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# Semidiscrete shocks for the full velocity difference model

N. El Khatib<sup>1</sup>, N. Forcadel<sup>2</sup>, M. Zaydan<sup>1</sup>

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## Abstract

In this paper, we consider the full velocity difference model for traffic flow and we study the existence and uniqueness of traveling wave solutions. First, using the monotony of the car's interdistance, we derive necessarily conditions for the existence of such solutions. Then, in the framework of viscosity solutions, we construct a traveling solution by considering an approximate non-local Hamilton-Jacobi equation on a bounded domain. This traveling waves solution can be interpreted as a phase transition between a congested state and a free-flow one.

**AMS Classification:** 74J40, 90B20, 35D40.

**Keywords:** traffic model, viscosity solution, semi-discrete shocks, strong comparison principle, Hamilton-Jacobi equation.

## 1 Introduction

Nowadays, modeling traffic flows and vehicle trajectories became a necessity in order to minimize its negative impacts such as emissions and congestion. There is an urgent need to develop traffic models that accurately represent the behavior of vehicles on urban road networks.

Macroscopic shock wave can be defined as discontinuous change in the characteristics of the traffic kinematic. According to Lighthill and Whitham [20] shocks in traffic can occur in the case of an accident, a reduction in number of lanes, an entrance ramp, or abrupt breaking. The construction of traveling shock profiles for finite difference schemes approximating hyperbolic conservation laws was studied in many work, we refer for example to [21, 22, 23, 27, 18]. At the microscopic scale, authors in [14] constructed shock solutions for a first order microscopic model exploiting its connection with a macroscopic model. Up to our knowledge, there are very few works on the existence of traveling waves in traffic at the microscopic scale, and it seems that (in addition to [14]) this subject was studied only in [28, 29, 26] where authors constructed discrete traveling wave profiles which are local attractors for the solution of a local and non-local follow-the-leader model.

One of the most famous microscopic models is the following type model ("car-following model") which describes how the vehicles adapt their position, their speed or their acceleration according to the surrounding vehicles. In this type of model, the behavior of the driver depends on the situation in front of him: if he is not preceded by another vehicle, he circulates freely ("free flow"). Otherwise, he must adapt his driving behavior depending on the distance with the "leading" vehicle. Several car-following models were proposed with the intention to describe precisely the driver's behaviors, see [25, 24, 4].

In [2], Bando et al. introduced the optimal velocity model (OVM), a straightforward car-following model that accurately captured many characteristics of real traffic flows [7]. The optimal

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velocity function, which depends on the headway distance, is used in this model to characterize each vehicle, and each driver controls the velocity based on this function. The jamming transition is found to be well described by the OVM.

In light of the empirical data, Helbing and Tilch calibrated the OVM [16]. According to the comparison with field data, the OVM exhibits rapid acceleration and unrealistic deceleration. In order to enhance the OVM, Helbing and Tilch proposed in the same paper a generalized force model. When the following vehicle's velocity exceeds that of the leader, a velocity difference term is considered in the model (that is negative velocity difference). The simulation findings demonstrate that the GFM and the empirical data correspond well. However, Treiber et al.'s descriptions of traffic phenomena (see [30]) cannot be explained by either the OVM or the GFM. If the cars ahead of it are moving considerably more quickly, the vehicle would not brake, even if its headway is smaller than the safe distance.

In [19], the authors noted that the GFM exhibited poor delay time of car motion and kinematic wave speed at jam density because positive velocity differences were disregarded. By taking both positive and negative velocity differences into account, Jiang et al. proposed in [19] a new microscopic model called full velocity difference model (FVDM). The numerical investigations indicated that the FVDM could describe the phase transition of traffic flow and estimate the evolution of traffic congestion.

In this paper, we construct traveling solutions for the FVDM which can be seen as phase transition between a congested state and a free-flow one. The FVDM is given by

$$\ddot{U}_i(t) = \kappa (V(U_{i+1}(t) - U_i(t)) - \dot{U}_i(t)) + \lambda(U'_{i+1}(t) - U'_i(t)), \quad (1.1)$$

where  $U_i$  denotes the position of the  $i$ -th vehicle,  $\dot{U}_i$  its velocity,  $\ddot{U}_i$  its acceleration,  $V : \mathbb{R} \rightarrow \mathbb{R}$  is the optimal velocity function and  $\kappa, \lambda$  are constant sensitivity coefficients.

For model (1.1), we construct a particular type of solutions called semi-discrete shocks. Before we give the definition of these solutions, let us explain the motivation of our work. Let us consider the macroscopic model

$$\chi_t = V(\chi_y) \quad \text{for } t > 0, y \in \mathbb{R}. \quad (1.2)$$

It was shown that (1.2) can be rigorously derived from the microscopic model (1.1) (see [11] or [12] with local perturbation considering the Eulerian coordinates). Let  $a < b$  and define the constants  $\frac{1}{T} = \frac{V(a) - V(b)}{a - b}$  and  $c = \frac{aV(b) - bV(a)}{a - b}$ . The function

$$\chi(t, y) = \min(ay + tV(a), by + tV(b)) = \begin{cases} ay + tV(a) & \text{if } y > -\frac{t}{T} \\ by + tV(b) & \text{if } y < -\frac{t}{T} \\ ct & \text{if } y = -\frac{t}{T} \end{cases} \quad (1.3)$$

is a viscosity solution of (1.2) if and only if

$$c + \frac{p}{T} \leq V(p) \quad \text{for any } p \in [a, b]. \quad (1.4)$$

The function  $\chi$  can be interpreted as a "shock" since its left and right derivatives at the point  $-\frac{t}{T}$  are different. This means that the spacing after the shock (resp. before the shock) is  $a$  (resp.  $b$ ) and the speed of propagation of the shock is  $c$ . Finally, let us remark that  $\chi$  is a traveling wave since we can write

$$\chi(t, y) = \chi\left(0, y + \frac{t}{T}\right) + ct \quad (1.5)$$

with  $\chi(0, y) = \min(ay, by)$ . Aim of this work is to construct solutions of (1.1) which can be seen as the discrete analogue of the function  $\chi$ . We look for particular shock solutions  $U_i$  of (1.1) satisfying

$$\begin{cases} U_{i+1}(t) - U_i(t) \rightarrow b & \text{as } i \rightarrow -\infty, \\ U_{i+1}(t) - U_i(t) \rightarrow a & \text{as } i \rightarrow +\infty. \end{cases} \quad (1.6)$$

Moreover, we will prove that the interdistance  $U_{i+1} - U_i$  is strictly decreasing (see Theorem 2.6). The traffic interpretation of (1.6) is that a shock occurred at the microscopic level and the interdistance is  $b$  far before the shock, and is  $a$  far after it. This shock represents for example the position of a traffic jam tail.

Following the definition of  $\chi$  in (1.5), we construct a similar solution at the microscopic level by considering

$$U_i(t) = u\left(i + \frac{t}{T}\right) + ct$$

with

$$\begin{cases} u(y+1) - u(y) \rightarrow b & \text{as } y \rightarrow -\infty, \\ u(y+1) - u(y) \rightarrow a & \text{as } y \rightarrow +\infty. \end{cases} \quad (1.7)$$

Finally, let us recall that at the macroscopic scale, (1.4) is a necessarily condition to prove that  $\chi$  is a solution. At the microscopic scale, we will provide a necessarily condition to ensure the existence of traveling solution satisfying (1.7) (see Theorem 2.6). We obtain our results in the framework of viscosity solutions and we refer the reader to reference [5, 3, 6, 8] for a full presentation of this theory.

## 2 Main results

In the rest of the paper, we will work with an equivalent formulation of (1.1). We borrow the idea from [12, 10] and consider for all  $i \in \mathbb{Z}$ ,

$$\Xi_i(t) = U_i(t) + \frac{1}{\alpha} \dot{U}_i(t) \quad \text{with} \quad \alpha = \frac{\kappa + \lambda}{2}.$$

Using this new function, we obtain the following system of ODEs equivalent to (1.1) for all  $i \in \mathbb{Z}$ , for all  $t \in (0, +\infty)$ ,

$$\begin{cases} \dot{U}_i(t) = \alpha(\Xi_i(t) - U_i(t)), \\ \dot{\Xi}_i(t) = \alpha(U_i(t) - \Xi_i(t)) + \frac{\kappa}{\alpha} V(U_{i+1}(t) - U_i(t)) + \lambda(\Xi_{i+1} - U_{i+1}). \end{cases} \quad (2.1)$$

We look for particular shock solutions of (1.1) of the form

$$\begin{aligned} U_i(t) &= u\left(i + \frac{t}{T}\right) + ct, \\ \Xi_i(t) &= \xi\left(i + \frac{t}{T}\right) + ct, \end{aligned}$$

where  $(u, \xi)$  solves

$$\begin{cases} \frac{1}{T}u'(y) + c = \alpha(\xi(y) - u(y)), \\ \frac{1}{T}\xi'(y) + c = \alpha(u(y) - \xi(y)) + \frac{\kappa}{\alpha}V(u(y+1) - u(y)) + \lambda(\xi(y+1) - u(y+1)), \end{cases} \quad (2.2)$$

and

$$\begin{cases} u(y+1) - u(y) \rightarrow b & \text{and } \xi(y+1) - \xi(y) \rightarrow b & \text{as } y \rightarrow -\infty, \\ u(y+1) - u(y) \rightarrow a & \text{and } \xi(y+1) - \xi(y) \rightarrow a & \text{as } y \rightarrow +\infty. \end{cases}$$

To obtain our results, we need the following assumptions on the optimal velocity function  $V$  and the parameters  $\kappa$  and  $\lambda$ .

**Assumptions (A) on  $V$  and  $s$ .**

- (A1) (Regularity)

$$V \in C^1(\mathbb{R}), V' \in L^\infty(\mathbb{R}).$$

- (A2) (Monotonicity)

$$V' > 0 \text{ on } \mathbb{R}.$$

- (A3) (Strict chord inequality) There exists  $a, b \in \mathbb{R}$  such that

$$\begin{cases} \frac{p}{T} + c \leq V(p) & \text{for } p \in \mathbb{R} \text{ if and only if } p \in [a, b], \\ \text{with equality} & \text{if and only if } p \in [a, b], \end{cases}$$

with

$$\frac{1}{T} = \frac{V(b) - V(a)}{b - a} \text{ and } c = \frac{bV(a) - aV(b)}{b - a}.$$

- (A4) (Non degeneracy)

$$V'(b) < \frac{1}{T} < V'(a).$$

- (A5) (Monotonicity) We assume that  $\kappa > \lambda > 0$  and that

$$\begin{cases} \frac{(\kappa + \lambda)^2}{\kappa} > 4 \max_{p \in [a, b]} V'(p), \\ \frac{\lambda(\kappa + \lambda)}{\kappa} < 2 \min_{p \in [a, b]} V'(p). \end{cases}$$

**Remark 2.1** (Comments on assumptions (A)). *The regularity assumption (A1) provides regular viscosity solutions. We will show (see Theorem 2.6) that our shock solutions exist if and only if assumption (A3) is satisfied. We will use assumption (A4) to get exponential asymptotics of the solution at infinity. We add assumption (A5) to obtain that the functions*

$$\begin{cases} f : z_1 \mapsto \alpha z_1 + \frac{\kappa}{\alpha} V(z_2 - z_1) \\ g : z_2 \mapsto \frac{\kappa}{\alpha} V(z_2 - z_1) - \lambda z_2 \end{cases}$$

*are strictly increasing if  $z_2 - z_1 \in [a, b]$ . We will use assumptions (A2)-(A5) to get strong comparison results for system (2.2) and to prove the monotonicity of the interdistance.*

**Remark 2.2.** As example of general functions  $V$  satisfying (A3) and (A4), we can consider  $V \in C^1(\mathbb{R})$  and strictly concave. We recall that

$$\frac{1}{T} = \frac{V(b) - V(a)}{b - a}.$$

Using the mean value theorem, there exists  $e \in (a, b)$  such that  $\frac{1}{T} = V'(e)$ . Using that  $V$  is strictly concave, we have

$$V'(a) > V'(e) > V'(b)$$

and the function  $p \mapsto V(p) - \frac{p}{T} - c$  is strictly increasing in  $[a, e]$  and strictly decreasing in  $(e, b]$ .

**Remark 2.3** (Examples of optimal velocities satisfying (A3) and (A4)). The Greenshields velocity [15] given by

$$V(p) = V_{max} \left( 1 - \frac{1}{\rho_{max} p} \right) \quad (2.3)$$

satisfies (A3) and (A4) since it's strictly concave for  $p > 0$ . Similarly, the Bando optimal velocity (see [2]) function given by

$$V(p) = V_1 + V_2 \tanh(C_1(p - l_c) - C_2) \quad (2.4)$$

where  $V_1, V_2, C_1$  and  $C_2$  are positive parameters is strictly concave for  $p > \frac{C_2}{C_1} + l_c$ .

**Remark 2.4** (Examples of optimal velocities satisfying (A5)). For any  $a, b > 0$  (resp.  $a, b > \frac{C_2}{C_1} + l_c$ ), taking a sensitivity  $\kappa$  large enough, assumption (A5) is satisfied for the velocity defined in (2.3) (resp in (2.4)). For the Bando velocity (2.4), authors in [16] carried out a calibration of the parameters with respect to the empirical data and they adopted the following parameters values:

$$\kappa = 0.85s^{-1}, \quad V_1 = 6.75m/s, \quad V_2 = 7.91m/s, \quad C_1 = 0.13m^{-1}, \quad C_2 = 1.57.$$

Taking  $\lambda$  as in [19], i.e.  $\lambda = 0.5s^{-1}$ , assumption (A5) is satisfied for  $\frac{C_2}{C_1} + l_c \approx 17.08m < a \leq 23m$  and  $b \geq 25m$ .

The main result of this paper is the following theorem.

**Theorem 2.5.** *i) (Existence.) Assume that (A) holds for some  $a, b \in \mathbb{R}$ . There exists a solution  $(u, \xi)$  of (2.2) such that  $u, \xi \in C^2(\mathbb{R})$  satisfy for some constant  $C > 0$ ,*

$$\begin{cases} |u(y) - \bar{u}(y)| \leq C, \\ |\xi(y) - \bar{u}(y)| \leq C, \end{cases} \quad (2.5)$$

with

$$\bar{u}(y) = \begin{cases} ay & \text{if } y \geq 0, \\ by & \text{if } y < 0. \end{cases} \quad (2.6)$$

Moreover, we have that

$$\begin{cases} a \leq G(y) \leq b, a \leq H(y) \leq b \text{ and } \frac{a}{\alpha} \leq F(y) \leq \frac{b}{\alpha} \text{ for all } y \in \mathbb{R}, \\ u'(+\infty) = \xi'(+\infty) = G(+\infty) = H(+\infty) = a, \\ u'(-\infty) = \xi'(-\infty) = G(-\infty) = H(-\infty) = b \end{cases}$$

with

$$G(y) = u(y+1) - u(y), H(y) = \xi(y+1) - \xi(y) \text{ and } F(y) = \xi(y) - u(y).$$

ii) (Uniqueness.) The solution  $(u, \xi)$  is unique (up to translation and addition of constants) among the solutions  $(v, \chi)$  with  $v, \chi \in C^2(\mathbb{R})$  such that

$$a \leq v(y+1) - v(y) \leq b \text{ for all } y \in \mathbb{R} \text{ and}$$

$$|v - \bar{u}| \leq C \text{ and } |\chi - \bar{u}| \leq C$$

for some constant  $C > 0$ .

