



WELL-POSEDNESS OF THE COAGULATION-FRAGMENTATION EQUATION WITH SIZE DIFFUSION

Philippe Laurençot, Christoph Walker

► To cite this version:

Philippe Laurençot, Christoph Walker. WELL-POSEDNESS OF THE COAGULATION-FRAGMENTATION EQUATION WITH SIZE DIFFUSION. *Differential and integral equations*, 2022, 35 (3-4), pp.211–240. hal-03381379

HAL Id: hal-03381379

<https://hal.science/hal-03381379>

Submitted on 16 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

WELL-POSEDNESS OF THE COAGULATION-FRAGMENTATION EQUATION WITH SIZE DIFFUSION

PHILIPPE LAURENÇOT AND CHRISTOPH WALKER

ABSTRACT. Local and global well-posedness of the coagulation-fragmentation equation with size diffusion are investigated. Owing to the semilinear structure of the equation, a semigroup approach is used, building upon generation results previously derived for the linear fragmentation-diffusion operator in suitable weighted L_1 -spaces.

1. INTRODUCTION

The coagulation-fragmentation equation with size diffusion

$$\partial_t \phi(t, x) = D \partial_x^2 \phi(t, x) + \mathcal{F}(\phi(t, \cdot))(x) + \mathcal{K}(\phi(t, \cdot))(x), \quad (t, x) \in (0, \infty)^2, \quad (1.1a)$$

$$\phi(t, 0) = 0, \quad t > 0, \quad (1.1b)$$

$$\phi(0, x) = f(x), \quad x \in (0, \infty), \quad (1.1c)$$

where

$$\mathcal{F}(\phi)(x) := -a(x)\phi(x) + \int_x^\infty a(y)b(x, y)\phi(y) \, dy$$

and $\mathcal{K}(\phi) := \mathcal{K}(\phi, \phi)$ with

$$\mathcal{K}(\phi, \psi)(x) := \frac{1}{2} \int_0^x k(y, x-y)\phi(y)\psi(x-y) \, dy - \phi(x) \int_0^\infty k(x, y)\psi(y) \, dy,$$

describes the dynamics of the size distribution function $\phi = \phi(t, x) \geq 0$ of particles of size $x \in (0, \infty)$ at time $t > 0$. Particles modify their sizes according to three different mechanisms: random fluctuations, here accounted for by size diffusion at a constant diffusion rate $D > 0$ (hereafter normalized to $D = 1$), spontaneous fragmentation with overall fragmentation rate $a \geq 0$ and daughter distribution function $b \geq 0$, and binary coalescence with coagulation kernel $k \geq 0$. Nucleation is not taken into account in this model, an assumption which leads to the homogeneous Dirichlet boundary condition (1.1b) at $x = 0$. Let us recall that the coagulation-fragmentation equation without size diffusion, corresponding to setting $D = 0$ in (1.1a), arises in several fields of physics (grain growth, aerosol and raindrops formation, polymer and colloidal chemistry) and biology (hematology, animal grouping) and has been studied extensively in the mathematical literature since the pioneering works

Date: October 16, 2021.

2020 Mathematics Subject Classification. 45K05 - 47D06 - 47J35 - 47N50.

Key words and phrases. coagulation-fragmentation - size diffusion - well-posedness - semigroup - global existence.

by Melzak [27] and McLeod [26], see the books and review articles [2, 6, 11, 12, 14, 15, 23, 29, 32] and the references therein. The fragmentation equation with size diffusion ($k = 0$) has been introduced more recently to describe the growth of microtubules [19], plankton cells [13], and ice crystals [18, 24]. That merging could also play a role in the dynamics of the latter is suggested and investigated in [17, 28] and the aim of this paper is to provide some mathematical insight into this model. Specifically, we shall study the well-posedness of (1.1) in a suitable functional setting for a class of rate coefficients a , b , and k which will be described below.

To this aim, let us first recall that, owing to the semilinear structure of (1.1a), a classical approach to well-posedness relies on semigroup theory and involves two steps: one first shows that the linear part of Equation (1.1) (including size diffusion and fragmentation, along with the boundary condition (1.1b)) generates a semigroup in a suitable function space. One subsequently establishes the well-posedness of (1.1) by a fixed point procedure applied to the associated Duhamel formula taking into account Lipschitz properties of the coagulation term. This approach has proven successful for the coagulation-fragmentation equation ($D = 0$) since the seminal work of Aizenman & Bak [1], see [7, 8, 10, 12, 25] and the references therein. It also leads to the well-posedness of coagulation-fragmentation equations with growth/decay (where the diffusion $D\partial_x^2 f$ is replaced by $\mp \partial_x(gf)$ for some growth rate $g = g(x) \geq 0$), see [9].

Coming back to the coagulation-fragmentation equation with size diffusion (1.1), we have performed a rather complete study of the generation properties of its linear part in [22] and shown that it generates a positive analytic semigroup in $L_1((0, \infty), (x + x^m)dx)$ for any $m > 1$, thereby setting the stage for the study of the well-posedness of (1.1) which is our main concern herein. Beforehand, let us set up some notation and make precise the assumptions on the rate coefficients a , b , and k that we shall use throughout the paper.

Notation and Assumptions. We suppose that the overall fragmentation rate a satisfies

$$a \in L_{\infty,loc}([0, \infty)), \quad a \geq 0 \text{ a.e. in } (0, \infty). \quad (1.2)$$

The daughter distribution function b is a non-negative measurable function on $(0, \infty)^2$ satisfying

$$\int_0^y xb(x, y) \, dx = y, \quad y \in (0, \infty), \quad (1.3)$$

and there is $\delta_2 \in (0, 1)$ such that

$$(1 - \delta_2)y^2 \geq \int_0^y x^2 b(x, y) \, dx, \quad y \in (0, \infty). \quad (1.4)$$

Recall that (1.3) guarantees that there is no loss of matter during fragmentation, while (1.4) implies that the distribution of the sizes of the fragments resulting from the breakup of a particle of size y is not too concentrated around y . As for the coagulation kernel k , we assume that there are $0 \leq \theta_0 < \theta \leq 1$, $m > 1$, and $k_* > 0$ such that

$$0 \leq k(x, y) = k(y, x) \leq k_* \frac{\ell(x)\ell(y)}{x + y + (x + y)^m}, \quad (x, y) \in (0, \infty)^2, \quad (1.5a)$$

where

$$\ell(x) := \begin{cases} x^{1-2\theta_0}, & x \in (0, 1), \\ (1 + a(x))^\theta x^m, & x > 1. \end{cases} \quad (1.5b)$$

As usual for coagulation-fragmentation equations, the analysis is performed in weighted L_1 -spaces, which we introduce next. For $r \in \mathbb{R}$, we set

$$X_r := L_1((0, \infty), x^r dx), \quad \|f\|_{X_r} := \int_0^\infty x^r |f(x)| dx, \quad f \in X_r,$$

and define the moment $M_r(f)$ of order r of f by

$$M_r(f) := \int_0^\infty x^r f(x) dx,$$

so that $\|f\|_{X_r} = M_r(|f|)$. We then define the following spaces, which are at the heart of our analysis, namely

$$E_0 := X_1 \cap X_m = L_1((0, \infty), (x + x^m) dx)$$

equipped with the norm $\|\cdot\|_{E_0} := \|\cdot\|_{X_1} + \|\cdot\|_{X_m}$ and

$$Y := L_1((0, \infty), \ell(x) dx) = X_{1-2\theta_0} \cap L_1((0, \infty), (1 + a(x))^\theta (x + x^m) dx), \quad (1.6)$$

with norm

$$\|f\|_Y := \int_0^\infty \ell(x) |f(x)| dx, \quad f \in Y,$$

recalling that the parameters (θ_0, θ, m) and the weight ℓ are defined in (1.5a) and (1.5b), respectively.

We next introduce the linear operator

$$\mathbb{A}f := \partial_x^2 f + \mathcal{F}(f), \quad f \in \text{dom}(\mathbb{A}),$$

where

$$\text{dom}(\mathbb{A}) := \{f \in E_0 : \partial_x^2 f \in E_0, af \in E_0, f(0) = 0\},$$

and we define the graph norm of $f \in \text{dom}(\mathbb{A})$ by

$$\|f\|_{\mathbb{A}} := \|f\|_{E_0} + \|\partial_x^2 f\|_{E_0} + \|af\|_{E_0}.$$

The operator \mathbb{A} includes the linear terms on the right-hand side of (1.1a) (diffusion + fragmentation) and generates an analytic semigroup in E_0 [22]. We shall recall its properties later, wherever needed. Setting

$$E_1 := (\text{dom}(\mathbb{A}), \|\cdot\|_{\mathbb{A}}),$$

we finally introduce the complex interpolation spaces

$$E_\xi := [E_0, E_1]_\xi, \quad \xi \in (0, 1), \quad (1.7)$$

and denote the corresponding norm by $\|\cdot\|_{E_\xi}$. The positive cone E_ξ^+ of E_ξ is then

$$E_\xi^+ := \{f \in E_\xi : f \geq 0 \text{ a.e. in } (0, \infty)\}.$$

Now, the coagulation-fragmentation equation with size diffusion (1.1) can be reformulated as the semilinear Cauchy problem

$$\frac{d\phi}{dt}(t) = \mathbb{A}\phi(t) + \mathcal{K}(\phi(t)), \quad t > 0, \quad \phi(0) = f, \quad (1.8)$$

and we first prove that (1.8) is locally well-posed in E_ξ for appropriate values of ξ .

Theorem 1.1. *Suppose that the rate coefficients a , b , and k satisfy (1.2), (1.3), (1.4), and (1.5a) with parameters $0 \leq \theta_0 < \theta < 1$ and $m > 1$, and consider $\xi \in [0, 1)$ with $2\theta < 1 + \xi$.*

(a) Local Existence: *Given $f \in E_\xi$, the Cauchy problem (1.8) has a unique maximal strong solution*

$$\phi = \phi(\cdot; f) \in C([0, t^+(f)), E_\xi) \cap C((0, t^+(f)), E_1) \cap C^1((0, t^+(f)), E_0)$$

with $t^+(f) \in (0, \infty]$ such that, if $\xi < \theta$, then

$$\lim_{t \rightarrow 0} t^{\theta - \xi} \|\phi(t)\|_{E_\theta} = 0.$$

Moreover,

$$M_1(\phi(t)) = M_1(f), \quad t \in [0, t^+(f)). \quad (1.9)$$

(b) Continuous Dependence: *The mapping $[(t, f) \mapsto \phi(t; f)]$ defines a semiflow on E_ξ .*

(c) Positivity: *If $f \in E_\xi^+$, then $\phi(t; f) \geq 0$ for $t \in [0, t^+(f))$.*

(d) Global Existence Criterion: *Let $f \in E_\xi$. If there are $0 < t_0 < T \wedge t^+(f)$ such that*

$$\sup_{t_0 < t < T \wedge t^+(f)} \|\phi(t)\|_Y < \infty, \quad (1.10)$$

then $t^+(f) \geq T$. In particular, $t^+(f) = \infty$ if (1.10) holds true for arbitrary $0 < t_0 < T$. Furthermore, if there is $K_ > 0$ such that*

$$0 \leq k(x, y) \leq K_* \frac{\ell(x)(y + y^m) + \ell(y)(x + x^m)}{x + y + (x + y)^m}, \quad (x, y) \in (0, \infty)^2, \quad (1.11)$$

and

$$\sup_{0 < t < T \wedge t^+(f)} \|\phi(t)\|_{E_0} < \infty, \quad T > 0, \quad (1.12)$$

then $t^+(f) = \infty$.

