A Simplicial approach to the Sheaf Theoretic construction of Intersection Cohomology

Sebastian Andrés Cea Garcia

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Abstract

Intersection (co)homology is a way to enhance classical (co)homology, allowing us to use a famous result called Poincaré duality on a large class of spaces known as stratified pseudomanifolds. There is a theoretically powerful way to arrive at intersection (co)homology by a classifying sheaves that satisfy what are called the Deligne axioms.

There is a successful way to construct a simplicial intersection (co)homology, exposed in the works of D. Chataur, D. Tanré and M. Saralegi-Araguren, but a simplicial manifestation of the Deligne axioms has remained under shadows until now.

This paper draws on constructions made by these authors, showing a simplicial manifestation of the Deligne axioms. This consists on presenting categories of "simplicial sheaves", localizing them appropriately and then stating "simplicial Deligne axioms". All this for different simplicial structures one can encounter. We finalize by presenting sheaves that satisfy the axioms on simplicial complexes.

Contents

1 Introduction

Today's thesis places us on a collective journey as old as human thought itself, as geometric *figures* had been objects of wonder and study for millennia. On this journey, mathematics have provided a precise and extremely effective (not to mention beautiful) language to do this. Nowadays, we have come far beyond what our ancestors could have dreamed of, reaching out to infinite dimensions and to "shapes" that can only abstractly be called shapes.

One of the main objects of study in geometry are those which extend the idea of smooth curves and surfaces, the *locally n-euclidean* spaces, better known as **manifolds**. Their pleasant *local* $\mathbb{R}^n$ structure provide home for differential forms, tangle bundles and other $\mathbb{R}^n$-type theories, reaching out into a full variety of applications all over mathematics and fields beyond. Their presence is ubiquitous in many areas of mathematics. It should not be a surprise then that one mayor program on mathematics is to

Classify manifolds up to homeomorphism.

Classifying topological spaces is not an easy endeavor. Given the complexity of information that they hold, a "direct approach" can easily leave a human in agony. One efficient way to do it instead is paying attention to certain characteristics of the spaces, constructing what we call *invariants*. These are assignments into simple objects such as numbers or better yet modules over a ring R, that do not depend on the homeomorphism class. Perhaps the most celebrated invariant is *(co)homology*.

$$H_k, H^k : \mathbf{Top} \to R\text{-mod}$$

Which is defined by using certain shapes *inside* the space of study: cycles (that is, closed curves) in dimension 1, and *k-cycles* in dimension k.

In the paper that founded modern algebraic topology, [Poi00], Poincaré studied the intersection of cycles of complementary dimension on manifolds. He found that in a n-manifold M, if we consider cycles of dimensions i, j such that $i+j = n$, we obtain that their intersection in general position satisfies properties that in modern language is stated as to define a product

$$H_i(M) \times H_j(M) \xrightarrow{-\cap-} H_0(M) \xrightarrow{\in} \mathbb{Z}$$

Which is a non-degenerate bilinear form when tensored with $\mathbb{Q}$. This result is called *Poincaré duality*, and it is one of the most powerful results on the study of manifolds.

Now, the category of manifolds is beautiful but very fragile, as very simple and natural constructions can bring us out of it, for example

- Consider a torus that has a parametric equation given as

  $$T = \{(x, y, z) \in \mathbb{R}^3 | (x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2(x^2 + y^2)\}$$

  And consider the projection

  $$\begin{aligned} h : T &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto z \end{aligned}$$

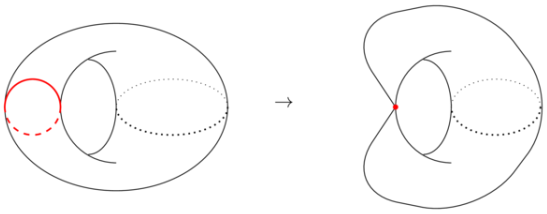
  At two points $z_0, z_1 \in \mathbb{R}$, we have that $h^{-1}(z_i)$ defines an eight-loop

We have that $h^{-1}(z_i)$ can not be a manifold because of the point in the middle, which is referred to as a **singularity**. Observe that this *almost manifold* figure is obtained by performing a pullback,

$$\begin{array}{ccc} h^{-1}(z_i) & \longrightarrow & T \\ \downarrow & \ulcorner & \downarrow{\scriptstyle h} \\ \{z_i\} & \hookrightarrow & \mathbb{R} \end{array}$$

Which is a very basic type of limit.

- Many times we want to perform the quotient of a manifold M by a submanifold $N \subset M$, contracting N into a point. By doing this we obtain a space M/N which is generally not a manifold, creating a singularity where N used to be. For example, if we quotient the torus by a circle as the following image shows, we will get a *pinched torus*.



This operation is obtained by performing a pushout,

$$\begin{array}{ccc} N & \hookrightarrow & M \\ \downarrow & \ulcorner & \downarrow \\ \{\star\} & \longrightarrow & M/N \end{array}$$

Which is a very basic type of colimit.

The constructions we obtain by performing this kind of (co)limits on manifolds can be vaguely described as *manifolds with singularities*.

If constructions as elemental as these take us out of the category, it should not be a surprise to know that the study of manifolds alone is not enough to classify manifolds. In fact, in the same way that polynomials with real coefficients cannot all have their roots in $\mathbb{R}$, there are invariants of manifolds whose study depends on the study of a larger classes of spaces: *manifolds with singularities*. It was D. Sullivan who remarked the importance of extending invariants from manifolds into these manifolds with singularities. One valuable invariant that he explicitly referred to is the *signature*. As it was posed in [Sul70], it will be necessary

To find a class of spaces with singularities for which the signature of manifolds extends as a cobordism invariant.

Since the signature problem is in a logical way contained by Poincaré duality, this problem sets up the scene for a more ambitious question.

Is there an invariant generalizing homology for which Poincaré duality holds on a class of spaces with singularities?

In the most famous paper of the subject, [GM80], M. Goresky and R. MacPherson successfully solved this question. As "spaces with singularities" they consider a particular class of stratified spaces, that is, with a chain of closed subspaces

$$X_0 \subset \cdots \subset X_n = X$$

With the idea that singularities are contained in X_{n-1}. These spaces are *almost manifolds* in the sense that $X - X_{n-1}$ is a n-manifold, dense in X. Furthermore each $X_k - X_{k-1}$ is a k-manifold. They also consider that these spaces should have a locally cone-like structure, meaning that each $x \in X_{n-k} - X_{n-k-1}$ has a neighborhood of the form $cL \times \mathbb{R}^{n-k}$ in a way compatible with the stratification. These spaces are called *stratified pseudomanifolds*.

Here, they consider functions $\bar{p} : \mathbb{N}_{\geq 2} \to \mathbb{Z}$ with $\bar{p}(2) = 0$ and $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k)+1$, called *GM perversities*, with which one discriminates k-cycles according to the dimension of intersection with the different strata X_j. The result is an invariant called **Intersection (co)homology**, typically denoted as $IH^{\bar{p}}_{\star}(X)$, that satisfies the following.

- It extends the notion of (co)homology in the sense that if X is a compact oriented manifold all the intersection (co)homology groups coincide with ordinary (co)homology. Furthermore, if X is a stratified pseudomanifold that is a normal space (recall that this is a space in which every two disjoint closed sets have disjoint open neighborhoods, such as all metric spaces and paracompact Hausdorff spaces. Every stratified pseudomanifold has a normalization).
  - For the top perversity $\bar{t} = (0, 1, 2, \dots)$ the intersection homology $IH^{\bar{t}}_i(X)$ is the homology $H_i(X)$.
  - For the zero perversity $\bar{0} = (0, 0, 0, \dots)$, the intersection homology $IH^{\bar{0}}_i(X)$ is the cohomology $H^{n-i}(X)$.
- It is natural with respect to *stratified maps* (rather than continuous functions), which are continuous maps compatible with respect to the stratification. Then, rather than being invariant under homotopy like homology is, intersection (co)homology is invariant under *stratified homotopy*.
- It satisfies Mayer-Vietoris and the excision theorem.
- Most importantly, it satisfies Poincaré duality. If $\bar{p}$ and $\bar{q}$ are GM perversities with $\bar{p} + \bar{q} = \bar{t}$ then their intersection in general position defines a map

  $$IH^{\bar{p}}_i(M) \times IH^{\bar{q}}_j(M) \xrightarrow{-\cap-} H^{\bar{t}}_0(M) \xrightarrow{\in} \mathbb{Z}$$

  That is non-degenerate when tensored with $\mathbb{Q}$.

This was a huge success, because it not only solved the signature problem proposed by Sullivam, it not only extended Poincaré duality to a class of spaces that considers singularities, but it gave rise to a powerful invariant that can be used outside of the field that saw its birth. Two renowned examples of this are the following.

- All algebraic irreducible complex varieties can be given the structure of stratified pseudomanifolds. This is explained in details in ([KW06] 4.10).
- Intersection cohomology has been used to work on the Kazhdan–Lusztig conjecture (see for example [Bre02]). In ([KL80] 4.3) for example, D. Kazhdan and G. Lusztig prove that their polynomials have coefficients depending on the dimension of certain intersection cohomology groups.

All this makes intersection homology have an important role to play in the game of geometry. Now, it might be interesting to know that, just as it happens with homology, *there are multiple ways to construct intersection cohomology*, for example:

- The most classical way is what Goresky and MacPherson presented in [GM80], which considers *simplicial* chains on a *PL* stratified pseudomanifold X. (We explain this on section 2.1).
- In [Kin85], King gives a *singular* (hence non-simplicial) construction of intersection chains whose homology is intersection homology. (We explain this on section 2.1).
- In [Pol05], G. Pollini gives a construction of intersection homology using *differential forms*. The idea is that in certain good conditions one can follow the behaviour of a differentials from smooth strata to singular points via a *tubular neighborhoods* T_S and *retractions* π_S. One then discriminates i-differential forms according to the trivialization of i-infinitesimals on elements of T_S with a certain number of "infinitesimal directions" along fibers of π_S.
- In [CSAT20] , D. Chataur, M. Saralegui-Aranguren and D. Tanré construct intersection cohomology with a chain inspired in the *blown up* of (filtered) simplices, with a shape given in simplices as $N^\star_\sigma = C^\star(c\Delta_0) \otimes \cdots \otimes C^\star(c\Delta_{n-1}) \otimes C^\star(\Delta_n)$, where $C^\star$ is the classical cochain functor, and then naturally extended to the space studied. One then discriminates elements according to its behaviour in the $C^\star(c\Delta_k)$'s (very roughly speaking).

There are in fact plenty of models for intersection homology! This messy picture raises an important question

How to characterize intersection (co)homology axiomatically?

Here, the famous P. Deligne had an important role to play. Legend tells (in [Kle07]) that on the night of Halloween of 1976, not far from Paris, MacPherson was talking with Deligne about advances in mathematics. As MacPherson described intersection homology, an inspired intuition descended into Deligne's mind. Could it have been the spirit of the night? The goddess of mathematics

perhaps? Or just a strike of genius from a beautiful mind? We do not know. What is certain is that he wrote at that moment on flimsy paper (long gone by now) the formula that would change the history of intersection homology.

$$IH_i^{\bar{p}}(X) = \mathbb{H}^{2d-i}(\mathbf{IC}_{\bar{p}}^{\star}(X))$$

Meaning that intersection homology is the hypercohomology of the complex of sheaves with formula $\mathbf{IC}_{\bar{p}}^{\star}(X) = \tau_{\leq \bar{p}(2d)} R(\iota_{2d})_{\star} \dots \tau_{\leq \bar{p}(2)} R(\iota_2)_{\star} \mathbb{C}_{X - X_{2d-2}}$.

Generaly speaking, sheaf theory holds a key in the study of manifolds, since the *gluing conditions* that they have to satisfy traduce local information into global information. And in this case, the *derived category of sheaf complexes* held the key to characterize intersection homology.

After working for some time on these intuitions, Goresky and MacPherson presented in a second paper, [GM83], a list of axioms, called the **Deligne axioms**, which read in the original paper literally as follows.

Let $S^{\cdot}$ be a complex of sheaves on X, which is constructible with respect to the stratification $\{X_k\}$ and let $S_k^{\cdot} = S^{\cdot}|_{X_k - X_{n-k}}$. We shall say $S^{\cdot}$ satisfies the axioms [AX1] (with perversity $\bar{p}$ with respect to the stratification) provided

(a) Normalization: $S^{\cdot}|_{X-\Sigma} \simeq F[n]$, where F is a local system on the regular strata $X - \Sigma$.

(b) Lower bound: $H^i(S^{\cdot}) = 0$ for all $i < -n$.

(c) Vanishing condition: $H^m(S_{k+1}^{\cdot}) = 0$ for all $m > p(k) - n$.

(d) Attaching: $S^{\cdot}$ is $p(k) - n$ attached across each stratum of codimension k, i.e., the attaching maps

$$H^m(j_k^{\star} S_{k+1}^{\cdot}) \to H^m(j_k^{\star} R(\iota_k)_{\star} \iota_k^{\star} S_{k+1}^{\cdot})$$

are isomorphisms for all $k \geq 2$ and all $m \leq \bar{p}(k) - n$.

They prove that $I^p C^{\cdot}$, the chain complex whose homology is intersection homology, naturally defines a complex of sheaves that satisfies these axioms for the constant sheaf $F = \mathbb{R}_{X - X_{n-2}}$. Furthermore, they prove that

$$S^{\cdot} \text{ satisfies [AX 1]} \iff S^{\cdot} \simeq \tau_{\leq \bar{p}(n)-n} R(\iota_n)_{\star} \dots \tau_{\leq \bar{p}(2)-n} R(\iota_2)_{\star} F[n]$$

Where the previous isomorphism considered at the level of the derived category. This result is very important as **it characterizes up to quasi isomorphism all the complexes of sheaves whose hyperhomology is intersection homology**, furthermore giving a quasi isomorphism to the so called *Deligne sheaf*, which is given by the formula $\tau_{\leq \bar{p}(n)-n} R(\iota_n)_{\star} \dots \tau_{\leq \bar{p}(2)-n} R(\iota_2)_{\star} \mathbb{R}_{X - X_{n-2}}[n]$.

The Deligne axioms are an important cornerstone for developments that enrich the intersection homology theory. Some examples of why we want them are the following.

- It provides an easy to check list to see if one's complex of sheaves computes intersection homology.
- Furthermore, it gives a *map* between different models (sheaves) for intersection homology and the intersection sheaf.

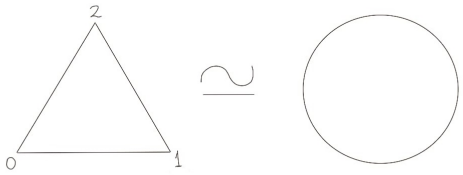
- Using sheaf theory provides an alternative way (sometimes a much easier way) to prove many properties. For example it was first proven in [GM83] using sheaves that intersection homology is a topological invariant (it does not depend on the stratification) before it was proven geometrically much later by H King in [Kin85].
- The use of sheaf theory also provides a way to study properties of intersection homology from a local point of view.
- The shape of the intersection sheaf $\tau_{\leq \bar{p}(n)-n} R(\iota_n)_\star \dots \tau_{\leq \bar{p}(2)-n} R(\iota_2)_\star \mathbb{R}_{X-\Sigma}[n]$ makes one think that we can extend information from the non-singular strata. This intuition is in fact correct.

For the reader who wants to follow this sheaf-theoric line of thought, on further developments and applications to algebraic geometry and other fields we recomend to start with [KW06].

The work of this thesis consists on painting the picture of Deligne axioms in the simplicial environment. To explain this further we need to turn the page and talk a bit about simplicial constructions.

As said before, topological spaces in general contain a lot of complex information and for this reason studying them directly can result very complicated. Even when invariants are introduced, those invariants can still contain too much complexity as to make calculations with them, for example homology works the infinite amount of continuous functions $\sigma : |\Delta^n| \to X$ in different dimensions, and isn't this more complicated than working in X itself? In most of the working cases the answer is **no**, because we do not need to consider *all* continuous functions $\sigma : |\Delta^n| \to X$ to calculate homology, but only the ones that are *representative*.

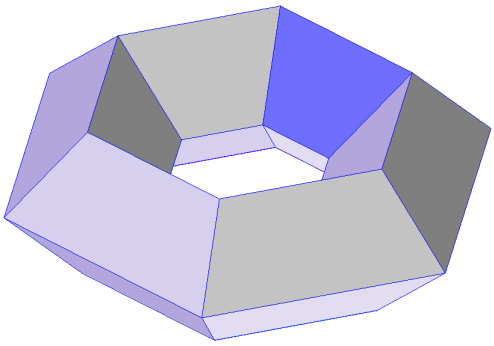
Homology works in such a way that a circle can be treated as a triangle



In such a way that, for example, instead of considering all the infinite continuous functions $\sigma : |\Delta^1| \to S^1$ we just need to work with the set of sides of a triangle $\{(0,1),(0,2),(1,2)\}$.

We suggest that the reader takes a minute to contemplate the magnitude of this.

We say that the circle was *triangulated* by a triangle. More generally, we can consider n-dimensional polyhedra which are called *simplicial complexes* in order to represent topological spaces. We say that a *triangulation* of X is a simplicial complex K together with an homeomorphism $|K| \xrightarrow{\simeq} X$, for example the triangulation of a torus might look as follows.



Although not all topological spaces can be triangulated, many important classes of spaces that are relevant in geometry can, such as smooth manifolds and low dimensional ones (see [Man14] for a quick survey on the subject).

Triangulation becomes specially beneficial in the case of compact spaces, where if a triangulation exists, it can be chosen to contain a finite amount of

simplices. Like in the case of S^1, this implies that we can obtain homology and other invariants by performing (or much better yet, by letting your computer perform) a finite calculation. We see that in the compact case the advantage of having a simplicial model is clear.

The notation $|-|$ is introduced following the fact that *simplicial complexes do not need to be defined as topological spaces*, and they in fact shine brighter when they are not. The brilliance of these structures is that they encompass information in a combinatorial way, that is, all the information needed to topologically reproduce a simplicial complex (up to homeomorphism) is how many simplices (n-dimensional triangles) one has and which simplices are face of each other. So a simplicial complex might be defined as follows.

Definition. *A **(abstract) simplicial complex** K is the information of a set (of vertices) $V(K)$ and a set (of simplices) $S(K) \subset \mathcal{P}(V(K)) - \emptyset$ such that*

1. *$\{v\} \in S(K)$ for all $v \in V(K)$*
2. *If $\sigma \in S(K)$ and $\tau \subset \sigma$ then $\tau \in S(K)$*

So a simplicial complex K is not a topological space, but we can construct one decoding the information, and this is called the *realization* $|K|$. The main *trick* that we do is to construct (co)homology on this category

$$H_k, H^k : \mathbf{Scx} \to R\text{-mod}$$

In a purely combinatorial way, to then prove that this way of doing it is equivalent to the topological counterpart, meaning that if $|K| \xrightarrow{\simeq} X$ is a triangulation, then

$$H_k(K) \simeq H_k(|K|) \simeq H_k(X)$$

We mentioned simplicial complexes as it is easier to illustrate our point, but in fact there are more simplicial constructions. Perhaps the most celebrated of these are simplicial sets, which is the presheaf category over finite ordered sets with increasing functions.

$$\mathbf{SSet} = \mathtt{Func}(\Delta^{op}, \mathbf{Set})$$

Simplicial sets paint the same picture with respect to (co)homology than simplicial complexes do. Having this success regarding homology an ambitious question arises.

How far can we get with simplicial constructions representing topological spaces?

It turns out that we can get very far. We are not going to review the history of simplicial sets here, but for example, in [Qui67], D. Quillen gave what is called a *Quillen equivalence*.

$$|-| : \mathbf{SSet} \leftrightarrows \mathbf{Top} : Sing$$

Meaning essentially that we can work out the homotopical theory of topological spaces using simplicial sets.

So coming back now to intersection (co)homology, the following question naturally arises

Is there a simplicial model for intersection (co)homology?

This question has been successfully answer in the works of D Chataur, M Saralegui-Aranguren and D Tanré (see for example [CSAT21]). Basically the way to do this is filtering a simplicial set with a map $X \to \Delta^n$ and then controlling (with a GM perversity $\bar{p}$) its simplices, by considering that the composition $\Delta^m \xrightarrow{\sigma} X \to \Delta^n$ induces a filtration on σ which is in correspondence with the intersection with the strata. By discriminating simplices by the dimensions of their filtration, they construct chain complex IC^p_Δ whose homology is called *simplicial intersection homology* which satisfies the same property as in homology

$$IH^{\bar{p}}_k(K) \simeq IH^{\bar{p}}_k(|K|) \simeq IH^{\bar{p}}_k(X)$$

For a filtered simplicial set representing of a stratified pseudomanifold $|K| \to X$. Now, it might be interesting to know that, just as it happens with homology, *there are multiple ways to construct* ***simplicial*** *intersection cohomology.*

- Intersection cochain complex IC^p_Δ, constructed as explained by discriminating simplices $\Delta^m \xrightarrow{\sigma} X \to \Delta^n$ by the dimension of their filtration.
- We can construct the *simplicial* blown up cochains in the same way that was explained before, with a shape given in simplices as $N^\star_\sigma = C^\star_\Delta(c\Delta_0) \otimes \cdots \otimes C^\star_\Delta(c\Delta_{n-1}) \otimes C^\star_\Delta(\Delta_n)$, where $C^\star$ is the classical simplicial cochain functor, and then naturally extended to the space studied. One then discriminates elements according to its behaviour in the $C^\star(c\Delta_k)$'s (very roughly speaking). This is exposed in [CSAT18b].
- One can generalize the previous construction and consider $N^\star_\sigma = F(c\Delta_0) \otimes \cdots \otimes F(c\Delta_{n-1}) \otimes F(\Delta_n)$ for any *universal system* $F : \Delta^{op} \to Ch(\mathbb{Q})$. In [CSAT18a], D. Chataur, M. Saralegui-Aranguren and D. Tanré prove that when taken $F = A_{PL}$, the cochains of *Sullivan polynomials*, we get simplicial intersection cohomology.

As we start to see that there are multiple models for simplicial intersection homology we might wonder, isn't this the same picture that we had in the topological setting? That is, we can ask

How to characterize *simplicial* intersection (co)homology axiomatically?

And this is the question that motivates this thesis. We have decided to follow on Deligne's intuition and Goresky and Macpherson's line of works, and state *simplicial Deligne axioms*, which would classify all *simplicial chains of sheaves* quasi isomorphic to a *simplicial Deligne sheaf* and compute simplicial intersection cohomology. To do this, the plan goes as follows.

1. Define a category of simplicial sheaves for the different simplicial structures one can encounter.
2. Localize this category with respect to *quasi isomorphisms* of sheaves (to be defined). This can be done either deriving an abelian category or by taking the homotopy category of a model category.

3. Stating a simplicial version of the Deligne axioms.

4. Finding acyclic sheaves that compute intersection homology and satisfy the axioms.

We make a summary of the constructions and results found.

1. Simplicial sheaves: We have found in fact two compatible candidates for the category of simplicial sheaves. If we consider $X \in \mathbf{SSet}$, we might take its category of simplices $\Delta \downarrow X$ and its category of subobjects $\mathtt{Sub}(X)$ and for an abelian or model category $\mathbf{A}$ define

$$\mathtt{Rep}(X) = \mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{A}) \quad \mathtt{PSh}_{\Delta}(X) = \mathtt{Func}(\mathtt{Sub}(X)^{op}, \mathbf{A})$$

Since $(X, \mathtt{Sub}(X))$ is a topology, $\mathtt{PSh}_{\Delta}(X)$ is the category of presheaves in the classical way, and we call $\mathtt{Sh}_{\Delta}(X)$ the category of sheaves. We define an adjoint pair for $f : X \to Y$

$$f_\star : \mathtt{Rep}(Y) \leftrightarrows \mathtt{Rep}(X) : f^\star$$

Where $f^\star F(\Delta^n \xrightarrow{\sigma} X) = F(\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y)$ and $f_\star$ is the right Kan extension along f. This adjoint pair is compatible with the ones on $\mathtt{Sh}_{\Delta}(X)$, which we call in the same way. There is a close relationship between $\mathtt{PSh}_{\Delta}(X)$ and $\mathtt{Rep}(X)$.

Theorem 1.1. *Given X a simplicial (or delta) set, there is an adjoint pair*

$$I : \mathtt{PSh}_{\Delta}(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$$

Where $I(F)(\sigma) = F(\bigcap_{\sigma \in Y} Y)$ and $\Gamma(F)(Y) = \varprojlim_{\sigma \in Y} F(\sigma)$, that satisfies

- *For each $\sigma \in X$ and $F \in \mathtt{PSh}(X)$ we have $I(F)(\sigma) = F_\sigma$, where F_σ is the stalk of F.*
- *Γ commutes with $f_\star$ and $f^\star$, I commutes with $f_\star$.*
- *For any $F \in \mathtt{Rep}(X)$ we have that $\Gamma(F)$ is a sheaf. Furthermore, the composition $\Gamma \circ I : \mathtt{PSh}_{\Delta}(X) \to \mathtt{Sh}_{\Delta}(X)$ is the sheafification functor.*
- *$I : \mathtt{Sh}_{\Delta}(X) \to \mathtt{Rep}(X)$ is fully faithful and $\Gamma : \mathtt{Rep}(X) \to \mathtt{Sh}_{\Delta}(X)$ is essentially surjective.*
- *If all simplices $\Delta^n \xrightarrow{\sigma} X$ are monomorphisms, then*

  $$I : \mathtt{Sh}_{\Delta}(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$$

  Is an equivalence of categories that commutes with $(f^\star, f_\star)$.

We have that the simplicial cochain sheaf complex $C^{\cdot}_{\Delta}$, as well as the complexes worked out by Chataur, Saralegui, Tanré are examples of simplicial sheaves.

Now, we might not always use on X the topology given by $\mathtt{Sub}(X)$. We call $\mathtt{Sh}^{\mathcal{T}}_{\Delta}(X)$ the sheaves over a topology $\mathcal{T} \xrightarrow{\iota_T} \mathtt{Sub}(X)$, and we have the following adjoints relating this with what we had

$$\mathtt{Rep}(X) \underset{I}{\overset{\Gamma}{\rightleftarrows}} \mathtt{PSh}_{\Delta}(X) \underset{Lan_{\iota_T}}{\overset{Ran_{\iota_T}}{\rightleftarrows}} \mathtt{PSh}^{T}_{\Delta}(X)$$

Where Lan_{ι_T} and Ran_{ι_T} are the left and right Kan extensions along ι_T. We have the formula $Lan_{\iota_T}F(Y) = \underset{Y \leq Z \in T}{\mathtt{colim}}\ F(Z)$. We still have here that

$$F \in \mathtt{Rep}(X) \Rightarrow \Gamma(F) \in \mathtt{Sh}^{T}_{\Delta}(X)$$

The theory just presented can be generalized to other presheaf categories such as **Set**, where we recover the classical sheaf theory. This allows us to use adjunctions such as $|-| : \mathbf{SSet} \leftrightarrows \mathbf{Top} : Sing$ to naturally define functors between simplicial and classical sheaves (such as the classical example $C_{\cdot}(X) = C_{\Delta}(Sing(X))$). We use this to define a functor

$$\Phi : \mathtt{Sh}(|X|) \to \mathtt{Sh}^{T}_{\Delta}(X)$$

Given by $\Phi(F)(Y) = \Gamma(|Y|, F) = \underset{|Y| \subset U}{\mathtt{colim}}\ F(U)$.

2. Localizing quasi isomorphisms: Following the inspiration from the topological setting, we say that $f : F \to G$ in $\mathtt{Rep}(X)$ (resp. in $\mathtt{Sh}(X)$) is a quasi isomorphism if and only if $f_\sigma : F(\sigma) \to G(\sigma)$ (resp. $f_\sigma : F_\sigma \to G_\sigma$) is a quasi isomorphism for all $\sigma \in X$. To localize with respect to these morphisms we present two alternatives.

- When $\mathtt{PSh}^{T}_{\Delta}(X) = \mathtt{Func}(T^{op}, Ch(\mathbf{A}))$ with $T \subset \mathtt{Sub}(X)$ topology and $\mathbf{A}$ an abelian category with enough injectives, we might take the derived category of $\mathtt{Sh}^{T}_{\Delta}(X)$ in exactly the same way as in the topological case. We call this $D_{\Delta}(X)$.
- When $\mathtt{Rep}(X) = \mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{M})$ with $\mathbf{M}$ a model category,since $\Delta \downarrow X$ is a *Reedy category*, we can consider the *Reedy model structure* on $\mathtt{Rep}(X)$, in which $f : F \to G$ is weak equivalence if it is so component wise. We call the homotopy category $Ho(X)$.

In general we have that

$$f : F \to G \text{ is a quasi isomorphism} \Leftrightarrow I(f) \text{ is a weak equivalence}$$

And in the case that all simplices of X are monomorphisms, these two categories are equivalent (over $Ch(R)$).

We present also fibrant sheaves, flasque sheaves (which are called *extendable* in the literature), injective sheaves and acyclic sheaves, and relations between them.

3. Stating the Deligne axioms: We give an abstract presentation of the Deligne axioms that can work in many contexts.

Given a chain of morphisms of categories

$$\mathbf{C}_0 \underset{j_0}{\overset{\iota_0}{\rightleftarrows}} \mathbf{C}_1 \underset{j_1}{\overset{\iota_1}{\rightleftarrows}} \dots \underset{j_{n-1}}{\overset{\iota_{n-1}}{\rightleftarrows}} \mathbf{C}_n \underset{j_n}{\overset{\iota_n}{\rightleftarrows}} \mathbf{C}_{n+1} = \mathbf{C} \tag{1}$$

And for a function $p : \mathbb{N} \to \mathbb{Z}$, endofunctors $\tau^i_{p(k)} : \mathbf{C}_i \to \mathbf{C}_i$ $k \in \{0, \dots, n\}, i \in \{k, k+1\}$, satisfying

i. $j_k \circ \iota_k \simeq 1_{\mathbf{C}_k}$ for all $k \in \{0, \dots, n\}$

ii. $\tau^k_{p(k+1)} \circ \tau^k_{p(k)} \simeq \tau^k_{p(k)}$ for all k

iii. $\tau^k_{p(k)} \circ \tau^k_{p(k)} \simeq \tau^k_{p(k)}$ for all k

iv. $j_k \circ \tau^k_{p(k)} \simeq \tau^{k-1}_{p(k)} \circ j_k$

We name $j^k = j_k \circ \dots \circ j_n : \mathbf{C} \to \mathbf{C}_k$ for $k \in \{0, \dots, n\}$ and $j^{n+1} = 1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$.

Definition. *In the setting just described, given $F_0 \in \boldsymbol{C}_0$ such that $\tau_{p(0)} F_0 \simeq F_0$ and $p : \mathbb{N} \to \mathbb{Z}$, we say that $A \in \boldsymbol{C}$ is of class Del_{F_0} if it satisfies*

(AX 1) $j^0 A \simeq F_0$

(AX 2) $j^{k+1} A \simeq \tau_{p(k)} \iota_k j^k A \ \forall k \in \{0, \dots, n\}$

And in this context we prove the following proposition.

Proposition 1.1.1. *$A \in \boldsymbol{C}$ is of class $Del_{F_0} \Leftrightarrow A \simeq \tau_{p(n)} \iota_n \dots \tau_{p(0)} \iota_0 F_0$*

So the axioms characterize an isomorphism class in $\mathbf{C}$. This abstract presentation is useful since it gives us freedom to

- Work in either $D_\Delta(X)$ and $Ho(X)$, which in particular means we have freedom over the model category if we find the proper truncation functors.
- Consider different simplicial structures, such as simplicial or delta sets and simplicial complexes. We can even go beyond simplicial and work in other presheaf categories.
- Work over different (simplicial) topologies and chains of subobjects.

For reasons explained in section 6.2, we consider simplicial complexes. More precisely,

1. We take X a simplicial complex such that $|X|$ is a PL stratified pseudomanifold with stratification $|X_0| \leq \dots \leq |X_n| = |X|$, with X_k is a subcomplex of X.
2. We suppose that X is subdivided two times, $X = Sd^2(X')$.
3. We take $T \subset \mathtt{Sub}(X)$ the topology generated by $\{ \bigcup_{\sigma < \tau} \mathtt{st}(b_\tau) | \sigma \in X' \}$, where b_τ corresponds to the barycenter of τ and $\mathtt{st}(b_\tau)$ to its star.
4. We take $U^\Delta_k = X -_\Delta X_{n-k} = \{\sigma \in X || \sigma | \cap X_{n-k} = \emptyset\}$.

For $p : \mathbb{N} \to \mathbb{Z}$ a GM perversity, we take the usual truncation functors as $\tau_{p(k)}$ and as for the chain of functors in (1) we take the derived functors of $(\iota_k^\star, (\iota_k)_\star)$, with $U_k \overset{\iota_k}{\hookrightarrow} U_{k+1}$. The simplicial Deligne axioms then read as follows

Definition. *$F \in D_\Delta(\mathcal{T})$ is said to satisfy the Δ-Deligne axioms if*

(AX 1) $F|_{U_0^\Delta} \simeq \mathbb{R}_{U_0^\Delta}$

(AX 2) $F|_{U_{k+1}^\Delta} \simeq \tau_{p(k)}(\iota_k)_\star F|_{U_k^\Delta}$ $\forall k \in \{0, \dots, n\}$

And we have as a consequence of proposition 1.1.1, this characterized by the Deligne sheaf.

Proposition 1.1.2. *The sheaf $F \in D_\Delta(\mathcal{T})$ satisfies the Δ-Deligne axioms if and only if $F \simeq \tau_{p(n)}\iota_n \dots \tau_{p(0)}\iota_0 \mathbb{R}_{U_0^\Delta}$.*

4. The main results: We now come to the final part which is to find sheaves that satisfy the simplicial Deligne axioms. Thanks to our functor $\Phi : \mathtt{Sh}(|X|) \to \mathtt{Sh}_\Delta^T(X)$, which takes soft sheaves to flasque ones (proposition 6.1.2), we have plenty of those.

We state our main results, which both come as corollaries of proposition 1.1.2 and the result that is proven in the last section.

$$\Phi(I^p C_\cdot) \text{ satisfies the } \Delta\text{-Deligne axioms.}$$

We have two consequences of this, the first one is that Φ takes Deligne sheaves to Δ-Deligne ones.

Theorem 1.2. *Given X a simplicial complex such that $|X|$ is a PL stratified pseudomanifold with stratification $|X_0| \leq \dots \leq |X_n| = |X|$, with X_k is a subcomplex of X. We suppose that X is subdivided two times, $X = Sd^2(X')$, and take the topology generated by $\{\bigcup_{\sigma<\tau} \mathtt{st}(b_\tau) | \sigma \in X'\}$. In this setting we have*

If $F \in D(|X|)$ satisfies the Deligne axioms, then $\Phi(F)$ satisfies Δ-Deligne axioms.

Now, since $IC_\cdot^{\bar{p}}$ is soft we have that $\Phi(IC_\cdot^{\bar{p}})$ is flasque, and this implies that

$$\mathbb{H}^\star(X, \Phi(IC_\cdot^{\bar{p}})) \simeq IH^{\bar{p}}(|X|)$$

That is, the hyperhomology of this sheaf corresponds to the intersection homology of the realization. This is a fact shared by all the sheaves that satisfy the Δ-Deligne axioms.

Theorem 1.3. *Consider a PL stratified pseudomanifold X with a compatible triangulation $|K| \xrightarrow{\simeq} X$. Take $K' = Sd^2(K)$ and the topology T generated by $\{\bigcup_{\sigma<\tau} \mathtt{st}(b_\tau) | \sigma \in K\}$ on K'. If $F \in D_\Delta(T)$ satisfies the Δ-Deligne axioms, then*

$$\mathbb{H}^\star(K', F) \simeq IH_p^\star(X)$$

This means that we have found a way to characterize in this case chains of sheaves that compute the classical intersection homology.

There are some results that for issues of time could not be explored here, as well as lines of work for the future and possible applications for this work. We mention a few of this.

- The theory of simplicial sheaves and simplicial Deligne axioms were at first thought to be stated for **simplicial sets**. Given the complexity of working geometrically on this environment, we switched to simplicial complexes, which use the same theory. The challenge of finding sheaves that satisfy the Δ-Deligne axioms in this context still remains.

- Everything up to the statement of the Deligne axioms can be worked for $Ho(X)$. This means that if we find the correct truncation functors, we can state the Deligne axioms for categories other than $Ch(\mathbf{A})$. One possibly nice option is to consider DGA's, which add a multiplicative structure to this theory.

- The theory of simplicial sheaves can be generalized to other presheaf categories, and by doing so, the simplest presheaf category **Set** recovers the classical sheaf theory. This opens the door for the study of a variety of objects in other categories.

- We worked using GM perversities mainly for comfort, but it might be easy to obtain all the results of this document for more general perversities.

- We used a functor $\Phi : \mathtt{Sh}(|X|) \to Sh^T_\Delta(X)$ naturally defined to prove that $F \in Del \Rightarrow \Phi(F) \in Del_\Delta$. One can naturally wonder about the question for the other way around, that is, to find a functor $\Psi : Sh^T_\Delta(X) \to \mathtt{Sh}(|X|)$ that transforms Δ-Deligne into Deligne sheaves. One possible line of work for this is to follow the construction made in [CSAT20] for the blown-up cochain sheaf.

