



**HAL**  
open science

## Reachability results for perturbed heat equations

Sylvain Ervedoza, Kévin Le Balc'H, Marius Tucsnak

► **To cite this version:**

Sylvain Ervedoza, Kévin Le Balc'H, Marius Tucsnak. Reachability results for perturbed heat equations. 2021. hal-03380745v1

**HAL Id: hal-03380745**

**<https://hal.science/hal-03380745v1>**

Preprint submitted on 15 Oct 2021 (v1), last revised 17 Nov 2022 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# REACHABILITY RESULTS FOR PERTURBED HEAT EQUATIONS

SYLVAIN ERVEDOZA, KÉVIN LE BALC'H, AND MARIUS TUCSNAK

ABSTRACT. This work studies the reachable space of infinite dimensional control systems which are null controllable in any positive time, the typical example being the heat equation controlled from the boundary or from an arbitrary open set. The focus is on the robustness of the reachable space with respect to linear or nonlinear perturbations of the generator. More precisely, our first main result asserts that this space is invariant under perturbations which are small (in an appropriate sense). A second main result asserts the invariance of the reachable space with respect to perturbations which are compact (again in an appropriate sense), provided that a Hautus type condition is satisfied. Moreover, our methods give precise information on the behavior of the reachable space when the generator is perturbed by a class of nonlinear operators. When applied to the classical heat equation, our results provide detailed information on the reachable space when the generator is perturbed by a small potential or by a class of non local operators, and in particular in one space dimension, we deduce from our analysis that the reachable space for perturbations of the 1-d heat equation is a space of holomorphic functions. We also show how our approach leads to reachability results for a class of semi-linear parabolic equations.

## CONTENTS

1. Introduction	2
2. Statement of the main results	3
2.1. Well-posed control systems and their reachable spaces	4
2.2. Main abstract results	5
3. Reachable spaces for the heat equation and its perturbations: state of the art	8
4. Applications to perturbed heat equations	12
4.1. The case of small potentials	12
4.2. Non-local perturbation operators with Neumann controls in 1-d	13
4.3. Non-local perturbation operators with distributed controls in multi-d: the general case	14
4.4. Non-local perturbation operators with distributed controls in multi-d: the case of a ball controlled from a ring	16
5. More background on infinite dimensional control systems	17
5.1. Complements on well-posed control systems	17

---

The three authors are partially supported by the Project TRECOS ANR-20-CE40-0009 funded by the ANR (2021–2024). The last two authors have benefited from financial support of the French State, managed by the French National Research Agency (ANR) in the frame of the “Investments for the future” Programme IdEx Bordeaux - SysNum (ANR-10-IDEX-03-02). The last author has been supported by the grant “Numerical simulation and optimal control in view of temperature regulation in smart buildings” of the Nouvelle Aquitaine Region.

5.2. The role of duality	18
5.3. Negative generators with compact resolvents	21
6. Well-posed control systems on $\text{Ran } \Phi_\tau$ : Proof of Theorem 2.1	22
7. Reachability results under small perturbations: Proof of Theorems 2.2 and 2.3	24
8. A Hautus test for approximate controllability	26
9. Proof of Theorem 2.4	31
10. Towards non autonomous or semi-linear systems	34
10.1. Abstract results	34
10.2. Applications to non autonomous or semilinear parabolic equations	37
Appendix A. Unique continuation properties for non-local linear perturbations	41
A.1. Checking condition (b) in Propositions 4.2 and 4.3	41
A.2. Unique continuation from the boundary in the 1-d case	43
Appendix B. Proof of Proposition 3.1	45
References	46

## 1. INTRODUCTION

Determining the reachable space of controlled dynamical systems is one of the major problems in control theory. This question is well understood for finite dimensional linear systems but it clearly requires further developments in an infinite dimensional context, namely for systems governed by partial differential equations. One of the most challenging cases is when the system is described by a parabolic type equation. Indeed, the well known smoothing effect appearing in this context is partially balanced by the influence of the input functions, which are generally supposed to be only in  $L^2$ . Giving a precise characterization of the states which can be reached in some time  $\tau > 0$  is thus a very difficult question: even for the constant coefficients heat equation in one space dimension and controlled from the boundary, the full characterization of the reachable space has been obtained only in the last years (see the brief description in Section 3 below of some of the relevant references). For the heat equation perturbed by a linear (local or non local) operator, assuming that these perturbations are very “smooth”, one can naturally conjecture that they will not affect the reachable space. However, the existing methods for the study of the reachable space for systems described by the constant coefficients heat equation seem hardly adaptable to the perturbed case. As far as we know, the only published work proposing a unified methodology to study the reachable spaces for systems described by the heat equation and a class of their perturbations is Laurent and Rosier [22].

On the other hand, the related problem of the robustness of the exact controllability property with respect to perturbations of the generator is well understood, even in a general abstract context. More precisely, robustness under small perturbations is proved in Hadd [16]. The case of compact perturbations is tackled in Cîndea and Tucsnak [5] and Duprez and Olive [8], following the compactness-uniqueness method introduced in Bardos, Lebeau and Rauch [2]. The methodology employed for exactly controllable systems is clearly not directly applicable to systems described by parabolic PDEs. As far as we know, for these systems there is no analogue of the small perturbation result in [16] and the only tentative to apply

compactness-uniqueness arguments is due to Fenández-Cara, Lu and Zuazua [12], where the authors discuss systems described by a class of non local perturbations of the heat equation.

The main contribution of the present paper is that we propose a quite complete perturbation theory for systems which are null controllable in any positive time, with emphasis on systems described by perturbed heat equations. We prove, in particular, that this theory has many common features with the similar one for exactly controllable systems (see [16], [5] or Duprez and Olive [8]). More precisely, a key point in our approach is to prove that for this class of problems the corresponding semigroup leaves the reachable space invariant and it acts continuously on this reachable space. Consequently, the considered systems become exactly controllable when restricted to their reachable spaces, fact which enables us to develop both small and compact perturbations techniques. These abstract theorems, when combined with recently obtained characterizations of the reachable spaces for systems described by the heat equation, give results which are either completely new or strongly improving those in the existing literature. As a consequence, we also often obtain that the null controllability property is invariant with respect to the considered class of perturbations.

We describe below the organization of the remaining part of this paper. In Section 2, after introducing the minimal necessary background on well posed infinite dimensional control systems, we give the statements of our main abstract results. Section 3 gives an overview of the recent advances on the characterization of the reachable spaces for the heat equation and its perturbations by complex analytic methods. In Section 4 we prove that, when applied to various systems described by perturbed heat equation, our main abstract theorems give a series of new and often sharp results. In Section 5 we begin to prepare the proofs of our main abstract results by providing some additional background on well posed infinite dimensional control systems. Section 6 is devoted to the proof of the fact that every system which is null-controllable in any positive time can be seen as an *exactly controllable one*, once it is restricted to its reachable space. In Section 7, we prove that the reachable space of a small-time null-controllable system is invariant with respect to “small” perturbations. In Section 8 we derive a Hautus test for the approximate controllability of compactly perturbed systems. The main result allowing the use of compactness-uniqueness techniques to study the reachable space for systems obtained by compact perturbations is proved in Section 9. Finally, Section 10 presents an approach, based on Section 6, to deal with time dependent or non-linear perturbations. In the Appendix, Section A is devoted to a discussion of the unique continuation properties related to non-local linear perturbations. Section B is dedicated to a proof of a technical result concerning the reachable space of the heat equation when the control domain is a neighborhood of the boundary.

## 2. STATEMENT OF THE MAIN RESULTS

In this section, we state our main abstract results. The corresponding proofs will be given in Sections 6, 7, 8 and 9. Giving the precise statements requires some basic concepts and results on well posed linear control systems, which are recalled in Subsection 2.1 below and which will be completed in Section 5, before detailing the main proofs.

**2.1. Well-posed control systems and their reachable spaces.** The concept of well posed linear control systems has been introduced in Weiss [35] under the name of *abstract control systems*. In this subsection we recall, following [35] and Tucsnak and Weiss [33, Chapters 2,3,4], [34], just those definitions and basic facts which are necessary for the statement of our main results. More background on well posed linear control systems will be provided in Section 5.

We first introduce the Hilbert spaces  $U$  (the input space) and  $X$  (the state space), which will be constantly identified with their duals. If there is no ambiguity, the inner products and the norms on  $X$  and  $U$  will be simply denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ , respectively.

**Definition 2.1.** A *well-posed linear control system with state space  $X$  and input space  $U$*  is a couple  $\Sigma = (\mathbb{T}, \Phi)$  of families of operators such that

- (1)  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  is a  $C^0$  semigroup of bounded linear operators on  $X$ ;
- (2)  $\Phi = (\Phi_t)_{t \geq 0}$  is a family of bounded linear operators from  $L^2([0, \infty); U)$  to  $X$ , called input maps, such that

$$(2.1) \quad \Phi_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{T}_t \Phi_{\tau} u + \Phi_t v \quad (t, \tau \geq 0, u, v \in L^2([0, \infty); U)),$$

where the  $\tau$ -concatenation of two signals  $u$  and  $v$ , denoted  $u \underset{\tau}{\diamond} v$ , is the function

$$u \underset{\tau}{\diamond} v = \begin{cases} u(t) & \text{for } t \in [0, \tau), \\ v(t - \tau) & \text{for } t \geq \tau. \end{cases}$$

Let  $A : \mathcal{D}(A) \rightarrow X$  be the generator of  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  on  $X$ . We denote by  $\mathbb{T}^*$  the adjoint semigroup, which is generated by the adjoint of  $A^*$  of  $A$ . The operator domain  $\mathcal{D}(A)$  when endowed with norm

$$(2.2) \quad \|z_0\|_{X_1}^2 = \|z_0\|^2 + \|Az_0\|^2 \quad (z_0 \in X_1),$$

is a Hilbert space. This Hilbert space is denoted by  $X_1$ . Similarly, we denote by  $X_1^d$  the Hilbert space obtained by endowing  $\mathcal{D}(A^*)$  with the norm

$$(2.3) \quad \|z_0\|_{X_1^d}^2 = \|z_0\|^2 + \|A^* z_0\|^2 \quad (z_0 \in X_1^d).$$

Let  $X_{-1}$  be the dual of  $X_1^d$  with respect to the pivot space  $X$ , so that  $X_1 \subset X \subset X_{-1}$ , with continuous and dense embeddings. Note that, for each  $k \in \{-1, 1\}$ , the original semigroup  $\mathbb{T}$  has a restriction (or an extension) to  $X_k$  that is the image of  $\mathbb{T}$  through the unitary operator  $(\beta I - A)^{-k} \in \mathcal{L}(X, X_k)$ , where  $\beta \in \rho(A)$  (the resolvent set of  $A$ ). We refer to [33, Remark 2.10.5] for a proof of the last statement. This restriction (or extension) will be still denoted by  $\mathbb{T}$ .

An important consequence of assumptions (1) and (2) in Definition 2.1 is that there exists a unique  $B \in \mathcal{L}(U, X_{-1})$ , called the *control operator* of  $\Sigma$ , such that

$$(2.4) \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma \quad (t \geq 0, u \in L^2([0, \infty); U)).$$

Notice that in the above formula,  $\mathbb{T}$  acts on  $X_{-1}$  and the integration is carried out in  $X_{-1}$ . The operator  $B$  can be found by

$$(2.5) \quad Bv = \lim_{t \rightarrow 0^+} \frac{1}{t} \Phi_t(\chi \cdot v) \quad (v \in U),$$

where  $\chi$  denotes the characteristic function of the interval  $[0, 1]$ . We mention that it follows from the above definitions that if  $(\mathbb{T}, \Phi)$  is a wellposed control system then for all  $u \in L^2([0, \infty); U)$ ,  $t \mapsto \Phi_t u$  is a continuous function from  $[0, \infty)$  to  $X$ .

From the above facts it follows that a well-posed control system can alternatively be described by a pair  $(A, B)$ , where  $A : \mathcal{D}(A) \rightarrow X$  generates a  $C^0$  semigroup  $\mathbb{T}$  on  $X$  and  $B \in \mathcal{L}(U, X_{-1})$  is an *admissible control operator* for  $\mathbb{T}$ . This latter property means that for some  $t > 0$ , the operator  $\Phi_t$  defined by (2.4) has its range contained in  $X$ . We refer again to [33, Sections 4 and 5] for more material on this concept. We just mention here that, for every  $\psi \in X$ , the associated *state trajectory*  $z(t) = \mathbb{T}_t\psi + \Phi_t u$  satisfies  $z(0) = \psi$  and the usual differential equation  $\dot{z} = Az + Bu$ , see Proposition 5.1 below.

In the remaining part of this work we describe, according to our convenience, every well posed control system either by a couple  $(\mathbb{T}, \Phi)$  as in Definition 2.1 or by the couple  $(A, B)$ , where  $A$  is the generator of  $\mathbb{T}$  and  $B$  is the unique operator in  $\mathcal{L}(U, X_{-1})$  satisfying (2.4).

Given a well-posed control system  $\Sigma = (\mathbb{T}, \Phi)$  and  $t > 0$ , the *reachable space in time  $\tau$*  of  $\Sigma$  is defined as  $\text{Ran } \Phi_t$ . This space can be endowed with the norm induced from  $L^2([0, t]; U)$  which is

$$(2.6) \quad \|\eta\|_{\text{Ran } \Phi_t} = \inf_{\substack{u \in L^2([0, t]; U) \\ \Phi_t u = \eta}} \|u\|_{L^2([0, t]; U)} \quad (\eta \in \text{Ran } \Phi_t).$$

**Remark 2.1.** It is not difficult to check (see, for instance, Saitoh and Sawano [28, Theorem 2.36]) that, when endowed with the norm (2.6),  $\text{Ran } \Phi_t$  becomes a Hilbert space, isomorphic with the orthogonal space in  $L^2([0, t]; U)$  of  $\text{Ker } \Phi_t$ .

The notion of reachable space can be used to define the main three controllability concepts used in infinite dimensional system theory (note that in the finite dimensional case all these concepts coincide).

**Definition 2.2.** Let  $t > 0$  and let the pair  $(\mathbb{T}, \Phi)$  define a well-posed control system.

- The pair  $(\mathbb{T}, \Phi)$  is *exactly controllable in time  $t$*  if  $\text{Ran } \Phi_t = X$ .
- The pair  $(\mathbb{T}, \Phi)$  is *approximately controllable in time  $t$*  if  $\text{Ran } \Phi_t$  is dense in  $X$ .
- The pair  $(\mathbb{T}, \Phi)$  is *null-controllable in time  $t$*  if  $\text{Ran } \Phi_t \supset \text{Ran } \mathbb{T}_t$ .

We recall the following classical result, which goes back to Fattorini [11] and Seidman [29]. Following the ideas in [29], a very short proof is provided in Kellay, Normand and Tucsnak [21].

**Proposition 2.1.** *If the well-posed linear control system  $(\mathbb{T}, \Phi)$  is null controllable in any positive time then  $\text{Ran } \Phi_t$  does not depend on  $t > 0$ .*

**Remark 2.2.** Let  $\tau > 0$ . If  $(\mathbb{T}, \Phi)$  is null controllable in any positive time then we know from Proposition 2.1 that  $\text{Ran } \Phi_t = \text{Ran } \Phi_\tau$  for every  $t \in (0, \tau)$ . Moreover, it is easily checked that

$$(2.7) \quad \|\eta\|_{\text{Ran } \Phi_\tau} \leq \|\eta\|_{\text{Ran } \Phi_t} \quad (t \in (0, \tau), \eta \in \text{Ran } \Phi_\tau).$$

Using the closed graph theorem, it follows that the norms  $\|\cdot\|_{\text{Ran } \Phi_\tau}$  and  $\|\cdot\|_{\text{Ran } \Phi_t}$  are equivalent on  $\text{Ran } \Phi_\tau$ .

**2.2. Main abstract results.** We are now in a position to state our main abstract results. The first one says, roughly speaking, that every system which is null controllable in any positive time can be seen as an exactly controllable one (in the sense of Definition 2.2).

**Theorem 2.1.** *Let  $\Sigma = (\mathbb{T}, \Phi)$  be a well posed control system which is null controllable in any positive time. For  $\tau > 0$  we denote by  $\tilde{\mathbb{T}} = \left(\tilde{\mathbb{T}}_t\right)_{t \geq 0}$  the semigroup of operators defined by*

$$\tilde{\mathbb{T}}_t = \tilde{\mathbb{T}}_t|_{\text{Ran } \Phi_\tau}, \quad (t \geq 0).$$

*Then the family  $\tilde{\mathbb{T}} = (\mathbb{T}_t|_{\text{Ran } \Phi_\tau})_{t \geq 0}$  does not depend on the choice of  $\tau > 0$ , and forms a  $C^0$  semigroup on  $\text{Ran } \Phi_\tau$ . Moreover, the couple  $\tilde{\Sigma} = (\tilde{\mathbb{T}}, \Phi)$  determines a well-posed control system with state space  $\text{Ran } \Phi_\tau$  and input space  $U$ . Finally, this system is exactly controllable in any positive time.*

This result, proved in Section 6, allows us to tackle linear perturbations  $P$  of the generator  $A$  in a manner which is quite close of the pretty simple methodology employed for exactly controllable systems, see [16, 5, 8]. More precisely, assume that  $P \in \mathcal{L}(X)$  and let  $\mathbb{T}^P$  be the strongly continuous semigroup generated by  $A + P$  (see Proposition 7.1 below). Then, according to, for instance, [33, Corollary 5.5.1],  $\Sigma^P = (\mathbb{T}^P, \Phi^P)$ , where  $\mathbb{T}^P$  is the semigroup generated by  $A + P$  and

$$(2.8) \quad \Phi_t^P u = \int_0^t \mathbb{T}_{t-\sigma}^P B u(\sigma) d\sigma \quad (t \geq 0, u \in L^2([0, \infty); U)),$$

defines a well posed control system with state space  $X$  and input space  $U$ .

We can now state our two first main abstract results, which will be proved in Section 7 below.

**Theorem 2.2.** *Assume that  $\Sigma$  is null controllable in any positive time and that  $P \in \mathcal{L}(X) \cap \mathcal{L}(\text{Ran } \Phi_{\tau_0})$  for some  $\tau_0 > 0$ . Then the corresponding input maps  $\Phi^P$  satisfy*

$$(2.9) \quad \Phi_\tau^P \in \mathcal{L}(L^2([0, \tau]; U), \text{Ran } \Phi_{\tau_0}), \quad (\tau > 0),$$

and

$$(2.10) \quad \text{Ran } \Phi_\tau^P \subset \text{Ran } \Phi_{\tau_0} \quad (\tau > 0).$$

**Theorem 2.3.** *Assume that  $\Sigma$  is null controllable in any positive time and that  $P \in \mathcal{L}(X) \cap \mathcal{L}(\text{Ran } \Phi_{\tau_0})$  for some  $\tau_0 > 0$ .*

*Then there exists  $\delta_{\tau_0} > 0$  such that if*

$$(2.11) \quad \|P\|_{\mathcal{L}(\text{Ran } \Phi_{\tau_0})} \leq \delta_{\tau_0},$$

then

$$\text{Ran } \Phi_{\tau_0}^P = \text{Ran } \Phi_{\tau_0}.$$

**Remark 2.3.** We emphasize that, in Theorem 2.3, the parameter  $\delta_{\tau_0}$  depends on the time horizon  $\tau_0$ , and it is not clear if the condition (2.11) for  $\tau_0$  would imply (2.11) for other values of  $\tau$ . Accordingly, within the setting of Theorem 2.3, we do not know if  $\text{Ran } \Phi_\tau^P$  is independent of  $\tau > 0$ . In fact, we do not even know if the assumptions of Theorem 2.3 are sufficient to guarantee that  $\text{Ran } \mathbb{T}_\tau^P \subset \text{Ran } \Phi_{\tau_0}^P$ , i.e. the null-controllability of  $(A + P, B)$  at time  $\tau_0$ .

We will see below in Remark 2.5 that, provided an additional technical assumption (namely (2.15) below), one can recover that  $\text{Ran } \Phi_\tau^P = \text{Ran } \Phi_\tau$  for all  $\tau > 0$  and that  $\text{Ran } \mathbb{T}_\tau^P \subset \text{Ran } \Phi_\tau^P$  for all  $\tau > 0$ .

Note that the result above does not require any special property (like analyticity) of the semigroup  $\mathbb{T}$ . This means, in particular, that it can be applied to systems

which are exactly controllable in arbitrarily small time (like certain systems described by the Schrödinger equations, see [33, Section 11.6] for examples of this type). However, applying Theorem 2.3 in this case does not bring any novelty with respect to [16]. Oppositely, as it will be detailed in Section 4, Theorems 2.2 and 2.3 lead to new results when applied to systems described by perturbations of the heat equations or, more generally, in the case when the generator  $A$  is a negative operator (this means that  $A$  is self-adjoint and  $\langle A\varphi, \varphi \rangle < 0$  for every  $\varphi \in \mathcal{D}(A) \setminus \{0\}$ ) and has compact resolvents. Moreover, in this case we can prove a third abstract result, inspired by the compactness-uniqueness methodology introduced in [2]. To give the precise statement of this result we remind the reader that if  $A$  is negative and has compact resolvents then  $A$  is diagonalizable with an orthonormal basis  $(\varphi_k)_{k \in \mathbb{N}}$  of eigenvectors, such that the corresponding family of eigenvalues  $(-\lambda_k)_{k \in \mathbb{N}}$  satisfies  $\lambda_k > 0$  for every  $k \in \mathbb{N}$  and  $\lim_{k \rightarrow \infty} \lambda_k = \infty$  (see, for instance, [33, Proposition 3.2.12]). Moreover, we have (see, for instance, [33, Proposition 3.2.9])

$$\mathcal{D}(A) = \left\{ z \in X \mid \sum_{k \in \mathbb{N}} \lambda_k^2 |\langle z, \varphi_k \rangle|^2 < \infty \right\},$$

$$Az = - \sum_{k \in \mathbb{N}} \lambda_k \langle z, \varphi_k \rangle \varphi_k \quad (z \in \mathcal{D}(A)),$$

and

$$(2.12) \quad \mathbb{T}_t z = \sum_{k \in \mathbb{N}} \exp(-\lambda_k t) \langle z, \varphi_k \rangle \varphi_k, \quad (t \geq 0, z \in X).$$

Within this context, the scale of Hilbert spaces

$$X_1 \subset X \subset X_{-1},$$

introduced in Subsection 2.1, can be completed to a scale  $(X_\alpha)_{\alpha \in \mathbb{R}}$  as follows: for every  $\alpha \geq 0$ , we set  $X_\alpha := \mathcal{D}((-A)^\alpha)$ , equipped with the norm

$$(2.13) \quad \|\psi\|_\alpha^2 := \sum_{k \in \mathbb{N}} \lambda_k^{2\alpha} |\langle \psi, \varphi_k \rangle|^2.$$

For  $\alpha > 0$ , the space  $X_{-\alpha}$  is defined as the dual space of  $X_\alpha$  with respect to the pivot space  $X$ . Equivalently,  $X_{-\alpha}$  is the completion of  $X$  for the norm

$$\|\psi\|_{-\alpha}^2 = \sum_{k \in \mathbb{N}} \lambda_k^{-2\alpha} |\langle \psi, \varphi_k \rangle|^2.$$

We then have this following result, which follows from Proposition 9.2 and Corollary 9.1 stated and proved in Sections 8 and 9:

**Theorem 2.4.** *Suppose that  $A < 0$  has compact resolvents,  $B \in \mathcal{L}(U, X_{-\alpha})$  for some  $\alpha \in [0, 1/2]$ , and that  $(A, B)$  is null-controllable in any time  $\tau > 0$ . Moreover, assume that  $P \in \mathcal{L}(X)$  is such that, for some  $\tau > 0$ , the operator  $L_\tau \in \mathcal{L}(L^2([0, \tau]; X), X)$  defined by*

$$(2.14) \quad L_\tau v = \int_0^\tau \mathbb{T}_{\tau-\sigma} P v(\sigma) \, d\sigma \quad (v \in L^2([0, \tau]; X)),$$

*satisfies the condition*

$$(2.15) \quad \text{Ran} (L_\tau|_{L^2([0, \tau]; X_{1-\alpha-\varepsilon})}) \subset \text{Ran} \Phi_\tau,$$

*for some  $\varepsilon \in (0, 1 - \alpha]$ .*



Finally, suppose that the pair  $(A + P, B)$  satisfies the Hautus type condition

$$(2.16) \quad \text{Ker}(sI - A - P^*) \cap \text{Ker} B^* = \{0\} \quad (s \in \mathbb{C}).$$

Then for every  $\tau > 0$  we have  $\text{Ran } \Phi_\tau^P = \text{Ran } \Phi_\tau$ , and  $\text{Ran } \mathbb{T}_\tau^P \subset \text{Ran } \Phi_\tau^P$ , that is the system  $(A + P, B)$  is null-controllable in any time  $\tau > 0$ .