**Normalization.** Up to consider a new velocity function

$$\tilde{V}(p) = T(V(p) - c),$$

and replacing  $V$  by  $\tilde{V}$ , we can assume that

$$T = 1 \text{ and } c = 0.$$

This allows us to consider assumptions (A) for  $T = 1$  and  $c = 0$ . Till the end of this paper, we will consider system (2.2) for  $T = 1$  and  $c = 0$  which gives us the following equations

$$\begin{cases} u'(y) = \alpha(\xi(y) - u(y)) & y \in \mathbb{R}, \\ \xi'(y) = \alpha(u(y) - \xi(y)) + \frac{\kappa}{\alpha}V(u(y+1) - u(y)) + \lambda(\xi(y+1) - u(y+1)) & y \in \mathbb{R}. \end{cases} \quad (2.7)$$

## 2.1 Steps to prove Theorem 2.5

In this paragraph, we give the steps to prove Theorem 2.5:

- 1) First, we provide necessarily and sufficient conditions for existence of shock solutions. This result can be seen as a justification of our choice to impose assumption (A3).

**Theorem 2.6.** [Classification of the solutions.] Assume (A1), (A2). Let  $u, \xi \in C^2(\mathbb{R})$  be such that  $(u, \xi)$  is a solution of (2.7) such that  $G(y) = u(y+1) - u(y)$ ,  $H(y) = \xi(y+1) - \xi(y)$  and  $F(y) = \xi(y) - u(y)$  are bounded functions. Let  $s_1, s_2 \in \mathbb{R}$  with  $s_1 < s_2$  such that  $s_1 \leq G(y) \leq s_2$  for all  $y \in \mathbb{R}$ .

Assume that (A5) holds for  $p \in [s_1, s_2]$ , i.e.  $\kappa > \lambda > 0$  and

$$\begin{cases} \frac{(\kappa + \lambda)^2}{\kappa} > 4 \max_{p \in [s_1, s_2]} V'(p), \\ \frac{\lambda(\kappa + \lambda)}{\kappa} < 2 \min_{p \in [s_1, s_2]} V'(p). \end{cases}$$

Then, there exists  $\bar{a}, \bar{b} \in \mathbb{R}$  such that

$$\begin{cases} u'(+\infty) = \xi'(+\infty) = G(+\infty) = H(+\infty) = \bar{a}, \\ u'(-\infty) = \xi'(-\infty) = G(-\infty) = H(-\infty) = \bar{b}. \end{cases}$$

Moreover, if  $\bar{a} < \bar{b}$ , then we have  $u'$  is strictly decreasing and

$$p \leq V(p) \text{ for } p \in [\bar{a}, \bar{b}] \text{ with equality if and only if } p = \bar{a}, \bar{b}. \quad (2.8)$$

If  $\bar{a} > \bar{b}$ , then we have  $u'$  is strictly increasing and

$$p \geq V(p) \text{ for } p \in [\bar{b}, \bar{a}] \text{ with equality if and only if } p = \bar{a}, \bar{b}. \quad (2.9)$$

If  $\bar{a} = \bar{b}$ , then we have  $u'$  is constant.

**Remark 2.7.** We would like to mention that Theorem 2.6 is a mathematical result and it's not related to traffic modeling. In this theorem, we do not provide the existence of shock solution. Our existence result is provided in Theorem 2.5. Since our work is applied to traffic modeling, we choose  $a < b$  in Theorem 2.5. To be more clear, let  $\chi$  be the solution of (1.2). We recall that  $\chi_y = \frac{1}{\rho}$  where  $\rho$  is the car's density and is the solution of the scalar conservation law

$$\rho_t + (f(\rho))_x = 0 \quad (2.10)$$

where the flux is  $f(p) = pV\left(\frac{1}{p}\right)$  (see [11] or [12]). It's known that in the theory of hyperbolic conservation laws only upward jumps in the density are admissible and that's why we define (1.3) for  $a < b$  and in Theorem 2.5, we construct shock solution where the distance  $U_{i+1} - U_i$  travels from state  $b$  to state  $a$ .

- 2) Assume that (A) holds for some  $a, b \in \mathbb{R}$ . Let  $(u, \xi)$  be a viscosity solution of (2.7) such that for all  $y \in \mathbb{R}$ , we have

$$a \leq G(y) \leq b, \quad a \leq H(y) \leq b, \quad \frac{a}{\alpha} \leq F(y) \leq \frac{b}{\alpha} \quad (2.11)$$

and  $u, \xi$  are globally Lipschitz functions such that their respective Lipschitz constants  $L_u$  and  $L_\xi$  satisfy

$$a \leq L_u \leq b \text{ and } 2a - b \leq L_\xi \leq 2b - a. \quad (2.12)$$

Thanks to Proposition 3.3, we have  $u, \xi \in C^2(\mathbb{R})$ . From Theorem 2.6, we know that

$$\begin{cases} u'(+\infty) = \xi'(+\infty) = G(+\infty) = H(+\infty) \in \{a, b\}, \\ u'(-\infty) = \xi'(-\infty) = G(-\infty) = H(-\infty) \in \{a, b\}, \end{cases} \quad (2.13)$$

and

$u'$  is strictly decreasing or  $u'$  is constant.

To obtain our result (Theorem 2.5), we will construct a viscosity solution  $(u, \xi)$  of (2.2) such that (2.11) and (2.12) hold. Moreover, our solution satisfies the following: for any  $y_0 \in (-1, -1/2)$ ,

$$u(y_0 + 1) - u(y_0) \geq a + \delta \tanh(\gamma(y_0 + 1)) > a \quad (2.14)$$

and



i) there exists  $y \in (-1, 1/2)$ ,

$$u(y+1) - u(y) \leq b + \delta (\tanh(\gamma y) - \tanh(\gamma(y+1))) < b \quad (2.15)$$

or there exists  $y_0 \in \left(0, \min\left(-\frac{1}{\alpha} + 1, \frac{1}{2}\right)\right)$ ,

ii)

$$\xi(y_0) - \xi(y_0 - 1) \leq b + \delta \tanh(\gamma(y_0 - 1)) + \frac{\delta \gamma (1 - \tanh^2(\gamma(y_0 - 1)))}{\alpha} < b. \quad (2.16)$$

Inequalities (2.14), (2.15) and (2.16) ensure that  $u'$  is strictly decreasing and that limit at  $+\infty$  (resp.  $-\infty$ ) in (2.13) is  $a$  (resp.  $b$ ). The construction of this solution is done in Section 5 where we consider an approximated operator and then we pass to the limit. This operator will become local at infinity and this will allow us to construct particular sub-solution and super-solution. To complete the proof of Theorem 2.5 (see Section 6), we prove (2.5) by using the non degeneracy assumption (A4). Finally, the uniqueness of the solution is obtained using the strong comparison principle (see Proposition 3.6).

**Remark 2.8** (Comparison with the work [14]). *In [14], authors obtained similar results for the Newell's model. To be more precise, they proved a classification theorem which provides necessarily conditions for the existence of the shock solutions and then they constructed these solutions. It's not surprising that the same result is obtained for the microscopic model (1.1) because the one in [14] is also strongly connected to (1.2) (see [13] or [9]). From a traffic modeling point of view, our model can describe the traffic with higher precision since we consider the acceleration. From a mathematical point of view, the proofs of our results require new ideas and the proof of [14] can not be adapted to our work since we consider a system of ODE (2.7).*

**Organization of the paper.** In section 3, we define the viscosity solutions of (2.7) and give strong comparison principle results. In section 4, we prove Theorem 2.6. To do this, we will show that the interdistances are monotone (see Proposition 4.1). In section 5, we prove Theorem 2.5. The idea is to consider a new non-local operator for which we can construct sub and super solutions (and then a solution by Perron method). We then use the stability of viscosity solutions to obtain the results in Theorem 2.5. Finally, in section 6, we prove the exponential behavior of the solution at  $\pm\infty$  and its uniqueness.

### 3 Viscosity solution

In this section, we first give the definition of viscosity solutions of (2.7). We then state different comparison principles.

#### 3.1 Definition of viscosity solutions

**Definition 3.1.** *Let  $u, \xi$  be two functions such that  $u, \xi \in L_{loc}^\infty(\mathbb{R})$ . We define the upper and lower semi-continuous envelopes of  $u$  and  $\xi$  by*

$$u^*(t, x) = \limsup_{s \rightarrow t, y \rightarrow x} u(s, y) \text{ and } u_*(t, x) = \liminf_{s \rightarrow t, y \rightarrow x} u(s, y)$$

$$\xi^*(t, x) = \limsup_{s \rightarrow t, y \rightarrow x} \xi(s, y) \text{ and } \xi_*(t, x) = \liminf_{s \rightarrow t, y \rightarrow x} \xi(s, y).$$

1) *We say that  $(u, \xi)$  is a viscosity sub-solution (resp. super-solution) of (2.7) if for all test function  $\varphi \in C^1(\mathbb{R})$  such that  $u^* - \varphi$  attains a local maximum (resp.  $u_* - \varphi$  attains a local minimum) at some point  $x_0$ , we have*

$$\varphi'(x_0) \leq \alpha(\xi(x_0) - u^*(x_0)) \text{ (resp. } \varphi'(x_0) \geq \alpha(\xi(x_0) - u_*(x_0)) \text{)}$$

and if for all test function  $\varphi \in C^1(\mathbb{R})$  such that  $\xi^* - \varphi$  attains a local maximum (resp.  $\xi_* - \varphi$  attains a local minimum) at some point  $x_0$ , we have

$$\begin{aligned} \varphi'(x_0) &\leq \alpha(u^*(x_0) - \xi^*(x_0)) + \frac{\kappa}{\alpha}V(u^*(x_0+1) - u^*(x_0)) + \lambda(\xi^*(x_0+1) - u^*(x_0+1)) \\ (\text{resp. } \varphi'(x_0) &\geq \alpha(u_*(x_0) - \xi_*(x_0)) + \frac{\kappa}{\alpha}V(u_*(x_0+1) - u_*(x_0)) + \lambda(\xi_*(x_0+1) - u_*(x_0+1)). \end{aligned}$$

2) We say that  $(u, \xi)$  is a viscosity solution of (2.7) if  $u, \xi \in L_{loc}^\infty(\mathbb{R})$  and  $(u, \xi)$  is a subsolution and supersolution of (2.7).

**Proposition 3.2** (Comparison Principle). *Assume (A1). Let  $(u, \xi)$  be a viscosity sub-solution and  $(v, \chi)$  be a viscosity super-solutions of (2.7). We assume that there exists  $K > 0$  such that*

$$(u - v)(y) \leq K(|y| + 1) \text{ and } (\xi - \chi)(y) \leq K(|y| + 1).$$

Then, we have

$$u(y) \leq v(y) \text{ and } \xi(y) \leq \chi(y) \text{ for all } y \in \mathbb{R}.$$

*Proof.* The proof of this result is classical and we refer the reader to [3, 5] for the proof details.  $\square$

**Proposition 3.3.** *Assume (A1). Let  $(u, \xi)$  be a viscosity solution of (2.7) such that  $u$  and  $\xi$  are globally Lipschitz continuous and  $\xi - u$  is globally bounded. Then  $u, \xi \in C^2(\mathbb{R})$ .*

*Proof.* The proof of this result is a simple adaptation of the proof of Proposition 3.2 in [14]. For the reader convenience, we explain shortly the idea of the proof. We set for  $y \in \mathbb{R}$ ,

$$\begin{cases} f(y) = \int_0^y \alpha(\xi(z) - u(z))dz + u(0) \\ g(y) = \int_0^y \left( \alpha(u(z) - \xi(z)) + \frac{\kappa}{\alpha}V(u(z+1) - u(z)) + \lambda(\xi(z+1) - u(z+1)) \right) dz + \xi(0). \end{cases}$$

We remark that  $f, g \in C^1(\mathbb{R})$  and  $(f, g)$  is a solution of (2.7). Hence, it's also a viscosity solution of (2.7). Using that  $u$  and  $\xi$  are globally Lipschitz functions and that  $\xi - u$  is bounded, there exists  $K > 0$  such that  $|u(y) - f(y)| \leq K|y|$  and  $|\xi(y) - g(y)| \leq K|y|$ . Hence, using the Proposition 3.2, we can prove that  $u = f$  and  $\xi = g$ . Finally, using that  $u, \xi, V \in C^1(\mathbb{R})$ , we deduce that  $u, \xi \in C^2(\mathbb{R})$ .  $\square$

## 3.2 Strong comparison principle

Since we will use the strong comparison principle for two systems ((2.7) and (4.3)), we will state the following lemma for a general function  $L$ .

**Lemma 3.4.** *[Partial strong comparison principle] Let  $s_1$  and  $s_2$  be two constants such that  $s_1 < s_2$  and  $L : \mathbb{R}^3 \rightarrow \mathbb{R}$  be function such that*

- 1)  $z_1 \mapsto L(z_1, z_2, z_3)$  is globally Lipschitz w.r.t  $z_1$  s.t  $\|L_{z_1}\|_\infty \leq \alpha$  and
- 2)  $z_2 \mapsto L(z_1, z_2, z_3)$  and  $z_3 \mapsto L(z_1, z_2, z_3)$  are non-decreasing if  $z_2 - z_1 \in [s_1, s_2]$ .

We consider the following equation

$$\begin{cases} u'(y) = \alpha(\xi(y) - u(y)) & y \in \mathbb{R}, \\ \xi'(y) = \alpha(u(y) - \xi(y)) + L(u(y), u(y+1), \xi(y+1)) & y \in \mathbb{R}. \end{cases} \quad (3.1)$$

Let  $(u^1, \xi^1)$  and  $(u^2, \xi^2)$  be respectively a sub-solution and a super-solution of (3.1). We assume that for  $i = 1, 2$  and for  $y \in \mathbb{R}$ ,

$$s_1 \leq u^i(y+1) - u^i(y) \leq s_2,$$

and that

$$\begin{cases} u^1 \leq u^2 & \text{on } \mathbb{R}, \\ \xi^1 \leq \xi^2 & \text{on } \mathbb{R}. \end{cases}$$

We then have the following:

i) if  $u^1(x_0) = u^2(x_0)$  for some  $x_0 \in \mathbb{R}$ , then  $u^1(y) = u^2(y)$  for all  $y \leq x_0$ .

ii) If  $\xi^1(x_0) = \xi^2(x_0)$  for some  $x_0 \in \mathbb{R}$ , then  $\xi^1(y) = \xi^2(y)$  for all  $y \leq x_0$ .