A striking difference between Theorem 1.1 and similar results on coagulation-fragmentation equations (possibly with growth or decay, but without size diffusion) is the underlying functional framework [1, 8, 9, 12, 25]. Indeed, coping with the coagulation term $\mathcal{K}(\phi)$ requires in general that ϕ belongs to $L_1((0, \infty), (1 + x^m)dx)$, a property which is provided here by the diffusion when starting from the smaller space E_0 .

Remark 1.2. Unfortunately, we do not have a precise characterization of the space E_ξ for $\xi \in (0, 1)$, besides some embedding in a weighted L_1 -space as in Lemma 2.2 below. Nevertheless, if $\theta \in (0, 1/2)$, then we can take $\xi = 0$ in Theorem 1.1, which then provides the well-posedness of (1.1) in $E_0 = L_1((0, \infty), (x + x^m)dx)$. In the same vein, the global existence criterion (1.12) only requires an a priori estimate in the norm of E_0 .

The proof of Theorem 1.1 relies on a more or less standard approach, involving time-weighted spaces, see [33], [34, Chapter 4] (and, e.g., [5] for an application of this approach to the coagulation-fragmentation equation with spatial diffusion). It strongly relies on the above mentioned fact that the operator \mathbb{A} generates a positive analytic semigroup in E_0 , see [22, Theorem 1.1] and Proposition 2.1 below, and the Lipschitz continuity of the coagulation operator \mathcal{K} in Y stemming from either (1.5a) or (1.11). This approach is adapted to the present situation and allows one to consider initial values in E_ξ for $\xi < \theta$, bearing in mind that the coagulation term is not necessarily well-defined in that space. We provide the proof of Theorem 1.1 in Section 2.

Let us collect a few comments on the assumptions (1.5a) and (1.11) on the coagulation kernel k .

(i) If there are $\alpha \in (1/2, 1]$ and $K > 0$ such that

$$k(x, y) \leq K(1 + a(x))^\alpha(1 + a(y))^\alpha, \quad (x, y) \in (0, \infty)^2, \quad (1.13)$$

then k satisfies (1.5a) with $\theta_0 = 1/2$, $\theta = \alpha$, and any $m > 1$. The assumption (1.13) is introduced in [7, 8] to study the local well-posedness of the coagulation-fragmentation equation (without diffusion), see also [12].

(ii) Let k be a coagulation kernel satisfying (1.5a) with parameters (θ_0, θ, m) . Then it also satisfies (1.5a) with parameters (θ_0, θ, m') for any $m' > m$.

(iii) According to (1.5a), the coagulation kernel k may feature a singularity for small sizes. In particular, given $-1 < \alpha \leq 0 \leq \beta < 1$, the choice

$$k(x, y) = (x^\alpha + y^\alpha)(x^\beta + y^\beta), \quad (x, y) \in (0, \infty)^2,$$

complies with (1.5a) for $\theta_0 = (1 - \alpha)/2 \geq 1/2$, provided that $(y \mapsto y^\beta(1 + a(y))^{-\theta})$ is bounded on $(1, \infty)$, the parameter $m > 1$ being arbitrary. This example includes Smoluchowski's celebrated coagulation kernel corresponding to $\beta = -\alpha = 1/3$ [30, 31].

(iv) It is easy to check that (1.11) implies (1.5a) with $k_* = 2K_*$.

(v) When $\theta_0 \in [0, 1/2]$, the assumption (1.11) implies that $k \in L_\infty((0, 1) \times (1, \infty))$. This excludes coagulation kernels satisfying

$$k(x, y) = K[1 + a(x))^\alpha + (1 + a(y))^\alpha], \quad (x, y) \in (0, \infty)^2. \quad (1.14)$$

Such an assumption is known to guarantee global existence of classical solutions for the case without size diffusion, see [7, 12].

Building upon Theorem 1.1 (d), we supplement Theorem 1.1 with a global existence result for a specific class of coagulation kernels.

Theorem 1.3 (Global Existence). *Suppose that the rate coefficients a , b , and k satisfy (1.2), (1.3), (1.4), and (1.11) with parameters $0 \leq \theta_0 < \theta < 1$ and $m > 1$ and consider $\xi \in [0, 1)$ with $2\theta < 1 + \xi$. Assume further that $\theta_0 \in [0, 1/2]$ and that there is $k_0 > 0$ such that*

$$k(x, y) \leq k_0 \frac{xy}{x+y} [(1+a(x))^\theta + (1+a(y))^\theta], \quad (x, y) \in (1, \infty)^2. \quad (1.15)$$

Then $t^+(f) = \infty$ for all $f \in E_\xi^+$.

Observe that (1.15) is stronger than (1.11) for $(x, y) \in (1, \infty)^2$ (and somehow corresponds to taking $m = 1$ in (1.11) and the definition (1.5b) of ℓ). Guided by (1.12), the proof of Theorem 1.3 relies on an a priori estimate in E_0 , which is derived in Section 3.

From now on, we assume that the rate coefficients a , b , and k satisfy (1.2), (1.3), (1.4), and (1.5a) with fixed parameters $0 \leq \theta_0 < \theta \leq 1$ and $m > 1$.

2. WELL-POSEDNESS

2.1. Diffusion & Fragmentation. In view of (1.8) and according to the above discussion, we begin with the linear terms on the right-hand side of (1.1a) and first recall the generation properties of \mathbb{A} in E_0 established in [22].

Proposition 2.1. *The operator \mathbb{A} generates a positive analytic semigroup $(U(t))_{t \geq 0}$ on E_0 with*

$$M_1(U(t)f) = M_1(f), \quad t \geq 0, \quad f \in E_0. \quad (2.1)$$

In addition, $(U(t))_{t \geq 0}$ is an analytic semigroup on E_ξ for every $\xi \in (0, 1)$.

Proof. The first statement in Proposition 2.1 follows from [22, Theorem 1.1] and implies the second one, according to [3, Theorem 6] or [4, II.Theorem 2.1.3]. \square

As already mentioned, the functional framework we shall work with involves the interpolation spaces $(E_\xi)_{\xi \in (0, 1)}$, for which a simple characterization is not obvious to derive. Nevertheless, we identify a weighted L_1 -space in which E_θ embeds continuously.

Lemma 2.2. *The embedding of the interpolation space E_θ in the weighted L_1 -space Y defined in (1.6) is continuous.*

Proof. Since

$$E_1 \hookrightarrow L_1((0, \infty), (1+a(x))(x+x^m)dx)$$

and

$$\begin{aligned} [L_1((0, \infty), (x+x^m)dx), L_1((0, \infty), (1+a(x))(x+x^m)dx)]_\theta \\ \doteq L_1((0, \infty), (1+a(x))^\theta(x+x^m)dx), \end{aligned}$$

it readily follows that

$$E_\theta = [E_0, E_1]_\theta \hookrightarrow L_1((0, \infty), (1+a(x))^\theta(x+x^m)dx).$$

Next, recall from [22, Lemma 2.1] that $E_1 \hookrightarrow X_r$ for $r \in (-1, 1)$. Therefore,

$$E_\theta \hookrightarrow [X_1, X_r]_\theta = X_{1+\theta(r-1)}, \quad r \in (-1, 1),$$

and the choice $r = 1 - (2\theta_0/\theta) \in (-1, 1)$ completes the proof. \square

We next derive some positivity properties for the semigroups generated by specific perturbations of \mathbb{A} on E_0 , which are required later on to establish the non-negativity of solutions to (1.1).

Lemma 2.3. *Let $\gamma > 0$ and*

$$V(x) := \gamma \frac{\ell(x)}{x + x^m}, \quad x > 0,$$

where ℓ is defined in (1.5b). Then $\mathbb{A}_V := \mathbb{A} - V$ with $\text{dom}(\mathbb{A}_V) = \text{dom}(\mathbb{A})$ generates a positive analytic semigroup on E_0 .

Proof. Note that $V := (f \mapsto Vf) \in \mathcal{L}(E_\theta, E_0)$ due to Lemma 2.2. Since V is thus \mathbb{A} -bounded with a zero \mathbb{A} -bound and \mathbb{A} generates an analytic semigroup on E_0 , it follows from [16, Theorem III.2.10] that \mathbb{A}_V with $\text{dom}(\mathbb{A}_V) = \text{dom}(\mathbb{A})$ generates an analytic semigroup on E_0 .

It remains to prove that this semigroup is positive. To this end, pick $\lambda > 0$ sufficiently large and $g \in E_0$ with $g \leq 0$. Then there is $u \in \text{dom}(\mathbb{A})$ such that

$$(\lambda - \mathbb{A} + V)u = g.$$

Multiplying this identity by $x \text{sign}_+(u(x))$ and integrating over $(0, \infty)$ give

$$\begin{aligned} \int_0^\infty x (\lambda + V(x)) u_+(x) dx - \int_0^\infty x \text{sign}_+(u(x)) \partial_x^2 u(x) dx \\ = \int_0^\infty x \text{sign}_+(u(x)) \mathcal{F}(u)(x) dx + \int_0^\infty x g(x) \text{sign}_+(u(x)) dx. \end{aligned}$$

On the one hand, using (1.3) and Fubini's theorem,

$$\begin{aligned} \int_0^\infty x \text{sign}_+(u(x)) \mathcal{F}(u)(x) dx &= \int_0^\infty x \int_x^\infty a(y) b(x, y) \text{sign}_+(u(x)) u(y) dy dx \\ &\quad - \int_0^\infty x a(x) u_+(x) dx \\ &\leq \int_0^\infty a(y) u_+(y) \int_0^y x b(x, y) dx dy - \int_0^\infty x a(x) u_+(x) dx \leq 0. \end{aligned}$$

On the other hand,

$$- \int_0^\infty x \text{sign}_+(u(x)) \partial_x^2 u(x) dx \geq \int_0^\infty \partial_x u_+(x) dx = 0$$

by Kato's inequality [21, Lemma A] and the boundary condition $u(0) = 0$, the functions u and $\partial_x u$ vanishing sufficiently rapidly at infinity by [22, Lemma 2.1]. Collecting these inequalities and using

the non-positivity of g , we end up with

$$\int_0^\infty x (\lambda + V(x)) u_+(x) dx \leq 0.$$

Consequently, $u \leq 0$. We have thus shown that the operator $\mathbb{A}_V = \mathbb{A} - V$ is resolvent positive; that is, the semigroup generated by \mathbb{A}_V is positive. \square

2.2. Coagulation. We next focus on the coagulation term on the right-hand side of (1.1a) and study the Lipschitz continuity of the coagulation operator \mathcal{K} , noting that it is bilinear.