**Remark 2.4.** Using Theorem 2.1, it is clear that a sufficient condition to guarantee that the operator  $L_\tau$  in (2.14) satisfies (2.15) is  $P \in \mathcal{L}(X_{1-\alpha-\varepsilon}, \text{Ran } \Phi_\tau)$ , where  $\alpha \in [0, 1/2]$  and  $\varepsilon \in (0, 1 - \alpha]$  are as in Theorem 2.4.

**Remark 2.5.** When  $A < 0$  has compact resolvents, if  $P \in \mathcal{L}(X)$  is such that the condition (2.15) holds for  $L_{\tau_0}$  as in (2.14) for some  $\tau_0 > 0$ , and  $P \in \mathcal{L}(\text{Ran } \Phi_{\tau_0})$  has a small enough  $\mathcal{L}(\text{Ran } \Phi_{\tau_0})$ -norm, then for all  $\tau > 0$ ,  $\text{Ran } \Phi_\tau^P = \text{Ran } \Phi_\tau$  and  $\text{Ran } \mathbb{T}_\tau^P \subset \text{Ran } \Phi_\tau^P$ .

Indeed, applying Theorem 2.3, when  $P \in \mathcal{L}(X) \cap \mathcal{L}(\text{Ran } \Phi_{\tau_0})$  has a small enough  $\mathcal{L}(\text{Ran } \Phi_{\tau_0})$ -norm, we have  $\text{Ran } \Phi_{\tau_0}^P = \text{Ran } \Phi_{\tau_0}$ . Accordingly, approximate controllability holds for  $\Sigma^P = (\mathbb{T}^P, \Phi^P)$  at time  $\tau_0$ , and the Hautus type condition (2.16) thus holds, so that Theorem 2.4 applies.

### 3. REACHABLE SPACES FOR THE HEAT EQUATION AND ITS PERTURBATIONS: STATE OF THE ART

To apply the abstract theorems in Section 2 to systems described by perturbations of the heat equation it is essential to have detailed information on the reachable space of the unperturbed systems. It turns out that most of the relevant literature on the reachable spaces for systems described by heat equations with various types of controls is very recent. This is why we felt necessary to insert here a section devoted to the description of these results, with focus on those which we use to illustrate when applying our main abstract results to perturbed heat equations.

To our knowledge, the work studying the reachable space of the heat equation in this direction goes back to the famous article of Fattorini and Russell [10]. In this paper the authors prove that the one-dimensional heat equation is small-time null-controllable thus implying the invariance of the reachable space with respect to the time horizon (see Proposition 2.1) and they also give examples of reachable states, see [10, Theorem 3.3]. The small-time null-controllability of this type of systems has been extended to several space dimensions and arbitrary control regions in the works of Fursikov and Imanuvilov [14] and Lebeau and Robbiano [23]. After the publication of these papers the literature on the null controllability of systems governed by parabolic PDEs became overwhelming, see, for instance, Barbu [1] for a recent review of this topic. Significant improvements of the results in [10] on the reachable space for systems governed by parabolic PDEs came much later, beginning with the paper of Martin, Rosier and Rouchon [24]. A little bit after that, results published in the series of papers Dardé and Ervedoza [7], Hartmann, Kellay and Tucsnak, [17], Kellay, Normand and Tucsnak [21], Orsoni [25] and Hartmann and Orsoni [19] lead to a full characterization of the reachable space for system governed by the constant coefficients heat equation in one space dimension and controlled from the boundary. We describe below, for later use, one of these reachability results for Neumann boundary controls.

Before going into the details, let us introduce some notation.

Let  $n \geq 1$  be an integer.

For  $\Omega$  an open set of  $\mathbb{C}^n$ , the space of holomorphic functions on  $\Omega$  we denote

$$\text{Hol}(\Omega) := \{f : \Omega \rightarrow \mathbb{C}^n ; f \text{ is holomorphic on } \Omega\},$$

and the Bergman space on  $\Omega$  is denoted by  $A^2(\Omega) := \text{Hol}(\Omega) \cap L^2(\Omega)$ .

For  $x_0 \in \mathbb{R}^n$ ,  $r > 0$ ,  $s_0 \in \mathbb{C}$ , we note  $B(x_0, r) := \{x \in \mathbb{R}^n ; |x - x_0| < r\}$ ,  $B(r) := B(0, r)$  and  $B_{\mathbb{C}}(s_0, r) := \{s \in \mathbb{C} ; |s - s_0| < r\}$ .

Consider the control system described by

$$(3.1) \quad \begin{cases} \frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) = 0 & (t \geq 0, x \in (0, \pi)), \\ \frac{\partial z}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial z}{\partial x}(t, \pi) = u_{\pi}(t) & (t \geq 0), \\ z(0, x) = 0 & (x \in (0, \pi)), \end{cases}$$

with state trajectory  $z$  and input function  $u = \begin{bmatrix} u_0 \\ u_{\pi} \end{bmatrix}$ . The corresponding input maps  $(\Phi_{\tau}^{(0, \pi), N})_{\tau \geq 0}$  (recall the definition in Subsection 2.1 for a precise definition of this concept) are defined by

$$(3.2) \quad \Phi_{\tau}^{(0, \pi), N} u = z(\tau, \cdot) \quad (\tau \geq 0, u \in L^2([0, \infty); \mathbb{C}^2)).$$

It is known (see, for instance, [33, Chapter 10]) that the first two equations in (3.1) determine a well posed control system, with state space  $X = L^2[0, \pi]$  and input space  $U = \mathbb{C}^2$ , in the sense defined in Subsection 2.1. Consequently,  $\Phi_{\tau}^{(0, \pi), N}$  is a linear bounded map from  $L^2([0, \infty); \mathbb{C}^2)$  to  $L^2[0, \pi]$ . For  $\tau > 0$  the range of this map, denoted by  $\text{Ran } \Phi_{\tau}^{(0, \pi), N}$  is the *reachable space at time  $\tau$  of the control system (3.1)*.

The following result can be easily deduced from [19, Corollary 3.2].

**Theorem 3.1.** *The reachable space of the system described by (3.1) is independent of the time horizon  $\tau > 0$  and, for all  $\tau > 0$ ,*

$$(3.3) \quad \text{Ran } \Phi_{\tau}^{(0, \pi), N} = \left\{ \eta \in A^2(D) \mid \frac{d\eta}{ds} \in A^2(D) \right\},$$

where

$$(3.4) \quad D = \{s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x\}.$$

Moreover, for every  $\tau > 0$  the norm induced on  $\text{Ran } \Phi_{\tau}^{(0, \pi), N}$  by the Sobolev norm of  $W^{1,2}(D)$  is equivalent to the norm in  $\text{Ran } \Phi_{\tau}^{(0, \pi), N}$  (recall the general definition (2.6) of the norm in the reachable space).

In the case of systems described by the heat equation in several space dimensions with control acting on the boundary, the first description of the reachable space has been recently given in Strohmaier and Waters [31]. To give the precise statement of this result we need more notation. Let  $n \in \mathbb{N}^*$  and  $\Omega \subset \mathbb{R}^n$  be an open bounded connected set with boundary  $\partial\Omega$  of class  $C^2$ . Consider the control system

$$(3.5) \quad \begin{cases} \frac{\partial z}{\partial t}(t, x) - \Delta z(t, x) = 0 & (t \geq 0, x \in \Omega), \\ z(t, x) = u(t, x) & (t \geq 0, x \in \partial\Omega), \\ w(0, x) = 0 & (x \in \Omega). \end{cases}$$

The corresponding input maps  $(\Phi_\tau^{\Omega,D})_{\tau \geq 0}$  are defined by

$$\Phi_\tau^{\Omega,D} u = z(\tau, \cdot) \quad (\tau \geq 0, u \in L^2([0, \infty); L^2(\partial\Omega))).$$

It is known, see for instance, [33, Section 10.7] that

$$\Phi_\tau^{\Omega,D} \in \mathcal{L}(L^2([0, \infty); L^2(\partial\Omega)), W^{-1,2}(\Omega)).$$

Characterizing the reachable space  $\text{Ran } \Phi_\tau^{\Omega,D}$  by an equality similar to (3.3) seems an extremely difficult question. However, if  $\Omega$  is a ball, this space can be sandwiched between two spaces of holomorphic functions. More precisely, the following result has been obtained in [31]:

**Theorem 3.2.** *Assume that  $R > 0$  and that  $\Omega = B(R)$ . For  $r > 0$ , let us define*

$$\mathcal{E}(B(r)) = \{x + iy \in \mathbb{C}^n, \text{ with } x, y \in \mathbb{R}^n, x \in B(r), |y| < d(x, \partial B(r))\}.$$

*Then the reachable space of the system described by (3.5) is independent of the time horizon  $\tau > 0$ , and for every  $\tau > 0$  we have*

$$(3.6) \quad \bigcup_{r > R} A^2(\mathcal{E}(B(r))) \subset \text{Ran } \Phi_\tau^{\Omega,D} \subset \bigcap_{r < R} A^2(\mathcal{E}(B(r))).$$

**Remark 3.1.** The results in [31] are not stated exactly under the aforementioned form, since they state that, when  $\Omega = B(R)$ , if a state belongs to  $\text{Ran } \Phi_\tau^{\Omega,D}$ , then it belongs to  $\text{Hol}(\mathcal{E}(B(r)))$  for every  $r < R$ , and that if a state belongs to  $\text{Hol}(\mathcal{E}(B(r_0)))$  for some  $r_0 > R$ , then it is reachable. Still, it is clear that the inclusions (3.6) are equivalent to the above mentioned results from [31].

The reachable space for a system described by the heat equation with control localized in an open subset inside the considered spatial domain has not, as far as we know, been explicitly considered in the literature until very recently, see Remark 3.2 below. However, in Fenandez-Cara and Zuazua [13, Section 6] we can find a result giving some interesting information in this direction. More precisely, denoting by  $\Omega$  a smooth bounded domain of  $\mathbb{R}^n$ , and  $\mathcal{O}$  a non-empty open subset of  $\Omega$ , we consider the heat equation in  $\Omega$  controlled from  $\mathcal{O}$ :

$$(3.7) \quad \begin{cases} \frac{\partial z}{\partial t}(t, x) - \Delta z(t, x) = u(t, x) \mathbb{1}_{\mathcal{O}}(x) & (t \geq 0, x \in \Omega), \\ z(t, x) = 0 & (t \geq 0, x \in \partial\Omega), \\ z(0, x) = 0 & (x \in \Omega), \end{cases}$$

where  $\mathbb{1}_{\mathcal{O}}$  is the indicator function of the set  $\mathcal{O}$ . As for the previous examples, for every  $\tau > 0$  we define the input map  $\Phi_\tau^{\Omega, \mathcal{O}}$  by

$$\Phi_\tau^{\Omega, \mathcal{O}} u = z(\tau, \cdot) \quad (\tau \geq 0, u \in L^2([0, \tau]; L^2(\Omega))).$$

The result below can be easily obtained, via a duality argument, from those in [13, Section 6].

**Theorem 3.3.** *Let  $A_0 : \mathcal{D}(A_0) \rightarrow L^2(\Omega)$  be the operator defined by*

$$\mathcal{D}(A_0) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad A_0 \phi = -\Delta \phi.$$

*Let us denote by  $(\mathbb{H}_\tau)_{\tau \geq 0}$  the  $C^0$  semigroup on  $L^2(\Omega)$  generated by  $-A_0^{\frac{1}{2}}$ .*

*Then the reachable space  $\text{Ran } \Phi_\tau^{\Omega, \mathcal{O}}$  of the system described by (3.7) is independent of the time horizon  $\tau > 0$ , and there exists a positive constant  $\tau_0$ , depending on  $\Omega$  and  $\mathcal{O}$ , such that for every  $\tau > 0$  we have*

$$\text{Ran } \mathbb{H}_{\tau_0} \subset \text{Ran } \Phi_\tau^{\Omega, \mathcal{O}}.$$

**Remark 3.2.** Note that very recently, Chen and Rosier [4] consider the reachable states for the distributed control of the heat equation on an interval and they prove that the reachable states are in the Sobolev space  $H^1$  and that they have complex analytic extensions on squares whose horizontal diagonals are regions where no control is applied.

From Theorem 3.2, one can obtain a more precise result than Theorem 3.3 in the particular situation where  $\Omega$  is a ball and the control domain  $\mathcal{O}$  is a neighborhood of the boundary  $\partial\Omega$ .

**Proposition 3.1.** *Let  $0 < R_0 < R_1$ ,  $\Omega = B(R_1)$  and  $\mathcal{O} = B(R_1) \setminus \overline{B(R_0)}$ . Then the reachable space  $\text{Ran } \Phi_\tau^{\Omega, \mathcal{O}}$  of the system described by (3.7) is independent of the time horizon  $\tau > 0$  and we have*

$$\cup_{r \in (R_0, R_1)} \tilde{A}^2(\mathcal{E}(B(r))) \subset \text{Ran } \Phi_\tau^{\Omega, \mathcal{O}},$$

where for every  $r \in (R_0, R_1)$  we have denoted by  $\tilde{A}^2(\mathcal{E}(B(r)))$  the set of functions  $f \in H_0^1(B(R_1))$  such that  $f|_{B(r)}$  can be extended as a function of  $A^2(\mathcal{E}(B(r)))$ .

The proof of Proposition 3.1 is based on a cut-off argument and is postponed in Appendix B.

As far as we know, the only work tackling the question of the determination of the reachable space for perturbed heat equations is [22]. This paper considers systems described by the one dimensional heat equation with perturbations of holomorphic type (linear or nonlinear) and provides results asserting that the states which can be extended to functions which are holomorphic in a sufficiently large disk can be reached by  $L^2$  boundary controls. The method proposed in [22] is based on considering the control problem as an ill-posed Cauchy problem in the  $x$  variable. Let us state precisely one of their results.

We consider a system obtained by perturbing the first equation in (3.1) by a term of the form  $Pz$ , with  $P \in \mathcal{L}(L^2[0, \pi])$ . More precisely, this system is described by the equations

$$(3.8) \quad \begin{cases} \frac{\partial z^P}{\partial t}(t, x) - \frac{\partial^2 z^P}{\partial x^2}(t, x) = (Pz^P)(t, x) & (t \geq 0, x \in (0, \pi)), \\ \frac{\partial z^P}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial z^P}{\partial x}(t, \pi) = u_\pi(t) & (t \geq 0), \\ z(0, x) = 0 & (x \in (0, \pi)). \end{cases}$$

The corresponding input maps  $(\Phi_\tau^{P, (0, \pi), N})_{\tau \geq 0}$  are defined by

$$(3.9) \quad \Phi_\tau^{P, (0, \pi), N} u = z^P(\tau, \cdot) \quad (\tau \geq 0, u \in L^2([0, \infty); \mathbb{C}^2)).$$

As far as we know, the only available result on the reachable spaces  $(\text{Ran } \Phi_\tau^{P, (0, \pi), N})_{\tau \geq 0}$ , when  $P$  is a general potential (when  $P(x) = x^2$ , we refer to the recent results [18]), of the system defined by equations (3.8) is a consequence of the results in [22] and can be stated as follows.

**Proposition 3.2.** *Let  $\hat{R} := (2\pi)e^{(2e)^{-1}}$ ,  $b > \hat{R}$  and  $P \in \mathcal{L}(L^2[0, \pi])$  be defined by*

$$(3.10) \quad (P\psi)(x) = q(x)\psi(x) \quad (\psi \in L^2(0, \pi)),$$

where  $q$  admits an holomorphic extension on  $B_{\mathbb{C}}(\pi/2, b)$ .

Then for every  $\tau > 0$  the range  $\text{Ran } \Phi_\tau^{P,(0,\pi),N}$  of the map defined in (3.9) contains all the functions  $z_1 \in L^2(0, \pi)$  which admit an holomorphic extension on  $B_{\mathbb{C}}(\pi/2, R)$  for some  $R > \hat{R}$ .

**Remark 3.3.** The statement given in [22] slightly differs from Proposition 3.2, but we claim that Proposition 3.2 is a straightforward consequence of Theorem 1.1 in [22] according to the following remarks:

- For  $s_0 \in \mathbb{C}$ ,  $R > 0$ , a function  $z \in \text{Hol}(B_{\mathbb{C}}(s_0, R))$  automatically admits a power series expansion  $z(s) = \sum_s \alpha_n \frac{(s-s_0)^n}{n!}$  which is convergent in  $B_{\mathbb{C}}(s_0, R')$  for any  $R' \in (0, R)$ , and the coefficients  $(\alpha_n)_{n \in \mathbb{N}}$  satisfy

$$(\alpha_n (R')^n / n!)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$$

(in fact, the sequence  $(\alpha_n (R')^n / n!)_{n \in \mathbb{N}}$  is bounded by  $\|z\|_{L^\infty(B_{\mathbb{C}}(s_0, R'))}$  from Cauchy's formula).

- The homogeneity of the linear equation (3.7) allows to pass from local results, concerning states which are small in some norms, to global results (in contrast with semilinear cases, which are also handled in Theorem 1.1 in [22] for which a smallness assumption is needed).

#### 4. APPLICATIONS TO PERTURBED HEAT EQUATIONS

In this section we state and prove several results obtained by applying Theorems 2.3 and 2.4 to systems governed by heat equations with various type of perturbations. For some of these systems we improve the existing results on the robustness of reachability and null controllability in presence of perturbations. Other results in this section consider systems for which the reachable spaces are studied here for the first time.

**4.1. The case of small potentials.** As a first application we complete and improve (at least for “small” potentials) the result in Proposition 3.2. More precisely, we give the following characterization of  $\text{Ran } \Phi_\tau^{P,(0,\pi),N}$ , under weaker regularity assumptions on the potential  $q$ :

**Theorem 4.1.** *Let  $P \in \mathcal{L}(L^2[0, \pi])$  be defined by (3.10) and let  $D$  be the set defined in (3.4). Then there exists  $\varepsilon > 0$  such that, if  $q$  can be extended to a function holomorphic on  $D$  with  $\|q\|_{W^{1,p}(D)} \leq \varepsilon$  for some  $p > 2$ , then the reachable set  $\text{Ran } \Phi_\tau^{P,(0,\pi),N}$  for the equation (3.8) is independent from  $\tau > 0$ , and for every  $\tau > 0$ ,*

$$(4.1) \quad \text{Ran } \Phi_\tau^{P,(0,\pi),N} = \text{Ran } \Phi_\tau^{(0,\pi),N} = \left\{ \eta \in A^2(D) \mid \frac{d\eta}{ds} \in A^2(D) \right\}.$$

*Proof.* We apply Theorem 2.3, with the following choice of the spaces and operators

- The state space is  $X = L^2[0, \pi]$  and the control space is  $U = \mathbb{C}^2$ ;
- The semigroup generator  $A : \mathcal{D}(A) \rightarrow X$  is defined by  $A = \frac{d^2}{dx^2}$ , with

$$\mathcal{D}(A) = \left\{ z \in H^2(0, \pi) \text{ with } \frac{dz}{dx}(0) = \frac{dz}{dx}(\pi) = 0 \right\};$$

- The control operator  $B$  is defined by

$$B \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = -u_0 \delta_0 + u_\pi \delta_\pi \quad (u_0, u_\pi \in L^2([0, \infty); \mathbb{C})),$$

where  $\delta_0, \delta_\pi$  are the Dirac masses concentrated at  $x = 0$  and  $x = \pi$ , respectively;

- The perturbation operator  $P$  is defined by

$$P\psi = q\psi \quad (\psi \in L^2[0, \pi]).$$

Indeed, by combining Theorem 2.3 and Theorem 3.1, to prove that  $\text{Ran } \Phi_\tau^{P,(0,\pi),N} = \text{Ran } \Phi_\tau^{(0,\pi),N}$  for some  $\tau$  provided  $\|q\|_{W^{1,p}(D)}$  is small enough, it suffices to check that for every  $\tau > 0$  there exists  $c_\tau > 0$  such that

$$\|P\psi\|_{\text{Ran } \Phi_\tau^{P,(0,\pi),N}} \leq c_\tau \|q\|_{W^{1,p}(D)} \|\psi\|_{\text{Ran } \Phi_\tau^{P,(0,\pi),N}} \quad (\tau > 0, \psi \in L^2[0, \pi]).$$

This last inequality can be easily checked by combining some elementary Sobolev embeddings with Theorem 3.1.

It then remains to use the well-known fact that heat equations with  $L^\infty$  potentials are null-controllable in any time  $\tau > 0$  (see for instance [14]) to conclude that  $\text{Ran } \Phi_\tau^{P,(0,\pi),N}$  does not depend on  $\tau > 0$  (recall Proposition 2.1) and thus that the identity (4.1) holds for all  $\tau > 0$ .  $\square$

**4.2. Non-local perturbation operators with Neumann controls in 1-d.** Our second application concerns again the reachable space of the system described by (3.8), this time with a non local perturbation operator  $P$ . As far as we know, such perturbations have not yet considered in the literature in the case of parabolic systems controlled from the boundary. More precisely, the operator  $P$  is defined by

$$(4.2) \quad (P\eta)(x) = \int_0^\pi K(x, y)\eta(y) \, dy \quad (\eta \in L^2[0, \pi], x \in (0, \pi)),$$

with  $K \in L^2([0, \pi] \times [0, \pi])$ . We stress here the fact that this second application characterizes the reachable set of the perturbed equation under a unique continuation assumption of the elliptic nonlocal operator associated to (3.8) and (4.2), but without smallness assumption.

**Proposition 4.1.** *Let  $P$  be defined by (4.2) with  $K \in L^2([0, \pi] \times [0, \pi])$ . Assume that for a.e.  $y \in [0, \pi]$  the map*

$$x \mapsto K(x, y) \quad (x \in [0, \pi]),$$

*extends to an element of  $\text{Hol}(D) \cap W^{1,2}(D)$ , where  $D$  is defined by (3.4), and  $K$  belongs to  $L^2_y([0, \pi]; W^{1,2}_x(D))$ . Moreover, suppose that for every  $s \in \mathbb{C}$ , the only function  $\psi \in H^2(0, \pi)$  satisfying*

$$(4.3) \quad \begin{cases} -\frac{d^2\psi}{dx^2}(x) - s\psi(x) = \int_0^\pi \overline{K(y, x)}\psi(y) \, dy, & (x \in [0, \pi]), \\ \psi(0) = \frac{d\psi}{dx}(0) = 0, \\ \psi(\pi) = \frac{d\psi}{dx}(\pi) = 0, \end{cases}$$

*is  $\psi = 0$ .*

*Then the reachable spaces of the system (3.8)–(4.2) do not depend on time and they satisfy (4.1) for every  $\tau > 0$ . Furthermore, the system (3.8)–(4.2) is null-controllable in any positive time.*

*Proof.* It is not difficult to check that the considered system satisfies the assumptions in Theorem 2.4 and Remark 2.4 with  $X$ ,  $U$ ,  $A$  and  $B$  as in the proof of Theorem 4.1 and with  $P$  defined by (4.2) satisfying  $P \in \mathcal{L}(X, \text{Ran } \Phi_\tau^{(0,\pi),N})$ .  $\square$

**Remark 4.1.** For examples of kernels  $K$  which satisfy or not the last assumption in Proposition 4.1 (i.e., the fact that the only solution of (4.3) is  $\psi = 0$ ), we refer to the discussion in Subsection A.2.

