Sequential Decision-Making Under Uncertainty Using Hybrid Probability-Possibility Functions

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Didier Dubois[1], Hélène Fargier[1], Romain Guillaume[1(✉)], and Agnès Rico[2]

[1] IRIT - CNRS, Université de Toulouse, Toulouse, France
{dubois,fargier,romain.guillaume}@irit.fr
[2] Eric, Université de Lyon, Lyon, France
agnes.rico@univ-lyon1.fr

Abstract. Probabilistic and possibilistic models of sequential decision problems are known to possess good behavioral and algorithmic properties. In this paper, the range of models of problems of sequential decision under uncertainty that are dynamically consistent, consequentialist and allow for tree reduction is enlarged by considering a representation of uncertainty that is both probabilistic and possibilistic. The corresponding utility functional is expected utility for highly likely states, and an optimistic or pessimistic possibility-based criterion for unlikely states.

Keywords: Decision tree · Possibility theory · Fuzzy measures · Belief functions

1 Introduction

In sequential decision making a strategy is a conditional plan that assigns a (possibly non deterministic) action to each state where a decision has to be made (also called "decision node"). Each strategy leads to a compound lottery, following Von Neuman and Morgenstern's terminology [12] - roughly, it is a tree representing the different possible scenarios, and thus the different possible final states that the plan/strategy may reach. The optimal strategy is then the one that minimizes a criterion whose value depends on utilities of final states and the resulting compound lottery.

Three assumptions are desirable in order to accept an optimal strategy without questioning its meaning. Those assumptions are:

- *Dynamic Consistency*: when reaching a decision node, following an optimal strategy, the best decision at this node is the one that had been considered so when computing this strategy, i.e. prior to applying it.
- *Consequentialism*: the best decision at each step of the problem only depends on potential consequences at this point.
- *Tree Reduction*: a compound lottery is equivalent to a simple one, assigning probabilities to final states.

https://doi.org/10.1007/978-3-030-85529-1_5

Those three assumptions are instrumental to enable an optimal strategy to be computed using dynamic programming [10].

When the decision maker is able to provide a probability distribution on the possible states and considers that the utilities are additive, a classical approach is the one based on expected utility. Under this strong assumption, the above three assumptions are satisfied.

When the problem is pervaded with possibilistic uncertainty, two families of criteria make sense: a Sugeno integral-based criterion [5] and the Choquet integral [6]. The criterion based on Choquet integral turns out to be incompatible with the above assumptions: it may happen that none of the optimal strategies is dynamically consistent nor consequentialist The possibilistic criteria based on Sugeno integral do not meet such difficulties. Nor does the generalization of both optimistic and pessimistic possibilistic Sugeno proposed by [7]. It aggregates optimistic and pessimistic criteria by means of a uninorm [13], a semi-group operation whose identity plays the role of a degree of optimism. Contrary to other criteria that account for a degree of optimism, like the Hurwicz criterion, the criterion proposed in [7] satisfies the three assumptions governing a good behavior of the decision tree.

In the present paper, we are looking for new decision criteria, beyond expected utility and possibilistic integrals, that can apply to decision trees and respect the three properties recalled above (*Dynamic Consistency, Consequentialism* and *Tree Reduction*).

2 Decision Trees

A convenient language to introduce sequential decision problems is the one of decision trees [10]. This framework proposes an explicit graphical model, representing each possible scenario by a path from the root to the leaves of a tree. Formally, a decision tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ is such that $\mathcal{N}$ contains three kinds of nodes (see Fig. 1 for an example):

- $\mathcal{D}$ is the set of decision nodes (depicted by rectangles).
- $\mathcal{LN}$ is the set of leaves, that represent final states in $\mathcal{S}$; such states can be evaluated by a utility function: $\forall s_i \in \mathcal{S}$, $u(s_i)$ is the degree of satisfaction of eventually being in state s_i (of reaching the corresponding node in $\mathcal{LN}$). For the sake of simplicity we assume, without loss of generality, that only leaf nodes are attached utilities.
- $\mathcal{X}$ is the set of chance nodes (depicted by circles).

For any node $n_i \in \mathcal{N}$, $Succ(n_i) \subseteq \mathcal{N}$ denotes the set of its children. In a decision tree, for any decision node $d_i, Succ(d_i) \subseteq \mathcal{X}$: $Succ(d_i)$ is the set of actions that can be chosen when d_i is reached. For any chance node $x_i, Succ(x_i) \subseteq \mathcal{LN} \cup \mathcal{D}$: $Succ(x_i)$ is the set of possible outcomes of action x_i - either a leaf node is observed, or a decision node is reached (and then a new action should be chosen).

Solving a decision tree amounts to building a *strategy*, i.e. a function δ that associates to each decision node d_i an action (i.e. a chance node) in $Succ(d_i)$:

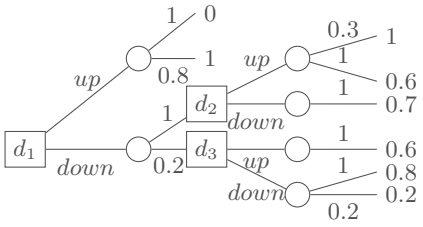

Fig. 1. A possibilistic decision tree.