There is also an advantage on using a simplicial topology, which is that the stalks are very simple. In fact, for the topology we used, and many others, they correspond to evaluations on certain complexes.

$$F_\sigma = F(\mathtt{fst}(\sigma'))$$

Since the Deligne axioms can be check at the level of stalks, the characterization we provide can simplify checking of the axioms by a great amount, specially for specific compact stratified manifolds.

Outline of the thesis:

We begin chapter 2 explaining in section 2.1 the classical setting presented in [GM80] and [GM83], from the construction of intersection (co)homology in a purely geometrical way, to the sheaf theoretic construction with the statement of the Deligne axioms. We then move in section 2.2 to present the construction of simplicial intersection (co)homology, presenting as well the sheaves that will serve us as the motivating examples for the developement of simplicial sheaves.

Chapter 3 goes deeply into the study of the different simplicial structures that we work with: Simplicial sets, Delta sets and Simplicial complexes, developing the theory from scratch in a particular way that is specially fitted to work with what comes afterwards.

Chapter 4 enters into the construction of simplicial sheaves, presenting the two categories $\mathtt{Rep}(X)$ and $\mathtt{Sh}_\Delta(X)$ that we mentioned together with their morphisms $(f^\star, f_\star)$ and their main properties. In section 4.2 we explore the relationship between these categories of simplicial sheaves, defining the adjunction

$I : \mathtt{PSh}_\Delta(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$ and proving the properties of theorem 1.1, and then we explain in section 4.2.3 how to relate this when having topologies other than $\mathtt{Sub}(X)$. In section 4.3 we generalize the theory so that it encompass the classical theory of sheaves as well, to then give in section 4.4 a natural way to construct functors (in particular the functor of the main theorem) between simplicial and classical sheaves.

In chapter 5 we localize these categories with respect to quasi isomorphisms, constructing an homotopy category for $\mathtt{Rep}(X)$ and a derived category for $\mathtt{Sh}_\Delta(X)$, we give relations between them, and then we give the results regarding acyclic sheaves. In section 5.3, we explain how to apply this theory to simplicial complexes, which are what we finally work with in chapter 6.

In chapter 6 we state the simplicial Deligne axioms and prove theorem 1.2 and corollary 1.3. In section 6.1 we present the abstract form of the Deligne axioms mentioned, which works many in different contexts. In section 6.2 we describe the setting in which we state the simplicial Deligne axioms, state the axioms, and give the functor $\Phi : \mathtt{Sh}(|X|) \to \mathtt{Sh}_\Delta(X)$ along with its main properties. Finally in section 6.3 we state the main result and corollary, and complete the proof in section 6.4.

We give an Appendix containing theoretical background that the reader might not have as familiar knowledge, and that is important to know as they are used without delicacy throughout this document. This corresponds to Kan extensions in section 8.1, model categories and Reedy model structure in section 8.2.1 and derived categories (of abelian categories with enough injectives) in section 8.3.

We assume the reader is familiar with undergraduate-level knowledge, as well as the basic knowledge on algebraic topology (we refer the reader to [Hat02] otherwise) and category theory (we refer the reader to [ML98] otherwise).

We shall now begin.

2 The classical theory

Before getting deep into waters, we will contextualize and present the territory in which we step on. This is to say, we will explain in section 2.1 what is intersection (co)homology, presenting the construction given by King in [Kin85] and by Goresky and MacPherson in [GM80], to then make a review on sheaf theory and state the *Deligne axioms*, satisfied by the *Deligne sheaf*, as it is done in [GM83]. We then explain briefly in section 2.2.1 the theory of simplicial intersection homology, based on the works of D. Chataur, M. Saralegui-Aranguren and D. Tanré, and give the examples that motivate the theory of simplicial sheaves.

2.1 The classical theory of Deligne axioms

During this section, we present the theory we pretend to emulate in simplicial terms: the Deligne axioms. We begin by showing the construction of intersection (co)homology for (PL) stratified pseudomanifolds, then construct the derived category of sheaves to finally state the Deligne axioms, explaining that they are a way to classify sheaves whose hyperhomology is intersection homology. We will closely follow classical references on the subject, mainly [Fri20] and [Ban07].

2.1.1 Intersection homology

Intersection (co)homology is basically a kind of (co)homology that we obtain by cutting off some chains (instead of let them all run freely as in (co)homology), by the use of a function called perversity. This discrimination follows decompositions of topological space into layers

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

Spaces endowed with this kind of decomposition are called *filtered spaces*. Although intersection (co)homology can be defined in these spaces, we will not in general get good results (particularly Poincaré duality) if we do not demand better conditions. For stating Poincaré duality, certain manifold structures should be considered. What we arrive at is something that could be very loosely described as "manifolds with singularities" with a locally conelike structure. There are called *stratified pseudomanifolds*.

Recall that **the (open) cone** of a space Y corresponds to the quotient $cY = Y \times [0,1)/(x,0) \sim (y,0)$.

Definition. *A 0-dimensional* ***stratified pseudomanifold*** *is a numerable set with discrete topology.*

An n-dimensional ***stratified pseudomanifold*** *is a paracompact Hausdorff space X with a filtration of closed subspaces $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{n-2} = X_{n-1} \subset X_n = X$ such that*

1. *Every non empty set $X_{n-k} - X_{n-k-1}$ is a manifold of dimension $n-k$ (called an open stratum of X)*
2. *The top stratum $X - X_{n-2}$ is dense in X*
3. *For each point $x \in X_{n-k} - X_{n-k-1}$ there is an open neighborhood U of x in X, a compact topological stratified pseudomanifold L of dimension $k-1$*

with stratification $\{L_{k-1-j}\}$ *and an homeomorphism* $\phi : U \to \mathbb{R}^{n-k} \times cL$ *which is stratum preserving (ie the restriction* $\phi : U \cap X_{n-j} \to \mathbb{R}^{n-k} \times cL_{k-j-1}$ *is a homeomorphism)*

The connected components of $X_k - X_{k-1}$ *are called* ***strata*** *(of dimension* k*), the sets* $\mathbb{R}^{n-k} \times cL$ *are called* ***distinguished neighborhoods*** *of* $x \in X$ *and the stratified pseudomanifolds* L *are called* ***links*** *of* $x \in X$*.*

We will give some examples of stratified pseudomanifolds to hopefully clarify the definition just presented.

Examples:

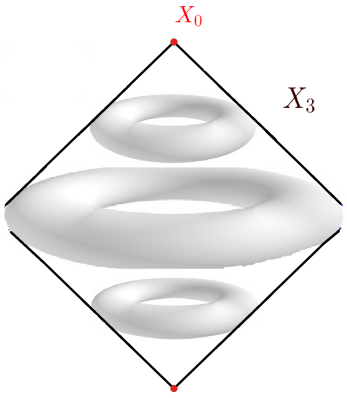
1. A manifold M of dimension n is an example of a stratified pseudomanifold of dimension n. We set $X_n = M$ and $X_k = \emptyset$ for all $k \in \{0, \dots, n-1\}$. We name this filtration as $\emptyset \subset M$. Observe that each point has distinguished neighborhood $\mathbb{R}^n \times c\emptyset$, with $\emptyset$ having dimension -1.

2. All irreducible complex algebraic or analytic varieties can be filtered so as to be stratified pseudomanifolds. The explanation of this can be found in ([KW06] Section 4.10).

3. The (open) cone $cM = M \times [0,1)/\sim$ of a manifold M of dimension n is a stratified pseudomanifold of dimension $n+1$, with stratification given by $\{\mathtt{c}\} \subset cM$, where $\mathtt{c}$ is the cone point.

   Similarly, the suspension of a compact connected manifold is a stratified pseudomanifold with the stratification given by $\{n, s\} \subset SM$. We have in fact that $SM - \{n, s\} = (-1, 1) \times M$ a manifold, and each point in here has a distinguished neighborhood given by $\emptyset \times \mathbb{R}^{n+1}$, where as n and s have distinguished neighborhoods $\mathbb{R}^0 \times cM$, with M being their link, filtered as in the first example.

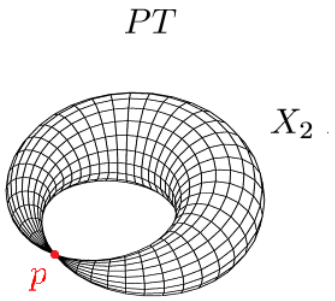
   We can visualize for example the suspension of a torus as follows



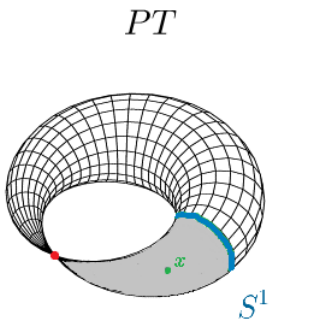
4. Generalizing the previous example, given a stratified pseudomanifold X of dimension n, its cone cX is a stratified pseudomanifold, with the stratification given by $(cX)_k = cX_{k-1}$. Observe that X_0 is the cone point. To

see how this yields a stratified pseudomanifold, suppose that $x \in X$ has a distinguished neighborhood $\mathbb{R}^k \times cl$, then if we form $(0,1) \times X$ with the filtration of the fourth example, each point $(t,x) \in (0,1) \times X$ will have a neighborhood filtered homeomorphic to $((0,1) \times \mathbb{R}^k) \times cL \cong \mathbb{R}^{k+1} \times cL$, as in the fourth example. Any point other than the cone point will have this kind of neighborhood, whereas the coin point will have $\mathbb{R}^0 \times cX$ as distinguished neighborhood.

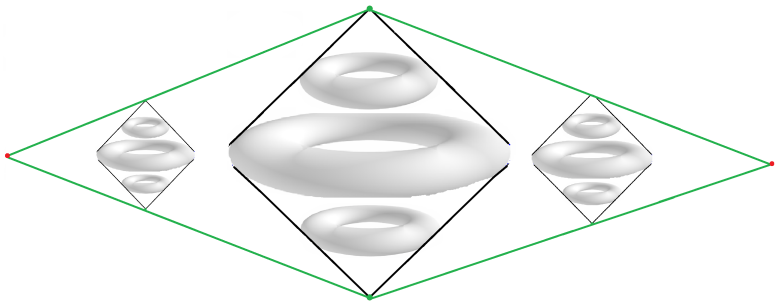
5. The pinched torus



is a stratified pseudomanifold of dimension 2, with $X_0 = \{p\}$ and $X_1 = X_2 = PT$ (we name this as $\{p\} \subset PT$). Here, the links of a point $x \in X_2 - X_0$ are S^1, whereas the distinguished neighborhoods are given by $\mathbb{R}^0 \times cS^1$, with p in the cone point, as the following image shows



6. The suspension of a stratified pseudomanifold goes in the same fashion as the cone in example 4. We can visualize for example the double suspension of a torus as follows.



7. Given a stratified pseudomanifold X of dimension n and a manifold M of dimension m, $X \times M$ is a stratified pseudomanifold of dimension $n+m$ with $(X \times M)_k = X_{k-m} \times M$ for $k \geq m$ and empty in other dimensions. Each point $(m, x) \in M \times X$ will have a neighborhood filtered homeomorphic to $(\mathbb{R}^m \times \mathbb{R}^k) \times cL \simeq \mathbb{R}^{k+m} \times cL$. In particular, if L is a link of x in X, then L will also be a link of (m, x) in $M \times X$.

To construct intersection (co)homology, the first thing at hand is to filtrate chains. This will be done by a function called **perversity**.

Definition. *We define a* ***GM-perversity*** *as a function* $\bar{p} : \mathbb{N}_{\geq 2} \to \mathbb{Z}$ *such that* $\bar{p}(2) = 0$ *and* $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$.

Perversities are commonly denoted in sequence form $(p(2), p(3), \dots)$. The idea of the inequality $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k)+1$ is that the sequence increases by 0 or 1 on each step. We might notice that GM perversities form a partially order abelian monoid $(\bar{P}, \leq, +)$, where $\bar{p} \leq \bar{q} \Leftrightarrow \bar{p}(k) \leq \bar{q}(k)$ for all $k \in \mathbb{N}_{\geq 2}$ and sum defined component-wise. We name the four most important GM perversities.

- The maximal element of $\bar{P}$ is called the **top perversity**, and is given by $\bar{t} = (2, 3, 4, \dots)$.
- The minimal element of $\bar{P}$ is called the **zero perversity** and it is given by $\bar{0} = (0, 0, 0, \dots)$.
- We name the **lower middle perversity** $\bar{n} = (0, 0, 1, 1, 2, 2, \dots)$ and the **upper middle perversity** $\bar{m} = (0, 1, 1, 2, 2, 3, 3, \dots)$.

With this at hand we can define the chain complex to which we are going to take homology.

Definition. *Given a stratified pseudomanifold* X *and a GM-perversity* $\bar{p}$

- *A continuous function* $\sigma : |\Delta^i| \to X$ *is* $\bar{p}$***-allowable*** *if for all strata* $S \subset X_{n-k} - X_{n-k-1}$ *and for all* $k \geq 2$, *we have*

$$\sigma^{-1}(S) \subset \{i - k + \bar{p}(k) \text{ skeleton of } \Delta^i\}$$

- *A chain* $c \in C_i(X)$ *is of* ***p-intersection*** *if all the simplices in* c *and* ∂c *are* $\bar{p}$*-allowable.*

Then the **intersection chain complex** corresponds to

$$IC_i^{\bar{p}}(X) = \{c \in C_i(X) | \text{ c is of } \bar{p}\text{-intersection}\}$$

The **intersection (co)homology** groups are the (co)homology groups of this complex

$$IH_i^{\bar{p}}(X) = H_i(IC_\cdot^{\bar{p}}(X)) \quad IH^i_{\bar{p}}(X) = H^i(IC_\cdot^{\bar{p}}(X))$$

We will work mainly with **PL**-stratified pseudomanifolds. PL spaces are topological spaces endowed with a family of compatible triangulations, with the idea in mind that these triangulations are arbitrarily refined. Then PL-stratified pseudomanifolds are stratified pseudomanifolds that are PL spaces in a compatible way. We start by defining triangulations.

Definition. *Given X a topological space*

- *A **triangulation of** X is a pair $T = (K, h)$, with K being a simplicial complex and $h : |K| \to X$ a homeomorphism.*
- *A **subdivision of a triangulation** $T = (K, h)$ is a triangulation $T' = (K', h)$ with K' being a subdivision of K. We write $T' \lhd T$ when T' is a subdivision of T.*
- *Two subdivisions T and S are said to have **a common subdivision** if there are subdivisions $T' \lhd T$ and $S' \lhd S$ with $S' \simeq T'$ (where $(K, h) \simeq (K', h')$ when $h' \circ h^{-1}$ is an isomorphism).*

For the definition of simplicial complex, realization and subdivision, the reader can flash forward to 3.2.3, where we review these definitions from scratch at the beginning of the section. We can now define a PL space.

Definition. *We define components of the category of PL spaces*

- *A **PL space** is a second countable Hausdorff space X together with a family of triangulations $\mathcal{T}$ satisfying*
  1. *For all $T \in \mathcal{T}$ we have $T' \lhd T \Rightarrow T' \in \mathcal{T}$.*
  2. *For all $T, S \in \mathcal{T}$ there is a common subdivision of T and S.*
- *A **PL map** $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is a continuous function $f : X \to Y$ such that for any triangulations $|K| \xrightarrow{h} X$ and $|L| \xrightarrow{j} Y$ there is a subdivision $K' \lhd K$ such that $j^{-1} f h$ takes each simplex of K' linearly into a simplex of L.*

  *A **PL homeomorphism** is a PL map with an inverse that is a PL map.*
- *A subspace Y of a PL space X is a **PL subspace** if the inclusion $Y \hookrightarrow X$ is a PL map. Observe that this means that a closed PL subspace is just a subcomplex of some triangulation of X.*

Now we move into the definition of a PL stratified pseudomanifold. First thing to do is to consider filtrations on PL spaces by closed PL subspaces

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{n-1} \subset X_n$$

We call these **PL filtered spaces**. Thanks to ([Fri20] Lemma 2.5.12), we can assume that each triangulation of the PL space is compatible with the filtration, in the sense that it contains X_k as subcomplexes.

Definition. *A **PL stratified pseudomanifold** X is a stratified pseudomanifold such that*

1. *X is a PL filtered space.*
2. *The strata of X are PL manifolds (this is, each point has a neighborhood PL homeomorphic to $|\Delta^n|$).*
3. *Each point has a distinguished neighborhood $N \cong \mathbb{R}^i \times cL$ such that the link L is a PL stratified pseudomanifold and the filtered homeomorphism $N \to \mathbb{R}^i \times cL$ is a PL map.*

With this definition at hand we can define PL chain complexes, PL intersection chains, and PL intersection cohomology for PL stratified pseudomanifolds.

One delicate point when defining PL chain complexes is that there are in fact different ways to do it. There is one way in which we allow infinite chains, called *Borel-Moore PL chain complexes*, and one that works only with finite chains, which closely mirrors the non PL construction we made before. To work out the theory of Deligne axioms we are going to need the complexes to be *sheaves*, and in particular we need a restriction morphims $F(X) \to F(U)$, which is obtainable in either one of two ways.

- We work with Borel-Moore PL chains, allowing infinite chains and constructing *Borel-Moore* intersection homology.
- We work with the classical PL intersection chains, that is, with finite chains, and then we take the dual. In other words, we consider the PL intersection **co**chain functor, constructing intersection cohomology.

The first option is the one that the main literature on the subject take, since we obtain a sheaf that satisfies the Deligne axioms on any commutative ring. As for the second option presented, it is possible that this sheaf satisfies the Deligne axioms, but only when we are working over a field or with restrictive conditions on the links. This issue has however been resolved in the works of D Chataur, M Saralegi-Aranguren and D Tanré, using what is called the *blown up cochain* functor, which allows to work on any commutative ring (see [CSAT18b]). We will make a sketchy construction of this sheaf at the end of section 2.2.1.

We will define now the Borel-Moore PL chain complexes. Let X be a PL stratified pseudomanifold and $T \in \mathcal{T}$ a triangulation. A **simplicial m-chain** is a function

$$c : \{\sigma \in T | \sigma \text{ is an oriented simplex}\} \to \mathbb{Z}$$

With $c(-\sigma) = -c(\sigma)$. Observe that the function manifestation implies that we allow infinite chains. We also define $\mathtt{supp}(c) = \{\sigma | c(\sigma) \neq 0\}$, called the support of c, and $|c| = \bigcup_{\sigma \in \mathtt{supp}(c)} |\sigma|$.

We give a special name to the abelian group of simplicial m-chains.

$$C_m^T(X) = \{c : \{\sigma \in T | \sigma \text{ is an oriented simplex}\} \to \mathbb{Z} | c(-\sigma) = -c(\sigma)\}$$

Now, if $T' \lhd T$ then we have a map

$$\mathcal{Y}_{T,T'} : C_m^T(X) \to C_m^{T'}(X)$$

That takes $c \in C_m^T(X)$ to $c' \in C_m^{T'}(X)$ defined by

$$c'(\sigma') = \begin{cases} c(\sigma) & \text{if } \sigma' \text{ is contained in the (compatibly oriented) } m\text{-simplex } \sigma \in T \\ 0 & \text{if } \sigma' \text{ is not contained in any } m\text{-simplex of } T \end{cases}$$

With this maps we can consider the colimit

Definition. *Given $(X, \mathcal{T})$ a PL stratified pseudomanifold, we define the group of* ***PL m-chains*** *as*

$$C_m(X) = \operatorname*{colim}_{T \in \mathcal{T}} C_m^T(X)$$

Observe that this means that we are taking the equivalence classes of m-chains, where $c \in C_m^T(X)$ and $c' \in C_m^{T'}(X)$ are equivalent if T and T' have a common subdivision T'' such that $\mathcal{Y}_{T,T''}(c) = \mathcal{Y}_{T',T''}(c')$.

Then the **Borel-Moore homology** of X corresponds to the homology of this complex

$$H_i(X) = H_i(C_\cdot(X))$$

Now we can define the *intersection* versions of these. Consider a GM-perversity p. We call a chain $c \in C_m(X)$ to be **of p-intersection** if all their simplices $\sigma \in \mathtt{supp}(c)$ satisfy

$$\mathtt{dim}(\sigma \cap X_{n-k}) \leq m - k + \bar{p}(k) \ \forall k \geq 2$$

And the PL intersection chain complex, which will be a mayor object of study in this document, will correspond to the p-intersection chains.

Definition. *Given X a PL stratified pseudomanifold and $\bar{p}$ a GM perversity we define the* ***Borel-Moore PL intersection chain complex*** *as*

$$IC_m^{\bar{p}}(X) = \{c \in C_m(X) | c \text{ is of } \bar{p}\text{-intersection}\}$$

And we define the ***Borel-Moore PL intersection homology*** *as the homology of this complex*

$$IH_i^{\bar{p}}(X) = H_i(IC_\cdot^{\bar{p}}(X))$$

We will use the notation $IC_\cdot^{\bar{p}}(X)$ and $IH_i^{\bar{p}}(X)$ from now on to refer to the Borel-Moore PL chain complex and intersection homology respectively.

The functor $IC_m^{\bar{p}}$ (on the following chapter we will see it is a *sheaf*) is a particularly good one, since it is *soft* and satisfies the *Deligne axioms*. Of course, it is not the only sheaf with these characteristics, as we will mention soon. We will now consider a couple of examples of intersection homology.

Examples:

1. Consider X a PL stratified pseudomanifold and $X \times \mathbb{R}$ stratified as the example given before. For a chain $c \in IC_m^{\bar{p}}(X)$ there is a $(m+1)$-chain $\mathbb{R} \times c$ that satisfies

$$\begin{aligned} \dim(|\mathbb{R} \times c| \cap (\mathbb{R} \times X)_{n+1-k}) &= \dim(\mathbb{R} \times (|c| \cap X_{n-k})) \\ &\leq 1 + \dim(|c| \cap X_{n-k}) \\ &\leq m + 1 - k - \bar{p}(k) \end{aligned}$$

And similarly $\dim(|\mathbb{R} \times c|) \leq i - k + \bar{p}(k)$, which means that $\mathbb{R} \times c \in IC^{\bar{p}}_{i+1}(\mathbb{R} \times X)$. This defines a chain map

$$IC^{\bar{p}}_{\cdot}(X) \to IC^{\bar{p}}_{\cdot}(\mathbb{R} \times X)[1]$$

(See 8.3.1 for the corresponding definitions) Whose corresponding map in homology is an isomorphism

$$IH^{\bar{p}}_{i}(X) \xrightarrow{\simeq} IH^{\bar{p}}_{i+1}(\mathbb{R} \times X)$$

As it is shown in ([Ba08] Chapter II.). Observe that we can iterate this process to obtain

$$IH^{\bar{p}}_{i}(X) \xrightarrow{\simeq} IH_{i+k}(\mathbb{R}^k \times X)$$

2. Consider X a PL stratified pseudomanifold and cX the cone. We will sketch the computation of the intersection homology of cX, giving ([Ban07] Example 4.1.15) and ([Ba08] Chapter II.) as references. Given a chain $\epsilon \in IC^{\bar{p}}_{i-1}(X)$, we denote $c\epsilon$ the cone on the chain. For $j < k$ it will satisfy

$$\dim(|c\epsilon| \cap (cX)_{k-j}) \leq 1 + \dim(|\epsilon| \cap X_{k-j-1}) \leq i - j + \bar{p}(j)$$

And similarly for $\partial c\epsilon$. When $j = k$ we have $\dim(|c\epsilon| \cap (cX)_0) = \dim(c\epsilon| \cap \{c\}) = \dim(\{c\}) = 0$. And we obtain the following situation

$$\begin{aligned}
&i > k - \bar{p}(k) \Rightarrow c\epsilon \in IC^{\bar{p}}_{i}(cX) \\
&i = k - \bar{p}(k) \Rightarrow c\epsilon \in IC^{\bar{p}}_{i}(cX) \text{ precisely when } \epsilon \text{ is a cycle.} \\
&i < k - \bar{p}(k) \Rightarrow c\epsilon \notin IC^{\bar{p}}_{i}(cX)
\end{aligned}$$

This will make the coning $\epsilon \mapsto c\epsilon$ define a chain map

$$\tau_{\geq k-\bar{p}(k)-1} IC^{\bar{p}}_{\cdot}(cX) \xrightarrow{c} IC^{\bar{p}}_{\cdot}(cX)[1]$$

Which induces an isomorphism on homology as shown in the references. With this we obtain what is called the **cone formula**:

$$IH^{\bar{p}}_{i}(cX) = \begin{cases} IH^{\bar{p}}_{i-1}(X) & \text{if } i \geq k - \bar{p}(k) \\ 0 & \text{if } i < k - \bar{p}(k) \end{cases}$$

This last formula, the cone formula, is at the heart of Deligne axioms. Basically these axioms are a theoretically correct but kind of convoluted way of classifying sheaves that satisfy the cone formula.

We give a property of intersection (co)homology that we are going to use in the proof of the main result. Turns out that maps induced on intersection (co)homology *are not* invariant under homotopy, but rather it is invariant under *stratified homotopy*. Moreover, intersection (co)homology is natural with respect to *stratified* maps rather than continuous functions like (co)homology.

Definition. *Given X and Y PL stratified pseudomanifolds, a* ***stratified map*** *$f : X \to Y$ is a continuous function such that*

- *For each stratum $T \subset Y$, $f^{-1}(T)$ is union of strata of X.*

- *For all strata* $S \subset X$

$$\bar{p}(\texttt{codim}(S)) - \texttt{codim}(S) \leq \bar{p}(\texttt{codim}(f(S)) - \texttt{codim}(f(S))$$

We have that a stratified map induces a map on the intersection cochains

$$f : IC^{\bar{p}}_{\cdot}(X) \to IC^{\bar{p}}_{\cdot}(Y)$$

which induces a map on intersection (co)homology, as it is shown in ([Fri20] Proposition 4.1.6).

Definition. *Given* X, Y *PL stratified pseudomanifolds. We give* $I = [0,1]$ *the trivial filtration and* $I \times X$ *the product filtration. In this setting we define the following.*

- *A* ***stratified homotopy*** *is a stratified map* $H : I \times X \to Y$
- *Two stratified maps* $f, g : X \to Y$ *are* ***stratified homotopic*** *if there is a stratified homotopy* H *with* $f = H|_{\{0\}\times X}$ *and* $g = H|_{\{1\}\times X}$.

The following proposition is proved in ([Fri20] Proposition 4.1.10).

Proposition 2.0.1. *If the stratified maps* $f, g : X \to Y$ *are stratified homotopic, then they induced chain homotopic maps* $IC^{\bar{p}}_{\cdot}(X) \to IC^{\bar{p}}_{\cdot}(Y)$ *and hence*

$$f = g : IH^{\bar{p}}_{\cdot}(X) \to IH^{\bar{p}}_{\cdot}(Y)$$

This has as a corollary that a stratified homotopy equivalence f defines an isomorphism in intersection homology (when $f(X_k) \subset Y_k$), as it is shown in ([Fri20] Corollary 4.1.11).

2.1.2 Derived category of sheaves

Deligne axioms are a way to characterize *constructions* that compute intersection (co)homology. Of course the word "construction" is not specific at all, and we want to specify. The key observation on this matter is that the intersection chains presented in the last section form a *sheaf*, and the category of sheaves is an abelian category with enough injectives, hence we can pleasantly (really, it could be much worse) construct its derived category and attempt to classify sheaves that compute intersection (co)homology up to quasi isomorphism. We begin by reminding what sheaves are, and then we move into their derived category to finally arrive into the statement of Deligne axioms. We strongly suggest the reader to read the appendix 8.3.1 for a clear understanding on abelian categories and derived categories and functors, since those results will be used here without forbearance.

Basic constructions

A presheaf can be thought of as a representation of a topological space.

Definition. *For a topological space* X*, the category of* ***presheaves*** *over* X *correspond to the functor category*

$$\texttt{PSh}(X) = \texttt{Func}(Op(X)^{op}, \boldsymbol{A})$$

Where $Op(X)$ *is the category associated with the poset of open sets of* X *with inclusion.*

The category $\mathbf{A}$ can be any category of the readers liking. Normally one takes **Set** or R-mod for some commutative ring. For now, we will work with an abelian category $\mathbf{A}$ with enough injectives (such as R-mod). Observe that this means that

$$\texttt{PSh}(X) \text{ is an abelian category with enough injectives.}$$

And the (co)limits are computed componentwise.

Definition. *Given $x \in X$ and $F \in \texttt{PSh}(X)$, we define the* ***stalk*** *of F at x as*

$$F_x = \operatorname*{\texttt{colim}}_{x \in U} F(U)$$

This loosely speaking means *how F behaves locally around x*. Stalks are at the heart of sheaf theory, for reasons that will become clear soon.

Notation: Given $F \in \texttt{PSh}(X)$ and U, V open sets.

- We call an element $s \in F(U)$ a **section** of F on U.
- When $U \subset V$ and $s \in F(V)$, one usually writes $F(U \subset V)(s)$ as $s|_U$. The morphisms $F(U \subset V)$ are called **restriction morphims**, whereas $s|_U$ is called the **restriction of s to U**.
- For $x \in U \subset X$ we name

$$\begin{aligned} F(U) &\longrightarrow F_x \\ s &\longmapsto s_x \end{aligned}$$

  s_x is called the **germ** of s at x.
- We call **the section of** F on an open U to $\Gamma(U, F) := F(U)$.

As pleasant as presheaves are for working at the categorical level (they form a functor category!), they can be improved into being compatible with many of the things we want to do in topology (and algebraic geometry). It is particularly useful in topology to *obtain global information from local information*, as many constructions are made locally. Loosely speaking, presheaves that do this global extraction from the local are called *sheaves*.

Definition. *A presheaf F is said to be a* ***sheaf*** *if it satisfies the two gluing properties*

1. *Consider an open set $U \subset X$ and open cover $U = \bigcup U_i$. If $s, t \in F(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.*
2. *Consider an open set $U \subset X$ and open cover $U = \bigcup U_i$. If we have $s_i \in F(U_i)$ for all i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there is $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i.*

We call $\texttt{Sh}(X)$ the full subcategory of $\texttt{PSh}(X)$ whose objects are sheaves, and we call

$$\iota_{Sh} : \texttt{Sh}(X) \hookrightarrow \texttt{PSh}(X)$$

The inclusion of categories.

We loosely said that sheaves are representations that can extract global information from the local (they satisfy the gluing conditions), and stalks are local behaviour around points. This suggests that we should be able to extract the information of a sheaf from its stalks, and this is reflected in the following proposition proven in ([Sch11] Proposition 5.2.4).

Proposition 2.0.2. *Given $f : F \to G$ a morphism of sheaves*

- *f is a monomorphism $\iff$ $f_x : F_x \to G_x$ is injective for all $x \in X$.*
- *f is an isomorphism $\iff$ $f_x : F_x \to G_x$ is an isomorphism for all $x \in X$.*

As it might be expected, it easier to find presheaves than it is to find sheaves. Luckily, there is a canonical way to construct sheaves from presheaves, which gives us in fact a left adjoint of the inclusion $\mathtt{Sh}(X) \overset{\iota_{Sh}}{\hookrightarrow} \mathtt{PSh}(X)$.

Theorem 2.1. *The inclusion ι_{Sh} has a left adjoint*

$$(-)^\sharp : \mathtt{PSh}(X) \leftrightarrows \mathtt{Sh}(X) : \iota_{Sh}$$

In particular we have a morphism $\alpha : F \to F^\sharp$ for each presheaf. This morphism is an isomorphism in the stalks $\alpha_x : F_x \xrightarrow{\simeq} F^\sharp_x$.

Proof. We construct $F^\sharp$ as follows

$$F^\sharp(U) = \{(s^u)_{u\in U} \in \prod F_u | \forall u \in U \exists V \subset U \text{ with } u \in V \text{ and } \exists t \in F(V) \\ \text{such that } s^v = t_v \forall v \in V\}$$

And the morphism $F \to F^\sharp$ sends $s \in F(U)$ to $(s_x)_{x\in U} \in F^\sharp(U)$. The verifications are straight forward. □

We call $F^\sharp$ the **sheafification** of F. Sheaves are all over the place in mathematics. We are going to see just a few examples

Examples: Consider X a topological space.

1. Many different set of functions one can think of, such as
   - $\mathcal{F}(U, R) = \{f : U \to R | f \text{ function}\}$ for some ring R.
   - $Co(U, \mathbb{R}) = \{f : U \to \mathbb{R} | f \text{ is continuous}\}$.
   - $C^k(U, \mathbb{R}) = \{f : U \to \mathbb{R} | f \text{ is of class } C^k\}$ for $k \in \mathbb{N} \cup \{\infty\}$.
   - $Hol(U, \mathbb{C}) = \{f : U \to \mathbb{R} | f \text{ is holomorphic}\}$.

   All define sheaves, with the restriction morphisms being restrictions of functions (hence the name *restrictions*). op
2. The most trivial example one can think of a presheaf is the **constant presheaf**, which is defined on an object $M \in \mathbf{A}$ as

$$\underline{\mathrm{M}} : Op(X)^{op} \longrightarrow \mathbf{A} \\ U \longmapsto M$$

   It is not hard to see that this is not a sheaf in general. We call its sheafification **the constant sheaf**, and denote it as $\underline{\mathrm{M}}$.

3. Given $(X, \mathcal{T})$ a PL stratified pseudomanifold, recall the construction of the group of **PL m-chains** as

$$C_m(X) = \operatorname*{colim}_{T \in \mathcal{T}} C_m^T(X)$$

With $C_m^T(X) = \{c : \{\sigma \in T | \sigma \text{ is an oriented simplex}\} \to \mathbb{Z} | c(-\sigma) = -c(\sigma)\}$ and the maps defined in the obvious way.

An open set $U \subset X$ has an induced PL structure $(U, \mathcal{T}_U)$ making the inclusion $U \hookrightarrow X$ a PL map (we take $S \in \mathcal{T}_U$ if there is a subdivision of S that has each simplex linearly contained in a simplex of some $T \in \mathcal{T}$).

For a chain $c \in C_m(X)$, choose $T \in \mathcal{T}_X$ and $S \in \mathcal{T}_U$ such that c is simplicial with respect to T and every simplex of S is contained in some simplex of T. We define $c|_U \in C_m(U)$ as the class of

$$c|_U(\sigma') = \begin{cases} c(\sigma) & \text{if } \sigma' \text{ is contained in the (compatibly oriented) } m\text{-simplex } \sigma \in T \\ 0 & \text{if } \sigma' \text{ is not contained in any } m\text{-simplex of } T \end{cases}$$

Observe that this is just a convoluted way to say that we are restricting the simplices of c to U. Observe that in fact $|c|_U| = |c| \cap U$. We can define in the same way restriction maps

$$C_i(U) \to C_i(V)$$

When $V \subset U$. It is not hard to see that this defines a sheaf.

4. We continue with the construction of the previous example, introducing now a perversity. Consider $(X, \mathcal{T})$ a PL stratified pseudomanifold and p a GM perversity. An open set $U \subset X$ induces an stratification given by the intersections $U_k = X_k \cap U$. Observing that for $V \subset U$ the map

$$IC_m^{\bar{p}}(U) \hookrightarrow C_m(U) \to C_m(V)$$

Has its image in $IC_m^{\bar{p}}(V)$ we define the map

$$IC_m^{\bar{p}}(U) \to IC_m^{\bar{p}}(V)$$

Which once again it is not hard to see that defines a sheaf.

5. Intersection blown up cochains $\mathcal{N}_{\bar{p}}^\star$, which we will define at the end of section 2.2.1 defines a sheaf.

We give to the sheaf $IC_m^p(-)$ a very annoying and highly confusing, but historically compatible, notation.

Definition. *The sheaf $U \mapsto IC_m^p(U)$ is called the p **intersection chain sheaf** on X on dimension $m \in \mathbb{N}$, and it is denoted in a confusing way as $\boldsymbol{IC}_p^{-m}(X)$*

In other words

$$\Gamma(U, \mathbf{IC}_p^{-m}(X)) = IC_m^p(U)$$

This notation is made to agree with the notation used in the original paper [GM83]. We also chose to include this notation to highlight the importance of

this sheaf, as it is the main example between the chains of sheaves that compute intersection homology.

Coming back to sheaves now, in ([Sch11] Theorem 5.3.3 and Propositions 5.3.4, 5.3.7) P. Schapira makes a brilliant work in summoning up the main properties of the main constructions for sheaves with respect to (co)limits. In particular to determine how to perform (co)limits in $\mathtt{Sh}(X)$, the exactness of the main functors and *abelianess* of $\mathtt{Sh}(X)$. We copy paste his exposition in our language here, as the author could not have made a better job. The proof can be followed in the reference.

Theorem 2.2. *Given X a topological space.*

1. *The category $\mathtt{Sh}(X)$ admits limits and these commute with ι_{Sh} (as ι_{Sh} has a left adjoint). This means that the limit of sheaves, which is computed componentwise in $\mathtt{PSh}(X)$*

$$(\lim_I F_i)(U) = \lim_I F_i(U)$$

   Is a sheaf.