**4.3. Non-local perturbation operators with distributed controls in multi-d: the general case.** Let us now take  $\Omega$  a smooth bounded domain of  $\mathbb{R}^n$  and  $\mathcal{O}$  a nonempty open set contained in  $\Omega$  and let  $\mathbb{1}_{\mathcal{O}}$  be the indicator function of the set  $\mathcal{O}$ . We consider the system

$$(4.4) \quad \begin{cases} \frac{\partial z^P}{\partial t}(t, x) - \Delta z^P(t, x) = (Pz^P)(t, x) + u(t, x)\mathbb{1}_{\mathcal{O}}(x) & (t \geq 0, x \in \Omega), \\ z^P(t, x) = 0 & (t \geq 0, x \in \partial\Omega), \\ z^P(0, x) = 0 & (x \in \Omega), \end{cases}$$

where the operator  $P$  is given through a kernel function  $K \in L^2(\Omega \times \Omega)$  by the formula:

$$(4.5) \quad (P\eta)(x) = \int_{\Omega} K(x, y)\eta(y) dy \quad (\eta \in L^2(\Omega), x \in \Omega).$$

The corresponding input maps  $(\Phi_\tau^{P,\Omega,\mathcal{O}})_{\tau \geq 0}$  are defined by

$$\Phi_\tau^{P,\Omega,\mathcal{O}} u = z^P(\tau, \cdot) \quad (\tau \geq 0, u \in L^2([0, \infty); \mathcal{O})).$$

Decomposing the kernel  $K$  on an orthonormal (in  $L^2(\Omega)$ ) basis of eigenfunctions  $(\varphi_n)_{n \in \mathbb{N}}$  of the Laplace operator, it follows that there exists coefficients  $(k_{n,m})_{n,m \in \mathbb{N}} \in \ell^2(\mathbb{N}^2)$  such that

$$(4.6) \quad K(x, y) = \sum_{n,m \in \mathbb{N}} k_{n,m} \varphi_n(x) \varphi_m(y), \quad (x \in \Omega, y \in \Omega).$$

We then have the following result:

**Proposition 4.2.** *Let  $P$  be defined by (4.5) with kernel  $K$  satisfying the representation (4.6).*

*Let  $\tau_0 > 0$  be given by Theorem 4.2, and assume*

$$(4.7) \quad \sum_{n \in \mathbb{N}} \exp(2\sqrt{\lambda_n} \tau_0) \sum_{m \in \mathbb{N}} \frac{|k_{n,m}|^2}{\lambda_n \lambda_m} < \infty.$$

*Then the following assertions are equivalent:*

- (a) *for all  $\tau > 0$ ,  $\text{Ran } \Phi_\tau^{P,\Omega,\mathcal{O}} = \text{Ran } \Phi_\tau^{\Omega,\mathcal{O}}$ ,*
- (b) *for every  $s \in \mathbb{C}$ , the only function  $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$  satisfying*

$$\begin{aligned} -\Delta \psi(x) - s\psi(x) &= \int_{\Omega} \overline{K(y, x)} \psi(y) dy & (x \in \Omega), \\ \psi(x) &= 0 & (x \in \mathcal{O}), \end{aligned}$$

*is  $\psi = 0$ .*

*Furthermore, if condition (b) is satisfied, system (4.4) is null-controllable in any positive time, that is, for all  $\tau > 0$ ,  $\text{Ran } \mathbb{T}_\tau^P \subset \text{Ran } \Phi_\tau^{P,\Omega,\mathcal{O}}$ .*

**Remark 4.2** (Comparison with [12]). Condition (4.7) coincides with the condition (5) in [12]. Compared to [12], the novelty of the above result is twofold: first, it assumes only a unique continuation result (item (b)) instead of the analyticity of the function  $x \mapsto \overline{K}(y, x)$  for all  $y \in \Omega$  as in [12]; second, while [12] only states the null-controllability of the perturbed system, the above result provides a full characterization of its reachable space (note that, although this is not explicitly mentioned in [12], the equation (16) in [12] actually already contains this type of conclusion).

To summarize: the sufficient condition given in [12] for getting null-controllability is stronger than the one we propose. As in our work, although not explicitly stated in [12], this condition guarantees that the reachable set of the heat equation with non-local term in space  $P$  coincides with the reachable set of the heat equation.

*Proof.* The above system fits into our abstract framework by setting  $A = \Delta$  in  $X = L^2(\Omega)$  with domain  $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$  and  $B = \mathbf{1}_\Omega$ , with  $U = L^2(\Omega)$ .

The proof thus consists in checking that condition (4.7) implies the condition (2.15) for  $L_\tau$  defined by (2.14), so that Theorem 4.2 will be a consequence of Theorem 2.4, with  $\alpha = 0$  and  $\varepsilon = 1/2$ .

Using Theorem 3.3, for  $f \in L^2([0, \tau]; X_{1/2})$ , writing  $f_m(s) = \langle f(s), \varphi_m \rangle_X$  for  $m \in \mathbb{N}$  and  $s \in [0, \tau]$ , we have for some positive constant  $C > 0$

$$\begin{aligned}
\|L_\tau f\|_{\text{Ran } \Phi_\tau^{\Omega, \mathcal{O}}}^2 &\leq C \|L_\tau f\|_{\text{Ran } \mathbb{H}_{\tau_0}}^2 \\
&\leq \sum_n e^{2\sqrt{\lambda_n}\tau_0} \left| \sum_m k_{n,m} \int_0^\tau f_m(s) e^{-\lambda_n(\tau-s)} ds \right|^2 \\
&\leq \sum_n e^{2\sqrt{\lambda_n}\tau_0} \left( \left( \sum_m \frac{|k_{n,m}|^2}{\lambda_m} \right) \left( \sum_m \lambda_m \left| \int_0^\tau f_m(s) e^{-\lambda_n(\tau-s)} ds \right|^2 \right) \right) \\
&\leq C \sum_n e^{2\sqrt{\lambda_n}\tau_0} \left( \left( \sum_m \frac{|k_{n,m}|^2}{\lambda_m} \right) \left( \sum_m \frac{\lambda_m}{\lambda_n} \int_0^\tau |f_m(s)|^2 ds \right) \right) \\
&\leq C \left( \sum_n e^{2\sqrt{\lambda_n}\tau_0} \sum_m \frac{|k_{n,m}|^2}{\lambda_m \lambda_n} \right) \|f\|_{L^2([0, \tau]; X_{1/2})}^2.
\end{aligned}$$

Note that we have used Cauchy-Schwarz inequality in the third and fourth inequalities above. Accordingly,  $L_\tau \in \mathcal{L}(L^2([0, \tau]; X_{1/2}), \text{Ran } \Phi_\tau^{\Omega, \mathcal{O}})$ , and Theorem 2.4 applies.  $\square$

**Remark 4.3.** It would be natural to prove Theorem 4.2 using Remark 2.4 and the sufficient condition  $P \in \mathcal{L}(X_{1/2}, \text{Ran } \Phi_\tau^{\Omega, \mathcal{O}})$  corresponding to the choices  $\alpha = 0$  and  $\varepsilon = 1/2$ . This would yield the same result under the slightly stronger condition

$$\sum_n e^{2\sqrt{\lambda_n}\tau_0} \sum_m \frac{|k_{n,m}|^2}{\lambda_m} < +\infty,$$

instead of (4.7).



Indeed, for  $\eta \in X_{1/2} = H_0^1(\Omega)$ , we have

$$\begin{aligned} \|P\eta\|_{\text{Ran } \Phi_\tau^{\Omega, \mathcal{O}}}^2 &\leq \|P\eta\|_{\text{Ran } \mathbb{H}_{\tau_0}}^2 \leq \sum_n e^{2\sqrt{\lambda_n}\tau_0} \left| \sum_m k_{n,m}\eta_m \right|^2 \\ &\leq C \left( \sum_n e^{2\sqrt{\lambda_n}\tau_0} \sum_m \frac{|k_{n,m}|^2}{\lambda_m} \right) \|\eta\|_{X_{1/2}}^2. \end{aligned}$$

**4.4. Non-local perturbation operators with distributed controls in multi-d: the case of a ball controlled from a ring.** We now assume that the domain  $\Omega = B(R_1) \subset \mathbb{R}^n$  and  $\mathcal{O} = B(R_1) \setminus \overline{B(R_0)}$  for some  $R_0$  and  $R_1$  with  $0 < R_0 < R_1$  and we show that in this case the assumptions on the kernel  $K$  can be considerably weakened with respect to those in the previous section.

In view of Proposition 3.1, it seems natural to study the case of a perturbation operator  $P$  given through a kernel function  $K$  as in (4.5), and to assume the following assumptions:

$$(4.8) \quad \begin{cases} \forall y \in \Omega, K(\cdot, y) \in H_0^1(B(R_1)) \\ \text{and } \int_{\Omega} \|\nabla_x K(x, \cdot)\|_{H_y^{-1}(\Omega)}^2 dx < \infty, \\ \\ \exists r \in (R_0, R_1), \\ \forall y \in \Omega, K(\cdot, y)|_{B(r)} \text{ has an holomorphic extension in } \mathcal{E}(B(r)) \\ \text{and } \int_{x \in \mathcal{E}(B(r))} \int_{y \in \Omega} |K(x, y)|^2 dy dx < \infty. \end{cases}$$

We have the following result.

**Proposition 4.3.** *Let  $\Omega = B(R_1)$  and  $\mathcal{O} = B(R_1) \setminus \overline{B(R_0)}$  for some constants  $R_0$  and  $R_1$  with  $0 < R_0 < R_1$ .*

*Let  $P$  be defined by (4.5) with kernel  $K$  satisfying (4.8).*

*Then the conclusion of Theorem 4.2 holds true.*

*Proof.* Again, the controlled system under consideration fits the abstract framework developed in Section 2, and using Remark 2.4 and Theorem 2.4, it is sufficient to check that  $P \in \mathcal{L}(L^2(B(R_1)), \text{Ran } \Phi_\tau^{\Omega, \mathcal{O}})$ .

Due to the form of the kernel and the assumption (4.8), we have that for all  $g \in L^2(B(R_1))$ ,  $Pg|_{B(r)}$  admits an holomorphic expansion on  $\mathcal{E}(B(r))$  and

$$\begin{aligned} \|Pg|_{\mathcal{E}(B(r))}\|_{A^2(\mathcal{E}(B(r)))}^2 &= \int_{x \in \mathcal{E}(B(r))} \left| \int_{y \in \Omega} K(x, y)g(y) dy \right|^2 dx \\ &\leq \left( \int_{x \in \mathcal{E}(B(r))} \int_{y \in \Omega} |K(x, y)|^2 dy dx \right) \int_{y \in \Omega} |g(y)|^2 dy. \end{aligned}$$

Similarly, for  $g \in H_0^1(B(R_1))$ , according to assumptions (4.8),  $Pg \in H_0^1(B(R_1))$  and

$$\begin{aligned} \|Pg\|_{H_0^1(B(R_1))}^2 &= \int_{x \in \Omega} \left| \int_{y \in \Omega} \nabla_x K(x, y)g(y) dy \right|^2 dx \\ &\leq \left( \int_{x \in \Omega} \|\nabla_x K(x, \cdot)\|_{H_y^{-1}(\Omega)}^2 dx \right) \int_{y \in \Omega} |\nabla_y g(y)|^2 dy. \end{aligned}$$

In particular, we have, for all  $g \in H_0^1(B(R_1))$ ,  $Pg \in \tilde{A}^2(\mathcal{E}(B(r)))$  and satisfies

$$\|Pg\|_{H_0^1(B(R_1))} + \|Pg|_{\mathcal{E}(B(r))}\|_{A^2(\mathcal{E}(B(r)))} \leq C\|g\|_{H_0^1(B(R_1))} = C\|g\|_{X_{1/2}}.$$

Proposition 3.1 then implies that  $P \in \mathcal{L}(L^2(B(R_1)), \text{Ran } \Phi_\tau^{\Omega, \mathcal{O}})$  as announced.  $\square$

## 5. MORE BACKGROUND ON INFINITE DIMENSIONAL CONTROL SYSTEMS

In this section we develop the background on well posed linear control systems, whose presentation has been initiated in Section 2.1. We first recall, without proofs but with appropriate references, some known results which will be used in the proofs of our main theorems. Moreover, we state and prove several results which, although quite simple, seem new.

We continue to use the notation and assumptions introduced in Section 2.1. We recall, in particular, that  $X$  and  $U$  are Hilbert spaces, denoting the state and the control space, respectively, and that a well-posed control system can be described either by a couple  $\Sigma = (\mathbb{T}, \Phi)$  of families of operators or by  $\Sigma = (A, B)$ , where  $A$  is the generator of  $\mathbb{T}$  and  $B$  is the control operator, see Definition 2.1.

**5.1. Complements on well-posed control systems.** We begin by recalling (see, for instance, [33, Proposition 4.2.5]) that the families  $\mathbb{T}$  and  $\Phi$  can also be seen as the solution operators for the initial value problem

$$(5.1) \quad \dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0,$$

in the following sense:

**Proposition 5.1.** *Let  $\tau > 0$ . Then for every  $z_0 \in X$  and every  $u \in L^2([0, \tau]; U)$ , the initial value problem (5.1) has a unique solution*

$$z \in C^0([0, \tau]; X) \cap H^1((0, \tau); X_{-1}).$$

*This solution is given by*

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \quad (t \in [0, \tau]).$$

The two results below are essentially known in the control theoretic community, but we did not find them explicitly stated in the existing literature. Therefore, with no claim of originality, we give a short proof.

**Proposition 5.2.** *For every  $\tau > 0$ , we have that  $\Phi_\tau \in \mathcal{L}(L^2[0, \tau]; U), \text{Ran } \Phi_\tau$  is onto and*

$$(5.2) \quad \|\Phi_\tau\|_{\mathcal{L}(L^2([0, \tau]; U), \text{Ran } \Phi_\tau)} = 1.$$

*Proof.* From Definition 2.1, we obviously have the inequality

$$\|\Phi_\tau\|_{\mathcal{L}(L^2([0, \tau]; U), \text{Ran } \Phi_\tau)} \leq 1.$$

Moreover, if  $\eta \in \text{Ran } \Phi_\tau \setminus \{0\}$  there exists a sequence  $(u_n)_{n \geq 0}$  in  $(L^2([0, \tau]; U) \setminus \{0\})^{\mathbb{N}}$  such that  $\Phi_\tau u_n = \eta$  for every  $n \in \mathbb{N}$  and  $\|u_n\|_{L^2([0, \tau]; U)} \rightarrow \|\eta\|_{\text{Ran } \Phi_\tau}$  as  $n \rightarrow \infty$ . We thus have that

$$\lim_{n \rightarrow \infty} \frac{\|\Phi_\tau u_n\|_{\text{Ran } \Phi_\tau}}{\|u_n\|_{L^2([0, \tau]; U)}} = 1,$$

and, consequently, (5.2).  $\square$

**Remark 5.1.** Since  $\Phi_\tau \in \mathcal{L}(L^2([0, \tau]; U), \text{Ran } \Phi_\tau)$  is onto, a classical consequence of the closed graph theorem (see, for instance, [33, Section 12.1]) yields the existence of an operator  $\Gamma_\tau \in \mathcal{L}(\text{Ran } \Phi_\tau, L^2([0, \tau]; U))$  such that

$$(5.3) \quad \Phi_\tau \Gamma_\tau = \mathbb{I}_{\text{Ran } \Phi_\tau}.$$

We also add the following useful property on the input maps:

**Proposition 5.3.** *For  $\tau > 0$ , for every  $0 \leq \sigma \leq \tau$ , for every  $u \in L^2([0, \infty); U)$ ,*

$$\|\Phi_\sigma u\|_{\text{Ran } \Phi_\tau} \leq \|u\|_{L^2([0, \sigma]; U)}.$$

*Proof.* For every  $u \in L^2([0, \infty); U)$  and every  $\sigma \in [0, \tau]$  we have

$$\Phi_\sigma u = \Phi_\tau v_\sigma,$$

where

$$v_\sigma(s) = \begin{cases} 0 & \text{for } s \in [0, \tau - \sigma], \\ u(s - \tau + \sigma) & \text{for } s \in [\tau - \sigma, \tau]. \end{cases}$$

Accordingly,  $\|\Phi_\sigma u\|_{\text{Ran } \Phi_\tau} = \|\Phi_\tau v_\sigma\|_{\text{Ran } \Phi_\tau} \leq \|v_\sigma\|_{L^2([0, \tau]; U)} = \|u\|_{L^2([0, \sigma]; U)}$  as announced.  $\square$

We also need the following result:

**Proposition 5.4.** *If  $(\mathbb{T}, \Phi)$  is null controllable in some time  $\tau > 0$  then  $\mathbb{T}_\tau \in \mathcal{L}(X, \text{Ran } \Phi_\tau)$ . Moreover, if  $(\mathbb{T}, \Phi)$  is null controllable in any time  $\tau > 0$  then for every  $t \in (0, \tau]$  we have  $\mathbb{T}_t \in \mathcal{L}(X, \text{Ran } \Phi_\tau)$ .*

*Proof.* According to a classical result, see, for instance, [32, Proposition 2.4], the null controllability in time  $\tau$  of the pair  $(\mathbb{T}, \Phi)$  implies the existence of an operator  $\mathbb{F}_\tau \in \mathcal{L}(X, L^2([0, \tau]; U))$  such that

$$\mathbb{T}_\tau + \Phi_\tau \mathbb{F}_\tau = 0.$$

The above formula implies that

$$\|\mathbb{T}_\tau \psi\|_{\text{Ran } \Phi_\tau} \leq \|\mathbb{F}_\tau\|_{\mathcal{L}(X, L^2([0, \tau]; U))} \|\psi\|_X \quad (\psi \in X),$$

so that indeed  $\mathbb{T}_\tau \in \mathcal{L}(X, \text{Ran } \Phi_\tau)$ .

If  $(\mathbb{T}, \Phi)$  is null controllable in any positive time then we know from Remark 2.2 that  $\|\cdot\|_{\text{Ran } \Phi_\tau}$  and  $\|\cdot\|_{\text{Ran } \Phi_t}$  are equivalent norms on  $\text{Ran } \Phi_\tau$ . Combining the equivalence of these norms with the fact that  $\mathbb{T}_t \in \mathcal{L}(X, \text{Ran } \Phi_t)$  we obtain the second assertion of the proposition.  $\square$

**5.2. The role of duality.** In this subsection we give the dual versions of some of the results in Sections 2.1 and 5.1. Throughout this subsection we continue to use the notation  $X$  and  $U$  for the state and input space, respectively. The operator  $A : \mathcal{D}(A) \rightarrow X$  is still supposed to be the generator of the  $C^0$  semigroup  $\mathbb{T}$  and  $X_1, X_1^d$  stand for the Hilbert spaces obtained by endowing  $\mathcal{D}(A)$  and  $\mathcal{D}(A^*)$  with the norms defined by (2.2) and (2.3), respectively. Moreover,  $Y$  designs another Hilbert space (the output space) and  $C \in \mathcal{L}(X_1, Y)$  is an *admissible observation operator* for  $\mathbb{T}$ . The admissibility assumption means that for some  $\tau > 0$ , the operator  $\Psi_\tau$  defined by

$$(5.4) \quad (\Psi_\tau z_0)(t) = C \mathbb{T}_t z_0 \quad (t \in [0, \tau], z_0 \in X_1),$$

has an extension to an operator  $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau], Y))$ . We refer to [33, 36] for more material on this concept. Here we only mention that it follows from the

admissibility assumption that  $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau]; Y))$  holds for all  $\tau \geq 0$ . The operators  $\Psi_\tau$  are called output maps corresponding to the pair  $(A, C)$ .

The concepts of admissible control operator and admissible observation operator are dual. More precisely, according to results in [33, Section 4.4] we have:

**Proposition 5.5.** *The operator  $B \in \mathcal{L}(U, X_{-1})$  is an admissible control operator for  $\mathbb{T}$  if and only if  $B^* \in \mathcal{L}(X_1^d, U)$  is an admissible observation operator for  $\mathbb{T}^*$ . In this case, for every  $\tau > 0$  the adjoint  $\Phi_\tau^* \in \mathcal{L}(X, L^2[0, \tau]; U)$  of the operator  $\Phi_\tau$  introduced in (2.4) is given by*

$$(5.5) \quad (\Phi_\tau^* \eta)(t) = (\Psi_\tau^d \eta)(\tau - t) \quad (t \in [0, \tau], \eta \in X),$$

where  $(\Psi_\tau^d)_{\tau \geq 0}$  are the output maps corresponding to the pair  $(A^*, B^*)$ .

By combining the above result and Proposition 4.3.4 in [33] it is easily checked that we have

**Proposition 5.6.** *If  $B$  be an admissible control operator for  $\mathbb{T}$  and  $\eta \in \mathcal{D}(A^*)$  then the map  $t \mapsto (\Phi_\tau^* \eta)(t)$  belongs to  $H^1((0, \tau); U)$  and we have*

$$\frac{d}{dt} ((\Phi_\tau^* \eta)(t)) = (\Phi_\tau^*(A^* \eta))(t) \quad \text{in } \mathcal{D}'(0, \tau).$$

We next recall the definition of the main observability concepts in infinite dimensional systems theory.

**Definition 5.1.** Let  $A$  be the generator of the  $C^0$  semigroup  $\mathbb{T}$  on  $X$  and let  $C \in \mathcal{L}(X_1, Y)$  be an admissible observation operator. Let  $(\Psi_\tau)_{\tau > 0}$  be the output maps corresponding to  $(A, C)$ .

- The pair  $(A, C)$  is *exactly observable in time  $\tau$*  if there exists a  $k_\tau > 0$  such that  $\|\Psi_\tau z_0\| \geq k_\tau \|z_0\|$  for all  $z_0 \in X$ .
- The pair  $(A, C)$  is *approximately observable in time  $\tau$*  if  $\text{Ker } \Psi_\tau = \{0\}$ .
- The pair  $(A, C)$  is *final state observable in time  $\tau$*  if there exists a  $k_\tau > 0$  such that  $\|\Psi_\tau z_0\| \geq k_\tau \|\mathbb{T}_\tau z_0\|$  for all  $z_0 \in X$ .

**Remark 5.2.** If  $A$  generates an analytic semigroup  $\mathbb{T}$  on  $X$ , then the approximate observability in some time  $\tau > 0$  of the pair  $(A, C)$  is equivalent to the approximate observability in any time  $\tau > 0$  of the pair  $(A, C)$ . This directly comes from the fact that, for  $z_0 \in X$ ,  $t \mapsto C\mathbb{T}_t z_0$  is analytic on  $(0, \infty)$  with values in  $X_1$ ; hence, if for some  $\tau_0 > 0$ ,  $\Psi_{\tau_0} z_0 = 0$ , then for all  $\tau > 0$ ,  $\Psi_\tau z_0 = 0$ .

According to a classical result (see, for instance, [33, Theorem 11.2.1]), the observability concepts defined above correspond by duality to the controllability concepts introduced in Subsection 2.1, as made precise in the proposition below.