$\delta(d_i)$ is the action to be executed when decision node d_i is reached. Let Δ be the set of strategies that can be built for $\mathcal{T}$. We shall also consider the subtree $\mathcal{T}_n$ of $\mathcal{T}$ rooted at node $n \in \mathcal{T}$, and denote by Δ_n its strategies: they are subtrategies of the strategies of Δ.

The satisfaction of the decision maker for a consequence (on a leaf of the tree) is captured by a utility degree on a totally ordered scale. The scale $[0, 1]$ is generally chosen for these degrees, but any ordered set can be used.

Here, we focus on the case where the information at each chance node is fully captured by a distribution over outcomes of the chance nodes, namely a possibility distribution π and/or a probability distribution p. When bearing on the leaf nodes, such distributions define simple lotteries on the utility degrees. More formally:

- A simple probabilistic lottery L^p [12]: is a probability distribution p on a set of utility degrees, $\Lambda = \{\lambda_1, ..., \lambda_n\}$. The probabilistic lotteries will be written as $L^p = <p_1/\lambda_1, ..., p_n/\lambda_n>$ with $p_i \in [0, 1], \sum_{i=1}^{n} p_i = 1$.
- A simple possibilistic lottery L^π [5] is a normalized possibility distribution π on a set of utility degrees, Λ, both being expressed in the same ordered scale. The possibilistic lotteries will be written as $L^p = <\pi_1/\lambda_1, ..., \pi_n/\lambda_n>$ with $\pi_i \in [0, 1], \max_{i=1}^{n} \pi_i = 1$.

In a lottery L^π (resp. L^p), the value π_i (resp. p_i) is the possibility (resp. probability) degree of getting utility λ_i according to the decision strategy captured by the lottery. For the sake of brevity, the λ_i's such that $\pi_i = 0$ (resp. $p_i = 0$) are often omitted in the notation of a lottery (e.g., $(<1/0.8>)$ denotes the lottery that provides utility 0.8 for sure, all the other utility degrees being impossible).

The expected utility of probabilistic simple lotteries was proposed by Von Neuman and Morgernstern [12] as a decision criterion under risk: $E(L^p) = \sum_{\lambda_i \in \Lambda} \lambda_i \cdot p_i$. Dubois and Prade [5] proposed to use optimistic and pessimistic possibilistic criteria, denoted by U^{Pes} and U^{Opt} in this paper, to evaluate the global utility of a possibilistic lottery using $U^{Pes}(L^\pi) = \min_{\lambda_i \in \Lambda} \max(1 - \pi_i, \lambda_i)$, and $U^{Opt}(L^\pi) = \max_{\lambda_i \in \Lambda} \min(\pi_i, \lambda_i)$.

Let us now consider full-fledged strategies. A strategy in Δ can be viewed as a connected subtree of $\mathcal{T}$ where there is exactly one edge (and thus one chance node) left at each decision node - skipping the decision nodes, we get a chance tree or, using von Neuwman and Morgernstern's terminology, a compound lottery:

like simple lotteries, which are distributions over utilities, compound lotteries are distributions over (simple or compound) lotteries.

The idea is to define a simple lottery equivalent to the original, compound one and to apply the decision criterion to this simple lottery.

Definition 1 (*Reduction*^{p})**.** *For any probabilistic compound lottery of the form* $L^p = <p_1/L^p_1, \ldots, p_m/L^p_m>$, $Red^p(L^p)$ *is the simple lottery that associates to each* λ_i *the probability degree* $\overline{p}_i = \sum_{j=1,\ldots,m} p_j \cdot p^j_i$, *where* p^j_i *denotes the probability of getting* λ_i *though lottery* L^p_j *and* p_j *the probability of getting* L^p_j.

Definition 2 (*Reduction*$^\pi$)**.** *For any possibilistic compound lottery of the form* $L^\pi = <\pi_1/L^\pi_1, \ldots, \pi_m/L^\pi_m>$, $Red^\pi(L^\pi)$ *is the simple lottery that associates to each* λ_i *the possibility degree* $\overline{\pi}_i = \max_{j=1,\ldots,m} \min(\pi_j, \pi^j_i)$, *where* π^j_i *denotes the possibility of getting* λ_i *though lottery* L^π_j *and* π_j *the possibility of getting* L^π_j.

The principle of lottery reduction allows the comparison of compound lotteries: L is preferred to L' iff its reduction is preferred to the one of L' and optimality can then be soundly defined:

– $\delta \in \Delta$ is optimal for a decision tree $\mathcal{T}$ iff $\forall \delta' \in \Delta, C(Red(L_\delta)) \succeq C(Red(L_{\delta'}))$,

for a criterion C. The principle of monotonicity and the one of decomposition [12] are valid for expected utility, and also for U^{Pes} and U^{Opt} in a weak form:

Definition 3 (Weak monotonicity). *A preference criterion* C *over possibilistic/probabilistic lotteries is said to be weakly monotonic iff whatever* L, L' *and* L'', *whatever a normalized possibility/probability distribution* w, v:

$$C(L) \leq C(L') \Rightarrow C(<w/L, v/L''>) \leq C(<w/L', v/L''>). \quad (1)$$

Importanty, all approaches that satisfy weak monotonicity, and in particular in the approach considered in this paper, also satisfy *Dynamic Consistency*, *Consequentialism* and *Tree Reduction*. This guarantees coherence with the intuition of rationality; this is also important from the algorithmic point of view, since it allows to find an optimal strategy by dynamic programming.