2. *The category $\mathtt{Sh}(X)$ admits colimits, which correspond to the sheafification of the colimit taken in $\mathtt{PSh}(X)$.*

3. *Expanding on the last point, the sheafification functor $(-)^\sharp$ commutes with colimits (as it has a right adjoint). This means that for $\{F_i\}_{i\in I}$ an inductive system of sheaves*

$$(\operatorname*{colim}_I F_i)^\sharp \simeq \operatorname*{colim}_I F_i^\sharp$$

4. *The category $\mathtt{Sh}(X)$ is an abelian category with enough injectives.*

5. *The functor $\iota_{Sh} : \mathtt{Sh}(X) \to \mathtt{PSh}(X)$ is fully faithful and left exact.*

6. *Colimits are exact in $\mathtt{Sh}(X)$.*

7. *The sheafification functor $(-)^\sharp$ is exact.*

8. *For $x \in X$, its stalk define a functor $(-)_x : \mathtt{Sh}(X) \to \boldsymbol{A}$ which is exact and commutes with $(-)^\sharp$. In particular $\mathtt{Ker}(\phi)_x \simeq \mathtt{Ker}(\phi_x)$ and $\mathtt{Coker}(\phi)_x \simeq \mathtt{Coker}(\phi_x)$.*

9. *A complex of sheaves $F \to G \to H$ is exact $\iff$ $F_x \to G_x \to H_x$ is exact for all $x \in X$.*

10. *For an open set $U \subset X$, the functor of sections*

$$\begin{aligned} \Gamma(U, -) : \mathtt{Sh}(X) &\longrightarrow \boldsymbol{A} \\ F &\longmapsto \Gamma(U, F) = F(U) \end{aligned}$$

    Is left exact.

We shall be aware that ι_{Sh} is not exact, which makes $\Gamma(U,-)$ fail its exactness. This means also that the cokernels of sheaf are not the component wise cokernels but rather their sheafification. The fact that $\mathtt{Sh}(X)$ has enough injectives can be found in ([Sch11] Theorem 6.1.2).

The direct and inverse images $f_\star$ and $f^\star$

It is very important throughout the theoretical frameworks to have a *natural* way to relate sheaves on different topological spaces that is compatible with the relation between these topological spaces.

One expects for example that given a subspace $Y \subset X$ there is a *restriction* (that is, a precomposition with the inclusion) that would take sheaves in $\mathtt{Sh}(X)$ to sheaves in $\mathtt{Sh}(Y)$. This restriction is trivial to define, although the question arises as to whether restricting gives a sheaf. One might also ask, is there a "good" way to *induce* sheaves in $\mathtt{Sh}(X)$ from the sheaves over its subspace $\mathtt{Sh}(Y)$?

Observe that this is the same sort of question that we ask for group representations (see appendix 8.1). In general what we are wondering for is that given $f : X \to Y$ a morphism of topological spaces, we have some sort of *precomposition functor* $\mathtt{Sh}(Y) \to \mathtt{Sh}(X)$. Furthermore, we wish for the existence of an *adjoint* of this sort of precomposition $\mathtt{Sh}(X) \to \mathtt{Sh}(Y)$. Greatfully our prays are answered successfully by the goddess of mathematics.

Definition. *Given $f : X \to Y$ a continuous function*

1. *For $F \in \mathtt{PSh}(Y)$ we define $f^{-1}F \in \mathtt{PSh}(X)$ by $f^{-1}F(U) = \underset{f(U)\subset V \text{ open}}{\mathtt{colim}} F(V)$*
2. *We define the **inverse image** of a functor by f to be*

$$\begin{aligned} f^\star : \mathcal{Sh}(Y) &\longrightarrow \mathcal{Sh}(X) \\ F &\longmapsto (f^{-1}F)^\sharp \end{aligned}$$

This construction reflects one important philosophy of sheaf theory. We would like to create a *precomposition with f* functor, something like "$f^\star F(U) = F(f(U))$", but since $f(U)$ is not necessarily open, we perform the best alternative solution (which is a Kan extension). Afterwords, since what we obtain performing $f^{-1}F$ is not necessarily a sheaf, we sheafificate. The adjoint of $f^\star$ is surprisingly simple.

Definition. *For a continuous function $f : X \to Y$ we define the **direct image** functor*

$$f_\star : \mathcal{Sh}(X) \to \mathcal{Sh}(Y)$$

By setting $f_\star G(V) = G(f^{-1}(V))$

It is straightforward to check that $f_\star G$ is a sheaf.

Once again, P. Schapira brilliantly exposes in ([Sch11] Theorem 5.5.7 and Proposition 5.5.8) the main properties that we desire for the direct and inverse images. We copy his exposition here.

Theorem 2.3. *Given $f : X \to Y$ and $g : Y \to Z$ continuous functions.*

1. *The direct and inverse images are an adjoint pair*

$$f^\star : \mathcal{Sh}(Y) \leftrightarrows \mathcal{Sh}(X) : f_\star$$

2. *$f_\star$ commutes with limits, in particular it is left exact.*

3. *$f^\star$ commutes with colimits.*

4. *$(f^\star F)_x \simeq F_{f(x)}$ for all $x \in X$. This means that $f^\star$ is exact.*

5. *$(g \circ f)_\star = g_\star \circ f_\star$ and $(g \circ f)^\star = f^\star \circ g^\star$.*

6. *$f_\star$ takes injectives into injectives.*

Proof. The adjunction comes directly from the Kan extension (co)limit formulas. Then the commutation of $f_\star$ and $f^\star$ with (co)limits is obvious since they are an adjoint pair. We have that $(f^{-1}F)_x \simeq F_{f(x)}$ by working with the colimits and then we use the commutation of the sheafification and the stalk to conclude that $(f^\star F)_x \simeq F_{f(x)}$, then the exactness of $f^\star$ comes from the tenth statement on theorem 2.2. It is obviuos that $(g \circ f)_\star = g_\star \circ f_\star$, and then $(g \circ f)^\star = f^\star \circ g^\star$ follows from adjunction. Finally, the fact that $f_\star$ takes injectives into injectives comes from the fact that it has an exact left adjoint. □

Homological considerations

We pay now particular attention to the fact that $\mathtt{Sh}(X)$ is an abelian category with enough injectives. This means that all the results from appendix 8.3 hold for this category. This means that we are now considering *chains of sheaves* on the category $Ch(\mathtt{Sh}(X))$.

$$\cdots \to F^{n-1} \xrightarrow{d^{n-1}} F^n \xrightarrow{d^n} F^{n+1} \to \dots$$

For these chains of sheaves we can consider their (co)homology.

$$H^k(F^\cdot)$$

Which in this case is a functor that can be obtained as the sheafification of the component wise (co)homology

$$(U \mapsto H^k(F(U)))^\sharp$$

The neccesity of the sheafification is due to the naive cokernel not usually being a sheaf. Observe however that since $(-)_x$ is exact for all $x \in X$ we have that

$$H^k(F)_x \simeq H^k(F_x)$$

We can define the truncation and shift functors, in the same way as in appendix 8.3.

Definition. *Given X a topological space and an integer $m \in \mathbb{Z}$, we define*

- *The **shift functor** as the endofunctor*

$$\begin{aligned}(-)[m] \colon Ch(\mathtt{Sh}(X)) &\longrightarrow Ch(\mathtt{Sh}(X)) \\ (F^\cdot, d_F^\cdot) &\longmapsto (F[m]^\cdot, d_{F[m]}^\cdot)\end{aligned}$$

  Where $(F[m])^n = F^{n+p}$ and $d^n_{F[m]} = (-1)^m d_F^{n+p}$. For a morphism $f : F^\cdot \to G^\cdot$ we define

$$f[m] : F[m]^\cdot \to G[m]^\cdot$$

  by $(f[m])^n = f^{n+m}$

- *The **truncation functors** as the endofunctors*

$$\tau_{\leq m}, \tau_{\geq m} : Ch(\mathcal{Sh}(X)) \to Ch(\mathcal{Sh}(X))$$

Defined on a chain $F^\cdot = \cdots \to F^{m-1} \xrightarrow{d^{m-1}} F^m \xrightarrow{d^m} F^{m+1} \to \dots$ *as*

  - $\tau_{\leq m}(F^\cdot) = \cdots \to F^{m-2} \to F^{m-1} \to \mathcal{Ker}(d^m) \to 0 \to \dots$
  - $\tau_{\geq m}(F^\cdot) = \cdots \to 0 \to \mathcal{Coker}(d^{m-1}) \to F^{m+1} \to F^{m+2} \to \dots$

And they are defined in morphisms in the obvious way.

For these functors, we have proposition 8.5.6 satisfied, in particular the shift and truncation functors shift and truncate homology.

$$H^k(F^\cdot[m]) = H^{k+m}(F^\cdot)$$

$$H^k(\tau_{\leq m}F^\cdot) = \begin{cases} H^k(F^\cdot) & \text{if } k \leq m \\ 0 & \text{if } k > m \end{cases}$$

$$H^k(\tau_{\geq m}F^\cdot) = \begin{cases} H^k(F^\cdot) & \text{if } k \geq m \\ 0 & \text{if } k < m \end{cases}$$

And they are well defined in the derived category as remarked in section 8.3.3. The fact that $\mathtt{Sh}(X)$ has enough injectives means that each functor has an injective resolution

$$F \to I_F$$

Which defines a functor from $\mathtt{Sh}(X)$ to the derived category $D_+(\mathtt{Sh}(X))$, which we call $D(X)$.

$$\mathcal{I} : \mathtt{Sh}(X) \to D(X)$$

We can also *derive functors*. Recall that $(-)^\sharp$, $(-)_x$ and $f^\star$ are exact functors, so their derivatives are themselves. We also saw that $\Gamma(U, -)$ and $f_\star$ are left exact, and as such we will consider their *right derivatives* $R\Gamma(U, -)$ and $Rf_\star$.

The right derivative of $\Gamma(U, -)$ on a sheaf $F \in \mathtt{Sh}(X)$ has a special name.

Definition. *For an open set $U \subset X$, the right derivative of $\Gamma(U, -)$ on a sheaf $F \in \mathcal{Sh}(X)$ is called the **hyperhomology** of F on U and it is denoted as $\mathbb{H}^k(U, F)$.*

And it is calculated by taking resolutions.

$\mathbb{H}^k(U, F) = R^k\Gamma(U, F) = H^k(I_F^\cdot(U))$ with I_F an injective resolution of F.

The right derivative of $f_\star$ receives no special name, although it might be worth noting that it defines an adjunction

$$f^\star : D(Y) \leftrightarrows D(X) : Rf_\star$$

We will normally wish to have better sets of sheaves to perform the computations of these derived functors, since being injective is a hard property to find and prove. The following conditions present alternatives for injectivity.

For a subset $S \subset X$ we define the section of F in S as (the Kan extension)

$$\Gamma(S, F) = \operatorname*{colim}_{S \subset U} F(U)$$

We will call $\Gamma(S, F) = F(S)$ when there is no risk of confusion.

Definition. *Given $F \in \mathcal{Sh}(X)$ we say that*

1. *F is **flasque** if $F(X) \to F(U)$ is surjective for all $U \subset X$ open.*
2. *F is **soft** if $F(X) \to F(Y)$ is surjective for all $Y \subset X$ closed.*
3. *F is **c-soft** if $F(X) \to F(K)$ is surjective for all $K \subset X$ compact.*

It is proven in ([Ive86] Theorem 3.4) that flasque sheaves are $\Gamma(U, -)$-acyclic and hence they can be used to compute hyper(co)homology. It is proven in ([Bre97] Theorem II.9.11) that for paracompact spaces soft sheaves are $\Gamma(U, -)$-acyclic.

2.1.3 Deligne axioms

We have now all the ingredients to state the Deligne axioms. As said before, the idea of this axioms is to have a set of properties that characterize up to quasi isomorphism all the chains of sheaves whose hyperhomology is intersection homology. This means that we are working in the derived category of the abelian category of sheaves, which we called as $D(X)$.

Throughout this section, let X be a PL stratified pseudomanifold and $\bar{p}$ a GM-perversity. We begin by focusing on $\mathbf{IC}^{\cdot}_p(X)$.

We have seen that $\mathbf{IC}^{-m}_p(X)$ defines a sheaf for all $m \in \mathbb{N}$. What should be of no surprise now is that it defines a *chain complex* of sheaves, considering the differentials

$$\partial^m_U : IC^{\bar{p}}_m(U) \to IC^{\bar{p}}_{m-1}(U)$$

Defined in the usual way. Observe that since we are using the notation "$-m$", the resulting chain ascends in degree

$$\cdots \to \mathbf{IC}^{-m}_{\bar{p}}(X) \to \mathbf{IC}^{-m+1}_{\bar{p}}(X) \to \dots$$

Observe that this also means that the homology is written in "negative terms". With this we mean that for $k \geq 0$ we have that

$$H^{-k}(\mathbf{IC}^{\cdot}_{\bar{p}}(X)) = H^k(U \mapsto IC^{\bar{p}}_{\cdot}(U))$$

Now, for all $m \in \mathbb{N}$, the sheaf $\mathbf{IC}^{-m}_{\bar{p}}(X)$ is soft, as it proven in ([Ban07] Proposition 4.1.19).

Proposition 2.3.1. *The sheaf $\boldsymbol{IC}^{-m}_{\bar{p}}(X)$ is soft for all $m \in \mathbb{N}$.*

This has as an immediate corollary that its hyperhomology corresponds to the (Borel-Moore) PL intersection homology of the space.

Theorem 2.4. *Given X be a PL stratified pseudomanifold and $\bar{p}$ a GM-perversity, we have that*

$$\mathbb{H}^i(X, \boldsymbol{IC}^{\cdot}_{\bar{p}}(X)) \simeq IH^{\bar{p}}_i(X)$$

And more generally $\mathbb{H}^i(U, \mathbf{IC}^{\cdot}_{\bar{p}}(X)) \simeq IH^{\bar{p}}_i(U)$. We are ready now to state the Deligne axioms.

Consider X be a PL stratified pseudomanifold and $\bar{p}$ a GM-perversity. We call $U_k = X - X_k$, $i_k : U_k \hookrightarrow U_{k+1}$ and $j_k : U_{k+1} - U_k \hookrightarrow U_{k+1}$, and denote $F^{\cdot}|_Y = (\iota_Y)^\star(F)$ for $Y \overset{\iota_Y}{\hookrightarrow} X$. We state the Deligne axioms as they appear in ([Ban07] Definition 4.1.27).

Definition. *We say that a complex of sheaves $A^{\cdot}$ in the derived category $D(X)$* ***satisfies the Deligne axioms*** *if it satisfies the following four properties.*

(AX 0) $A^{\cdot}|_{U_2} \simeq \underline{R}_{U_2}[n]$ *the constant sheaf on* U_2 *shifted by* n.

(AX 1) $H^{-i}(A^{\cdot}) = 0$ *for all* $i > n$.

(AX 2) $H^{-i}(A^{\cdot}|_{U_{k+1}}) = 0$ *for all* $i < n - \bar{p}(k), k \geq 2$.

(AX 3) $H^{-i}(j_k^{\star}A^{\cdot}|_{U_{k+1}}) \to H^{-i}(j_k^{\star}R(i_k)_{\star}i_k^{\star}A^{\cdot}|_{U_{k+1}})$ *are isomorphisms for all* $i \geq n - \bar{p}(k)$ *and* $k \geq 2$.

It is easy to see that

$$S^{\cdot} = \tau_{\leq \bar{p}(n)-n}R(i_n)_{\star}\tau_{\leq \bar{p}(n-1)-n}R(i_{n-1})_{\star}\ldots\tau_{\leq \bar{p}(2)-n}R(i_2)_{\star}\underline{\mathrm{R}}_{U_2}[n]$$

satisfies the Deligne axioms. Moreover,

Lemma 2.4.1. *If $A^{\cdot}$ satisfies the Deligne axioms then*

$$A^{\cdot} \simeq S^{\cdot}$$

In the derived category $D(X)$.

Which means that the chains of sheaves that satisfy the Deligne axioms define a class of quasi isomorphism.

Proposition 2.4.1. $\boldsymbol{IC}^{\cdot}_{\bar{p}}$ *satisfies the Deligne axioms.*

Therefore, any complex of $\Gamma(U, -)$-acyclic sheaves that satisfies the Deligne axioms can be used to calculate the intersection (co)homology of a PL stratified pseudomanifold.

The proof of this last proposition is long and it goes beyond the intent of this descriptive section. The key factor of this proof is what is proven in ([Ban07] Proposition 4.1.21), that for all distinguished neighborhoods $x \in \mathbb{R}^{n-k} \times cL$ the colimit morphism $\mathbf{IC}^{-m}_{\bar{p}}(X)(\mathbb{R}^{n-k} \times cL) \to \mathbf{IC}^{-m}_{\bar{p}}(X)_x$ induces an insomorphism.

$$IH^{\bar{p}}_i(\mathbb{R}^{n-k} \times cL) \xrightarrow{\simeq} H^{-i}(\mathbf{IC}^{-m}_{\bar{p}}(X))_x$$

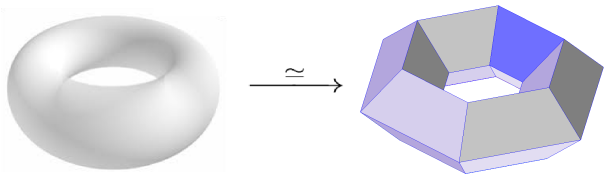
This means that $\mathbf{IC}^{-m}_{\bar{p}}(X)$ *locally satisfies cone formula.*

$$H^{-j}(\mathbf{IC}^{\cdot}_{\bar{p}}(X))_x \simeq IH^{\bar{p}}_j(\mathbb{R}^{n-k} \times cL) \simeq \begin{cases} IH^{\bar{p}}_{j-(n-k+1)}(L) & \text{if } j \geq n - \bar{p}(k) \\ 0 & \text{if } j < n - \bar{p}(k) \end{cases}$$

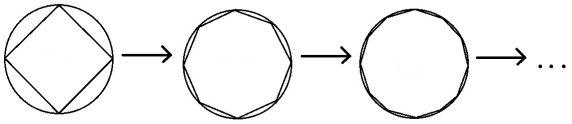
Which is basically what we need in order to obtain the axioms. This is shown in great detail throughout ([Ban07] Section 4.1.4).

2.2 Simplicial framework

As mentioned in the introduction, it has been a long tradition on algebraic topology to represent topological spaces using simplicial structures.



This tradition dates back even to the beginnings of calculus, as mathematicians thought about a good way to approximate smooth shapes with polygons.



And continuous functions with polynomials. This idea is then naturally generalized into polyhedra and simplicial complexes triangulating topological spaces, and then abstracted into considering simplicial and delta sets. One triumph of topology is the fact that now the spaces are *homeomorphic* to (the realization of) their representants in the simplicial world.

As the reader probably knows, the main idea of performing these representations is that simplicial structures are much easier to work with, since they contain much less information. So in some sense, it is a way to encompass the infinite information that a topological space contains into finite information, as it can be mathematically seen for example when defining the chain complex

$$C_n(X) = \mathbb{Z}\{\sigma : |\Delta^n| \to X | \sigma \text{ continuous}\}$$

which has an infinite base (hence, complicated calculus of the homology), vs the chain of a simplicial representation

$$C_n^{\Delta}(X) = \mathbb{Z}X_n$$

whose base contains in most practical cases finite simplices.

Flashing forward into modern times, Quillen famously expressed that there is a Quillen equivalence

$$|-| : \mathbf{SSet} \leftrightarrows \mathbf{Top} : Sing$$

Which means that the homotopical information about topological spaces can be successfully be worked out in terms of simplicial sets. This is a lot to say, since a lot of topology consists in the study of homotopy.

Simplicial sets are a beautiful thing, and this is because their definition can be made by taking functors to **Set** from one of the most basic objects in mathematics: ordered finite sets. There is a curious (meta)property in life that the more basic an object is, the more probability it has to appear in many different places (this is what happens to archetypes and also to natural numbers). Simplicial sets in fact appear as models to define ∞-categories.

Anyways, coming back to this relation **Top** $\leftrightarrows$ **SSet**, and considering what we have written in the last section, we are left to wonder

Can we state all this in simplicial terms?

That is, is there a simplicial intersection (co)homology? Are there simplicial sheaves? Is there a simplicial statement of the Deligne axioms?

The first question has been answered successfully in the works of D. Chataur, M. Saralegui-Aranguren and D. Tanré. The second question has been somewhat studied but not with a clear theory, and the third one is the main purpose of this thesis.

We are going to review now the construction of the simplicial intersection (co)homology to then state the results we are looking for. The main references for this chapters are as said, the works of D. Chataur, M. Saralegui-Aranguren and D. Tanré, mainly [CSAT18a], [CSAT21] and [CSAT20].

2.2.1 Simplicial Intersection Homology

We will explore in detail simplicial sets during the following chapter, recalling now very quickly the main definitions. Simplicial sets are objects on the presheaf category over finite ordered sets.

Definition. *We call Δ the category whose*

- *Objects are finite (ordered) sets $[n] = \{0, \dots, n\}$.*
- *Morphisms $f : [n] \to [m]$ are increasing functions.*

The category of simplicial sets corresponds to the presheaf category of Δ.

$$\mathbf{SSet} = [\Delta^{op}, \mathbf{Set}]$$

We call $X_n = X([n]) \simeq \hom(\Delta^n, X)$, with $\Delta^n = \hom_\Delta(-, [n])$. Observe that elements of Δ^n can be written as lists $(a_0, \dots, a_m)$ with $0 \leq a_i \leq a_{i+1} \leq n$, representing a m-face of Δ^n. We have a functor

$$|-| : \mathbf{SSet} \to \mathbf{Top}$$

Defined in Δ as $|\Delta^n| = \{\sum_{i=0}^n t_i \hat{e}_i | \sum_{i=0}^n t_i = 1\}$, with $\{\hat{e}_0, \dots, \hat{e}_n\}$ being a geometrically independent subset of $\mathbb{R}^n$, and in **SSet** as $|X| = \underset{\sigma:\Delta^n \to X}{\texttt{colim}} |\Delta^n|$. This functor has a right adjoint

$$Sing : \mathbf{Top} \to \mathbf{SSet}$$

Defined by $Sing(X)_n = \{\sigma : |\Delta^n| \to X | \sigma \text{ is continuous}\}$, and $Sing(\delta)(\sigma) = \sigma \circ |\delta|$ for $\delta \in \hom_\Delta([n], [m])$.

Given a ring R, we defined the homology of a simplicial set X. Set

$$C_n^\Delta(X) = RX_n$$

The group generated by X_n, with the differentials $\partial(\sigma) = \sum_k (-1)^k d_k(\sigma)$ (with $d_k = X((0, \dots, n) \mapsto (0, \dots, k-1, k+1, \dots, n))$). this defines a chain complex and we can take its (co)homology

$$H_i^\Delta(X) = H_i(C_\cdot^\Delta(X))$$

Which actually extends the homology of simplicial sets in the sense that given W a topological space, we have that

$$H_i^\Delta(Sing(W)) = H_i(W)$$

And it is also true that

$$H_i^{\Delta}(X) = H_i(|X|)$$

Giving us a great model to study topological spaces using simplicial sets. This construction can be mimed for other simplicial structures such as simplicial complexes.

What could now be the intersection (co)homology on this setting? Folowing the previous sections what we first need to do is to consider *stratified simplicial sets*. For this one could attempt to define simply $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n$, but this runs on some order problems, the best way to *stratify* X is to consider a morphism

$$X \to \Delta^n$$

Thinking X_k as the pullback of the inclusion of the face $(0, \ldots, k) \hookrightarrow \Delta^n$ along $X \to \Delta^n$. This defines a category

Definition. *$\Delta^{[n]}_{\mathcal{F}}$ is the category in which*

- *Objects are joins of n simplices, denoted as $\Delta^{(j_0,\cdots,j_n)}$ or $\Delta^{j_0} \star \cdots \star \Delta^{j_n}$, with $j_i \geq -1$ and the conventions $\Delta^{-1} = \emptyset$ and $X \star \emptyset = X$*
- *Morphisms $\sigma : \Delta^{(j_0,\cdots,j_n)} \to \Delta^{(k_0,\cdots,k_n)}$ are joins of maps of the shape $\sigma = \star_{i=0}^{n} \sigma_i$, with each $\sigma_i : \Delta^{j_i} \to \Delta^{k_i}$ an order preserving map, or the map $\emptyset \to \Delta^{k_i}$*

In other words, $\Delta^{[n]}_{\mathcal{F}} = \Delta \downarrow \Delta^n$, we call the category of **filtered simplicial sets** to the presheaf category over $\Delta^{[n]}_{\mathcal{F}}$

$$\mathbf{SSet}^{[n]}_{\mathcal{F}} = [(\Delta^{[n]}_{\mathcal{F}})^{op}, \mathbf{Set}]$$

Which is just $\mathbf{SSet} \downarrow \Delta^n$.

Consider now a GM perversity $\bar{p} : \mathbb{N} \to \mathbb{Z}$. We want to give a simplex $\sigma \in X$ a number representing its dimension of intersection with each X_k. Observe that the morphism

$$\Delta^m \xrightarrow{\sigma} X \to \Delta^n$$

Gives a stratification for σ, that is, $\sigma : \Delta^{(j_0,\cdots,j_n)} \to X$ in $\mathbf{SSet}^{[n]}_{\mathcal{F}}$. We define the **dimension** of a stratified simplex $\Delta^{(j_0,\cdots,j_n)}$ as

$$\dim(\Delta^{(j_0,\cdots,j_n)}) = j_0 + \cdots + j_n - n$$

Definition. *Consider $\sigma : \Delta^{(j_0,\cdots,j_n)} \to X$ in $\boldsymbol{SSet}^{[n]}_{\mathcal{F}}$, we call*

- *The **perverse degree** of σ is the $(n+1)$-tuple $|\sigma| = (|\sigma|_0, \ldots, |\sigma|_n)$, where*

$$|\sigma|_k = \begin{cases} \dim(\Delta^{j_0,\ldots,j_{n-k}}) & \text{if } \Delta^{j_0,\ldots,j_{n-k}} \neq \emptyset \\ -\infty & \text{if } \Delta^{j_0,\ldots,j_{n-k}} = \emptyset \end{cases}$$

- *We say that σ is $\bar{p}$**-allowable** if*

$$|\sigma|_k \leq \dim(\Delta^{(j_0,\cdots,j_n)}) - k + \bar{p}(k)$$

For all $k \in \{0, \ldots, n\}$.

- *We say that $c \in C_m^\Delta$ is $\bar{p}$-intersection if all its simplices are $\bar{p}$-allowable.*

Then we set the **simplicial intersection chain complex** to be defined on dimension m as

$$IC_m^{\bar{p}}(X) = \{c \in C_m^\Delta(X) | c \text{ and } \partial c \text{ are } \bar{p}\text{-intersection}\}$$

And with this we can define the **simplicial intersection (co)homology**.

$$IH_i^{\bar{p}}(X) = H_i(IC_\cdot^{\bar{p}}(X))$$

This once again is compatible with the topological counterpart, since if we have a stratified pseudomanifold W we can define $Sing(W)_{\bar{p}} \in \mathbf{SSet}_{\mathcal{F}}^{[n]}$ as

$$Sing_{\bar{p}}(X) = \{\sigma : \Delta^{j_0} \star \cdots \star \Delta^{j_n} \to X | \sigma \text{ continuous and } \sigma^{-1}(W_k) = \Delta^{j_0} \star \cdots \star \Delta^{j_k}\}$$

And we obtain that

$$IH_i^{\bar{p}}(Sing_{\bar{p}}(W)) = IH_i^{\bar{p}}(W)$$

A similar construction can be done for simplicial complexes, although in this case things simplify greatly since there are no order issues. This implies that we can take without a problem stratifications like $X_0 \leq \cdots \leq X_n$, and build intersection (co)homology closely miming the topological construction.

There are alternative ways to define simplicial intersection (co)homology which enhance the theory. We are going to briefly explain a particularly good one called the blown-up intersection cohomology, constructed in [CSAT18b].

Consider $C_\Delta^\star$ the dual of $C_\star^\Delta$, for any $\sigma : \Delta^{(j_0, \cdots, j_n)} \to X$ a **regular** simplex (that is, such that $\Delta^{j_n} \neq \emptyset$) we define

$$\mathcal{N}_\sigma^\star = C_\Delta^\star(c\Delta^{j_0}) \otimes \cdots \otimes C_\Delta^\star(c\Delta^{j_{n-1}}) \otimes C_\Delta^\star(\Delta^{j_n})$$

For $X \in \mathbf{SSet}_{\mathcal{F}}^{[n]}$, call $FSimp(X)$ the full subcategory of $\Delta_{\mathcal{F}}^{[n]} \downarrow X$ corresponding to regular simplices. Call

$$\mathcal{N}^\star(X) = \lim_{\sigma \in FSimp(X)} \mathcal{N}_\sigma^\star$$

This is called the **blown-up cochains** of X. We can make this complex *perverse* by introducing a degree.

Definition. *Given $\sigma : \Delta^{(j_0, \cdots, j_n)} \to X$ a regular simplex and $k \in \{1, \ldots, n\}$, we might consider the restriction*

$$r_k : \mathcal{N}_\sigma^\star \to C_\Delta^\star(c\Delta^{j_0}) \otimes \cdots C_\Delta^\star(c\Delta^{j_{n-k-1}}) \otimes C_\Delta^\star(\Delta^{j_{n-k}}) \otimes C_\Delta^\star(c\Delta^{j_{n-k+1}}) \cdots \otimes C_\Delta^\star(\Delta^{j_n})$$

For each $\gamma \in \mathcal{N}_\sigma^\star$ the cochain $r_k(\gamma) = \sum a_i \otimes b_i$ with $\{a_i\}$ being a basis of $C_\Delta^\star(c\Delta^{j_0}) \otimes \cdots \otimes C_\Delta^\star(c\Delta^{j_{n-k}})$ and $b_i \in C_\Delta^\star(c\Delta^{j_{n-k+1}}) \otimes \cdots \otimes C_\Delta^\star(\Delta^{j_n})$. We set

$$|\gamma|_k = \begin{cases} \max\{\deg(b_i) | b_i \neq 0\} & \text{if } r_k(\gamma) \neq 0 \\ -\infty & \text{if } r_k(\gamma) = 0 \end{cases}$$

*And we call the k-**th perverse degree** of a cochain $c \in \mathcal{N}^\star(X)$ to*

$$|c|_k = \sup\{|c_\sigma|_k | \sigma \in FSimp(X)\}$$

Where c_σ corresponds to the image of c on the colimit.

With this we can define the intersection blown-up cochains. We say that $c \in \mathcal{N}^\star(X)$ is $\bar{p}$**-allowable** if $|c|_k \leq \bar{p}(k)$ for all k.

Definition. *We call* $\bar{p}$***-intersection blown-up cochains*** *the complex*

$$\mathcal{N}^\star_{\bar{p}}(X) = \{c \in \mathcal{N}^\star(X) | c \text{ and } \partial c \text{ are } \bar{p}\text{-allowable}\}$$

And we call **(simplicial)** $\bar{p}$**-intersection blown-up cohomology** the homology of $\mathcal{N}^\star_{\bar{p}}(X)$. There are multiple advantages of considering the blown-up cochain complexes instead of the intersection simplicial cochains or even over the intersection cochains. The first main reason is that the blown-up intersection cohomology has a product, which is something that does not happen with simplicial intersection cohomology. More specifically, the collection $\{\mathcal{N}^\star_{\bar{p}}(X)\}_{\bar{p} \text{ perversity}}$ is a *perverse differential graded algebra* with a product

$$- \cup - : \mathcal{N}^\star_{\bar{p}}(X) \otimes \mathcal{N}^\star_{\bar{q}}(X) \to \mathcal{N}^\star_{\bar{p}+\bar{q}}(X)$$

Which induces a product on the homology.

The second important fact is that we can construct a classical sheaf in a natural way from $\mathcal{N}^\star_{\bar{p}}$, that satisfies the Deligne axioms for any conmutative ring (in contrast with the sheaf of intersection cochains $IC^\cdot_{\bar{p}}$ which only satisfies them when we are working over a field or with restrictive conditions on the links). We can perform this construction by setting **filtered simplex** to be a singular simplex $\sigma : |\Delta^k| \to X$ of a (PL) stratified pseudomanifold X, such that $\sigma^{-1}(X_j)$ a face of Δ^k. This induces a filtration $\Delta^{(j_0,\cdots,j_n)}$, and we call σ **regular** if $j_n \geq 0$. We then define as before

$$\mathcal{N}^\star_\sigma = C^\star_\Delta(c\Delta^{j_0}) \otimes \cdots \otimes C^\star_\Delta(c\Delta^{j_{n-1}}) \otimes C^\star_\Delta(\Delta^{j_n})$$

We call $FSimp(X)$ the category of regular simplices of X, and take

$$\mathcal{N}^\star(X) = \lim_{\sigma \in FSimp(X)} \mathcal{N}^\star_\sigma$$

We construct $\mathcal{N}^\star_{\bar{p}}$ as before. Now to make this into a sheaf, consider for a given open set $V \subset X$ and $\mathcal{U}$ an open cover of V, the chain $\mathcal{N}^{\star,\mathcal{U}}_{\bar{p}}(V)$, constructed in ([CSAT18b] Definition 9.6) by considering regular simplices σ with $\texttt{Im}(\sigma) \subset U$ for some $U \in \mathcal{U}$.

If $\mathcal{U}'$ is a refinement of $\mathcal{U}$, we have a restriction map $\mathcal{N}^{\star,\mathcal{U}}_{\bar{p}}(V) \to \mathcal{N}^{\star,\mathcal{U}'}_{\bar{p}}(V)$, with these maps we can take the colimit over `Cov(V)`, the category of open coverings of V. The **sheaf of blown-up cochains** $\mathbf{N}^\star_{\bar{p}}$ is defined in the sections as

$$\mathbf{N}^\star_{\bar{p}}(V) = \underset{\mathcal{U} \in \texttt{Cov}(V)}{\texttt{colim}} \mathcal{N}^{\star,\mathcal{U}}_{\bar{p}}(V)$$

It is proven in ([CSAT20] Proposition 2.6) that $\mathbf{N}^\star_{\bar{p}}$ is a complex of soft sheaves, having as a corollary that

$$\mathbb{H}(X, \mathbf{N}^\star_{\bar{p}}) \simeq IH^\star_{\bar{p}}(X, R)$$

And then in ([CSAT20] Theorem A), they prove that this sheaf satisfies the Deligne axioms as stated in [Fri10].

2.2.2 What we are after

With the construction of simplicial intersection (co)homology, one basically translates into the simplicial world section 2.1.1. The idea now is to translate into simplicial terms the rest of the sections of 2.1. That is to say,

- Define simplicial sheaves.
- Localize this category with respect to *quasi isomorphisms* of sheaves (to be defined). This should be done either deriving an abelian category or by taking the homotopy category of a model category.
- Stating a simplicial version of the Deligne axioms.
- Finding acyclic sheaves that satisfy the axioms.

3 Simplicial-type categories

As said in the introduction of section 2.2, simplicial structures follow a tradition of having *simple* structures to represent topological spaces up to homeomorphism

$$|X| \xrightarrow{\simeq} X$$

This has been successfully made for years to even come to a way to express that the homotopy structure is fully preserved by the iconic adjunction

$$|-| : \mathbf{SSet} \leftrightarrows \mathbf{Top} : Sing$$

It is of great importance for us to engage into a detailed study of the simplicial structures, since they are at the heart of our research (they are *the* objects to study), and as Taoist masters would say, a tree with strong roots will not be taken down easily.

Even though the theory of simplicial (and delta) sets and even more so the theory of simplicial complexes have been around for a while, the author will dare to consider this chapter as an important part of the thesis, since the theory is presented in a way that is in tune with the theory that follows. That is, the definitions and results given here are specially made in a way that makes the result and proofs in the following chapters feel straight forward (with the exception of the last result). Furthermore, we give proof of all the results (except the ones in section 3.2.3) because giving references will result in a hugely confusing exercise for both the author and the reader, as we develop a sort of autodidact presentation.

Because of that, we highly suggest the reader either to not skip this chapter, or to come back at it instead of visiting the literature (except for section 3.2.3), since here in a way we might say that we are developing our own simplicial language. We encourage the reader however to visit the literature to develop a wider vision on the subject, classical references include [FP90], [GJ99], and as a great introduction one can read [Fri12]. Furthermore, section 3.2.3 *is* based on [RS72].

Having that say, the author does **not** claim originality on the results given on this section, since (in the author's opinion) they can probably be found elsewhere with a different presentation, and also does not believe to hold any sort of superiority with the theory here presented, as it is simply a way to present things so that what follows feels natural and easy.

3.1 Simplicial and Delta Sets

There are multiple ways to introduce the intuition behind simplicial sets, since the subject has gone far and wide in appearances. According to wikipedia, "simplicial sets can be viewed as a higher-dimensional generalization of directed multigraphs". We are however taking the viewpoint of combinatorial representatives of topological spaces.

We start by recapitulating some results about simplicial sets that will be useful in the development of our theory.