*Proof.* Let  $w(y) = u^2(y) - u^1(y) \geq 0$  and  $\psi(y) = \xi^2(y) - \xi^1(y) \geq 0$ . Since  $(u^2, \xi^2)$  is a super-solution and  $(u^1, \xi^1)$  is a sub-solution of (3.1), then using the doubling of variable method (see [1]), we have for all  $y \in \mathbb{R}$ ,

$$\begin{cases} w'(y) \geq \alpha(\psi(y) - w(y)) \\ \psi'(y) \geq \alpha(w(y) - \psi(y)) + L(u^2(y), u^2(y+1), \xi^2(y+1)) - L(u^1(y), u^1(y+1), \xi^1(y+1)). \end{cases}$$

Using that  $L$  is non-decreasing w.r.t  $z_2$  and  $z_3$ , we get

$$\psi'(y) \geq \alpha(w(y) - \psi(y)) + L(u^2(y), u^1(y+1), \xi^1(y+1)) - L(u^1(y), u^1(y+1), \xi^1(y+1)).$$

Now, since  $L$  is globally Lipschitz w.r.t  $z_1$  with a Lipschitz constant equal to  $\alpha$ , we obtain

$$\psi'(y) \geq \alpha(w(y) - \psi(y)) - \alpha w(y) = -\alpha\psi(y).$$

We deduce that for all  $y \in \mathbb{R}$ ,

$$\begin{cases} w'(y) \geq \alpha(\psi(y) - w(y)) \geq -\alpha w(y) \\ \psi'(y) \geq -\alpha\psi(y). \end{cases} \quad (3.2)$$

Let  $y_0 \leq x_0$ : using the comparison principle for "ODE", we deduce that for all  $y \geq y_0$ ,

$$\begin{cases} w(y) \geq w(y_0)e^{-\alpha(y-y_0)} \\ \psi(y) \geq \psi(y_0)e^{-\alpha(y-y_0)}. \end{cases} \quad (3.3)$$

Assume that  $\psi(x_0) = 0$ . Taking  $y = x_0$  in the second inequality in (3.3), we get that  $\psi(y_0) \leq 0$ . Using that  $\psi \geq 0$ , we deduce  $\psi(y_0) = 0$ . In the same way, if  $w(x_0) = 0$ , we get  $w(y_0) = 0$ .  $\square$

As a consequence of the preceding lemma, we can prove the following results.

**Proposition 3.5.** Let  $s_1, s_2$  and  $L$  defined as in Lemma 3.4. Let  $(u^1, \xi^1)$  and  $(u^2, \xi^2)$  be two viscosity solutions of (3.1) such that for  $i = 1, 2$ ,

$$s_1 \leq u^i(y+1) - u^i(y) \leq s_2.$$

We assume that

$$\begin{cases} \lim_{|y| \rightarrow +\infty} (u^1(y) - u^2(y)) \leq 0, \\ \lim_{|y| \rightarrow +\infty} (\xi^1(y) - \xi^2(y)) \leq 0. \end{cases} \quad (3.4)$$

Then we have,

$$\begin{cases} u^1 \leq u^2 & \text{on } \mathbb{R}, \\ \xi^1 \leq \xi^2 & \text{on } \mathbb{R}. \end{cases}$$

*Proof.* We define

$$M = \sup_{y \in \mathbb{R}} \{u^1(y) - u^2(y), \xi^1(y) - \xi^2(y)\}.$$

We want to prove that  $M \leq 0$ . Assume by contradiction that  $M > 0$ . Using (3.4), we deduce that  $M$  is reached at some point  $y_0$ .

**Case 1:**  $M = \xi^1(y_0) - \xi^2(y_0)$ . We define

$$\begin{cases} \bar{u}^2(y) = u^2(y) + M, \\ \bar{\xi}^2(y) = \xi^2(y) + M. \end{cases}$$

For all  $y \in \mathbb{R}$ , we have  $\bar{u}^2(y) \geq u^1(y)$ ,  $\bar{\xi}^2(y) \geq \xi^1(y)$  and  $\bar{\xi}^2(y_0) = \xi^1(y_0)$ . Using that  $(\bar{u}^2, \bar{\xi}^2)$  is a solution of (2.7) and Lemma 3.4, we get

$$\bar{\xi}^2(y) = \xi^1(y) \quad \text{for all } y \leq y_0.$$

Taking  $y \rightarrow -\infty$ , and using the second limit in (3.4), we get  $M \leq 0$  which gives a contradiction.

**Case 2:**  $M = u^1(y_0) - u^2(y_0)$ . Similar to case 1. □

**Proposition 3.6** (Strong comparison principle.). *Let  $s_1, s_2$  and  $L$  defined as in Lemma 3.4, and assume that*

$$z_2 \mapsto L(z_1, z_2, z_3) \text{ is strictly increasing if } z_2 - z_1 \in [s_1, s_2].$$

*Let  $(u^1, \xi^1)$  and  $(u^2, \xi^2)$  be respectively a sub-solution and a super-solution of (3.1). We assume that for  $i = 1, 2$  and for  $y \in \mathbb{R}$ ,*

$$s_1 \leq u^i(y+1) - u^i(y) \leq s_2,$$

*and that for all  $y \in \mathbb{R}$ ,*

$$\begin{cases} u^1(y) \leq u^2(y), \\ \xi^1(y) \leq \xi^2(y). \end{cases}$$

*If for some  $x_0$ , we have  $u^1(x_0) = u^2(x_0)$  and  $\xi^1(x_0) = \xi^2(x_0)$ , then we have for all  $y \in \mathbb{R}$ ,*

$$\begin{cases} u^1(y) = u^2(y), \\ \xi^1(y) = \xi^2(y). \end{cases}$$

*Proof.* Let  $w(y) = u^2(y) - u^1(y)$  and  $\psi(y) = \xi^2(y) - \xi^1(y)$ . From Lemma 3.4, we have that  $w(y) = \psi(y) = 0$  if  $y \leq x_0$ . It remains to prove that

$$w(y) = \psi(y) = 0 \quad \text{for } y \geq x_0.$$

Using that  $\psi$  reaches a global minimum at  $x_0$ , we have

$$\begin{aligned} 0 &\geq \alpha(w(x_0) - \psi(x_0)) + L(u^2(x_0), u^2(x_0+1), \xi^2(x_0+1)) - L(u^1(x_0), u^1(x_0+1), \xi^1(x_0+1)) \\ &\geq L(u^1(x_0), u^2(x_0+1), \xi^1(x_0+1)) - L(u^1(x_0), u^1(x_0+1), \xi^1(x_0+1)) \end{aligned}$$

where we use that  $w(x_0) = \psi(x_0) = 0$  and that  $L$  is non-decreasing w.r.t.  $z_3$ . Using that  $L$  is strictly increasing w.r.t.  $z_2$ , we obtain

$$u^2(x_0 + 1) = u^1(x_0 + 1).$$

Hence, the function  $w$  reaches a global minimum at  $x_0 + 1$  which implies

$$0 \geq w'(x_0 + 1) \geq \alpha(\psi(x_0 + 1) - w(x_0 + 1)) = \alpha\psi(x_0 + 1).$$

We deduce that  $w(x_0 + 1) = \psi(x_0 + 1) = 0$ . Similarly, we can prove that

$$w(y) = \psi(y) = 0 \text{ for } y = x_0 + k \text{ with } k \in \mathbb{N} - \{1\}.$$

Proceeding as in the proof of Lemma 3.4, we can prove that for  $k \in \mathbb{N} \cup \{0\}$ ,

$$w(y) = \psi(y) = 0 \text{ for } y \in [x_0 + k, x_0 + k + 1].$$

□

**Remark 3.7.** We will use Lemma 3.4, Proposition 3.5 and Proposition 3.6 when proving Theorem 2.6 and Theorem 2.5. To be more precise, let  $s_1 < s_2$  and let  $u, \xi \in C^2(\mathbb{R})$  such that  $(u, \xi)$  is a solution of (2.7) with

$$s_1 \leq u(y + 1) - u(y) \leq s_2 \text{ for all } y \in \mathbb{R}.$$

If we set

$$L(z_1, z_2, z_3) = \frac{\kappa}{\alpha}V(z_2 - z_1) + \lambda(z_3 - z_2), \quad (3.5)$$

and if we assume that (A5) holds replacing  $a$  by  $s_1$  and  $b$  by  $s_2$ , then  $L$  satisfies the assumptions in Proposition 3.6.

In the same way, we will have to consider  $G(y) = u(y + 1) - u(y)$  and  $H(y) = \xi(y + 1) - \xi(y)$ . Then,  $(G, H)$  is a solution of

$$\begin{cases} G'(y) = \alpha(H(y) - G(y)) & y \in \mathbb{R}, \\ H'(y) = \alpha(G(y) - H(y)) + \frac{\kappa}{\alpha}(V(G(y + 1)) - V(G(y))) + \lambda(H(y + 1) - G(y + 1)) & y \in \mathbb{R}. \end{cases}$$

In that case, if we set

$$L(z_1, z_2, z_3) = \frac{\kappa}{\alpha}(V(z_2) - V(z_1)) + \lambda(z_3 - z_2) \quad (3.6)$$

then under assumption (A5) (with  $a = s_1$  and  $b = s_2$ ),  $L$  satisfies the assumptions in Proposition 3.6.

The next proposition is useful in two directions. First, using this proposition, we can prove the uniqueness up to translation and addition of constants of the solution (part *ii*) in Theorem 2.5. Secondly, when proving Theorem 2.6, we will use it to prove that

$$\begin{cases} p < V(p) & \text{for } p \in (\bar{a}, \bar{b}) \text{ if } \bar{a} < \bar{b} \text{ and} \\ p > V(p) & \text{for } p \in (\bar{b}, \bar{a}) \text{ if } \bar{a} > \bar{b}. \end{cases}$$

**Proposition 3.8.** Assume (A1). Let  $s_1, s_2 \in \mathbb{R}$  with  $s_1 < s_2$ . Let  $(u, \xi)$  be a solution of (2.7) such that  $u, \xi \in C^2(\mathbb{R})$ . We assume that there exists  $C > 0$  such that

$$\begin{cases} |u(y) - \hat{u}(y)| \leq C, \\ |\xi(y) - \hat{\xi}(y)| \leq C \end{cases}$$

with  $\hat{u}(y) = \min(s_1y, s_2y)$ . Assume that  $u$  is concave and that

$$u'(-\infty) = s_2 \quad \text{and} \quad u'(+\infty) = s_1. \quad (3.7)$$

Then, there exists constants  $c_1, c_2$  such that  $(\tilde{u}, \tilde{\xi})$  defined by

$$\begin{cases} \tilde{u}(y) = c_1 + u(y + c_2) \\ \tilde{\xi}(y) = c_1 + \xi(y + c_2) \end{cases}$$

is solution of (2.7). Moreover, we have that

$$\begin{cases} \lim_{|y| \rightarrow +\infty} (\tilde{u}(y) - \hat{u}(y)) = 0 \\ \lim_{|y| \rightarrow +\infty} (\tilde{\xi}(y) - \hat{\xi}(y)) = 0 \end{cases} \quad (3.8)$$

where  $\hat{\xi}(y) = \min(s_1y + \frac{s_1}{\alpha}, s_2y + \frac{s_2}{\alpha})$ .

*Proof.* We define  $\tilde{u}(y) = c_1 + u(y + c_2)$  and  $\tilde{\xi}(y) = c_1 + \xi(y + c_2)$  where  $c_1$  and  $c_2$  are constants to be chosen later. We can easily verify that  $(\tilde{u}, \tilde{\xi})$  is a solution of (2.7). Let  $\phi(y) = u(y) - \hat{u}(y)$  with

$$\hat{u}(y) = \begin{cases} s_1y & y \geq 0, \\ s_2y & y < 0. \end{cases}$$

Using that  $|\phi(y)| \leq C$  and  $\phi$  is concave, we deduce that  $\lim_{|y| \rightarrow +\infty} \phi(y)$  exists. We denote by

$$c^+ = \lim_{y \rightarrow +\infty} \phi(y) \quad \text{and} \quad c^- = \lim_{y \rightarrow -\infty} \phi(y).$$

Then, choosing  $(c_1, c_2)$  as the solution of the following system,

$$\begin{cases} c_1 + c^+ + s_1c_2 = 0 \\ c_1 + c^- + s_2c_2 = 0, \end{cases}$$

we obtain  $\lim_{|y| \rightarrow +\infty} (\tilde{u}(y) - \hat{u}(y)) = 0$ . To obtain the second claim in (3.8), it's enough to remark using the first equation in (2.7) that  $\tilde{u}'(y) = \alpha(\tilde{\xi}(y) - \tilde{u}(y))$  which implies

$$\begin{aligned} \lim_{y \rightarrow +\infty} (\tilde{\xi}(y) - \hat{\xi}(y)) &= \lim_{y \rightarrow +\infty} \left( \tilde{u}(y) + \frac{\tilde{u}'(y)}{\alpha} - s_1y - \frac{s_1}{\alpha} \right) = 0, \\ \lim_{y \rightarrow -\infty} (\tilde{\xi}(y) - \hat{\xi}(y)) &= \lim_{y \rightarrow -\infty} \left( \tilde{u}(y) + \frac{\tilde{u}'(y)}{\alpha} - s_2y - \frac{s_2}{\alpha} \right) = 0 \end{aligned}$$

where we use (3.7). □

## 4 Proof of Theorem 2.6

In this section, we prove Theorem 2.6. We will first prove that if the interdistances  $u(y+1) - u(y)$  and  $\xi(y+1) - \xi(y)$  are bounded, then they are monotone. Let  $(u, \xi)$  be a solution of (2.7) such that  $u, \xi \in C^2(\mathbb{R})$ . For  $y \in \mathbb{R}$ , we define the functions

$$G(y) = u(y+1) - u(y) \quad (4.1)$$

$$H(y) = \xi(y+1) - \xi(y). \quad (4.2)$$

The couple  $(G, H)$  satisfies

$$\begin{cases} G'(y) = \alpha(H(y) - G(y)) & y \in \mathbb{R}, \\ H'(y) = \alpha(G(y) - H(y)) + \frac{\kappa}{\alpha}(V(G(y+1)) - V(G(y))) + \lambda(H(y+1) - G(y+1)) & y \in \mathbb{R}. \end{cases} \quad (4.3)$$

We have the following proposition.

**Proposition 4.1.** *Assume (A1)-(A2). Let  $G, H \in C^2(\mathbb{R})$  be bounded functions such that  $(G, H)$  is a solution of (4.3). Let  $s_1, s_2 \in \mathbb{R}$  with  $s_1 < s_2$  such that  $s_1 \leq G(y) \leq s_2$  for all  $y \in \mathbb{R}$ . Assume that (A5) holds replacing  $a$  by  $s_1$  and  $b$  by  $s_2$ . Then we have that*

$$\begin{cases} G \text{ and } H \text{ are strictly increasing} & \text{or} \\ G \text{ and } H \text{ are strictly decreasing} & \text{or} \\ G \text{ and } H \text{ are constant functions.} & \end{cases} \quad (4.4)$$

**Remark 4.2.** *As consequence of (A5) (with  $a = s_1$  and  $b = s_2$ ), we have*

$$z \mapsto \alpha z - \frac{\kappa}{\alpha}V(z) \text{ and } z \mapsto \frac{\kappa}{\alpha}V(z) - \lambda z \text{ are strictly increasing functions in } [s_1, s_2].$$

To prove Proposition 4.1, we need the following lemma whose proof is postponed.

**Lemma 4.3.** *Under the assumption of the previous proposition, we have the following result: let  $y_0 \in \mathbb{R}$ . We have*

$$\begin{cases} \max(G(y), H(y)) \leq \max(G(y_0), H(y_0)) & \text{for all } y \geq y_0 & \text{or} \\ \min(G(y), H(y)) \geq \min(G(y_0), H(y_0)) & \text{for all } y \geq y_0. \end{cases} \quad (4.5)$$

*Proof of Proposition 4.1.* We will first show that the function  $G$  is monotone.