In the following we consider decision trees where uncertainty is captured by set functions that are more general than probability and possibility measures, while taking into account a degree of optimism of the decision maker, although without giving up the monotonicity principle.

3 Hybrid Possibility-Probability Measures

The three previous criteria (expected utility, pessimistic and optimistic possibilistic utility functionals) are particular instances of generalized integrals (Choquet integral for expected utility, Sugeno integral for the other ones) based on fuzzy measures.

Definition 4. *A fuzzy measure is a set function* $\mu : 2^S \to [0,1]$*, satisfying the following axioms:*

- $\mu(\emptyset) = 0$*;* $\mu(S) = 1$ *(limit conditions)*
- $\mu(A \cup B) \geq \mu(A)$ *(monotony)*

Fuzzy measures include, among others, probability measures, necessity measures and possibility measures.[1]

However, we look for special fuzzy measures representable by means of lotteries (or, equivalently, by distributions on utility values). Hence they must be *decomposable*:

Definition 5 [4]**.** *A decomposable fuzzy measure is a fuzzy measure* μ *for which there is a t-conorm*[2] S *such that* $\mu(A \cup B) = S(\mu(A), \mu(B))$ *whenever* A *and* B *are disjoint.*

Possibility measures are max-decomposable (for the t-conorm max), while probability measures are additively decomposable using the Łukasiewicz t-conorm $\min(a+b, 1)$. To define lottery reduction, we also need a generalization of the notion of independence between events.

Definition 6. *Let* T *be a triangular norm. Two events* A *and* B *are said to be* T*-separable with respect to a fuzzy measure* μ *if and only if* $\mu(A \cap B) = T(\mu(A), \mu(B))$ *for a t-norm* T*.*

If A and B are disjoint events T-separable from another event C, then, since $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, a decomposable fuzzy measure should satisfy

$$T(S(\mu(A), \mu(B)), \mu(C)) = S(T(\mu(A), \mu(C)), T(\mu(B), \mu(C)),$$

which requires a property called Conditional Distributivity of T over S. This property is essential for enabling the reduction of generalised lotteries.

Definition 7. *A t-norm* T *is conditionally distributive over a t-conorm* S *if for all* $x, y, z \in [0,1]$ *we have*

$$(CD)\, T(x, S(y,z)) = S(T(x,y), T(x,z)), \ \textit{whenever } S(y,z) < 1.$$

In [3], it has been proved that the only family of pairs (t-conorm, t-norm) that satisfy the condition (CD) are built from the pairs $(\max, T)$ and $(\min(a+b,1), \times)$ [3,9], where T is an arbitrary continuous t-norm. It is a parametric family of pairs denoted by (S^α, T^α), with parameter α, where S^α (resp. T^α) is the ordinal sum [9] of max and $\min(x+y, 1)$ (resp. T and product) represented on Fig. 2 for $T = \min$.

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[1] Given from a possibility distribution π over a set S, the possibility and the necessity of any event $A \subseteq S$ are defined by $\Pi(A) = \max_{s \in A} \pi(s), N(A) = 1 - \Pi(\bar{A}) = 1 - \max_{s \notin A} \pi(s)$.

[2] A t-conorm is a non-decreasing semi-group operation on $[0,1]$ with identity 0 and absorbing element 1. A t-norm is a non-decreasing semi-group operation on $[0,1]$ with identity 1 and absorbing element 0. T-norms and t-conorms are gradual models of conjunction and disjunction. See [9] for more details.

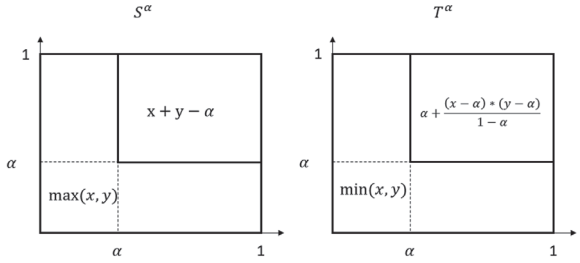

Fig. 2. S^α and T^α

The pairs (S^α, T^α) where T^α is distributive on S^α are thus of the form

$$S^\alpha(x, y) = \begin{cases} x + y - \alpha \text{ if } x > \alpha, y > \alpha \\ \max(x, y) \end{cases} \tag{2}$$

$$T^\alpha(x, y) = \begin{cases} \alpha + \frac{(x-\alpha)(y-\alpha)}{1-\alpha} \text{ if } x > \alpha, y > \alpha \\ \min(x, y) \text{ otherwise.} \end{cases} \tag{3}$$