Definition. *We call* Δ *the category whose*

- *Objects are finite sets* $[n] = \{0, \dots, n\}$

- *Morphisms $f : [n] \to [m]$ are increasing functions*

We call $\tilde{\Delta}$ the category with same objects and strictly increasing functions as morphisms.

The category of simplicial sets corresponds to the presheaf category of Δ.

$$\mathbf{SSet} = [\Delta^{op}, \mathbf{Set}]$$

And the category of delta sets corresponds to the presheaf category of $\tilde{\Delta}$.

$$\mathbf{DSet} = [\tilde{\Delta}^{op}, \mathbf{Set}]$$

We choose to make this chapter about simplicial sets mainly because they are more popular, but keep in mind that everything that we will say about simplicial sets is valid for delta sets except for some points here and there that will be remark when it corresponds.

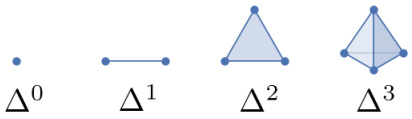
For $X \in \mathbf{SSet}$ we call $X_n = X([n])$. We have important examples of simplicial sets.

Examples:

1. The representables

$$\begin{aligned} \Delta^n = \hom_\Delta(-, [n]) : \Delta^{op} &\longrightarrow \mathbf{Set} \\ [m] \xrightarrow{\delta} [k] &\longmapsto \hom_\Delta([k], [n]) \xrightarrow{-\circ\delta} \hom_\Delta([m], [n]) \end{aligned}$$

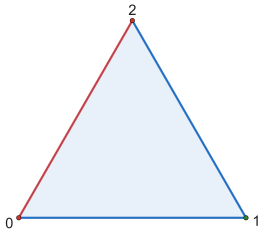
   are simplicial sets. These are better visualize as the following picture shows



   Observe that each simplex $\sigma \in \hom_\Delta([m], [n])$ can be seen as a list of numbers $(a_0, \ldots, a_m)$, with $0 \leq a_0 \leq \ldots a_m \leq n$ (with the inequality being strict on **DSet**). With this presentation, given $\delta : [k] \to [m]$, we have that

$$\begin{aligned} \Delta^n(\delta) : \Delta^n_m &\longrightarrow \Delta^n_k \\ (a_0, \ldots, a_m) &\longmapsto (a_{\delta(0)}, \ldots, a_{\delta(m)}) \end{aligned}$$

   These simplices will stand for faces of the corresponding simplex Δ^n. For example, in the picture of Δ^2, the simplex $(0, 2)$ corresponds to the upper left face



The relation between the abstract presentation $\Delta^n = \hom(-,[n])$ and the corresponding picture can be done explicitly by means of *realizing* Δ^n, that is, we set

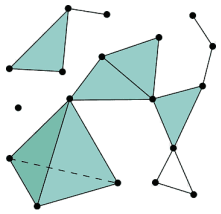
$$|\Delta^n| = \{\sum_{i=0}^{n} t_i \hat{e}_i | \sum_{i=0}^{n} t_i = 1\}$$

With $\{\hat{e}_0, \dots, \hat{e}_n\}$ being a geometrically independent (that is, $\{\hat{e}_0 - \hat{e}_j\}_{j\in\{1,\dots,n\}}$ linearly independent) subset of $\mathbb{R}^n$. We can give $|\Delta^n|$ the induced topology on $\mathbb{R}^n$, and we can also set for $\delta : [n] \to [m]$ a continuous function defined by $|\delta|(\sum_{i=0}^{n} t_i \hat{e}_i) = \sum_{i=0}^{n} t_i \hat{e}_{\delta(i)}$. On this way we have defined a functor

$$|-| : \Delta \to \mathbf{Top}$$

called the **realization functor**.

2. In fact it will be clear when we see simplicial complexes that any picture of simplices glued by faces



Will yield a simplicial set, where X_n is going to be given as the set of n-simplices of the picture. There is a way to *realize* X, relating the abstract shape with the picture, and in fact there is a realization functor

$$|-| : \mathbf{SSet} \to \mathbf{Top}$$

which we will define at the end of the subsection.

3. Given a topological space X, we can define its **Singular space** $Sing(X) \in \mathbf{SSet}$ by

$$Sing(X)_n = \{\sigma : |\Delta^n| \to X | \sigma \text{ is continuous}\}$$

And for any $\delta : [n] \to [m]$, we define $Sing(\delta)(\sigma) = \sigma \circ |\delta|$. Singular spaces are the canonical way to construct simplicial sets from topological spaces, making in fact a functor.

$$Sing : \mathbf{Top} \to \mathbf{SSet}$$

Which sends a continuous function $f : X \to Y$ to the composition map

$$\sigma \mapsto f \circ \sigma$$

The singular space functor is at the heart of homology theory, and it is not only adjoint of, but actually makes a Quillen equivalence with, the realization functor.

$$|-| : \mathbf{SSet} \leftrightarrows \mathbf{Top} : Sing$$

4. Given a partial order $(P, \leq)$, we can define its **nerve** $\mathcal{N}(P) \in \mathbf{SSet}$ by setting

   - $\mathcal{N}(P)_n = \{f : [n] \to P | f \text{ is order preserving}\}$
   - For $\delta : [n] \to [m]$, we take $\mathcal{N}(P)(\delta)(f) = f \circ \delta$

   That is, n-simplices are lists $(a_0, \ldots, a_n)$ with $a_i \in P$ and $a_0 \leq \cdots \leq a_n$, where as $\mathcal{N}(P)(\delta)$ sends $(a_0, \ldots, a_n) \mapsto (a_{\delta(0)}, \ldots, a_{\delta(n)})$. Observe the similarity with the description of Δ^n.

5. Generalizing this idea, for a category $\mathbf{C}$ we can define its **nerve** $\mathcal{N}(\mathbf{C}) \in \mathbf{SSet}$, by considering the poset $[n]$ as a category and setting

   - $\mathcal{N}(\mathbf{C})_n = \{F : [n] \to \mathbf{C} | F \text{ is a functor}\}$
   - For $\delta : [n] \to [m]$, we take $\mathcal{N}(P)(\delta)(F) = F \circ \delta$

   That is, the n-simplices are given by chains of n composable morphisms $c_0 \xrightarrow{f_0} \ldots \xrightarrow{f_{n-1}} c_n$, and we perform $\delta : [n] \to [m]$ by extracting f_i's from or adding identities to the chain according to δ.

As a presheaf category, **SSet** enjoys desirable properties that makes it be in many senses like componentwise sets, for example

Proposition 3.0.1. *For a simplicial (or delta) set X we have that*

1. *Commutativity of diagrams can be checked component-wise.*
2. *Limits and colimits exist in **SSet** and are computed component-wise.*
3. *A morphism $f : X \to Y$ is monomorphism (resp. epimorphism) if and only if $f_n : X_n \to Y_n$ is injective (resp. surjective) for all $n \in \mathbb{N}$*

Another important feature of **SSet** as a presheaf category to consider is the **Yoneda embedding**

$$\begin{aligned} h_\Delta : \Delta &\longrightarrow \mathbf{SSet} \\ [n] &\longmapsto \hom_\Delta(-, [n]) \end{aligned}$$

That is, $h_\Delta([n]) = \Delta^n$. The Yoneda lemma tells us that

$$X_n = \hom(\Delta^n, X)$$

In other words, we can identify elements of sets $\sigma \in X_n$ with morphisms $\sigma : \Delta^n \to X$. This is done in the following way:

- To $\sigma \in X_n$ we associate the morphism $\hat{\sigma} : \Delta^n \to X$ given in the component $[m]$ by

$$\begin{aligned} \hat{\sigma}_m : \hom_\Delta([m], [n]) &\longrightarrow X_m \\ ([m] \xrightarrow{\delta} [n]) &\longmapsto X(\delta)(\sigma) \end{aligned}$$

- To a morphism $\hat{\sigma} : \Delta^n \to X$ we associate the element $\sigma = \hat{\sigma}(Id_{[n]}) \in X_n$

Sometimes we will denote simplices with a hat, as $\hat{\sigma}$, to emphasize that we are considering them as morphisms, but most of the time we use indistinguishably $\sigma \in X$ to refer to either a set element and a morphism.

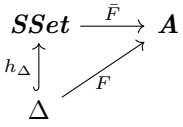
The Yoneda lemma has as a consequence that any $X \in \mathbf{SSet}$ is the colimit of its representables.

Proposition 3.0.2. *For any $X \in \boldsymbol{SSet}$*

$$X \simeq \underset{\Delta^n \xrightarrow{\sigma} X}{\mathtt{colim}}\ \Delta^n$$

This result in fact is a particular case of a more general one.

Proposition 3.0.3. *Given $\boldsymbol{A}$ a (co)complete category and a functor $F : \Delta \to \boldsymbol{A}$, there is a unique (up to isomorphism) functor $\bar{F} : \boldsymbol{SSet} \to \boldsymbol{A}$ such that the following diagram commutes.*



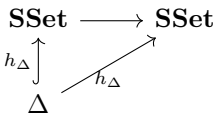
Where $\bar{F}(X) = Lan_h(F)(X) = \underset{\Delta^n \xrightarrow{\sigma} X}{\mathtt{colim}} F(\Delta^n)$

Proof. This is a result given by Kan extensions (see 8.3.1), using that $\mathbf{A}$ is (co)complete, Δ is small and h_Δ is fully faithful. □

This process let us naturally define functors by extending them from the basic level of Δ. One important example of this is process is the realization functor.

Examples:

1. We can extend the Yoneda embedding along itself



   With which we obtain that $Lan_{h_\Delta} h_\Delta(X) \simeq \underset{\Delta^n \xrightarrow{\sigma} X}{\mathtt{colim}}\ \Delta^n$. Using [Leh14], we obtain that $Lan_{h_\Delta} h_\Delta(X) = 1_{\mathbf{SSet}}$, hence obtaining that for all $X \in \mathbf{SSet}$

$$X \simeq \underset{\Delta^n \xrightarrow{\sigma} X}{\mathtt{colim}}\ \Delta^n$$

2. We define the **realization functor**

$$\begin{aligned} |-| : \mathbf{SSet} &\longrightarrow \mathbf{Top} \\ X &\longmapsto |X| = \underset{\Delta^n \xrightarrow{\sigma} X}{\mathtt{colim}} |\Delta^n| \end{aligned}$$

   By extending the corresponding functor $|-| : \Delta \to \mathbf{Top}$ defined in the examples before, in the representables as $|\Delta^n| = \{\sum_{i=0}^n t_i \hat{e}_i | \sum_{i=0}^n t_i = 1\}$, and in morphisms $\delta : [n] \to [m]$ as $|\delta|(\sum_{i=0}^n t_i \hat{e}_i) = \sum_{i=0}^n t_i \hat{e}_{\delta(i)}$.

By considering opposite categories, we can state a version of the result for contravariant functors, for which the formula looks like $\bar{F}(X) = \lim_{\Delta^n \xrightarrow{\sigma} X} F(\Delta^n)$.

This previous result should hint at the structural importance of the simplex category $(\Delta \downarrow X)^{op}$, as it not only reconstructs X itself as a colimit, it can also be used to extend any functor to a (co)complete category. It is a loose conjecture to say that just like a group is well represented in **Cat** by a groupoid with one object, a simplicial set X will be well represented by its simplex category $(\Delta \downarrow X)^{op}$, which is somewhat backed up by the following proposition.

Proposition 3.0.4. *The functor $\Delta \downarrow - : \boldsymbol{SSet} \to \boldsymbol{Cat}$ is faithful*

We will save the full definition of $\Delta \downarrow -$ and the proof of the proposition to subsection 2.2 and focus now on a definition of simplicial sets as sets which will bring a remark of vital importance for this theory.

3.1.1 Simplicial Sets as Sets

In this subsection we will explore the nature of simplicial sets as sets, and particularly to the union and intersection of subsimplicial sets which will be basic for the later topologies to consider. We also will spend some time with the `Im` operation, which is an important corner stone in what is to come, to finish the subsection with a version of the complement suited for simplicial structures that we will make us of when working with the Deligne axioms.

First let us define simplicial sets as sets with extra structure.

Definition. *A simplicial set X is a disjoint union of sets $X = \bigsqcup_{n\in\mathbb{N}} X_n$ together with functions $d_i^{n+1} : X_{n+1} \to X_n$ and $s_i^n : X_n \to X_{n+1}$ for $n \in \mathbb{N}$and$i \in \{0, \dots, n\}$, satisfying*

- $d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1}$ *for* $i < j$
- $s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n$ *for* $i \leq j$
- *Combining d and s*

$$d_i^n s_j^n = \begin{cases} 1 & \text{if } i \in \{j, j+1\} \\ s_{j-1}^n d_i^n & \text{if } i < j \\ s_j^n d_{i-1}^n & \text{if } i > j+1 \end{cases}$$

Delta sets can be defined in an almost identical way by stating the same definition without the s_i functions.

One normally calls the d_i^n's and s_j^n's as d_i and s_i, or as d_i^X and s_i^X if there is need to clarify the simplicial set. Of course this definition is equivalent to the definition of a simplicial set as objects of a presheaf category.

Observe that the functions d_i^n and s_i^n generate all $\{X(\delta) | \delta \in \mathtt{Morf}(\Delta)\}$. Let us now define morphisms between simplicial sets in this context in a way that is equivalent to the definition as natural transformations.

Definition. *A morphism of simplicial sets is a function $f : X \to Y$ such that*

- $f(X_n) \subset Y_n$ *for all* $n \in \mathbb{N}$

- $f \circ d_i^{n,X} = d_i^{n,Y} \circ f$ *for all* $n \in \mathbb{N}$ *and* $i \in \{0, \dots, n\}$
- $f \circ s_i^{n,X} = s_i^{n,Y} \circ f$ *for all* $n \in \mathbb{N}$ *and* $i \in \{0, \dots, n\}$

For such a morphism, we call $f_n = f|_{X_n}$.

The main advantage that we are going to take from this viewpoint is that simplicial sets can be regarded as sets, and therefore we can attempt to perform set operations on them. Before getting into this, we will define a facedegeneracy relation.

Definition. *Consider a simplicial (or delta) set* X *and* $\sigma, \tau \in X$*, we say that*

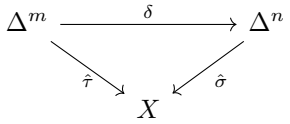
1. τ *is a* ***face*** *of* σ *if it is obtained from* σ *by performing a composition of* d_i*'s.*
2. τ *is a* ***degeneracy*** *of* σ *if it is obtained from* σ *by performing a composition of* s_i*'s.*
3. *We write* $\tau < \sigma$ *whenever* $\tau = X(\delta)(\sigma)$ *for some* $\delta \in \texttt{Morf}(\Delta)$

In other words, $\tau < \sigma$ means that τ is obtained from σ by performing a composition of d_i's and s_i's, so τ is a face or a degeneracy of a face of σ. Observe that in delta sets, $\tau < \sigma$ just means that τ is a face of σ.

This relation is a very special one in X and it will be in the core of many results to come. We state some properties of this relation.

Proposition 3.0.5. *Given* $X, Y \in$ ***SSet*** *and* $X', Y' \in$ ***DSet***

1. *For* $\tau \in X_m$ *and* $\sigma \in X_n$ *(or* $\tau \in X'_m$ *and* $\sigma \in X'_n$*), we have that* $\tau < \sigma$ *if and only if there is* $\delta : \Delta^m \to \Delta^n$ *such that the following commutes*



   That is, if $\tau \in (\Delta \downarrow X) \downarrow \sigma$.

2. $<$ *is a preorder on* X
3. $<$ *is a partial order on* X'
4. *For* $f : X \to Y$ *a morphism (or* $f : X' \to Y'$*) we have that* $\tau < \sigma \Rightarrow f(\tau) < f(\sigma)$

Proof. 1. This follows from applying the identification of Yoneda $X(\delta)(\tau) = \hat{\tau}(\delta)$ to say that $\sigma = \tau \circ h(\delta) \Leftrightarrow X(\delta)(\tau) = \sigma$.

2. The relation $<$ is reflexive since for any $\sigma \in X_n$ we have that $\sigma = X(1_{[n]})(\sigma)$, and it is transitive since if $\tau = X(\delta)(\sigma)$ and $\eta = X(\delta')(\tau)$, then $\eta = X(\delta')(X(\delta)(\sigma)) = X(\delta \circ \delta')(\sigma)$.

3. In a delta set, if $\sigma = X'(\delta)(\tau)$ and $\tau = X'(\delta')(\sigma)$, we have that $\delta \circ \delta' : [n] \to [n]$ is an increasing function, so it must be that $\delta \circ \delta' = 1_{[n]}$. Similarly, $\delta' \circ \delta = 1_{[m]}$. This means that δ and δ' are bijections, which means that $[n] = [m]$ and that $\delta = \delta' = 1_{[n]}$, so $\sigma = \tau$.

4. This is because if $\tau = X(\delta)(\sigma)$ then $f(\tau) = f(X(\delta)(\sigma)) = Y(\delta)(f(\sigma))$

□

Interestingly, the category associated to this preorder is not $\Delta \downarrow X$ in general, which will in turn both complicate and enrich the theoretical framework to come.

Let us define what a subsimplicial set is.

Definition. *A sub simplicial (or delta) set of a simplicial (or delta) set X is a subset $Y \subset X$ that satisfies*

1. *Y is a simplicial set. That is, $Y = \bigsqcup_{n\in\mathbb{N}} Y_n$ and there exist d_i^Y's and s_i^Y's satisfying the equations above.*
2. *We have compatibility with X*
   - *$Y_n \subset X_n$ for all $n \in \mathbb{N}$.*
   - *$d_i^Y = d_i^X|_Y$ and $s_i^Y = s_i^X|_Y$ for all i.*

We write $Y \leq X$ whenever $Y \subset X$ is a sub simplicial (or delta) set.

Observe that this is equivalent to say that the inclusion function $\iota_Y : Y \hookrightarrow X$ is a morphism. The relation $\leq$ forms a partial order, which in turn forms a category. We will call this category as $\mathtt{Sub}_{SSet}(X)$ or simply as $\mathtt{Sub}(X)$, a category which will be of great importance later on.

We have a very useful way to determine if a subset is a subsimplicial (or delta) set.

Proposition 3.0.6. *Given $Y \subset X$, we have that*

- *If $Y \leq X$ then $\forall \sigma \in Y$ we have $\tau < \sigma \Rightarrow \tau \in Y$*
- *If $\forall \sigma \in Y$ we have $\tau < \sigma \Rightarrow \tau \in Y$, then we can define a simplicial set structure on Y by setting*
  - *$Y_n = Y \cap X_n$ for all $n \in \mathbb{N}$*
  - *$Y(\delta) = X(\delta)|_Y$ for all $\delta \in \mathtt{Morf}(\Delta)$*

  With which we have $Y \leq X$

Proof. The first statement is true because if $\sigma \in Y$ and $\sigma < \tau$ then $\tau = X(\delta)(\iota_Y(\sigma)) = \iota_Y(Y(\delta)(\sigma))$ so $\tau \in Y$. Inversely, the property given assures that $\mathtt{Im}(X(\delta)|_Y) \subset Y$ and then $X(\delta)(\iota_Y(\sigma)) = Y(\delta)(\sigma) = \iota_Y(Y(\delta)(\sigma))$ gives that ι_Y is a morphism. □

So in the following proposition, we will determine whether some subset is a subsimplicial set by proving if it satisfies this last condition, and then we will assume that it inherits the simplicial structure described.

Proposition 3.0.7. *Let X be a simplicial set, we have that*

1. *For $Y, Z \leq X$ subsimplical sets, $Y \cap Z \leq X$ with $(Y \cap Z)_n = Y_n \cap Z_n$*
2. *For $\{Y^i\}_{i\in I}$ a family of subsimplical sets of X, $\bigcup_{i\in I} Y^i \leq X$ with $(\bigcup_{i\in I} Y^i)_n = \bigcup_{i\in I} Y_n^i$ and $\bigcap_{i\in I} Y^i \leq X$ with $(\bigcap_{i\in I} Y^i)_n = \bigcap_{i\in I} Y_n^i$*

3. *For a morphism $f : X \to X'$ of simplicial sets, $\mathtt{Im}(f) \leq X'$ with $\mathtt{Im}(f)_n = \mathtt{Im}(f_n)$*

4. *For a morphism $f : X \to X'$ of simplicial sets and subsimplicial sets $Z \leq X$ and $Z' \leq X'$ we have that $f^{-1}(Z') \leq X$ and $f(Z) = \mathtt{Im}(f \circ \iota_Z) \leq X'$ (where $\iota_Z : Z \hookrightarrow X$ is the inclusion)*

From 1. and 2. of this proposition we have an interesting corollary.

Corollary 3.0.1. *$(X, \mathtt{Sub}(X))$ is a topological space.*

And we also have of course many topological spaces whose open sets are all subsimplicial sets. Let us see a proof of the proposition now.

Proof. 1. We use that the intersection is a pullback and limits in **SSet** are compute componentwise as limits in sets.

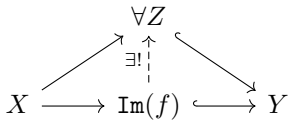
2. Given $\sigma \in \bigcup_{i\in I} Y^i$, say $\sigma \in Y^{i_0}$. Then for any $\tau < \sigma$, since $Y^{i_0} \leq X$, we have $\tau \in Y^{i_0} \subset \bigcup_{i\in I} Y^i$, and $(\bigcup_{i\in I} Y^i)_n = X_n \cap \bigcup_{i\in I} Y^i = \bigcup_{i\in I} X_n \cap Y^i = \bigcup_{i\in I} Y^i_n$. The proof for the intersection is analogous.

3. If $\sigma \in \mathtt{Im}(f)$ and $\tau < \sigma$, we have that $\tau = X(\delta)(\sigma) = X(\delta)(f(\eta)) = f(X'(\delta)(\eta))$ and therefore $\tau \in \mathtt{Im}(f)$

4. If $f(\sigma) \in Z'$ and $\tau < \sigma$ then $f(\tau) < f(\sigma)$ by the previous observation and since $Z' \leq X$ then $f(\tau) \in Z'$. The second part of this statement is a consequence of 3.

□

We will now take a closer look into $\mathtt{Im}$. Every morphism $f : X \to Y$ has a canonical decomposition $X \xrightarrow{f} \mathtt{Im}(f) \hookrightarrow Y$ which satisfies the universal property of image



Which in turn comes from the corresponding universal property in **Set**. Observe that this means that when f is a monomorphism, then $\mathtt{Im}(f) \simeq X$.

There is one special case for which the image will be very important, that is, for a simplex $\sigma : \Delta^n \to X$. There is a lot to say about $\mathtt{Im}(\sigma)$. In a very specific sense, it is the best way to get a subsimplicial set out of the simplex.

Definition. *For a subset $S \subset X$ we call*

$$< S >= \bigcap_{S \subset Y \leq X} Y$$

*The **simplicial set generated by** S.*

And we have the following properties of $\mathtt{Im}(\sigma)$.

Proposition 3.0.8. *Let $\sigma : \Delta^n \to X$ be an element of X and $Y \leq X$, then*

1. *$\sigma \in Y$ if and only if $\mathtt{Im}(\sigma) \leq Y$*

2. $< \sigma >= \mathit{Im}(\sigma)$

3. *If* σ *is mono then* $\mathit{Im}(\sigma) \simeq \Delta^n$

4. $Y = \bigcup_{\sigma \in Y} \mathit{Im}(\sigma)$

Proof. The first and third statements come directly from the universal property of the image. Observe first that $\sigma = \hat{\sigma}(Id_n) \in \mathtt{Im}(\sigma)$, then if $\mathtt{Im}(\sigma) \leq Y$ we will have that $\sigma \in Y$. On the other hand, if $\sigma \in Y$ that means we have the decomposition $\sigma = \Delta^n \xrightarrow{\sigma} Y \hookrightarrow X$ so by the universal property there is a monomorphism $\mathtt{Im}(\sigma) \hookrightarrow Y$.

For the second statement, since $\sigma \in \mathtt{Im}(\sigma)$ then $< \sigma > \leq \mathtt{Im}(\sigma)$, and by the first statement any $Y \leq X$ containing σ will satisfy $\mathtt{Im}(\sigma) \leq Y$, hence $\mathtt{Im}(\sigma) \leq \bigcap_{\sigma \in Y \leq X} Y =< \sigma >$.

The last statement comes directly from the first. □

Observe that since the Yoneda embedding gives that $X(\delta)(\sigma) = \hat{\sigma}(\delta)$, we have that $\tau < \sigma \Leftrightarrow \tau \in \mathtt{Im}(\sigma)$, and therefore as a set

$$< \sigma >= \mathtt{Im}(\sigma) = \{\tau \in X | \tau < \sigma\}$$

So $\mathtt{Im}(\sigma) \leq X$ is the subsimplicial set formed by all faces (and degeneracies of faces) of σ. Observe that this means that we can consider σ to be non-degenerate when writing $\mathtt{Im}(\sigma)$.

This also means that $\mathtt{Im}$ reflects $<$ very well, since

$$\tau < \sigma \Leftrightarrow \tau \in \mathtt{Im}(\sigma) \Leftrightarrow \mathtt{Im}(\tau) \leq \mathtt{Im}(\sigma)$$

Since we are going to use this latter, we will write it as a proposition, together with one other useful result.

Proposition 3.0.9. *Let* $X \in \boldsymbol{SSet}$ *and* $\sigma, \tau \in X$*. Let* $f : X \to Y$ *be a simplicial set morphism, then*

1. $\tau < \sigma$ *if and only if* $\mathit{Im}(\tau) \leq \mathit{Im}(\sigma)$

2. *The morphism* $\hat{f(\sigma)}$ *associated to the element* $f(\sigma)$ *is the composition* $\hat{f(\sigma)} = f \circ \hat{\sigma}$

3. $f(\mathit{Im}(\sigma)) = \mathit{Im}(f(\sigma))$

Proof. We already have the first statement. For the second statement and third, first note that $\hat{f(\sigma)}(\delta) = Y(\delta)(f(\sigma)) = f(X(\delta)(\sigma)) = f(\hat{\sigma}(\delta))$, so $\hat{f(\sigma)} = f \circ \hat{\sigma}$.

Given $f(\hat{\sigma}(\delta)) \in f(\mathtt{Im}(\sigma))$, we have that $f(\hat{\sigma}(\delta)) = \hat{f(\sigma)}(\delta) \in \mathtt{Im}(f(\sigma))$ and vice versa. □

The next operation we might want to perform is complement, and particularly for us this operation has relevance since when working with Deligne axioms we are considering open sets $U_k = X - X_{n-k}$ obtained by subtracting strata.

Sadly, the subtraction of simplicial sets as sets $X - Y$ is not a simplicial set except for insipid cases. However, the closest possible option, which we will denote as

$$X -_\Delta Y = \{\sigma \in X | \mathtt{Im}(\sigma) \cap Y = \emptyset\}$$

is a simplicial set and has many of the properties that the set complement has, situation that gets even better in simplicial complexes when we subdivide.

Proposition 3.0.10. *Given $Y, Y', Z, Z' \leq X$*

1. $Y -_\Delta Z = \bigcup\{W \subset Y - Z | W \leq X\} \leq Y$
2. $Y -_\Delta \emptyset = Y$ *and* $Y -_\Delta Y = \emptyset$ *(this is a version of* $X^c = \emptyset$ *and* $\emptyset^c = X$*)*
3. *If* $Z \leq Z'$ *then* $Y -_\Delta Z' \leq Y -_\Delta Z$
4. $Y -_\Delta Z = Y \cap (X -_\Delta Z)$
5. *Morgan Laws:*
   (a) $X -_\Delta (Y \cup Z) = (X -_\Delta Y) \cap (X -_\Delta Z)$
   (b) $(X -_\Delta Y) \cup (X -_\Delta Z) \leq X -_\Delta (Y \cap Z)$
6. $Z -_\Delta Y \leq (X -_\Delta Y) -_\Delta (X -_\Delta Z)$ *(in particular* $Z \leq X -_\Delta (X -_\Delta Z)$*. Equality does not hold as we will see)*
7. *If* $Y \leq Z$ *then* $Y -_\Delta Z = \emptyset$
8. *If* $Z' \leq Z$ *then* $Y -_\Delta Z = (Y -_\Delta Z') -_\Delta Z$
9. $(X -_\Delta Y) -_\Delta Z = X -_\Delta (Y \cup Z)$
10. *Others*
    (a) $Y \cap (Z -_\Delta Z') = (Y \cap Z) -_\Delta (Y \cap Z')$
    (b) $(Z \cup Z') -_\Delta Y = (Z -_\Delta Y) \cup (Z' -_\Delta Y)$
    (c) $(Z \cap Z') -_\Delta Y = (Z -_\Delta Y) \cap (Z' -_\Delta Y)$
    (d) $Y -_\Delta (Y \cap Z) = Y -_\Delta Z = (Y \cup Z) -_\Delta Z$
    (e) $(Y -_\Delta Y') \cap (Z -_\Delta Z') = (Z -_\Delta Y') \cap (Y -_\Delta Z')$

Proof. 1. It is obvious that $Y -_\Delta Z \subset Y$. To see that it is a simplicial set notice that if $\sigma \in Y -_\Delta Z$ and $\tau < \sigma$ then $\mathtt{Im}(\tau) \leq \mathtt{Im}(\sigma)$ and then $\mathtt{Im}(\sigma) \cap Z \subset \mathtt{Im}(\tau) \cap Z = \emptyset$.

We see now that $Y -_\Delta Z$ is optimal. It is obvious that $Y -_\Delta Z \subset Y - Z$. Furthermore if we have a simplicial set $W \subset Y - Z$ and $\sigma \in W$, we will have that $\mathtt{Im}(\sigma) \leq W \subset Y - Z$ and therefore $\mathtt{Im}(\sigma) \cap Z = \emptyset$ and then $\sigma \in Y -_\Delta Z$.

2. $Y -_\Delta \emptyset = \{\sigma \in Y | \mathtt{Im}(\sigma) \cap \emptyset = \emptyset\} = Y$ and $Y -_\Delta Y \subset Y - Y = \emptyset$
3. This comes from $\mathtt{Im}(\sigma) \cap Z \subset \mathtt{Im}(\sigma) \cap Z' = \emptyset$
4. This comes directly from the definitions.
5. The first Morgan law comes from the fact that $\mathtt{Im}(\sigma) \cap (Z \cup Y) = (\mathtt{Im}(\sigma) \cap Z) \cup (\mathtt{Im}(\sigma) \cap Y)$ is empty if only if $\mathtt{Im}(\sigma) \cap Y$ and $\mathtt{Im}(\sigma) \cap Z$ are empty. As for the second, if let us say $\mathtt{Im}(\sigma) \cap Y$ is empty, then $\mathtt{Im}(\sigma) \cap Y \cap Z$ is empty.

6. Given $\sigma \in Z$ with $\texttt{Im}(\sigma) \cap Y = \emptyset$, we need to prove that $\texttt{Im}(\sigma) \cap (X -_\Delta Z) = \emptyset$. Say that $\tau \in \texttt{Im}(\sigma) \cap (X -_\Delta Z)$, since $\sigma \in Z$ then $\texttt{Im}(\sigma) \leq Z$ and $\tau \in \texttt{Im}(\sigma) \Rightarrow \texttt{Im}(\tau) \leq \texttt{Im}(\sigma) \leq Z$, which contradicts that $\tau \in X -_\Delta Z$ (notice that $\texttt{Im}(\tau)$ is not empty since $\tau \in \texttt{Im}(\tau)$)

7. This comes from the fact that $\sigma \in Z$ implies that $\texttt{Im}(\sigma) \cap Z \neq \emptyset$ (notice that $\texttt{Im}(\sigma)$ is not empty since $\sigma \in \texttt{Im}(\sigma)$)

8. Since $Y -_\Delta Z' \leq Y$ we have that $(Y -_\Delta Z') -_\Delta Z \leq Y -_\Delta Z$. On the other direction, given $\sigma \in Y -_\Delta Z$, we have that $\sigma \in Y$ and $\texttt{Im}(\sigma) \cap Z = \emptyset$, which implies that $\texttt{Im}(\sigma) \cap Z' = \emptyset$ since $Z' \leq Z$, and these three last properties mean that $\sigma \in (Y -_\Delta Z') -_\Delta Z$.

9. This comes from the fact that $\texttt{Im}(\sigma) \cap (Z \cup Y) = (\texttt{Im}(\sigma) \cap Z) \cup (\texttt{Im}(\sigma) \cap Y) = \emptyset$ if and only if $\texttt{Im}(\sigma) \cap Z = \emptyset$ and $\texttt{Im}(\sigma) \cap Y = \emptyset$.

10. For statement (a), one inclusion comes from $\texttt{Im}(\sigma) \cap (Y \cap Z') \subset \texttt{Im}(\sigma) \cap Z'$ and the other inclusion follows from the fact that if $\sigma \in Y \cap Z \subset Y$ then $\texttt{Im}(\sigma) \leq Y$ and therefore $\texttt{Im}(\sigma) \cap Y \cap Z' = \emptyset \Rightarrow \texttt{Im}(\sigma) \cap Z' = \emptyset$.

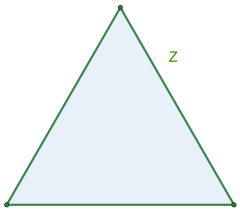
    Statemens (b), (c) and (e) are straightforward. For the first equality of (e), one inclusion is a consequence of 3. using $Y \cap Z \leq Z$ and the second inclusion comes from the fact that if $\sigma \in Y$ then $\texttt{Im}(\sigma) \leq Y$ and then $\texttt{Im}(\sigma) \cap Y \cap Z = \emptyset$ implies $\texttt{Im}(\sigma) \cap Z = \emptyset$. For the second equality $Y -_\Delta Z \leq (Y \cup Z) -_\Delta Z$ is obvious and for the other inclusion notice that if $\texttt{Im}(\sigma) \cap Z = \emptyset$ then σ is not in Z since $\sigma \in \texttt{Im}(\sigma)$.

□

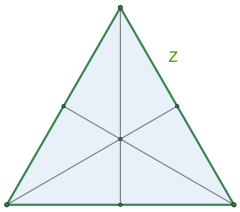
In other words, $-_\Delta$ functions as $-$ except for three important details, which is that in general we are missing the following

1. $X -_\Delta (Y \cap Z) \leq (X -_\Delta Y) \cup (X -_\Delta Z)$
2. $X -_\Delta (X -_\Delta Z) \leq Z$
3. $Y -_\Delta Z = \emptyset \Longrightarrow Y \leq Z$

A counterexample for the Morgan law is to take $X = \Delta^1$ and Y and Z as the two points of its boundary $\partial\Delta^1$. As for the other two properties $\partial\Delta^2 \leq \Delta^2$ provide counterexamples.



The problem is that when we perform $X -_\Delta Z$, together with Z we are also taking off an aura around it (technically a star), which in the example $\partial\Delta^2 \leq \Delta^2$ will take off all Δ^2. This problem can be solved by subdividing, as we will see in the simplicial complex section.



3.1.2 The category of simplices

Let X be a simplicial (or delta) set. As mentioned before, the comma category $(\Delta \downarrow X)^{op}$ (for sheaf theoretic reasons we are more interested in the opposite category) will reflect structurally most of the features of a simplicial (or delta) set X, note for example that as a set

$$X = \bigsqcup_{n\in\mathbb{N}} X_n = \bigsqcup_{n\in\mathbb{N}} \hom(\Delta^n, X) = Obj((\Delta \downarrow X)^{op})$$

And it will also mirror, as we saw with the preorder we introduced $<$, the face/degeneracy structure of X. We will get ambitious and present the functor $\Delta \downarrow - : \mathbf{SSet} \to \mathbf{Cat}$ as an excuse to call the category $\mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{A})$ as the category of representations of X over $\mathbf{A}$.

Let us start by recalling the objects and morphisms of this category

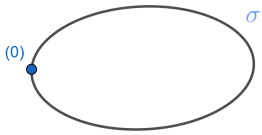
- Objects of $\Delta \downarrow X$ are morphisms $\sigma : \Delta^n \to X$ (that is, by Yoneda lemma, elements $\sigma \in X$
- Morphisms are commutative diagrams $\hom_{\Delta\downarrow X}(\sigma, \tau) = \{\delta \in \mathtt{Morf}(\Delta) | X(\delta)(\tau) = \sigma\} = \hom_{(\Delta\downarrow X)^{op}}(\tau, \sigma)$

Observe that the definition of morphisms is not exactly direct from the definition of a comma category. In fact we used the Yoneda identification $X(\delta)(\tau) = \hat{\tau}(\delta)$ to say that $\sigma = \tau \circ h(\delta) \Leftrightarrow X(\delta)(\tau) = \sigma$.

Presenting the morphisms this way has the advantage of making sense with respect to what we have seen, since $X(\delta)(\sigma) = \tau$ means that $\tau < \sigma$, so

$$\tau < \sigma \iff \hom_{\Delta\downarrow X}(\tau, \sigma) \neq \emptyset$$

In ordinary words, the fact that exists an arrow $\tau \to \sigma$ in $\Delta \downarrow X$ means that τ is a face (or degeneracy of face) of σ. So $\hom_{(\Delta\downarrow X)^{op}}(\sigma, \tau)$ contains the information of whether $\tau < \sigma$, but it goes further as to have the information about *all the ways in which this can happen*, which might sound weird when thinking about simplicial complexes since on a simplicial complex $\tau < \sigma$ can be face of σ in a unique way, but in simplicial sets it is a rather common situation as in the following picture



The fact on simplicial complexes that $\tau < \sigma$ can be face of σ only in one way can be encapsulated in the following statement.