Lemma 2.4. *The operator \mathcal{K} belongs to $\mathcal{L}^2(Y, E_0)$, the space Y being defined in (1.6), and*

$$M_1(\mathcal{K}(\phi)) = 0, \quad \phi \in Y. \quad (2.2)$$

Furthermore, if k satisfies additionally (1.11), then

$$\|\mathcal{K}(\psi, \phi)\|_{E_0} \leq \frac{3K_*}{2} (\|\psi\|_Y \|\phi\|_{E_0} + \|\psi\|_{E_0} \|\phi\|_Y), \quad (\psi, \phi) \in Y^2. \quad (2.3)$$

Proof. Given $(\psi, \phi) \in Y^2$, Fubini's theorem and (1.5a) imply that

$$\begin{aligned} \|\mathcal{K}(\psi, \phi)\|_{E_0} &\leq \frac{1}{2} \int_0^\infty (x + x^m) \int_0^x k(y, x - y) |\psi(y)| |\phi(x - y)| dy dx \\ &\quad + \int_0^\infty (x + x^m) |\psi(x)| \int_0^\infty k(x, y) |\phi(y)| dy dx \\ &\leq \frac{3}{2} \int_0^\infty \int_0^\infty k(x, y) [x + y + (x + y)^m] |\psi(x)| |\phi(y)| dy dx \\ &\leq \frac{3k_*}{2} \left(\int_0^\infty \ell(x) |\psi(x)| dx \right) \left(\int_0^\infty \ell(y) |\phi(y)| dy \right) \\ &\leq \frac{3k_*}{2} \|\psi\|_Y \|\phi\|_Y, \end{aligned} \quad (2.4)$$

which proves the claim $\mathcal{K} \in \mathcal{L}^2(Y, E_0)$. A classical computation based on Fubini's theorem then gives (2.2).

Assume next that k satisfies additionally (1.11). Then, for $(\psi, \phi) \in Y^2$, the estimate (2.4) entails

$$\begin{aligned} \|\mathcal{K}(\psi, \phi)\|_{E_0} &\leq \frac{3}{2} \int_0^\infty \int_0^\infty k(x, y) [x + y + (x + y)^m] |\psi(x)| |\phi(y)| dy dx \\ &\leq \frac{3K_*}{2} \int_0^\infty \int_0^\infty [\ell(x)(y + y^m) + \ell(y)(x + x^m)] |\psi(x)| |\phi(y)| dy dx \\ &\leq \frac{3K_*}{2} (\|\psi\|_Y \|\phi\|_{E_0} + \|\psi\|_{E_0} \|\phi\|_Y), \end{aligned}$$

which completes the proof of Lemma 2.4. \square

2.3. Proof of Theorem 1.1. Having established the above preliminary results, we are now in a position to begin the proof of Theorem 1.1. Since we aim at handling initial values with mild regularity, we introduce the following time-weighted spaces. Given $T > 0$, a Banach space E , and $\mu \in \mathbb{R}$, we denote the Banach space of all functions $u \in C((0, T], E)$ such that $(t \mapsto t^\mu u(t))$ is bounded on $(0, T]$ by $BC_\mu((0, T], E)$, equipped with the norm

$$u \mapsto \|u\|_{BC_\mu((0, T], E)} := \sup_{t \in (0, T]} \{t^\mu \|u(t)\|_E\}.$$

We write $C_\mu((0, T], E)$ for the closed linear subspace thereof consisting of all $u \in BC_\mu((0, T], E)$ satisfying additionally $t^\mu u(t) \rightarrow 0$ in E as $t \rightarrow 0$. Note that $BC_\nu((0, T], E) \hookrightarrow BC_\mu((0, T], E)$ for $\nu \leq \mu$ with

$$\|u\|_{BC_\mu((0, T], E)} \leq T^{\mu-\nu} \|u\|_{BC_\nu((0, T], E)}, \quad u \in BC_\nu((0, T], E). \quad (2.5)$$

We denote the open and closed balls of E centered at $f \in E$ and of radius R by $\mathbb{B}_E(f, R)$ and $\mathbb{B}_E(f, R)$, respectively.

To start with, we analyze how the semigroup $(U(t))_{t \geq 0}$ generated by \mathbb{A} on E_0 , see Proposition 2.1, acts on $C_\mu((0, T], E_\theta)$. The subsequent Lemma 2.5 and Lemma 2.6 are more or less implicitly contained in the proof of [34, Theorem 4.1] with the difference that domains of fractional powers are used instead of interpolation spaces.

Lemma 2.5. *Let $0 \leq \xi < \eta \leq 1$ with $(\xi, \eta) \neq (0, 1)$ and $0 < T \leq T_0$. There is $c_0(T_0, \xi, \eta) > 0$ such that, if $f \in E_\xi$, then*

$$Uf := (t \mapsto U(t)f) \in C_{\eta-\xi}((0, T], E_\eta) \quad \text{with} \quad \|Uf\|_{C_{\eta-\xi}((0, T], E_\eta)} \leq c_0(T_0, \xi, \eta) \|f\|_{E_\xi}.$$

Proof. Recall that there is $c_0(T_0, \xi, \eta) > 0$ such that

$$t^{\eta-\xi} \|U(t)\|_{\mathcal{L}(E_\xi, E_\eta)} + \|U(t)\|_{\mathcal{L}(E_\eta)} + t \|\mathbb{A}U(t)\|_{\mathcal{L}(E_\eta)} \leq c_0(T_0, \xi, \eta), \quad t \in (0, T_0], \quad (2.6)$$

see [4, II.Lemma 5.1.3]. Thus, since $(U(t))_{t \geq 0}$ is a strongly continuous semigroup on E_η , see Proposition 2.1, it readily follows from (2.6) that

$$(t \mapsto U(t)f) \in BC_{\eta-\xi}((0, T], E_\eta) \quad \text{with} \quad \|Uf\|_{BC_{\eta-\xi}((0, T], E_\eta)} \leq c_0(T_0, \xi, \eta) \|f\|_{E_\xi}$$

for all $f \in E_\xi$ and $T \in (0, T_0]$.

We next recall that E_η is dense in E_ξ since $[\cdot, \cdot]_\xi$ is an admissible interpolation functor. Hence, given $\varepsilon > 0$, there is $g \in E_\eta$ such that

$$\|f - g\|_{E_\xi} \leq \frac{\varepsilon}{c_0(T_0, \xi, \eta)}.$$

Therefore, by (2.6),

$$\begin{aligned} t^{\eta-\xi} \|U(t)f\|_{E_\eta} &\leq t^{\eta-\xi} \|U(t)(f - g)\|_{E_\eta} + t^{\eta-\xi} \|U(t)g\|_{E_\eta} \\ &\leq t^{\eta-\xi} \|U(t)\|_{\mathcal{L}(E_\xi, E_\eta)} \|f - g\|_{E_\xi} + t^{\eta-\xi} \|U(t)\|_{\mathcal{L}(E_\eta)} \|g\|_{E_\eta} \\ &\leq \varepsilon + c_0(T_0, \xi, \eta) \|g\|_{E_\eta} t^{\eta-\xi}, \end{aligned}$$

so that, since $\eta > \xi$,

$$\limsup_{t \rightarrow 0} t^{\eta-\xi} \|U(t)f\|_{E_\eta} \leq \varepsilon.$$

We then let $\varepsilon \rightarrow 0$ to complete the proof. \square

The next step is to elucidate the behavior of $(U(t))_{t \geq 0}$ when involved in a convolution with respect to time. To this end, given $T > 0$ and $u : (0, T] \rightarrow E_0$, we set

$$U \star u(t) := \int_0^t U(t-s)u(s) \, ds, \quad t \in (0, T],$$

whenever this integral makes sense.

Lemma 2.6. *Consider $(2\eta, \nu) \in [0, 1)^2$ and $0 < T \leq T_0$. Then*

$$[(u, v) \mapsto U \star \mathcal{K}(u, v)] \in \mathcal{L}^2(C_\eta((0, T], E_\theta), C_{2\eta+\nu-1}((0, T], E_\nu)) \quad (2.7)$$

and

$$[(u, v) \mapsto U \star \mathcal{K}(u, v)] \in \mathcal{L}^2(BC((0, T], E_\theta), BC_{\nu-1}((0, T], E_\nu)), \quad (2.8)$$

where θ is defined in (1.5) and the norms of the above bilinear form depend only on T_0 , η , and ν .

Proof. Pick $\mu \in [0, 1)$ and $u \in BC_\mu((0, T], E_0)$. Let $t \in (0, T]$. By (2.6),

$$\begin{aligned} \|U \star u(t)\|_{E_\nu} &\leq \int_0^t \|U(t-s)\|_{\mathcal{L}(E_0, E_\nu)} \|u(s)\|_{E_0} \, ds \\ &\leq c_0(T_0, 0, \nu) \int_0^t (t-s)^{-\nu} s^{-\mu} \, ds \|u\|_{BC_\mu((0, t], E_0)} \\ &= c_0(T_0, 0, \nu) t^{1-\nu-\mu} \mathbf{B}(1-\nu, 1-\mu) \|u\|_{BC_\mu((0, t], E_0)}, \end{aligned} \quad (2.9)$$

where \mathbf{B} denotes the Beta function. Therefore, since

$$\|u\|_{BC_\mu((0, t], E_0)} \leq \|u\|_{BC_\mu((0, T], E_0)},$$

we infer from (2.9) that

$$[u \mapsto U \star u] \in \mathcal{L}(BC_\mu((0, T], E_0), BC_{\mu+\nu-1}((0, T], E_\nu)). \quad (2.10)$$

Assume then that $u \in C_\mu((0, T], E_0)$. This property, along with (2.9), readily implies that $U \star u$ belongs to $C_{\mu+\nu-1}((0, T], E_\nu)$ and we conclude that

$$[u \mapsto U \star u] \in \mathcal{L}(C_\mu((0, T], E_0), C_{\mu+\nu-1}((0, T], E_\nu)). \quad (2.11)$$

Now, if $(u, v) \in (BC_\rho((0, T], E_\theta))^2$ for some $\rho \in \mathbb{R}$, then Lemma 2.2 and Lemma 2.4 entail that, for $t \in (0, T]$,

$$\begin{aligned} t^{2\rho} \|\mathcal{K}(u(t), v(t))\|_{E_0} &\leq t^{2\rho} \|\mathcal{K}\|_{\mathcal{L}^2(Y, E_0)} \|u(t)\|_Y \|v(t)\|_Y \\ &\leq c_1^2 \|\mathcal{K}\|_{\mathcal{L}^2(Y, E_0)} \|u\|_{BC_\rho((0, t], E_\theta)} \|v\|_{BC_\rho((0, t], E_\theta)}, \end{aligned}$$

where c_1 is the norm of the continuous embedding of E_θ in Y , see Lemma 2.2. Consequently,

$$\mathcal{K}(u, v) \in BC_{2\rho}((0, T], E_0). \quad (2.12)$$

Also, if $u \in C_\rho((0, T], E_\theta)$ or $v \in C_\rho((0, T], E_\theta)$, then

$$\mathcal{K}(u, v) \in C_{2\rho}((0, T], E_\theta). \quad (2.13)$$

We then combine (2.11) (with $\mu = 2\eta$) and (2.13) (with $\rho = \eta$) to derive (2.7), while (2.8) follows from (2.10) (with $\mu = 0$) and (2.12) (with $\rho = 0$). \square

We are now ready to provide the proof of Theorem 1.1.