An S^α-decomposable fuzzy measure, denoted by ρ^α, is a hybrid function between possibility and probability measures defined as follows:

Definition 1 (Hybrid π-p measure [3]). *A hybrid possibility-probability measure ρ^α is a fuzzy measure such that, for disjoint sets A and B:*

$$\rho^\alpha(A \cup B) = S^\alpha(\rho^\alpha(A), \rho^\alpha(B)) \tag{4}$$

The fuzzy measure ρ^α is clearly a probability measure (rescaled on $[\alpha, 1]$) for likely events, and a possibility measure if one of the events is unlikely. The hybrid π-p measure is an example of level-dependent capacity [8]. For events that can be considered independent A, B, we have

$$\rho^\alpha(A \cap B) = T^\alpha(\rho^\alpha(A), \rho^\alpha(B)) \tag{5}$$

The limit condition $\rho^\alpha(\Omega) = 1$ comes down to enforcing, for any event A and its complement A^c, the duality condition $\rho^\alpha(A) + \rho^\alpha(A^c) = 1 + \alpha$, when $\min(\rho^\alpha(A), \rho^\alpha(A^c)) > \alpha$ and $\max(\rho^\alpha(A), \rho^\alpha(A^c)) = 1$ otherwise.

Because it is decomposable, ρ^α is completely defined by a distribution of weights ρ^α on the singletons of the referential S. Applying (4) to the union of singletons yields the normalization condition suggested in [3]:

Proposition 1. *A distribution ρ^α on S defines a normalised S^α-decomposable fuzzy measure if and only if*

$$\sum_{s:\rho^\alpha(s)>\alpha} \rho^\alpha(s) = 1 + (card(C_\alpha^+) - 1) \times \alpha \tag{6}$$

where $C_\alpha^+ = \{s, \rho^\alpha(s) > \alpha\}$.

Proof (not given in [3]). Let $C_\alpha^+ = \{s, \rho^\alpha(s) > \alpha\}$. Then, by associativity and commutativity we have $\rho^\alpha(S) = S_s^\alpha(\rho^\alpha(s)) = \max(\max_{s\notin C_\alpha^+} \rho^\alpha(s), S^\alpha_{s\in C_\alpha^+}(\rho^\alpha(s))$, and the latter term is $S^\alpha_{s\in C_\alpha^+}(\rho^\alpha(s)) = \sum_{s\in C_\alpha^+} \rho^\alpha(s) - (card(C_\alpha^+) - 1)\alpha > \alpha \geq \max_{s\notin C_\alpha^+} \rho^\alpha(s)$. Clearly $S_s^\alpha(\rho^\alpha(s)) = S^\alpha_{s\in C_\alpha^+}(\rho^\alpha(s)) = 1$ yields condition (6).

Note that $C_\alpha^+ \neq \emptyset$. For otherwise, $\rho^\alpha(S) < 1$. If $card(C_\alpha^+) = 1$, then $\rho^\alpha(S) = \max_s \rho^\alpha(s) = 1$, i.e. ρ^α is a possibility measure. If $card(C_\alpha^+) = n$, then $\rho^\alpha(S) = 1$ reads $\sum_s \rho^\alpha(s) - (n-1)\alpha = 1$, hence $\sum_s^n \rho^\alpha(s) = 1 + (n-1)\alpha$ and the fuzzy measure ρ^α is additive.

Example 1. Consider the $S^{0.7}$-decomposable fuzzy measure $\rho^{0.7}$ on the set $\{s_a, s_b, s_c, s_d\}$, defined by the distribution $\rho^{0.7}(s_a) = 0.5$ $\rho^{0.7}(s_b) = 0.5$ $\rho^{0.7}(s_c) = 0.8$ $\rho^{0.7}(s_d) = 0.9$. Here, $C_\alpha^+ = \{s_c, s_d\}$. One can check that $\rho^{0.7}(\Omega) = 0.8 + 0.9 - 0.7 = 1$. It is normalized.

More generally it is easy to express the value of ρ^α on general events:

$$\rho^\alpha(A) = \begin{cases} \sum_{s\in A} \rho^\alpha(s) - \alpha(card(A) - 1) \text{ if } A \subseteq C_\alpha^+, \\ \max_{s\in A} \rho^\alpha(s) \text{ if } A \subseteq \overline{C_\alpha^+}, \\ \sum_{s\in A\cap C_\alpha^+} \rho^\alpha(s) - \alpha(card(A \cap C_\alpha^+) - 1) \text{ otherwise.} \end{cases}$$

Note that the third case ($C_\alpha^+ \cap A \neq \emptyset$) covers the first. We can moreover prove that any measure ρ^α is a plausibility measure in the sense of Shafer [11] obtained as a probabilistic mixture between a possibility measure and a probability measure.