Proposition 3.0.11. *$\sigma \in X$ is monomorphism if and only if $\#\hom_{\Delta\downarrow X}(\tau, \sigma) \leq 1$ for all $\tau \in X$*

Proof. Suppose that $\sigma : \Delta^n \to X$ is a monomorphism and let $\delta_1, \delta_2 \in \hom_{\Delta\downarrow X}(\tau, \sigma)$ for some $\tau \in X$. Then $\sigma \circ h(\delta_1) = \tau = \sigma \circ h(\delta_2) \Rightarrow h(\delta_1) = h(\delta_2) \Rightarrow \delta_1 = \delta_2$ since σ is monomorphism.

For the other direction, we can prove that $\sigma_k : \Delta^n_k \to X_k$ is injective for each $k \in \mathbb{N}$. So let $\delta_1, \delta_2 \in \Delta^n_k$ and suppose $\sigma(\delta_1) = \sigma(\delta_2)$, which is to say that $X(\delta_1)(\sigma) = X(\delta_2)(\sigma)$. This means that $\delta_1, \delta_2 \in \hom(X(\delta_1)(\sigma), \sigma)$ and then we conclude using the hypothesis that $\delta_1 = \delta_2$ □

Recall that if $\sigma : \Delta^n \to X$ is a monomorphism then $\texttt{Im}(\sigma) \simeq \Delta^n$, which would not happen in just any case (for example in the picture before $|\texttt{Im}(\sigma)| \simeq S^1$ so it cannot be Δ^1). We make a special definition for this kind of simplex

Definition. *We call a simplex $\sigma : \Delta^n \to X$ geo friendly if $\mathit{Im}(\sigma) \simeq \Delta^n$*

So any monomorphic simplex is geo friendly. We will see soon that actually geo friendly simplicies are monomorphic.

There seems to be a consistency that a geo friendly element brings to its faces/degeneracies which we will shortly explore now.

We present two main candidates to be "the category of face/degeneracies" of a simplex $\tau : \Delta^n \to X$, which are $\Delta \downarrow \texttt{Im}(\tau)$ and $(\Delta \downarrow X) \downarrow \tau$. Both of them contain information about $\{\sigma \in X | \sigma < \tau\}$, but with a subtle (and as it turns out mayor) difference: The objects of $(\Delta \downarrow X) \downarrow \tau$ are tuples (σ, α) with $\sigma < \tau$ and $\alpha : \sigma \to \tau$. In other words the information that objects of $(\Delta \downarrow X) \downarrow \tau$ carry includes in which way σ is a face (or degeneration of a face) of τ. This means that $\Delta \downarrow \texttt{Im}(\tau)$ is the category of face/degeneracies with the information on how the faces are faces, information that $(\Delta \downarrow X) \downarrow \tau$, and in fact there is a forgetful functor

$$\textsc{f} : (\Delta \downarrow X) \downarrow \tau \to \Delta \downarrow \texttt{Im}(\tau)$$

that takes (σ, α) to σ.

Now, when τ is a monomorphism, the hole "in which way $\sigma < \tau$" thing becomes immaterial, and this is shown in the following result

Proposition 3.0.12. *$\tau \in X$ is geo friendly if and only if we have an isomorphism of categories $(\Delta \downarrow X) \downarrow \tau \simeq \Delta \downarrow \mathit{Im}(\tau)$*

A counterexample for $(\Delta \downarrow X) \downarrow \tau \simeq \Delta \downarrow \texttt{Im}(\tau)$ being always true is a non-geometrical simplex such as the one in the picture shown before.

For τ a monomorphism, an isomorphism $(\Delta \downarrow X) \downarrow \tau \simeq \Delta \downarrow \mathtt{Im}(\tau)$ is actually provided by F, but we will not prove that here (directly at least). We will intead proceed to the proof of the proposition starting from the following lemma which reflects the fact that working in a "τ environment" feels like working on $\Delta^{\mathtt{dim}(\tau)}$.

Lemma 3.0.1. *For any simplex $\tau : \Delta^n \to X$ we have an isomorphism*

$$(\Delta \downarrow X) \downarrow \tau \simeq \Delta \downarrow \Delta^n$$

Then the result follows from the fact that $X \simeq Y \Leftrightarrow \Delta \downarrow X \simeq \Delta \downarrow Y$, which will hopefully be obvious soon. Let us proof the lemma now.

Proof. We use a forgetful functor $\Phi : (\Delta \downarrow X) \downarrow \tau \to \Delta \downarrow \Delta^n$ that takes $(\sigma, \alpha) \mapsto \alpha$ and that takes a morphism $\delta : (\sigma, \alpha) \to (\sigma', \alpha')$ to $\delta : \alpha \to \alpha'$. From the construction it is obvious that it is functorial.

What it's not very obvious is that it has an inverse $\Psi : \Delta \downarrow \Delta^n \to (\Delta \downarrow X) \downarrow \tau$ that acts on objects as $\Psi(\alpha) = (\tau \circ \alpha, \alpha)$ and on morphisms as $\Psi(\alpha \xrightarrow{\delta} \alpha') = (\tau \circ \alpha, \alpha) \xrightarrow{\delta} (\tau \circ \alpha', \alpha')$. Ψ is well defined since $\alpha = \alpha' \circ \delta \Rightarrow \tau \circ \alpha = \tau \circ \alpha' \circ \delta$ and by construction it is obvious that Ψ is functorial.

Now $\Phi \circ \Psi = Id$ since on objects $\Phi(\Psi(\alpha)) = \Phi((\tau \circ \alpha, \alpha)) = \alpha$ and on morphisms $\Phi(\Psi(\delta)) = \Phi(\delta) = \delta$. $\Psi \circ \Phi = Id$ since on objects $\Psi \circ \Phi(\sigma, \alpha) = \Psi(\alpha) = (\tau \circ \alpha, \alpha)$ and $\tau \circ \alpha = \sigma$ since that is what (σ, α) means (and on morphisms $\Psi \circ \Phi(\delta) = \Psi(\delta) = \delta$.

Then the categories are isomorphic.

□

We can encapsulate all our discussion on the following theorem.

Theorem 3.1. *For a simplex $\tau : \Delta^n \to X$ the following are equivalent*

1. *τ is a monomorphism*
2. *$\mathtt{Im}(\tau) \simeq \Delta^n$ (that is, τ is geo friendly)*
3. *$\#\hom_{\Delta \downarrow X}(\sigma, \tau) \leq 1$ for all $\sigma \in X$*
4. *$(\Delta \downarrow X) \downarrow \tau \simeq \Delta \downarrow \tau$*

Proof. We are only left to prove that a geo friendly simplex is a monomorphism. Suppose that we have an isomorphism $l : \mathtt{Im}(\tau) \xrightarrow{\sim} \Delta^n$ and consider $\tau' : \Delta^n \xrightarrow{\tau} \mathtt{Im}(\tau) \xrightarrow{l} \Delta^n$. We have that τ' is epimorphism since $\Delta^n \xrightarrow{\tau} \mathtt{Im}(\tau)$ is, and since l is an isomorphism τ' mono $\Leftrightarrow \tau$ mono.

In other words we moved the proof to $\tau' : \Delta^n \twoheadrightarrow \Delta^n$, and now the proof is easy. Since τ' is epimorphism, $\tau'(1_{[n]}) = 1_{[n]}$, which means that τ' is monomorphism. □

A delta set X is called geo friendly if all its elements are geo friendly. A simplicial set X is called geo friendly if all non-degenerate simplices are geo friendly (notice that degeneracies are not going to be monomorphisms). Simplicial complexes are an example of geo friendly simplicial structures as we will see in section 3.2.

Geo friendly delta sets behave particularly well. For example a morphism of geo friendly delta sets is just a function $f : X \to Y$ such that $\tau < \sigma \Rightarrow f(\tau) < f(\sigma)$ for all $\tau, \sigma \in X$. They are going to be top of the class in our theory.

3.1.3 The functor $\Delta \downarrow - : \mathbf{SSet} \to \mathbf{Cat}$

Last, but definitely not least, on this section we prove the result we left on suspense some pages ago, that the functor $\Delta \downarrow - : \mathbf{SSet} \to \mathbf{Cat}$ is faithful.

Perhaps the most useful characteristic on this result is the action of this functor on morphisms. Given a morphism $f : X \to Y$ we construct a very natural functor $\Delta \downarrow f : \Delta \downarrow X \to \Delta \downarrow Y$ that takes a simplex $\sigma : \Delta^n \to X$ to the composition $f \circ \sigma$ (and a morphism $\delta : \sigma \to \tau$ to $\delta : f \circ \sigma \to f \circ \tau$). The functoriality of $\Delta \downarrow -$ is clear.

Notice that since $\widehat{f(\sigma)} = f \circ \hat{\sigma}$, $\Delta \downarrow f$ is basically just applying f plus arranging names according to Yoneda, and for this reason when there is no risk of confusion we will call $\Delta \downarrow f$ simply as f. The fact that it takes morphisms of $\Delta \downarrow X$ to themselves means that when $\sigma < \tau$ we will have that $f(\sigma)$ is a face (or degeneration of a face) of $f(\tau)$ *in the same way* that σ is of τ, which is the stronger version of the statement $\sigma < \tau \Rightarrow f(\sigma) < f(\tau)$.

It should not be a mystery then that $\Delta \downarrow f$ inherits the properties of f.

Proposition 3.1.1. *Let $f : X \to Y$ a morphism, then*

1. *$\Delta \downarrow f$ is faithful.*
2. *$\Delta \downarrow f = \Delta \downarrow g \Longrightarrow f = g$*
3. *If f is a monomorphism then $\Delta \downarrow f$ is fully faithful (and injective on objects).*
4. *If f is an isomorphism then $\Delta \downarrow f$ is an isomorphism of categories.*

Proof. The first statement is obvious since $\Delta \downarrow f$ takes morphism to themselves.

For the second statement, notice that $\Delta \downarrow f = \Delta \downarrow g$ is saying that for all $\sigma \in X$ we have $f \circ \sigma = g \circ \sigma$ which translated by Yoneda means that $f(\sigma) = g(\sigma)$.

The third statement is true because if we have $f \circ \sigma \xrightarrow{\delta} f \circ \tau$ then $f \circ \sigma \circ \delta = f \circ \tau$ and then if f is mono $\sigma \circ \delta = \tau$ which is to say $\sigma \xrightarrow{\delta} \tau$.

For the last statement we have that $\Delta \downarrow f^{-1}$ is an inverse. □

Corollary 3.1.1. $X \simeq Y \Longleftrightarrow \Delta \downarrow X \simeq \Delta \downarrow Y$

Proof. The direction $\Rightarrow$ is direct from the last proposition. As for the other direction notice that $X \simeq \underset{\sigma \in \Delta \downarrow X}{\mathtt{colim}} \Delta^{\dim(\sigma)} \simeq \underset{\sigma \in \Delta \downarrow Y}{\mathtt{colim}} \Delta^{\dim(\sigma)} \simeq Y$ □

Finally, notice that the second statement of the proposition is saying that the functor $\Delta \downarrow - : \mathbf{SSet} \to \mathbf{Cat}$ is faithful, and by the corollary it is essencially injective on objects.

It would be "crème de la crème" if on top this functor was fully faithful. This is not the case sadly, as for example a constant functor $cte : \Delta \downarrow X \to \Delta \downarrow Y$ or other functors that don't respect dimension do not have preimage.

A very interesting question to ask is whether $\Delta \downarrow -$ has adjoints. It would be particularly nice to have a right adjoint, since as soon seen one mayor object of interest for us is the functor category $\mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{A})$, and if $\Delta \downarrow -$ has a right adjoint, say F, then we would have bijections

$$\mathtt{Func}(\Delta \downarrow X, \mathbf{A}) \simeq \hom_{\mathbf{SSet}}(X, \mathrm{F}(\mathbf{A}))$$

natural in X and $\mathbf{A}$. So we could for example calculate $\mathrm{F}(Ch(R))$ and then have a way to study these functors using simplicial set theory, which is much simplier.

The author has no answer to this question since it was never necessary to answer it and it runs into set-size issues. Therefore it remains a conjecture.

3.2 Simplicial complexes and PL topology

Simplicial complexes are much easier to work with than simplicial or even delta sets. This is essentially due to their geometrical nature and the fact that each simplex is defined *uniquely* by the set of its vertices, making many calculations that in simplicial sets would represent a pain feel easy.

We present the framework and basic results needed to work on the last chapter.

3.2.1 Simplicial complexes as simplicial (or delta) sets

There are many ways to define what is a simplicial complex. Essentially, they are a bunch of simplices sticked together by faces, which is encapsulated in our first definition. Let us make a couple of definitions first.

We call **topological simplex** of dimension $n \in \mathbb{N}$ a topological space linearly homeomorphic to

$$\{\sum_{i=0}^{n} \lambda_i e_i | \sum \lambda_i = 1\}$$

Where $\{e_0, \dots, e_n\}$ are a geometrically independent set of elements of $\mathbb{R}^n$. A **face** of such simplex, written as $\tau < \sigma$, is the convex hull of a subset of $\{e_0, \dots, e_n\}$.

Definition. *A **simplicial complex** K is a set of topological simplices in some euclidean space $\mathbb{R}^N$ such that*

1. *If $\sigma \in K$ and $\tau < \sigma$ then $\tau \in K$*
2. *For $\sigma, \tau \in K$ the topological space $\sigma \cap \tau$ is a face of σ and of τ*

And we call $|K| = \bigcup K$ the **realization** of K.

As said, simplicial complexes are a way to combinatorially encapsulate certain information of a topological space, so it should be no surprise that there is a combinatorial way to define a simplicial complex, sometimes called **abstract simplicial complex**.

Definition. *A **(abstract) simplicial complex** K is the information of a set (of vertices) $V(K)$ and a set (of simplices) $S(K) \subset \mathcal{P}(V(K)) - \emptyset$ such that*

1. *$\{v\} \in S(K)$ for all $v \in V(K)$*
2. *If $\sigma \in S(K)$ and $\tau \subset \sigma$ then $\tau \in S(K)$*

The idea is that each set $\sigma \in S(K)$ corresponds to a simplex of K and the elements of $\sigma \subset V(K)$ are its vertices. We say that τ is a **face** of σ (written as $\tau < \sigma$) if $\tau \subset \sigma$. Observe that this forms a partial order.

The equivalence of the definitions is standard knowledge and we assume that the reader has no problem going between them. The simplicial complexes form a category that we will call **Scx**, with the following morphisms

Definition. *A **morphism** of simplicial complexes $f : K \to L$ is a function $f : V(K) \to V(L)$ such that $f(\sigma) \in S(L)$ for all $\sigma \in K$*

Given a morphism $f : K \to L$ we can define $\bar{f} : S(K) \to S(L)$ as $\bar{f}(\sigma) = f(\sigma)$. We will treat $\bar{f}$ and f indistinguishably.

There is a way to identify a simplicial complex as a particular kind of simplicial (or delta) set. To see this we will recapitulate a way to obtain a simplicial complex from a simplicial (or delta) set and viceversa.

First it will be useful to give a name to the function

$$\begin{aligned} \delta_k : [0] &\longrightarrow [n] \\ 0 &\longmapsto k \end{aligned}$$

Now, given a simplicial (or delta) set X we can define a simplicial complex $sc(X)$ given by

- $V(sc(X)) = X_0$
- $S(sc(X)) = \{\{\sigma(\delta_0), \ldots, \sigma(\delta_n)\} | \sigma \in X\}$

Observe that this in fact defines a functor $sc : \mathbf{SSet} \to \mathbf{Scx}$ with $sc(f) = f_0$.

To go in the other direction is less canonical, in the sense that it is very messy to give a functor $\mathbf{Scx} \to \mathbf{SSet}$. Instead, we assume that $K \in \mathbf{Scx}$ is given with an order of the vertices $V(K)$

Then we can construct a simplicial set $ss(K)$ by setting

- $ss(K)_n = \{(v_0, \ldots, v_n) \in V(K)^{n+1} | \{v_0, \ldots, v_n\} \in S(K) \text{ and } v_i \leq v_{i+1} \forall i\}$.
- For an arrow $\alpha : [m] \to [n]$ we have the function $ss(K)(\alpha) : ss(K)_n \to ss(K)_m$ given by $ss(K)(\alpha)(v_0, \ldots, v_n) = (v_{\alpha(0)}, \ldots, v_{\alpha(m)})$.

And we can construct a delta set $ds(K)$ by setting

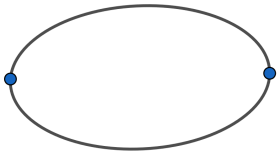
- $ds(K)_n = \{(v_0, \ldots, v_n) \in V(K)^{n+1} | \{v_0, \ldots, v_n\} \in S(K) \text{ and } v_i < v_{i+1} \forall i\}$.
- For an arrow $\alpha : [m] \to [n]$ we have the function $ds(K)(\alpha) : ds(K)_n \to ds(K)_m$ given by $ds(K)(\alpha)(v_0, \ldots, v_n) = (v_{\alpha(0)}, \ldots, v_{\alpha(m)})$.

With this, we can define a simplicial complex from a simplicial (or delta) set.

Definition. *We say that a simplicial (or delta) set X is **modeled with a simplicial complex** if $\exists K \in \mathbf{Scx}$ with an order in its vertices, such that $ss(K) \simeq X$ (or $sd(K) \simeq X$).*

We will normally abuse language and say that certain simplicial set *is* a simplicial complex, even though technically it is not.

In a way, simplicial complexes are simplicial (or delta) sets whose simplices can be stated as an array of its vertices, which is something we usually do not have in general for simplicial or delta sets, even for geo friendly ones such as



This observation is given shape in the following construction.

Construction: Given X a simplicial (or delta) set, we construct a simplicial (or delta) set $\bar{X}$ given by

- $\bar{X}_n = \{(\sigma(\delta(0)), \dots, \sigma(\delta(n))) \in X([0])^{n+1} | \sigma \in X_n\}$
- For an arrow $\alpha : [m] \to [n]$ we have the function $\bar{X}(\alpha) : \bar{X}_n \to \bar{X}_m$ given by $\bar{X}(\alpha)(\sigma(\delta(0)), \dots, \sigma(\delta(n))) = (\sigma(\delta(\alpha(0))), \dots, \sigma(\delta(\alpha(m))))$

And there is an obvious epimorphism $X \to \bar{X}$ which sends a simplex σ to $(\sigma(\delta(0)), \dots, \sigma(\delta(\dim(\sigma))))$, and we have the following property.

Proposition 3.1.2. *$X \to \bar{X}$ is an isomorphism $\Leftrightarrow$ X is a simplicial complex.*

Proof. This is just the fact that choosing the right order, $\bar{X} \simeq ss(sc(X))$ (or $\bar{X} \simeq ds(sc(X))$ for a delta set), together with $sc(ss(K)) = K$. □

So on a way, we can define a simplicial complex as a special kind of simplicial (or delta) set, one for which all simplices can be identified uniquely from its vertices. One might expect that this vertex-identifiable shape for simplices will also mean that faces are faces in a unique way, and this is true.

Proposition 3.1.3. *Given a simplicial (or delta) set X, if X is a simplicial complex then it is geo friendly.*

Proof. Let $\sigma : \Delta^n \to X$ be a non-degenerate simplex. We can assume that $\sigma = (\sigma_0, \dots, \sigma_n)$ with $\sigma_0 < \cdots < \sigma_n$ (being non degenerate implies that all σ_i's are different). Let $\alpha, \alpha' : [n] \to [m]$ such that $\sigma(\alpha) = \sigma(\alpha')$, this means that $\sigma_{\alpha(k)} = \sigma_{\alpha'(k)}) =$ for all $k \in \{0, \dots, n\}$

Since all σ_i's are different we conclude that $\alpha(k) = \alpha'(k)$ for all $k \in \{0, \dots, n\}$ and so $\alpha = \alpha'$ □

So simplicial complexes are geo friendly. The other direction is false as the previous image shows. Now we have enough material to establish $(S(K), \subset)$ as a category of simplices.

Proposition 3.1.4. *For a simplicial complex K we have that*

$$(S(K), \subset) \simeq \Delta \downarrow ds(K)$$

Proof. Since $ds(K)$ is geo friendly, $\Delta \downarrow ds(K)$ is a poset, so an isomorphism is a bijection that respects orders. The isomorphism is very obvious, it sends $\sigma \subset \tau$ to $\sigma \to \tau$. Given the definition of $ds(K)$ one can easily see that this defines a bijection between $S(K)$ and $ds(K)$ (seen as a set if you will). □

3.2.2 Subdivision, $-_\Delta$ and other operations in Scx

The fact that simplicial complexes are a special kind of simplicial (or delta) set, allows us to define union, intersection, image and simplicial complement using the same definitions that we gave in section 3.1.1.

In particular, simplicial complexes can be regarded as its set of simplices. This is direct for the geometric definition, and for the abstract definition we will regard $S(K)$ as K. Having that said, we might treat $S(K)$ as K when there is no risk of confusion.

We can also define a product as the cell complex $\{\sigma \times \tau | \sigma \in K, \tau \in L\}$ subdivided in a way that it is a simplicial complex (see section 3.2.3 on how to do this).

The abstract way of defining the product goes through the product in the category of simplicial sets, we will describe it explicitly.

Definition. *Given K and L simplicial complexes with their vertices ordered. We define $L \times K$ as follows*

- $V(K \times L) = V(K) \times V(L)$
- $S(K \times L) = \{\{(v_0, w_0), \ldots, (v_n, w_n)\} | \{v_0, \ldots, v_n\} \in S(K), \{w_0, \ldots, w_n\} \in S(L)$ *and* $v_0 \leq \cdots \leq v_n, w_0 \leq \cdots \leq w_n\}\}$

And we have that $|K \times L| \simeq |K| \times |L|$.

Observe that in a simplicial complex the image of a simplex has a very familiar shape

$$\texttt{Im}(\sigma) = \mathcal{P}(\sigma)$$

With vertices given by $V(\texttt{Im}(\sigma)) = \sigma$.

With this, the definition of $-_\Delta$ also simplifies, as the following lemma shows

Lemma 3.1.1. *For $Z \leq K$ simplicial complex and $\sigma \in K$ a simplex*

$$\mathit{Im}(\sigma) \cap Z = \emptyset \iff \sigma \cap V(Z) = \emptyset$$

Proof. Left to right direction is just taking $V(\texttt{Im}(\sigma) \cap Z) = V(\emptyset) = \emptyset$

For the opposite direction, consider $\tau \subset \sigma$ and suppose that $\tau \in Z$.

Now, $\tau \in Z$ means that $\tau \subset Z$, and then $\tau \in \sigma \cap V(Z)$ which is empty by hypothesis. This is a contradiction since from the definition we take simplices to be non empty. □

So we have that

$$Y -_\Delta Z = \{\sigma \in Y | \sigma \cap Z_0 = \emptyset\}$$

We now present a way to generate a particular kind of subsimplicial complex starting from a set of vertices

Definition. *Given K a simplicial complex and a set of vertices $A \subset V(K)$, we define $\mathcal{G}(A) \leq K$ as*

- $V(\mathcal{G}(A)) = A$
- $S(\mathcal{G}(A)) = \{\sigma \in S(K) | \sigma \subset A\}$

Since $\subset$ is transitive we will have that $\mathcal{G}(A)$ is a simplicial complex. We present some properties of this construction

Proposition 3.1.5. *Given K a simplicial complex, the construction above defines a function $\mathcal{G} : \mathcal{P}(V(K)) \to \mathit{Sub}(K)$ that satisfies*

1. $V(\mathcal{G}(A)) = A$
2. $A = B \Leftrightarrow \mathcal{G}(A) = \mathcal{G}(B)$ *(that is, $\mathcal{G}$ is injective)*
3. $A \subset B \Rightarrow \mathcal{G}(A) \subset \mathcal{G}(B)$

4. $Z \leq \mathcal{G}(V(Z))$ *for all* $Z \leq K$
5. $\bigcap_{i \in I} \mathcal{G}(A_i) = \mathcal{G}(\bigcap_{i \in I} A_i)$
6. $\bigcup_{i \in I} \mathcal{G}(A_i) \leq \mathcal{G}(\bigcup_{i \in I} A_i)$
7. *For* $f : K \to L$ *a morphism,* $\mathcal{G}(f(A)) \leq f(\mathcal{G}(A))$
8. *For* $f : K \to L$ *a morphism,* $\mathcal{G}(f^{-1}(A)) = f^{-1}(\mathcal{G}(A))$
9. $\mathcal{G}(A \times B) = \mathcal{G}(A) \times \mathcal{G}(B)$
10. $\mathcal{G}(A - B) = \mathcal{G}(A) -_\Delta \mathcal{G}(B)$

Proof. The first statement is definition, and the second statement comes from this since $\mathcal{G}(A) = \mathcal{G}(B) \Rightarrow A = V(\mathcal{G}(A)) = V(\mathcal{G}(B)) = B$. The third statement is true because if $\sigma \in \mathcal{G}(A)$ then $\sigma \subset A \subset B$ so $\sigma \in \mathcal{G}(B)$. The forth statement is obvious as all simplices of Z have vertices in Z.

The fifth statement comes from the set theoric property that $\sigma \in \bigcap A_i \Leftrightarrow \sigma \in A_i \ \forall i$, and similarly the sixth statement comes from $\sigma \in A_i$ for some i then $\sigma \in \bigcup A_i$. For the seventh statement, if we have $f(\sigma) \in f(\mathcal{G}(A))$, $\sigma \in \mathcal{G}(A)$ means that $\sigma \subset A$ and then $f(\sigma) \subset f(A)$ so $f(\sigma) \in \mathcal{G}(f(A))$. As for the eighth statement, $\sigma \in \mathcal{G}(f^{-1}(A)) \Leftrightarrow \sigma \subset f^{-1}(A) \Leftrightarrow f(\sigma) \subset A \Leftrightarrow f(\sigma) \in \mathcal{G}(A) \Leftrightarrow \sigma \in f^{-1}(\mathcal{G}(A))$.

For the ninth statement, consider $\sigma = \{(v_0, w_0), \ldots, (v_n, w_n)\} \in K \times K'$ (that is, $\{v_0, \ldots, v_n\} \in K$, $\{w_0, \ldots, w_n\} \in K'$ and $v_i \leq v_{i+1}$, $w_i \leq w_{i+1}$ for all i). If $\sigma \in \mathcal{G}(A) \times \mathcal{G}(B)$ then $\{v_0, \ldots, v_n\} \in \mathcal{G}(A)$ and $\{w_0, \ldots, w_n\} \in \mathcal{G}(B)$, so $(v_i, w_i) \in A \times B$ for all i, so $\sigma \in \mathcal{G}(A \times B)$. On the other direction, if $\sigma \in \mathcal{G}(A \times B)$ then $v_i \in A$ and $w_i \in B$ for all i which means that $\{v_0, \ldots, v_n\} \in \mathcal{G}(A)$ and $\{w_0, \ldots, w_n\} \in \mathcal{G}(B)$. We conclude that $\sigma \in \mathcal{G}(A) \times \mathcal{G}(B)$.

For the last statement, given $\sigma \in \mathcal{G}(A) -_\Delta \mathcal{G}(B)$ we have that $\sigma \in \mathcal{G}(A)$ so $\sigma \subset A$, and we also have that $\sigma \cap V(\mathcal{G}(B)) = \sigma \cap B = \emptyset$. This means that $\sigma \subset A - B$.

For the other direction, consider $\sigma \in \mathcal{G}(A - B)$, this is $\sigma \subset A - B$, which implies that $\sigma \subset A$ (so $\sigma \in \mathcal{G}(A)$ and also it means that $\sigma \cap B = \sigma \cap V(\mathcal{G}(B)) = \emptyset$. □

Of course we will not always have that $Z = \mathcal{G}(Z)$, as for example

We give a special name for complexes in the image of $\mathcal{G}$

Definition. *A simplicial complex* $Z \leq K$ *is called* ***fat*** *if* $Z = \mathcal{G}(V(Z))$

A good example of a fat subsimplicial complex is $\texttt{Im}(\sigma)$, since

$$\texttt{Im}(\sigma) = \mathcal{G}(\sigma)$$

The proof of this is simply that $\tau \in \mathcal{G}(\sigma) \Rightarrow \tau \subset \sigma \Rightarrow \tau < \sigma$. Observe that this means that $\mathcal{G}(V(Z)) \cap \texttt{Im}(\sigma) = \mathcal{G}(\sigma \cap V(Z)) = \mathcal{G}(\tau) = \texttt{Im}(\tau)$ for some $\tau < \sigma$. We will state this as a corollary.

Corollary 3.1.2. *For any $\sigma \in K$ and fat subcomplex $Z = \mathcal{G}(V(Z)) \leq K$ we have that*

$$Z \cap \mathit{Im}(\sigma) = \mathit{Im}(\tau)$$

For some $\tau < \sigma$.

We also present a lemma that will use in the proof of the main result. If one is not careful at all, one might be tempted to think that a bijection $f : A \to B$ defines an isomorphism $f : \mathcal{G}(A) \to \mathcal{G}(B)$. This is of course false as $\mathcal{G}(A)$ and $\mathcal{G}(B)$ depend on the complexes in which they are defined. However, we do have the following lemma.

Lemma 3.1.2. *Given K and L simplicial complexes and consider $A \subset V(K)$ and $B \subset V(L)$. Consider a bijection $f : A \to B$ with inverse $g : B \to A$*

If $f : \mathcal{G}(A) \to \mathcal{G}(B)$ and $g : \mathcal{G}(B) \to \mathcal{G}(A)$ define morphisms, then they are inverse of each other. In particular

$$f : \mathcal{G}(A) \xrightarrow{\simeq} \mathcal{G}(B)$$

is an isomorphism.

Proof. With this number of hypothesis one would expect that the proof is very easy, and it is. It is just the fact that $f(g(\sigma)) = f(f^{-1}(\sigma)) = \sigma$ and $g(f(\sigma)) = g(g^{-1}(\tau)) = \tau$ (here f^{-1} and g^{-1} refer to the preimage). □

We will now define the barycentric subdivision.

Definition. *Given a simplicial complex K we define its **subdivision** $Sd(K)$ as the following (abstract) simplicial complex*

- $V(Sd(K)) = S(K)$
- $S(Sd(K)) = \{\{\sigma_0, \ldots, \sigma_n\} \in \mathcal{P}(S(K)) | \sigma_0 \subset \cdots \subset \sigma_n\}$

We call the **barycenter** of $\sigma \in K$ to the vertex $\{\sigma\}$. Now, notice that given $Z \leq K$, the subdivision $Sd(Z)$ defines a subcomplex of $Sd(K)$, that is, we have a functor $Sd : \texttt{Sub}(K) \to \texttt{Sub}(Sd(K))$. We will state some properties of this functor.

Proposition 3.1.6. *Given $K \in \mathbf{Scx}$, the functor $Sd : \mathtt{Sub}(K) \to \mathtt{Sub}(Sd(K))$ satisfies*

1. *$Z = Y \Leftrightarrow Sd(Z) = Sd(Y)$ (that is, Sd is injective)*
2. *$Z \leq Y \Rightarrow Sd(Z) \leq Sd(Y)$*
3. *$Sd(\bigcup Y_i) = \bigcup (Sd(Y_i))$*

4. $Sd(\bigcap Y_i) = \bigcap Sd(Y_i)$

5. $\{\sigma_0, \dots, \sigma_n\} \in Sd(Z) \Leftrightarrow \sigma_n \in Z$ *(where we suppose that* $\sigma_0 \subset \cdots \subset \sigma_n$*)*

6. $Sd(Z) = \mathcal{G}(V(Sd(Z)) = \mathcal{G}(Z)$ *(it is fat)*

Proof. The first statement comes from $Z = V(Sd(Z)) = V(Sd(Y)) = Y$. The fifth statement comes from the fact that for $Z \leq K$ we have $\tau \subset \sigma \in Z \Rightarrow \tau \in Z$ together with $\sigma_0 \subset \cdots \subset \sigma_n$. The second statement is clear using the fifth.

The third statement comes as follows $\{\sigma_0, \dots, \sigma_n\} \in \bigcup Sd(Y_i) \Leftrightarrow \exists i_0$ such that $\{\sigma_0, \dots, \sigma_n\} \in Sd(Y_{i_0}) \Leftrightarrow \exists i_0$ such that $\sigma_n \in Y_{i_0} \Leftrightarrow \sigma_n \in \bigcup Y_i \Leftrightarrow \{\sigma_0, \dots, \sigma_n\} \in Sd(\bigcup Y_i)$

Similarly $\{\sigma_0, \dots, \sigma_n\} \in \bigcap Sd(Y_i) \Leftrightarrow \forall i \ \{\sigma_0, \dots, \sigma_n\} \in Sd(Y_i) \Leftrightarrow \forall i \ \sigma_n \in Y_i \Leftrightarrow \sigma_n \in \bigcap Y_i \Leftrightarrow \{\sigma_0, \dots, \sigma_n\} \in Sd(\bigcap Y_i)$

For the last statement, consider $\{\sigma_0, \dots, \sigma_n\} \in \mathcal{G}(V(Sd(Z))) = \mathcal{G}(Z)$, we have that $\{\sigma_0, \dots, \sigma_n\} \subset V(\mathcal{G}(Z)) = Z$ so $\{\sigma_0, \dots, \sigma_n\} \in Sd(Z)$ □

We will usually abuse notation and call Z to $Sd(Z)$ when there is no risk of confusion. We will also suppose that when writing $\{\sigma_0, \dots, \sigma_n\} \in Sd(K)$ we have that $\sigma_0 \subset \cdots \subset \sigma_n$.

We are of course missing still the most important property of subdivision

Proposition 3.1.7. $|K| \cong |Sd(K)|$

There is one important operation that we are missing. As it was promised, we will now solve the issues we were having with $-_\Delta$, namely, the three properties that were different from the ones in **Set**.

- $X -_\Delta (X -_\Delta Z) \leq Z$
- $Y -_\Delta Z = \emptyset \Longrightarrow Y \leq Z$
- $X -_\Delta (Y \cap Z) \leq (X -_\Delta Y) \cup (X -_\Delta Z)$

What solves this is the fact that the functor Sd transforms the condition of $-_\Delta$ in a much more set-theoretic relate-able condition

Lemma 3.1.3. *For* X *simplicial complex and* $Y, Z \leq X$*, given* $\{\sigma_0, \dots, \sigma_n\} \in Sd(X)$ *we have*

$$\{\sigma_0, \dots, \sigma_n\} \in Sd(Y) -_\Delta Sd(Z) \Longleftrightarrow \{\sigma_0, \dots, \sigma_n\} \subset Y - Z$$

Proof. $\{\sigma_0, \dots, \sigma_n\} \in Sd(Y) -_\Delta Sd(Z) \Leftrightarrow \{\sigma_0, \dots, \sigma_n\} \subset Y$ and $\{\sigma_0, \dots, \sigma_n\} \cap Z = \emptyset \Leftrightarrow \sigma_i \in Y - Z \ \forall i$ □

Now we see the fixed properties for $-_\Delta$

Proposition 3.1.8. *Given* X *a simplicial complex, we have*

1. $Sd(X) -_\Delta (Sd(X) -_\Delta Sd(Z)) \leq Sd(Z)$
2. $Sd(Y) -_\Delta Sd(Z) = \emptyset \Longrightarrow Sd(Y) \leq Sd(Z)$
3. $Sd(X) -_\Delta (Sd(Y) \cap Sd(Z)) \leq (Sd(X) -_\Delta Sd(Y)) \cup (Sd(X) -_\Delta Sd(Z))$

Proof. For the first two properties we use that $Sd(Z) = \mathcal{G}(Z)$ for all $Z \leq X$.

So $Sd(X) -_\Delta (Sd(X) -_\Delta Sd(Z)) = \mathcal{G}(X) -_\Delta (\mathcal{G}(X) -_\Delta \mathcal{G}(Z)) = \mathcal{G}(X - (X - Z)) = \mathcal{G}(Z) = Sd(Z)$

And $Sd(Y) -_\Delta Sd(Z) = \mathcal{G}(Y) -_\Delta \mathcal{G}(Z) = \mathcal{G}(Y - Z) = \emptyset = \mathcal{G}(\emptyset)$, then $Y - Z = \emptyset$ and therefore $Y \subset Z$ which implies $Sd(Y) \leq Sd(Z)$

The Morgan law is slightly more subtle. Consider $\{\sigma_0, \ldots, \sigma_n\} \in Sd(X) -_\Delta (Sd(Y) \cap Sd(Z)) = Sd(X) -_\Delta (Sd(Y \cap Z))$, which says that $\forall i\ \sigma_i \notin Z \cap Y$.

