$ computed using 100 Monte Carlo simulations with different forgetting factors are compared in Fig. 6. On the one hand, smooth mode probability estimates can be obtained for a large forgetting factor. On the other hand, a small forgetting factor allows more abrupt changes in the mode probabilities to be tracked. The RMSEs of the position estimates obtained using the different approaches are shown in Fig. 7 for the x - and y -axis. The position estimation accuracy obtained with the proposed approach is better than that obtained with the other two approaches. This improvement results from a better estimation of the system mode parameters yielding better state estimation.

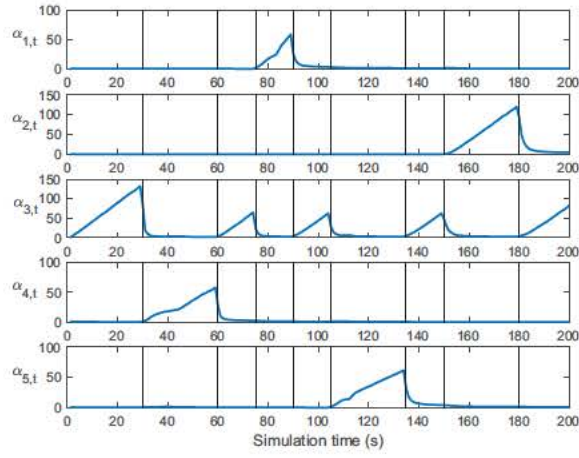


Figure 5: Mean of the estimates $\hat{\alpha}_t$ (100 Monte Carlo runs).

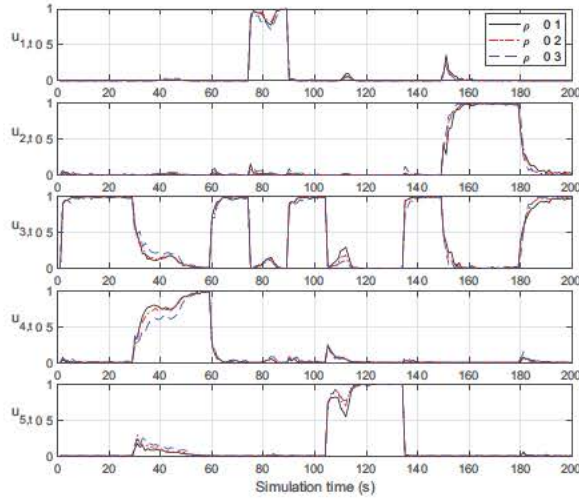


Figure 6: Mean of the estimates \hat{u}_t (100 Monte Carlo runs) with different forgetting factors.

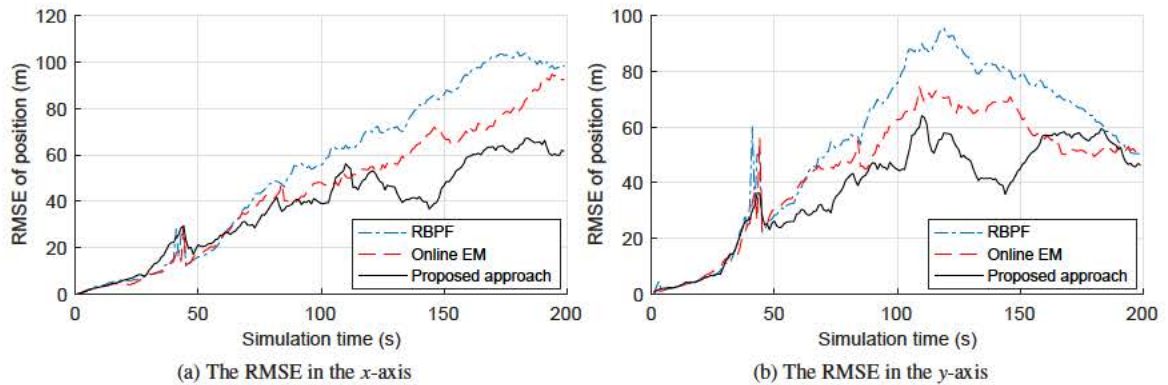


Figure 7: The RMSE of position estimates in the $x - y$ plane.