Proposition 2. *For any hybrid possibility-probability function ρ^α, there exists a possibility measure Π with possibility distribution π and a probability measure P with distribution p such that $\rho^\alpha(s) = \alpha\pi(s) + (1-\alpha)p(s)$ where $\forall s, \pi(s) < 1$ implies $p(s) = 0$. Moreover $\forall A, \rho^\alpha(A) = \alpha \max_{s\in A} \pi(s) + (1-\alpha)\sum_{s\in A} p(s)$.*

Proof. For events, if $A \cap C_\alpha^+ = \emptyset$, $\rho^\alpha(A) = \alpha \max_{s\in A} \pi(w) = \alpha\Pi(A) + (1-\alpha)P(A)$ since $P(A) = 0$. Otherwise, if $A \cap C_\alpha^+ \neq \emptyset$:

$$\begin{aligned} \rho^\alpha(A) &= \sum_{s\in A\cap C_\alpha^+} \rho^\alpha(s) - \alpha(card(A \cap C_\alpha^+) - 1) \\ &= \sum_{s\in A\cap C_\alpha^+} (\alpha(1 - p(s)) + p(s)) - \alpha(card(A \cap C_\alpha^+) - 1) \\ &= \alpha card(A \cap C_\alpha^+) - \alpha P(A \cap C_\alpha^+) + P(A \cap C_\alpha^+) - \alpha(card(A \cap C_\alpha^+) - 1) \\ &= \alpha + (1-\alpha)P(A \cap C_\alpha^+) = \alpha\Pi(A) + (1-\alpha)P(A) \end{aligned}$$

since $\Pi(A) = 1$ and $p(w) = 0$ for $s \notin C_\alpha^+$. □

Note that the dual of ρ^α ($\overline{\rho}^\alpha(A) = 1 - \rho^\alpha(\overline{A})$) is a convex combination of a necessity measure and a probability measure. Indeed, we have $\overline{\rho}^\alpha(A) = 1 -$

$\alpha\Pi(\overline{A})-(1-\alpha)P(\overline{A}) = \alpha(1-\Pi(\overline{A}))+(1-\alpha)(1-P(\overline{A})) = \alpha N(A)+(1-\alpha)P(A)$. When the possibility distribution is vacuous ($C_\alpha^+ = S$), it is a special case of Shafer discounting scheme [11], and is known in the imprecise probability literature as the linear-vacuous model [1]. In particular, $\rho^\alpha = \alpha\pi + (1-\alpha)p$ is a normalized ρ^α distribution.

Hybrid π-p distributions combine models of two extreme behaviors in uncertain contexts, namely when the knowledge can be expressed through a possibility distribution and when it is can be expressed through a probability distribution. Hybrid distributions behave like probabilities only on the states with maximum possibility. As seen above the hybrid model must satisfy the constraint $P(s) = 0$ if $\pi(s) < 1, \forall s$. It is clear that C_α^+ is the core of π. Then from both distributions and a given threshold α, we can build the hybrid one ρ^α using a weighted average $\rho^\alpha(s) = \alpha\pi(s)+(1-\alpha)p(s)$. The idea is that the decision-maker provides a standard probability distribution (e.g., frequencies) on normal states. Then she considers that there is a subjective probability α that the actual state has not been observed, and defines a possibility measure on the states of zero probability, an idea in agreement with the handling of zero probability events in De Finetti approach to subjective probability (see [2]).

4 Decision Making on Hybrid π-p Decision Trees

Back to our problematics of decision making under uncertainty, let us consider hybrid π-p simple lotteries $L = <\rho/\lambda_1, ..., \rho_n/\lambda_n>$. We define two utility functionals ES^{Opt} and ES^{Pes} based on such hybrid π-p distributions:

$$ES^{Opt}(L) = S^\alpha_{i=1,...,n}(T^\alpha(\rho_i^{\alpha,L}, \lambda_i)) \tag{7}$$

$$ES^{Pes}(L) = 1 - S^\alpha_{i=1,...,n}(T^\alpha(\rho_i^{\alpha,L}, 1-\lambda_i)) \tag{8}$$

Considering a lottery $L^\rho = <\rho_1/\lambda_1, ..., \rho_n/\lambda_n>$ we define a lottery $(1-L)^\rho$ by $(1-L)^\rho = <\rho_1/(1-\lambda_1), ..., \rho_n/(1-\lambda_n)>$. We have the following semi-duality relation

$$ES^{Pes}(L^\rho) = 1 - ES^{Opt}((1-L)^\rho). \tag{9}$$

This property will be useful for some proofs in the following. For the sake of brevity, we will drop α from S^α and T^α in the sequel.

4.1 Hybrid Utility Functional and Decision Maker Behavior

Let us rewrite $ES^{Opt}(L)$ and $ES^{Pes}(L)$ more explicitly so as to lay bare its meaning.