Now suppose $\{\sigma_0, \ldots, \sigma_n\} \notin Sd(X) -_\Delta Sd(Y)$, then for some i_0 we have that $\sigma_{i_0} \in Y$, then as $Y \leq X$ $\sigma_0, \ldots, \sigma_{i_0} \in Y$ and therefore $\sigma_0, \ldots, \sigma_{i_0} \notin Z$. This means, since $Z \leq X$, that $s_k \notin Z$ for $k > i_0$, with which we conclude that $\{\sigma_0, \ldots, \sigma_n\} \in Sd(X) -_\Delta Sd(Z)$ □

We finish this section with an abstract definition of the star and the link.

Definition. *Given X a simplicial complex and $\sigma \in X$*

- $st(\sigma) = \bigcup_{\sigma < \tau} \mathtt{Im}(\tau)$
- $lk(\sigma) = st(\sigma) -_\Delta \mathtt{Im}(\sigma)$

3.2.3 Geometrical viewpoint on Simplicial Complexes

We will now present a different viewpoint on simplicial complexes, presented in [RS72], which enriches the theory giving it a geometrical flavour, and also establishes a couple of properties of simplicial complexes, necessary to complete the proof of our main theorem.

As mentioned before, a **simplicial complex** is formed *sticking* topological simplices. We call **topological simplex** of dimension $n \in \mathbb{N}$ a topological space linearly homeomorphic to

$$\{\sum_{i=0}^{n} \lambda_i v_i | \sum \lambda_i = 1\}$$

Where $\{v_0, \ldots, v_n\}$ are a geometrically independent set of elements of $\mathbb{R}^n$. The elements v_i are called **vertices** of the simplex, and we call a **face** of such simplex, written as $\tau < \sigma$, to the convex hull of a subset of $\{v_0, \ldots, v_n\}$.

Definition. *A **simplicial complex** K is a set of topological simplices in some euclidean space $\mathbb{R}^N$ such that*

1. *If $\sigma \in K$ and $\tau < \sigma$ then $\tau \in K$*
2. *For $\sigma, \tau \in K$ the topological space $\sigma \cap \tau$ is a face of σ and of τ*

And we call $|K| = \bigcup K$ the **realization** of K. All simplicial complexes we will consider are **locally finite**, meaning that each $x \in |K|$ has a neighborhood that intersects finitely many simplices.

Simplicial complexes are equivalent to what is called *euclidean polyhedra*, which are locally cone-like spaces. For this reason it should not be surprising that they can easily form PL stratified pseudomanifolds. Let us begin by saying what cones are.

Definition. *Given $B, A_0, \ldots, A_m \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we define their* ***join*** *as*

$$A_0 A_1 \ldots A_m = \{\sum_{i=0}^{n} \lambda_i a_i | \ \ \textit{min}(\lambda_i) \geq 0, \sum \lambda_i = 1, a_i \in A_i\}$$

By convention $A\emptyset = A$. The join $\{a\}B := aB$ is a ***cone*** *if all its elements can be written uniquely as $\lambda_0 a + \lambda_1 b$, with $\lambda_0 + \lambda_1 = 1$, $\lambda_0, \lambda_1 \geq 0$ and $b \in B$.*

Definition. *We define the category of euclidean polyhedra* ***Poly*** *as follows.*

1. *A set $P \subset \mathbb{R}^n$ is a* ***polyhedron*** *if for each point $x \in P$ there is a cone neighborhood $x \in N = aL$, with L compact.*

2. *A (continuous) map $f : P \to Q$ is called* ***piecewise-linear*** *(denoted as PL) if each $a \in P$ has a neighborhood $N = aL \subset P$ such that*

   $$f(\lambda_0 a + \lambda_1 x) = \lambda_0 f(a) + \lambda_1 f(x)$$

   For all $x \in L$ and $\lambda_0 + \lambda_1 = 1$, $\lambda_0, \lambda_1 \geq 0$.

We name $N = aL := N_a(P)$ a ***star*** *of a in P and $L := L_a(P)$ a* ***link*** *of a in P*

We give some examples

Examples:

1. $\mathbb{R}^n$ is a polyhedron, as well as linear subspaces of $\mathbb{R}^n$.

2. A n-simplex is a polyhedron.

3. An open subset of a polyhedron is a polyhedron. As a consequence, the intersection of finitely many polyhedra is a polyhedron. The product $P \times Q$ of polyhedra is also a polyhedron.

4. The isomorphisms on **Poly** are called **PL homeomorphisms**, and correspond to homeomorphisms which are PL. Obseve that the function inverse of a PL homeomorphism is its inverse in the category.

5. A polyhedron M is called **PL manifold** of dimension n if each point $x \in M$ has a neighborhood PL homeomorphic to $\mathbb{R}^n$. A PL manifold is called a n**-ball** if it is PL homeomorphic to $[-1, 1]^n$, and it is called n**-sphere** if it is PL homeomorphic to $\partial[-1, 1]^n$.

6. The union of polyhedra $P = \bigcup P_i$ is a polyhedron when the union is **locally finite**, meaning that each point $x \in P$ has a neighborhood intersecting finitely many P_i's.

   Suppose that each P_i is compact. If $f : P \to Q$ is a map with $f|_{P_i}$ is PL, then f is PL.

7. The last example means that given K a finite dimensional (that is $|K|$ can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$) locally finite simplicial complex. The space $|K|$ is a polyhedron.

The following proposition corresponds to ([RS72] Theorem 2.2).

Proposition 3.1.9. *A compact polyhedra is a finite union of simplices. In general, a polyhedron is the locally finite union* $\bigcup P_i$ *of simplices.*

This *will* mean that polyhedra are actually (finite dimensional and locally finite) simplicial complexes. To see this, Rourke and Sanderson in our reference define *cell complexes*, which are a generalization of simplicial complexes.

Given $A \subset \mathbb{R}^n$, we call

$$< A >= \bigcap_{A \subset V \subset_{\text{Affine}} \mathbb{R}^n} V$$

And we say that A *spans* $< A >$.

Definition. *A* ***linear*** n***-cell****, or just a* ***cell****, is a convex compact polyhedron which spans a space of dimension* n*.*

We give some examples.

Examples:

1. An n-simplex is an n-cell, as well as the cube $[-1, 1]^n$.
2. The intersection and product of cells is a cell
3. The join $a_0 \dots a_m$ of m points in $\mathbb{R}^n$ is a cell called the cell **spanned** by $\{a_0, \dots, a_m\}$.

We define *face* and *vertex* of a cell now. For a point in a n-cell $x \in C$, call $< x, C >$ the union of lines l through x such that $l \cap C$ is an arc with $x \in int(l \cap C)$. By convexity $< x, C >$ is an affine subspace of $\mathbb{R}^n$. If there are no such lines we say that $< x, C >= x$ and call it a **vertex**, otherwise the cell $< x, C > \cap C$ is called a **face** of C and denoted $C_x < C$. We have the following proposition for faces.

Proposition 3.1.10. *Given* C *a* n*-cell*

1. C *is spanned by finitely many vertices.*
2. $F < C$ *is spanned by a subset of the vertices that span* C*.*
3. *If* $F < D < C$ *then* $F < C$*.*
4. *If* $F, D < C$ *then* $F \cap D < C$*.*
5. *For* $x \in C$*,* $C = xB$ *the cone with* $B = \bigcup \{F < C | x \notin F\}$*.*

This is proven in ([RS72] Appendix of Chapter 2).

Definition. *A* ***cell complex*** K *is a collection of cells in* $\mathbb{R}^n$ *such that*

- *If* $C \in K$ *and* $B < C$*, then* $B \in K$
- *If* $B, C \in K$ *then* $B \cap C$ *is face of* B *and* C

The realization $|K|$ *of* K *is the union of its cells.*

So in particular a simplicial complex is a cell complexes whose cells are simplices. One key property is that cell complex can be *subdivided* into a simplicial complex.

Definition. *Given L and K cell complexes, we say that L is a **subdivision** of K if*

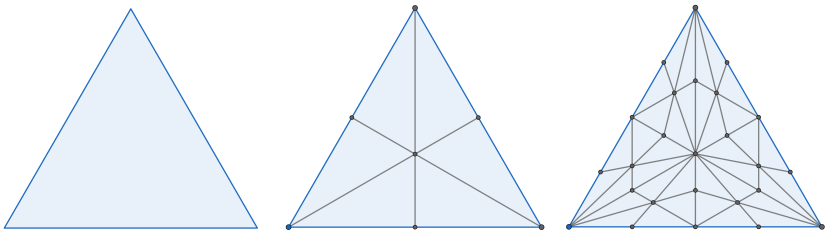
- $|L| = |K|$
- *Each cell of L is contained in some cell of K*

We denote this as $L \lhd K$.

For a simplicial complex K, the complex $Sd(K)$ seen in the last subsection corresponds to a particular subdivision called **barycentric subdivision**. This subdivision can be constructed by taking for all $\sigma = v_0 v_1 \dots v_{n-1} \in K$ their **barycenter** $b(\sigma) = 1/n \sum_i v_i \in \sigma$, and then considering

$$Sd(K) = \{b(\sigma_0)b(\sigma_1)\dots b(\sigma_m) | \sigma_0 < \sigma_1 < \dots < \sigma_m \in K\}$$

We give for example the barycentric and double barycentric subdivision of Δ^2



Δ^2 $Sd(\Delta^2)$ $Sd^2(\Delta^2)$

Now we connect cell complexes with simplicial complexes with the following result corresponding to ([RS72] Proposition 2.9).

Proposition 3.1.11. *A cell complex can be subdivided into a simplicial complex without introducing any new vertices.*

This means that cell complexes are simplicial complexes up to subdivision. The step through cell complexes might seem at first glance too much bother, but it is actually worthy as it gives powerful results.

Proposition 3.1.12. *Given K and L simplicial complexes and $f : |K| \to |L|$ a PL map, we have the following*

1. *If $|L| \subset |K|$, there are simplicial subdivisions $L' < L$ and $K' < K$ such that $L' \subset K'$.*
2. *If f is PL, there are simplicial subdivisions $L' < L$ and $K' < K$ such that $f : |K'| \to |L'|$ is simplicial (that is, for all $\sigma \in K$ we have $f(\sigma) \in L$ and $f|_\sigma$ is linear).*

And we have the following (see [RS72] Theorem 2.11).

Theorem 3.2. *For any polyhedron P there is a simplicial complex K with $P \simeq |K|$.*

This means that in a topological sense simplicial complexes are locally cone-like spaces.

We finish this section with a result that is fundamental for the proof of the main result. We begin by defining the link and the star of a point.

Definition. *Consider K a cell complex and $a \in K$ a vertex, we define the following subcomplexes.*

- *The **star** of a corresponds to $\mathtt{st}(a, K) = \{A \in K | \exists B \in K \text{ with } A < B \text{ and } a \in B\}$*
- *The **link** of a corresponds to $\mathtt{lk}(a, K) = \{A \in K | aA \in K \text{ and } a \notin A\}$*

We have that

$$\mathtt{st}(a, K) = a\mathtt{lk}(a, K)$$

. Hence $|\mathtt{st}(a, K)|$ and $|\mathtt{lk}(a, K)|$ are a star and a link of a in $|K|$.

Proposition 3.2.1. *Given K a cell complex and $a \in K$, then for any $K' \lhd K$ we have*

$$|\mathtt{lk}(a, K)| \cong |\mathtt{lk}(a, K')|$$

Proof. We use **pseudoradial projection** to prove this. For any A simplex of K', consider

$$A^+ = \{\lambda_0 a + \lambda_1 b | b \in A, \lambda_0 \leq 1, \lambda_1 \geq 0, \lambda_0 + \lambda_1 = 1\}$$

Then $M = \{A^+ \cap B | B \in \mathtt{lk}(a, K)\}$ is a simplicial subdivision of $\mathtt{lk}(a, K)$. Furthermore, the radial projection $|\mathtt{lk}(a, K')| \to |M|$ maps simplices homeomorphically to simplices and hence determines a simplicial isomorphism. □

We have a corollary of this

Corollary 3.2.1. *A linear n-cell is a n-ball.*

4 Simplicial Sheaves

We begin now to *simplicialize* the theory of the Deligne axioms, with the first point in the agenda: Constructing what can be called *simplicial sheaves*.

There can be multiple ways to do this. One first impulse is to get into the fancy world of Grothendieck topologies (using the theory briefly explained in [McF12] and more thoroughly developed in sources such as [Joh02]). As it turns out, this is not the best option, not only for the complexity of the theory, but also because the non-trivial Grothendieck topologies on Δ (and therefore in $\Delta \downarrow X$) give us very restrictive conditions on a presheaf for being sheaf. Since these conditions are not necessary for the development of the theory, as there are better ways to construct simplicial sheaves, we opt for not entering this world.

A second approach comes from the examples on the works D. Chataur, M. Saralegi-Aranguren and D. Tanré, particularly the blown-up cochains (see section 2.2.1) which is defined in filtered simplices $\sigma : \Delta^{(j_0,\cdots,j_n)} \to X$ as

$$\mathcal{N}^\star_\sigma = C^\star_\Delta(c\Delta^{j_0}) \otimes \cdots \otimes C^\star_\Delta(c\Delta^{j_{n-1}}) \otimes C^\star_\Delta(\Delta^{j_n})$$

And then "extended" into complete sections by taking the limit.

$$\mathcal{N}^\star(X) = \lim_{\sigma \in FSimp(X)} \mathcal{N}^\star_\sigma$$

This in rough terms (that we will specify and theorize appropriately) suggests the idea that we are considering $\mathcal{N}^\star$ as a functor

$$\mathcal{N}^\star : (\Delta \downarrow X)^{op} \to Ch(R)$$

And then extending it *somehow* along the Yonneda embedding $h_\Delta : \Delta \hookrightarrow \mathbf{SSet}$, obtaining the sort of limit defining the value on the sections. This leads us to consider the category

$$\mathtt{Func}((\Delta \downarrow X)^{op}, Ch(R))$$

As the main object of study. This object of study can also be seen as *representations* of the simplicial set X (recall the discussion on how $\Delta \downarrow X$ is a good representant in **Cat** of X), which gives a motivation since presheaves $\mathtt{Func}(Op(Y)^{op}, R\text{-mod})$ can be seen as representations of a topological space.

Now, the image category $Ch(R)$ *can* be abstracted to be any abelian or model category $\mathbf{A}$ (defined in the appendix), which *will* in time endow $\mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{A})$ with a model or derived category.

This turns out to be a good way to go for us, but it is not complete, as more subtlety becomes necessary. We have that the extension along the Yoneda embedding

$$\begin{array}{ccc} \mathbf{SSet}^{op} & \xrightarrow{\exists! \bar{F}} & \mathbf{A} \\ {\scriptstyle h_\Delta}\uparrow & \nearrow_{\forall F} & \\ \Delta^{op} & & \end{array} \tag{2}$$

With which we construct the *section functor* $\Gamma(F)(Z) = \lim_{\sigma \in Z} F(\sigma)$, hints on the category $\mathtt{Func}(\mathtt{Sub}(X)^{op}, \mathbf{A})$ as a force to be recon with. In fact, it turns out that there is a close relationship between $\mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{A})$ and $\mathtt{Func}(\mathtt{Sub}(X)^{op}, \mathbf{A})$, furthermore, the latest category can be seen as *presheaves*

in a topological sense, and we can consider sheaf conditions and a sheafification functor. We get that the representations are actually sheaves in $\texttt{Func}(\texttt{Sub}(X)^{op}, \mathbf{A})$.

The subtle point is that the topology considered in $\texttt{Func}(\texttt{Sub}(X)^{op}, \mathbf{A})$ is the discrete one, and it can be made more subtle, and as it happens, it is useful to do so to state the Deligne axioms.

4.1 The category Rep(X)=$\texttt{Func}((\Delta \downarrow X)^{op}, A)$

Hopefully it is clear now that $\Delta \downarrow X$ represents in a faithful way the characteristics of a simplicial or delta set X. As it is common in mathematics we can extract valuable information about the object in question X via representing in some easy to digest category, which is a common practice in algebraic settings.

Even "outside of algebra", going into topology, one might say that sheaves serve that roll, of course with a gluing condition added. Theory has gone far into endow them with a derived category structure that provides them with a natural homological category structure, and theory has gone wide into abstracting the concept to fit environments outside the usual set-topology into the realm of topos.

Today we are making sheaf theory take a step into the simplicial world, starting with the study of the category $\texttt{Func}((\Delta \downarrow X)^{op}, \mathbf{A})$, which inspired in the precedent arguments we are calling the category of representations of X. This category delivers more that one might expect at first glance. We have a rich enough variety of examples that make this category worthy of study, we have model category structures for it and it can be related in a very natural and easy way into the category of functors from subsimplicial structures of X, which better emulates topological sheaf theory.

4.1.1 Definition and first remarks

A representation of a simplicial or delta set X on a category $\mathbf{A}$ is a functor

$$F : (\Delta \downarrow X)^{op} \to \mathbf{A}$$

We call this category $\texttt{Rep}_{\mathbf{A}}(X)$, $\texttt{Rep}_R(X)$ for $\mathbf{A} = R$-mod, or just $\texttt{Rep}(X)$ if there is no risk of confusion. We will work with $\mathbf{A}$ a (co)complete category, which normally will be an abelian and/or model category.

We have that $\texttt{Rep}(X)$ enjoys functor category properties (see appendix 8.3) such as

- Limits and colimits exist and are computed component-wise.
- If $\mathbf{A}$ is abelian then $\texttt{Rep}_{\mathbf{A}}(X)$ is abelian.
- If $\mathbf{A}$ has enough injectives then $\texttt{Rep}_{\mathbf{A}}(X)$ has enough injectives.

A good variety of examples can be created from *universal* functors

$$F : \Delta^{op} \to \mathbf{A}$$

The name universal is because of the independance from a particular simplicial (or delta) set X. To get to a representation of X we precompose with the

forgetful functor

$$Q^X : (\Delta \downarrow X)^{op} \longrightarrow \Delta^{op}$$
$$(\Delta^n \xrightarrow{\sigma} X) \longmapsto \Delta^n$$

This means that from a universal functor $F : \Delta^{op} \to \mathbf{A}$, we construct

$$\bar{F} : (\Delta \downarrow X)^{op} \to \mathbf{A}$$

Given by the formula $\bar{F}(\Delta^n \xrightarrow{\sigma} X) = F(\Delta^n)$. We will denote $\bar{F}$ simply as F.

Observation: Notice that since the categories Δ^{op} and $(\Delta \downarrow X)^{op}$ are small and $\mathbf{A}$ is (co)complete, this process composing with Q^X has left and right adjoints given by the Kan extensions. In particular it preserves limits and colimits.

4.1.2 $f_\star$ and $f^\star$

Recall from section 3.1.3 that for a morphism $f : X \to Y$ we have a functor $f : (\Delta \downarrow X)^{op} \to (\Delta \downarrow Y)^{op}$ which is basically applying f up to Yoneda embedding. By precomposing with this functor we obtain a natural way to compare $\texttt{Rep}(X)$ and $\texttt{Rep}(Y)$

$$\texttt{Func}((\Delta \downarrow Y)^{op}, \mathbf{A}) \xrightarrow{-\circ f} \texttt{Func}((\Delta \downarrow X)^{op}, \mathbf{A})$$

which we call $f^\star$. Then

$$f^\star F(\Delta^n \xrightarrow{\sigma} X) = F(\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y)$$

Now, since the categories $(\Delta \downarrow -)^{op}$ are small and $\mathbf{A}$ is (co)complete, precomposing with f has left and right adjoints given by its Kan extensions

$$\texttt{Rep}(Y) \underset{\xleftarrow{Lan_f}}{\overset{\xleftarrow{Ran_f}}{\xrightarrow{-\circ f}}} \texttt{Rep}(X)$$

We are particularly interested in the right Kan extension which we call $f_\star$. One formula for this functor is

$$f_\star(F)(\Delta^m \xrightarrow{\tau} Y) = \lim_{f(\sigma) \to \tau} F(\sigma)$$

That is, we are taking the limit over the σ's such that $f(\sigma)$ is a face (or degeneracy of a face) of τ. This limit is taken in the environment of the category $(\Delta \downarrow X) \downarrow \tau$.

Observe that this is the best possible approximation we have to something like $F(f^{-1}(\tau))$, which would emulate the classical theory in terms of $(f^\star, f_\star)$.

There is an alternative version of $f_\star$, more based on a geometrical point of view, which we expose now.

We can construct $f_\star$ by sending an object $F \in \texttt{Rep}(X)$ to $f_\star F : (\Delta \downarrow Y)^{op} \to \mathbf{A}$, which sends an object $\Delta^m \xrightarrow{\tau} Y$ to $\lim_{\sigma \in \Delta \downarrow f^{-1}(\Delta^m)} F(\Delta^n \xrightarrow{\sigma} f^{-1}(\Delta^m) \xrightarrow{\tilde{\tau}} X)$

(that is, $f_\star F(\Delta^m \xrightarrow{\tau} Y) = \lim_{\Delta\downarrow f^{-1}(\Delta^m)} \tilde{\tau}^\star F$), where $f^{-1}(\Delta^m)$ and $\tilde{\tau}$ are notation for what is obtained from the pullback

$$\begin{array}{ccc} f^{-1}(\Delta^m) & \xrightarrow{f^\tau} & b \\ {\scriptstyle\tilde{\tau}}\downarrow & \ulcorner & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{3}$$

We will call the limit $\lim_{\Delta\downarrow f^{-1}(\Delta^m)} \tilde{\tau}^\star F$ simply as $\lim \tilde{\tau}^\star F$, and the cone mophisms of the limit as $l^\tau_\sigma : \lim \tilde{\tau}^\star F \to F(\Delta^n \xrightarrow{\sigma} f^{-1}(\Delta^m) \xrightarrow{\tilde{\tau}} X)$.

For the action of $f_\star F$ on a morphism

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\alpha} & \Delta^{m\prime} \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\tau'} & \\ & Y & \end{array}$$

consider that the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xleftarrow{\tau} & \Delta^m \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle\alpha} \\ X & \xrightarrow[f]{} & Y & \xleftarrow[\tau']{} & \Delta^{m\prime} \end{array}$$

induces a morphism $\bar{\alpha} : f^{-1}(\Delta^m) \to f^{-1}(\Delta^{m\prime})$ such that the cube

$$\begin{array}{ccccc} & & f^{-1}(\Delta^m) & \longrightarrow & b \\ & {\scriptstyle\bar{\alpha}}\swarrow & \downarrow{\scriptstyle\tilde{\tau}} & {\scriptstyle\alpha}\swarrow & \downarrow{\scriptstyle\tau} \\ f^{-1}(\Delta^{m\prime}) & \longrightarrow & b' & & \\ \downarrow{\scriptstyle\tilde{\tau}'} & & \downarrow & {\scriptstyle\tau'}\downarrow & \\ X & = & X & \longrightarrow & Y \\ & \searrow & & \searrow & \| \\ X & & \xrightarrow[f]{} & & Y \end{array} \tag{4}$$

commutes. Since in particular $\tilde{\tau}\sigma = \tilde{\tau}'\alpha\sigma$ for any $\sigma : \Delta^n \to f^{-1}(\Delta^m)$, the morphisms $l^{\tau'}_{\bar{\alpha}\sigma} : \lim \tilde{\tau}'^\star F \xrightarrow{l^{\tau'}_{\bar{\alpha}\sigma}} \tilde{\tau}^\star F(\sigma)$ are a cone of $\lim \tilde{\tau}^\star F$, which gives the morphism

$$f_\star F(\alpha) : \lim \tilde{\tau}'^\star F \to \lim \tilde{\tau}^\star F$$

This morphism is such that the following diagram commutes

$$\begin{array}{ccc} & {\scriptstyle l^{\tau'}_{\bar{\alpha}\sigma}}\nearrow & F(\Delta^n \xrightarrow{\sigma} f^{-1}(\Delta^m) \xrightarrow{\tilde{\tau}} X) \\ \lim \tilde{\tau}'^\star F & \xrightarrow[f_\star F(\alpha)]{} \lim \tilde{\tau}^\star F & \nearrow{\scriptstyle l^\tau_\sigma} \end{array} \tag{5}$$

For a morphism $\delta : F \to F'$ in $Sh(X)$, we define the natural transformation $f_\star(\delta)$ whose components $f_\star(\delta)_\tau : \lim \tilde{\tau}^\star F \to \lim \tilde{\tau}^\star F'$ are induced by the natural transformations $\tilde{\tau}^\star(\delta) : \tilde{\tau}^\star F \to \tilde{\tau}^\star F'$, that is

$$f_\star(\delta)_\tau = \lim_{\sigma \in \Delta \downarrow f^{-1}(\Delta^m)} \delta_{\tilde{\tau}\sigma}$$

where $\tau : \Delta^m \to Y$.

We prove that the second version of $f_\star$ is equivalent to the first one given by showing that it is also a right adjoint of $f^\star$

Proposition 4.0.1. *$(f^\star, f_\star)$ is an adjoint pair*

Proof. We give the bijection

$$\Phi : Hom_{Sh(X)}(f^\star G, F) \leftrightarrows Hom_{Sh(Y)}(G, f_\star F) : \Psi$$

For $\phi : f^\star G \to F$, we define $\Phi(\phi) : G \to f_\star F$ in a component $\Delta^m \xrightarrow{\tau} Y$ as the morphism $\Phi(\phi)_\tau : G(\tau) \to \lim \tilde{\tau}^\star F$ whose cone morphism component in $\sigma : \Delta^n \to f^{-1}(\Delta^m)$ is given by

$$G(\Delta^m \xrightarrow{\tau} Y) \xrightarrow{G(f^\tau \sigma)} G(\Delta^n \xrightarrow{\tilde{\tau}\sigma} X \xrightarrow{f} Y) \xrightarrow{\phi_{\tilde{\tau}\sigma}} F(\Delta^n \xrightarrow{\sigma} f^{-1}(\Delta^m) \xrightarrow{\tilde{\tau}} X)$$

(Observe that these form a cone, since if we have a morphism $\mu : \sigma \to \sigma'$ in $\Delta \downarrow f^{-1}(\Delta^m)$, the naturality of ϕ gives $F(\mu)\phi_{\tilde{\tau}\sigma'} = \phi_{\tilde{\tau}\sigma} G(\mu)$, which implies that $F(\mu)\phi_{\tilde{\tau}\sigma'} G(f^\tau \sigma') = \phi_{\tilde{\tau}\sigma} G(f^\tau \sigma' \mu) = \phi_{\tilde{\tau}\sigma} G(f^\tau \sigma)$).

For $\rho : G \to f_\star F$, the component $\sigma : a \to X$ of $\Psi(\rho)$ is the composition

$$\Psi(\rho)_\sigma = G(a \xrightarrow{f\sigma} Y) \xrightarrow{\rho_{f\sigma}} \lim \tilde{f\sigma}^\star F \xrightarrow{l_\eta^{f\sigma}} F(a \xrightarrow{\eta} f^{-1}(a) \xrightarrow{\tilde{f\sigma}} X) = F(\sigma)$$

where η is the morphism to the pullback of $f\sigma$, considering the commutative square

$$\begin{array}{ccc} a & \xrightarrow{Id_a} & a \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle f\sigma} \\ X & \xrightarrow[f]{} & Y \end{array}$$

In the following we prove that Φ and Ψ are well defined, that they are inverse to each other and that the bijection is natural respect to F and G.

- $\Phi(\phi)$ is a natural transformation for any $\phi : f^\star G \to F$.

  Given

  $$\begin{array}{ccc} \Delta^m & \xrightarrow{\alpha} & \Delta^{m\prime} \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\tau'} & \\ & Y & \end{array}$$

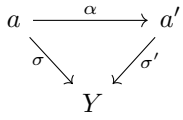
  From 4 we see that $\tilde{\tau}'\bar{\alpha} = \tilde{\tau}$, and $\alpha f^\tau = f^{\tau'}\bar{\alpha}$, therefore we have for any $\sigma : \Delta^n \to f^{-1}(\Delta^m)$ the following commutative square

  $$\begin{array}{ccc} G(\tau') & \xrightarrow{\phi_{\bar{\tau}'\bar{\alpha}\sigma} G(f^{\tau'}\bar{\alpha}\sigma)} & F(\bar{\tau}'\bar{\alpha}\sigma) \\ {\scriptstyle G(\alpha)}\downarrow & & \| \\ G(\tau) & \xrightarrow{\phi_{\bar{\tau}\sigma} G(f^\tau \sigma)} & F(\bar{\tau}\sigma) \end{array}$$

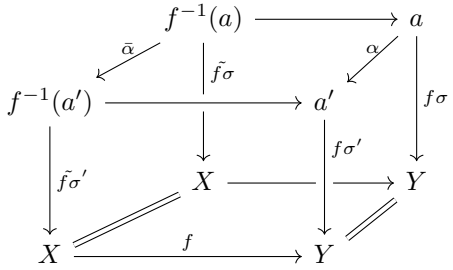
Therefore $l^{\tau'}_{\bar{\alpha}\sigma}\Phi(\phi)_{\tau'} = l^{\tau}_{\sigma}\Phi(\phi)_{\tau}G(\alpha)$. By 5 we have that $l^{\tau'}_{\bar{\alpha}\sigma} = l^{\tau}_{\sigma}f_\star F(\alpha)$, and so $l^{\tau}_{\sigma}\Phi(\phi)_{\tau}G(\alpha) = l^{\tau}_{\sigma}f_\star F(\alpha)\Phi(\phi)_{\tau'}$, for all σ. This gives the naturality of $\Phi(\phi)$.

- $\Psi(\rho)$ is a natural transformation for any $\rho : G \to f_\star F$

  Given



  Let $\eta : a \to f^{-1}(a)$ and $\eta' : a' \to f^{-1}(a')$ be the morphisms for σ and σ' as in 4.1.2. We have a commutative cube as in 4



  Observe that $\bar{\alpha}\eta = \eta'\alpha$. This can be seen by components. Let $x \in Obj(\Delta)$ and $j \in a(x)$, then $j \overset{\eta_x}{\mapsto} (\sigma_x(j), j) \overset{\bar{\alpha_x}}{\mapsto} (\sigma_x(j), \alpha_x(j)) = (\sigma'_x\alpha_x(j), \alpha_x(j)) = \eta'_x(\alpha_x(j)$.

  Consider the following diagram

$$\begin{array}{ccccc} G(f\sigma') & \xrightarrow{\rho_{f\sigma'}} & \lim \tilde{f\sigma}'^{\star}F & \xrightarrow{l^{f\sigma'}_{\eta'}} & F(\tilde{f\sigma}'\eta') \\ \downarrow{\scriptstyle G(\alpha)} & & \downarrow{\scriptstyle f_\star F(\alpha)} & & \downarrow{\scriptstyle F(\alpha)} \\ G(f\sigma) & \xrightarrow{\rho_{f\sigma}} & \lim \tilde{f\sigma}^{\star}F & \xrightarrow{l^{f\sigma}_{\eta}} & F(\tilde{f\sigma}\eta) \end{array}$$

  The square in the left of the diagram commutes by naturality of ρ. For the square in the right, notice that the equation $\bar{\alpha}\eta = \eta'\alpha$ together with the fact that the limit is a cone gives us $l^{f\sigma'}_{\bar{\alpha}\eta} = F(\alpha)l^{f\sigma'}_{\eta'}$, and diagram 5 (taking τ as $f\sigma$) gives $l^{f\sigma}_{\eta}f_\star F(\alpha) = l^{f\sigma'}_{\bar{\alpha}\eta} = F(\alpha)l^{f\sigma'}_{\eta'}$.

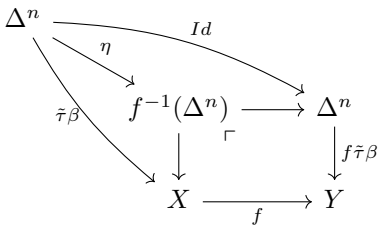
- $\Psi \circ \Phi = Id$.

  This is just the fact that for $\sigma : a \to X$ and η as in 4.1.2, $\Psi(\Phi(\phi))_\sigma = l^{f\sigma}_{\eta}\Phi(\phi)_{f\sigma} = \phi_{\tilde{f\sigma}\eta}G(f^{f\sigma}\eta) = G(Id)\phi_\sigma = \phi_\sigma$, for any $\phi : f^\star G \to F$.

- $\Phi \circ \Psi = Id$.

  Given $\rho : G \to f_\star F$, $\tau : \Delta^m \to Y$ and $\beta : \Delta^n \to f^{-1}(\Delta^m)$ (with $f^{-1}(\Delta^m)$ as in the diagram 3). We name η the morphism to the pullback of $f\tilde{\tau}\beta$

along f, that is, the morphism satisfying the commutative diagram



Since the component β of $\Phi(\Psi(\rho))_\tau$ is given by $\Psi(\rho)_{\tilde{\tau}\beta}G(f^\tau\beta) = l_\eta^{f\tilde{\tau}\beta}\rho_{f\tilde{\tau}\beta}G(f^\tau\beta)$, what we need to prove is that

$$l_\beta^\tau\rho_\tau = l_\eta^{f\tilde{\tau}\beta}\rho_{f\tilde{\tau}\beta}G(f^\tau\beta)$$

The naturality of ρ gives us $\rho_{f\tilde{\tau}\beta}G(f^\tau\beta) = f_\star F(f^\tau\beta)\rho_\tau$, so we just need to prove that

$$l_\beta^\tau = l_\eta^{f\tilde{\tau}\beta}f_\star F(f^\tau\beta)$$

Which comes from the diagram 5. We just need to prove that $\beta = \overline{f^\tau\beta}\eta$, for which we proceed component-wise: Let $x \in Obj(\Delta)$ and $j \in c(x)$, we have that $j \overset{\eta_x}{\mapsto} (\tilde{\tau}_x\beta_x(j), j) \overset{\overline{f_x^\tau\beta_x}}{\mapsto} (\tilde{\tau}_x\beta_x(j), f_x^\tau\beta_x(j)) = \beta_x(j)$.

- Ψ is natural respect to F and G.

  Let $F, F' \in ShX$ and $G, G' \in ShY$. For $\phi : F \to F'$, $\delta : G' \to G$, $\rho : G \to f_\star F$ and $\sigma : a \to X$ we need to prove that

  $$\phi_\sigma\Psi_{F,G}(\rho)\delta_{f\sigma} = \Psi_{F',G'}(f_\star(\phi)\rho\delta)_\sigma$$

  $\Psi_{F',G'}(f_\star(\phi)\rho\delta)_\sigma = (l_\eta^{f\sigma})'f_\star(\phi)_{f\sigma}\rho_{f\sigma}\delta_{f\sigma}$, and $\Psi_{F,G}(\rho) = l_\eta^{f\sigma}\rho_{f\sigma}$, with η as in 4.1.2. So we just need to prove that

  $$(l_\eta^{f\sigma})'f_\star(\phi)_{f\sigma} = \phi_\sigma l_\eta^{f\sigma}$$

  $f_\star(\phi)_{f\sigma}$ is the morphism induced by the natural transformation $\tilde{f\sigma}^\star(\phi) : \tilde{f\sigma}^\star F \to \tilde{f\sigma}^\star F'$, which satisfies the following commutative diagram

  $$\begin{array}{ccc} \lim \tilde{f\sigma}^\star F & \xrightarrow{l_\eta^{f\sigma}} & F(\sigma) \\ {\scriptstyle f_\star(\phi)_{f\sigma}}\downarrow & & \downarrow{\scriptstyle \tilde{f\sigma}^\star(\phi)_\eta} \\ \lim \tilde{f\sigma}^\star F' & \xrightarrow{(l_\eta^{f\sigma})'} & F'(\sigma) \end{array}$$

  So $(l_\eta^{f\sigma})'f_\star(\phi)_{f\sigma} = \tilde{f\sigma}^\star(\phi)_\eta l_\eta^{f\sigma} = \phi_{\tilde{f\sigma}\eta}l_\eta^{f\sigma} = \phi_\sigma l_\eta^{f\sigma}$.

□

We recapitulate some properties of the pair $(f_\star, f^\star)$. Observe how this property mimics the property given before for classical sheaves (theorem 2.3).

Proposition 4.0.2. *Given a morphism $f : X \to Y$ between simplicial (or delta) sets, we have the following*

i. $f^\star : \texttt{Rep}(Y) \leftrightarrows \texttt{Rep}(X) : f_\star$ *is an adjoint pair*

ii. $f^\star$ *commutes with limits and colimits. In particular for* $\boldsymbol{A}$ *abelian, it is exact.*

iii. $f^\star F(\tau) \simeq F(f(\tau))$ *for all* $\tau \in X$

iv. $f_\star$ *commutes with limits, in particular for* $\boldsymbol{A}$ *abelian, it is left exact.*

v. For an inclusion $\iota : X \hookrightarrow Y$ *we have* $\iota_\star F(\sigma) = F(\sigma)$ *for all* $\sigma \in X$*, furthermore the morphism* $\iota^\star \iota_\star \to Id$ *is an isomorphism.*

vi. $(f \circ g)_\star = f_\star \circ g_\star$ *and* $(f \circ g)^\star = g^\star \circ f^\star$

Proof. The first, second and fourth statements are direct from the construction as Kan extensions. The third statement is obvious by Yoneda embedding identification. For the last statement the commutativity of $(-)^\star$ is obvious and the commutativity of $(-)_\star$ follows by adjunction.