5. Conclusion

This paper considered conjugate prior distributions for the system mode parameters of a jump Markov nonlinear system instead of using the Markovian assumption classically used in this context. A variational marginalized particle filter was then investigated for obtaining the joint posterior distribution of the state and unknown mode parameters. A simulation study was conducted for a modified nonlinear benchmark model and for more realistic manoeuvring target tracking scenarios. The proposed approach was compared to the state-of-the-art, namely the RBPF-based online EM algorithm and the RBPF algorithm, providing accurate estimates of the system mode parameters at the price of a higher computational cost. In particular, the proposed algorithm was shown to be robust to uncertainties about the mode transition probabilities, improving state estimation accuracy when these transition probabilities are imperfectly known.

We think that the proposed approach could be interesting for jump semi-Markov systems, especially when the transition probabilities of the embedded Markov chain are time-varying leading to discrete-time non-homogeneous jump semi-Markov linear systems (JS-MLSs). For instance, the variational Bayesian marginalized particle filter could be investigated for estimating the time-varying transition probability matrix of non-homogeneous JS-MLSs, in the case where the distributions of sojourn times for different system modes are independent of the jump instants, as in [39]. Since time-varying transition probabilities are associated with a convex polytope, the application of the proposed approach to a nonhomogeneous Markov chain with a polytope-structured TPM [40] would also be interesting to investigate.

In order to obtain a closed-form solution for the unknown system mode parameters of a jump Markov nonlinear system, the extended factorized approximation was used for maximizing the cost function of interest. Recent work on stochastic variational inference might solve this problem by applying stochastic optimization to this cost function [41]. This work is currently under investigation.

Appendix A. The lower-bound $\tilde{\mathcal{L}}$ for the variational objective function \mathcal{L}

The log-inverse-beta function being convex over its range, the multivariate log-inverse-beta function is a convex function of $\ln \alpha_{k,t}$ when $\sum_{m=1, m \neq k}^K \alpha_{m,t} > 1$ [42]. Thus the multivariate log-inverse-beta function $\text{MLIB}(\alpha_t)$ satisfies the following inequality according to its first-order Taylor expansion for $(\ln \alpha_{1,t}, \dots, \ln \alpha_{K,t})^\top$ around $(\ln \bar{\alpha}_{1,t}, \dots, \ln \bar{\alpha}_{K,t})^\top$

$$\text{MLIB}(\alpha_t) \geq \text{MLIB}(\bar{\alpha}_t) + \sum_{k=1}^K \left\{ \Psi \left(\sum_{m=1}^k \bar{\alpha}_{m,t} + \sum_{l=k+1}^K \alpha_{l,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\ln \alpha_{k,t} - \ln \bar{\alpha}_{k,t}), \quad (\text{A.1})$$

where $\bar{\alpha}_{k,t} = \mathbb{E}[\alpha_{k,t}]$ denotes the expected value of $\alpha_{k,t}$, $k = 1, \dots, K$ and $\Psi(\cdot)$ is the digamma function. Replacing (A.1) in (28), $\mathbb{E}_{\theta_t} [\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})]$ appearing in the variational objective function \mathcal{L} satisfies the following inequality

$$\begin{aligned} & \mathbb{E}_{\theta_t} [\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] \geq \\ & \text{MLIB}(\bar{\alpha}_t) + \sum_{k=1}^K \mathbb{E}_{\alpha_t} \left[\left\{ \Psi \left(\sum_{m=1}^k \bar{\alpha}_{m,t} + \sum_{l=k+1}^K \alpha_{l,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\ln \alpha_{k,t} - \ln \bar{\alpha}_{k,t}) \right] + \mathcal{R} \end{aligned} \quad (\text{A.2})$$