Proposition 3

$$ES^{Opt}(L) = \begin{cases} U^{Opt}(L) \text{ if } \nexists i \text{ s.t. } \lambda_i > \alpha \text{ with } \rho_i^\alpha > \alpha \\ E^{Opt}(L) = \alpha + \frac{\sum_{i|\lambda_i \rho_i^\alpha > \alpha}(\lambda_i - \alpha)(\rho_i^\alpha - \alpha)}{1-\alpha} \text{ otherwise} \end{cases} \tag{10}$$

$$ES^{Pes}(L) = \begin{cases} U^{Pes}(L) \text{ if } \nexists i \text{ s.t. } \lambda_i < 1-\alpha \text{ with } \rho_i^\alpha > \alpha \\ E^{Pes}(L) = 1-\alpha - \frac{\sum_{i|1-\lambda_i,\rho_i^\alpha>\alpha}(1-\lambda_i-\alpha)(\rho_i^\alpha-\alpha)}{1-\alpha} \text{ otherwise} \end{cases} \quad (11)$$

Proof. If $\nexists i$ s.t. $\lambda_i > \alpha$ with $\rho_i^\alpha > \alpha$ then $\forall i$, $T = \min$ is used and $S = \max$ too. So we are back to the $U^{Opt}(L)$ criterion. If there is only one i s.t. $\lambda_i > \alpha$ with $\rho_i^\alpha > \alpha$, then $ES^{Opt}(L) = T^\alpha(\rho_i^\alpha, \lambda_i) = \alpha + \frac{(\lambda_i-\alpha)(\rho_i^\alpha-\alpha)}{1-\alpha}$. For the general case where there are several i such that $\min(\rho_i^{\alpha,L}, \lambda_i) > \alpha$, say a set I^+ of indices then $T(\rho_i^{\alpha,L}, \lambda_i) > \alpha, i \in I^+$ only. Then $ES^{Opt}(L) = \sum_{i\in I^+}(\alpha + \frac{(\lambda_i-\alpha)(\rho_i^\alpha-\alpha)}{1-\alpha}) - (card(I^+)-1)\alpha = \alpha + \sum_{i\in I^+}(\frac{(\lambda_i-\alpha)(\rho_i^\alpha-\alpha)}{1-\alpha})$. Using semi-duality between ES^{Pes} and ES^{Opt} we get the expression of the former. □

From Proposition 3 it is easy to check that:

Proposition 4. *$ES^{Opt}(L) \leq \alpha$ iff $ES^{Opt}(L) = U^{Opt}(L)$.*
Likewise, $ES^{Pes}(L) \geq 1-\alpha$ iff $ES^{Pes}(L) = U^{Pes}(L)$.

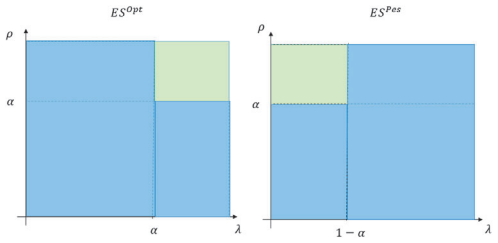

Fig. 3. $ES^{Opt}(L)$ and $ES^{Pes}(L)$

In other words, the criterion $ES^{Opt}(L)$ is possibilistic optimistic ($= U^{Opt}(L)$) so long as entries (utilities or plausibilities) are below the threshold α (distribution included in blue area on Fig. 3). Otherwise, we get an expected value over states with plausibilities and utilities greater than α (see green area in Fig. 3, left). Likewise, with $ES^{Pes}(L)$, we get an expected value over states with utility less than $1-\alpha$ and with high enough plausibility i.e. greater than α. We get the pessimistic possibilistic criterion $U^{Pes}(L)$ otherwise (with either high utilities or low plausibilities); see green area in Fig. 3 right side.

Example 2. let D_1, D_2 and D_3 be decisions with $\rho^{0.70}$ distribution on $\{a, b, c, d\}$ with $\lambda_a = 0.2$, $\lambda_b = 0.6$, $\lambda_c = 0.8$, $\lambda_d = 1$, $D_1 = <0.7/\lambda_a,\ 0.9/\lambda_b, 0.6/\lambda_c,\ 0.5/\lambda_d>$, $D_2 = <0.75/\lambda_a,\ 0.90/\lambda_b,\ 0.75/\lambda_c, 0.5/\lambda_d>$ and $D_3 = <0.75/\lambda_a,\ 0.85/\lambda_b,\ 0.75/\lambda_c, 0.75/\lambda_d>$. If the DM is optimistic, we can see that the D_1 is in the possibility area since $\rho_{D_1}^{0.7}(c)$ and $\rho_{D_1}^{0.7}(c)$ are ≤ 0.7 while D_2 and D_3 are in EU area ES^{Opt}. We have $D_3 \succ D_2 \succ D_1$. Note that D_3 is preferred to D_2 since the expected utility to be in the green square Fig. 3 is greater. If the DM is pessimistic, D_1 is in possibility area while D_2 and D_3 are in EU area since $\rho_{D_2}^{0.7}(a) = \rho_{D_2}^{0.7}(a) < 1-\lambda$ and the preference relation is $D_1 \succ D_2 \sim D_3$.