Proof of Theorem 1.1. Step 1: Local Existence. Let $T_0 > 0$ be arbitrary. We consider $\xi \in [0, 1)$ with $2\theta < 1 + \xi$ and $f \in E_\xi$, but handle the ranges $\xi \in [0, \theta)$ and $\xi \in [\theta, 1)$ differently. In the following, c and $(c_i)_{i \geq 2}$ denote positive constants depending only on θ_0 , θ , m , T_0 , and ξ . Dependence upon additional parameters will be indicated explicitly.

Assume first that $\xi \in [0, \theta)$ and let $T \in (0, T_0]$. We note that (2.7) (with $2\eta = 2(\theta - \xi) < 1$ and $\nu = \theta$) implies that,

$$[(u, v) \mapsto U \star \mathcal{K}(u, v)] \in \mathcal{L}^2(C_{\theta-\xi}((0, T], E_\theta), C_{3\theta-2\xi-1}((0, T], E_\theta)), \quad (2.14)$$

with a norm depending only on θ , T_0 , and ξ , while the constraint $2\theta < 1 + \xi$ and (2.5) guarantee

$$C_{3\theta-2\xi-1}((0, T], E_\theta) \hookrightarrow C_{\theta-\xi}((0, T], E_\theta) \quad (2.15)$$

with norm bounded by $T^{1+\xi-2\theta}$. Introducing

$$Z_T := C_{\theta-\xi}((0, T], E_\theta),$$

it follows from (2.14) and (2.15) that there is $c_2 > 0$ such that

$$\|U \star \mathcal{K}(\phi)\|_{Z_T} \leq c_2 T^{1+\xi-2\theta} \|\phi\|_{Z_T}^2, \quad \phi \in Z_T, \quad (2.16)$$

and

$$\|U \star \mathcal{K}(\phi) - U \star \mathcal{K}(\psi)\|_{Z_T} \leq c_2 T^{1+\xi-2\theta} (\|\phi\|_{Z_T} + \|\psi\|_{Z_T}) \|\phi - \psi\|_{Z_T}, \quad (\phi, \psi) \in Z_T^2. \quad (2.17)$$

Given $R \geq \|f\|_{E_\xi}$, we choose $T_1 = T_1(R) \in (0, T_0]$ such that

$$4(1 + c_0(T_0, \xi, \theta)R)^2 c_2 T_1^{1+\xi-2\theta} \leq 1,$$

the constant $c_0(T_0, \xi, \theta)$ being defined in Lemma 2.5, and infer from Lemma 2.5 (with $\eta = \theta$) and the choice of R that

$$\|Uf\|_{Z_{T_1}} \leq c_0(T_0, \xi, \theta)R. \quad (2.18)$$

We now define

$$F(\phi) := Uf + U \star \mathcal{K}(\phi), \quad \phi \in \bar{\mathbb{B}}_{Z_{T_1}}(Uf, 1).$$

On the one hand, it follows from (2.16) and (2.18) that $F(\phi) \in Z_{T_1}$ and, by the choice of T_1 ,

$$\begin{aligned} \|F(\phi) - Uf\|_{Z_{T_1}} &\leq \|U \star \mathcal{K}(\phi)\|_{Z_{T_1}} \leq c_2 T_1^{1+\xi-2\theta} \|\phi - Uf + Uf\|_{Z_{T_1}}^2 \\ &\leq c_2 T_1^{1+\xi-2\theta} (1 + c_0(T_0, \xi, \theta)R)^2 \leq 1, \end{aligned}$$

so that $F(\phi) \in \bar{\mathbb{B}}_{Z_{T_1}}(Uf, 1)$. On the other hand, for $(\phi, \psi) \in (\bar{\mathbb{B}}_{Z_{T_1}}(Uf, 1))^2$, we deduce from (2.17) that

$$\begin{aligned} \|F(\phi) - F(\psi)\|_{Z_{T_1}} &= \|U \star \mathcal{K}(\phi) - U \star \mathcal{K}(\psi)\|_{Z_{T_1}} \\ &\leq c_2 T_1^{1+\xi-2\theta} (\|\phi\|_{Z_{T_1}} + \|\psi\|_{Z_{T_1}}) \|\phi - \psi\|_{Z_{T_1}} \\ &\leq 2c_2 T_1^{1+\xi-2\theta} (1 + c_0(T_0, \xi, \theta)R) \|\phi - \psi\|_{Z_{T_1}} \\ &\leq \frac{1}{2} \|\phi - \psi\|_{Z_{T_1}}. \end{aligned} \quad (2.19)$$

Consequently, F is a strict contraction on $\bar{\mathbb{B}}_{Z_{T_1}}(Uf, 1)$, so that Banach's fixed point theorem yields a unique fixed point $\phi = \phi(\cdot; f) \in \bar{\mathbb{B}}_{Z_{T_1}}(Uf, 1)$ of F ; that is, for $t \in (0, T_1]$,

$$\phi(t) = (Uf + U \star \mathcal{K}(\phi))(t) = U(t)f + \int_0^t U(t-s) \mathcal{K}(\phi)(s) ds \quad (2.20)$$

and ϕ is thus a mild solution to (1.8) in E_θ on $(0, T_1]$. From (2.5), (2.7) (with $\eta = \theta - \xi$ and $\nu = \xi$), and the constraint $2\theta < 1 + \xi$, we deduce that

$$U \star \mathcal{K}(\phi) \in C_{2\theta-\xi-1}((0, T_1], E_\xi) \hookrightarrow C_0((0, T_1], E_\xi),$$

which – together with (2.20) and the continuity property $Uf \in C([0, T_1], E_\xi)$ due to $f \in E_\xi$ and Proposition 2.1 – ensures that $\phi \in C([0, T_1], E_\xi)$. Moreover, for $(f_1, f_2) \in (\mathbb{B}_{E_\xi}(0, R))^2$, the integral formulation (2.20), along with Lemma 2.5 (with $\eta = \theta$) and (2.19), yields

$$\begin{aligned} \|\phi(\cdot; f_1) - \phi(\cdot; f_2)\|_{Z_{T_1}} &\leq \|U(f_1 - f_2)\|_{Z_{T_1}} + \|U \star \mathcal{K}(\phi(\cdot; f_1)) - U \star \mathcal{K}(\phi(\cdot; f_2))\|_{Z_{T_1}} \\ &\leq c_0(T_0, \xi, \theta) \|f_1 - f_2\|_{E_\xi} + \frac{1}{2} \|\phi(\cdot; f_1) - \phi(\cdot; f_2)\|_{Z_{T_1}}, \end{aligned}$$

and thus

$$\|\phi(\cdot; f_1) - \phi(\cdot; f_2)\|_{Z_{T_1}} \leq 2c_0(T_0, \xi, \theta) \|f_1 - f_2\|_{E_\xi}. \quad (2.21)$$

Using again the embedding $C_{2\theta-\xi-1}((0, T_1], E_\xi) \hookrightarrow C_0((0, T_1], E_\xi)$, see (2.5), it follows from Proposition 2.1, (2.7) (with $\eta = \theta - \xi$ and $\nu = \xi$), (2.20), and (2.21) that

$$\begin{aligned} \|\phi(\cdot; f_1) - \phi(\cdot; f_2)\|_{C([0, T_1], E_\xi)} &\leq \|U(f_1 - f_2)\|_{C([0, T_1], E_\xi)} + \|U \star \mathcal{K}(\phi(\cdot; f_1)) - U \star \mathcal{K}(\phi(\cdot; f_2))\|_{C_0((0, T_1], E_\xi)} \\ &\leq c \|f_1 - f_2\|_{E_\xi} + c (\|\phi(\cdot; f_1)\|_{Z_{T_1}} + \|\phi(\cdot; f_2)\|_{Z_{T_1}}) \|\phi(\cdot; f_1) - \phi(\cdot; f_2)\|_{Z_{T_1}} \\ &\leq c \|f_1 - f_2\|_{E_\xi}. \end{aligned}$$

This estimate entails local uniqueness of solutions to (1.8) and we can then extend $\phi(\cdot; f)$ to a unique mild solution on a maximal interval of existence $[0, t^+(f))$ by classical arguments.

Finally, the conservation of mass stated in (1.9) is a consequence of (2.1), (2.2), and (2.20).

For the case $f \in E_\xi$ with $\xi \in [\theta, 1)$, one proceeds as above but performs the fixed point argument for F in $C([0, T_1], E_\xi)$ (instead of Z_{T_1}) with the help of (2.8) (instead of (2.7)).