with

$$\mathcal{R} = \mathbb{E}_{\theta_t} \left[\sum_{k=1}^K \ln \left\{ u_{k,t}^{\alpha_{k,t}-1} \gamma(\alpha_{k,t} | a_{k,t}, b_{k,t}) [u_{k,t} g_k(\mathbf{y}_t | \mathbf{x}_t^i) f_k(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i)]^{\mathbb{I}(r_i=k)} \right\} \right], \quad (\text{A.3})$$

where \mathcal{R} denotes the remaining unchanged parts in $\mathbb{E}_{\theta_t} [\ln p(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})]$. Considering that $\{\alpha_{1,t}, \dots, \alpha_{K,t}\}$ are parameters of the digamma function, the expectation operation in the right hand side (RHS) of (A.2) is not tractable. However, $\Psi(\cdot)$ and $\ln(\cdot)$ are concave functions leading to $\Psi(\mathbb{E}[x]) \geq \mathbb{E}_x[\Psi(x)]$ and $\ln(\mathbb{E}[x]) \geq \mathbb{E}_x[\ln(x)]$ by using

Jensen's inequality. Thus, the expectation operation in the RHS of (A.2) can be written as follows [42]

$$\begin{aligned}
& \mathbb{E}_{\alpha_t} \left[\left\{ \Psi \left(\sum_{m=1}^k \bar{\alpha}_{m,t} + \sum_{l=k+1}^K \alpha_{l,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\ln \alpha_{k,t} - \ln \bar{\alpha}_{k,t}) \right] \\
&= \left\{ \mathbb{E}_{\alpha_t} \left[\Psi \left(\sum_{m=1}^k \bar{\alpha}_{m,t} + \sum_{l=k+1}^K \alpha_{l,t} \right) \right] - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\mathbb{E}_{\alpha_{k,t}} [\ln \alpha_{k,t}] - \ln \bar{\alpha}_{k,t}) \\
&\geq \left\{ \Psi \left(\sum_{m=1}^k \bar{\alpha}_{m,t} + \mathbb{E}_{\alpha_t} \left[\sum_{l=k+1}^K \alpha_{l,t} \right] \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\mathbb{E}_{\alpha_{k,t}} [\ln \alpha_{k,t}] - \ln \bar{\alpha}_{k,t}) \\
&= \left\{ \Psi \left(\sum_{k=1}^k \bar{\alpha}_{k,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\mathbb{E}_{\alpha_{k,t}} [\ln \alpha_{k,t}] - \ln \bar{\alpha}_{k,t})
\end{aligned} \tag{A.4}$$

where $\mathbb{E}_{\alpha_t} \left[\Psi \left(\sum_{m=1}^k \bar{\alpha}_{m,t} + \sum_{l=k+1}^K \alpha_{l,t} \right) \right] \leq \Psi \left(\sum_{m=1}^k \bar{\alpha}_{m,t} + \mathbb{E}_{\alpha_t} \left[\sum_{l=k+1}^K \alpha_{l,t} \right] \right)$ and $\mathbb{E}_{\alpha_{k,t}} [\ln \alpha_{k,t}] - \ln \bar{\alpha}_{k,t} \leq 0$. Using (A.4) in (A.2), the lower-bound $\tilde{\mathcal{L}}$ for the variational objective function \mathcal{L} can be defined as follows

$$\tilde{\mathcal{L}} = \mathbb{E}_{\theta_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] - \mathbb{E}_{\theta_t} [\ln q(\boldsymbol{\theta}_t)], \tag{A.5}$$

where

$$\mathbb{E}_{\theta_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] = \text{MLIB}(\bar{\alpha}_t) + \sum_{k=1}^K \left\{ \Psi \left(\sum_{k=1}^k \bar{\alpha}_{k,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\mathbb{E}_{\alpha_{k,t}} [\ln \alpha_{k,t}] - \ln \bar{\alpha}_{k,t}) + \mathcal{R}. \tag{A.6}$$