**Step 1:  $G$  is monotone.** Assume that  $G$  has a left strict local maximum at  $x_0$ , i.e, there exists  $r > 0$  such that

$$G(x_0) > G(y) \text{ for } y \in (x_0 - r, x_0). \quad (4.6)$$

We claim that

$$G(x_0) \leq G(y) \text{ for } y \geq x_0. \quad (4.7)$$

Assume by contradiction that there exists  $y_0 > x_0$  such that  $G(x_0) > G(y_0)$ . Using the continuity of the function  $G$ , we can choose  $z_0 \in (x_0 - r, x_0)$  such that

$$G(z_0) > G(y_0). \quad (4.8)$$

Using that  $G'(x_0) = 0$  (local maximum) and  $G'(y) \geq 0$  for  $y \in (x_0 - r, x_0)$ , we have

$$G'(x_0) = \alpha(H(x_0) - G(x_0)) = 0, \quad (4.9)$$

$$G'(y) = \alpha(H(y) - G(y)) \geq 0 \text{ for } y \in (x_0 - r, x_0) \quad (4.10)$$

which implies that

$$G(x_0) = \min(G(x_0), H(x_0)) > G(z_0) = \min(G(z_0), H(z_0)). \quad (4.11)$$

Using the second inequality in (4.5), we get for all  $y \geq z_0$ ,

$$\min(G(y), H(y)) \geq G(z_0) = \min(G(z_0), H(z_0)) \quad (4.12)$$

which implies that  $G(y_0) \geq G(z_0)$  and we get a contradiction using (4.8).

In the same way (using the first inequality in (4.5)), we can prove that if  $G$  has a left strict local minimum at  $x_0$  i.e, there exists  $r > 0$  such that

$$G(x_0) < G(y) \text{ for } y \in (x_0 - r, x_0),$$

then,

$$G(x_0) \geq G(y) \text{ for } y \geq x_0. \quad (4.13)$$

We deduce that  $G$  is monotone.

**Step 2:** ( $G' \geq 0$  and  $H' \geq 0$ ) or ( $G' \leq 0$  and  $H' \leq 0$ ). Assume that  $G' \geq 0$ . Let  $\eta > 0$ . We claim that for  $x \geq y$ ,

$$H(x) \geq H(y) - \eta.$$

We define

$$M = \sup_{x \geq y} \{H(y) - H(x) - \eta\}.$$

By contradiction, assume that  $M > 0$ . We then define

$$M_\beta = \sup_{x \geq y} \{H(y) - H(x) - \beta x^2 - \beta y^2 - \eta\}.$$

For  $\beta$  small enough, and since  $M > 0$ , we have  $M_\beta > 0$ . Using the fact that  $H$  is bounded, we have

$$H(y) - H(x) - \beta x^2 - \beta y^2 \longrightarrow -\infty \text{ if } |x| \rightarrow +\infty \text{ or } |y| \rightarrow +\infty.$$

We deduce that  $M_\beta$  is reached at some point  $(x_\beta, y_\beta)$ . In addition, there exists a constant  $C$  such that,

$$0 < H(y_\beta) - H(x_\beta) - \beta x_\beta^2 - \beta y_\beta^2 \leq C - \beta x_\beta^2 - \beta y_\beta^2$$

which implies

$$\beta |x_\beta| \leq \sqrt{\beta C} \text{ and } \beta |y_\beta| \leq \sqrt{\beta C}.$$

Since  $M_\beta > 0$ , we also have that  $x_\beta > y_\beta$ . Writing the viscosity inequalities, we obtain

$$\begin{aligned} 2\beta x_\beta + 2\beta y_\beta &\leq \alpha (G(y_\beta) - H(y_\beta)) + \frac{\kappa}{\alpha} (V(G(y_\beta + 1)) - V(G(y_\beta))) + \lambda (H(y_\beta + 1) - G(y_\beta + 1)) \\ &\quad - \alpha (G(x_\beta) - H(x_\beta)) - \frac{\kappa}{\alpha} (V(G(x_\beta + 1)) - V(G(x_\beta))) - \lambda (H(x_\beta + 1) - G(x_\beta + 1)). \end{aligned}$$

Using that  $G(y_\beta) \leq G(x_\beta)$ ,  $G(y_\beta + 1) \leq G(x_\beta + 1)$  and Remark 4.2, we get

$$2\beta x_\beta + 2\beta y_\beta \leq \alpha (H(x_\beta) - H(y_\beta)) + \lambda (H(y_\beta + 1) - H(x_\beta + 1)). \quad (4.14)$$

By definition of  $M_\beta$ , we know that

$$H(y_\beta) - H(x_\beta) + 2\beta(x_\beta + y_\beta + 1) \geq H(y_\beta + 1) - H(x_\beta + 1). \quad (4.15)$$

Injecting (4.15) in (4.14), we obtain

$$2\beta x_\beta + 2\beta y_\beta \leq (\alpha - \lambda)(H(x_\beta) - H(y_\beta)) + o(\beta) \leq -\eta(\alpha - \lambda) + o(\beta).$$

Taking  $\beta$  to zero, we get a contradiction and we deduce that  $M \leq 0$ . Finally, taking  $\eta$  to zero, we get  $H' \geq 0$ . In the same way, we can prove that if  $G' \leq 0$ , then  $H' \leq 0$ .



**Step 3: Strict Monotony.** Assume that  $G' \geq 0$  and  $H' \geq 0$  on  $\mathbb{R}$ . We will show that

$G$  and  $H$  are strictly increasing or  
 $G$  and  $H$  are constant functions.

**Step 3.1: Constant left implies constant right.** Assume that there exists  $r > 0$  and  $x_0 \in \mathbb{R}$  such that

$$G(x) = G(x_0) \text{ for } x \in (x_0 - r, x_0).$$

We will prove that  $G(x) = G(x_0)$  for all  $x \geq x_0$ . Let  $x_1 \in (x_0 - r, x_0)$ . We have  $G'(x_1) = 0$  ( $G$  is constant on  $(x_0 - r, x_0)$ ) which implies (using the first equation in (4.3))

$$G(x_1) = G(x_0) = H(x_1) = H(x_0). \quad (4.16)$$

This implies that

$$\max(G(x_0), H(x_0)) \leq \max(G(x_1), H(x_1)).$$

Using Lemma 4.3, we obtain for all  $x \geq x_1$ ,

$$\max(G(x), H(x)) \leq \max(G(x_1), H(x_1)) = G(x_0)$$

which implies  $G(x) \leq G(x_0)$ . Using that  $G$  is non-decreasing, we have  $G(x) \geq G(x_0)$  for  $x \geq x_0$  and thus  $G(x) = G(x_0)$  for  $x \geq x_0$ .

We now prove the same result for  $H$ . Assume that there exists  $r > 0$  and  $x_0 \in \mathbb{R}$  such that

$$H(x) = H(x_0) \text{ for } x \in (x_0 - r, x_0).$$

We will prove that  $H(x) = H(x_0)$  for all  $x \geq x_0$ . Let  $x_1 \in (x_0 - r, x_0)$ . We have  $H'(x) = 0$  for  $x \in [x_1, x_0]$  which implies

$$\begin{aligned} 0 = H'(x) &= \alpha(G(x) - H(x)) + \frac{\kappa}{\alpha}(V(G(x+1)) - V(G(x))) + \lambda(H(x+1) - G(x+1)) \\ &\geq \alpha(G(x) - H(x)) + \frac{\kappa}{\alpha}(V(G(x+1)) - V(G(x))) + \lambda(H(x) - G(x+1)) \\ &\geq \alpha(G(x) - H(x)) + \lambda(H(x) - G(x)) \\ &= (\alpha - \lambda)(G(x) - H(x)) \end{aligned}$$

where we use  $H' \geq 0$ ,  $\lambda > 0$  in the second line and  $G' \geq 0$ ,  $z \mapsto \frac{\kappa}{\alpha}V(z) - \lambda z$  is strictly increasing in the third line. We deduce that  $\max(G(x_1), H(x_1)) = H(x_1) = H(x_0) = \max(G(x_0), H(x_0))$ . Proceeding as above, we get the desired result.

**Step 3.2: Conclusion.** Using step 3.1, we remark that if  $G$  is not strictly increasing and  $G$  is not constant then  $G$  has a global maximum at some point  $x_0$  with

$$\begin{cases} G(x) = G(x_0) & \text{if } x \geq x_0, \\ G(x) < G(x_0) & \text{if } x < x_0. \end{cases}$$

For  $x < x_0$ , we have  $H(x) \leq H(x_0)$ . In addition, for  $x \geq x_0$ , we have

$$0 = G'(x) = \alpha(H(x) - G(x)).$$

This implies that  $H(x) = H(x_0) = G(x_0) = G(x)$  for all  $x \geq x_0$ . We deduce that  $H$  has a global maximum at  $x_0$ . Using the strong comparison principle (Proposition 3.6) and that  $(G(x_0), H(x_0))$  is a solution of (4.3), we get  $G(x) = G(x_0)$  for all  $x \in \mathbb{R}$  which gives a contradiction. Similarly, if  $H$  is not strictly increasing and  $H$  is not constant then  $H$  has a global maximum at some point  $x_1$  with

$$\begin{cases} H(x) = H(x_1) & \text{if } x \geq x_1, \\ H(x) < H(x_1) & \text{if } x < x_1. \end{cases}$$

For  $x < x_1$ , we have

$$G'(x) = \alpha(H(x) - G(x)) \geq 0$$

which implies  $G(x) < H(x_1)$ . In addition, for  $x \geq x_1$ , we have

$$G'(x) = \alpha(H(x_1) - G(x)) \geq 0 \quad (4.17)$$

and

$$\begin{aligned} 0 = H'(x) &= \alpha(G(x) - H(x_1)) + \frac{\kappa}{\alpha}(V(G(x+1)) - V(G(x))) + \lambda(H(x_1) - G(x+1)) \\ &\leq \left(\frac{\kappa}{\alpha}V(G(x+1)) - \lambda G(x+1)\right) - \left(\frac{\kappa}{\alpha}V(H(x_1)) - \lambda H(x_1)\right) \end{aligned} \quad (4.18)$$

where we use that  $G(x) \leq H(x_1)$  for  $x \geq x_1$  and that  $z \mapsto \alpha z - \frac{\kappa}{\alpha}V(z)$  is strictly increasing.

Using that  $z \mapsto \frac{\kappa}{\alpha}V(z) - \lambda z$  is strictly increasing and (4.17), we deduce that  $G(x+1) = H(x_1)$  for  $x \geq x_1$ . Injecting the last equality in the first equation in (4.18), we get

$$0 = (\alpha G(x) - \frac{\kappa}{\alpha}V(G(x))) - (\alpha H(x_1) - \frac{\kappa}{\alpha}V(H(x_1)))$$

which implies  $G(x) = H(x_1)$  for all  $x \geq x_1$ . Therefore, we have now

$$\begin{cases} G(x) \leq G(x_1) & \text{if } x \leq x_1 \\ G(x) = G(x_1) & \text{if } x > x_1. \end{cases}$$

This means that  $G$  has a global maximum at  $x_1$  and we get contradiction as in the beginning of this step.  $\square$

We turn now to the proof of Lemma 4.3.

*Proof of Lemma 4.3.* We will prove the first inequality of (4.5), the second one can be done similarly. Let  $x_0, x_1 \in \mathbb{R}$  be such that  $x_1 > x_0$  and

$$\max(H(x_1), G(x_1)) \leq \max(H(x_0), G(x_0)). \quad (4.19)$$

Let  $\eta > 0$  and  $m = \max(H(x_0), G(x_0))$ . We define

$$M = \sup_{x \geq x_0} \{H(x) - m - \eta, G(x) - m - \eta\}.$$

By contradiction, we assume that  $M > 0$ .

**Case 1:**  $M$  is reached for some point  $\bar{x} > x_0$ .

**Case 1.1:**  $M = G(\bar{x}) - m - \eta$ . Writing the viscosity inequality, we get

$$0 \leq \alpha(H(\bar{x}) - G(\bar{x})).$$

Using that  $G(\bar{x}) \geq H(\bar{x})$ , we get a contradiction in the above inequality if  $G(\bar{x}) > H(\bar{x})$ . If  $G(\bar{x}) = H(\bar{x})$ , we deduce that  $M = H(\bar{x}) - m - \eta$  and so we can write the viscosity inequality using the function  $H$  at the point  $\bar{x}$  and we get

$$0 \leq \frac{\kappa}{\alpha} (V(G(\bar{x} + 1)) - V(G(\bar{x}))) + \lambda(H(\bar{x} + 1) - G(\bar{x} + 1)).$$

Using that  $G(\bar{x}) \geq H(\bar{x} + 1)$  and  $G(\bar{x}) \geq G(\bar{x} + 1)$ , we get a contradiction in the above inequality if  $G(\bar{x}) > G(\bar{x} + 1)$ . If  $G(\bar{x}) = G(\bar{x} + 1)$ , we deduce that  $M = G(\bar{x} + 1) - m - \eta$  and so we can write the viscosity inequality using the function  $G$  at the point  $\bar{x} + 1$ . Continuing in the same way, we construct a sequence  $x_n = \bar{x} + n$  such that  $M = G(x_n) - m - \eta = H(x_n) - m - \eta$ . We then define the following functions,

$$\begin{cases} G_n(x) = G(x + x_n) - m, \\ H_n(x) = H(x + x_n) - m. \end{cases}$$

Using the fact that  $G$  and  $H$  are bounded Lipschitz continuous functions, we have (up to passing to the limit on a subsequence)

$$\begin{cases} G_n \rightarrow G_\infty, \\ H_n \rightarrow H_\infty. \end{cases}$$

The stability of viscosity solutions implies (see [3]) that  $(G_\infty, H_\infty)$  solves (4.3). Let  $x \in \mathbb{R}$ : for  $n$  large enough, we have  $x + x_n \geq x_0$ . Using the definition of  $M$ , we have

$$G_n(0) = G(x_n) - m \geq G_n(x) = G(x + x_n) - m.$$

This implies that for any  $x \in \mathbb{R}$ , we have

$$\begin{cases} G_\infty(0) \geq G_\infty(x), \\ H_\infty(0) \geq H_\infty(x). \end{cases}$$

Using the strong comparison principle (Proposition 3.6) and that  $(G_\infty(0), H_\infty(0))$  solves (4.3), we get for all  $x \in \mathbb{R}$

$$\begin{cases} G_\infty(x) = G_\infty(0) \geq \eta > 0 \\ H_\infty(x) = H_\infty(0) \geq \eta > 0. \end{cases}$$

Equation (4.19) implies that  $H_n(x_1 - x_n) \leq 0$  and  $G_n(x_1 - x_n) \leq 0$ . Taking  $n \rightarrow +\infty$ , we get a contradiction.

**Case 1.2:**  $M = H(\bar{x}) - m - \eta$ . We proceed as above. The difference is that we will begin writing the viscosity solution satisfied by  $H$  at  $\bar{x}$ .

We deduce that  $M \leq 0$ . Sending  $\eta$  to zero, we get the desired result.