For the fifth statement we have that $\iota_\star F(\tau) = \lim_{\iota(\sigma)\to\tau} F(\sigma) \simeq F(\tau)$ since τ is final in $(\Delta \downarrow X) \downarrow \tau$ and ι is fully faithful. The isomorphism to the identity follows from this and the third statement. □

Notice how the property $f^\star F(\tau) \simeq F(f(\tau))$ is analogous to the property $f^\star F_x \simeq F_{f(x)}$ that we find in the topological setting. This analogy is supported by the theory that we are going to develop in the next subsection.

4.2 Δ-sheaves

Even when the closest representation in the category world of a simplicial (or delta) set X is its simplex category $\Delta \downarrow X$, there are some limitations or questions that arise when we want to perform sheaf theory. The first challenge is how to talk about sections. Inspired by the universal property

$$\begin{array}{ccc} \mathbf{SSet}^{op} & \xrightarrow{\exists! \bar{F}} & \mathbf{A} \\ {\scriptstyle h_\Delta}\uparrow & \nearrow_{\forall F} & \\ \Delta^{op} & & \end{array} \tag{6}$$

we can say that for $Y \leq X$ its section should be $\lim_{\Delta\downarrow Y} F$, and that we have for $Z \leq Y$ morphisms $F(Y) \to F(Z)$ given by the limit morphisms. Counting this, do gluing conditions hold for this limit morphisms?

This treatment also suggests that by performing these limits is actually transforming the area of work from simplices in $\Delta \downarrow X$ to subsimplicial sets $\texttt{Sub}(X)$, which would have a more geometrical tone in where we can more directly emulate this and other aspects of sheaf theory. This intuition is not only true, but also unfolds a rich theory of simplicial sheaves.

4.2.1 Presheaves on $\texttt{Sub}_\Delta(X)$

We begin with the key observation that follows 3.0.7 that there are topological spaces on a simplicial (or delta) set X as a set (observe that we said X and not

$|X|$) whose open sets are subsimplicial sets. The most obvious and the one that this section takes on focus is the set of simplices

$$\mathtt{Sub}(X) = \{Y \subset X | Y \text{ is a simplicial set}\}$$

As $(X, \mathtt{Sub}(X))$ is a topological space, we can consider presheaves and sheaves for it. We will call these categories $\mathtt{PSh}_\Delta(X)$ and $\mathtt{Sh}_\Delta(X)$ respectively. So for example the category of presheaves is just the category of functors

$$F : (\mathtt{Sub}(X)^{op}, \leq) \to \mathbf{A}$$

We observe that even though we inherit all the theory of sheaves done for topological spaces, one should take some care as the topology itself is bad behaved. For example this topological space is not Hausdorff since one cannot separate a simplex from its faces. Furthermore, the closed sets are not simplicial sets, which creates certain awkwardness in the theory. However (and perhaps surprisingly) this topological space(s) are good for our theory.

Observe that the last statement of 3.0.8 implies the following result.

Lemma 4.0.1. *For X a simplicial set, $\{\mathtt{Im}(\sigma) | \sigma \in X\}$ is a base for the topology $\mathtt{Sub}(X)$.*

We begin by asking the question for the stalks. If we have good memory we would recall that $\mathtt{Im}(\sigma)$ was called the best possible subsimplicial set associated to $\sigma \in X$. Observe that since $\mathtt{Sub}(X)$ is closed under intersection we have that

$$F_\sigma = \operatorname*{colim}_{\sigma \in Z} F(Z) \simeq F(\bigcap_{\sigma \in Z} Z) = F(\mathtt{Im}(\sigma))$$

which should hint that the stalks are in fact a very natural composition functor. In fact notice that since $\sigma < \tau \Rightarrow \mathtt{Im}(\sigma) \leq \mathtt{Im}(\tau)$ we have that

$$\mathtt{Im} : (\Delta \downarrow X)^{op} \to \mathtt{Sub}(X)^{op}$$

is a functor, and we can consider the precomposition functor

$$- \circ \mathtt{Im} : \mathtt{Func}(\mathtt{Sub}(X)^{op}, \mathbf{A}) \to \mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{A})$$

which is important enough as to be called with one letter

$$I : \mathtt{PSh}_\Delta(X) \to \mathtt{Rep}(X)$$

We will soon say more about this functor, for now observe that since the categories $(\Delta \downarrow X)^{op}$ and $\mathtt{Sub}(X)^{op}$ are small, and $\mathbf{A}$ is (co)complete, we have that the stalk functor I has left and right adjoints given by its Kan extensions.

We will be interested in the right Kan extension, whose formula is (given $F \in \mathtt{Rep}(X)$ and $Y \leq X$)

$$Ran_{\mathtt{Im}} F(Y) = \lim_{Y \downarrow \mathtt{Im}} F \circ Q^Y = \lim_{\sigma \in Y} F(\sigma) = \lim_{\Delta \downarrow Y^{op}} F$$

(where the second equality follows from $\sigma \in Y \Leftrightarrow \mathtt{Im}(\sigma) \leq Y$) As it follows the previous intuition, we will call $Ran_{\mathtt{Im}}$ the section functor and denote it as $\Gamma := Ran_{\mathtt{Im}}$, and we denote $\Gamma(Y, F) = \Gamma(F)(Y) = \lim_{\sigma \in Y} F(\sigma)$ the section on Y of the representation F.

Observe that for $F : \Delta^{op} \to \mathbf{A}$ we have that

$$\Gamma(Y, F) = \lim_{\Delta^n \xrightarrow{\sigma} Y} F(\Delta^n)$$

Which corresponds to the universal property (6).

We also have that through Γ, all our examples be also examples of objects of $\mathtt{PSh}_\Delta(X)$ (and will in fact be examples of $\mathtt{Sh}_\Delta(X)$ as we will soon see). In this case for example C_Δ takes its usual shape.

Observe that given its limit formula, for $\mathbf{A}$ abelian we have that for $Y \leq X$ the functor $\Gamma(Y, -) : \mathtt{Rep}(X) \to \mathbf{A}$ is left exact.

There are some natural questions that arise with the functors just presented. Is I (fully) faithful? How good is the adjunction $I \dashv \Gamma$? Is there a condition for $\Gamma(F)$ to be a sheaf? We will attend these questions on the following subsection.

We are going to consider morphisms $f : X \to Y$ which by 3.0.7 are continous and open functions, and this makes $(f^\star, f_\star)$ have a very simple shape at the level of presheaves

$$f_\star F(Z) = F(f^{-1}(Z))$$

and

$$f^\star F(Z) = F(f(Z))$$

which are in turn composition functors themselves and hence preserve limits and colimits (in $\mathtt{PSh}_\Delta(X)$).

Now, in going to sheaves, observe that $f^\star F$ is a sheaf if F is a sheaf. On the other hand $f_\star$ is not so lucky, although $f_\star F$ will be separated presheaf if F is, we do not have guaranty that $f_\star F$ is a sheaf when F is, so we sheafificate afterwards. Observe that $f_\star$ takes sheaves to sheaves when f is a monomorphism.

Also observe that since $f^\star F_\sigma = F(f(\mathtt{Im}(\sigma))) = F(\mathtt{Im}(f(\sigma))) = F_{f(\sigma)}$, we have that $f^\star$ is exact.

On the other hand, $f_\star$ has no reason to be right exact, even in good cases such as X being geo-friendly and f being a monomorphism. One context in which this functor is exact is if X is a simplicial complex and $f : \mathcal{G}(V(Z)) \hookrightarrow X$ is an inclusion from a fat subcomplex (this follows directly from corollary 3.1.2). It is easy to see however that $f_\star$ is left exact.

The following proposition explores the relation between these $(f^\star, f_\star)$ and the ones we had on the previous subsection

Proposition 4.0.3. *For a morphism $f : X \to Y$ the following diagrams commute*

- $$\begin{array}{ccc} \mathtt{PSh}_\Delta(X) & \xrightarrow{f_\star} & Psh_\Delta(Y) \\ {\scriptstyle\Gamma}\uparrow & & \uparrow{\scriptstyle\Gamma} \\ \mathtt{Rep}(X) & \xrightarrow{f_\star} & \mathtt{Rep}(Y) \end{array}$$

- $$\begin{array}{ccc} \mathtt{PSh}_\Delta(X) & \xleftarrow[f^\star]{} & Psh_\Delta(Y) \\ {\scriptstyle\Gamma}\uparrow & & \uparrow{\scriptstyle\Gamma} \\ \mathtt{Rep}(X) & \xleftarrow[f^\star]{} & \mathtt{Rep}(Y) \end{array}$$

•

$$\begin{array}{ccc} \mathcal{PSh}_\Delta(X) & \underset{f^\star}{\leftarrow} & Psh_\Delta(Y) \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I} \\ \mathcal{Rep}(X) & \underset{f^\star}{\leftarrow} & \mathcal{Rep}(Y) \end{array}$$

Proof. For the commutativity of $f^\star$ and Γ consider that

$$\Gamma f^\star G(Z) = \lim_{\sigma \in Z} G(f(\sigma)) = \lim_{\tau \in f(Z)} G(\tau) = f^\star \Gamma G(Z)$$

The commutativity of $f^\star$ and I goes as follows

$$f^\star I G(\sigma) = G(\mathtt{Im}(f(\sigma)) = G(f(\mathtt{Im}(\sigma))) = I f^\star G(\sigma)$$

And then the last commutativity follows from adjunction: as $\hom(F, f_\star \Gamma G) \simeq \hom(I f^\star F, G) \simeq \hom(f^\star I F, G) \simeq \hom(F, \Gamma f_\star G)$ then we can conclude that $f_\star \Gamma = \Gamma f_\star$ □

In general we do not have that $f_\star$ commutes with I, and this is given by the difference between the categories $\Delta \downarrow \mathtt{Im}(\tau)$ and $(\Delta \downarrow X) \downarrow \tau$ that we discussed before. If we were to solvent this problem, say for example by saying that X is a geo friendly delta set, then we would have the missing commutativity.

4.2.2 $Sh_\Delta(X)$ vs $\mathtt{Rep}(X)$

For this subsection we suppose that **A** is abelian. We will now explore further the adjunction

$$I : \mathtt{PSh}_\Delta(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$$

which in turn will give us a relation with sheaves, as the next proposition suggests

Proposition 4.0.4. *For any $F \in \mathcal{Rep}(X)$ we have that $\Gamma(F)$ is a sheaf.*

Proof. We use the formula for the limit on abelian categories, for $Y, Z \leq X$

$$\Gamma(F)(Y) = \{(s_\sigma)_{\sigma \in Y} \in \prod_{\sigma \in Y} F(\sigma) | F(\phi)(s_\sigma) = s_\tau \ \forall \tau \xrightarrow{\phi} \sigma\}$$

and for $Z \overset{\iota}{\hookrightarrow} Y$ we have $\Gamma(F)(Y) \to \Gamma(F)(Z)$ given by

$$(s_\sigma)_{\sigma \in Y}|_Z = (s_{\iota(\tau)})_{\tau \in Z} = (s_\tau)_{\tau \in Z}$$

Consider $Y = \bigcup_{i \in I} Y_i$. We will prove the two gluing conditions

(G1) Consider $s = (s_\sigma)_{\sigma \in Y} \in \Gamma(F)(Y)$ with $s|_{Y_i} = 0$ for all $i \in I$, then we have that $s_\tau = 0 \ \forall \tau \in Y_i$ and $\forall i \in I$. This means that $s_\sigma = 0$ for all $\sigma \in Y$.

(G2) Consider for all $i \in I$, $s^i \in \Gamma(F)(Y_i)$ with $s^i_{Y_i \cap Y_j} = s^j_{Y_i \cap Y_j}$ for all i, j

We define $s \in \Gamma(F)(Y)$ by $s_\sigma = s^i_\sigma$ whenever $\sigma \in Y_i$. By the equality $s^i_{Y_i \cap Y_j} = s^j_{Y_i \cap Y_j}$, s is well defined, and to see that $s \in \Gamma(F)(Y)$ we need to check that for a morphism $\tau \xrightarrow{\phi} \sigma$, $F(\phi)(s_\sigma) = s_\tau$. For this, notice that the existence of a morphism $\tau \to \sigma$ means that $\tau < \sigma$, so if $\sigma \in Y_i$ then $\tau \in Y_i$ and we use the equation $F(\phi)(s_\sigma) = s_\tau$ from $\Gamma(F)(Y_i)$. Finally, the equation $s|_{Y_i} = s^i$ is obviuos from the definition.

□

So the adjunction $I : \mathtt{PSh}_\Delta(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$ is actually an adjunction at the level of sheaves

$$I : \mathtt{Sh}_\Delta(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$$

Now, we can also consider $\Gamma \circ I : \mathtt{PSh}_\Delta(X) \to \mathtt{Sh}_\Delta(X)$. A audacious question here is whether $\Gamma \circ I$ coincides with the sheafification. The answer is yes.

Proposition 4.0.5. *For all $F \in \mathtt{PSh}_\Delta(X)$ we have that $\Gamma \circ I(F)$ is the sheafification of F.*

Proof. We have that $\Gamma(I(F))$ is given on sections by

$$\begin{aligned}\Gamma(I(F))(Y) &= \{(s_\sigma)_{\sigma\in Y} \in \prod_{\sigma\in Y} F(\mathtt{Im}(\sigma)) | I(F)(\phi)(s_\sigma) = s_\tau \ \forall \tau \xrightarrow{\phi} \sigma\} \\ &= \{(s_\sigma)_{\sigma\in Y} \in \prod_{\sigma\in Y} F(\mathtt{Im}(\sigma)) | s_\sigma|_{\mathtt{Im}(\tau)} = s_\tau \ \forall \tau < \sigma\}\end{aligned}$$

Since $I(F)(\phi) = F(\mathtt{Im}(\sigma) \leq \mathtt{Im}(\tau))$ for $\sigma \xrightarrow{\phi} \tau$. Meanwhile the sheafification is given on the sections by

$$\begin{aligned}F^\sharp(Y) = \{(s_\sigma)_{\sigma\in Y} \in \prod_{\sigma\in Y} F(\mathtt{Im}(\sigma)) | \forall \sigma \in Y \exists Z \leq Y \text{ with } \sigma \in Z \text{ and} \\ \exists t \in F(Z) \text{ such that } t_\eta = s_\eta \forall \eta \in Z\}\end{aligned}$$

These sets are equal

- $\subset$ Take $(s_\sigma)_{\sigma\in Y} \in F^\sharp(Y)$, and consider $\sigma \to \tau$ in Y. For τ there is $Z \leq Y$ with $\tau \in Z$‘ and $\exists t \in F(Z)$ such that $t_\eta = s_\eta \forall \eta \in Z$. In particular, $\sigma, \tau \in Z$, so $t|_{\mathtt{Im}(\tau)} = s_\tau$ and $t|_{\mathtt{Im}(\sigma)} = s_\sigma$, then

$$s_\tau|_{\mathtt{Im}(\sigma)} = t|_{\mathtt{Im}(\tau)}|_{\mathtt{Im}(\sigma)} = t|_{\mathtt{Im}(\sigma)} = s_\sigma$$

- $\supset$ Consider $(s_\sigma)_{\sigma\in Y} \in \Gamma(I(F))(Y)$, for $\sigma \in Y$ consider $Z = \mathtt{Im}(\sigma)$ and $t = s_\sigma \in \mathtt{Im}(\sigma)$. We have that $\forall \eta \in \mathtt{Im}(\sigma)$ there is a morphism $\eta \to \sigma$, to $s_\sigma|_{\mathtt{Im}(\eta)} = s_\eta$

And it is easy to see that the morphisms are the same by construction. We conclude that

$$\Gamma(I(F)) = F^\sharp$$

For all $F \in \mathtt{PSh}_\Delta(X)$ □

In particular we will get that

$$\Gamma \circ I(F) \simeq F \text{ for all } F \in \mathtt{Sh}_\Delta(X)$$

which suggests good properties for the adjunction $I : \mathtt{Sh}_\Delta(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$. In fact we will have the following

Proposition 4.0.6. *For X a simplicial (or delta) set we have the following*

1. $\Gamma : \mathtt{Rep}(X) \to \mathtt{Sh}_\Delta(X)$ *is essentially surjective.*

2. $I : \mathtt{Sh}_\Delta(X) \to \mathtt{Rep}(X)$ *is fully faithful.*

3. *If all* $\sigma \in X$ *are monomorphisms then* (I, Γ) *is an equivalence of categories.*

Proof. 1. This is just the fact that $\Gamma \circ I(F) \simeq F$ for all $F \in \mathtt{Sh}_\Delta(X)$

2. I is faithful since $f_{\mathtt{Im}(\sigma)} = g_{\mathtt{Im}(\sigma)}$ means that (f and g have the same stalks and therefore are equal. To see that I is full, consider $g : I(F) \to I(G)$. We want $f : F \to G$ with $f_{\mathtt{Im}(\sigma)} = g_\sigma$, and to obtain this we shall use the gluing properties.

   Consider $Y \leq X$ and $s \in F(Y)$. Observe that $Y = \bigcup_{\sigma \in Y} \mathtt{Im}(\sigma)$, and consider $(g_\sigma(s|_{\mathtt{Im}(\sigma)}))_{\sigma \in Y} \in \prod_{\sigma \in Y} G(\mathtt{Im}(\sigma))$, since $g_\sigma(s|_{\mathtt{Im}(\sigma)})|_{\mathtt{Im}(\tau) \cap \mathtt{Im}(\sigma)} = g_\tau(s|_{\mathtt{Im}(\tau)})|_{\mathtt{Im}(\tau) \cap \mathtt{Im}(\sigma)}$ (this is because for $\eta \in \mathtt{Im}(\tau) \cap \mathtt{Im}(\sigma)$ $g_\sigma(s|_{\mathtt{Im}(\sigma)})|_{\mathtt{Im}(\eta)} = g_\eta(s|_{\mathtt{Im}(\eta)}) = g_\tau(s|_{\mathtt{Im}(\tau)})|_{\mathtt{Im}(\eta)}$) we have that there is (a unique) $t \in G(Y)$ with $t|_{\mathtt{Im}(\sigma)} = g_\sigma(s|_{\mathtt{Im}(\sigma)})$ for all $\sigma \in Y$. We define

   $$f_Y(s) = t$$

   We obtain from the unicity of t that f is a linear function on each section. Furthermore, it is obvious from the construction that $f_{\mathtt{Im}(\sigma)} = g_\sigma$ for all $\sigma \in X$.

   We are left to prove that it is a morphism of sheaves, so consider $Z \leq Y \leq X$. We need to prove that for $s \in F(Y)$ we have that $f_Z(s|_Z) = f_Y(s)|_Z$. Given $s \in F(Y)$, observe that for all $\sigma \in Z$ we have that

   $$f_Y(s)|_Z|_{\mathtt{Im}(\sigma)} = f_Y(s)|_{\mathtt{Im}(\sigma)} = g_\sigma(s|_{\mathtt{Im}(\sigma)}) = f_Z(s|_Z)|_{\mathtt{Im}(\sigma)}$$

   Here the last equality uses that $s|_Z|_{\mathtt{Im}(\sigma)} = s|_{\mathtt{Im}(\sigma)}$. So using that $Z = \bigcup_{\sigma \in Z} \mathtt{Im}(\sigma)$ and the gluing condition we conclude that $f_Z(s|_Z) = f_Y(s)|_Z$

3. It will suffice to prove that $I(\Gamma(F)) \simeq F$ for all $F \in \mathtt{Rep}(X)$. Observe that by proposition 3.1 we have that for all $\sigma \in X$

   $$I(\Gamma(F))(\sigma) = \lim_{\Delta \downarrow \mathtt{Im}(\sigma)} F = \lim_{(\Delta \downarrow X) \downarrow \sigma} F \simeq F(\sigma)$$

   The last isomorphism corresponds to the limit morphism l^σ_σ, and it is an isomorphism since σ is final in $(\Delta \downarrow X) \downarrow \sigma$. To prove that this gives an morphism of sheaves, we first give some names. Consider a morphism $\sigma \to \tau$ in $\Delta \downarrow X$

   - We call $l^\alpha_\eta : \lim_{\mathtt{Im}(\alpha)} \to F(\eta)$ the limit morphisms. These satisfy that $F(\eta \to \eta') \circ l^\alpha_{\eta'} = l^\alpha_\eta$
   - We call $l^{\sigma\tau} : \lim_{\mathtt{Im}(\tau)} F \to \lim_{\mathtt{Im}(\sigma)} F$ the morphism induced by the inclusion of categories $\Delta \downarrow \iota : \Delta \downarrow \mathtt{Im}(\sigma) \to \Delta \downarrow \mathtt{Im}(\tau)$. This morphism satisfies that $l^\sigma_\eta \circ l^{\sigma\tau} = l^\tau_\eta$ for all $\eta < \sigma$. Observe that $l^{\sigma\tau} = \Gamma(F)(\mathtt{Im}(\sigma) \leq \mathtt{Im}(\tau)) = I(\Gamma(F))(\sigma \to \tau)$.

So to prove that $(l^\sigma_\sigma)_{\sigma\in X}$ defines a morphism, we need to prove that for all $\sigma \to \tau$ we have $F(\sigma \to \tau)\circ l^\tau_\tau = l^\sigma_\sigma \circ l^{\sigma\tau}$, which follows easily from what we have

$$F(\sigma \to \tau) \circ l^\tau_\tau = l^\tau_\sigma = l^\sigma_\sigma \circ l^{\sigma\tau}$$

□

We finalize this section given some examples of simplicial sheaves. Recall from section 2.2.1 that a filtration of a simplicial set X is a morphism $X \to \Delta^n$, and these define a category $\mathbf{SSet}^{[n]}_{\mathcal{F}} = \mathbf{SSet} \downarrow \Delta^n = \mathtt{Func}((\Delta^{[n]}_{\mathcal{F}})^{op}, \mathbf{Set})$. Each element $\sigma : \Delta^m \to X$ has a filtration $\sigma : \Delta^{(j_0,\cdots,j_n)} \to X$ induced by the composition $\Delta^m \xrightarrow{\sigma} X \to \Delta^n$, and such element is called *regular* if $j_n > -1$.

Examples

1. For an object $M \in \mathbf{A}$, the easiest example to consider is the constant functor

$$\begin{aligned} \underline{\mathrm{M}} : \Delta^{op} &\longrightarrow \mathbf{A} \\ [n] &\longmapsto M \end{aligned}$$

This defines a representation $\underline{\mathrm{M}} : (\Delta \downarrow X)^{op} \to \mathbf{A}$ that sends each simplex to M and each morphism to 1_M. We also have a functor $\bar{M} : \mathtt{Sub}(X)^{op} \to \mathbf{A}$ given by $\bar{M} = \Gamma(\underline{\mathrm{M}})$. This defines a sheaf which we call the **constant sheaf**.

Observe that $\bar{M}$ *is not* the presheaf that sends any sub simplicial set to M (what we call **constant presheaf**), since for instance

$$\bar{M}(\Delta^0 \sqcup \Delta^0) = \underset{\sigma\in\Delta^0\sqcup\Delta^0}{\mathtt{colim}} M = M \oplus M$$

We have instead that $\bar{M}$ *is the sheafification* of the constant presheaf.

2. Given X a simplicial (or delta) set and a ring R, we set

$$C_n(X) = RX_n$$

The group generated by X_n, with the differentials $\partial(\sigma) = \sum_k (-1)^k d_k(\sigma)$ this defines a chain complex in $Ch(R)$. Consider the dual

$$C^n(X) = \hom_R(C_n(X), R)$$

For any morphism $f : X \to Y$, the precomposition with f defines a morphism

$$\begin{aligned} \bar{f} : C^n(Y) &\longrightarrow C^n(X) \\ (C_n(Y) \xrightarrow{\alpha} R) &\longmapsto \bar{f}(\alpha) : \sigma \in X_n \mapsto \alpha(f(\sigma)) \end{aligned}$$

This assignment makes C^n into a functor

$$C^n : \mathbf{SSet}^{op} \to R\text{-mod}$$

Furthermore, if we define differentials as it is common $\delta = \partial^\star$, we obtain a functor

$$C^\cdot : \mathbf{SSet}^{op} \to Ch(R)$$

It is not hard to see that these functors are in fact the extension along the Yoneda embedding $\Delta^{op} \hookrightarrow \mathbf{SSet}^{op}$ of

$$\begin{aligned} C^n : \Delta^{op} &\longrightarrow R\text{-mod} \\ [n] &\longmapsto C^n(\Delta^n) \end{aligned}$$

From which we would define $C^n \in \mathtt{Rep}(X)$ and its corresponding $C^n \in Sh_\Delta(X)$ with image in either R-mod or $Ch(R)$.

3. For $\mathbb{K}$ a field of characteristic 0, we might construct what are called the *Sullivan polynomials* A_{PL} (whose construction and properties are given in detail in [FHT01] Chapter II.10). We set a functor

$$A_{PL} : \Delta^{op} \to Ch(\mathbb{K})$$

Defined on a representable Δ^n to be

$$A_{PL}(\Delta^n) = \bigwedge(t_0, \cdots, t_n, y_0, \cdots, y_n)/(\sum t_i - 1, \sum y_j)$$

With differencials given by $dt_i = y_i$ and $dy_i = 0$, and degrees $|y_i| = 1$ and $|t_i| = 0$. The action on $\delta : [n] \to [m]$ is defined by $A_{PL}(\delta)(t_k) = t_{\delta(k)}$. This functor can be extended to a representation

$$A_{PL} : (\Delta \downarrow X)^{op} \to Ch(\mathbb{K})$$

And furthermore to a simplicial sheaf

$$A_{PL} : \mathtt{Sub}(X)^{op} \to Ch(\mathbb{K})$$

By taking $A_{PL}(Y) = \lim_{\sigma:\Delta^n \to Y} A_{PL}(\Delta^n)$.

It is proven in ([FHT01] Theorem 10.5) that there is a quasi isomorphism $A_{PL} \to C^\cdot_\Delta$, hence, A_{PL} computes the classical cohomology, with the advantage that A_{PL} is commutative.

4. We can generalize the construction of the **blown-up cochains** given in secion 2.2.1. For a simplicial set with a filtration $X \to \Delta^n$, consider $FSimp(X)$ the full subcategory of $(\Delta \downarrow X)^{op}$ with regular simplices as objects. Given a functor $F : \Delta^{op} \to Ch(R)$ we define

$$\mathcal{N}^\star_{(-)} : FSimp(X) \to Ch(R)$$

By sending σ to $\mathcal{N}^\star_\sigma = F(c\Delta^{j_0}) \otimes \cdots \otimes F(c\Delta^{j_{n-1}}) \otimes F(\Delta^{j_n})$. This gives a *regular* representation which can be extended to a simplicial sheaf by considering

$$\Gamma(\mathcal{N}^\star_{(-)})(Y) = \lim_{\sigma \in FSimp(Y)} \mathcal{N}^\star_\sigma$$

In particular blown-up cochains serve as an example of our theory.

5. We can take the *intersection version* of $C^\Delta_\star$ on $X \in \mathbf{SSet}^{[n]}_{\mathcal{F}}$ by considering the k-th perverse degree of a filtered simplex $\sigma : \Delta^{(j_0,\cdots,j_n)} \to X$ to be

$$|\sigma|_k = \begin{cases} \dim(\Delta^{j_0,\ldots,j_{n-k}}) & \text{if } \Delta^{j_0,\ldots,j_{n-k}} \neq \emptyset \\ -\infty & \text{if } \Delta^{j_0,\ldots,j_{n-k}} = \emptyset \end{cases}$$

We say that σ is p-**allowable** if $|\sigma|_k \leq \dim(\Delta^{(j_0,\cdots,j_n)}) - k + p(k)$ for all $k \in \{0,\ldots,n\}$, and we say that $c \in C_m^\Delta$ is p-intersection if all its simplices are p-allowable.

We construct a functor

$$IC_p^\star : \Delta^{op} \to Ch(R)$$

By taking the dual of $IC_m^p(\Delta^k) = \{c \in C_m^\Delta(\Delta^k) | c \text{ and } \partial c \text{ are } p\text{-intersection}\}$ with the differentials induced by the ones in $C_\star^\Delta$. This ones again defines a representation that is extended to a simplicial sheaf

$$IC_p^\star : \mathtt{Sub}(X)^{op} \to Ch(R)$$

6. We can also take the *intersection version* of the construction of example 3. (in particular of the blown-up cochain), by defining as in section 2.2.1

   Definition. *Given $\sigma : \Delta^{(j_0,\cdots,j_n)} \to X$ a regular simplex and $k \in \{1,\ldots,n\}$, we might consider the restriction*

   $$r_k : \mathcal{N}_\sigma^\star \to F(c\Delta^{j_0}) \otimes \cdots F(c\Delta^{j_{n-k-1}}) \otimes F(\Delta^{j_{n-k}}) \otimes F(c\Delta^{j_{n-k+1}}) \cdots \otimes F(\Delta^{j_n})$$

   For each $\gamma \in \mathcal{N}_\sigma^\star$ the cochain $r_k(\gamma) = \sum a_i \otimes b_i$ with $\{a_i\}$ being a basis of $F(c\Delta^{j_0}) \otimes \cdots \otimes F(c\Delta^{j_{n-k}})$ and $b_i \in F(c\Delta^{j_{n-k+1}}) \otimes \cdots \otimes F(\Delta^{j_n})$. We set

   $$|\gamma|_k = \begin{cases} \mathtt{max}\{\mathtt{deg}(b_i) | b_i \neq 0\} & \text{if } r_k(\gamma) \neq 0 \\ -\infty & \text{if } r_k(\gamma) = 0 \end{cases}$$

   *And we call the k-**th perverse degree** of a cochain $c \in \mathcal{N}^\star(X)$ to*

   $$|c|_k = \mathtt{sup}\{|c_\sigma|_k | \sigma \in FSimp(X)\}$$

   Where c_σ corresponds to the image of c on the colimit.

   With this we can define the intersection blown-up cochains. We say that $c \in \mathcal{N}^\star(X)$ is p-**allowable** if $|c|_k \leq p(k)$ for all k. We define a simplicial presheaf

   $$\mathcal{N}_p^\star : \mathtt{Sub}(X)^{op} \to Ch(R)$$

   by setting $\mathcal{N}_p^\star(Y) = \{c \in \mathcal{N}^\star(Y) | c \text{ and } \partial c \text{ are } p\text{-allowable}\}$.

4.2.3 Topologies other than $\mathtt{Sub}(X)$

Even beautiful as it is, the category $(\mathtt{Sub}(X), \leq)$ is not always the best environment to work on. We might wish for example that the stalks correspond to evaluation on stars rather than evaluation on images. For our work in fact it will be necessary to select a different topology.

Now, a topology can actually be composed by any subsets, but being faithful to simplicial structures, we are going to be interested only in the topologies whose open sets are taken from $\mathtt{Sub}(X)$.

So, given one such topology $T \subset \mathtt{Sub}(X)$ we will have an inclusion that is in turn a fully faithful functor of categories $\iota_T : (T, \leq) \to (\mathtt{Sub}(X), \leq)$ which will give a restriction functor

$$- \circ \iota_T : \mathtt{Func}(\mathtt{Sub}(X)^{op}, \mathbf{A}) \to \mathtt{Func}(T^{op}, \mathbf{A})$$

We will call $\mathtt{PSh}_\Delta^T$ and $\mathtt{Sh}_\Delta^T$ the categories of presheaves and sheaves on T. This restriction functor as it is a precomposition functor (and we have that the source categories are small plus that $\mathbf{A}$ is (co)complete) will have left and right adjoints given by the Kan extensions

$$\mathtt{PSh}_\Delta(X) \underset{\overleftarrow{Lan_{\iota_T}}}{\overset{\overleftarrow{Ran_{\iota_T}}}{\overset{\longleftarrow}{-\circ\iota_T}}} \mathtt{PSh}_\Delta^T(X)$$

Here, the left Kan extension has a very familiar formula

$$Lan_{\iota_T}F(Y) = \underset{Y\leq Z\in T}{\mathtt{colim}}\ F(Z)$$

The Kan diagram give us a way to relate to $\mathtt{Rep}(X)$

$$\mathtt{Rep}(X) \underset{I}{\overset{\Gamma}{\rightleftarrows}} \mathtt{PSh}_\Delta(X) \underset{\overleftarrow{Lan_{\iota_T}}}{\overset{\overleftarrow{Ran_{\iota_T}}}{\overset{\longleftarrow}{-\circ\iota_T}}} \mathtt{PSh}_\Delta^T(X)$$

In particular we have an adjunction to presheaves $I \circ Lan_{\iota_T} \dashv (- \circ \iota_T) \circ \Gamma$. One might wonder if this adjunction is valid for sheaves. The answer is no, because Lan_{ι_T} does not in general take sheaves into sheaves. However, the restriction functor $- \circ \iota_T$ does take sheaves into sheaves, and hence we have that

$$F \in \mathtt{Rep}(X) \Rightarrow \Gamma(F) \in \mathtt{Sh}_\Delta^T(X)$$

4.3 Generalization to presheaf categories

This section follows from the observation that what we have developed for simplicial and delta sets can actually be developed for other presheaf categories. We are in particular interested in the simpliest of presheaf categories, that is $\mathbf{Set} = \mathtt{Func}(\cdot^{op}, \mathbf{Set})$.

We describe very briefly how the theory will go for a presheaf category different from the ones we have seen. As the reader will notice, the theory is the same as for simplicial or delta sets. The proofs of what follows are the same as the corresponding proofs of simplicial or delta sets.

4.3.1 Generalization of the theory in a nutshell

Given $\mathbf{C}$ a small category, we call $\mathbf{C}^\vee = \mathtt{Func}(\mathbf{C}^{op}, \mathbf{Set})$ the presheaf category associated to it. An object $X \in \mathbf{C}^\vee$ has a corresponding set given by $X = \coprod_{c\in\mathbf{C}} X(c)$. A subobject is a subset $Y \subset X$ such that the inclusion $Y \hookrightarrow X$ is a morphism. We have that $\bigcup Y_i$, $\bigcap Y_i$, $\mathtt{Im}(f)$, $f(Y)$, $f^{-1}(Z)$ are subobjects, so in particular $\mathtt{Sub}_\mathbf{C}(X)$ is a topology on X. A morphism $f : X \to Y$ on $\mathbf{C}^\vee$ defines $f : \mathbf{C} \downarrow X \to \mathbf{C} \downarrow Y$ by composition. We define

$$\mathtt{Rep}(X) = \mathtt{Func}((\mathbf{C} \downarrow X)^{op}, \mathbf{A})$$

And we call $f^\star = - \circ f : \mathtt{Rep}(Y) \to \mathtt{Rep}(X)$ for $f : X \to Y$. As before, it has left and right adjoints. We call $f_\star = Ran_f$ its right adjoint.

Since $\mathtt{Sub}_{\mathbf{C}}(X)$ defines a topology, we can consider (pre)sheaves over it, which we call $\mathtt{PSh}_{\mathbf{C}}(X)$ and $\mathtt{Sh}_{\mathbf{C}}(X)$. We have as before a functor

$$\mathtt{Im} : \mathbf{C} \downarrow X \to \mathtt{Sub}_{\mathbf{C}}(X)$$

with which we define $I = - \circ \mathtt{Im}$, and we have that for all $F \in \mathtt{PSh}(X)$ and $\sigma \in X$

$$I(F)(\sigma) = F(\mathtt{Im}(\sigma)) = F(\bigcap_{\sigma \in Y} Y) = \operatorname*{\mathtt{colim}}_{\sigma \in Y} F(Y) = F_\sigma$$

that is, evaluations on $I(F)$ correspond to stalks.

$\Gamma(F)(Y) = \lim_{(\mathbf{C}\downarrow Y)^{op}} F$ defines the right adjoint of I. In $\mathtt{Sh}_{\mathbf{C}}(X)$ there are $f_\star$ and $f^\star$ as before which commute with (I, Γ) in the way shown in proposition 4.0.3.

We have $\forall F \in \mathtt{Rep}(X)$ that $\Gamma(F) \in \mathtt{Sh}_{\mathbf{C}}(X)$, furthermore, $\Gamma(I(F))$ is the sheafification of F. We also have that $I : \mathtt{Sh}_{\mathbf{C}}(X) \to \mathtt{Rep}(X)$ is fully faithful, and if X is geo friendly (that is, all its elements are monomorphisms) then

$$I : \mathtt{Sh}_{\mathbf{C}}(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$$

is an equivalence of categories.

Other than **SSet** and **DSet**, the category in which we are interested to apply this setting is the easiest of all presheaf categories, that is **Set**

4.3.2 The case of the category Set

The category **Set** as a presheaf category corresponds to $\mathtt{Func}(\cdot^{op}, \mathbf{Set})$, where $\cdot$ is the category with one object and one morphism.

We have that for a set A, the category $\cdot \downarrow A$ is just A, that is, the category whose objects are elements of A and has only identity morphisms. This implies that for all $a, b \in A$ we have $\sharp \hom(a, b) \leq 1$, so all sets are geo friendly.

The subobject relation $\leq$ corresponds to the subset relation $\subset$, and given $a \in A$ we have

$$\mathtt{Im}(a) = \{a\} \subset A$$

A function $f : X \to Y$ defines $f : \cdot \downarrow X \to \cdot \downarrow Y$ which is itself. The category

$$\mathtt{Rep}(X) = \mathtt{Func}((\cdot \downarrow X)^{op}, \mathbf{A})$$

has functors $F : X \to \mathbf{A}$ which assign to elements $x \in X$ objects $F(x) \in \mathbf{A}$ that have no relationship between them. A universal functor

$$F : \cdot \to \mathbf{A}$$

is just an object $F(\cdot) = M \in \mathbf{A}$. It defines $F : X \to \mathbf{A}$ as the constant $F(x) = M$ for all $x \in X$.