Step 2: Regularity. Given $\xi \in [0, 1)$ with $2\theta < 1 + \xi$ and $f \in E_\xi$, we now check that $\phi = \phi(\cdot; f)$ is a classical solution to (1.8) in E_0 on $(0, t^+(f))$. To this end, let $\nu \in (\max\{\theta, \xi\}, 1)$ and $T \in (0, t^+(f))$. If $\xi \in [0, \theta)$, then $\phi \in C_{\theta-\xi}((0, T], E_\theta)$. Hence, Lemma 2.5 (with $\eta = \nu$) and (2.7) (with $\eta = \theta - \xi$) ensure that $Uf \in C_{\nu-\xi}((0, T], E_\nu)$ and $U \star \mathcal{K}(\phi) \in C_{2(\theta-\xi)+\nu-1}((0, T], E_\nu)$, respectively. Similarly, if $\xi \in [\theta, 1)$, then $\phi \in C([0, T], E_\theta)$ and it follows from Lemma 2.5 (with $\eta = \nu$) and from (2.8) that $Uf \in C_{\nu-\xi}((0, T], E_\nu)$ and $U \star \mathcal{K}(\phi) \in C_{\nu-1}((0, T], E_\nu)$, respectively. In both cases, since $T < t^+(f)$ is arbitrary, we then deduce from (2.20) that

$$\phi \in C((0, t^+(f)), E_\nu).$$

Next, given $\varepsilon \in (0, t^+(f))$, we define $\phi_\varepsilon(\cdot) := \phi(\varepsilon + \cdot)$ and

$$h_\varepsilon := \mathcal{K}(\phi_\varepsilon) \in C([0, t^+(f) - \varepsilon), E_0),$$

the latter being a consequence of Lemma 2.2, Lemma 2.4, and the regularity of ϕ . According to (2.20), ϕ_ε is the unique mild solution in E_0 to the linear problem

$$\frac{d\phi_\varepsilon}{dt}(t) = \mathbb{A}\phi_\varepsilon(t) + h_\varepsilon(t), \quad t \in [0, t^+(f) - \varepsilon), \quad \phi_\varepsilon(0) = \phi(\varepsilon) \in E_\nu,$$

so that $\phi_\varepsilon \in C^{\nu-\theta}([0, t^+(f) - \varepsilon), E_\theta)$ by [4, II.Theorem 5.3.1]. This last property, along with Lemma 2.2 and Lemma 2.4, entails that

$$h_\varepsilon \in C^{\nu-\theta}([0, t^+(f) - \varepsilon), E_0),$$

and [4, II.Theorem 1.2.1] implies that the mild solution ϕ_ε is actually a strong solution with

$$\phi_\varepsilon \in C^1((0, t^+(f) - \varepsilon), E_0) \cap C((0, t^+(f) - \varepsilon), E_1).$$

As $\varepsilon \in (0, t^+(f))$ is arbitrary, we conclude that ϕ satisfies

$$\phi \in C^1((0, t^+(f)), E_0) \cap C((0, t^+(f)), E_1),$$

and is thus a strong solution to (1.8) on $(0, t^+(f))$.

Step 3: Continuous Dependence. Let $f_0 \in E_\xi$ and $t_0 \in (0, t^+(f_0))$ be arbitrary. We fix $t_* \in (t_0, t^+(f_0))$ and $R > 0$ such that $\phi([0, t_*]; f_0) \subset \mathbb{B}_{E_\xi}(f_0, R)$. By **Step 1**, there are $T_1 = T_1(2R + \|f_0\|_{E_\xi}) > 0$ and $\kappa_0 \geq 1$ such that

$$T_1 < t^+(f) \quad \text{for any } f \in \bar{\mathbb{B}}_{E_\xi}(f_0, 2R) \quad (2.22)$$

and

$$\|\phi(t; f_1) - \phi(t; f_2)\|_{E_\xi} \leq \kappa_0 \|f_1 - f_2\|_{E_\xi}, \quad t \in [0, T_1], \quad (f_1, f_2) \in (\bar{\mathbb{B}}_{E_\xi}(f_0, 2R))^2. \quad (2.23)$$

Let $N_1 \in \mathbb{N} \setminus \{0\}$ be such that $(N_1 - 1)T_1 < t_* \leq N_1 T_1$. We claim that there exists $k_0 \geq 1$ such that

- (α) $t_* < t^+(f_1)$ for each $f_1 \in \mathbb{B}_{E_\xi}(f_0, R\kappa_0^{1-N_1})$,
- (β) $\|\phi(t; f_1) - \phi(t; f_0)\|_{E_\xi} \leq k_0 \|f_1 - f_0\|_{E_\xi}$ for $0 \leq t \leq t_*$ and $f_1 \in \mathbb{B}_{E_\xi}(f_0, R\kappa_0^{1-N_1})$.

Indeed, let $f_1 \in \mathbb{B}_{E_\xi}(f_0, R\kappa_0^{1-N_1}) \subset \mathbb{B}_{E_\xi}(f_0, R)$. Either $t_* \leq T_1$ (i.e., $N_1 = 1$) and the properties (α) and (β) follow from (2.22) and (2.23). Or $T_1 < t_*$, from which we deduce that $N_1 \geq 2$ and $\phi(T_1; f_0) \in \mathbb{B}_{E_\xi}(f_0, R)$. Consequently, by (2.23),

$$\begin{aligned} \|\phi(T_1; f_1) - f_0\|_{E_\xi} &\leq \|\phi(T_1; f_1) - \phi(T_1; f_0)\|_{E_\xi} + \|\phi(T_1; f_0) - f_0\|_{E_\xi} \\ &< \kappa_0 \|f_1 - f_0\|_{E_\xi} + R \leq R\kappa_0^{2-N_1} + R \leq 2R, \end{aligned}$$

so that $\phi(T_1; f_1) \in \mathbb{B}_{E_\xi}(f_0, 2R)$. Recalling (2.22), one has $T_1 < t^+(\phi(T_1; f_i))$ for $i = 0, 1$, while uniqueness of solutions to (1.8) entails that $\phi(t; \phi(T_1; f_i)) = \phi(t + T_1; f_i)$ for $0 \leq t \leq T_1$ and $i = 0, 1$. An application of (2.23) then gives

$$\|\phi(t + T_1; f_1) - \phi(t + T_1; f_0)\|_{E_\xi} \leq \kappa_0 \|\phi(T_1; f_1) - \phi(T_1; f_0)\|_{E_\xi} \leq \kappa_0^2 \|f_1 - f_0\|_{E_\xi}$$

for $t \in [0, T_1]$. If $N_1 = 2$, then the properties (α) and (β) are proved with $k_0 = \kappa_0^2$. Otherwise, we proceed by induction to deduce (α) and (β) after $N_1 - 1$ iterations. In particular, the property (α) implies that $(0, t_*) \times \mathbb{B}_{E_\xi}(f_0, R\kappa_0^{1-N_1})$ is a neighborhood of (t_0, f_0) in

$$\mathcal{D} := \{(t, f) : 0 \leq t < t^+(f), f \in E_\xi\};$$

that is, \mathcal{D} is open in $\mathbb{R}^+ \times E_\xi$. Owing to this feature and (β) , it is now immediate that the map $\phi \in C(\mathcal{D}, E_\xi)$ defines a semiflow in E_ξ .

Step 4: Global Existence Criterion. Let $f \in E_\xi$ and consider $0 < t_0 < T$ such that $t_0 < t^+(f)$ and the corresponding solution $\phi = \phi(\cdot; f)$ to (1.8) satisfies

$$\sup_{t_0 < t < T \wedge t^+(f)} \|\phi(t)\|_Y < \infty. \quad (2.24)$$

Assume for contradiction that $t^+(f) < T$. Lemma 2.4 and (2.24) then entail that

$$\|\mathcal{K}(\phi(s))\|_{E_0} \leq M_0, \quad t_0 < s < t^+(f),$$

for some $M_0 > 0$. We readily obtain from (2.6) and (2.20) that, for $t \in [t_0, t^+(f))$,

$$\begin{aligned} \|\phi(t)\|_{E_\xi} &\leq \|U(t - t_0)\phi(t_0)\|_{E_\xi} + \int_{t_0}^t \|U(t - s)\mathcal{K}(\phi)(s)\|_{E_\xi} ds \\ &\leq c_0(T_0, \xi, \xi) \|\phi(t_0)\|_{E_\xi} + c_0(T_0, 0, \xi) \int_{t_0}^t (t - s)^{-\xi} \|\mathcal{K}(\phi)(s)\|_{E_0} ds \\ &\leq c_0(T_0, \xi, \xi) \|\phi(t_0)\|_{E_\xi} + \frac{c_0(T_0, 0, \xi) M_0}{1 - \xi} T_0^{1-\xi} =: R_0. \end{aligned}$$

Step 1 now implies that there is $T_1 = T_1(R_0) > 0$ such that ϕ exists at least on $[s, s + T_1]$ for every $t_0 < s < t^+(f)$, contradicting the maximality of $t^+(f)$. Consequently, $t^+(f) \geq T$ as claimed and (1.10) is proved.

Moreover, if (1.11) and (1.12) are valid, then it follows from Lemma 2.2 and Lemma 2.4 that, for any $T > 0$, there is $M(T) > 0$ such that

$$\|\mathcal{K}(\phi(t))\|_{E_0} \leq M(T) \|\phi(t)\|_{E_\theta}, \quad 0 < t < T \wedge t^+(f).$$

Combining this estimate with (2.6) and (2.20), we find, for $t \in (0, T \wedge t^+(f))$,

$$\begin{aligned} \|\phi(t)\|_{E_\theta} &\leq \|U(t)f\|_{E_\theta} + \int_0^t \|U(t-s)\mathcal{K}(\phi)(s)\|_{E_\theta} ds \\ &\leq c_0(T, 0, \theta)t^{-\theta}\|f\|_{E_0} + c_0(T, 0, \theta)M(T) \int_0^t (t-s)^{-\theta}\|\phi(s)\|_{E_\theta} ds. \end{aligned}$$

We then apply the singular Gronwall inequality, see [4, II.Theorem 3.3.1] or [20, Lemma 7.1.1] to conclude that ϕ satisfies (1.10) (with any $t_0 \in (0, T \wedge t^+(f))$). Consequently, $t^+(f) \geq T$ and, as $T > 0$ is arbitrary, $t^+(f) = \infty$.

Step 5: Positivity. We finally provide a proof of the non-negativity of solutions to (1.8) emanating from non-negative initial values. To this end, let us first note that (1.5a) and Lemma 2.2 imply that, for $\psi \in E_\theta^+$,

$$\begin{aligned} 0 &\leq \int_0^\infty k(x, y)\psi(y) dy \leq \int_0^\infty k_* \frac{\ell(x)\ell(y)}{x+y+(x+y)^m} \psi(y) dy \\ &\leq k_* \|\psi\|_Y \frac{\ell(x)}{x+x^m} \\ &\leq c_3 \|\psi\|_{E_\theta} \frac{\ell(x)}{x+x^m}, \quad x > 0. \end{aligned}$$

Assume first that the initial value f belongs to E_θ^+ and fix $T \in (0, t^+(f))$. By **Step 1**,

$$\phi = \phi(\cdot; f) \in C([0, t^+(f)), E_\theta)$$

and we may pick $R \geq \|\phi\|_{L^\infty((0, T), E_\theta)}$. Setting

$$V(x) := c_3 R \frac{\ell(x)}{x+x^m}, \quad x > 0,$$

the above upper bound entails that, for $\psi \in \bar{\mathbb{B}}_{E_\theta}(0, R) \cap E_\theta^+$ and $x \in (0, \infty)$,

$$[V\psi + \mathcal{K}(\psi)](x) \geq (V\psi)(x) - \psi(x) \int_0^\infty k(x, y)\psi(y) dy \geq 0.$$

Consider now the Cauchy problem

$$\frac{d\psi}{dt} = \mathbb{A}_V \psi + \mathcal{K}(\psi) + V\psi, \quad t > 0, \quad \psi(0) = f, \quad (2.25)$$

with $\mathbb{A}_V = \mathbb{A} - V$. Since \mathbb{A}_V generates a positive analytic semigroup on E_0 according to Lemma 2.3 and

$$\|V\psi\|_{E_0} \leq c_3 R \|\psi\|_Y, \quad \psi \in Y,$$

we may proceed as in **Step 1** (with $\xi = \theta$) to obtain a unique mild solution $\psi \in C([0, T_1], E_\theta)$ to (2.25) for some $T_1 > 0$, which satisfies additionally $\psi(t) \in E_\theta^+$ for $t \in [0, T_1]$. Clearly, $\psi = \phi$ on $[0, T \wedge T_1]$ by uniqueness of mild solutions to (1.8), and we iterate this argument to show that $\phi(t) \in E_\theta^+$ for $t \in [0, T]$. Since $T < t^+(f)$, the proof is complete for initial values in E_θ^+ .