**Case 2:  $M$  is not reached.** In this case, there exists a sequence  $x_n \rightarrow +\infty$  such that

$$\begin{cases} G(x_n) - m - \eta \rightarrow M \text{ or} \\ H(x_n) - m - \eta \rightarrow M. \end{cases} \quad (4.20)$$

We define

$$\begin{cases} G_n(x) = G(x + x_n) - m - \eta, \\ H_n(x) = H(x + x_n) - m - \eta. \end{cases}$$

Up to subsequence, we have  $G_n \rightarrow G_\infty$  and  $H_n \rightarrow H_\infty$ . Assume that the second line in (4.20) is true (if the first one is true, we proceed similarly and even simpler): this implies that for all  $x \in \mathbb{R}$ ,

$$H_\infty(0) \geq H_\infty(x) \quad \text{and} \quad H_\infty(0) \geq G_\infty(x). \quad (4.21)$$

We have that

$$0 = H'_\infty(0) = \alpha(G_\infty(0) - H_\infty(0)) + \frac{\kappa}{\alpha}(V(G_\infty(1)) - V(G_\infty(0))) + \lambda(H_\infty(1) - G_\infty(1)). \quad (4.22)$$

Using (4.21) and the fact that  $\alpha z - \frac{\kappa}{\alpha}V(z)$  is strictly increasing, we get

$$0 \leq \frac{\kappa}{\alpha}(V(G_\infty(1)) - V(H_\infty(0))) + \lambda(H_\infty(0) - G_\infty(1)).$$

We deduce using (4.21) that  $G_\infty(1) = H_\infty(0)$ . Injecting the last equality in (4.22), we get

$$0 \leq \alpha(G_\infty(0) - H_\infty(0)) + \frac{\kappa}{\alpha}(V(H_\infty(0)) - V(G_\infty(0))).$$

Using again strict monotony, we get that  $H_\infty(0) = G_\infty(0)$ . Finally, the strict comparison principle implies

$$\begin{cases} G_\infty(x) = G_\infty(0) > 0, \\ H_\infty(x) = H_\infty(0) > 0. \end{cases}$$

This contradicts (4.19) and implies that  $M \leq 0$ . □

We are now ready to prove Theorem 2.6.

*Proof of Theorem 2.6.* Using Proposition 4.1, we deduce that the limit of  $G$  and  $H$  at  $\pm\infty$  exist: there exist  $\bar{a}, \tilde{a}, \bar{b}$  and  $\tilde{b}$  such that

$$\begin{aligned} G(+\infty) &= \bar{a}, & G(-\infty) &= \bar{b}, \\ H(+\infty) &= \tilde{a}, & H(-\infty) &= \tilde{b}. \end{aligned}$$

**Case 1: if  $G$  and  $H$  are strictly decreasing.** In this case  $\bar{a} < \bar{b}$  and  $\tilde{a} < \tilde{b}$ .

**Step 1:  $F$  is strictly decreasing.** We have for all  $y \in \mathbb{R}$ ,

$$F'(y) = -2\alpha F(y) + \frac{\kappa}{\alpha}V(G(y)) + \lambda F(y+1). \quad (4.23)$$

Using that  $F$  is bounded, we can proceed as in step 2 of the proof of Proposition 4.1 to prove that  $F'(y) \leq 0$  for all  $y \in \mathbb{R}$ . Let us assume by contradiction that  $F$  is not decreasing: there exists an open interval  $I$  such that  $F$  is constant in  $I$ . Let  $x_1 < x_2$  be two points in  $I$ : we have  $F'(x_1) = F'(x_2) = 0$  and  $F(x_1) = F(x_2)$ . Using (4.23), we obtain

$$\frac{\kappa}{\alpha}V(G(x_1)) + \lambda F(x_1+1) = \frac{\kappa}{\alpha}V(G(x_2)) + \lambda F(x_2+1)$$

which gives a contradiction because  $G$  is strictly decreasing and  $V$  is strictly increasing. We deduce that  $F$  is strictly decreasing.

**Step 2: Limits at  $\pm\infty$ .** Using that  $u'(y) = \alpha F(y)$ , we deduce that  $u'$  is strictly decreasing. Therefore  $u'(\pm\infty)$  exist and

$$\begin{cases} u'(+\infty) = G(+\infty) = \bar{a}, \\ u'(-\infty) = G(-\infty) = \bar{b}. \end{cases}$$

We have

$$\begin{aligned} u'(y+1) - u'(y) &= \alpha(\xi(y+1) - \xi(y)) - \alpha(u(y+1) - u(y)) \\ &= \alpha(H(y) - G(y)). \end{aligned}$$

Taking  $y$  to  $+\infty$ , we get

$$0 = \alpha(\bar{a} - \bar{a}),$$

and so  $\bar{a} = \bar{a}$ . Moreover, we have

$$\begin{aligned} \xi'(y) &= \alpha(u(y) - \xi(y)) + \frac{\kappa}{\alpha}V(u(y+1) - u(y)) + \lambda(\xi(y+1) - u(y+1)) \\ &= -u'(y) + \frac{\kappa}{\alpha}V(u(y+1) - u(y)) + \lambda\frac{u'(y+1)}{\alpha}. \end{aligned}$$

Taking  $y$  to  $+\infty$ , we get

$$\xi'(+\infty) = -\bar{a} + \frac{\kappa V(\bar{a}) + \lambda\bar{a}}{\alpha}.$$

Finally, using that  $\xi'(+\infty) = H(+\infty)$ , we deduce that  $-\bar{a} + \frac{\kappa V(\bar{a}) + \lambda\bar{a}}{\alpha} = \bar{a} = \bar{a}$ . Recalling that  $\alpha = \frac{\lambda + \kappa}{2}$ , we obtain that  $V(\bar{a}) = \bar{a}$ . Similarly, we can prove that  $\xi'(-\infty)$  exists and  $\xi'(-\infty) = \bar{b}$ .

**Step 3:  $p \leq V(p)$  if  $p \in [\bar{a}, \bar{b}]$ .** We define the functions

$$\hat{u}(t, y) = u(t + y) \quad \text{and} \quad \hat{\xi}(t, y) = \xi(t + y).$$

We remark that  $(\hat{u}, \hat{\xi})$  is a viscosity solution of

$$\begin{cases} \hat{u}_t(t, y) = \alpha \left( \hat{\xi}(t, y) - \hat{u}(t, y) \right), \\ \hat{\xi}_t(t, y) = \alpha \left( \hat{u}(t, y) - \hat{\xi}(t, y) \right) + \frac{\kappa}{\alpha}V(\hat{u}(t, y+1) - \hat{u}(t, y)) + \lambda(\hat{\xi}(t, y+1) - \hat{u}(t, y+1)). \end{cases}$$

We then rescale  $\hat{u}$  and  $\hat{\xi}$  in the following way

$$\hat{u}^\varepsilon(t, y) = \varepsilon \hat{u} \left( \frac{t}{\varepsilon}, \frac{y}{\varepsilon} \right) \quad \text{and} \quad \hat{\xi}^\varepsilon(t, y) = \varepsilon \hat{\xi} \left( \frac{t}{\varepsilon}, \frac{y}{\varepsilon} \right).$$

As  $\varepsilon$  goes to zero, we have that  $\hat{u}^\varepsilon, \hat{\xi}^\varepsilon \rightarrow u^0$  with

$$u^0(t, y) = \begin{cases} \bar{a}(y+t) & \text{if } y+t \geq 0, \\ \bar{b}(y+t) & \text{if } y+t < 0. \end{cases}$$

In [11], authors obtained a homogenization result for a traffic model (Theorem 1.3) considering  $n$  different velocities and sensitivities. Using the same arguments and even simpler (since we have

one velocity  $V$ ), we can prove the same homogenization result:  $\hat{u}^\varepsilon$  and  $\hat{\xi}^\varepsilon$  both converge uniformly towards  $u^0$  which is a viscosity solution of

$$u_t^0 = V(u_y^0).$$

Testing  $u^0$  from above with any test function of the form

$$\varphi(y+t) \quad \text{where } \bar{a} \leq \varphi'(0) \leq \bar{b},$$

we deduce that

$$p \leq V(p) \quad \text{for all } p \in [\bar{a}, \bar{b}] \text{ with equality for } p = \bar{a}, \bar{b}.$$

**Step 4:**  $p < V(p)$  if  $p \in (\bar{a}, \bar{b})$ . Assume by contradiction that there exists  $\bar{d} \in (\bar{a}, \bar{b})$  such that  $V(\bar{d}) = \bar{d}$ . Using Proposition 3.8, there exists  $(\tilde{u}, \tilde{\xi})$  solution of (2.7) such that

$$\lim_{|y| \rightarrow +\infty} (\tilde{u} - \bar{u}) = 0 \quad \text{and} \quad \lim_{|y| \rightarrow +\infty} (\tilde{\xi} - \bar{\xi}) = 0$$

with  $\bar{u}(y) = \min(\bar{a}y, \bar{b}y)$  and  $\bar{\xi}(y) = \min\left(\bar{a}y + \frac{\bar{a}}{\alpha}, \bar{b}y + \frac{\bar{b}}{\alpha}\right)$ . Using that  $V(\bar{d}) = \bar{d}$ , we define the following solution of (2.7),

$$v(y) = \bar{d}y \quad \text{and} \quad \chi(y) = \bar{d}y + \frac{\bar{d}}{\alpha}.$$

Moreover, we have

$$\lim_{|y| \rightarrow +\infty} (\tilde{u} - v) = -\infty \quad \text{and} \quad \lim_{|y| \rightarrow +\infty} (\tilde{\xi} - \chi) = -\infty. \quad (4.24)$$

Using Proposition 3.5, we obtain

$$\begin{cases} \tilde{u} \leq v & \text{on } \mathbb{R}, \\ \tilde{\xi} \leq \chi & \text{on } \mathbb{R}. \end{cases}$$

Using (4.24), we remark that  $m_1 = \min_{\mathbb{R}}(v - \tilde{u})$  and  $m_2 = \min_{\mathbb{R}}(\chi - \tilde{\xi})$  exist. For  $m = \min(m_1, m_2)$ , we have for all  $y \in \mathbb{R}$ ,

$$\tilde{u} + m \leq v, \quad (4.25)$$

$$\tilde{\xi} + m \leq \chi \quad (4.26)$$

with equality at some point  $x_0$  in (4.25) or (4.26). Assume that  $\tilde{\xi}(x_0) + m = \chi(x_0)$  (we proceed similarly if  $\tilde{u}(x_0) + m = v(x_0)$ ). Using Lemma 3.4, we get that

$$\tilde{\xi}(y) + m = \chi(y) \quad \text{for all } y \leq x_0.$$

Taking  $y \rightarrow -\infty$ , we get a contradiction and this implies that  $V(\bar{d}) > \bar{d}$ .

**Case 2: if  $G$  and  $H$  are strictly increasing.** Similar to the above case.

**Case 3: if  $G$  and  $H$  are constant functions.** We obtain that  $u'$  and  $\xi'$  are constant functions.  $\square$

## 5 Proof of Theorem 2.5

In this section, we prove the main result of this paper, Theorem 2.5. The idea of the proof is to construct the solution for a suitable non-local operator and then to pass to the limit.

*Proof of Theorem 2.5.* We will construct viscosity solutions  $(u, \xi)$  of (2.7) such that

$$G(y) = u(y+1) - u(y), \quad H(y) = \xi(y+1) - \xi(y)$$

are bounded functions on  $\mathbb{R}$ . In fact, using the classification theorem (see Theorem 2.6), and using assumption (A3), we get that  $G$  and  $H$  are strictly decreasing and

$$\begin{cases} G(y) \rightarrow a, & H(y) \rightarrow a & \text{as } y \rightarrow +\infty, \\ G(y) \rightarrow b, & H(y) \rightarrow b & \text{as } y \rightarrow -\infty \end{cases}$$

with  $a = \inf_{\mathbb{R}} G = \inf_{\mathbb{R}} H$  and  $b = \sup_{\mathbb{R}} G = \sup_{\mathbb{R}} H$ .

**Step 1: Construction of the approximated solution.** We modify the function  $V$  on  $\mathbb{R}$  in the following way: let  $\tilde{V} : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\tilde{V} = V$  in  $[a, b]$  and  $\tilde{V}(p) = a$  if  $p < a$ ,  $\tilde{V}(p) = b$  if  $p > b$ . To simplify the notations, we assume in the rest of the proof that  $\tilde{V} = V$ . For  $\eta > 0$  small, we define  $T : \mathbb{R} \rightarrow \mathbb{R}$  by

$$T(p) = \begin{cases} a - \eta & \text{if } p \leq a - \eta, \\ p & \text{if } a - \eta < p < b + \eta, \\ b + \eta & \text{if } p \geq b + \eta. \end{cases} \quad (5.1)$$

Let  $R > 0$ . We consider the following operators:

$$\begin{cases} G_R^1(y, u, w, p) &= \psi_R(y)T(\alpha(w - u)) + (1 - \psi_R(y))V(p), \\ G_R^2(y, u, v, w, z, p) &= \psi_R(y) \left( -T(\alpha(w - u)) + \frac{\kappa}{\alpha}V(v - u) + \frac{\lambda}{\alpha}T(\alpha(z - v)) \right) + (1 - \psi_R(y))V(p) \end{cases} \quad (5.2)$$

with  $\psi_R \in C^\infty(\mathbb{R})$  defined by

$$\psi_R(y) = \begin{cases} 1 & \text{if } |y| < R, \\ 0 & \text{if } |y| > R + 1. \end{cases}$$

We introduced the function  $T$  to obtain Lipschitz functions  $u_R, \xi_R$  and bounded interdistance. Remark that if  $w - u, z - v \in \left[ \frac{a}{\alpha}, \frac{b}{\alpha} \right]$  and if  $v - u \in [a, b]$ , then as  $R \rightarrow +\infty$ , the operators  $G_R^1$  and  $G_R^2$  converge to the right hand side term in equation (2.7), i.e.

$$\begin{cases} G_R^1(y, u, w, p) \rightarrow \alpha(w - u) & \text{as } R \rightarrow +\infty, \\ G_R^2(y, u, v, w, z, p) \rightarrow \alpha(u - w) + \frac{\kappa}{\alpha}V(v - u) + \lambda(z - v) & \text{as } R \rightarrow +\infty. \end{cases}$$

We will consider the following equation for  $y \in \mathbb{R}$ ,

$$\begin{cases} u'_R(y) &= G_R^1(y, u_R(y), \xi_R(y), u'_R(y)), \\ \xi'_R(y) &= G_R^2(y, u_R(y), u_R(y+1), \xi_R(y), \xi_R(y+1), \xi'_R(y)). \end{cases} \quad (5.3)$$

**Proposition 5.1.** *There exists a viscosity solution  $(u_R, \xi_R)$  of (5.3). Moreover,  $u_R$  and  $\xi_R$  are Lipschitz continuous.*

*Proof.* Let  $l$  be a big positive number,  $l \gg R$ . We define the following functions

$$u_R^+(y) = \begin{cases} ay & \text{if } y \geq 0, \\ by & \text{if } y < 0, \end{cases} \quad \xi_R^+(y) = \begin{cases} ay + \frac{a}{\alpha} & \text{if } y \geq -\frac{1}{\alpha}, \\ by + \frac{b}{\alpha} & \text{if } y < -\frac{1}{\alpha}, \end{cases} \quad (5.4)$$

$$u_R^-(y) = \begin{cases} a(y+l) - bl & \text{if } -l < y < 0, \\ by + (a-b)l & \text{if } 0 \leq y < l, \\ by & \text{if } y \leq -l, \\ ay & \text{if } y \geq l, \end{cases} \quad \xi_R^-(y) = \begin{cases} a(y+l) - bl + \frac{a}{\alpha} & \text{if } -l - \frac{1}{\alpha} < y < 0, \\ by + (a-b)l + \frac{b}{\alpha} & \text{if } -\frac{1}{\alpha} \leq y < l - \frac{1}{\alpha}, \\ by + \frac{b}{\alpha} & \text{if } y \leq -l - \frac{1}{\alpha}, \\ ay + \frac{a}{\alpha} & \text{if } y \geq l - \frac{1}{\alpha}. \end{cases} \quad (5.5)$$