**Proposition 5.7.** *Assume that  $B \in \mathcal{L}(U, X_{-1})$  is an admissible control operator for  $\mathbb{T}$ , the semigroup generated by  $A$ , and let  $\tau > 0$ .*

- (1) *The pair  $(A, B)$  is exactly controllable in time  $\tau$  if and only if  $(A^*, B^*)$  is exactly observable in time  $\tau$ .*
- (2) *The pair  $(A, B)$  is approximately controllable in time  $\tau$  if and only if  $(A^*, B^*)$  is approximately observable in time  $\tau$ .*
- (3) *The pair  $(A, B)$  is null-controllable in time  $\tau$  if and only if  $(A^*, B^*)$  is final state observable in time  $\tau$ .*

We next consider a second notion of duality for the operator  $\Phi_\tau$  introduced in (2.4). Let us remind that for the first one, appearing in Proposition 5.5 we have

considered that  $\Phi_\tau \in \mathcal{L}(L^2([0, \tau]; U), X)$  and we have identified  $X$  and  $L^2([0, \tau]; U)$  with their duals, obtaining the operator  $\Phi_\tau^* \in \mathcal{L}(L^2([0, \tau]; U), X)$ , called adjoint of  $\Phi_\tau$ . Remarking that  $\Phi_\tau \in \mathcal{L}(L^2([0, \tau]; U), \text{Ran } \Phi_\tau)$  for every  $\tau > 0$  (recall Remark 5.2), we can also define a slightly different concept.

**Definition 5.2.** Assume that the pair  $(A, B)$  is approximately controllable in time  $\tau$ , so that the dual  $(\text{Ran } \Phi_\tau)'$  of  $\text{Ran } \Phi_\tau$  with respect to the pivot space  $X$  is well defined and we have

$$\text{Ran } \Phi_\tau \subset X \subset (\text{Ran } \Phi_\tau)',$$

with continuous and dense inclusions.

The dual  $\Phi_\tau' \in \mathcal{L}((\text{Ran } \Phi_\tau)', L^2([0, \tau]; U))$  of the operator  $\Phi_\tau$  introduced in (2.4) is defined by

$$\langle \Phi_\tau u, \eta \rangle_{\text{Ran } \Phi_\tau, (\text{Ran } \Phi_\tau)'} = \langle u, (\Phi_\tau)'\eta \rangle_{L^2([0, \tau]; U)},$$

for every  $u \in L^2([0, \tau]; U)$  and  $\eta \in (\text{Ran } \Phi_\tau)'$ .

**Remark 5.3.** From the above definition it clearly follows that, provided that  $(A, B)$  is approximately controllable in time  $\tau > 0$ , the operator  $\Phi_\tau'$  is an extension of the operator  $\Phi_\tau^*$ . Thus, according to Proposition 5.5 we have:

$$(\Phi_\tau' \eta)(t) = (\Psi_\tau^d \eta)(\tau - t) \quad (t \in [0, \tau], \eta \in X).$$

The norm in the space  $(\text{Ran } \Phi_\tau)'$  can be characterized as follows:

**Proposition 5.8.** Assume that  $(A, B)$  is approximately controllable in some time  $\tau > 0$ . Then

$$(5.6) \quad \|\eta\|_{(\text{Ran } \Phi_\tau)'} = \|\Phi_\tau^* \eta\|_{L^2([0, \tau]; U)} \quad (\eta \in X).$$

*Proof.* We first remark that for every  $\xi \in \text{Ran } \Phi_\tau$ , there exists a sequence  $(u_n^\xi)$  in  $L^2([0, \tau]; U)$  such that  $\Phi_\tau u_n^\xi = \xi$  for every  $n \in \mathbb{N}$  and

$$\lim_{n \rightarrow \infty} \|u_n^\xi\|_{L^2([0, \tau]; U)} = \|\xi\|_{\text{Ran } \Phi_\tau}.$$

We next note that for every  $\eta \in X$  and  $\xi \in \text{Ran } \Phi_\tau \setminus \{0\}$  we have

$$\begin{aligned} \frac{1}{\|\xi\|_{\text{Ran } \Phi_\tau}} \langle \xi, \eta \rangle_{\text{Ran } \Phi_\tau, (\text{Ran } \Phi_\tau)'} &= \frac{1}{\|\xi\|_{\text{Ran } \Phi_\tau}} \langle u_n^\xi, \Phi_\tau^* \eta \rangle_{L^2([0, \tau]; U)} \\ &\leq \frac{1}{\|\xi\|_{\text{Ran } \Phi_\tau}} \lim_{n \rightarrow \infty} \|u_n^\xi\|_{L^2([0, \tau]; U)} \|\Phi_\tau^* \eta\|_{L^2([0, \tau]; U)} \quad (n \in \mathbb{N}). \end{aligned}$$

Passing to the limit when  $n \rightarrow \infty$  in the last inequality we obtain that for every  $\eta \in X$  and  $\xi \in \text{Ran } \Phi_\tau \setminus \{0\}$  we have

$$\frac{1}{\|\xi\|_{\text{Ran } \Phi_\tau}} \langle \xi, \eta \rangle_{\text{Ran } \Phi_\tau, (\text{Ran } \Phi_\tau)'} \leq \|\Phi_\tau^* \eta\|_{L^2([0, \tau]; U)}.$$

The last estimate clearly implies that

$$\|\eta\|_{(\text{Ran } \Phi_\tau)'} \leq \|\Phi_\tau^* \eta\|_{L^2([0, \tau]; U)} \quad (\eta \in X).$$

The above inequality, combined with the fact, following from (5.2), that

$$\|\Phi_\tau^* \eta\|_{L^2([0, \tau]; U)} \leq \|\eta\|_{(\text{Ran } \Phi_\tau)'} \quad (\eta \in X),$$

implies the conclusion (5.6).  $\square$

We end this subsection with the following result:

**Proposition 5.9.** *Assume that  $(A, B)$  is approximately controllable in time  $\tau > 0$ . Then there exists an operator*

$$(5.7) \quad W_\tau \in \mathcal{L}(L^2([0, \tau]; U), (\text{Ran } \Phi_\tau)')$$

such that

$$W_\tau \Phi_\tau^* \eta = \eta \quad (\eta \in X).$$

*Proof.* Let  $\Gamma_\tau \in \mathcal{L}(\text{Ran } \Phi_\tau, L^2([0, \tau]; U))$  be the operator in Remark 5.1 and let  $W_\tau \in \mathcal{L}((L^2([0, \tau]; U), (\text{Ran } \Phi_\tau)'))$  be the dual of  $\Gamma_\tau$ , defined by

$$\langle W_\tau u, \eta \rangle_{(\text{Ran } \Phi_\tau)', \text{Ran } \Phi_\tau} = \langle u, \Gamma_\tau \eta \rangle_{L^2([0, \tau]; U)} \quad (u \in L^2([0, \tau]; U), \eta \in \text{Ran } \Phi_\tau).$$

Thus, according to (5.3), we have

$$W_\tau \Phi_\tau' = \mathbb{I}_{(\text{Ran } \Phi_\tau)'}$$

The conclusion (5.7) follows now by recalling from Remark 5.3 that  $\Phi_\tau' \eta = \Phi_\tau^* \eta$  for every  $\eta \in X$ .  $\square$

**5.3. Negative generators with compact resolvents.** When  $A$  is negative, it is well-known that it generates an analytic semigroup  $\mathbb{T}$  on  $X$ .

In particular, there is a simple sufficient condition for the admissibility of an observation operator.

**Proposition 5.10.** *Let  $A$  be a negative operator on  $X$ . If  $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$  (recall the definition of  $X_{\frac{1}{2}}$  in (2.13)), then  $C$  is an admissible observation operator for the semigroup  $\mathbb{T}$  generated by  $A$  on  $X$ . Moreover, the output maps corresponding to the pair  $(A, C)$  satisfy*

$$(5.8) \quad (\Psi_\tau z_0)(t) = C \mathbb{T}_t z_0 \quad (t \in [0, \tau] \text{ a.e.}, z_0 \in X).$$

*Proof.* The admissibility of  $C$  is, with our assumptions, a classical result (see, for instance, [33, Proposition 5.1.3]).

In order to prove (5.8) we note that from (2.12) it follows that for every  $z_0 \in X$  the map  $t \mapsto \mathbb{T}_t z_0$  is in  $L^2([0, \tau]; X_{\frac{1}{2}})$ . Setting

$$\left( \tilde{\Psi}_\tau z_0 \right)(t) = C \mathbb{T}_t z_0 \quad (t \in [0, \tau] \text{ a.e.}, z_0 \in X)$$

It follows that the map  $\tilde{\Psi}_\tau \in \mathcal{L}(X, L^2([0, \tau]; Y))$  and

$$\tilde{\Psi}_\tau z_0 = \Psi_\tau z_0 \quad (z_0 \in \mathcal{D}(A)).$$

The above relation, (5.4) and the density of  $\mathcal{D}(A)$  in  $X$  imply the conclusion (5.8).  $\square$

For later use, we also recall two classical results. The first, see, for instance, Bensoussan et al. [3, Section II.1.3.6] is the so-called maximal regularity property:

**Theorem 5.1.** *Let  $\tilde{A}$  be the generator of an analytic semigroup  $\mathbb{S}$  on an Hilbert space  $X$ . For  $\alpha \in [0, 1]$ , we introduce  $\tilde{X}_\alpha = [X, \mathcal{D}(\tilde{A})]_\alpha$ , where  $[\cdot, \cdot]_\alpha$  stands for the interpolation space of order  $\alpha$  between  $X$  and  $\mathcal{D}(\tilde{A})$  (see [3, Section II.1.4.7]), and for  $\alpha \in [-1, 0]$ ,  $\tilde{X}_\alpha = \tilde{X}'_{-\alpha}$  with  $X$  identified with its dual.*

*Then for all  $\alpha \in [-1, 0]$  and  $\tau > 0$ , if  $f \in L^2([0, \tau]; \tilde{X}_\alpha)$  and  $z_0 \in \tilde{X}_{\alpha+1/2}$ , the solution  $z$  of*

$$\dot{z}(t) = \tilde{A}z(t) + f(t) \quad (t \in [0, \tau]), \quad \text{with } z(0) = z_0,$$

equivalently given by

$$z(t) = \mathbb{S}_t z_0 + \int_0^t \mathbb{S}_{t-\sigma} f(\sigma) d\sigma \quad (t \in [0, \tau]),$$

belongs to  $L^2([0, \tau]; \tilde{X}_{\alpha+1}) \cap H^1((0, \tau); \tilde{X}_\alpha)$  and satisfies

$$\|z\|_{L^2([0, \tau]; \tilde{X}_{\alpha+1}) \cap H^1((0, \tau); \tilde{X}_\alpha)} \leq C \left( \|z_0\|_{\tilde{X}_\alpha} + \|f\|_{L^2([0, \tau]; \tilde{X}_\alpha)} \right).$$

Remark that in [3, Section II.1.3.6], the above theorem is stated with  $\alpha = 0$ . Here, we use the fact that, if  $\mathbb{S}$  is an analytic semigroup on  $X$ , it is also an analytic semigroup on  $\tilde{X}_\alpha$  for any  $\alpha \in [0, 1]$ , and thus by duality, it can be extended as an analytic semigroup on  $\tilde{X}_\alpha$  for any  $\alpha \in [-1, 0]$ .

Also, when  $A$  is negative, the spaces  $X_\alpha$  defined in (2.13) and  $\tilde{X}_\alpha$  obviously coincide for all  $\alpha \in [-1, 1]$ . In fact, when  $A$  is negative, Theorem 5.1 also holds for any  $\alpha \in \mathbb{R}$ . We chose not to state the above result with such generality when  $A$  is the generator of an analytic semigroup to avoid additional notation.

We then recall the following standard compactness arguments, known as Aubin-Lions lemma (see, for instance, Simon [30]):

**Theorem 5.2.** *Let  $H_0 \subset H_1 \subset H_2$  be three Hilbert spaces, which are continuously embedded one into the other, such that the embedding of  $H_0$  to  $H_1$  is compact. Then, for all  $\tau > 0$ , the set  $L^2([0, \tau]; H_0) \cap H^1((0, \tau); H_2)$  is compactly embedded in  $L^2([0, \tau]; H_1)$ .*

*In particular, when  $\tilde{A}$  is the generator of an analytic semigroup on  $X$  and has compact resolvent in  $X$ , for all  $\alpha \in [-1, 0]$  and  $\tau > 0$ , for all  $\varepsilon > 0$ , defining  $\tilde{X}_\alpha$  as in Theorem 5.1, the set  $L^2([0, \tau]; \tilde{X}_{\alpha+1}) \cap H^1((0, \tau); \tilde{X}_\alpha)$  is compactly embedded in  $L^2([0, \tau]; \tilde{X}_{\alpha+1-\varepsilon})$ .*

## 6. WELL-POSED CONTROL SYSTEMS ON $\text{Ran } \Phi_\tau$ : PROOF OF THEOREM 2.1

Throughout this section we still assume that  $X$  and  $U$  are Hilbert spaces and that  $\Sigma = (\mathbb{T}, \Phi)$  is a well posed control system with state space  $X$  and input space  $U$ , in the sense of Definition 2.1.

The main aim of this section is to prove Theorem 2.1. Informally, this theorem says that if  $\Sigma$  is null controllable in any positive time then the system  $\tilde{\Sigma} = (\tilde{\mathbb{T}}, \Phi)$ , where  $\tilde{\mathbb{T}}$  is the restriction of  $\mathbb{T}$  to  $\text{Ran } \Phi_\tau$  defines an exactly controllable system, with state space  $\text{Ran } \Phi_\tau$  and input space  $U$ , in the sense of Definition 2.2. Thus every system which is null controllable in any time can be seen as an exactly controllable one, provided that the state space is appropriately chosen.

The key point consists in establishing that the restriction of  $\mathbb{T}$  to  $\text{Ran } \Phi_\tau$  forms a  $C^0$  semigroup on  $\text{Ran } \Phi_\tau$ . The proof of this fact is essentially based on the following result, stating that the restriction  $\tilde{\mathbb{T}}$  is bounded (for small times).

**Proposition 6.1.** *Assume that  $(\mathbb{T}, \Phi)$  is null controllable in any positive time. Then for every  $\tau > 0$  there exists a constant  $c_\tau > 0$  such that*

$$(6.1) \quad \|\mathbb{T}_t\|_{\mathcal{L}(\text{Ran } \Phi_\tau)} \leq c_\tau \quad (t \in (0, \tau]).$$

*Proof.* Using the equivalence (see Remark 2.2) of the norms  $\|\cdot\|_{\text{Ran } \Phi_\tau}$  and  $\|\cdot\|_{\text{Ran } \Phi_{2\tau}}$  on  $\text{Ran } \Phi_\tau$  it follows that there exists  $c_\tau > 0$  such that

$$(6.2) \quad \|\mathbb{T}_t \eta\|_{\text{Ran } \Phi_\tau} \leq c_\tau \|\mathbb{T}_t \eta\|_{\text{Ran } \Phi_{2\tau}} \quad (t \in (0, \tau], \eta \in \text{Ran } \Phi_\tau).$$

On the other hand, it is not difficult to see that

$$(6.3) \quad \|\mathbb{T}_t \eta\|_{\text{Ran } \Phi_{2\tau}} \leq \|\eta\|_{\text{Ran } \Phi_{2\tau-t}} \quad (t \in (0, \tau], \eta \in \text{Ran } \Phi_\tau).$$

Indeed, if  $u \in L^2([0, 2\tau - t]; U)$  is such that  $\Phi_{2\tau-t} u = \eta$  then  $\tilde{u} \in L^2([0, 2\tau]; U)$  defined by

$$\tilde{u}(t) = \begin{cases} u(t) & (t \in [0, 2\tau - t]), \\ 0 & (t \in (2\tau - t, 2\tau]), \end{cases}$$

satisfies  $\Phi_{2\tau} \tilde{u} = \mathbb{T}_t \eta$  and  $\|\tilde{u}\|_{L^2([0, 2\tau]; U)} = \|u\|_{L^2([0, 2\tau-t]; U)}$ . So we get

$$\|\mathbb{T}_t \eta\|_{\text{Ran } \Phi_{2\tau}} = \|\Phi_{2\tau} \tilde{u}\|_{\text{Ran } \Phi_{2\tau}} \leq \|\tilde{u}\|_{L^2([0, 2\tau]; U)} = \|u\|_{L^2([0, 2\tau-t]; U)},$$

for every  $u \in L^2([0, 2\tau - t]; U)$  such that  $\Phi_{2\tau-t} u = \eta$ , then by taking the infimum of  $u \in L^2([0, 2\tau - t]; U)$  such that  $\Phi_{2\tau-t} u = \eta$  in both sides of the previous inequality we obtain (6.3). We can thus combine the two inequalities (6.2) and (6.3) to obtain that

$$\|\mathbb{T}_t \eta\|_{\text{Ran } \Phi_\tau} \leq c_\tau \|\eta\|_{\text{Ran } \Phi_{2\tau-t}} \quad (t \in (0, \tau], \eta \in \text{Ran } \Phi_\tau).$$

Since  $2\tau - t \geq \tau$  we can combine the last estimate and (2.7) to obtain (6.1).  $\square$

*Proof of Theorem 2.1.* We first remark that the space  $\text{Ran } \Phi_\tau$  is independent from the choice of  $\tau > 0$  since  $\Sigma = (\mathbb{T}, \Phi)$  is null controllable in any positive time, see Proposition 2.1. Consequently, the family  $\tilde{\mathbb{T}} = (\mathbb{T}_t|_{\text{Ran } \Phi_\tau})_{t \geq 0}$  does not depend on the choice of  $\tau > 0$ .

The fact that  $\tilde{\mathbb{T}}$  is a semigroup of bounded operators on  $\text{Ran } \Phi_\tau$  follows from Proposition 5.4 and the fact that  $\mathbb{T}$  is a semigroup of operators on  $X$ . The strong continuity of  $\tilde{\mathbb{T}}$  follows from Proposition 6.1, the fact that  $\text{Ran } \Phi_\tau$  is a Hilbert space (see Remark 2.1) and from Corollary 2.4 from Hille [20]. Using next Proposition 5.2 and the identity (2.1) of Definition 2.1 it follows that indeed  $\tilde{\Sigma}$  is a well posed control system with state space  $\text{Ran } \Phi_\tau$  and input space  $U$ .

Finally, the above facts clearly imply that  $\tilde{\Sigma}$  satisfies the Definition 2.2 of an exactly controllable system in any positive time.  $\square$

**Remark 6.1.** Since  $\text{Ran } \Phi_\tau$  is a closed subspace of  $X$  which, by Theorem 2.1, is invariant for the semigroup  $\mathbb{T}$ , it follows (see, for instance, Engel and Nagel [9, p.61]) that the generator of the semigroup  $\tilde{\mathbb{T}}$  is the operator  $\tilde{A}$  defined by  $\mathcal{D}(\tilde{A}) = \mathcal{D}(A) \cap \text{Ran } \Phi_\tau$  and

$$\tilde{A}z_0 = Az_0 \quad (z_0 \in \mathcal{D}(\tilde{A})).$$

**Remark 6.2.** When the semigroup  $\mathbb{T}$  is further assumed to be an analytic semigroup on the space  $X$ , an interesting open question is whether the semigroup  $\tilde{\mathbb{T}}$  is analytic on  $\text{Ran } \Phi_\tau$ . We do not have any clear guess on the answer to the question above. We think however that a positive answer would be quite surprising in view of the results in Rebarber and Weiss [27] (see e.g. Corollary 1.7 in [27]), where it is shown that if  $(A, B)$  is exactly controllable, with  $A$  having a Riesz basis of eigenvectors, generating an analytic semigroup and with a finite dimensional input space, then the eigenvalues of  $A$  have to grow faster than every polynomial. On the other hand, it would be quite natural that the semigroup  $\tilde{\mathbb{T}}$  retains some of the smoothing properties of  $\mathbb{T}$ .

For later use, we also make explicit an extra useful property, which is a direct consequence of Theorem 2.1 and of [33, Proposition 4.2.4]:



**Proposition 6.2.** *With the assumptions of Theorem 2.1, for every  $\tau > 0$ , for every  $u \in L^2([0, \infty); U)$ , the map  $\sigma \mapsto \Phi_\sigma u$  is continuous from  $[0, \infty)$  to  $\text{Ran } \Phi_\tau$ .*

7. REACHABILITY RESULTS UNDER SMALL PERTURBATIONS: PROOF OF THEOREMS 2.2 AND 2.3

We continue to assume that  $X$  and  $U$  are Hilbert spaces and that  $\Sigma = (\mathbb{T}, \Phi)$  is a well posed control system with state space  $X$  and input space  $U$ , in the sense of Definition 2.1. Let  $A : \mathcal{D}(A) \rightarrow X$  be the generator of the  $C^0$  semigroup  $\mathbb{T}$  on  $X$  and let  $B \in \mathcal{L}(U, X_{-1})$  be the corresponding control operator, defined by (2.5), so that  $B$  is an admissible control operator for  $\mathbb{T}$ . Throughout this section we assume that  $P \in \mathcal{L}(X)$ .

We first recall a classical perturbation result, see, for instance, Pazy [26, Section 3.1, Theorem 1.1, Equation (1.2)]:

**Proposition 7.1.** *The operator  $A + P : \mathcal{D}(A) \rightarrow X$  is the generator of a strongly continuous semigroup  $\mathbb{T}^P$  on  $X$ . For every  $\tau > 0$ , this semigroup can be defined as the sum in  $C^0([0, \tau]; \mathcal{L}(X))$  of the series*

$$\mathbb{T}_t^P = \sum_{n \geq 0} S_n(t) \quad (t \in [0, \tau]),$$

where the sequence  $(S_n)$  is defined by  $S_0 = \mathbb{T}$  and

$$(7.1) \quad S_{n+1}(t)\psi = \int_0^t \mathbb{T}_{t-\sigma} P S_n(\sigma)\psi \, d\sigma \quad (t \geq 0, \psi \in X).$$

Moreover,  $\mathbb{T}^P$  satisfies the integral equation

$$(7.2) \quad \mathbb{T}_t^P \psi = \mathbb{T}_t \psi + \int_0^t \mathbb{T}_{t-\sigma} P \mathbb{T}_\sigma^P \psi \, d\sigma \quad (\psi \in X, t \geq 0).$$

**Remark 7.1.** We can invert the roles of  $A$  and  $A + P$  in (7.2) to obtain the formula

$$(7.3) \quad \mathbb{T}_t^P \psi = \mathbb{T}_t \psi + \int_0^t \mathbb{T}_{t-\sigma}^P P \mathbb{T}_\sigma \psi \, d\sigma \quad (\psi \in X, t \geq 0).$$

Using the above proposition it follows that:

**Proposition 7.2.** *With the above notation and assumptions, the pair  $(A + P, B)$  defines a well posed linear control system. The corresponding input maps  $\Phi^P$  satisfy*

$$(7.4) \quad \Phi_\tau^P u = \Phi_\tau u + K_\tau u \quad (\tau > 0, u \in L^2([0, \tau]; U)),$$

where  $K_\tau \in \mathcal{L}(L^2([0, \tau]; U), X)$  is defined by

$$K_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma}^P P \Phi_\sigma u \, d\sigma \quad (\tau > 0, u \in L^2([0, \tau]; U)).$$

*Proof.* For  $t \geq 0$  and  $u \in L^2([0, \infty); U)$  we set  $w(t) = \Phi_t u$  and  $\xi(t) = \Phi_t^P u$ . According to Proposition 5.1, the functions  $w$  and  $\xi$  are the solutions (in the sense made precise in [33, Section 4.1]) of the initial value problems

$$\dot{w}(t) = Aw(t) + Bu(t) \quad (t \geq 0), \quad w(0) = 0,$$

and

$$\dot{\xi}(t) = (A + P)\xi(t) + Bu(t) \quad (t \geq 0), \quad \xi(0) = 0.$$

Noting that  $\xi = w + \varphi$ , where

$$\dot{\varphi}(t) = (A + P)\varphi(t) + Pw(t) \quad (t \geq 0), \quad \varphi(0) = 0,$$

we obtain that

$$\xi(\tau) = w(\tau) + \int_0^\tau \mathbb{T}_{\tau-\sigma}^P Pw(\sigma) d\sigma.$$

The last formula and (2.8) imply the conclusion (7.4).  $\square$

The conclusion of the proposition below says that the first assertion in Theorem 2.1 is robust with respect to a class of perturbations of the generator.