4.2 Decision Trees with Hybrid π-p Distributions

Consider now decisions tree. We now know how to compare simple lotteries. In order to compare strategies, i.e. compound lotteries, we define a principle of reduction of hybrid π-p compound lotteries:

Definition 8 (*Reduction*$^\rho$)**.** *For any compound lottery of the form* $L = <\rho_1^\alpha/L_1^{\rho^\alpha}, \cdots, \rho_m^\alpha/L_m^{\rho^\alpha}>$, $Red^\rho(L)$ *is the simple lottery that associates to each* λ_i *the weight* $\forall\lambda_i, \overline{\rho}_i^\alpha = S_{j=1}^m T(\rho_j^\alpha, \rho_i^{\alpha,j})$, *where* $\rho_i^{\alpha,j}$ *denotes the confidence value of getting* λ_i *though lottery* $L_j^{\rho^\alpha}$ *and* ρ_j^α *the confidence value of getting* $L_j^{\rho^\alpha}$.

It is easy to get the following result:

Proposition 5

$$\overline{\rho}_i^\alpha = \begin{cases} \max_{j=1,\ldots,m} \min(\rho_j^\alpha, \rho_i^{\alpha,j}) \text{ if } \nexists j \text{ s.t. } \rho_j^\alpha > \alpha,\ \rho_i^{\alpha,j} > \alpha \\ \alpha + \frac{\sum_{j|\rho_j^\alpha, \rho_j^{\alpha,i} > \alpha} (\rho_j^\alpha - \alpha)(\rho_j^{\alpha,i} - \alpha)}{1-\alpha} \text{ otherwise} \end{cases} \tag{12}$$

The following proposition states the main result of this paper:

Proposition 6. *Given a decision tree with* ρ^α *measure, the hybrid criteria* ES^{Opt} *and* ES^{Pes} *satisfy the weak monotonicity property.*

Proof. We propose a proof for ES^{Opt}, the proof is similar for ES^{Pes}. For each lottery L, we have two cases: (a) $\nexists i$ such that $\rho_i > \alpha$ and $\lambda_i > \alpha$ and (b) $\exists i$ such that $\rho_i > \alpha$ and $\lambda_i > \alpha$. Let us explore all possible configurations.

1. L, L' and L'' are in (a): $\forall i$ the t-norm min and t-conorm max are used so we are in possibility case and the weak monotonicity property holds.
2. L and L' in (a) and L'' in (b), we need to distinguish two cases:
   i) if $v \leq \alpha$ then we are again in the case with t-norm min and t-conorm max so the weak monotonicity property holds.
   ii) if $v > \alpha$ then $<u/L, v/L''>$ and $<u/L', v/L''>$ are in (b) so $ES^{Opt}(<u/L, v/L''>) = ES^{Opt}(<u/L', v/L''>)$ so the weak monotonicity is satisfied.
3. L, L' and L'' in (b), we need to distinguish three cases:
   i) if $v \leq \alpha$ and $u > \alpha$ then $<u/L, v/L''>$ and $<u/L', v/L''>$ are in (b) so $ES^{Opt}(<u/L, v/L''>) = \alpha + \frac{\sum_{i|\lambda_i, T^P(\rho_i^{\alpha,L}, u) > \alpha} (\lambda_i - \alpha) \times (T^P(\rho_i^{\alpha,L}, u) - \alpha)}{1-\alpha}$
   $= \alpha + (u - \alpha)\frac{\sum_{i|\lambda_i, \rho_i^{\alpha,L} > \alpha} (\lambda_i - \alpha) \times (\rho_i^{\alpha,L} - \alpha)}{(1-\alpha)^2} \leq ES^{Opt}(<u/L', v/L''>)$
   $= \alpha + (u - \alpha)\frac{\sum_{i|\lambda_i, \rho_i^{\alpha,L'} > \alpha} (\lambda_i - \alpha) \times (\rho_i^{\alpha,L'} - \alpha)}{(1-\alpha)^2}$ so the weak monotonicity is satisfied.
   ii) if $v > \alpha$ and $u \leq \alpha$ we are in a similar situation as in 2) ii)
   iii) if $v > \alpha$ and $u > \alpha$ then we have
   $ES^{Opt}(<u/L, v/L''>) = (u - \alpha)\frac{ES^{Opt}(L)}{(1-\alpha)} + (v - \alpha)\frac{ES^{Opt}(L'')}{(1-\alpha)} - \alpha \leq$
   $ES^{Opt}(<u/L', v/L''>) = (u - \alpha)\frac{ES^{Opt}(L')}{(1-\alpha)} + (v - \alpha)\frac{ES^{Opt}(L'')}{(1-\alpha)} - \alpha$