By simple computations, we can verify that  $(u_R^+, \xi_R^+)$  is a viscosity super solution of (5.3) and that  $(u_R^-, \xi_R^-)$  is a viscosity sub solution of (5.3). By Perron method [17], we can construct a viscosity solution  $(u_R, \xi_R)$  of (5.3) satisfying for  $y \in \mathbb{R}$ ,

$$\begin{aligned} u_R^- &\leq u_R \leq u_R^+ \\ \xi_R^- &\leq \xi_R \leq \xi_R^+. \end{aligned}$$

In particular, from the definition of the sub and super solutions, (see (5.5) and (5.4)), we remark that

$$u_R(y) = \begin{cases} ay & \text{if } y \geq l, \\ by & \text{if } y \leq -l, \end{cases} \quad \xi_R(y) = \begin{cases} ay + \frac{a}{\alpha} & \text{if } y \geq l - \frac{1}{\alpha}, \\ by + \frac{b}{\alpha} & \text{if } y \leq -l - \frac{1}{\alpha}. \end{cases}$$

Moreover, using the definition of  $V$  and  $T$ , the functions  $u_R$  and  $\xi_R$  are lipshitz and satisfy

$$\begin{cases} a - \eta & \leq u'_R(y) \leq b + \eta, \\ 2a - b - \left(\frac{\lambda}{\alpha} + 1\right)\eta & \leq \xi'_R(y) \leq 2b - a + \left(\frac{\lambda}{\alpha} + 1\right)\eta. \end{cases}$$

□

**Remark 5.2.**  $(u_R^-, \xi_R^-)$  is a viscosity sub-solution of (5.3) because the operator is local for  $y \geq l - \frac{1}{\alpha}$  and  $y \leq -l$ .

**Step 2: Bounds for  $\xi_R - u_R$  and  $u'$ .**

**Proposition 5.3.** For all  $y \in \mathbb{R}$ , we have

$$\frac{a}{\alpha} \leq \xi_R(y) - u_R(y) \leq \frac{b}{\alpha}. \quad (5.6)$$

In particular, this implies that for all  $y \in \mathbb{R}$ ,

$$a \leq u'_R(y) \leq b. \quad (5.7)$$

*Proof.* We will prove the upper bound. The lower bound can be done in the same way. We define

$$M = \sup_{y \in \mathbb{R}} \left\{ \xi_R(y) - u_R(y) - \frac{b}{\alpha} \right\}.$$

Assume by contradiction that  $M > 0$ . For  $y > l$ , we have

$$\xi_R(y) - u_R(y) - \frac{b}{\alpha} = \frac{a}{\alpha} - \frac{b}{\alpha} < 0.$$



Similarly, For  $y < l - \frac{1}{\alpha}$ , we have

$$\xi_R(y) - u_R(y) - \frac{b}{\alpha} = \frac{b}{\alpha} - \frac{b}{\alpha} = 0.$$

We deduce that  $M$  is reached at some point  $\bar{y} \in \left[ l - \frac{1}{\alpha}, l \right]$ . We then define

$$M_\varepsilon = \sup_{x, y \in B_r(\bar{y})} \left\{ \xi_R(x) - u_R(y) - \frac{b}{\alpha} - \frac{(x-y)^2}{2\varepsilon} - (x-\bar{y})^2 \right\}$$

where  $B_R(\bar{y})$  is the open ball of radius  $r$ . We have  $M_\varepsilon \geq M > 0$  and  $M_\varepsilon$  is reached at some point  $(x, y)$ . Moreover, we have

$$0 \leq C_R - \frac{b}{\alpha} - \frac{(x-y)^2}{2\varepsilon}$$

where we use that  $\xi_R$  and  $u_R$  are continuous. This implies that  $|x-y| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Let  $\bar{x}$  be the common limit of  $x$  and  $y$  as  $\varepsilon$  goes to zero. We have

$$M \leq M_\varepsilon \leq \xi_R(x) - u_R(y) - \frac{b}{\alpha} - (x-\bar{y})^2.$$

Taking  $\varepsilon$  to zero, we get

$$\begin{aligned} M &\leq \xi_R(\bar{x}) - u_R(\bar{x}) - \frac{b}{\alpha} - (\bar{x} - \bar{y})^2 \\ &\leq M - (\bar{x} - \bar{y})^2 \end{aligned}$$

which implies that  $\bar{x} = \bar{y}$ . Writing the viscosity inequalities, we get

$$\begin{aligned} 2(x-\bar{y}) &\leq G_R^2 \left( x, u_R(x), u_R(x+1), \xi_R(x), \frac{x-y}{\varepsilon} + 2(x-\bar{y}) \right) - G_R^1 \left( y, u_R(y), \xi_R(y), \frac{x-y}{\varepsilon} \right) \\ &= \psi_R(x) \left( -T(\alpha(\xi_R(x) - u_R(x))) + \frac{\kappa}{\alpha} V(u_R(x+1) - u_R(x)) + \frac{\lambda}{\alpha} T(\alpha(\xi_R(x+1) - u_R(x+1))) \right) \\ &\quad + (1 - \psi_R(x)) V \left( \frac{x-y}{\varepsilon} + 2(x-\bar{y}) \right) - \psi_R(y) T(\alpha(\xi_R(y) - u_R(y))) - (1 - \psi_R(y)) V \left( \frac{x-y}{\varepsilon} \right) \\ &\leq \psi_R(x) \left( -T(\alpha(\xi_R(x) - u_R(x))) + \frac{\kappa}{\alpha} V(u_R(x+1) - u_R(x)) + \frac{\lambda}{\alpha} T(\alpha(\xi_R(x+1) - u_R(x+1))) \right) \\ &\quad - \psi_R(y) T(\alpha(\xi_R(y) - u_R(y))) + b|\psi'_R|_\infty |x-y| + 2\|V'\|_\infty |x-y| \end{aligned}$$

where we use in the last line the fact that  $V \leq b$  and  $V, \psi_R$  are lipshitz. Taking  $\varepsilon$  to zero, we get

$$0 \leq \psi_R(\bar{y}) \left( -2T(\alpha(\xi_R(\bar{y}) - u_R(\bar{y}))) + \frac{\kappa}{\alpha} V(u_R(\bar{y}+1) - u_R(\bar{y})) + \frac{\lambda}{\alpha} T(\alpha(\xi_R(\bar{y}+1) - u_R(\bar{y}+1))) \right).$$

Using that  $\xi_R(\bar{y}) - u_R(\bar{y}) \geq \xi_R(\bar{y}+1) - u_R(\bar{y}+1)$ , we obtain

$$0 \leq \psi_R(\bar{y}) \left( \left( \frac{\lambda}{\alpha} - 2 \right) T(\alpha(\xi_R(\bar{y}) - u_R(\bar{y}))) + \frac{\kappa}{\alpha} V(u_R(\bar{y}+1) - u_R(\bar{y})) \right).$$

Using that  $2\alpha > \lambda$ ,  $\alpha = \frac{\kappa + \lambda}{2}$ ,  $\xi_R(\bar{y}) - u_R(\bar{y}) > \frac{b}{\alpha}$  and the definition of  $T$  (see (5.1)), we get a contradiction. □

**Step 3: Control of the oscillations of  $u_R$  and  $\xi_R$ .** The goal of the next two propositions is to exploit the definition of the sub-solution  $(u_R^-, \xi_R^-)$  and the super-solution  $(u_R^+, \xi_R^+)$  at infinity in order to ensure that  $u_R(y+1) - u_R(y)$  and  $\xi_R(y+1) - \xi_R(y)$  are not constant functions equal to  $a$  or  $b$ .

**Proposition 5.4.** *Let  $(u_R, \xi_R)$  be the solution of (5.3) provided by Proposition 5.1. Let  $y_0 \in (-1, -1/2)$  and let  $\gamma, \delta \in (0, 1)$  small such that*

$$\begin{cases} \delta \left(1 + \frac{\gamma}{\alpha}\right) \leq y_0(a-b), & \gamma \leq \frac{\alpha}{2}, & \min_{p \in [a, a+2\delta]} V'(p) > 1 \\ \min_{p \in [a, a+2\delta]} V'(p) > \frac{\gamma \left(2\alpha(1 + \tanh(\gamma)) \left(1 + \frac{\gamma}{\alpha}\right) - \lambda(1 - \tanh(\gamma))\right)}{\kappa \tanh(\gamma)}. \end{cases}$$

Then, we have

$$\begin{cases} u_R(y) - u_R(y_0) \geq a(y - y_0) + \delta \tanh(\gamma y), \\ \xi_R(y) - u_R(y_0) \geq a(y - y_0) + \delta \tanh(\gamma y) + \frac{a + \delta\gamma(1 - \tanh^2(\gamma y))}{\alpha} \end{cases}$$

for all  $y \geq y_0$ .

*Proof.* The proof of Proposition 5.4 is similar and even simpler than the proof of the next proposition.  $\square$

**Proposition 5.5.** *Let  $(u_R, \xi_R)$  be the solution of (5.3) provided by Proposition 5.1. Let  $\delta, \gamma \in (0, 1)$  small such that*

$$\frac{b-a}{\alpha} > \delta, \quad b > b - 2\delta > a, \quad 1 > \max_{p \in [b-2\delta, b]} V'(p) \quad (5.8)$$

and

$$\gamma < \frac{\alpha}{2}, \quad \frac{\gamma \left(2\alpha(1 - \tanh(\gamma)) \left(1 - \frac{\gamma}{\alpha}\right) - \lambda(1 + \tanh(\gamma))\right)}{\kappa \tanh(\gamma)} > \max_{p \in [b-2\delta, b]} V'(p). \quad (5.9)$$

We have the following:

i) either there exists  $y \in (-1, 1/2)$  such that

$$u_R(y+1) - u_R(y) \leq b + \delta (\tanh(\gamma y) - \tanh(\gamma(y+1))) \quad (5.10)$$

or

ii) for all  $y_0 \in (0, 1/2)$ , we have

$$\begin{cases} \xi_R(y_0) - u_R(y) \leq b(y_0 - y) + \delta \tanh(\gamma y) + \frac{b}{\alpha}, \\ \xi_R(y_0) - \xi_R(y) \leq b(y_0 - y) + \delta \tanh(\gamma y) + \frac{\delta\gamma(1 - \tanh^2(\gamma y))}{\alpha}. \end{cases} \quad (5.11)$$

for all  $y \leq y_0$ .

**Remark 5.6.** *The second condition in (5.9) is well defined since we have*

$$\lim_{\gamma \rightarrow 0} \frac{\gamma \left(2\alpha(1 - \tanh(\gamma)) \left(1 - \frac{\gamma}{\alpha}\right) - \lambda(1 + \tanh(\gamma))\right)}{\kappa \tanh(\gamma)} = 1.$$

*Proof of Proposition 5.5.* Assume that  $i$  is not true, i.e for all  $y \in (-1, 1/2)$ ,

$$u_R(y+1) - u_R(y) > b + \delta (\tanh(\gamma y) - \tanh(\gamma(y+1))). \quad (5.12)$$

We will prove that  $ii$  is true. Let  $y_0 \in (0, 1/2)$ . We define the following functions

$$\begin{cases} \varphi_1(y) = \xi_R(y_0) - u_R(y) - b(y_0 - y) - \delta \tanh(\gamma y) - \frac{b}{\alpha}, \\ \varphi_2(y) = \xi_R(y_0) - \xi_R(y) - b(y_0 - y) - \delta \tanh(\gamma y) - \frac{\delta \gamma (1 - \tanh^2(\gamma y))}{\alpha}. \end{cases}$$

We claim that  $M = \sup_{y \leq y_0} (\varphi_1(y), \varphi_2(y)) \leq 0$ . Assume by contradiction that  $M > 0$ . We will first show that  $M$  is reached at some point  $x$ . In fact, if  $y < -l$ , and using that  $u_R(y) = by$ , we get

$$\begin{aligned} \varphi_1(y) &= \xi_R(y_0) - by_0 - \delta \tanh(\gamma y) - \frac{b}{\alpha} \\ &\leq ay_0 + \frac{a}{\alpha} - by_0 - \frac{b}{\alpha} + \delta \\ &\leq 0 \end{aligned}$$

where we use in the second line that  $\xi_R(y_0) \leq \xi_R^+(y_0)$  and (5.8) in the third line. Similarly, using  $\xi_R(y) = by + \frac{b}{\alpha}$  for  $x \leq -l - \frac{1}{\alpha}$ , we have

$$\begin{aligned} \varphi_2(y) &= \xi_R(y_0) - by_0 - \frac{b}{\alpha} - \delta \tanh(\gamma y) - \frac{\delta \gamma (1 - \tanh^2(\gamma y))}{\alpha} \\ &\leq ay_0 + \frac{a}{\alpha} - by_0 - \frac{b}{\alpha} + \delta \\ &\leq 0 \end{aligned}$$

where we use (5.8) in the third line.