Finally, the just established non-negativity extends to arbitrary initial values in E_ξ^+ with $\xi \in (0, \theta)$ by a density argument, thanks to the continuous dependence of $\phi(\cdot; f)$ on the initial value f established in **Step 3**. \square

3. GLOBAL EXISTENCE

In this section, the rate coefficients a , b , and k are assumed to satisfy (1.2), (1.3), (1.4), (1.11), and (1.13) with parameters $0 \leq \theta_0 < \theta < 1$, $\theta_0 \in [0, 1/2]$, and $m > 1$. We recall that (1.11) implies that k satisfies also (1.5a) with $k_* = 2K_*$. Throughout this section, κ and $(\kappa_i)_{i \geq 1}$ denote positive constants depending only on a , b , k , θ_0 , and θ . Dependence upon additional parameters will be indicated explicitly.

Let $r > 1$. We first recall that, owing to (1.4),

$$\delta_r := \inf_{y>0} \left\{ 1 - \frac{1}{y^r} \int_0^y x^r b(x, y) dx \right\} \in (0, 1), \quad (3.1)$$

with $\delta_r \in [\delta_2, 1)$ for $r \geq 2$ and $\delta_r \in [1 - (1 - \delta_2)^{r-1}, 1)$ for $r \in (1, 2)$, see [11, Theorem 5.1.47]. We next define the weight

$$w_r(x) := \begin{cases} \frac{r-1}{2}x^3, & x \in [0, 1], \\ x^r + \frac{r-3}{2}x, & x \in (1, \infty), \end{cases}$$

and notice that $w_r \in C^1([0, \infty)) \cap C^2((0, \infty) \setminus \{1\})$ is nonnegative and increasing on $[0, \infty)$. Moreover,

$$\frac{x + x^r}{2} \leq w_r(x) + rx, \quad w_r(x) \leq x^r + rx, \quad x \in (0, \infty). \quad (3.2)$$

We now estimate how w_r acts on the three mechanisms (fragmentation, diffusion, and coagulation) involved in (1.1) and begin with the contribution of the fragmentation term.

Lemma 3.1. *Consider $\psi \in E_1^+$ and $r \in (1, m]$. Then*

$$\int_0^\infty w_r(x) \mathcal{F}(\psi)(x) dx \leq -\delta_r \int_1^\infty x^r a(x) \psi(x) dx + \int_1^\infty xa(x) \psi(x) dx.$$

Proof. Owing to the definition of w_r ,

$$\begin{aligned} \int_0^\infty w_r(x) \mathcal{F}(\psi)(x) dx &= -\frac{r-1}{2} \int_0^1 a(y) \psi(y) \left[y^3 - \int_0^y x^3 b(x, y) dx \right] dy \\ &\quad - \int_1^\infty a(y) \left(y^r + \frac{r-3}{2}y \right) \psi(y) dy \\ &\quad + \frac{r-1}{2} \int_1^\infty a(y) \psi(y) \int_0^1 x^3 b(x, y) dx dy \\ &\quad + \int_1^\infty a(y) \psi(y) \int_1^y \left(x^r + \frac{r-3}{2}x \right) b(x, y) dx dy. \end{aligned}$$

By (1.3),

$$\int_0^y x^3 b(x, y) dx \leq y^2 \int_0^y x b(x, y) dx \leq y^3, \quad y > 0,$$

which, together with the constraint $r > 1$ and the non-negativity of ψ , implies that the first term on the right-hand side of the above identity is non-positive. Consequently, using once more (1.3), as well as (3.1),

$$\begin{aligned} \int_0^\infty w_r(x) \mathcal{F}(\psi)(x) dx &\leq - \int_1^\infty a(y) y^r \psi(y) dy - \frac{r-3}{2} \int_1^\infty a(y) \psi(y) \int_0^1 x b(x, y) dx dy \\ &\quad + \frac{r-1}{2} \int_1^\infty a(y) \psi(y) \int_0^1 x^3 b(x, y) dx dy \\ &\quad + \int_1^\infty a(y) \psi(y) \int_0^y x^r b(x, y) dx dy - \int_1^\infty a(y) \psi(y) \int_0^1 x^r b(x, y) dx dy \\ &= - \int_1^\infty a(y) \psi(y) \left[y^r - \int_0^y x^r b(x, y) dx \right] dy \\ &\quad + \int_1^\infty a(y) \psi(y) \int_0^1 \left(\frac{r-1}{2} x^3 - \frac{r-3}{2} x - x^r \right) b(x, y) dx dy \\ &\leq -\delta_r \int_1^\infty a(y) y^r \psi(y) dy \\ &\quad + \int_1^\infty a(y) \psi(y) \int_0^1 \left(\frac{r-1}{2} x^3 - \frac{r-3}{2} x \right) b(x, y) dx dy. \end{aligned}$$

Since $r > 1$, another use of (1.3) gives

$$\int_0^1 \left(\frac{r-1}{2} x^3 - \frac{r-3}{2} x \right) b(x, y) dx \leq \int_0^1 x b(x, y) dx \leq y, \quad y > 1,$$

which completes the proof. \square

We now turn to the contribution of the diffusion term.

Lemma 3.2. *Consider $\psi \in E_1^+$ and $r \in (1, m]$. Then*

$$- \int_0^\infty w_r(x) \partial_x^2 \psi(x) dx \geq -3r M_1(\psi) - r^2 \int_1^\infty x^{r-2} \psi(x) dx.$$

Proof. First, recalling that

$$\partial_x \psi(x) = - \int_x^\infty \partial_y^2 \psi(y) dy, \quad \psi(x) = \int_x^\infty (y-x) \partial_y^2 \psi(y) dy, \quad x \in (0, \infty),$$

by [22, Lemma 2.1], the properties $\psi \in E_1$ and $r \in (1, m]$ ensure that

$$\lim_{x \rightarrow \infty} |w_r(x) \partial_x \psi(x)| \leq (1+r) \lim_{x \rightarrow \infty} \int_x^\infty y^r |\partial_y^2 \psi(y)| dy = 0$$

and

$$\lim_{x \rightarrow \infty} |\partial_x w_r(x) \psi(x)| \leq r \lim_{x \rightarrow \infty} \int_x^\infty y (1 + y^{r-1}) |\partial_y^2 \psi(y)| dy = 0.$$

Therefore, since $w_r \in C^1([0, \infty))$ with $w_r(0) = \partial_x w_r(0) = 0$,

$$\begin{aligned} - \int_0^\infty w_r(x) \partial_x^2 \psi(x) dx &= - \left[w_r(x) \partial_x \psi(x) \right]_{x=0}^{x=\infty} + \left[\partial_x w_r(x) \psi(x) \right]_{x=0}^{x=\infty} - \int_0^\infty \psi(x) \partial_x^2 w_r(x) dx \\ &= -3(r-1) \int_0^1 x \psi(x) dx - r(r-1) \int_1^\infty x^{r-2} \psi(x) dx \\ &\geq -3r M_1(\psi) - r^2 \int_1^\infty x^{r-2} \psi(x) dx, \end{aligned}$$

as claimed. \square

We finally estimate the contribution of the nonlinear coagulation term which is, without much surprise, harder to handle.

Lemma 3.3. *Consider $\psi \in E_1^+$ and $r \in (1, m]$. Then*

$$\begin{aligned} \int_0^\infty w_r(x) \mathcal{K}(\psi)(x) dx &\leq \kappa_1(r) (M_1(\psi) + M_{1+(r-2)_+}(\psi)) \int_1^\infty x^r (1 + a(x))^\theta \psi(x) dx \\ &\quad + \kappa_1(r) M_1(\psi)^2. \end{aligned}$$

Proof. A standard computation gives

$$\int_0^\infty w_r(x) \mathcal{K}(\psi)(x) dx = \frac{1}{2} \int_0^\infty \int_0^\infty [w_r(x+y) - w_r(x) - w_r(y)] k(x, y) \psi(x) \psi(y) dy dx,$$

and we proceed differently for the contributions of large sizes, of small sizes, and the interactions between small and large sizes. We start with the former and infer from (1.15) and [12, Lemma 7.4.4] that

$$\begin{aligned} I_1 &:= \frac{1}{2} \int_1^\infty \int_1^\infty [w_r(x+y) - w_r(x) - w_r(y)] k(x, y) \psi(x) \psi(y) dy dx \\ &= \frac{1}{2} \int_1^\infty \int_1^\infty [(x+y)^r - x^r - y^r] k(x, y) \psi(x) \psi(y) dy dx \\ &\leq \frac{\max\{r, 2^r - 2\}}{2} \int_1^\infty \int_1^\infty \frac{x^r y + x y^r}{x+y} k(x, y) \psi(x) \psi(y) dy dx \\ &\leq \max\{r, 2^r - 2\} k_0 \int_1^\infty \int_1^\infty \left[\frac{x^{r+1} y^2}{(x+y)^2} + \frac{x^2 y^{r+1}}{(x+y)^2} \right] (1 + a(x))^\theta \psi(x) \psi(y) dy dx \\ &\leq 2^r k_0 \int_1^\infty \int_1^\infty [x^r y + x^2 y (x+y)^{r-2}] (1 + a(x))^\theta \psi(x) \psi(y) dy dx. \end{aligned}$$

At this point, either $r \in (1, 2]$ and

$$\begin{aligned} I_1 &\leq 2^r k_0 \int_1^\infty \int_1^\infty [x^r y + x^r y] (1 + a(x))^\theta \psi(x) \psi(y) \, dy dx \\ &\leq 2^{r+1} k_0 M_1(\psi) \int_1^\infty x^r (1 + a(x))^\theta \psi(x) \, dx. \end{aligned} \quad (3.3a)$$