**Proposition 7.3.** *Assume that  $\Sigma$  is null controllable in any positive time and that  $P \in \mathcal{L}(X) \cap \mathcal{L}(\text{Ran } \Phi_{\tau_0})$  for some (and hence all)  $\tau_0 > 0$ . Then, for all  $\tau > 0$ , the restriction  $\mathbb{T}^P$  of  $\mathbb{T}^P$  to  $\text{Ran } \Phi_\tau$  is a  $C^0$  semigroup on  $\text{Ran } \Phi_\tau$ , which is independent of  $\tau > 0$ . Moreover, the generator  $\tilde{A}^P$  of  $\mathbb{T}^P$  is given by*

$$(7.5) \quad \mathcal{D}(\tilde{A}^P) = \mathcal{D}(A) \cap \text{Ran } \Phi_\tau$$

$$(7.6) \quad \tilde{A}^P \eta = \tilde{A} \eta + P \eta \quad (\eta \in \mathcal{D}(\tilde{A}^P)).$$

Finally, there exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that

$$(7.7) \quad \sup_{t \in [0, \tau]} \|\mathbb{T}_t^P\|_{\mathcal{L}(\text{Ran } \Phi_{\tau_0})} \leq M e^{(\omega + M\|P\|_{\mathcal{L}(\text{Ran } \Phi_{\tau_0})})t}.$$

*Proof.* First, thanks to Theorem 2.1 and to an elementary result in semigroup theory (see, for instance, [26, Corollary 1.4]) we know that there exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that

$$(7.8) \quad \|\tilde{\mathbb{T}}_t\|_{\mathcal{L}(\text{Ran } \Phi_\tau)} \leq M \exp(\omega t) \quad (t \geq 0).$$

Let  $(S_n)$  be the sequence defined in (7.1). For each  $n \in \mathbb{N}$  and  $t \geq 0$  we denote by  $\tilde{S}_n(t)$  the restriction of  $S_n(t)$  to  $\text{Ran } \Phi_\tau$ . By induction, using (7.1) and (7.8), we have that  $\tilde{S}_n(t) \in \mathcal{L}(\text{Ran } \Phi_\tau)$  for every  $n \in \mathbb{N}$  and  $t \geq 0$ .

We then prove below by induction that

$$(7.9) \quad \|\tilde{S}_n(t)\|_{\mathcal{L}(\text{Ran } \Phi_\tau)} \leq \exp(\omega t) \frac{M^{n+1} t^n \|P\|_{\mathcal{L}(\text{Ran } \Phi_\tau)}^n}{n!} \quad (n \in \mathbb{N}, t \geq 0).$$

Indeed, the above inequality reduces to (7.8) for  $n = 0$ . Assuming next that (7.9) holds for some  $n \in \mathbb{N}$  it follows from (7.1) that for every  $t \geq 0$  and

$$\|\tilde{S}_{n+1}(t)\|_{\mathcal{L}(\text{Ran } \Phi_\tau)} \leq \exp(\omega t) M^{n+2} \|P\|_{\mathcal{L}(\text{Ran } \Phi_\tau)}^{n+1} \int_0^t \frac{\sigma^n}{n!} d\sigma,$$

which clearly implies that (7.9) holds with  $n$  replaced by  $n + 1$  and thus for all  $n \in \mathbb{N}$ .

We next note that (7.9) implies that there exists a  $C^0$  semigroup  $\mathbb{S}$  on  $\text{Ran } \Phi_\tau$  such that for all  $T > 0$ ,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\mathbb{S}_t \eta - \sum_{k=0}^n \tilde{S}_k(t) \eta\|_{\text{Ran } \Phi_\tau} = 0 \quad (\eta \in \text{Ran } \Phi_\tau).$$

On the other hand, it follows from Proposition 7.1 that for all  $T > 0$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\mathbb{T}_t^P \eta - \sum_{k=0}^n \tilde{S}_k(t) \eta\|_X = 0 \quad (\eta \in \text{Ran } \Phi_\tau).$$

The last two formulas and the continuous embedding  $\text{Ran } \Phi_\tau \subset X$  imply that  $\tilde{\mathbb{T}}^P$  obtained by restricting  $\mathbb{T}^P$  to  $\text{Ran } \Phi_\tau$  is equal to  $\mathbb{S}$ ; then it is indeed a  $C^0$  semigroup on  $\text{Ran } \Phi_\tau$ .

Moreover, the fact that the generator  $\tilde{A}^P$  of  $\tilde{\mathbb{T}}^P$  is described by (7.5) and (7.6) follows from [9, Corollary p.61].

Finally, the bound (7.7) directly follows from (7.9).  $\square$

We are now in a position to give the proof of Theorem 2.2.

*Proof of Theorem 2.2.* Propositions 7.3 and 6.2 imply that for every  $\tau \in [0, \tau_0]$  we have

$$\int_0^\tau \mathbb{T}_{\tau-\sigma}^P P \Phi_\sigma u \, d\sigma \in \text{Ran } \Phi_{\tau_0} \quad (u \in L^2([0, \tau]; U)),$$

and that for every  $\tau \in [0, \tau_0]$  and  $u \in L^2([0, \tau_0]; U)$  we have

$$(7.10) \quad \left\| \int_0^\tau \mathbb{T}_{\tau-\sigma}^P P \Phi_\sigma u \, d\sigma \right\|_{\text{Ran } \Phi_{\tau_0}} \leq \tau \sup_{t \in [0, \tau]} \|\mathbb{T}_t^P\|_{\mathcal{L}(\text{Ran } \Phi_{\tau_0})} \|P\|_{\mathcal{L}(\text{Ran } \Phi_{\tau_0})} \|u\|_{L^2([0, \tau_0]; U)}.$$

The above estimate, combined with (7.4) and the fact that for every  $\tau \in (0, \tau_0]$  we have  $\text{Ran } \Phi_\tau = \text{Ran } \Phi_{\tau_0}$ , (with equivalent norms) implies that indeed we have (2.9). A direct consequence of this fact is the inclusion (2.10).  $\square$

*Proof of Theorem 2.3.* To prove Theorem 2.3 it suffices to show that there exists  $\delta_{\tau_0} > 0$  such that (2.11) implies that

$$\text{Ran } \Phi_{\tau_0}^P \supset \text{Ran } \Phi_{\tau_0}.$$

To this aim it suffices to note that from (7.4) and (7.10), for every  $\varepsilon > 0$  there exists  $\delta_{\tau_0} > 0$  (depending also on  $\varepsilon$ ) such that if (2.11) holds then

$$\|\Phi_{\tau_0}^P - \Phi_{\tau_0}\|_{\mathcal{L}(L^2([0, \tau_0]; U), \text{Ran } \Phi_{\tau_0})} \leq \varepsilon.$$

Since the subset formed by the surjective operators in  $\mathcal{L}(L^2([0, \tau_0]; U), \text{Ran } \Phi_{\tau_0})$  is open, it follows that the condition (2.11) for  $\delta_{\tau_0} > 0$  small enough implies that  $\Phi_{\tau_0}^P$  is surjective and maps indeed  $L^2([0, \tau_0]; U)$  onto  $\text{Ran } \Phi_{\tau_0}$ , which ends the proof of Theorem 2.3.  $\square$

## 8. A HAUTUS TEST FOR APPROXIMATE CONTROLLABILITY

In this section we study the behavior of some of the admissibility and controllability properties of a system with negative generator when this generator is perturbed by a class of bounded operators. We give, in particular, sufficient conditions for the validity of a Hautus test for the approximate observability and approximate controllability of the perturbed system. This study will be continued in the next section, where we give the proof of the abstract perturbation result stated in Theorem 2.4.

*Notation:* Throughout this section we continue to use the notation and the assumptions introduced in Section 5 and more particularly the ones of Subsection 5.3. More precisely, we assume that  $X$ ,  $U$  and  $Y$  are Hilbert spaces,  $A : \mathcal{D}(A) \rightarrow X$  is supposed to be negative with compact resolvents,  $(X_\alpha)_{\alpha \in \mathbb{R}}$  is the scale of Hilbert spaces introduced in (2.13),  $B \in \mathcal{L}(U, X_{-1})$  and  $C \in \mathcal{L}(X_1, Y)$  are admissible control, respectively observation, operators for the semigroup  $\mathbb{T}$  generated by  $A$ .

Moreover,  $(\Phi_\tau)_{\tau \geq 0}$  are the input maps corresponding to the pair  $(A, B)$ , whereas  $(\Psi_\tau)_{\tau \geq 0}$  are the output maps corresponding to  $(A, C)$ .

We first note the following classical results, see, for instance, Pazy [26, Section 3.2, Theorem 2.1]:

**Proposition 8.1.** *Suppose that  $A < 0$  and let  $P \in \mathcal{L}(X)$ . Then  $A + P$  generates an analytic semigroup  $\mathbb{T}^P$  on  $X$ .*

Furthermore, when  $P \in \mathcal{L}(X)$ , we easily check that  $\mathcal{D}(A + P) = \mathcal{D}(A)$ , so that the space  $\tilde{X}_1$  corresponding to the generator  $\tilde{A} = A + P$  of the analytic semigroup  $\mathbb{S} = \mathbb{T}^P$  in  $X$ , defined in Theorem 5.1, coincide with the spaces  $X_1$  in (2.13). This implies that for any  $\alpha \in [-1, 1]$ , the spaces  $\tilde{X}_\alpha$  corresponding to the generator  $\tilde{A} = A + P$  defined in Theorem 5.1 coincide with the spaces  $X_\alpha$  in (2.13). We can thus use Theorem 5.1 for both semigroups  $\mathbb{T}$  and  $\mathbb{T}^P$  on the spaces  $X_\alpha$  defined in (2.13).

We then state the following regularity result for the perturbed semigroup  $\mathbb{T}^P$ :

**Proposition 8.2.** *Suppose that  $A < 0$  and let  $P \in \mathcal{L}(X)$ . For every  $\psi \in X$ , the linear map  $\psi \mapsto \mathbb{T}_t^P \psi$  is bounded from  $X$  to  $L^2([0, \tau]; X_{\frac{1}{2}})$ .*

*Proof.* From Theorem 5.1, the map

$$f \mapsto \int_0^t \mathbb{T}_{t-\sigma} f(\sigma) d\sigma,$$

is linear and bounded from  $L^2([0, \tau]; X)$  to  $L^2([0, \tau]; X_{\frac{1}{2}})$ . This, combined with the fact that the map

$$\psi \mapsto \mathbb{T}_t^P \psi \quad (t \in [0, \tau]),$$

is linear and continuous from  $X$  to  $L^2([0, \tau]; X)$ , together with (7.2) yields the conclusion of this proposition.  $\square$

For the remaining part of this section we found more convenient to state and prove our results in the case of observation systems  $(A, C)$  and then use duality to go back to controllability properties.

We first note that a consequence of Proposition 8.2 is that the result in Proposition 5.10 is still valid for systems with the generator obtained by a bounded perturbation of a negative operator with compact resolvents.

**Proposition 8.3.** *Suppose that  $A < 0$  and assume that  $Q \in \mathcal{L}(X)$  and  $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$ . Then  $C$  is an admissible observation operator for the semigroup  $\mathbb{T}^Q$  generated by  $A + Q$  and the corresponding output maps  $\Psi_\tau^Q$  satisfy*

$$(\Psi_\tau^Q z_0)(t) = C \mathbb{T}_t^Q z_0 \quad (t \in [0, \tau] \text{ a.e., } z_0 \in X).$$

We next prove the following preliminary result:

**Proposition 8.4.** *Suppose that  $A < 0$ ,  $C \in \mathcal{L}(X_\alpha, Y)$  for some  $\alpha \in [0, 1/2]$  and that  $Q \in \mathcal{L}(X)$  is such that there exist  $M > 0$  and  $\tau > 0$  with*

$$(8.1) \quad \int_0^\tau \|Q \mathbb{T}_t \eta\|_{X_{-1+\alpha}}^2 dt \leq M^2 \int_0^\tau \|C \mathbb{T}_t \eta\|_Y^2 dt \quad (\eta \in \mathcal{D}(A)).$$

Then

$$(8.2) \quad \int_0^\tau \|C \mathbb{T}_t^Q \eta\|_Y^2 dt \leq M^2 \int_0^\tau \|C \mathbb{T}_t \eta\|_Y^2 dt \quad (\eta \in \mathcal{D}(A)).$$

*Proof.* Using formula (7.3) with  $Q$  instead of  $P$  it follows that

$$(8.3) \quad \mathbb{T}_t^Q \eta = \mathbb{T}_t \eta + \int_0^t \mathbb{T}_{t-\sigma}^Q Q \mathbb{T}_\sigma \eta \, d\sigma \quad (\eta \in X, t \geq 0).$$

Since for every  $\eta \in \mathcal{D}(A)$  the map  $\sigma \mapsto Q \mathbb{T}_\sigma \eta$  belongs to  $L^2([0, \tau]; X_{-1+\alpha})$  with  $\|Q \mathbb{T}_t \eta\|_{L^2([0, \tau]; X_{-1+\alpha})} \leq M \|C \mathbb{T}_t \eta\|_{L^2([0, \tau]; Y)}$ , from (8.1) and the maximal regularity results for  $\mathbb{T}^Q$  (recall Theorem 5.1) it follows that

$$\left\| \int_0^t \mathbb{T}_{t-\sigma}^Q Q \mathbb{T}_\sigma \eta \, d\sigma \right\|_{L^2([0, \tau]; X_\alpha)} \leq M \|C \mathbb{T}_t \eta\|_{L^2([0, \tau]; Y)} \quad (\eta \in \mathcal{D}(A)).$$

Accordingly, since  $C \in \mathcal{L}(X_\alpha, Y)$ , we have

$$\left\| C \int_0^t \mathbb{T}_{t-\sigma}^Q Q \mathbb{T}_\sigma \eta \, d\sigma \right\|_{L^2([0, \tau]; Y)} \leq M \|C \mathbb{T}_t \eta\|_{L^2([0, \tau]; Y)}.$$

The conclusion (8.2) follows now easily from (8.3).  $\square$

We can now state and prove the main result of this section, which gives sufficient conditions allowing to test the approximate observability of the pair  $(A+Q, C)$ , with  $Q \in \mathcal{L}(X)$ , by a Hautus type condition completely similar to the one encountered in the classical finite dimensional case.

**Proposition 8.5.** *Suppose that  $A < 0$  and has compact resolvents,  $C \in \mathcal{L}(X_\alpha, Y)$  for  $\alpha \in [0, 1/2]$  and that  $(A, C)$  is approximately observable in some (thus any) time  $\tau > 0$ . Moreover, suppose that  $Q \in \mathcal{L}(X)$  is such that there exist  $M_0 > 0$  and  $\tau_0 > 0$  such that for every  $\eta \in \mathcal{D}(A)$  we have*

$$(8.4) \quad \int_0^{\tau_0} \|Q \mathbb{T}_t \eta\|_{X_{-1+\alpha+\varepsilon}}^2 \, dt \leq M^2 \int_0^{\tau_0} \|C \mathbb{T}_t \eta\|_Y^2 \, dt,$$

for some  $\varepsilon \in (0, 1 - \alpha]$ . Moreover, assume that

$$(8.5) \quad \text{Ker}(sI - A - Q) \cap \text{Ker} C = \{0\} \quad (s \in \mathbb{C}).$$

Then the pair  $(A + Q, C)$  is approximately observable in any time  $\tau > 0$ .

**Remark 8.1.** In order to prove Proposition 8.5 it is convenient to introduce some new notation and to rephrase one of our previous results. Let  $\eta \in X$ , and let us define, for every  $t \geq 0$ ,

$$(8.6) \quad \varphi(t) = \mathbb{T}_t^Q \eta, \quad p(t) = \mathbb{T}_t \eta, \quad \zeta(t) = \int_0^t \mathbb{T}_{t-\sigma}^Q Q \mathbb{T}_\sigma \eta \, d\sigma.$$

Rephrasing (8.3), we have  $\varphi = p + \zeta$  and the proof of Proposition 8.4 implies:

$$(8.7) \quad \|\zeta\|_{L^2([0, \tau]; X_\alpha)} + \|C \zeta\|_{L^2([0, \tau]; Y)} \leq M \|C p\|_{L^2([0, \tau]; Y)}.$$

We are now in a position to prove Proposition 8.5.

*Proof of Proposition 8.5.* According to Definition 5.1 and to Proposition 8.3 it suffices to prove that the set

$$(8.8) \quad \mathcal{N}_\tau = \left\{ \eta \in X \mid C \mathbb{T}_t^Q \eta = 0 \text{ for } t \in [0, \tau] \text{ a.e.} \right\}$$

reduces to the trivial one, i.e., that

$$(8.9) \quad \mathcal{N}_\tau = \{0\}.$$

The proof of (8.9) is divided into four steps.

In the *first step of our proof* we introduce the input maps  $\Phi^d = (\Phi_t^d)_{t \geq 0}$  of the system  $(A, C^*)$  (with state space  $X$  and input space  $Y$ ). From Proposition 5.10 and Proposition 5.5 it follows that  $C^* \in \mathcal{L}(Y, X_{-\alpha})$  is an admissible control operator for  $\mathbb{T}$ , then we have that  $\Phi_\tau^d \in \mathcal{L}(L^2([0, \tau]; Y), X)$ . Moreover, by Proposition 5.7, the pair  $(A, C^*)$  is approximately controllable in time  $\tau$ .

A *second step of our proof* consists in showing that the space  $\mathcal{N}_\tau$  is finite dimensional. Knowing from the first step that  $\text{Ran } \Phi_\tau^d$  is dense in  $X$ , let  $(\text{Ran } \Phi_\tau^d)'$  be the dual of  $\text{Ran } \Phi_\tau^d$  with respect to the pivot space  $X$ .

From (5.6), (5.5) and (5.4), we recall that we have for every  $\eta \in (\text{Ran } \Phi_\tau^d)'$ ,

$$\|\eta\|_{(\text{Ran } \Phi_\tau^d)'} = \|(\Phi_\tau^d)^* \eta\|_{L^2([0, \tau]; Y)} = \|C\mathbb{T}_{\tau-} \eta\|_{L^2([0, \tau]; Y)} = \|C\mathbb{T} \eta\|_{L^2([0, \tau]; Y)}.$$

Note that, since condition (8.1) is stronger than (8.4), the result of Proposition 8.4 holds. Thus, recalling (5.6),  $C\mathbb{T}^Q$  can be extended uniquely as a continuous operator from  $(\text{Ran } \Phi_\tau^d)'$  to  $L^2([0, \tau]; Y)$ . For the sake of simplicity, this extension will be still denoted by  $C\mathbb{T}^Q$ .

On the other hand, using the notation in Remark 8.1, it follows from (8.7) that the linear map  $\eta \mapsto \zeta$  from  $X$  to  $L^2([0, \tau]; X_\alpha)$  can be uniquely extended to a linear bounded map from  $(\text{Ran } \Phi_\tau^d)'$  to  $L^2([0, \tau]; X_\alpha)$ , and similarly,  $\eta \mapsto C\mathbb{T}_t^Q \eta$  can be extended to  $(\text{Ran } \Phi_\tau^d)'$  to  $L^2([0, \tau]; Y)$ .

Let

$$\widetilde{\mathcal{N}}_\tau = \left\{ \eta \in (\text{Ran } \Phi_\tau^d)' \mid C\mathbb{T}_t^Q \eta = 0 \text{ for } t \in [0, \tau] \text{ a.e.} \right\},$$

and let  $(\eta_j)$  be a sequence in  $\widetilde{\mathcal{N}}_\tau$  such that

$$(8.10) \quad \|\eta_j\|_{(\text{Ran } \Phi_\tau^d)'} \leq 1 \quad (j \in \mathbb{N}).$$

For each  $j \in \mathbb{N}$  we denote by  $(\varphi_j)$  and  $(\zeta_j)$  the functions given by (8.6) with  $\eta = \eta_j$ , so that

$$(8.11) \quad C\mathbb{T}_t \eta_j + C\zeta_j(t) = 0 \quad (t \in [0, \tau] \text{ a.e.}).$$

At this stage we remark that (8.10) and (5.6) imply that

$$\|C\mathbb{T}_t \eta_j\|_{L^2([0, \tau]; Y)} \leq 1 \quad (j \in \mathbb{N}).$$

The above estimate and (8.4) imply that the sequence  $(Q\mathbb{T}_t \eta_j)$  contains a subsequence which is weakly convergent in  $L^2([0, \tau]; X_{-1+\alpha+\varepsilon})$ . Using the maximal regularity property for  $\mathbb{T}^Q$  (recall Theorem 5.1) it follows that the sequence  $(\zeta_j)$  contains a subsequence which is weakly convergent to some  $\zeta \in L^2([0, \tau]; X_{\alpha+\varepsilon}) \cap H^1((0, \tau); X_{-1+\alpha+\varepsilon})$ . The Aubin-Lions lemma (Theorem 5.2) then gives that the sequence  $(\zeta_j)$  is strongly convergent to  $\zeta \in L^2([0, \tau]; X_\alpha)$ . This convergence, the fact that  $C \in \mathcal{L}(X_\alpha, Y)$  and (8.11) imply that, up to the extraction of a subsequence, we have

$$\lim_{j \rightarrow \infty} \int_0^\tau \|C\zeta_j(t) - C\zeta(t)\|_Y^2 dt = 0,$$

so that

$$C\mathbb{T}_t \eta_j + C\zeta(t) \rightarrow 0 \quad \text{in } L^2([0, \tau]; Y).$$

Hence the map  $t \mapsto C\mathbb{T}_{\tau-t} \eta_j + C\zeta(\tau-t)$  converges to 0 in  $L^2([0, \tau]; Y)$  when  $j \rightarrow \infty$ . Using the operator  $W_\tau$  constructed in Proposition 5.9 it follows that the sequence  $(\eta_j)$  contains a subsequence which converges strongly in  $\widetilde{\mathcal{N}}_\tau$ , endowed with the norm  $\|\cdot\|_{(\text{Ran } \Phi_\tau^d)'}$ , towards  $-W_\tau(t \mapsto C\zeta(\tau-t))$ . We have thus shown that the unit ball

of  $\widetilde{\mathcal{N}}_\tau$ , endowed with the norm  $\|\cdot\|_{(\text{Ran } \Phi_\tau)^\vee}$ , is relatively compact so that indeed  $\widetilde{\mathcal{N}}_\tau$  is finite dimensional. Since  $\mathcal{N}_\tau \subset \widetilde{\mathcal{N}}_\tau$ , this obviously implies that  $\mathcal{N}_\tau$  is of finite dimension.