**Case 1: if  $M = \varphi_1(x)$ .** We first show that  $x \neq y_0$ . If  $x = y_0$ , we get

$$0 < \xi_R(y_0) - u_R(y_0) - \delta \tanh(\gamma y_0) - \frac{b}{\alpha} \leq -\delta \tanh(\gamma y_0) < 0$$

where we use (5.6). Writing the viscosity super-solution inequality, we have

$$\begin{aligned} b - \delta \gamma (1 - \tanh^2(\gamma x)) &\geq \psi_R(x) \alpha (\xi_R(x) - u_R(x)) \\ &\quad + (1 - \psi_R(x)) V(b - \delta \gamma (1 - \tanh^2(\gamma x))). \end{aligned} \quad (5.13)$$

Using that  $\varphi_1(x) \geq \varphi_2(x)$ , we have

$$\xi_R(x) - u_R(x) \geq \frac{b - \delta \gamma (1 - \tanh^2(\gamma x))}{\alpha}.$$

Injecting the above inequality in (5.13), we get

$$\begin{aligned} b - \delta \gamma (1 - \tanh^2(\gamma x)) &\geq \psi_R(x) (b - \delta \gamma (1 - \tanh^2(\gamma x))) \\ &\quad + (1 - \psi_R(x)) V(b - \delta \gamma (1 - \tanh^2(\gamma x))). \end{aligned} \quad (5.14)$$

Moreover, using (5.8) and that  $V(p) > p$  for  $p \in (a, b)$ , we obtain a contradiction in (5.14) if  $x < -R$ . It remains to treat the case where  $x \geq -R$ . In that case and again using (5.14), we

obtain  $\xi_R(x) - u_R(x) = \frac{b - \delta\gamma(1 - \tanh^2(\gamma x))}{\alpha}$  which implies that  $M$  is reached for  $\varphi_2(x)$ . Writing the viscosity inequality, we get

$$b - f(\delta, \gamma) \geq \psi_R(x) \left( \alpha(u_R(x) - \xi_R(x)) + \frac{\kappa}{\alpha} V(u_R(x+1) - u_R(x)) + \lambda(\xi_R(x+1) - u_R(x+1)) \right) + (1 - \psi_R(x))V(b - f(\delta, \gamma)) \quad (5.15)$$

with

$$f(\delta, \gamma) = \delta\gamma(1 - \tanh^2(\gamma x)) \left( 1 - \frac{2\gamma \tanh(\gamma x)}{\alpha} \right). \quad (5.16)$$

Using that  $\varphi_2(x) \geq \varphi_1(x)$ , we get

$$u_R(x) \geq \xi_R(x) + \frac{\delta\gamma(1 - \tanh^2(\gamma x)) - b}{\alpha}. \quad (5.17)$$

**If  $x + 1 \leq y_0$ :** using that  $\varphi_2(x) \geq \varphi_1(x+1)$  and  $\varphi_2(x) \geq \varphi_2(x+1)$ , we have

$$u_R(x+1) \geq \xi_R(x) + b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+1))) + \frac{\delta\gamma(1 - \tanh^2(\gamma x)) - b}{\alpha} \quad (5.18)$$

and

$$\xi_R(x+1) \geq \xi_R(x) + b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+1))) + \frac{\delta\gamma}{\alpha} (\tanh^2(\gamma(x+1)) - \tanh^2(\gamma x)). \quad (5.19)$$

From (5.7), we know that  $u_R(x+1) - u_R(x) \in [a, b]$ . Hence, we can use assumption (A5) which provides that  $\alpha z_1 - \frac{\kappa}{\alpha} V(z_2 - z_1)$  and  $\frac{\kappa}{\alpha} V(z_2 - z_1) - \lambda z_2$  are strictly increasing. Injecting (5.17), (5.18) and (5.19) in (5.15), we obtain

$$b - f(\delta, \gamma) \geq \psi_R(x) \left( \delta\gamma(1 - \tanh^2(\gamma x)) - b + \frac{\kappa}{\alpha} V(b - g(\delta, \gamma)) + \frac{\lambda b}{\alpha} - \lambda h(\delta, \gamma) \right) + (1 - \psi_R(x))V(b - f(\delta, \gamma))$$

with

$$g(\delta, \gamma) = \delta (\tanh(\gamma(x+1)) - \tanh(\gamma x)), \quad (5.20)$$

$$h(\delta, \gamma) = \frac{\delta\gamma}{\alpha} (1 - \tanh^2(\gamma(x+1))). \quad (5.21)$$

To get a contradiction, we claim that

$$V(b - f(\delta, \gamma)) > b - f(\delta, \gamma) \quad (5.22)$$

and

$$\delta\gamma(1 - \tanh^2(\gamma x)) - b + \frac{\kappa}{\alpha} V(b - g(\delta, \gamma)) + \frac{\lambda b}{\alpha} - \lambda h(\delta, \gamma) > b - f(\delta, \gamma). \quad (5.23)$$

To get (5.22), it's sufficient to remark that  $0 \leq f(\delta, \gamma) \leq \delta\gamma \left( 1 + \frac{2\gamma}{\alpha} \right) \leq 2\delta$  where we use (5.8) and (5.9). Using the fact that  $V(p) > p$  for  $p \in (a, b)$ , we obtain (5.22). We will prove now (5.23). We have

$$V(b - g(\delta, \gamma)) = V(b) - g(\delta, \gamma)V'(p_0) = b - g(\delta, \gamma)V'(p_0) \quad (5.24)$$

with  $p_0 \in [b - g(\delta, \gamma), b]$ . Using (5.24), we obtain (5.23) if we have

$$V'(p_0) < \frac{\alpha (f(\delta, \gamma) + \delta\gamma(1 - \tanh^2(\gamma x)) - \lambda h(\delta, \gamma))}{\kappa g(\delta, \gamma)} \quad (5.25)$$

with  $f, g$  and  $h$  defined in (5.16), (5.20) and (5.21).

First, let us remark that

$$\begin{aligned} g(\delta, \gamma) &= \delta (\tanh(\gamma(x+1)) - \tanh(\gamma x)) \\ &= \frac{\delta \tanh(\gamma)(1 - \tanh^2(\gamma x))}{1 + \tanh(\gamma x) \tanh(\gamma)}. \end{aligned}$$

On one hand, we have

$$\begin{aligned} \frac{f(\delta, \gamma) + \delta\gamma(1 - \tanh^2(\gamma x))}{g(\delta, \gamma)} &= \frac{2\gamma(1 + \tanh(\gamma x) \tanh(\gamma))}{\tanh(\gamma)} \left(1 - \frac{\gamma \tanh(\gamma x)}{\alpha}\right) \\ &\geq \frac{2\gamma(1 - \tanh(\gamma))}{\tanh(\gamma)} \left(1 - \frac{\gamma}{\alpha}\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g(\delta, \gamma) &= \delta (\tanh(\gamma(x+1)) - \tanh(\gamma x)) \\ &= \frac{\delta \tanh(\gamma)(1 - \tanh^2(\gamma(x+1)))}{1 - \tanh(\gamma(x+1)) \tanh(\gamma)}. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{h(\delta, \gamma)}{g(\delta, \gamma)} &= \frac{\gamma(1 - \tanh(\gamma(x+1)) \tanh(\gamma))}{\alpha \tanh(\gamma)} \\ &\leq \frac{\gamma(1 + \tanh(\gamma))}{\alpha \tanh(\gamma)}. \end{aligned}$$

We deduce that

$$\begin{aligned} \frac{\alpha (f(\delta, \gamma) + \delta\gamma(1 - \tanh^2(\gamma x)) - \lambda h(\delta, \gamma))}{\kappa g(\delta, \gamma)} &\geq \frac{\gamma \left(2\alpha(1 - \tanh(\gamma)) \left(1 - \frac{\gamma}{\alpha}\right) - \lambda(1 + \tanh(\gamma))\right)}{\kappa \tanh(\gamma)} \\ &> V'(p_0) \end{aligned}$$

where we use (5.9) and this gives us the desired contradiction.

**If  $x + 1 > y_0$ .** In this case, we have  $-1 < y_0 - 1 < x < y_0 < 1/2$ . We proceed as above: the only difference is that we use (5.12).

**Case 2: if  $M = \varphi_2(x)$ .** We proceed as above after showing easily that  $x \neq y_0$ . □

**Step 4: passing to the limit.** We define the following functions:

$$\begin{cases} \bar{u}_R(y) = u_R(y) - u_R(0), \\ \bar{\xi}_R(y) = \xi_R(y) - \xi_R(0). \end{cases}$$

Using Ascoli-Arzelà theorem, and up to a sub-sequence, we have locally uniformly  $\bar{u}_R \rightarrow u$  and  $\bar{\xi}_R \rightarrow \xi$  as  $R \rightarrow +\infty$ . The stability of viscosity solutions implies that  $(u, \xi)$  is a viscosity solution of (2.7). Moreover, from Proposition 3.3, we know that  $u, \xi \in C^2(\mathbb{R})$ . Using steps 1 and 2, we have

$$\begin{cases} a \leq G(y) = u(y+1) - u(y) \leq b, \\ a \leq H(y) = \xi(y+1) - \xi(y) \leq b, \\ \frac{a}{\alpha} \leq F(y) = \xi(y) - u(y) \leq \frac{b}{\alpha}. \end{cases}$$

Using step 4, we have for  $y_0 \in (-1, -1/2)$

$$u(y_0+1) - u(y_0) \geq a + \delta \tanh(\gamma(y_0+1)) > a \quad (5.26)$$

and

i) there exists  $y \in (-1, 1/2)$ ,

$$u(y+1) - u(y) \leq b + \delta (\tanh(\gamma y) - \tanh(\gamma(y+1))) < b \quad (5.27)$$

or there exists  $y_0 \in \left(0, \min\left(-\frac{1}{\alpha} + 1, \frac{1}{2}\right)\right)$ ,

ii)

$$\xi(y_0) - \xi(y_0-1) \leq b + \delta \tanh(\gamma(y_0-1)) + \frac{\delta \gamma (1 - \tanh^2(\gamma(y_0-1)))}{\alpha} < b. \quad (5.28)$$

Using Theorem 2.6, we have that

$$\tilde{a} = u'(+\infty) = \xi'(+\infty) = G(+\infty) = H(+\infty) \leq \tilde{b} = u'(-\infty) = \xi'(-\infty) = G(-\infty) = H(-\infty) \quad (5.29)$$

with  $\tilde{a}, \tilde{b} \in \{a, b\}$ . If we have equality in (5.29), we deduce using Proposition 4.1 that  $G$  and  $H$  are constant which contradicts inequalities (5.26), (5.27) and (5.28). This implies that

$$a = u'(+\infty) = \xi'(+\infty) = G(+\infty) = H(+\infty) < b = u'(-\infty) = \xi'(-\infty) = G(-\infty) = H(-\infty).$$

Finally, using Proposition 6.1, there exists a constant  $C > 0$  such that

$$|u - \bar{u}| \leq C \quad \text{and} \quad |\xi - \bar{\xi}| \leq C.$$

**Remark 5.7.** We can easily show that for any  $x_0 \in \left(-1, \min\left(-\frac{1}{\alpha}, -\frac{1}{2}\right)\right)$ , we have

$$\frac{\tanh(\gamma x_0)}{\tanh^2(\gamma x_0) - 1} > \frac{\gamma}{\alpha} \quad \text{for any } \gamma > 0.$$

In fact, let  $f(\gamma) = \frac{\tanh(\gamma x_0)}{\tanh^2(\gamma x_0) - 1} - \frac{\gamma}{\alpha}$ . We have

$$f'(\gamma) = -\left(x_0 + 2x_0 \sinh^2(\gamma x_0) + \frac{1}{\alpha}\right).$$

In addition, we have  $f(0) = 0$  and  $f'(0) > 0$  because  $x_0 < -\frac{1}{\alpha}$ . Let us also mention that  $\alpha > 1$ . In fact, we know that  $\kappa$  and  $\alpha$  satisfies (A5). Hence, we have (for  $T = 1$ )

$$\frac{(\kappa + \lambda)^2}{4\kappa} > V'(a) > 1$$

and

$$\frac{\lambda(\kappa + \lambda)}{2\kappa} < V'(b) < 1.$$

This implies that

$$\kappa > 1 \text{ and } 2\sqrt{\kappa} - \kappa < \lambda < \frac{1}{2} \left( \sqrt{\kappa(\kappa + 8)} - \kappa \right).$$

$$\text{Therefore, } \alpha = \frac{\kappa + \lambda}{2} > \sqrt{\kappa} > 1.$$

□

## 6 Asymptotics and uniqueness

In this section, we complete the proof of Theorem 2.5 by studying the asymptotic behavior of the solution of (2.7) and then proving its uniqueness. We start with the asymptotic behavior. We have the following proposition.

**Proposition 6.1.** *Assume (A). Let  $(u, \xi)$  be a solution of (2.7) such that  $u, \xi \in C^2(\mathbb{R})$  and let  $G(y) = u(y+1) - u(y)$ ,  $H(y) = \xi(y+1) - \xi(y)$  and  $F(y) = \xi(y) - u(y)$ . We assume that  $G, H$  and  $F$  are non-increasing and that*

$$\begin{cases} u'(+\infty) = \xi'(+\infty) = G(+\infty) = H(+\infty) = a, \\ u'(-\infty) = \xi'(-\infty) = G(-\infty) = H(-\infty) = b \end{cases} \quad (6.1)$$

and that for  $y \in \mathbb{R}$ ,

$$a \leq G(y), H(y) \leq b, \quad (6.2)$$

and

$$\frac{a}{\alpha} \leq F(y) \leq \frac{b}{\alpha}. \quad (6.3)$$

Then there exists  $K, \gamma > 0$  and  $c_1, c_2 \in \mathbb{R}$  such that

$$\begin{cases} |u(y) - ay - c_1| \leq Ke^{-\gamma y} & \text{and} & |\xi(y) - ay - c_1| \leq Ke^{-\gamma y} & \text{for } y \geq 0, \\ |u(y) - by - c_2| \leq Ke^{\gamma y} & \text{and} & |\xi(y) - by - c_2| \leq Ke^{\gamma y} & \text{for } y \leq 0. \end{cases} \quad (6.4)$$

The proof of this proposition is a direct consequence of the following lemma

**Lemma 6.2.** *Assume (A). Let  $(u, \xi)$  be a solution of (2.7) satisfying (6.1) such that  $u, \xi \in C^2(\mathbb{R})$  and  $(G, H)$  satisfying (6.2). We remark that  $(G, H)$  satisfies*

$$\begin{cases} G'(y) = \alpha(H(y) - G(y)) & y \in \mathbb{R} \\ H'(y) = \alpha(G(y) - H(y)) + \frac{\kappa}{\alpha} (V(G(y+1)) - V(G(y))) + \lambda(H(y+1) - G(y+1)) & y \in \mathbb{R}. \end{cases} \quad (6.5)$$

Recalling that  $V'(a) > 1 > V'(b)$ , let  $\varepsilon > 0$  be small enough such that

$$\begin{cases} \min_{[a-(1+\varepsilon)\varepsilon, a+(1+\varepsilon)\varepsilon]} V'(p) > 1, \\ \max_{[b-(1+\varepsilon)\varepsilon, b+(1+\varepsilon)\varepsilon]} V'(p) < 1. \end{cases}$$

We have the following:

1) let  $\gamma \in (0, 1)$  be small enough such that

$$\min_{[a-(1+e)\varepsilon, a+(1+e)\varepsilon]} V'(p) > \frac{\gamma(-\lambda e^\gamma + 2\alpha + \gamma)}{\kappa(1 - e^{-\gamma})}.$$

Then there exists a constant  $C > 0$  such that for all  $y \geq 0$ ,

$$\begin{cases} G(y) \leq a + Ce^{-\gamma y}, \\ H(y) \leq a + Ce^{-\gamma y}. \end{cases} \quad (6.6)$$

2) Let  $\gamma \in (0, 1)$  be small enough such that

$$\max_{[b-(1+e)\varepsilon, b+(1+e)\varepsilon]} V'(p) < \frac{\gamma(\lambda e^\gamma - 2\alpha - \gamma)}{\kappa(1 - e^\gamma)}. \quad (6.7)$$

Then there exists a constant  $C > 0$  such that for all  $y \leq 0$ ,

$$\begin{cases} G(y) \geq b - Ce^{\gamma y}, \\ H(y) \geq b - Ce^{\gamma y}. \end{cases} \quad (6.8)$$

**Remark 6.3.** We recall that  $\alpha = \frac{\kappa + \lambda}{2}$ . Therefore, we have that

$$\lim_{\gamma \rightarrow 0} \frac{\gamma(\lambda e^\gamma - 2\alpha - \gamma)}{\kappa(1 - e^\gamma)} = 1.$$

*Proof of Lemma 6.2.* We will only prove part 2) since the proof of part 1) can be done in the same way (even simpler). Using (6.1), let  $y_0 < 0$  be such that for all  $y \leq y_0$ ,

$$\begin{cases} b - \varepsilon \leq G(y) \leq b, \\ b - \varepsilon \leq H(y) \leq b. \end{cases}$$

We will prove that for  $y \leq y_0$ ,

$$\begin{cases} G(y) \geq b - C_1 e^{\gamma(y-y_0)}, \\ H(y) \geq b - C_1 e^{\gamma(y-y_0)} - \frac{C_1 \gamma}{\alpha} e^{\gamma(y-y_0)} \end{cases} \quad (6.9)$$

with  $C_1 > b - a$ .