Or $r > 2$ and it follows from the basic inequality

$$(x + y)^{r-2} \leq 2^{r-2} (x^{r-2} + y^{r-2}), \quad (x, y) \in (0, \infty)^2,$$

that

$$\begin{aligned} I_1 &\leq 2^r k_0 \int_1^\infty \int_1^\infty [x^r y + 2^{r-2} (x^r y + x^2 y^{r-1})] (1 + a(x))^\theta \psi(x) \psi(y) \, dy dx \\ &\leq 2^{2r-1} k_0 \int_1^\infty \int_1^\infty [x^r y^{r-1} + x^r y^{r-1}] (1 + a(x))^\theta \psi(x) \psi(y) \, dy dx \\ &\leq 4^r k_0 M_{r-1}(\psi) \int_1^\infty x^r (1 + a(x))^\theta \psi(x) \, dx. \end{aligned} \quad (3.3b)$$

We now study the contribution involving only small sizes. To this end, we observe that

$$\partial_x^2 w_r(z) \leq \kappa_2(r) z \quad \text{for a.a. } z \in (0, 2)$$

with $\kappa_2(r) := r(r+3)2^{(r-3)+}$, so that, for $(x, y) \in (0, 1)^2$,

$$w_r(x+y) - w_r(x) - w_r(y) = \int_0^x \int_0^y \partial_x^2 w_r(x_* + y_*) \, dy_* dx_* \leq \kappa_2(r) xy(x+y).$$

Therefore, using also (1.5a),

$$\begin{aligned} I_2 &:= \frac{1}{2} \int_0^1 \int_0^1 [w_r(x+y) - w_r(x) - w_r(y)] k(x, y) \psi(x) \psi(y) \, dy dx \\ &\leq \frac{\kappa_2(r)}{2} \int_0^1 \int_0^1 xy(x+y) k(x, y) \psi(x) \psi(y) \, dy dx \\ &\leq \kappa_2(r) k_* \int_0^1 \int_0^1 x^{2-2\theta_0} y^{2-2\theta_0} \frac{x+y}{x+y+(x+y)^m} \psi(x) \psi(y) \, dy dx \\ &\leq \kappa_2(r) k_* \left(\int_0^1 x^{2-2\theta_0} \psi(x) \, dx \right)^2. \end{aligned}$$

Since $\theta_0 \in [0, 1/2]$, we have $x^{2-2\theta_0} \leq x$ for $x \in (0, 1)$ and conclude that

$$I_2 \leq \kappa_2(r) k_* M_1(\psi)^2. \quad (3.4)$$

We finally estimate the contribution to coagulation mixing small and large sizes. In that case, owing to the symmetry of k ,

$$\begin{aligned} I_3 &:= \frac{1}{2} \int_1^\infty \int_0^1 [w_r(x+y) - w_r(x) - w_r(y)] k(x, y) \psi(x) \psi(y) \, dy dx \\ &\quad + \frac{1}{2} \int_0^1 \int_1^\infty [w_r(x+y) - w_r(x) - w_r(y)] k(x, y) \psi(x) \psi(y) \, dy dx \\ &= \int_0^1 \int_1^\infty [w_r(x+y) - w_r(x) - w_r(y)] k(x, y) \psi(x) \psi(y) \, dy dx. \end{aligned}$$

Since

$$\begin{aligned} w_r(x+y) - w_r(x) - w_r(y) &= (x+y)^r - y^r + \frac{r-3}{2}x - \frac{r-1}{2}x^3 \\ &\leq (x+y)^r - x^r - y^r + x^r + \frac{r-3}{2}x \\ &\leq \max\{r, 2^r - 2\} \frac{x^r y + xy^r}{x+y} + rx \end{aligned}$$

for $(x, y) \in (0, 1) \times (1, \infty)$ by [12, Lemma 7.4.4], we deduce from (1.5a) and the above inequality

$$\begin{aligned} I_3 &\leq 2^r k_* \int_0^1 \int_1^\infty \left(\frac{x^r y + xy^r}{x+y} + x \right) \frac{x^{1-2\theta_0} (1+a(y))^\theta y^m}{x+y+(x+y)^m} \psi(x) \psi(y) \, dy dx \\ &\leq 2^r k_* \int_0^1 \int_1^\infty \left(\frac{2xy^r}{y} + x \right) x^{1-2\theta_0} (1+a(y))^\theta \psi(x) \psi(y) \, dy dx. \end{aligned}$$

Recalling that $\theta_0 \in [0, 1/2]$, we further obtain

$$\begin{aligned} I_3 &\leq 2^{r+2} k_* \int_0^1 \int_1^\infty xy^r (1+a(y))^\theta \psi(x) \psi(y) \, dy dx \\ &\leq 2^{r+2} k_* M_1(\psi) \int_1^\infty y^r (1+a(y))^\theta \psi(y) \, dy. \end{aligned} \tag{3.5}$$

Since

$$\int_0^\infty w_r(x) \mathcal{K}(\psi)(x) \, dx = I_1 + I_2 + I_3,$$

we collect (3.3), (3.4), and (3.5) to complete the proof of Lemma 3.3. \square

Collecting the outcome of Lemma 3.1, Lemma 3.2, and Lemma 3.3 leads us to the following estimate.

Lemma 3.4. *Consider $\psi \in E_1^+$ and $r \in (1, m]$. Then*

$$\begin{aligned} & \int_0^\infty w_r(x) [\mathbb{A}\psi(x) + \mathcal{K}(\psi)(x)] \, dx \\ & \leq \kappa_3(r) \left(1 + M_1(\psi) + M_{1+(r-2)_+}(\psi)\right)^{1/(1-\theta)} \left(1 + M_1(\psi) + \int_1^\infty w_r(x)\psi(x) \, dx\right), \end{aligned}$$

recalling that θ is defined in (1.5).

Proof. We infer from Lemma 3.1, Lemma 3.2, and Lemma 3.3 that

$$\begin{aligned} & \int_0^\infty w_r(x) [\mathbb{A}\psi(x) + \mathcal{K}(\psi)(x)] \, dx \\ & \leq -\delta_r \int_1^\infty x^r a(x)\psi(x) \, dx + \int_1^\infty x a(x)\psi(x) \, dx \\ & \quad + 3r M_1(\psi) + r^2 \int_1^\infty x^{r-2}\psi(x) \, dx \\ & \quad + \kappa_1(r) (M_1(\psi) + M_{1+(r-2)_+}(\psi)) \int_1^\infty x^r (1 + a(x))^\theta \psi(x) \, dx + \kappa_1(r) M_1(\psi)^2. \end{aligned} \tag{3.6}$$

On the one hand, it follows from (1.2) that, with $R_r^{r-1} := \max\{1, 2r/\delta_r\}$,

$$\begin{aligned} \int_1^\infty x a(x)\psi(x) \, dx & \leq \|a\|_{L^\infty(1, R_r)} \int_1^{R_r} x\psi(x) \, dx + \frac{1}{R_r^{r-1}} \int_{R_r}^\infty x^r a(x)\psi(x) \, dx \\ & \leq \kappa(r) M_1(\psi) + \frac{\delta_r}{2} \int_1^\infty x^r a(x)\psi(x) \, dx \end{aligned} \tag{3.7}$$

and

$$\int_1^\infty x^{r-2}\psi(x) \, dx \leq \int_1^\infty x^{1+(r-2)_+}\psi(x) \, dx \leq M_{1+(r-2)_+}(\psi). \tag{3.8}$$

On the other hand, we infer from Young's inequality that

$$\begin{aligned} & \kappa_1(r) (M_1(\psi) + M_{1+(r-2)_+}(\psi)) \int_1^\infty x^r (1 + a(x))^\theta \psi(x) \, dx \\ & \leq \kappa_1(r) (M_1(\psi) + M_{1+(r-2)_+}(\psi)) \int_1^\infty x^r \psi(x) \, dx \\ & \quad + \kappa_1(r) (M_1(\psi) + M_{1+(r-2)_+}(\psi)) \int_1^\infty x^r a(x)^\theta \psi(x) \, dx \\ & \leq \kappa_1(r) (M_1(\psi) + M_{1+(r-2)_+}(\psi)) \int_1^\infty x^r \psi(x) \, dx \\ & \quad + \frac{\delta_r}{2} \int_1^\infty x^r a(x)\psi(x) \, dx \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2}{\delta_r} \right)^{\theta/(1-\theta)} \kappa_1(r)^{1/(1-\theta)} (M_1(\psi) + M_{1+(r-2)_+}(\psi))^{1/(1-\theta)} \int_1^\infty x^r \psi(x) dx \\
& \leq \frac{\delta_r}{2} \int_1^\infty x^r a(x) \psi(x) dx + \kappa(r) (1 + M_1(\psi) + M_{1+(r-2)_+}(\psi))^{1/(1-\theta)} \int_1^\infty x^r \psi(x) dx.
\end{aligned}$$

Combining (3.6), (3.7), (3.8), and the above inequality leads us to

$$\begin{aligned}
& \int_0^\infty w_r(x) [\mathbb{A}\psi(x) + \mathcal{K}(\psi)(x)] dx \\
& \leq \kappa(r) (1 + M_1(\psi) + M_{1+(r-2)_+}(\psi))^{1/(1-\theta)} \left(1 + \int_1^\infty x^r \psi(x) dx \right).
\end{aligned}$$

We finally use (3.2) to complete the proof. \square

To finish off the proof of Theorem 1.3, we shall use Lemma 3.4 to derive time-dependent bounds on the solutions to (1.1) in E_0 . Bearing in mind that the first moment of solutions to (1.1) does not vary with time according to (1.9), it is then sufficient to obtain estimates in $L_1((0, \infty), w_m(x)dx)$. Observing that the right-side of the inequality in Lemma 3.4 depends linearly on ψ when $r \in (1, 2]$, a feature which is not available when $r > 2$, an application of Lemma 3.4 will only provide an estimate in $X_{\min\{m, 2\}}$ in a first step. As this is not sufficient to conclude when $m > 2$, we shall use an iterative procedure as in [7] in that case.