The *third step of the proof* begins by remarking that for every  $z_0 \in X$  and  $\delta > 0$  the map  $t \mapsto C\mathbb{T}_t^Q z_0$  is analytic from  $(\delta, \infty)$  to  $Y$ . Thus the set  $\mathcal{N}_\tau$  defined in (8.8) is equal to

$$\mathcal{N}_\tau = \left\{ \eta \in X \mid C\mathbb{T}_t^Q \eta = 0 \text{ for } t \in [0, \infty) \text{ a.e.} \right\}.$$

Using again the analyticity of the semigroup  $\mathbb{T}^Q$  it follows that  $\mathcal{N}_\tau$  satisfies (8.9) iff the set  $\mathcal{N}_\infty$  defined by

$$\mathcal{N}_\infty = \left\{ \psi \in \mathcal{D}((A+Q)^\infty) \mid C\mathbb{T}_t^Q \psi = 0 \text{ for } t \geq 0 \right\},$$

reduces to the trivial space, i.e., we have

$$\mathcal{N}_\infty = \{0\}.$$

The *fourth and last step of the proof* begins by remarking that  $(A+Q)\mathcal{N}_\infty \subset \mathcal{N}_\infty$ . Indeed, if  $\psi \in \mathcal{N}_\infty$  then  $\frac{d}{dt}(C\mathbb{T}_t^Q \psi) = 0$  for every  $t \geq 0$ , thus

$$C\mathbb{T}_t^Q (A^Q \psi) = 0 \quad (t \geq 0).$$

It follows that  $A^Q \psi = (A+Q)\psi \in \mathcal{N}_\infty$ . We end by a contradiction argument. Assume that  $\mathcal{N}_\infty \neq \{0\}$  and denote  $\mathcal{A}_\mathcal{N}^Q$  the restriction of  $A+Q$  to  $\mathcal{N}_\infty$ . Then  $\mathcal{A}_\mathcal{N}^Q \in \mathcal{L}(\mathcal{N}_\infty)$ . Since, as seen from the first step,  $\mathcal{N}_\infty$  is finite dimensional it follows that  $\mathcal{A}_\mathcal{N}^Q$  has at least one eigenvalue, which clearly contradicts (8.5).  $\square$

By duality, or more precisely by combining Proposition 8.5 and Proposition 5.7, we obtain:

**Proposition 8.6.** *Let  $\tau > 0$ . Suppose that  $A < 0$  has compact resolvents,  $B \in \mathcal{L}(U, X_{-\alpha})$  for  $\alpha \in [0, 1/2]$  and that  $(A, B)$  is approximately controllable in some (and hence all) time  $\tau > 0$ . Moreover, assume that  $P \in \mathcal{L}(X)$  is such that the operator  $L_\tau \in \mathcal{L}(L^2([0, \tau]; X), X)$  defined by*

$$(8.12) \quad L_\tau v = \int_0^\tau \mathbb{T}_{\tau-\sigma} P v(\sigma) d\sigma \quad (v \in L^2([0, \tau]; X)),$$

*satisfies the condition*

$$(8.13) \quad \text{Ran} (L_\tau|_{L^2([0, \tau]; X_{1-\alpha-\varepsilon})}) \subset \text{Ran } \Phi_\tau,$$

*for some  $\varepsilon \in (0, 1-\alpha]$ . Finally, suppose that the pair  $(A+P, B)$  satisfies the Hautus type condition*

$$(8.14) \quad \text{Ker}(sI - A - P^*) \cap \text{Ker } B^* = \{0\} \quad (s \in \mathbb{C}).$$

*Then the pair  $(A+P, B)$  is approximately controllable in any time  $\tau > 0$  and  $\text{Ran } \Phi_\tau^P \subset \text{Ran } \Phi_\tau$ .*

*Proof.* Assume that (8.13) holds. Let us compute the adjoint of

$$\hat{L} := L_\tau|_{L^2([0, \tau]; X_{1-\alpha-\varepsilon})} \in \mathcal{L}(L^2([0, \tau]; X_{1-\alpha-\varepsilon}), \text{Ran } \Phi_\tau).$$

We have that for every  $\eta \in \text{Ran } \Phi_\tau$ ,

$$\begin{aligned} & \left\langle \int_0^\tau \mathbb{T}_{\tau-\sigma} P v(\sigma) \, d\sigma, \eta \right\rangle_{(\text{Ran } \Phi_\tau)', \text{Ran } \Phi_\tau} = \int_0^\tau \langle \mathbb{T}_{\tau-\sigma} P v(\sigma), \eta \rangle_{(\text{Ran } \Phi_\tau)', \text{Ran } \Phi_\tau} \, d\sigma \\ &= \int_0^\tau \langle \mathbb{T}_{\tau-\sigma} P v(\sigma), \eta \rangle_{X, X} \, d\sigma \\ &= \int_0^\tau \langle v(\sigma), P^* \mathbb{T}_{\tau-\sigma}^* \eta \rangle_{X, X} \, d\sigma \\ &= \int_0^\tau \langle v(\sigma), P^* \mathbb{T}_{\tau-\sigma} \eta \rangle_{X_{1-\alpha-\varepsilon}, X_{-1+\alpha+\varepsilon}} \, d\sigma. \end{aligned}$$

By identification, the adjoint of  $\hat{L}$  is given, for  $\eta \in (\text{Ran } \Phi_\tau)'$ , by

$$\hat{L}^*(\eta) = (\sigma \mapsto P^* \mathbb{T}_{\tau-\sigma} \eta),$$

seen as an element of  $L^2([0, \tau]; X_{-1+\alpha+\varepsilon})$ .

Now, we recall that we have from (5.5) and (5.6)

$$\|\eta\|_{(\text{Ran } \Phi_\tau)'} = \|B^* \mathbb{T}_{\tau-\cdot} \eta\|_{L^2([0, \tau]; Y)}.$$

Therefore using the fact that  $\hat{L}^*$  belongs to  $\mathcal{L}((\text{Ran } \Phi_\tau)', L^2([0, \tau]; X_{-1+\alpha+\varepsilon}))$ , we have (8.4) with  $Q = P^*$ ,  $C = B^*$ . Moreover by using the duality Proposition 5.7, we have that  $(A, C)$  is approximately observable in time  $\tau$ . Finally, (8.14) exactly corresponds to (8.5). So by applying Proposition 8.5 and Proposition 5.7, we obtain that  $(A + P, B)$  is approximately controllable in any time  $\tau > 0$ .

Finally, using (5.6), we interpret (8.2) as  $(\text{Ran } \Phi_\tau)' \subset (\text{Ran } \Phi_\tau^P)'$ . By duality, this means  $\text{Ran } \Phi_\tau^P \subset \text{Ran } \Phi_\tau$  as announced.  $\square$

## 9. PROOF OF THEOREM 2.4

In this section we continue to use all the notation and assumptions described at the beginning of Section 8. We first prove the following result:

**Proposition 9.1.** *With the assumptions in Proposition 8.5, assume moreover that  $(A, C)$  is final state observable in any time  $\tau > 0$ . Then there exists a constant  $c_0 > 0$  such that*

$$(9.1) \quad \|C \mathbb{T}_t^Q \eta\|_{L^2([0, \tau]; Y)} \geq c_0 \|C \mathbb{T}_t \eta\|_{L^2([0, \tau]; Y)} \quad (\tau > 0, \eta \in X).$$

*Proof.* We argue by contradiction. If the conclusion of the proposition were false it would exist a sequence  $(\eta_j)$  formed of elements of  $\mathcal{D}(A)$  such that

$$(9.2) \quad \int_0^\tau \|C \mathbb{T}_t \eta_j\|_Y^2 \, dt = 1 \quad (j \in \mathbb{N}),$$

$$(9.3) \quad \lim_{j \rightarrow \infty} \int_0^\tau \|C \mathbb{T}_t^Q \eta_j\|_Y^2 \, dt = 0.$$

For each  $j \in \mathbb{N}$  we introduce  $p_j$ ,  $\varphi_j$  and  $\zeta_j$  as in (8.6) with  $\eta = \eta_j$ . With this notation, (9.2) and (9.3) can be rephrased as

$$(9.4) \quad \int_0^\tau \|C p_j(t)\|_Y^2 \, dt = 1 \quad (j \in \mathbb{N}),$$

$$(9.5) \quad \lim_{j \rightarrow \infty} \int_0^\tau \|C \varphi_j(t)\|_Y^2 \, dt = 0.$$



Since  $p_j = \varphi_j - \zeta_j$ , it follows that

$$(9.6) \quad \int_0^\tau \|Cp_j(t)\|_Y^2 dt \leq 2 \int_0^\tau \|C\varphi_j(t)\|_Y^2 dt + 2 \int_0^\tau \|C\zeta_j(t)\|_Y^2 dt \quad (j \in \mathbb{N}).$$

To obtain a contradiction it thus suffices to show that

$$(9.7) \quad \lim_{j \rightarrow \infty} \int_0^\tau \|C\zeta_j(t)\|_Y^2 dt = 0.$$

Indeed, assuming that (9.7) holds, we can combine (9.5) and (9.6) to obtain that

$$\lim_{j \rightarrow \infty} \int_0^\tau \|Cp_j(t)\|_Y^2 dt = 0,$$

which contradicts (9.4).

The remaining part of the proof is devoted to checking (9.7). To this aim we first note that using (9.4) and (8.4) it follows that

$$(9.8) \quad \sup_{j \in \mathbb{N}} \left( \int_0^\tau \|Qp_j(t)\|_{X_{-1+\alpha+\varepsilon}}^2 dt \right) < \infty.$$

Using the smoothing effect of the analytic semigroup  $\mathbb{T}^Q$  (or, more precisely, the maximal regularity property of the equation satisfied by  $\zeta_j$ , recall Theorem 5.1) it follows that the sequence  $(\zeta_j)$  is bounded in

$$L^2([0, \tau]; X_{\alpha+\varepsilon}) \cap H^1((0, \tau); X_{-1+\alpha+\varepsilon}).$$

Using next the Aubin-Lions compactness theorem (Theorem 5.2), it follows that, up to the extraction of a subsequence, we have

$$\|\zeta_j - \zeta\|_{L^2([0, \tau]; X_\alpha)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

This implies, in particular, that

$$(9.9) \quad \sup_{j \in \mathbb{N}} \left( \int_0^\tau \|\zeta_j(t)\|_{X_\alpha}^2 dt \right) < \infty. \quad (j \in \mathbb{N}),$$

On the other hand, from the identity  $\varphi_j = p_j + \zeta_j$  it follows that for every  $j \in \mathbb{N}$  and  $\delta \in (0, \tau)$  we have

$$(9.10) \quad \int_\delta^\tau \|\varphi_j(t)\|_{X_\alpha}^2 dt \leq 2 \int_\delta^\tau \|p_j(t)\|_{X_\alpha}^2 dt + 2 \int_\delta^\tau \|\zeta_j(t)\|_{X_\alpha}^2 dt.$$

Moreover, using the final state observability in any positive time of  $(A, C)$ , (9.4), and the analyticity of  $\mathbb{T}$ , it follows that

$$\sup_{j \in \mathbb{N}} \left( \int_\delta^\tau \|p_j(t)\|_{X_\alpha}^2 dt \right) < \infty.$$

Thus, by combining the last estimate with (9.9) and (9.10) we have, up to the extraction of a subsequence,

$$(9.11) \quad \varphi_j \rightarrow \varphi \quad \text{in } L^2([\delta, \tau]; X_\alpha) \text{ weakly,}$$

with  $\varphi$  satisfying

$$\dot{\varphi}(t) = (A + Q)\varphi(t) \quad (t \in (\delta, \tau)).$$

Moreover, from (9.5) it follows that

$$C\varphi(t) = 0 \quad (t \in (\delta, \tau)),$$

so that, by the conclusion of Proposition 8.5,  $\varphi = 0$  in  $(\delta, \tau)$ . Now, using the equation satisfied by  $\varphi_j$  and the fact that the semigroup  $\mathbb{T}^Q$  is analytic, the convergence (9.11) for any  $\delta > 0$  implies that for any  $\delta > 0$ ,

$$\varphi_j \rightarrow 0 \quad \text{in } L^2([\delta, \tau]; X) \text{ strongly.}$$

Accordingly,

$$(9.12) \quad \|Q\varphi_j\|_{L^2([\delta, \tau]; X)} \rightarrow 0 \quad (\delta > 0).$$

On the other hand, from (9.8) and (9.9) it follows that  $Q\varphi_j$  is bounded in  $L^2([0, \tau]; X_{-1+\alpha+\varepsilon})$ . Thus, up to the extraction of a subsequence, we have

$$Q\varphi_j \rightarrow \gamma \quad \text{in } L^2([0, \tau]; X_{-1+\alpha+\varepsilon}) \text{ weakly.}$$

The last convergence and (9.12) imply that

$$Q\varphi_j \rightarrow 0 \quad \text{in } L^2([0, \tau]; X_{-1+\alpha+\varepsilon}) \text{ weakly.}$$

This, combined with the fact, following from the equation satisfied by  $\zeta_j$  and the identity  $\varphi_j = \zeta_j + p_j$ , that

$$\dot{\zeta}_j(t) = A\zeta_j(t) + Q\varphi_j(t), \quad \zeta_j(0) = 0,$$

and the maximal regularity of the semigroup  $\mathbb{T}$  (recall Theorem 5.1) implies that  $\zeta_j$  converges to 0 weakly in  $L^2([0, \tau]; X_{\alpha+\varepsilon}) \cap H^1((0, \tau); X_{-1+\alpha+\varepsilon})$  as  $j \rightarrow \infty$ , and thus strongly in  $L^2([0, \tau]; X_\alpha)$  by the Aubin-Lions lemma (Theorem 5.2). In particular, this proves (9.7), and as mentioned at the beginning of the proof, this yields the announced contradiction.  $\square$

**Proposition 9.2.** *Suppose that  $A < 0$  has compact resolvents,  $B \in \mathcal{L}(U, X_{-\alpha})$  for  $\alpha \in [0, 1/2]$  and that  $(A, B)$  is null controllable in any time  $\tau > 0$ . Moreover, assume that  $P \in \mathcal{L}(X)$  is such that the operator  $L_\tau$ , defined in (8.12) as an operator in  $\mathcal{L}(L^2([0, \tau]; X), X)$  satisfies (8.13), and that (8.14) holds. Then  $\text{Ran } \Phi_\tau^P = \text{Ran } \Phi_\tau$  for every  $\tau > 0$ .*

*Proof.* Recalling identity (5.6), the inclusion  $\text{Ran } \Phi_\tau^P \supset \text{Ran } \Phi_\tau$  simply is the dual version of the inequality (9.1), corresponding to the choice  $C = B^*$  and  $Q = P^*$ .

Indeed, on one hand, we know that  $(A, B)$  is null controllable in any time  $\tau > 0$  so by Proposition 5.7,  $(A, B^*)$  is final state observable. Moreover the assumptions in Proposition 8.5 are all satisfied because (8.13), (8.14) hold taking  $C = B^*$ ,  $Q = P^*$ . Then one can apply Proposition 9.1.

The other inclusion  $\text{Ran } \Phi_\tau^P \subset \text{Ran } \Phi_\tau$  has already been proved in Proposition 8.6.  $\square$

Let us finally give the following result:

**Corollary 9.1.** *With the assumptions in Proposition 9.2, the pair  $(A + P, B)$  is null controllable in any time  $\tau > 0$ .*

*Proof.* Let  $\tau > 0$ .

For  $\psi \in X_{1/2}$ ,  $\sigma \mapsto \mathbb{T}_\sigma \psi \in L^2([0, \tau]; X_1)$  and assumption (8.13) implies that  $L_\tau(\sigma \mapsto \mathbb{T}_\sigma \psi)$  belongs to  $\text{Ran } \Phi_\tau$ .

Now, the identity (7.2) implies that for  $\psi \in X$ ,

$$\mathbb{T}_\tau^P \psi = \mathbb{T}_\tau \psi + L_\tau(\sigma \mapsto \mathbb{T}_\sigma \psi).$$

Therefore, for  $\psi \in X_{1/2}$ , each term in the right hand-side of this identity belongs to  $\text{Ran } \Phi_\tau$  (recall Proposition 5.4 for the first term), and  $\mathbb{T}_\tau^P \psi \in \text{Ran } \Phi_\tau$ .

Since  $\tau > 0$  is arbitrary and  $\mathbb{T}^P$  is an analytic semigroup, if  $\psi \in X$ ,  $\mathbb{T}_{\tau/2}^P \psi \in X_{1/2}$  and thus  $\mathbb{T}_{\tau}^P \psi = \mathbb{T}_{\tau/2}^P(\mathbb{T}_{\tau/2}^P \psi) \in \text{Ran } \Phi_{\tau/2} = \text{Ran } \Phi_{\tau}$ .

Using then Proposition 9.2, we easily conclude the null controllability of the pair  $(A + P, B)$ .  $\square$

*Proof of Theorem 2.4.* Theorem 2.4 directly follows from Proposition 9.2 and Corollary 9.1.  $\square$

## 10. TOWARDS NON AUTONOMOUS OR SEMI-LINEAR SYSTEMS

The results we presented in the previous sections are limited to perturbations preserving the time invariant and linear character of the original system. In the present section we show how our methods extend to some time dependent or non-linear perturbations of linear time invariant systems which are null controllable in any positive time. We first state some abstract results and then we apply them to systems described by non autonomous or semi-linear parabolic equations.

**10.1. Abstract results.** Throughout this subsection we assume that  $X$  and  $U$  are Hilbert spaces and that  $\Sigma = (\mathbb{T}, \Phi)$  is a well posed control system with state space  $X$  and input space  $U$ , in the sense of Definition 2.1.

We show, in particular, that the abstract result in Theorem 2.1 allows us to deduce some reachability results for perturbed semi-linear parabolic equations, see Corollary 10.1 below.

Before that we show that Theorem 2.1 implies, with appropriate assumptions, that the reachable space is not affected by the presence of a source term.

**Proposition 10.1.** *With the assumptions of Theorem 2.1, let  $\tau > 0$ . Then there exist a positive constant  $C > 0$  and a continuous linear map*

$$\mathcal{L} : \text{Ran } \Phi_{\tau} \times L^1([0, \tau]; \text{Ran } \Phi_{\tau}) \rightarrow L^2([0, \tau]; U)$$

such that for every  $\eta \in \text{Ran } \Phi_{\tau}$  and  $g \in L^1([0, \tau]; \text{Ran } \Phi_{\tau})$  the solution of

$$(10.1) \quad \dot{z}(t) = Az(t) + Bu(t) + g, \quad (t \in [0, \tau]), \quad z(0) = 0,$$

associated to the control  $u = \mathcal{L}(\eta, g)$ , satisfies  $z \in C^0([0, \tau]; \text{Ran } \Phi_{\tau})$ , together with

$$(10.2) \quad z(\tau) = \eta,$$

and

$$(10.3) \quad \|z\|_{C^0([0, \tau]; \text{Ran } \Phi_{\tau})} + \|u\|_{L^2([0, \tau]; U)} \leq C (\|\eta\|_{\text{Ran } \Phi_{\tau}} + \|g\|_{L^1([0, \tau]; \text{Ran } \Phi_{\tau})}).$$

*Proof.* The solution  $z$  of (10.1) is given by

$$z(\sigma) = \int_0^{\sigma} \mathbb{T}_{\sigma-s} g(s) ds + \Phi_{\sigma} u \quad (\sigma \in [0, \tau]).$$

Using Proposition 6.1, Theorem 2.1 and our assumptions on  $\eta$  and  $g$  we see that

$$\eta - \int_0^{\tau} \mathbb{T}_{\tau-s} g(s) ds \in \text{Ran } \Phi_{\tau},$$

and

$$\left\| \eta - \int_0^{\tau} \mathbb{T}_{\tau-s} g(s) ds \right\|_{\text{Ran } \Phi_{\tau}} \leq \|\eta\|_{\text{Ran } \Phi_{\tau}} + c_{\tau} \|g\|_{L^1([0, \tau]; \text{Ran } \Phi_{\tau})}.$$

Accordingly, with  $\Gamma_\tau$  the operator introduced in Remark 5.1, we can take

$$u = \Gamma_\tau \left( \eta - \int_0^\tau \mathbb{T}_{\tau-s} g(s) \, ds \right),$$

to obtain that

$$\Phi_\tau u = \eta - \int_0^\tau \mathbb{T}_{\tau-s} g(s) \, ds,$$

with the estimate

$$\|u\|_{L^2([0,\tau];U)} \leq C \left( \|\eta\|_{\text{Ran } \Phi_\tau} + \|g\|_{L^1([0,\tau];\text{Ran } \Phi_\tau)} \right),$$

for some constant  $C > 0$  (not depending on  $\eta$  and on  $g$ ).

The above estimate on the control function, together with Proposition 6.1 and Theorem 2.1, implies the estimate (10.3).  $\square$

As a corollary of Proposition 10.1 we can provide some “local” information on the reachable states in the presence of some nonlinear perturbations.

**Corollary 10.1.** *With the assumptions of Theorem 2.1, let  $\tau > 0$ . Moreover, suppose that  $f : C^0([0, \tau]; \text{Ran } \Phi_\tau) \rightarrow L^1([0, \tau]; \text{Ran } \Phi_\tau)$  is such that  $f(0) = 0$  and, for all  $z_1, z_2 \in C^0([0, \tau]; \text{Ran } \Phi_\tau)$  we have*

$$(10.4) \quad \begin{aligned} & \|f(z_1) - f(z_2)\|_{L^1([0,\tau];\text{Ran } \Phi_\tau)} \\ & \leq \|z_1 - z_2\|_{C^0([0,\tau];\text{Ran } \Phi_\tau)} \left( \varepsilon + C \| (z_1, z_2) \|_{(C^0([0,\tau];\text{Ran } \Phi_\tau))^2} \right), \end{aligned}$$

with  $C, \varepsilon$  positive constants.

Then there exists  $\varepsilon_0 > 0$  such that if (10.4) holds with  $\varepsilon < \varepsilon_0$ , then there exists  $\delta > 0$  such that for every  $\eta \in \text{Ran } \Phi_\tau$  satisfying  $\|\eta\|_{\text{Ran } \Phi_\tau} \leq \delta$ , there exist a control function  $u \in L^2([0, \tau]; U)$  and a controlled trajectory  $z \in C^0([0, \tau]; \text{Ran } \Phi_\tau)$  satisfying

$$(10.5) \quad \dot{z}(t) = Az(t) + Bu(t) + f(z)(t), \quad (t \in [0, \tau]), \quad z(0) = 0,$$

and the final condition (10.2).

When  $f$  is linear, i.e.  $f \in \mathcal{L}(C^0([0, \tau]; \text{Ran } \Phi_\tau), L^1([0, \tau]; \text{Ran } \Phi_\tau))$ , and satisfies  $\|f\|_{\mathcal{L}(C^0([0,\tau];\text{Ran } \Phi_\tau), L^1([0,\tau];\text{Ran } \Phi_\tau))} \leq \varepsilon$ , then the same conclusion holds for any  $\eta \in \text{Ran } \Phi_\tau$ .

*Proof.* The result is proved by a fixed point method. To this aim we define a map  $\Lambda : C^0([0, \tau]; \text{Ran } \Phi_\tau) \rightarrow C^0([0, \tau]; \text{Ran } \Phi_\tau)$ , which associates to every  $\hat{z} \in C^0([0, \tau]; \text{Ran } \Phi_\tau)$  the controlled state trajectory  $z = \Lambda(\hat{z})$  corresponding to the final state  $\eta$  and  $g = f(\hat{z})$ , given by Proposition 10.1.