4. L'' in (a) and L, L' in (b), we need to distinguish two cases: if $u > \alpha$ the proof is similar to the one in 3) i) and if $u \leq \alpha$ similar to the one in 2) i).
5. L, L'' in (a) and L' in (b), we need to distinguish two cases:
   i) if $u > \alpha$, $ES^{Opt}(<u/L, v/L''>) \leq \alpha$ and $ES^{Opt}(<u/L', v/L''>) > \alpha$.
   ii) if $u \leq \alpha$ then $ES^{Opt}(<u/L, v/L''>) \leq ES^{Opt}(<u/L', v/L''>)$ the proof is similar to the one in 2) i) since u/L' is in (a).
6. L in (a) and L', L'' in (b), we need to distinguish three cases:
   i) if $u \leq \alpha$ and $v > \alpha$ then the proof is similar to the one in 2) ii).
   ii) if $u > \alpha$ and $v \leq \alpha$ then $<u/L, v/L''>$ in (a) and $<u/L, v/L''>$ in (b) from Proposition 4 the weak monotonicity property holds.
   iii) if $u > \alpha$ and $v > \alpha$ then $ES^{Opt}(<u/L, v/L''>) = (v-\alpha)\frac{ES^{Opt}(L'')}{(1-\alpha)} \leq ES^{Opt}(<u/L', v/L''>) = (u-\alpha)\frac{ES^{Opt}(L')}{(1-\alpha)} + (v-\alpha)\frac{ES^{Opt}(L'')}{(1-\alpha)} - \alpha$ so the weak monotonicity property holds. □

From Proposition 6, when the decision maker provides a ρ-style decision tree, ES^{Opt} and ES^{Pes} satisfy the three basic properties required in the introduction: consequentialism, dynamic consistency and lottery reduction.

4.3 Composing Possibilistic and Probabilistic Lotteries

According to Proposition 2, a (compound) lottery can be viewed as two (compound) lotteries, a possibilistic one $L^{\pi} = <\pi_1/L_1^{\pi}, \ldots, \pi_m^{\pi}/L_m>$, and a probabilistic one $L^{p} = <p_1/L_1^{p}, \ldots, p_m/L_m^{p}>$, both on the same decision tree. Assume that $\pi_i < 1$ implies $p_i = 0, i = 1, \ldots m$. We are interested in merging them into a hybrid lottery, given the parameter α. Proposition 2 leads to define a fusion operation:

$$F(\pi, p, \alpha) = \begin{cases} \rho_i^{\alpha} = \alpha\pi_i \text{ if } \pi_i < 1, \\ \rho_i^{\alpha} = \alpha + (1-\alpha)p_i \text{ otherwise.} \end{cases} \tag{13}$$

Merging the possibility and probability distributions locally should be equivalent to merging the reduced lotteries at the global level. Moreover, the fusion operation must be distributive over the reduction operator.

Property 1 (Distributivity over reduction). *Let $L = (L^{\pi}, L^{p})$ be a pair of possibilistic and probabilistic compound lotteries on the same decision tree. Operator F is said to satisfy the distributivity property iff $F(Red^{\pi}(L^{\pi}), Red^{p}(L^{p}), \alpha) = Red^{\rho}(<F(\pi_1, p_1, \alpha)/F(L_1^{\pi}, L_1^{p}, \alpha), \ldots, F(\pi_m, p_m, \alpha)/F(L_m^{\pi}, L_m^{p}, \alpha)>)$.*

Proposition 7. *$F(\pi, p, \alpha)$, defined by Eq. (13), satisfies Property 1.*

Proof. When applying reduction to the probability tree followed by F, we obtain $\rho_i = \alpha + (1-\alpha)\sum_j p_j \times p_j^i, \forall i$ s.t. $\exists j$ with $p_j^i > 0, p_j > 0$ and $\rho_i = \alpha \max_j \min(\pi_j, \pi_j^i)$ otherwise. When applying F first: if $\exists j$ with $p_j^i > , p_j > 0$ then $\rho_j^{\alpha,i} > \alpha, \rho_j^{\alpha} > \alpha$. From Proposition 5, the ρ^{α}-reduction is $\overline{\rho}_i^{\alpha} = \alpha + (1-\alpha)\sum_j p_j \times p_j^i$. Otherwise, the ρ^{α}-reduction is $\max_{j=1,\ldots,n} \min(\rho_j^{\alpha,i}, \rho_j^{\alpha})$ with $\min(\rho_j^{\alpha,i}, \rho_j^{\alpha}) = \alpha\pi_j^i$ or $\alpha\pi_j$ so we obtain $\rho_i = \alpha \max_j \min(\pi_j, \pi_j^i)$. □

This result ensures the dynamic consistency of the hybrid π-p approach to sequential decision-making under uncertainty.

5 Conclusion

In this paper, we try to improve the range of decision trees that can be solved by dynamic programming and respect consequentialism as well as dynamic consistency, beyond standard probabilistic decision trees, and possibilistic ones. We have shown that everything relies on i) defining the uncertainty measure by means of a generalized weight distribution on utility values; 2) the possibility of reducing compound lotteries into simple ones; 3) the definition of a utility functional by means of a generalized integral. This paper proposes a solution to this problem, and shows that it leads to a very restricted family of decomposable measures with respect to a specific family of t-conorms, due to the conditional distributivity property required to ensure lottery reduction. The paper proves that the obtained utility functionals satisfy the weak monotonicity property, which ensures computability of optimal decisions via dynamic programming. The kind of uncertainty function laid bare in this study turns out to be a Shafer plausibility (resp. belief) function obtained as a convex mixture of probability and possibility (resp.necessity) functions, which opens the way to a natural interpretation of these uncertainty measures.

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