Appendix B. The expression of $\mathbb{E}_{\theta_t \setminus \mathcal{X}} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})]$

According to (A.5), the expectations of $\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})$ with respect to $\boldsymbol{\theta}_t$ except \mathcal{X} (where \mathcal{X} is equal to r_t, \mathbf{u}_t , and α_t , respectively) can be defined as follows

$$\mathbb{E}_{\theta_t \setminus r_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] = \sum_{k=1}^K \left\{ \mathbb{E} [\ln u_{k,t}] + \ln g(\mathbf{y}_t | \mathbf{x}_t^i, r_t^i) + \ln f(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i, r_t^i) \right\} \mathbb{I}(r_t = k) + C_1 \tag{B.1}$$

with

$$\begin{aligned}
C_1 &= \text{MLIB}(\bar{\alpha}_t) + \sum_{k=1}^K \left\{ \Psi \left(\sum_{k=1}^k \bar{\alpha}_{k,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\mathbb{E}_{\alpha_{k,t}} [\ln \alpha_{k,t}] - \ln \bar{\alpha}_{k,t}) \\
&\quad + \sum_{k=1}^K \mathbb{E}_{u_{k,t}, \alpha_{k,t}} \left[\ln \left(u_{k,t}^{\alpha_{k,t}-1} \gamma(\alpha_{k,t} | a_{k,t}, b_{k,t}) \right) \right],
\end{aligned} \tag{B.2}$$

where C_1 denotes terms independent of r_t .

$$\mathbb{E}_{\theta_t \setminus u_t} [\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \boldsymbol{\theta}_t | \mathbf{y}_{1:t-1})] = \sum_{k=1}^K \{ \mathbb{E} [\mathbb{I}(r_t = k)] + \mathbb{E} [\alpha_{k,t}] - 1 \} \ln u_{k,t} + C_2 \tag{B.3}$$

with

$$\begin{aligned}
C_2 &= \text{MLIB}(\bar{\alpha}_t) + \sum_{k=1}^K \left\{ \Psi \left(\sum_{k=1}^k \bar{\alpha}_{k,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right\} \bar{\alpha}_{k,t} (\mathbb{E}_{\alpha_{k,t}} [\ln \alpha_{k,t}] - \ln \bar{\alpha}_{k,t}) \\
&\quad + \sum_{k=1}^K \mathbb{E}_{r_t, \alpha_{k,t}} \left[\ln \left\{ \gamma(\alpha_{k,t} | a_{k,t}, b_{k,t}) \left[g_k(\mathbf{y}_t | \mathbf{x}_t^i) f_k(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i) \right]^{\mathbb{I}(r_t=k)} \right\} \right],
\end{aligned} \tag{B.4}$$

where C_2 denotes terms independent of \mathbf{u}_t .

$$\begin{aligned} & \mathbb{E}_{\theta_t \setminus \alpha_t} \left[\ln \tilde{p}(\mathbf{y}_t, \mathbf{x}_t^i, \theta_t | \mathbf{y}_{1:t-1}) \right] = \\ & \sum_{k=1}^K \left\{ a_{k,t} + \left[\Psi \left(\sum_{k=1}^k \bar{\alpha}_{k,t} \right) - \Psi(\bar{\alpha}_{k,t}) \right] \bar{\alpha}_{k,t} - 1 \right\} \ln \alpha_{k,t} - \sum_{k=1}^K (b_{k,t} - \mathbb{E}[\ln u_{k,t}]) \alpha_{k,t} + C_3 \end{aligned} \quad (\text{B.5})$$

with

$$C_3 = \sum_{k=1}^K \mathbb{E}_{r_t, u_{k,t}} \left[\ln \left\{ \left[u_{k,t} g_k(\mathbf{y}_t | \mathbf{x}_t^i) f_k(\mathbf{x}_t^i | \mathbf{x}_{t-1}^i) \right]^{\mathbb{I}(r_t=k)} \right\} \right], \quad (\text{B.6})$$

where C_3 denotes terms independent of α_t .

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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