According to Proposition 10.1 and using assumption (10.4), we obtain that for every  $\hat{z} \in C^0([0, \tau]; \text{Ran } \Phi_\tau)$  we have

$$\begin{aligned} \|\Lambda(\hat{z})\|_{C^0([0,\tau];\text{Ran } \Phi_\tau)} & \leq C \left( \|\eta\|_{\text{Ran } \Phi_\tau} + \|f(\hat{z})\|_{L^1([0,\tau];\text{Ran } \Phi_\tau)} \right) \\ & \leq C \|\eta\|_{\text{Ran } \Phi_\tau} + C\varepsilon \|\hat{z}\|_{C^0([0,\tau];\text{Ran } \Phi_\tau)} + C \|\hat{z}\|_{C^0([0,\tau];\text{Ran } \Phi_\tau)}^2. \end{aligned}$$

In particular, if  $\varepsilon > 0$  is such that  $C\varepsilon \leq 1/3$ , choosing  $R$  such that  $CR = 1/3$ , and  $\delta$  such that  $C\delta = R/3$ , we get that for every  $\eta$  satisfying  $\|\eta\|_{\text{Ran } \Phi_\tau} \leq \delta$ , for all  $\hat{z} \in C^0([0, \tau]; \text{Ran } \Phi_\tau)$  with  $\|\hat{z}\|_{C^0([0,\tau];\text{Ran } \Phi_\tau)} \leq R$ ,  $\|\Lambda(\hat{z})\|_{C^0([0,\tau];\text{Ran } \Phi_\tau)} \leq R/3 + R/3 + R/3 = R$ , i.e. the set  $B_R = \{\hat{z} \in C^0([0, \tau]; \text{Ran } \Phi_\tau) \text{ with } \|\hat{z}\|_{C^0([0,\tau];\text{Ran } \Phi_\tau)} \leq R\}$  is stable by  $\Lambda$ .

Furthermore, using assumption (10.4), for all  $\hat{z}_1, \hat{z}_2 \in B_R$ ,

$$\begin{aligned} & \|\Lambda(\hat{z}_1) - \Lambda(\hat{z}_2)\|_{C^0([0, \tau]; \text{Ran } \Phi_\tau)} \leq C \|f(\hat{z}_1) - f(\hat{z}_2)\|_{L^1([0, \tau]; \text{Ran } \Phi_\tau)} \\ & \leq \|\hat{z}_1 - \hat{z}_2\|_{C^0([0, \tau]; \text{Ran } \Phi_\tau)} \left( C\varepsilon + C \|(\hat{z}_1, \hat{z}_2)\|_{(C^0([0, \tau]; \text{Ran } \Phi_\tau))^2} \right) \\ & \leq \|\hat{z}_1 - \hat{z}_2\|_{C^0([0, \tau]; \text{Ran } \Phi_\tau)} \left( C\varepsilon + \sqrt{2}CR \right) \\ & \leq \|\hat{z}_1 - \hat{z}_2\|_{C^0([0, \tau]; \text{Ran } \Phi_\tau)} \left( \frac{1}{3} + \frac{\sqrt{2}}{3} \right). \end{aligned}$$

In particular,  $B_R$  is stable by  $\Lambda$ , and  $\Lambda$  is contractive there. Accordingly, by Banach-Picard fixed point theorem,  $\Lambda$  admits a unique fixed point in  $B_R$ , and by construction this fixed point satisfies (10.5) and (10.2) for some control function  $u \in L^2([0, \tau]; U)$ .

When  $f$  is linear it suffices to remark that for all  $\hat{z}_1, \hat{z}_2 \in C^0([0, \tau]; \text{Ran } \Phi_\tau)$ ,

$$\begin{aligned} & \|\Lambda(\hat{z}_1) - \Lambda(\hat{z}_2)\|_{C^0([0, \tau]; \text{Ran } \Phi_\tau)} \\ & \leq C \|f\|_{\mathcal{L}(C^0([0, \tau]; \text{Ran } \Phi_\tau), L^1([0, \tau]; \text{Ran } \Phi_\tau))} \|\hat{z}_1 - \hat{z}_2\|_{C^0([0, \tau]; \text{Ran } \Phi_\tau)}. \end{aligned}$$

In particular, if  $C\|f\|_{\mathcal{L}(C^0([0, \tau]; \text{Ran } \Phi_\tau), L^1([0, \tau]; \text{Ran } \Phi_\tau))} < 1$ , the map  $\Lambda$  is strictly contractive and has a unique fixed point by Banach-Picard fixed point theorem.  $\square$

**Remark 10.1.** Looking back to the above proof of Proposition 10.1, it is clear that the assumption (10.1) only needs to be valid in a neighborhood of 0 in  $C^0([0, \tau]; \text{Ran } \Phi_\tau)$ . Indeed, if the assumption (10.1) holds only for  $z \in C^0([0, \tau]; \text{Ran } \Phi_\tau)$  satisfying  $\|z\|_{C^0([0, \tau]; \text{Ran } \Phi_\tau)} \leq \rho$  for some positive constant  $\rho$ , it suffices to define the map  $\Lambda$  on  $B_\rho$ , and check that one can find  $R \leq \rho$  such that  $\Lambda$  is stable and contractive on  $B_R$ . Details are left to the reader.

A first application of Corollary 10.1 concerns the linear case and yields the following extension of Theorem 2.3 when the perturbation  $P$  is time dependent:

**Theorem 10.1.** *Assume that  $\Sigma$  is null controllable in any positive time and that  $P \in L^1([0, \tau_0]; \mathcal{L}(X)) \cap L^1([0, \tau_0]; \mathcal{L}(\text{Ran } \Phi_{\tau_0}))$  for some  $\tau_0 > 0$ .*

*Then there exists  $\delta_{\tau_0} > 0$  such that if*

$$(10.6) \quad \|P\|_{L^1([0, \tau_0]; \mathcal{L}(\text{Ran } \Phi_{\tau_0}))} \leq \delta_{\tau_0},$$

*then*

$$\text{Ran } \Phi_{\tau_0}^P = \text{Ran } \Phi_{\tau_0},$$

*where  $\text{Ran } \Phi_{\tau_0}^P$  denotes the set of all  $z(\tau_0)$  obtained by solving*

$$(10.7) \quad \dot{z}(t) = Az(t) + Bu(t) + P(t)z(t), \quad (t \in [0, \tau_0]), \quad z(0) = 0,$$

*for  $u \in L^2([0, \tau_0]; U)$ .*

*Proof.* The inclusion  $\text{Ran } \Phi_{\tau_0} \subset \text{Ran } \Phi_{\tau_0}^P$  follows for a small enough parameter  $\delta_{\tau_0}$  in (10.6) from Corollary 10.1 corresponding to  $f$  given by, for all  $z \in C^0([0, \tau_0]; \text{Ran } \Phi_\tau)$  and  $t \in [0, \tau_0]$ ,  $f(z)(t) = P(t)(z(t))$ .

The other inclusion is similar to the proof of Theorem 2.2. Indeed, for  $u \in L^2([0, \tau_0]; U)$ , we write the solution  $z$  of (10.7) as  $z = z_1 + z_2$ , where  $z_1$  solves

$$\dot{z}_1 = Az_1(t) + Bu(t), \quad (t \in [0, \tau_0]), \quad z_1(0) = 0,$$

i.e.  $z_1(t) = \Phi_t u$ , and  $z_2$  solves

$$(10.8) \quad \dot{z}_2(t) = (A + P(t))z_2 + P(t)z_1, \quad (t \in [0, \tau_0]), \quad z_2(0) = 0.$$

Using Proposition 6.2,  $z_1 \in C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})$ .

We then prove that  $z_2$  in  $C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})$  by constructing it through a convergent iteration argument. Indeed, we define  $z_{2,0} = 0$  and for  $n \in \mathbb{N}$ , we set

$$z_{2,n+1}(t) = \int_0^t \mathbb{T}_{t-s}(P(s)z_{2,n}(s) + P(s)z_1(s)) \, ds, \quad (t \in [0, \tau_0]).$$

Using then Theorem 2.1, a straightforward recurrence shows that for all  $n \in \mathbb{N}$ ,  $z_{2,n} \in C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})$ .

Using again Theorem 2.1, we also check that there is a constant  $C > 0$  such that for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} \|z_{2,n+2} - z_{2,n+1}\|_{C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})} \\ \leq C \|P\|_{L^1([0, \tau_0]; \mathcal{L}(\text{Ran } \Phi_{\tau_0}))} \|z_{2,n+1} - z_{2,n}\|_{C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})}. \end{aligned}$$

Accordingly, if  $\|P\|_{L^1([0, \tau_0]; \mathcal{L}(\text{Ran } \Phi_{\tau_0}))}$  is small enough, the sequence  $(z_{2,n})$  is a Cauchy sequence in  $C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})$ . In particular, its limit  $z_{2,\infty}$  also belongs to  $C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})$  and satisfies

$$z_{2,\infty}(t) = \int_0^t \mathbb{T}_{t-s}(P(s)z_{2,\infty}(s) + P(s)z_1(s)) \, ds, \quad (t \in [0, \tau_0]).$$

By uniqueness of the solutions of (10.8), we deduce  $z_{2,\infty} = z_2$  and thus, that  $z_2 \in C^0([0, \tau_0]; \text{Ran } \Phi_{\tau_0})$ . In particular,  $z(\tau_0) = z_1(\tau_0) + z_2(\tau_0)$  belongs to  $\text{Ran } \Phi_{\tau_0}$ , and thus  $\text{Ran } \Phi_{\tau_0}^P \subset \text{Ran } \Phi_{\tau_0}$ .  $\square$

## 10.2. Applications to non autonomous or semilinear parabolic equations.

We give below some applications of Corollary 10.1 and of Theorem 10.1 to systems described by parabolic PDEs. For the sake of simplicity, we focus on the case in which the unperturbed system is the one-dimensional heat equation controlled from both ends through Neumann boundary conditions, i.e. equation (3.1) with input maps  $\Phi_\tau^{(0,\pi),N}$  as in (3.2). However, the proposed methodology can be easily adapted to other situations in which we have detailed information on the reachable space of the unperturbed system.

We first remark that using Theorem 10.1, the results in Theorem 4.1 can be easily generalized to potentials depending on time:

**Theorem 10.2.** *Let  $D$  be the set defined in (3.4),  $\tau > 0$  and  $q \in L^1([0, \tau]; L^\infty[0, \pi])$ . Assume that for a.e.  $t \in [0, \tau]$ , the map  $x \mapsto q(t, x)$  can be extended to a function holomorphic on  $D$  with  $q \in L^1([0, \tau]; W^{1,p}(D))$  for some  $p > 2$ . Then there exists  $\varepsilon > 0$  such that if  $\|q\|_{L^1([0, \tau]; W^{1,p}(D))} \leq \varepsilon$ , then the set of all  $z(\tau, \cdot)$  obtained by solving*

$$\begin{cases} \frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) = q(t, x)z(t, x) & (t \geq 0, x \in (0, \pi)), \\ \frac{\partial z}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial z}{\partial x}(t, \pi) = u_\pi(t) & (t \geq 0), \\ z(0, x) = 0 & (x \in (0, \pi)), \end{cases}$$

for  $u_0, u_\pi \in L^2([0, \tau]; \mathbb{C})$  coincides with  $\text{Ran } \Phi_\tau^{(0,\pi),N}$  given in (3.3).

*Proof.* This is a straightforward adaptation of the proof of Theorem 4.1 by applying Theorem 10.1 instead of Theorem 2.3.  $\square$

We can also consider quadratic and non local perturbations. We give here two examples: for  $z \in C^0([0, \tau]; L^2[0, \pi])$ ,  $f_1$  and  $f_2$  are given by

$$(10.9) \quad f_1(z)(t, x) = \left( \int_0^\pi z(t, y) dy \right) z(t, x), \quad (t \in [0, \tau], x \in [0, \pi]),$$

$$(10.10) \quad f_2(z)(t, x) = \left( \int_0^t \int_0^\pi z(s, y) ds dy \right) z(t, x), \quad (t \in [0, \tau], x \in [0, \pi]).$$

As a consequence of Corollary 10.1, we obtain the following result:

**Theorem 10.3.** *Let  $\tau > 0$  and  $i \in \{1, 2\}$ . Then there exists  $\delta > 0$  such that for every  $\eta \in A^2(D)$  with  $\frac{d\eta}{ds} \in A^2(D)$  satisfying  $\|\eta\|_{W^{1,2}(D)} \leq \delta$ , there exist control functions  $u_0, u_\pi \in L^2([0, \tau]; \mathbb{C})$  such that the solution  $z$  of*

$$\begin{cases} \frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) = f_i(z)(t, x) & (t \geq 0, x \in (0, \pi)), \\ \frac{\partial z}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial z}{\partial x}(t, \pi) = u_\pi(t) & (t \geq 0), \\ z(0, x) = 0 & (x \in (0, \pi)), \end{cases}$$

with  $f_1$  and  $f_2$  defined in (10.9)-(10.10), satisfies the terminal condition  $z(\tau, \cdot) = \eta$ .

*Proof.* The proof simply consists in recalling (3.3) and checking that condition (10.4) holds with  $\varepsilon = 0$  for  $f_i$ . In both cases, this mainly consists in noticing that, for  $z \in A^2(D)$  with  $\frac{dz}{ds} \in A^2(D)$ ,  $\int_0^\pi z(y) dy$  is well-defined and satisfies

$$\left| \int_0^\pi z(y) dy \right| \leq C \|z\|_{W^{1,2}(D)},$$

as a consequence of standard trace results. Details are left to the reader.  $\square$

It would be natural to consider quadratic terms  $f(z) = z^2$  or polynomials with no constant and linear terms, but this clearly requires the reachable set to be an algebra, which is not the case of  $\text{Ran } \Phi_\tau^{(0,\pi),N}$  in (3.3). A natural strategy is thus to consider smoother controls, relying on [21, Proposition 5.1], which gives the reachable space of the system described by the 1-d heat equation with Dirichlet boundary controls when the controls are restricted to a space of smooth functions. This result can be adapted for Neumann boundary controls, yielding:

**Theorem 10.4.** *For  $\tau > 0$ , let  $H_L^1(0, \tau)$  be the set of all functions  $v \in H^1(0, \tau)$  satisfying  $v(0) = 0$ .*

*Then for every  $\tau > 0$  the range of the restriction of  $\text{Ran } \Phi_\tau^{(0,\pi),N}$  to  $H_L^1((0, \tau); \mathbb{C}^2)$  is  $A^{3,2}(D)$ , where*

$$A^{3,2}(D) = \left\{ \eta \in A^2(D), \frac{d^k \eta}{ds^k} \in A^2(D) \text{ for all } k \in \{1, 2, 3\} \right\}.$$

*Proof.* The proof of Theorem 10.4 can be obtained following line by line the proof of [21, Proposition 5.1], thus it is omitted here.  $\square$

The feature making the space  $A^{3,2}(D)$  suitable for the application we have in mind is that this space is an algebra (this follows by standard Sobolev's embedding since  $D$  is of dimension two). More precisely, we have the following result:

**Theorem 10.5.** *Let  $\tau > 0$ . Let  $f(t, x, s)$  be a function of  $t \in [0, \tau]$ ,  $x \in [0, \pi]$  and  $s \in \mathbb{R}$  such that  $f$  can be expanded as*

$$f(t, x, s) = \sum_{k=2}^{\infty} f_k(t, x) s^k, \quad (t \in [0, \tau], x \in [0, \pi], s \in \mathbb{R}),$$

with coefficients  $f_k(t, x) \in L^1([0, \tau]; A^{3,2}(D))$  satisfying for some  $\rho > 0$ ,

$$(10.11) \quad \sum_{k=2}^{\infty} k \|f_k\|_{L^1([0, \tau]; A^{3,2}(D))} \rho^k < \infty.$$

Then there exists  $\delta > 0$  such that for every  $\eta \in A^{3,2}(D)$ , satisfying  $\|\eta\|_{A^{3,2}(D)} \leq \delta$  there exist control functions  $u_0, u_\pi \in L^2([0, \tau]; \mathbb{C})$  such that the solution  $z$  of

$$(10.12) \quad \begin{cases} \frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) = f(t, x, z) & (t \geq 0, x \in (0, \pi)), \\ \frac{\partial z}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial z}{\partial x}(t, \pi) = u_\pi(t) & (t \geq 0), \\ z(0, x) = 0 & (x \in (0, \pi)), \end{cases}$$

satisfies the terminal condition  $z(\tau, \cdot) = \eta$ .

*Proof.* To fit the assumptions of Corollary 10.1, it is convenient to introduce the extended system

$$(10.13) \quad \begin{cases} \frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) = 0 & (t \geq 0, x \in (0, \pi)), \\ \frac{\partial z}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial z}{\partial x}(t, \pi) = u_\pi(t) & (t \geq 0), \\ \frac{du_0}{dt}(t) = v_0(t), \quad \frac{du_\pi}{dt}(t) = v_\pi(t), & (t \geq 0), \\ (z(0, x), u_0(0), u_\pi(0)) = (0, 0, 0) & (x \in (0, \pi)), \end{cases}$$

with control functions  $v_0$  and  $v_\pi$  in  $L^2[0, \tau]$ . Accordingly, the state space should now be considered as

$$X = \left\{ Z = \begin{bmatrix} z \\ u_0 \\ u_\pi \end{bmatrix}, z \in H^2(0, \pi), u_0, u_\pi \in \mathbb{C}, \text{ with } \frac{dz}{dx}(0) = u_0, \frac{dz}{dx}(\pi) = u_\pi \right\},$$

and the corresponding operator  $A$  is defined by

$$A = \begin{bmatrix} \frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with domain

$$\mathcal{D}(A) = \left\{ Z = \begin{bmatrix} z \\ u_0 \\ u_\pi \end{bmatrix} \in X, \text{ with } z \in H^4(0, \pi), \text{ and } \frac{d^3 z}{dx^3}(0) = \frac{d^3 z}{dx^3}(\pi) = 0 \right\}.$$

The control operator is then simply given by

$$Bv = \begin{bmatrix} 0 \\ v_0 \\ v_\pi \end{bmatrix} \quad \left( v = \begin{bmatrix} v_0 \\ v_\pi \end{bmatrix} \in \mathbb{C}^2 \right).$$



It is clear that the reachable space in time  $\tau$  of system (10.13) with controls  $v_0, v_\pi \in L^2[0, \tau]$  coincides with the reachable space in the same time of (3.1) with controls  $u_0, u_\pi \in H_L^1(0, \tau)$ . Accordingly, using Theorem 10.4, the reachable set  $\mathcal{R}$  for (10.13) with controls  $v_0, v_\pi \in L^2[0, \tau]$  is

$$\mathcal{R} = \left\{ Z = \begin{bmatrix} z \\ u_0 \\ u_\pi \end{bmatrix} \in A^{3,2}(D) \times \mathbb{C}^2 \text{ with } \frac{dz}{dx}(0) = u_0, \frac{dz}{dx}(\pi) = u_\pi \right\}.$$

System (10.12) should now be seen as a perturbation of the extended system (10.13).

The nonlinear perturbation  $F$  is given, for every  $Z = \begin{bmatrix} z \\ u_0 \\ u_\pi \end{bmatrix} \in C^0([0, \tau]; X)$  by

$$F(Z) = \begin{bmatrix} f(t, x, z) \\ \frac{\partial}{\partial x} \left( f(t, x, z(t, x)) \right) \Big|_{x=0} \\ \frac{\partial}{\partial x} \left( f(t, x, z(t, x)) \right) \Big|_{x=\pi} \end{bmatrix},$$

and we then check condition (10.4) for  $F$  for  $Z = [z, u_0, u_\pi]^{\text{tr}} \in C^0([0, \tau]; \mathcal{R})$  in a neighborhood of 0 (see Remark 10.1).

Using that  $A^{3,2}(D)$  is an algebra, there exists  $C_0 \geq 1$  such that for all  $z_1, z_2 \in A^{3,2}(D)$ ,  $\|z_1 z_2\|_{A^{3,2}(D)} \leq C_0 \|z_1\|_{A^{3,2}(D)} \|z_2\|_{A^{3,2}(D)}$ . We have that for all  $z_1, z_2$  in  $C^0([0, \tau]; A^{3,2}(D))$

$$\begin{aligned} \|f(t, x, z_1) - f(t, x, z_2)\|_{L^1([0, \tau]; A^{3,2}(D))} &\leq \sum_{k=2}^{+\infty} \|f_k(z_1^k - z_2^k)\|_{L^1([0, \tau]; A^{3,2}(D))} \\ &\leq C_0 \sum_{k=2}^{+\infty} \|f_k\|_{L^1([0, \tau]; A^{3,2}(D))} \|z_1^k - z_2^k\|_{C^0([0, \tau]; A^{3,2}(D))}. \end{aligned}$$

Then using that for all  $s_1$  and  $s_2$  in  $\mathbb{C}$ , and  $k \in \mathbb{N}$ ,

$$s_1^k - s_2^k = (s_1 - s_2) \left( \sum_{j=0}^{k-1} s_1^{k-1-j} s_2^j \right),$$

we deduce that for all  $k \in \mathbb{N}$ ,  $k \geq 2$ ,

$$\begin{aligned} \|z_1^k - z_2^k\|_{C^0([0, \tau]; A^{3,2}(D))} &\leq C_0 \|z_1 - z_2\|_{C^0([0, \tau]; A^{3,2}(D))} \sum_{j=0}^{k-1} C_0^{k-2} \|z_1\|_{C^0([0, \tau]; A^{3,2}(D))}^{k-1-j} \|z_2\|_{C^0([0, \tau]; A^{3,2}(D))}^j \\ &\leq C_0 \|z_1 - z_2\|_{C^0([0, \tau]; A^{3,2}(D))} |k| C_0^{k-1} \|(z_1, z_2)\|_{(C^0([0, \tau]; A^{3,2}(D)))}^{k-1}. \end{aligned}$$

Choosing  $\rho_0 > 0$  such that  $C_0 \rho_0 \leq \rho$  with  $\rho$  as in (10.11), we get, for all  $z_1$  and  $z_2$  in  $C^0([0, \tau]; A^{3,2}(D))$  satisfying  $\|z_1\|_{C^0([0, \tau]; A^{3,2}(D))} \leq \rho_0$  and  $\|z_2\|_{C^0([0, \tau]; A^{3,2}(D))} \leq$

$\rho_0$ ,

$$(10.14) \quad \|f(t, x, z_1) - f(t, x, z_2)\|_{L^1([0, \tau]; A^{3,2}(D))} \\ \leq C_0 \|z_1 - z_2\|_{C^0([0, \tau]; A^{3,2}(D))} \left( \sum_{k=2}^{\infty} k \|f_k\|_{L^1([0, \tau]; A^{3,2}(D))} \rho^{k-1} \right).$$

We then recall that standard trace estimates give that for  $z \in A^{3,2}(D)$ ,  $\frac{dz}{dx}(0)$  and  $\frac{dz}{dx}(\pi)$  are well-defined and satisfies

$$\left| \frac{dz}{dx}(0) \right| + \left| \frac{dz}{dx}(\pi) \right| \leq C \|z\|_{A^{3,2}(D)}, \quad (z \in A^{3,2}(D)).$$

Therefore, combining estimate (10.14) with these trace estimates, condition (10.4) holds for any  $Z_1, Z_2$  in  $C^0([0, \tau]; \mathcal{R})$  satisfying  $\|Z_1\|_{C^0([0, \tau]; \mathcal{R})} \leq \rho_0$  and  $\|Z_2\|_{C^0([0, \tau]; \mathcal{R})} \leq \rho_0$ .

Corollary 10.1 and Remark 10.1 then give the claimed result.  $\square$

Theorem 10.5 should be compared with the recent results of [22] dealing with the reachable set for non-linear heat equations. We do not recall here the main assumptions in [22] as they are quite technical, but we only underline the following elements:

- Theorem 1.1 in [22] handles nonlinearities depending analytically on  $z$  and  $\frac{\partial z}{\partial x}$ , whereas our Theorem 10.5 does not allow to deal with terms in  $\frac{\partial z}{\partial x}$ . Note that this is related to the fact that we do not know if the semigroup  $\tilde{\mathbb{T}}$  obtained by restricting the heat semigroup to the reachable space of the considered system is analytic or not, recall Remark 6.2.
- Theorem 1.1 in [22] requires coefficients which are independent of the time variable, while Theorem 10.5 allows coefficients in  $L^1$  with respect to the time variable.
- Theorem 1.1 in [22] also allows first order terms without any smallness condition on it, while, using our methodology, Theorem 10.5 could clearly be adapted only to the case of small potentials as in Theorem 10.2.
- Theorem 1.1 in [22] requires stronger analyticity conditions on the nonlinearities than the ones in Theorem 10.5.
- The set of reachable states described in [22] is much smaller than the one obtained in Theorem 10.5, since it is shown that data which admit an holomorphic extension on  $B_{\mathbb{C}}(\pi/2, R)$  for some  $R > \hat{R} = (2\pi)e^{(2e)^{-1}}$  are reachable. Theorem 10.5 only requires analyticity on the set  $D$ , which is much more natural in view of the reachable space for the linear heat equation (recall Theorem 3.1).