Proof of Theorem 1.3: $m \in (1, 2]$. Let $\xi \in [0, 1)$ with $2\theta < 1 + \xi$ and consider $f \in E_\xi^+$. According to Theorem 1.1, there is a unique classical solution $\phi = \phi(\cdot; f)$ to (1.1) defined on $[0, t^+(f))$ and satisfying

$$M_1(\phi(t)) = M_1(f) \leq \|f\|_{E_0}, \quad t \in [0, t^+(f)). \quad (3.9)$$

Let $T > 0$ and consider $t \in (0, T \wedge t^+(f))$. Then $\phi(t) \in E_1$ by Theorem 1.1 and we infer from (1.1), Lemma 3.4 with $r = m$, and (3.9) that

$$\frac{d}{dt} \int_0^\infty w_m(x) \phi(t, x) dx \leq \kappa(\|f\|_{E_0}) \left(1 + \int_0^\infty w_m(x) \phi(t, x) dx \right).$$

Integrating the above differential inequality and using (3.2) give

$$\begin{aligned}
\int_0^\infty w_m(x) \phi(t, x) dx & \leq \kappa(\|f\|_{E_0}) e^{\kappa(\|f\|_{E_0})t} \left(1 + \int_0^\infty w_m(x) f(x) dx \right) \\
& \leq \kappa(\|f\|_{E_0}) e^{\kappa(\|f\|_{E_0})t} (1 + m\|f\|_{E_0})
\end{aligned}$$

for $t \in (0, T \wedge t^+(f))$; that is, thanks to (3.2) and (3.9),

$$\|\phi(t)\|_{E_0} \leq 2 \int_0^\infty (w_m(x) + mx) \phi(t, x) dx \leq \kappa(\|f\|_{E_0}) e^{\kappa(\|f\|_{E_0})t}$$

for $t \in (0, T \wedge t^+(f))$. We now infer from Theorem 1.1 **(d)** that $t^+(f) = \infty$. \square

To reach higher values of m , we proceed along the lines of [7] and employ an iterative method.

Proof of Theorem 1.3: $m > 2$. Keeping the notation used in the proof of Theorem 1.3 for $m \in (1, 2]$, we introduce

$$\mu_r(T) := \sup_{t \in [0, T \wedge t^+(f))} \int_0^\infty (x + x^r) \phi(t, x) \, dx \in (0, \infty], \quad T > 0.$$

We first perform the same computation as in the previous proof with $m = 2$ to obtain that

$$\mu_2(T) < \infty, \quad T > 0. \quad (3.10)$$

We next take $r \in (2, m]$ and claim that, for each $T > 0$,

$$\mu_{r-1}(T) < \infty \implies \mu_r(T) < \infty. \quad (3.11)$$

Thus suppose that $\mu_{r-1}(T) < \infty$ for some $T > 0$. We then infer from Lemma 3.4, (3.9), and (3.11) that, for $t \in (0, T \wedge t^+(f))$,

$$\begin{aligned} & \frac{d}{dt} \int_0^\infty w_r(x) \phi(t, x) \, dx \\ & \leq \kappa_3(r) \left(1 + M_1(\phi(t)) + M_{1+(r-2)_+}(\phi(t))\right)^{1/(1-\theta)} \left(1 + M_1(\phi(t)) + \int_1^\infty w_r(x) \phi(t, x) \, dx\right) \\ & \leq \kappa_3(r) \left(1 + M_1(f) + M_{r-1}(\phi(t))\right)^{1/(1-\theta)} \left(1 + M_1(f) + \int_1^\infty w_r(x) \phi(t, x) \, dx\right) \\ & \leq \kappa(r, \|f\|_{E_0}) (1 + \mu_{r-1}(T))^{1/(1-\theta)} \left(1 + \int_0^\infty w_r(x) \phi(t, x) \, dx\right). \end{aligned}$$

After integrating with respect to time and using (3.2), we end up with

$$\int_0^\infty (x + x^r) \phi(t, x) \, dx \leq \kappa(r, \|f\|_{E_0}) e^{\kappa(r, \|f\|_{E_0})(1 + \mu_{r-1}(T))^{1/(1-\theta)} t}, \quad t \in (0, T \wedge t^+(f)).$$

Consequently, recalling that $\mu_{r-1}(T) < \infty$,

$$\mu_r(T) < \infty,$$

and (3.11) is proved.

We now introduce $r_i := m - \lfloor m \rfloor + i$ for $1 \leq i \leq \lfloor m \rfloor$ and note that $r_1 \in [1, 2]$. We then deduce from (3.9) and (3.10) that $\mu_{r_1}(T) < \infty$ for all $T > 0$. We next apply (3.11) recursively to obtain that $\mu_{r_i}(T) < \infty$ for all $T > 0$ and $1 \leq i \leq \lfloor m \rfloor$. In particular, $\mu_m(T) = \mu_{r_{\lfloor m \rfloor}}(T) < \infty$ for all $T > 0$ and we have established (1.12), thereby completing the proof. \square

ACKNOWLEDGMENTS

This work was done while PhL enjoyed the kind hospitality and support of the Institut für Angewandte Mathematik, Leibniz Universität Hannover.

REFERENCES

- [1] M. AIZENMAN AND T. A. BAK, *Convergence to equilibrium in a system of reacting polymers*, Comm. Math. Phys., 65 (1979), pp. 203–230.
- [2] D. J. ALDOUS, *Deterministic and stochastic models for coalescence (aggregation and coagulation): A review of the mean-field theory for probabilists*, Bernoulli, 5 (1999), pp. 3–48.
- [3] H. AMANN, *Semigroups and nonlinear evolution equations*, Linear Algebra Appl., 84 (1986), pp. 3–32.
- [4] H. AMANN, *Linear and quasilinear parabolic problems. Vol. I*, vol. 89 of Monographs in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.
- [5] H. AMANN AND CH. WALKER, *Local and global strong solutions to continuous coagulation-fragmentation equations with diffusion*, J. Differential Equations, 218 (2005), pp. 159–186.
- [6] J. BANASIAK, *Shattering and non-uniqueness in fragmentation models—an analytic approach*, Phys. D, 222 (2006), pp. 63–72.
- [7] ———, *Global solutions of continuous coagulation-fragmentation equations with unbounded coefficients*, Discrete Contin. Dyn. Syst. Ser. S, 13 (2020), pp. 3319–3334.
- [8] J. BANASIAK AND W. LAMB, *Analytic fragmentation semigroups and continuous coagulation-fragmentation equations with unbounded rates*, J. Math. Anal. Appl., 391 (2012), pp. 312–322.
- [9] J. BANASIAK AND W. LAMB, *Growth-fragmentation-coagulation equations with unbounded coagulation kernels*, Philos. Trans. Roy. Soc. A, 378 (2020), pp. 20190612, 22.
- [10] J. BANASIAK, W. LAMB, AND M. LANGER, *Strong fragmentation and coagulation with power-law rates*, J. Engrg. Math., 82 (2013), pp. 199–215.
- [11] J. BANASIAK, W. LAMB, AND PH. LAURENÇOT, *Analytic methods for coagulation-fragmentation models. Vol. I*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2020.
- [12] ———, *Analytic methods for coagulation-fragmentation models, Vol. II*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2020.
- [13] B. BASSE, G. C. WAKE, D. J. N. WALL, AND B. VAN BRUNT, *On a cell-growth model for plankton*, Math. Med. Biol., 21 (2004), pp. 49–61.
- [14] J. BERTOIN, *Random fragmentation and coagulation processes*, vol. 102 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2006.
- [15] P. B. DUBOVSKII, *Mathematical theory of coagulation*, vol. 23 of Lecture Notes Series, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1994.
- [16] K.-J. ENGEL AND R. NAGEL, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [17] J. FERKINGHOFF-BORG, M. H. JENSEN, J. MATHIESEN, AND P. OLESEN, *Diffusion, fragmentation and merging: rate equations, distributions and critical points*, Phys. D, 222 (2006), pp. 88–96.
- [18] J. FERKINGHOFF-BORG, M. H. JENSEN, J. MATHIESEN, P. OLESEN, AND K. SNEPPEN, *Competition between diffusion and fragmentation: An important evolutionary process of nature*, Phys. Rev. Lett., 91 (2003), p. 266103.
- [19] H. FLYVBJERG, T. E. HOLY, AND S. LEIBLER, *Stochastic dynamics of microtubules: A model for caps and catastrophes*, Phys. Rev. Lett., 73 (1994), pp. 2372–2375.
- [20] D. HENRY, *Geometric theory of semilinear parabolic equations*, vol. 840, Springer, Cham, 1981.
- [21] T. KATO, *Schrödinger operators with singular potentials*, Israel J. Math., 13 (1972), pp. 135–148.
- [22] PH. LAURENÇOT AND CH. WALKER, *The fragmentation equation with size diffusion: Well-posedness and long-term behavior*. arXiv: 2104.14798, 2021.
- [23] PH. LAURENÇOT AND S. MISCHLER, *On coalescence equations and related models*, in Modeling and computational methods for kinetic equations, Model. Simul. Sci. Eng. Technol., Birkhäuser Boston, Boston, MA, 2004, pp. 321–356.

- [24] J. MATHIESEN, J. FERKINGHOFF-BORG, M. H. JENSEN, M. LEVINSEN, P. OLESEN, D. DAHL-JENSEN, AND A. SVENSON, *Dynamics of crystal formation in the greenland NorthGRIP ice core*, J. Glaciol., 50 (2004), pp. 325–328.
- [25] D. J. MCLAUGHLIN, W. LAMB, AND A. C. MCBRIDE, *An existence and uniqueness result for a coagulation and multiple-fragmentation equation*, SIAM J. Math. Anal., 28 (1997), pp. 1173–1190.
- [26] J. B. MCLEOD, *On the scalar transport equation*, Proc. London Math. Soc. (3), 14 (1964), pp. 445–458.
- [27] Z. A. MELZAK, *A scalar transport equation*, Trans. Amer. Math. Soc., 85 (1957), pp. 547–560.
- [28] P. OLESEN, J. FERKINGHOFF-BORG, M. H. JENSEN, AND J. MATHIESEN, *Diffusion, fragmentation, and coagulation processes: Analytical and numerical results*, Phys. Rev. E, 72 (2005), p. 031103.
- [29] D. RAMKRISHNA, *Population balances. Theory and applications to particulate systems in engineering*, Academic Press, 2000.
- [30] M. v. SMOLUCHOWSKI, *Drei Vorträge über Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen*, Physik. Zeitschr., 17 (1916), pp. 557–571, 585–599.
- [31] ———, *Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen*, Zeitschrift f. phys. Chemie, 92 (1917), pp. 129–168.
- [32] J. A. D. WATTIS, *An introduction to mathematical models of coagulation-fragmentation processes: A discrete deterministic mean-field approach*, Phys. D, 222 (2006), pp. 1–20.
- [33] F. B. WEISSLER, *Local existence and nonexistence for semilinear parabolic equations in L^p* , Indiana Univ. Math. J., 29 (1980), pp. 79–102.
- [34] A. YAGI, *Abstract parabolic evolution equations and their applications*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UMR 5219, UNIVERSITÉ DE TOULOUSE, CNRS, F-31062 TOULOUSE CEDEX 9, FRANCE

Email address: `laurenco@math.univ-toulouse.fr`

LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ANGEWANDTE MATHEMATIK, WELFENGARTEN 1, D-30167 HANNOVER, GERMANY

Email address: `walker@ifam.uni-hannover.de`