If (6.9) is true, we obtain (6.8) for all  $y \leq 0$  because we can easily check that for  $y_0 < y \leq 0$ ,

$$\begin{aligned} G(y) &\geq a \geq b - C_1 \geq b - C_1 e^{\gamma(y-y_0)} \\ H(y) &\geq a \geq b - C_1 \geq b - C_1 \left(1 + \frac{\gamma}{\alpha}\right) e^{\gamma(y-y_0)}. \end{aligned}$$

We define the following functions

$$\begin{cases} \varphi_1(y) = b - C_1 e^{\gamma(y-y_0)} - G(y), \\ \varphi_2(y) = b - C_1 e^{\gamma(y-y_0)} - \frac{C_1 \gamma}{\alpha} e^{\gamma(y-y_0)} - H(y). \end{cases}$$

We then define

$$M = \sup_{y \leq y_0} (\varphi_1(y), \varphi_2(y)).$$

We will prove that  $M \leq 0$ . Assume by contradiction that  $M > 0$ . Using that  $G(y), H(y) \rightarrow b$  as  $y \rightarrow -\infty$ , we deduce that  $M$  is reached at some point  $x$ .



**Case 1: if  $M = \varphi_1(x)$ .** If  $x = y_0$ , we get

$$0 < b - C_1 - G(x) \leq b - a - C_1 < 0.$$

We deduce that  $x \neq y_0$  and writing the viscosity inequality, we get

$$-C_1\gamma e^{\gamma(x-y_0)} \geq \alpha(H(x) - G(x)).$$

Using that  $\varphi_1(x) \geq \varphi_2(x)$ , we obtain

$$-C_1\gamma e^{\gamma(x-y_0)} \geq \alpha(H(x) - G(x)) \geq -C_1\gamma e^{\gamma(x-y_0)}.$$

This means that  $H(x) = G(x) - \frac{C_1\gamma e^{\gamma(x-y_0)}}{\alpha}$  and  $M = \varphi_2(x) > 0$ . Writing the viscosity inequality, we get

$$-C_1\gamma e^{\gamma(x-y_0)} - C_1\frac{\gamma^2}{\alpha}e^{\gamma(x-y_0)} \geq \alpha(G(x) - H(x)) + \frac{\kappa}{\alpha}(V(G(x+1)) - V(G(x))) + \lambda(H(x+1) - G(x+1)).$$

We claim that  $x+1 < y_0$ . If  $x+1 \geq y_0$ , we get

$$C_1e^{\gamma(x-y_0)} + C_1\frac{\gamma}{\alpha}e^{\gamma(x-y_0)} \geq C_1e^{-\gamma} + C_1\frac{\gamma}{\alpha}e^{-\gamma} > b - a$$

for  $C_1$  big enough. This implies that

$$H(x) \geq a > b - C_1e^{\gamma(x-y_0)} - C_1\frac{\gamma}{\alpha}e^{\gamma(x-y_0)}$$

which contradicts the fact that  $\varphi_2(x) > 0$ . We deduce that  $x+1 < y_0$  and using that

$$\varphi_1(x) \leq \varphi_2(x), \varphi_1(x+1) \leq \varphi_2(x) \text{ and } \varphi_2(x+1) \leq \varphi_2(x)$$

we obtain that

$$\begin{cases} G(x) \geq H(x) + \frac{C_1\gamma}{\alpha}e^{\gamma(x-y_0)} \\ G(x+1) \geq H(x) + C_1e^{\gamma(x-y_0)}\left(\frac{\gamma}{\alpha} + 1 - e^\gamma\right), \\ H(x+1) \geq H(x) + C_1e^{\gamma(x-y_0)}(1 - e^\gamma)\left(\frac{\gamma}{\alpha} + 1\right). \end{cases}$$

Using assumptions (A2)-(A5) and the above inequalities, we get

$$\begin{aligned} & -C_1\gamma e^{\gamma(x-y_0)} - C_1\frac{\gamma^2}{\alpha}e^{\gamma(x-y_0)} \geq \\ & C_1\gamma e^{\gamma(x-y_0)} + \frac{\kappa}{\alpha}(V(H(x) + f(\gamma) + g(\gamma)) - V(H(x) + g(\gamma))) - \lambda e^\gamma g(\gamma) \end{aligned} \quad (6.10)$$

with

$$\begin{cases} f(\gamma) = C_1e^{\gamma(x-y_0)}(1 - e^\gamma), \\ g(\gamma) = \frac{C_1\gamma}{\alpha}e^{\gamma(x-y_0)}. \end{cases} \quad (6.11)$$

Using (6.10), we remark that we will get a contradiction if we prove

$$V(H(x) + f(\gamma) + g(\gamma)) - V(H(x) + g(\gamma)) > C_1\gamma e^{\gamma(x-y_0)}\left(\frac{\lambda e^\gamma}{\alpha} - \frac{\gamma}{\alpha} - 2\right). \quad (6.12)$$

We have

$$V(H(x) + f(\gamma) + g(\gamma)) - V(H(x) + g(\gamma)) = V'(p)f(\gamma)$$

with

$$p \in [H(x) + f(\gamma) + g(\gamma), H(x) + g(\gamma)].$$

To get (6.12), we have to prove that

$$V'(p) < \frac{\gamma(\lambda e^\gamma - 2\alpha - \gamma)}{\kappa(1 - e^\gamma)}. \quad (6.13)$$

Using that  $\varphi_2(x) > 0$ , we get

$$H(x) < b - C_1 e^{\gamma(x-y_0)} - \frac{\gamma}{\alpha} C_1 e^{\gamma(x-y_0)}.$$

Using that  $H(y) \geq b - \varepsilon$  for  $y \leq y_0$ , we deduce that

$$\varepsilon > C_1 e^{\gamma(x-y_0)} \left(1 + \frac{\gamma}{\alpha}\right).$$

Recalling the definition of  $f$  and  $g$  (see (6.11)), and using that  $b - \varepsilon \leq H(x) \leq b + \varepsilon$ , we obtain

$$\begin{aligned} H(x) + f(\gamma) + g(\gamma) &\geq b - (1 + e)\varepsilon, \\ H(x) + g(\gamma) &\leq b + 2\varepsilon. \end{aligned}$$

Using the condition on  $\gamma$  in (6.7), we get (6.13). We deduce that  $M \leq 0$  and in particular, we get (6.8).

**Case 2:**  $M = \varphi_2(x)$ . We proceed as above. □

*Proof of Proposition 6.1.* We will only prove the first line in (6.4). The proof of the second line is similar. We have for  $x \geq 0$ ,

$$\begin{aligned} \xi(x) - \xi(0) &= \int_0^x \xi'(s) ds = \int_0^x \alpha(u(s) - \xi(s)) + \frac{\kappa}{\alpha} V(u(s+1) - u(s)) + \lambda(\xi(s+1) - u(s+1)) ds \\ &\leq \int_0^x (\lambda - \alpha)(\xi(s) - u(s)) + \frac{\kappa}{\alpha} V(u(s+1) - u(s)) ds \\ &\leq (\lambda - \alpha) \int_0^x \frac{a}{\alpha} + \frac{\kappa}{\alpha} \int_0^x V(a + C e^{-\gamma s}) ds \\ &\leq (\lambda - \alpha) \frac{a}{\alpha} x + \frac{\kappa}{\alpha} \int_0^x V(a + C e^{-\gamma s}) - V(a) + V(a) ds \\ &\leq \frac{(\lambda - \alpha + \kappa)ax}{\alpha} + \frac{\kappa}{\alpha} \|V'\|_\infty C \int_0^x e^{-\gamma s} ds \\ &= ax + \frac{\kappa}{\alpha} \|V'\|_\infty \frac{C}{\gamma} (1 - e^{-\gamma x}) \end{aligned}$$

where we use that  $F(y) = \xi(y) - u(y)$  is non-increasing in the second line,  $\alpha > \lambda$  and (6.3) in the third line, the fact that  $V(a) = a$  in the fifth line and that  $\frac{\lambda - \alpha + \kappa}{\alpha} = 1$  in the last line. □

It remains to prove the uniqueness of the solution (part *ii*) in Theorem 2.5). This is a direct consequence of the following proposition.

**Proposition 6.4.** *Assume (A). Let  $(u, \xi)$  be a solution of (2.7) with  $u, \xi \in \mathcal{C}^2(\mathbb{R})$  such that  $u$  is concave and satisfying*

$$a \leq u(y+1) - u(y) \leq b \text{ for all } y \in \mathbb{R} \text{ and}$$

$$|u - \bar{u}| \leq C \text{ and } |\xi - \bar{\xi}| \leq C$$

where  $\bar{u}$  is defined in (2.6). Then,  $(u, \xi)$  is unique up to translation and addition of constants on  $\mathbb{R}$ .

*Proof.* Using Proposition 3.8, we construct a solution  $(\tilde{u}, \tilde{\xi})$  of (2.7) and satisfying

$$\begin{cases} \lim_{|y| \rightarrow +\infty} (\tilde{u}(y) - \bar{u}(y)) = 0, \\ \lim_{|y| \rightarrow +\infty} (\tilde{\xi}(y) - \bar{\xi}(y)) = 0. \end{cases}$$

Finally, applying Proposition 3.5 with  $s_1 = a, s_2 = b$  and

$$L(z_1, z_2, z_3) = \frac{\kappa}{\alpha} V(z_2 - z_1) + \lambda(z_3 - z_2),$$

we get the desired result. □

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## References

- [1] M. AL HAJ, N. FORCADEL, AND R. MONNEAU, *Existence and uniqueness of traveling waves for fully overdamped frenkel-kontorova models*, Archive for Rational Mechanics and Analysis, 210 (2013), pp. 45–99.
- [2] M. BANDO, K. HASEBE, A. NAKAYAMA, A. SHIBATA, AND Y. SUGIYAMA, *Dynamical model of traffic congestion and numerical simulation*, Physical review E, 51 (1995), p. 1035.
- [3] G. BARLES, *An introduction to the theory of viscosity solutions for first-order hamilton-jacobi equations and applications*, in Hamilton-Jacobi equations: approximations, numerical analysis and applications, Springer, 2013, pp. 49–109.
- [4] M. BRACKSTONE AND M. McDONALD, *Car-following: a historical review*, Transportation Research Part F: Traffic Psychology and Behaviour, 2 (1999), pp. 181–196.
- [5] M. G. CRANDALL, H. ISHII, AND P.-L. LIONS, *User’s guide to viscosity solutions of second order partial differential equations*, Bulletin of the American Mathematical Society, 27 (1992), pp. 1–67.
- [6] M. G. CRANDALL AND P.-L. LIONS, *Viscosity solutions of hamilton-jacobi equations*, Transactions of the American Mathematical Society, 277 (1983), pp. 1–42.

- [7] L. DAVIS, *Modifications of the optimal velocity traffic model to include delay due to driver reaction time*, Physica A: Statistical Mechanics and its Applications, 319 (2003), pp. 557–567.
- [8] J. DRONIOU AND C. IMBERT, *Solutions de viscosité et solutions variationnelle pour edp non-linéaires*, Cours de DEA, Département de Mathématiques, Montpellier II, (2002).
- [9] N. FORCADEL, C. IMBERT, AND R. MONNEAU, *Homogenization of fully overdamped frenkel-kontorova models*, Journal of Differential Equations, 246 (2009), pp. 1057–1097.
- [10] ———, *Homogenization of accelerated frenkel-kontorova models with  $n$  types of particles*, Transactions of the American Mathematical Society, 364 (2012), pp. 6187–6227.
- [11] N. FORCADEL AND W. SALAZAR, *Homogenization of second order discrete model and application to traffic flow*, Differential and Integral Equations, 28 (2015), pp. 1039–1068.
- [12] N. FORCADEL, W. SALAZAR, AND M. ZAYDAN, *Homogenization of second order discrete model with local perturbation and application to traffic flow*, Discrete and Continuous Dynamical Systems-Series A, 37 (2017), pp. 1437–1487.
- [13] ———, *Specified homogenization of a discrete traffic model leading to an effective junction condition*, Communications on Pure & Applied Analysis, 17 (2018), p. 2173.
- [14] A. GHORBEL AND R. MONNEAU, *Existence and nonexistence of semidiscrete shocks for a car-following model in traffic flow*, SIAM Journal on Mathematical Analysis, 46 (2014), pp. 3612–3639.
- [15] B. D. GREENSHIELDS, W. CHANNING, H. H. MILLER, ET AL., *A study of traffic capacity*, in Highway research board proceedings, vol. 1935, National Research Council (USA), Highway Research Board, 1935.
- [16] D. HELBING AND B. TILCH, *Generalized force model of traffic dynamics*, Physical review E, 58 (1998), p. 133.
- [17] H. ISHII, *Perron's method for hamilton-jacobi equations*, Duke math. J, 55 (1987), pp. 369–384.
- [18] G.-S. JIANG AND S.-H. YU, *Discrete shocks for finite difference approximations to scalar conservation laws*, SIAM journal on numerical analysis, 35 (1998), pp. 749–772.
- [19] R. JIANG, Q. WU, AND Z. ZHU, *Full velocity difference model for a car-following theory*, Physical Review E, 64 (2001), p. 017101.
- [20] M. J. LIGHTHILL AND G. B. WHITHAM, *On kinematic waves ii. a theory of traffic flow on long crowded roads*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 229 (1955), pp. 317–345.
- [21] T.-P. LIU AND S.-H. YU, *Continuum shock profiles for discrete conservation laws i: Construction*, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 52 (1999), pp. 85–127.
- [22] A. MAJDA AND J. RALSTON, *Discrete shock profiles for systems of conservation laws*, Communications on Pure and Applied Mathematics, 32 (1979), pp. 445–482.
- [23] D. MICHELSON, *Discrete shocks for difference approximations to systems of conservation laws*, Advances in Applied Mathematics, 5 (1984), pp. 433–469.
- [24] G. F. NEWELL, *Nonlinear effects in the dynamics of car following*, Operations research, 9 (1961), pp. 209–229.

- [25] L. A. PIPES, *An operational analysis of traffic dynamics*, Journal of applied physics, 24 (1953), pp. 274–281.
- [26] J. RIDDER AND W. SHEN, *Traveling waves for nonlocal models of traffic flow*, Discrete & Continuous Dynamical Systems, 39 (2019), p. 4001.
- [27] D. SERRE, *Remarks about the discrete profiles of shock waves*, Ecole Normale Supérieure de Lyon, 1995.
- [28] W. SHEN, *Traveling wave profiles for a follow-the-leader model for traffic flow with rough road condition*, Networks and Heterogeneous Media, 13 (2018), pp. 449–478.
- [29] W. SHEN AND K. SHIKH-KHALIL, *Traveling waves for a microscopic model of traffic flow*, Discrete & Continuous Dynamical Systems, 38 (2018), p. 2571.
- [30] M. TREIBER, A. HENNECKE, AND D. HELBING, *Derivation, properties, and simulation of a gas-kinetic-based, nonlocal traffic model*, Physical Review E, 59 (1999), p. 239.