## APPENDIX A. UNIQUE CONTINUATION PROPERTIES FOR NON-LOCAL LINEAR PERTURBATIONS

This appendix provides sufficient conditions and illustrative examples in which the last assumption in Proposition 4.1 or assumption (b) in Propositions 4.2 and 4.3 are verified.

### A.1. Checking condition (b) in Propositions 4.2 and 4.3.

A.1.1. *The case of a distributed control.* In the case of distributed controls as in (4.4), for each of Theorems 4.2 and 4.3, condition (b) reads as follows:

$$(A.1) \quad \begin{cases} \varphi \in L^2(\Omega) \\ s \in \mathbb{C} \end{cases} \quad \text{and} \quad \begin{cases} \Delta\varphi(x) + \int_{\Omega} \overline{K(y,x)}\varphi(y) \, dy = s\varphi(x), & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega, \\ \varphi = 0, & x \in \mathcal{O}, \end{cases} \\ \Rightarrow \varphi = 0 \text{ in } \Omega.$$

As we will see in the following, this unique continuation property is rather delicate to prove, and we shall provide below several sufficient conditions guaranteeing the unique continuation property.

**Proposition A.1.** *If for all  $y \in \overline{\Omega}$ ,  $x \mapsto K(y,x)$  is analytic, then the unique continuation property (A.1) holds.*

*Proof.* Indeed, if  $\varphi$  vanishes on  $\mathcal{O}$ , from the first equation of (A.1), we should have that

$$\forall x \in \mathcal{O}, \quad \int_{\Omega} \overline{K(y,x)}\varphi(y) \, dy = 0,$$

which entails from analyticity of all functions  $K(y, \cdot)$  for  $y \in \overline{\Omega}$  that

$$\forall x \in \Omega, \quad \int_{\Omega} \overline{K(y,x)}\varphi(y) \, dy = 0.$$

So the first equation and the third equation of (A.1) read as

$$\Delta\varphi(x) = s\varphi(x), \quad x \in \Omega \text{ and } \varphi = 0 \text{ in } \mathcal{O}.$$

Classical unique continuation properties for elliptic equations (see e.g. [33, Appendix IV]) then imply that  $\varphi = 0$  in  $\Omega$ , and concludes the proof of Proposition A.1.  $\square$

**Proposition A.2.** *If there exists a finite dimensional vector space  $V$  of  $L^2(\Omega)$  such that for all  $y \in \overline{\Omega}$ ,  $x \mapsto K(y,x) \in V$ , and such that*

$$(A.2) \quad \forall \psi \in V, \quad \psi = 0 \text{ in } \mathcal{O} \Rightarrow \psi = 0 \text{ in } \Omega,$$

*then the unique continuation property (A.1) holds.*

*In particular, if the kernel function  $K$  is of the form  $K(x,y) = g(x)h(y)$  with  $g$  and  $h$  in  $L^2(\Omega)$  and  $\|h\|_{L^2(\mathcal{O})} \neq 0$ , then the unique continuation property (A.1) holds.*

**Remark A.1.** The sufficient condition for unique continuation stated in Proposition A.2 is in fact closely related to the one used in the recent work of Guerrero and Takahashi [15], see in particular conditions (1.10), (1.11) and Remark 1.2 therein.

**Remark A.2.** In fact, the assumptions of Proposition A.2 can be slightly relaxed, by assuming, instead of  $V$  being of finite dimension, that the vector space  $V$  is a closed subspace of  $L^2(\Omega)$  for which the unique continuation property (A.2) holds.

*Proof.* For  $\varphi \in L^2(\Omega)$ , the function  $\psi$  defined by  $\psi(x) = \int_{\Omega} \overline{K(y,x)}\varphi(y) \, dy$  belongs to  $V$ . Thus, if  $\varphi$  satisfies the conditions of (A.1), necessarily, the function  $\psi$  vanishes in  $\mathcal{O}$ , and thus, according to property (A.2),  $\psi$  vanishes in the whole domain  $\Omega$ .

It follows then from classical unique continuation properties for elliptic equations (see e.g. [33, Appendix IV]) that  $\varphi = 0$  in  $\Omega$ , and this concludes the proof of Proposition A.1.  $\square$

**A.2. Unique continuation from the boundary in the 1-d case.** As we shall see next, checking condition (b) in Theorem 4.1 in the case of boundary control is much more delicate than in the case of distributed control, partly because we have strictly less information in such case.

To better explain it, we focus on the 1-d case set on  $\Omega = (0, \pi)$  with Neumann boundary conditions, observation at  $x = 0$ , meaning that the unique continuation property we are interested in is the following one:

$$(A.3) \quad \begin{cases} \varphi \in L^2(0, \pi) \\ s \in \mathbb{C} \end{cases} \quad \text{and} \quad \begin{cases} \frac{d^2\varphi}{dx^2}(x) + \int_0^\pi \overline{K(y, x)}\varphi(y) dy = s\varphi(x), & x \in (0, \pi), \\ \frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(\pi) = 0, \\ \varphi(0) = 0, \end{cases} \\ \Rightarrow \varphi = 0 \text{ in } (0, \pi).$$

Moreover, to clearly formulate our results, we restrict ourselves to more specific kernels  $k$  of the form

$$(A.4) \quad K(x, y) = g(x)h(y), \text{ for some } g \in L^2(0, \pi) \text{ and } h \in L^2(0, \pi).$$

In this case we have the following result:

**Theorem A.1.** *Let  $K$  be as in (A.4) for some  $g \in L^2(0, \pi)$  and  $h \in L^2(0, \pi)$ .*

*Then the unique continuation property (A.3) is satisfied if and only if there is no  $s \in \mathbb{C}$  such that*

$$(A.5) \quad \frac{\partial \varphi_h}{\partial x}(\pi, s) = 0, \text{ and } \int_0^\pi \varphi_h(x, s) \overline{g(x)} dx = 1,$$

where  $\varphi_h$  is defined on  $(0, \pi) \times \mathbb{C}$  by the formula

$$(A.6) \quad \varphi_h(x, s) = - \int_0^x \frac{\sinh(\sqrt{s}(x-x_0))}{\sqrt{s}} \overline{h(x_0)} dx_0.$$

**Remark A.3.** Let us remark that for all  $x \in (0, \pi)$ ,  $\varphi_h(x, \cdot)$  is a holomorphic function in  $\mathbb{C}$ , since for all  $a \in \mathbb{C}$ ,  $s \mapsto \sinh(\sqrt{s}a)/\sqrt{s}$  is holomorphic in  $\mathbb{C}$ , as it can be easily seen from the expansion

$$\frac{\sinh(\sqrt{s}a)}{\sqrt{s}} = \sum_{n=0}^{\infty} \frac{s^n a^{2n+1}}{(2n+1)!}.$$

Similarly, one easily gets that  $\frac{\partial \varphi_h}{\partial x}(\pi, \cdot)$  is a holomorphic function on  $\mathbb{C}$ .

Theorem A.1 then expresses the fact that the unique continuation property (A.3) holds if and only if the two holomorphic functions  $s \mapsto \partial_x \varphi_h(\pi, s)$  and  $s \mapsto \int_0^\pi \varphi_h(x, s) \overline{g(x)} dx - 1$  do not vanish simultaneously.

*Proof.* Before going to the proof of Theorem A.1, let us do two remarks.

First, for  $K$  of the form (A.4), condition (A.3) can be reformulated as follows:

$$\begin{cases} \varphi \in L^2[0, \pi] \\ s \in \mathbb{C} \end{cases} \quad \text{and} \quad \begin{cases} \frac{d^2\varphi}{dx^2}(x) + \overline{h(x)} \int_0^\pi \overline{g(y)}\varphi(y) dy = s\varphi(x), & x \in (0, \pi), \\ \frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(\pi) = 0, \\ \varphi(0) = 0, \end{cases} \\ \Rightarrow \varphi = 0 \text{ in } (0, \pi).$$

Second,  $\varphi_h$  given by (A.6) is the only solution of the problem

$$\begin{cases} \frac{d^2\varphi_h}{dx^2}(x, s) - s\varphi_h(x, s) = -\overline{h(x)}, & x \in (0, \pi), s \in \mathbb{C}, \\ \varphi_h(0, s) = \frac{d\varphi_h}{dx}(0, s) = 0, & s \in \mathbb{C}. \end{cases}$$

It is then obvious that, if there exists  $s \in \mathbb{C}$  such that both identities in (A.5) are satisfied, then the unique continuation property (A.3) is violated for this choice of  $s$  and  $\varphi = \varphi_h(\cdot, s)$ .

We thus focus on the other implication, and assume that the unique continuation property (A.3) is violated for some  $\varphi \in L^2[0, \pi] \setminus \{0\}$  and  $s \in \mathbb{C}$ . Then  $\varphi$  satisfies

$$\begin{cases} \frac{d^2\varphi}{dx^2}(x) + \overline{h(x)} \int_0^\pi \overline{g(y)}\varphi(y) dy = s\varphi(x), & x \in (0, \pi), \\ \frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(\pi) = 0, \\ \varphi(0) = 0. \end{cases}$$

If  $\int_0^\pi \overline{g(y)}\varphi(y) dy = 0$ , the classical unique continuation property for the Laplace operator with Neumann boundary conditions would yield a contradiction. We can thus assume that  $\int_0^\pi \overline{g(y)}\varphi(y) dy \neq 0$ . We then set

$$\varphi_0(x) = \frac{\varphi(x)}{\int_0^\pi \overline{g(y)}\varphi(y) dy} \quad (x \in (0, \pi)),$$

which, by construction, satisfies

$$(A.7) \quad \int_0^\pi \overline{g(y)}\varphi_0(y) dy = 1 \quad \text{and} \quad \frac{d\varphi_0}{dx}(\pi) = 0.$$

Moreover,  $\varphi_0$  also satisfies

$$\begin{cases} \frac{d^2\varphi_0}{dx^2}(x) + \overline{h(x)} = s\varphi_0(x), & x \in (0, \pi), \\ \frac{d\varphi_0}{dx}(0) = 0, \\ \varphi_0(\pi) = 0. \end{cases}$$

which implies that  $\varphi_0 = \varphi_h(\cdot, s)$ . Conditions (A.7) are thus equivalent to (A.5).  $\square$

As an example of application of Theorem A.1, let us point out that the unique continuation property (A.3) holds for the kernel function  $K(x, y) = \cos(x) \cos(y)$ .

Indeed, for such kernel, corresponding to  $K(x, y) = g(x)h(y)$  with  $g(x) = \cos(x)$  and  $h(y) = \cos(y)$ , one can compute explicitly  $\varphi_h(x, s)$ :

$$\varphi_h(x, s) = \frac{\cosh(\sqrt{s}x) - \cos(x)}{1 + s}, \quad (x \in (0, \pi), s \in \mathbb{C}).$$

We then obtain:

$$\begin{aligned} \frac{\partial\varphi_h}{\partial x}(\pi, s) &= \frac{\sqrt{s} \sinh(\pi\sqrt{s})}{s + 1}; \\ \int_0^\pi \varphi_h(x, s) \overline{g(x)} dx - 1 &= \frac{\pi(s + 1) + 2\sqrt{s} \sinh(\pi\sqrt{s})}{2(s + 1)^2} - 1. \end{aligned}$$

Accordingly, the function  $s \mapsto \frac{\partial \varphi_h}{\partial x}(\pi, s)$  vanishes if and only if  $s = -k^2$  for some  $k \in \mathbb{N} \setminus \{1\}$ , whereas one easily checks that the right hand side of the last formula cannot vanish for such  $s$ .

### APPENDIX B. PROOF OF PROPOSITION 3.1

*Proof.* We first remark that, for any smooth bounded domain  $\Omega$ ,  $\text{Ran } \Phi_\tau^{\Omega, \Omega} = H_0^1(\Omega)$ , as this can be easily deduced from the following observability inequality: for any  $w$  solving

$$(B.1) \quad \begin{cases} \frac{\partial w}{\partial t}(t, x) - \Delta_x w(t, x) = 0, & t \geq 0, x \in \Omega, \\ w(t, x) = 0, & t \geq 0, x \in \partial\Omega, \\ w(0, x) = w_0 & x \in \Omega, \end{cases}$$

for any  $\tau > 0$ , there exists  $C_\tau$  such that

$$\|w(0)\|_{H^{-1}(\Omega)} \leq C \|w\|_{L^2([0, \tau]; L^2(\Omega))}.$$

This observability estimate comes immediately by expressing the solutions of (B.1) on the basis of eigenfunctions of the Laplace operator and using their orthogonality properties.

For obtaining the reachability result  $\text{Ran } \Phi_\tau^{\Omega, \Omega} = H_0^1(\Omega)$  from there, one can see for instance [6, Proof of Theorem 2.42].

Accordingly, for  $f \in H_0^1(B(R_1))$ , there exists a control function  $u_1$  such that the solution  $w_1$  of

$$(B.2) \quad \begin{cases} \frac{\partial w_1}{\partial t}(t, x) - \Delta_x w_1(t, x) = u_1, & t \geq 0, x \in B(R_1), \\ w_1(t, x) = 0, & t \geq 0, x \in \partial B(R_1), \\ w_1(0, x) = 0 & x \in B(R_1), \end{cases}$$

satisfies  $w_1(\tau, \cdot) = f$  in  $B(R_1)$ .

Let  $r \in (R_0, R_1)$ , and  $f \in H_0^1(B(R_1))$  be such that  $f|_{B(r)}$  can be extended as a function of  $A^2(\mathcal{E}(B(r)))$ , and choose  $u_1$  and  $w_1$  as above.

We then take  $r_0 \in (R_0, r)$ . Then  $f|_{B(r_0)}$  is holomorphic on  $\mathcal{E}(B(r))$ . Thus, by Proposition 3.2, and in particular, there exists  $u_0 \in L^2([0, \tau]; L^2(\partial B(r_0)))$  such that the solution  $w_0$  of

$$\begin{cases} \frac{\partial w_0}{\partial t}(t, x) - \Delta_x w_0(t, x) = 0, & t \geq 0, x \in B(r_0), \\ w_0(t, x) = u_0, & t \geq 0, x \in \partial B(r_0), \\ w_0(0, x) = 0 & x \in B(r_0), \end{cases}$$

satisfies  $w_0(T, \cdot) = f$  in  $B(r_0)$ .

Accordingly, taking a cut-off function  $\zeta$  equal to one on  $B(R_0)$  and vanishing in a neighborhood of  $\overline{B(R_1)} \setminus \overline{B(r_0)}$ , we have that  $\tilde{w}_0 = \zeta w_0$  satisfies

$$\begin{cases} \frac{\partial \tilde{w}_0}{\partial t}(t, x) - \Delta_x \tilde{w}_0(t, x) = \tilde{u}_0, & t \geq 0, x \in B(R_1), \\ \tilde{w}_0(t, x) = 0, & t \geq 0, x \in \partial B(R_1), \\ \tilde{w}_0(0, x) = 0 & x \in B(R_1), \end{cases}$$

satisfies  $\tilde{w}_0(\tau, \cdot) = \zeta f$  in  $B(R_1)$ , with  $\tilde{u}_0 = [\Delta, \zeta]w_0 = -2\nabla w_0 \cdot \nabla \zeta - w_0 \Delta \zeta$ , which is localized in  $B(R_1) \setminus B(R_0)$ , and belongs to  $L^2([0, \tau]; L^2(B(R_1) \setminus B(R_0)))$  due to the regularizing effects of the heat equation.

We then set  $\tilde{w}_1 = (1 - \zeta)w_1$ , where  $w_1$  is given by (B.2). Easy computations show that  $\tilde{w}_1$  satisfies

$$\begin{cases} \frac{\partial \tilde{w}_1}{\partial t}(t, x) - \Delta_x \tilde{w}_1(t, x) = \tilde{u}_1, & t \geq 0, x \in B(R_1), \\ \tilde{w}_1(t, x) = 0, & t \geq 0, x \in \partial B(R_1), \\ \tilde{w}_1(0, x) = 0 & x \in B(R_1), \end{cases}$$

with  $\tilde{u}_1 = (1 - \zeta)u_1 - [\zeta, \Delta]w_1$  localized in  $B(R_1) \setminus B(R_0)$  and belongs to the space  $L^2([0, \tau]; L^2(B(R_1) \setminus B(R_0)))$ , while  $\tilde{w}_1(\tau, \cdot) = (1 - \zeta)f$  in  $B(R_1)$ .

Accordingly,  $f = \zeta f + (1 - \zeta)f = \tilde{w}_0(\tau, \cdot) + \tilde{w}_1(\tau, \cdot) \in \text{Ran } \Phi_\tau^{B(R_1), B(R_1) \setminus B(R_0)}$ , corresponding to the control function  $u = \tilde{u}_0 + \tilde{u}_1$ .  $\square$

#### REFERENCES

- [1] V. Barbu, *Controllability and stabilization of parabolic equations*, vol. 90 of Progress in Non-linear Differential Equations and their Applications, Birkhäuser/Springer, Cham, 2018, URL <https://doi-org.docelec.u-bordeaux.fr/10.1007/978-3-319-76666-9>, Subseries in Control.
- [2] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Control. and Optim.*, **30** (1992), 1024–1065.
- [3] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems*, 2nd edition, Systems & Control: Foundations & Applications, Birkhäuser, Boston, MA, 2007.
- [4] M. Chen and L. Rosier, Reachable states for the distributed control of the heat equation, *Preprint*: <https://hal.archives-ouvertes.fr/hal-03259878>.
- [5] N. Cîndea and M. Tucsnak, Internal exact observability of a perturbed Euler-Bernoulli equation, *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, **2** (2010), 205–221.
- [6] J.-M. Coron, *Control and Nonlinearity*, vol. 136 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007.
- [7] J. Darde and S. Ervedoza, On the reachable set for the one-dimensional heat equation, *SIAM Journal on Control and Optimization*, **56** (2018), 1692–1715.
- [8] M. Duprez and G. Olive, Compact perturbations of controlled systems, *Math. Control Relat. Fields*, **8** (2018), 397–410, URL <https://doi-org.docelec.u-bordeaux.fr/10.3934/mcrf.2018016>.
- [9] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [10] H. O. Fattorini and D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, *Arch. Rational Mech. Anal.*, **43** (1971), 272–292.
- [11] H. Fattorini, Reachable states in boundary control of the heat equation are independent of time, *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, **81** (1978), 71–77.
- [12] E. Fernández-Cara, Q. Lü and E. Zuazua, Null controllability of linear heat and wave equations with nonlocal spatial terms, *SIAM J. Control Optim.*, **54** (2016), 2009–2019, URL <https://doi.org/10.1137/15M1044291>.
- [13] E. Fernández-Cara and E. Zuazua, The cost of approximate controllability for heat equations: the linear case, *Adv. Differential Equations*, **5** (2000), 465–514.
- [14] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of Evolution Equations*, vol. 34 of Lecture Notes Series, Seoul National University Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

- [15] S. Guerrero and T. Takahashi, Controllability to trajectories of a Ladyzhenskaya model for a viscous incompressible fluid, 2021, URL <https://hal.archives-ouvertes.fr/hal-03140709>, Working paper or preprint.
- [16] S. Hadd, Exact controllability of infinite dimensional systems persists under small perturbations, *Journal of Evolution Equations*, **5** (2005), 545–555.
- [17] A. Hartmann, K. Kellay and M. Tucsnak, From the reachable space of the heat equation to Hilbert spaces of holomorphic functions, *J. Eur. Math. Soc. (JEMS)*, **22** (2020), 3417–3440, URL <https://doi.org/10.4171/jems/989>.
- [18] A. Hartmann and M.-A. Orsoni, Reachable space of the Hermite heat equation with boundary control, 2021, URL <https://hal.archives-ouvertes.fr/hal-03330217>, Working paper or preprint.
- [19] A. Hartmann and M.-A. Orsoni, Separation of singularities for the Bergman space and application to control theory, *Journal de Mathématiques Pures et Appliquées*, URL <https://hal.archives-ouvertes.fr/hal-03194639>.
- [20] S. C. Hille, Continuity of the restriction of  $C_0$ -semigroups to invariant Banach subspaces, *Integral Equations and Operator Theory*, **53** (2005), 597–601.
- [21] K. Kellay, T. Normand and M. Tucsnak, Sharp reachability results for the heat equation in one space dimension, *hal preprint: hal-02302165*, to appear in *Anal. PDE*.
- [22] C. Laurent and L. Rosier, Exact controllability of nonlinear heat equations in spaces of analytic functions, *arXiv preprint arXiv:1812.06637*.
- [23] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur, *Comm. Partial Differential Equations*, **20** (1995), 335–356.
- [24] P. Martin, L. Rosier and P. Rouchon, On the reachable states for the boundary control of the heat equation, *Appl. Math. Res. Express. AMRX*, 181–216.
- [25] M.-A. Orsoni, Reachable states and holomorphic function spaces for the 1-d heat equation, *Journal of Functional Analysis*, **280** (2021), 108852.
- [26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.
- [27] R. Rebarber and G. Weiss, Necessary conditions for exact controllability with a finite-dimensional input space, *Systems Control Lett.*, **40** (2000), 217–227, URL [https://doi.org/10.1016/S0167-6911\(00\)00029-3](https://doi.org/10.1016/S0167-6911(00)00029-3).
- [28] S. Saitoh and Y. Sawano, *Theory of reproducing kernels and applications*, Springer, 2016.
- [29] T. I. Seidman, Time-invariance of the reachable set for linear control problems, *J. Math. Anal. Appl.*, **72** (1979), 17–20.
- [30] J. Simon, Compact sets in the space  $L^p(0, T; B)$ , *Ann. Mat. Pura Appl. (4)*, **146** (1987), 65–96.
- [31] A. Strohmaier and A. Waters, Analytic properties of heat equation solutions and reachable sets, *arXiv preprint arXiv:2006.05762*.
- [32] G. Tenenbaum and M. Tucsnak, New blow-up rates for fast controls of schrödinger and heat equations, *Journal of Differential Equations*, **243** (2007), 70–100.
- [33] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009.
- [34] M. Tucsnak and G. Weiss, Well-posed systems—the LTI case and beyond, *Automatica*, **50** (2014), 1757–1779.
- [35] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control Optim.*, **27** (1989), 527–545.
- [36] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.*, **65** (1989), 17–43.

UNIV. BORDEAUX, CNRS, BORDEAUX INP, IMB, UMR 5251, F-33400 TALENCE, FRANCE  
 Email address: [sylvain.ervedoza@math.u-bordeaux.fr](mailto:sylvain.ervedoza@math.u-bordeaux.fr)

UNIV. BORDEAUX, CNRS, BORDEAUX INP, IMB, UMR 5251, F-33400 TALENCE, FRANCE  
 Email address: [kevin.le-balch@math.u-bordeaux.fr](mailto:kevin.le-balch@math.u-bordeaux.fr)

UNIV. BORDEAUX, CNRS, BORDEAUX INP, IMB, UMR 5251, F-33400 TALENCE, FRANCE  
 Email address: [marius.tucsnak@u-bordeaux.fr](mailto:marius.tucsnak@u-bordeaux.fr)