A function $f : X \to Y$ defines a functor $f^\star : \mathtt{Rep}(Y) \to \mathtt{Rep}(X)$ by $f^\star(F)(x) = F(f(x))$, which has a right adjoint $f_\star$ given by

$$f_\star G(y) = \lim_{f(x)=y} G(x) = \prod_{x \in f^{-1}(y)} G(x)$$

And they satisfy the properties of proposition 4.0.2.

We have that $\mathtt{Sub}(X) = \mathcal{P}(X)$ is the set of subsets of X, it forms a topology and therefore we can construct (pre)sheaves over it. Observe that these are sheaves in the most classical of ways, and since $\mathcal{P}(X)$ is the discrete topology, we shall expect toy-like characteristics for these (pre)sheaves.

We have as before that $\mathtt{Im} : X \to \mathcal{P}(X)$ (which sends $x \in X$ to $\{x\} \in \mathcal{P}(X)$) defines

$$I = - \circ \mathtt{Im} : \mathtt{PSh}(X) \to \mathtt{Rep}(X)$$

which acts as $I(F)(x) = F(\{x\}) = \lim_{x \in A} F(A) = F_x$, corresponding to the stalks. It has a right adjoint $\Gamma : \mathtt{Rep}(X) \to \mathtt{Sh}(X)$ which acts as follows

$$\Gamma(F)(Y) = \lim_{y \in Y} F(y) = \prod_{y \in Y} F(y)$$

This means that the sheafification of a presheaf has the simple formula of

$$F^\sharp(Y) = \prod_{y \in Y} F_y$$

And since all sets $X \in \mathbf{Set}$ are geo friendly, the adjunction

$$I : \mathtt{Sh}(X) \leftrightarrows \mathtt{Rep}(X) : \Gamma$$

is an equivalence of categories which completely commutes with $(f^\star, f_\star)$ as in proposition 4.0.3.

Although these results might not seem very impressive, when we introduce topologies subtler than $\mathcal{P}(X)$ (observe that this means working in the classical theory of sheaves) it can give interesting insights. For a topology T on X, the inclusion $\iota_T : T \hookrightarrow \mathcal{P}(X)$ defines a fully faithful functor

$$\iota_T : (T, \subset) \to (\mathcal{P}(X), \subset)$$

which defines $- \circ \iota_T : \mathtt{PSh}(X) \to \mathtt{Func}(T^{op}, \mathbf{A}) = \mathtt{PSh}^T(X)$. As T is small and $\mathbf{A}$ is complete and cocomplete, we have the triple of adjoints

$$\mathtt{PSh}(X) \underset{\xrightarrow{Lan_{\iota_T}}}{\overset{\xleftarrow{Ran_{\iota_T}}}{\xrightarrow[-\circ\iota_T]{}}} \mathtt{PSh}^T(X)$$

As ι_T is fully faithful, Ran_{ι_T} and Lan_{ι_T} are extensions, that is, for all $U \in T$ we have that $Ran_{\iota_T}F(U) = F(U) = Lan_{\iota_T}F(U)$

We have simple formulas for this extensions, for a subset $Y \subset X$

$$Ran_{\iota_T}(Y) = \lim_{U \subset Y} F(U) = F(\mathtt{int}(Y))$$

and

$$Lan_{\iota_T}F(Y) = \underset{Y \subset U}{\mathtt{colim}}\ F(U)$$

This last functor is very popular throughout sheaf theory. Basically any time we cannot define the evaluation of a sheaf on a subset of X we use Lan_{ι_T}. Let us see three examples of this usage

- The stalks of a (pre)sheaf are obtained as

$$F_x = Lan_{\iota_T} F(\{x\})$$

  which is also equal to $(Lan_{\iota_T} F)_x$.

- For a closed subset $Z \subset X$ the section of Z is given by $\Gamma(Z, F) = Lan_{\iota_T} F(Z)$

- For a continuous function $f : X \to Y$ we have that $f^\star F$ is the sheafification of the presheaf defined on the sections as

$$f^\cdot F(Y) = Lan_{\iota_T} F(f(Y))$$

Given the discussion of this subsection, we note that the generalization to presheaf categories of our theory is also one way to generalize the classical sheaf theory.

4.4 Functors between simplicial and classical sheaves

During this section, we wish to present natural functors between the simplicial and the topological environments at the level of presheaves and sheaves. Having both the classical and the simplicial theory of (pre)sheaves been placed under the same theory during the last section, we can make use of this machinery to state very natural ways to relate these theories via functors.

The key example motivating the following discussion corresponds to the most classical one

$$C^\cdot(X) = C_\Delta(Sing(X))$$

Observe that here we are obtaining $C^\cdot$ by applying the pre-composition functor

$$- \circ Sing : \mathtt{Func}(\mathtt{Sub}(Sing(X))^{op}, R\text{-mod}) \to \mathtt{Func}(O(X)^{op}, R\text{-mod})$$

to $C_\Delta \in \mathtt{PSh}_\Delta(X)$. This calls for a generalization.

In order to make the generalization we need to make a little detour into **C**-topologies.

Definition. *Given as in last section a small category* $\boldsymbol{C}$ *and an object* $X \in \boldsymbol{C}^\vee = \mathtt{Func}(\boldsymbol{C}^{op}, \mathtt{Set})$*, a* $\boldsymbol{C}$*-topological space over* X *is a set* $T \subset \mathtt{Sub}_{\boldsymbol{C}}(X)$ *that satisfies*

- $\emptyset, X \in T$
- $A, A' \in T \Rightarrow A \cap A' \in T$
- $\{A_i\}_{i \in I} \subset T \Rightarrow \bigcup_{i \in I} A_i \in T$

We also can define **C**-continuous functions

Definition. *Given* $X, Y \in \boldsymbol{C}^\vee$ *with topologies* T_X *and* T_Y *respectively, a morphism* $f : X \to Y$ *is* $\boldsymbol{C}$*-continuous if* $f^{-1}(A) \in T_X \ \forall A \in T_Y$

Given $Z \leq X$, the *topology induced by* Z is the optimal topology that makes the inclusion morphism $Z \hookrightarrow X$ $\mathbf{C}$-continuous.

We call $\mathbf{Top_C}$ the category whose objects are $\mathbf{C}$-topological spaces and whose morphisms are $\mathbf{C}$-continuous morphisms. We have a forgetful functor $\textsc{f} : \mathbf{Top_C} \to \mathbf{C}^\vee$ that takes (X, T_X) to X.

By composing with $\textsc{f}$, we can construct from a functor $\Psi : \mathbf{C}^\vee \to \mathbf{Top_D}$ a functor $\Psi : \mathbf{Top_C} \to \mathbf{Top_D}$. We are specifically planning to do this with the realization functor $|\cdot| : \mathbf{SSet} \to \mathbf{Top}$.

Now consider either a functor $\Psi : \mathbf{Top_C} \to \mathbf{D}$ or a functor $\Psi : \mathbf{Top_C} \to \mathbf{Top_D}$ such that $Z \leq Y \Rightarrow \Psi(Z) \leq \Psi(Y)$. For $Y \in T_X$, we give $\Psi(Y)$ the induced topology, as $\Psi(Y) \leq \Psi(X)$. We have that Ψ defines the precomposition functor

$$- \circ \Psi : \mathtt{Func}(\mathtt{Sub_D}(\Psi(X))^{op}, \mathbf{A}) \to \mathtt{Func}(T_X^{op}, \mathbf{A})$$

Since $\mathtt{Sub_D}(\Psi(X))^{op}$ is small and $\mathbf{A}$ is (co)complete, we have a triple of adjoints

$$\mathtt{Func}(\mathtt{Sub_D}(\Psi(X))^{op}, \mathbf{A}) \underset{\overleftarrow{Lan_\Psi}}{\overset{\overleftarrow{Ran_\Psi}}{\overset{\longrightarrow}{\underset{\overleftarrow{Lan_\Psi}}{-\circ\Psi}}}} \mathtt{Func}(T_X^{op}, \mathbf{A})$$

Now, if we set a topology $T_{\Psi(X)}$ on $\Psi(X)$ (which in the case of a functor $\Psi : \mathbf{Top_C} \to \mathbf{Top_D}$ comes automatically with Ψ) we can use the inclusion functor $(T_{\Psi(X)}, \leq) \overset{\iota}{\hookrightarrow} (\mathtt{Sub_D}(\Psi(X)), \leq)$ to get the adjoints

$$\mathtt{Func}(T_{\Psi(X)}^{op}, \mathbf{A}) \underset{\overrightarrow{Lan_\iota}}{\overset{\overrightarrow{Ran_\iota}}{\overset{\longleftarrow}{\underset{\overrightarrow{Lan_\iota}}{-\circ\iota}}}} \mathtt{Func}(\mathtt{Sub_D}(\Psi(X))^{op}, \mathbf{A}) \underset{\overleftarrow{Lan_\Psi}}{\overset{\overleftarrow{Ran_\Psi}}{\overset{\longrightarrow}{\underset{\overleftarrow{Lan_\Psi}}{-\circ\Psi}}}} \mathtt{Func}(T_X^{op}, \mathbf{A})$$

By composing these we get a pair of adjoints

$$\mathtt{PSh}_{\mathbf{D}}^{T_{\Psi(X)}}(\Psi(X)) \underset{(-\circ\Psi)\circ Lan_\iota}{\overset{(-\circ\iota)\circ Ran_\Psi}{\overset{\longleftarrow}{\longrightarrow}}} \mathtt{PSh}_{\mathbf{C}}^{T_X}(X)$$

$$\mathtt{PSh}_{\mathbf{D}}^{T_{\Psi(X)}}(\Psi(X)) \underset{(-\circ\iota)\circ Lan_\Psi}{\overset{(-\circ\Psi)\circ Ran_\iota}{\overset{\longrightarrow}{\longleftarrow}}} \mathtt{PSh}_{\mathbf{C}}^{T_X}(X)$$

We present the formulas in sections of each of these functors

- $(- \circ \Psi) \circ Ran_\iota F(Y) = \underset{\Psi(Y) \geq Z \in T_{\Psi(X)}}{\mathtt{lim}} F(Z)$
- $(- \circ \Psi) \circ Lan_\iota F(Y) = \underset{\Psi(Y) \leq Z \in T_{\Psi(X)}}{\mathtt{colim}} F(Z)$
- $(- \circ \iota) \circ Ran_\Psi F(Y) = \underset{\Psi(Z) \leq Y, Z \in T_X}{\mathtt{lim}} F(Z)$
- $(- \circ \iota) \circ Lan_\Psi F(Y) = \underset{Y \leq \Psi(Z), Z \in T_X}{\mathtt{colim}} F(Z)$

Observe that for the first two formulas, whenever we have that $\Psi(Y) \in T_{\Psi(Y)}$, then

$$(- \circ \Psi) \circ Ran_\iota F(Y) \simeq (- \circ \Psi) \circ Lan_\iota F(Y) \simeq F(\Psi(Y))$$

Similarly, for the last two formulas, when $Y = \Psi(W)$ for some $W \in T_X$ we have

$$(- \circ \iota) \circ Ran_\Psi F(\Psi(W)) \simeq (- \circ \iota) \circ Lan_\Psi F(\Psi(W)) \simeq F(W)$$

When required we can extend this constructions to sheaves by composing with the sheafification adjoint pair $((-)^\sharp, \iota_{Sh})$ as follows

$$\mathtt{Sh}_{\mathbf{D}}^{T_{\Psi(X)}}(\Psi(X)) \underset{(-)^\sharp}{\overset{\iota_{Sh}}{\rightleftarrows}} \mathtt{PSh}_{\mathbf{D}}^{T_{\Psi(X)}}(\Psi(X)) \rightleftarrows \mathtt{PSh}_{\mathbf{C}}^{T_X}(X) \underset{(-)^\sharp}{\overset{\iota_{Sh}}{\rightleftarrows}} \mathtt{Sh}_{\mathbf{C}}^{T_X}(X)$$

As for examples of this construction we shall consider the classical adjunction

$$|-| : \mathbf{SSet} \leftrightarrows \mathbf{Top} : Sing$$

When applying it to the functor $Sing : \mathbf{Top} \to \mathbf{SSet}$ we obtain the relation discussed at the beginning of the section

$$- \circ Sing : \mathtt{PSh}_\Delta(Sing(X)) \to \mathtt{PSh}(X)$$

As for $|-| : \mathbf{SSet} \to \mathbf{Top}$, we can define $|-| : \mathbf{Top}_\Delta \to \mathbf{Top}$ composing with the forgetful functor, and then apply the machinery just described. In particular we obtain for $X \in \mathbf{SSet}$

$$\Phi = (- \circ |-|) \circ Lan_\iota : \mathtt{PSh}^{T_{|X|}}(|X|) \to \mathtt{PSh}_\Delta(X)$$

Defined in the sections by

$$\Phi(F)(Y) = \underset{|Y| \subset U \in T_{|X|}}{\mathtt{colim}} F(U) = \Gamma(|Y|, F)$$

This functor has a right adjoint at the level of presheaves given again by the machinery presented in this section

$$\Phi'(F)(U) = \underset{|Z| \subset U, Z \leq X}{\mathtt{lim}} F(Z)$$

We will see in section 6.2.2 that given the correct setting, Φ defines a functor between sheaves (that is, if F is a sheaf then $\Phi(F)$ is a sheaf). Furthermore, this is the functor that we use to state the main theorem.

5 Adding model/derived structure

In this chapter we take the second step in our theory, which corresponds to localizing the categories just described with respect to *quasi isomorphisms*. So one first question is to define what these are, which is not a hard question given the fact that in the topological setting quasi isomorphisms can be stated stalk-wise

$$f : F \to G \text{ is a quasi isomorphism } \iff f_x : G_x \to F_x \text{ is a quasi isomorphism for all } x \in X$$

Since for us stalks are evaluations ($F(\sigma)$ or $F(\mathtt{Im}(\sigma))$) we are looking to invert the morphisms $f : F \to G$ such that $f_\sigma : F(\sigma) \to G(\sigma)$ (or $f_{\mathtt{Im}(\sigma)}$) are quasi isomorphisms for all $\sigma \in X$. As it turns out, there are different ways to do this. One easy way (easy because we have a huge lot of mathematical history supporting us) is to just define the derived category of sheaves for $\mathtt{Sh}_\Delta(X)$ exactly as we do for classical sheaves. This turns out to give a good construction and it is in fact what we are going to use.

The second option, that we will present anyways, is to work on $\mathtt{Rep}(X)$, but now considering a *model category* structure. As it is presented in the appendix 8.2.1, there is a way to construct a model category on the functor category $\mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{M})$ over a model category $\mathbf{M}$, when $(\Delta \downarrow X)^{op}$ is what is called a *Reedy category*. The model category for $\mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{M})$ considers as weak equivalences morphisms $f : F \to G$ such that $f_\sigma : F(\sigma) \to G(\sigma)$ is weak equivalence for all $\sigma \in X$, which is just what we want. This option is interesting as it present a way to extend sheaf theory into the terrain of model categories, opening up interesting avenues.

We will give throughout this section a description of both options presented and we will see a natural way in which they are compatible.

5.1 First remarks

Throughout this chapter $\mathbf{A}$ is going to be an abelian category with enough injectives, and $\mathbf{M}$ is going to be a model category. We begin by showing that

Proposition 5.0.1. *$\mathtt{PSh}_\Delta(X)$, $\mathtt{Sh}(X)$ and $\mathtt{Rep}(X)$ are abelian categories with enough injectives. Also if $F \in \mathtt{Rep}(X)$ is injective then $\Gamma(F) \in \mathtt{Sh}_\Delta(X)$ is injective.*

Proof. The first part of the statement is classical theory and can be found on Tohoku paper. As for the second part, it follows from the fact that Γ is right adjoint of the exact functor I. □

In case the reader is not convinced by the proof or cannot find the specific part on Tohoku that talks about this, we can establish the corresponding structure on $\mathtt{Rep}(X)$. As a functor category, we are going to obtain (co)limits, and in particular the (co)ker as well as the morphism $\mathtt{Im} \to \mathtt{coIm}$ component-wise. Injective objects in $\mathtt{Rep}(X)$ can be constructed as follows

Given $F : (\Delta \downarrow X)^{op} \to \mathbf{A}$, we have for all $\sigma \in X$ a immersion $F(\sigma) \overset{g_\sigma}{\hookrightarrow} I_\sigma$ into a injective $I_\sigma \in \mathbf{A}$. Now, notice that $F(\sigma) = ev_\sigma(F)$, and the evaluation is a composition functor

$$ev_\sigma : \mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{A}) \overset{-\circ\sigma}{\longrightarrow} \mathtt{Func}(\cdot, \mathbf{A})$$

Where $\cdot$ is the category with one object and one morphism, and $\sigma : \cdot \to (\Delta \downarrow X)^{op}$ is the functor that takes $\cdot$ to $\sigma \in (\Delta \downarrow X)^{op}$. As a composition functor between nice categories it will have Kan extensions. We are interested in the right Kan extension with formula (for $M \in \mathbf{A}$ and $\tau \in X$) given by

$$Ran_\sigma M(\tau) = \lim_{\tau \downarrow \sigma} M = \prod_{\hom(\tau,\sigma)} M$$

which counts using M in how many ways τ is a face (or degeneracy of a face) of σ. Now consider

$$\hom_{\mathbf{A}}(F(\sigma), I_\sigma) \xrightarrow{\psi^\sigma} \hom_{\mathtt{Rep}(X)}(F, Ran_\sigma(I_\sigma))$$

As Ran_σ is right adjoint of the exact functor ev_σ we have that $Ran_s(I_\sigma)$ is injective for all $\sigma \in X$ and then $\prod_{\sigma \in X} Ran_\sigma(I_\sigma)$ and we have the morphism from F

$$F \xrightarrow{\prod_{\sigma \in X} \phi^\sigma(g_\sigma)} \prod_{\sigma \in X} Ran_\sigma(I_\sigma)$$

which is monomorphism since each g_σ is monomorphism.

From the classical theory of sheaves we obtain the derived category $D_\Delta(X)$ as the derived category of $\mathtt{Sh}_\Delta(X)$. Observe that the quasi-isomorphisms have a interesting property

$$\begin{aligned} F \xrightarrow{f} G \text{ quasi-iso } &\Leftrightarrow F(\mathtt{Im}(\sigma)) \xrightarrow{f_{\mathtt{Im}(\sigma)}} G(\mathtt{Im}(\sigma)) \text{ quasi-iso } \forall \sigma \in X \\ &\Leftrightarrow I(F)(\sigma) \xrightarrow{I(f)_\sigma} I(G)(\sigma) \text{ quasi-iso } \forall \sigma \in X \end{aligned} \tag{7}$$

which hints that there is a connection between the homological theories of $\mathtt{Rep}(X)$ and $\mathtt{Sh}(X)$.

Now, from the classical theory of sheaves we obtain a jungle of $\Gamma(X, -)$-injective sheaves for $\mathtt{Sh}_\Delta(X)$, from which the most important for us will be injective and flasque sheaves. In this theory flasque (pre)sheaves have been called *extendable*

Definition. *$F : \mathtt{Sub}(X)^{op} \to \mathbf{A}$ is called extendable if $F(X) \to F(Y)$ is epimorphism $\forall Y \leq X$*

Of course as in the classical theory we will have that F injective $\Rightarrow$ F extendable. Examples of extendable sheaves are A_{PL} and C_Δ. One might wonder if there are acyclics for $\mathtt{Rep}(X)$ other than injectives, and the answer is yes.

5.2 Model category for Rep(X)

Following the equivalence (7), we want a nice category structure in which we can invert morphisms $F \xrightarrow{f} G$ such that $F(\sigma) \xrightarrow{f_\sigma} G(\sigma)$ is quasi-isomorphism for all $\sigma \in X$. There is such a structure.

The reader who does not know about Reedy model categories structures is welcome now to go to the last appendix, where we make a summary about Reedy categories and the model structure associated to it, as well as the fact

that $(\Delta \downarrow X)^{op}$ is a Reedy category together with a characterization for the fibrant objects.

In short, since $(\Delta \downarrow X)^{op}$ is a Reedy category, then $\mathtt{Func}((\Delta \downarrow X)^{op}, \mathbf{M})$ is endowed with a Reedy model category structure, which is a model category for which $F \xrightarrow{f} G$ is a weak equivalence iff $F(\sigma) \xrightarrow{f_\sigma} G(\sigma)$ is a weak equivalence in $\mathbf{M}$ for all $\sigma \in X$. Furthermore, a functor F in this model category is fibrant iff $F(\sigma) \to F(\partial\sigma)$ is a fibration for all $\sigma \in X$, where σ is the morphism $\partial\sigma : \partial\Delta^n \hookrightarrow \Delta^n \xrightarrow{\sigma} X$ in the presheaf category $\mathtt{Func}(((\Delta \downarrow X) \downarrow \sigma)^{op}, \mathbf{Set})$.

We are particularly interested in the model category $Ch^+(\mathbf{A})$, which has compatibility with the derived category mentioned before. Observe that now the equivalence (7) is saying that

$$f \text{ is quasi isomorphism } \Leftrightarrow I(f) \text{ is weak equivalence}$$

So the fully faithful functor I is reflecting the isomorphisms between $D_\Delta(X)$ and $Ho(\mathtt{Rep}(X))$

Now, following the next definition

Definition. *$F \in \mathtt{Rep}_{\boldsymbol{A}}(X)$ is called Δ-fibrant if $F(\sigma) \to F(\partial\sigma)$ is epimorphism for all $\sigma \in X$*

we have that $F : (\Delta \downarrow X)^{op} \to Ch^+(\mathbf{A})$ is fibrant if F^i is Δ-fibrant for all $i \in \mathbb{N}$

Observe the similarity here with extendable sheaves. It would be nice to have that $\Gamma(F)$ extendable implies F fibrant (or backwards). This result is obviously true for sheaves coming from $F : \Delta^{op} \to Ch^+(\mathbf{A})$ for which the fibrant condition is just $F(\Delta^n) \twoheadrightarrow F(\partial\Delta^n)$ for all $n \in \mathbb{N}$. The situation in general could be a little more complicated, although we will prove the following.

Proposition 5.0.2. *If $F \in \mathtt{Rep}(X)$ is injective then F is Δ-fibrant.*

To prove this we make a definition and a lemma on a very abstract setting. Let $\mathbf{C}$ be a small category.

Definition. *$F : \boldsymbol{C}^{op} \to \boldsymbol{A}$ is called $\boldsymbol{C}$-extendable if $F(c) \xrightarrow{F(f)} F(d)$ is epimorphism for all monomorphisms $d \xhookrightarrow{f} c$*

And we have the following result

Lemma 5.0.1. *If $F : \boldsymbol{C}^{op} \to \boldsymbol{A}$ is injective, then F is $\boldsymbol{C}$-extendable*

Proof. Take F injective and let us consider just as in the begining of the section the evaluation functor $ev_c : \mathtt{Func}(\mathbf{C}^{op}, \mathbf{A}) \xrightarrow{-\circ c} \mathtt{Func}(\cdot, \mathbf{A})$

Just as before we will have Kan extensions for $- \circ c$. We will be interested in Ran_c whose formula is given by $Ran_c(M)(d) = \prod_{\hom(c,d)} M$. We call

$$Ra(F) = \prod_{c \in \mathbf{C}} Ran_c(F(c)) : \mathbf{C}^{op} \to \mathbf{A}$$

Observe that $Ra(F)$ is extendable, since for a monomorphism $d' \xhookrightarrow{\alpha} d$ we have that $\hom(c,d) \xrightarrow{\alpha\circ -} \hom(c,d')$ is injective and therefore $\prod_{\hom(c,d)} M \to \prod_{\hom(c,d)} M$

is epimorphism for all $M \in \mathbf{A}$, from where we conclude that $Ra(F)(d) \twoheadrightarrow Ra(F)(d')$

We have just as before that there is a monomorphism $F \overset{f}{\hookrightarrow} Ra(F)$, and since F is injective, we have that f splits.

So there is $g : Ra(F) \twoheadrightarrow F$ such that $g \circ f = 1_{Ra(F)}$. So given $\alpha : d' \hookrightarrow d$ we have a natural square for g

$$\begin{array}{ccc} Ra(F)(d) & \xrightarrow{g_d} & F(d) \\ {\scriptstyle Ra(F)(\alpha)}\downarrow & & \downarrow{\scriptstyle F(\alpha)} \\ Ra(F)(d') & \xrightarrow{g_{d'}} & F(d') \end{array}$$

from which we deduce $F(d) \twoheadrightarrow F(d')$ □

Observe that $Ra(F)$ will give us flasque resolutions. Observe also that for a geo friendly delta set $Ra(F)$ simplifies greatly since $\#\hom(\sigma, \tau) \leq 1 \ \forall \sigma, \tau \in X$.

Now we prove proposition 5.0.2

Proof. Let $F \in \mathtt{Rep}(X)$ be injective, and call $\mathbf{D}$ the full subcategory of $\mathbf{SSet} \downarrow X$ with objects

$$Obj(\mathbf{D}) = Obj(\Delta \downarrow X) \cup \{\partial\sigma | \sigma \in X\}$$

We consider the right Kan extension of $\iota_{\mathbf{D}} : \Delta \downarrow X \hookrightarrow \mathbf{D}$. As it is the right adjoint of a exact functor, $Ran_{\iota_{\mathbf{D}}}$ takes injectives to injectives, and therefore $Ran_{\iota_{\mathbf{D}}}(F)$ is injective.

By the previous lemma we have that $Ran_{\iota_{\mathbf{D}}}(F)$ is $\mathbf{D}$-extendable, which will give us $F(\sigma) \twoheadrightarrow F(\partial\sigma)$ for all $\Delta^n \xrightarrow{\sigma} X$ as $\partial\Delta^n \to \Delta^n$ is always a monomorphism (observe that the formula to define $Ran_{\iota_{\mathbf{D}}}(\partial\sigma)$ coincides with $F(\partial\sigma)$). □

Observe that given these results, particularly the last one and 5.0.1, the notions of derived functor of the section functor $\Gamma(Y, -)$ are equivalent for $Ho(\mathtt{Rep}(X))$ and $D_\Delta(X)$, and we define the hypercohomology of a sheaf as the derived functor of the section functor $\Gamma(Y, -)$. This will give us a interesting example

Example:

- Consider $\underline{\mathbb{Q}}$ be the constant sheaf as in the example before and $X \in \mathbf{SSet}$. We want to calculate the hypercohomology $\mathbb{H}^*(X, \underline{\mathbb{Q}})$. We can easily check that $\underline{\mathbb{Q}}$ is not flasque, since for example, as $\partial\Delta^1$ is the discrete category with two points, we have that

  $$\Gamma(\underline{\mathbb{Q}})(\partial\Delta^1) = \mathbb{Q} \oplus \mathbb{Q}$$

  And there are no surjections $\mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}$.

  With the result shown in ([FHT01] Lemma 10.7), we directly deduce that $\underline{\mathbb{Q}} \to A_{PL}$ gives a flasque resolution of $\underline{\mathbb{Q}}$. Hence

  $$\mathbb{H}^i(X, \underline{\mathbb{Q}}) = H^i(A_{PL}(X)) = H^i(X)$$

  That is to say, the hyperhomology of the constant functor $\underline{\mathbb{Q}}$ is the classical cohomology.

- We can actually generalize this process, since $C^{\star}_{\Delta}(X)$ is a flasque resolution of the constant sheaf $\underline{R}$, *now for any commutative ring* R, that is

$$\mathbb{H}^i(X, \underline{\mathrm{R}}) = H^i(C^{\star}_{\Delta}(X)) = H^i(X, R)$$

  the hyperhomology of the constant functor $\underline{R}$ is the classical cohomology with coefficients in R.

5.3 The case of Simplicial Complexes

Simplicial complexes are better for our immediate purposes than simplicial or delta sets, given its simple geometrical nature. In order to continue into our results, we need to explain how to apply the theory exposed to a simplicial complex X.

In fact, there are multiple ways to do this

- A simplicial complex is an object of a presheaf category, as explained in `https://ncatlab.org/nlab/show/simplicial+complex`. We can apply the theory to this presheaf category using what we did in section 4.3.
- We can develop a theory for the poset category of simplices with faces $(S(X), <)$ analogous to the one maid for $\Delta \downarrow X$.
- We may regard the simplicial complex as a simplicial set using the construction shown in 3.2.1.
- We may regard the simplicial complex as a delta set using the construction shown in 3.2.1.

The first option is troublesome because simplicial complexes seen as presheaf objects contain much more information than it is necessary for the uses we give them. The second option is unnecessary since it is contained in the forth option by proposition 3.1.4.

Using the third or forth options will depend on the situation at hand. Simplicial sets have seen more use than Delta sets in general, however there is some advantage in using the later, which is that all simplices of a simplicial complex seen as a delta set are geo friendly as seen in proposition 3.1.3. This makes the theory for the category of simplices equivalent to the theory for subcomplexes.

Since we will work solely with the category of subcomplexes we can use either delta or simplicial sets.

6 Simplicial Deligne Axioms

Having built a theory of simplicial sheaves and then having localized this over quasi isomorphisms, we can move onto the statement and satisfaction of what could be called simplicial Deligne axioms. This is a ambiguous name, as we will see that there are many ways and different contexts in which the Deligne axioms can be stated. In fact, we begin the section by abstracting the statement of the axioms in a way that allows us to work in many scenarios, to then focus on one particular scenario in which the axioms are satisfied by desirable and well known sheaves. Turns out that, unsurprisingly perhaps, the discrete topology is not the best topology to work on, this being mainly because the stalks are too small ($F_\sigma = F(\mathtt{Im}(\sigma))$), so we give a better topology to work on, one in which stalks work as evaluation on *auras* around simplices. Working under this topology, we state the simplicial Deligne axioms as a natural mirror of the topological ones, and we use the functor

$$\Phi : \mathtt{Sh}(|X|) \to \mathtt{Sh}^{\mathcal{T}}_{\Delta}(X)$$

Constructed, as explained in section 4.4, naturally from the realization functor $|-|$, to find Δ-Deligne sheaves. This functor is special as it takes soft sheaves into flasque sheaves. In particular $\Phi(IC^{\bar{p}}_\cdot)$ is flasque, and this implies that

$$\mathbb{H}^\star(X, \Phi(IC^{\bar{p}}_\cdot)) \simeq IH^{\bar{p}}_\star(|X|)$$

We prove that $\Phi(IC^{\bar{p}}_\cdot)$ satisfies the simplicial Deligne axioms, and this brings as a consequence our two main results.

If F is a Deligne sheaf then $\Phi(F)$ satisfies de simplicial Deligne axioms.

And

If F satisfies the simplicial Deligne axioms, then $\mathbb{H}^\star(X, F) \simeq IH^{\bar{p}}_\star(|X|)$

6.1 Abstract presentation for Deligne axioms

We start by giving an abstract presentation on Deligne axioms that can serve us into stating this axioms in many contexts. This will give us the freedom to chose between the many simplicial and topological categories in play, to see which ones adapt better to the axioms. This abstraction of the Deligne axioms also will give us a particularly lissom presentation.

Sadly, the conexts in which the Deligne axioms are satisfied by "day to day" functors are not abundant, since the heart of the Deligne axioms lies in the *cone formula*, which is a rather geometrical construct. This means that we cannot go crazy into abstraction, at least for what the author is aware of.

Consider a chain of categories with morphisms

$$\mathbf{C}_0 \underset{j_0}{\overset{\iota_0}{\leftrightarrows}} \mathbf{C}_1 \underset{j_1}{\overset{\iota_1}{\leftrightarrows}} \cdots \underset{j_{n-1}}{\overset{\iota_{n-1}}{\leftrightarrows}} \mathbf{C}_n \underset{j_n}{\overset{\iota_n}{\leftrightarrows}} \mathbf{C}_{n+1} = \mathbf{C}$$

Such that $j_k \circ \iota_k \simeq 1_{\mathbf{C}_k}$ for all $k \in \{0, \dots, n\}$

We will name $j^k = j_k \circ \cdots \circ j_n : \mathbf{C} \to \mathbf{C}_k$ for $k \in \{0, \dots, n\}$ and $j^{n+1} = 1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$. This will act as our "projection to subspaces".

We need two more elements, first a sequence $p : \mathbb{N} \to \mathbb{Z}$ and secondly a "truncation functor", which in this case is a set of endofunctors

$$\tau^i_{p(k)} : \mathbf{C}_i \to \mathbf{C}_i \ k \in \{0, \dots, n\}, i \in \{k, k+1\}$$

that satisfy the following

i. $\tau^k_{p(k+1)} \circ \tau^k_{p(k)} \simeq \tau^k_{p(k)}$ for all k

ii. $\tau^k_{p(k)} \circ \tau^k_{p(k)} \simeq \tau^k_{p(k)}$ for all k

We will omit the superindex whenever the categories involved are clear. We will also ask a compatibility with the j_k functors

- $j_k \circ \tau^k_{p(k)} \simeq \tau^{k-1}_{p(k)} \circ j_k$

And with these hypothesis we can state the Deligne axioms

Now take $F_0 \in \mathbf{C}_0$ such that $\tau_{p(0)} F_0 \simeq F_0$. Having this setting, we state the Deligne axioms.

Definition. *We say that $A \in \boldsymbol{C}$ is of class Del_{F_0} if it satisfies*

(AX 1) $j^0 A \simeq F_0$

(AX 2) $j^{k+1} A \simeq \tau_{p(k)} \iota_k j^k A \ \forall k \in \{0, \dots, n\}$

Observe that in the second axiom $k = n$ means $A \simeq \tau_{p(n)} \iota_n j^n A$. We will abuse notation and say that $A \in Del_{F_0}$ (or simply $A \in Del$ if F_0 is clear from the context) whenever A is of class Del_{F_0}

Now let us call

$$P_{F_0} = \tau_{p(n)} \iota_n \dots \tau_{p(0)} \iota_0 F_0$$

We have the following theorem.

e

Theorem 6.1. $A \in Del_{F_0} \Leftrightarrow A \simeq P_{F_0}$

So the axioms characterize in fact an isomorphism class in $\mathbf{C}$.

Proof. First suppose that A satisfies the axioms, then

$$A \simeq \tau_{p(n)} \iota_n j^n A \simeq \tau_{p(n)} \iota_n \tau_{p(n-1)} \iota_{n-1} j^{n-1} A \simeq \dots \simeq \tau_{p(n)} \iota_n \dots \tau_{p(0)} \iota_0 j^0 A$$

and $j^0 A \simeq F_0$, so we conclude that $A \simeq P_{F_0}$

For the other direction we define

$$P_k = \tau_{p(k)} \iota_k \dots \tau_{p(0)} \iota_0 F_0$$

And we call $P_{-1} = F_0$. Observe that

- $P_{k+1} = \tau_{p(k+1)} \iota_{k+1} P_k$ and $P_n = P_{F_0}$
- $j_k P_k \simeq P_{k-1}$ for $k \in \{0, \dots, n\}$
- $j^k P_{F_0} \simeq P_{k-1}$ for $k \in \{0, \dots, n\}$

The first statement is direct from the definition of P_k and the third statement follows directly from the second.

For the second statement we separate cases

- For $k \geq 1$ we have $j_k P_k = j_k \tau_{p(k)} \iota_k P_{k-1} \simeq \tau_{p(k)} j_k \iota_k P_{k-1} \simeq \tau_{p(k)} P_{k-1} \simeq \tau_{p(k)} \tau_{p(k-1)} \iota_{k-1} P_{k-2} \simeq \tau_{p(k-1)} \iota_{k-1} P_{k-2} = P_{k-1}$
- For $k = 0$ we have $j_0 P_0 = j_0 \tau_{p(0)} \iota_0 F_0 \simeq \tau_{p(0)} j_0 \iota_0 F_0 \simeq \tau_{p(0)} F_0 \simeq F_0 = P_{-1}$

For $k = 0$ the statement $j^k P_{F_0} \simeq P_{k-1}$ is the first axiom, and the second axioms is satisfied as $j^{k+1} P_{F_0} \simeq P_k = \tau_{p(k)} \iota_k P_{k-1} \simeq \tau_{p(k)} \iota_k j^k P_{F_0}$ □

As the reader can imagine, there are many situations in which the conditions for stating the Deligne axioms are satisfied, as said, the hard element here is to find an object in $\mathbf{C}$ that would satisfy them.

To get the classical, topological, Deligne axioms we take for a chain of inclusions $U_0 \overset{\iota_0}{\hookrightarrow} U_1 \overset{\iota_1}{\hookrightarrow} \ldots \overset{\iota_n}{\hookrightarrow} U_n = X$ the derived categories $\mathbf{C}_k = D(U_k)$. We take $\tau_{p(k)}$ as truncations and for the functors (ι_k, j_k) we take the derived functors of $(\iota_k)^\star$ and $(\iota_k)_\star$.

Notice that with respect to what the statement of Deligne refers, we would not need to be in the derived categories, however we would loose track of the homological structure, which would make the axioms very insipid. We need to be wise in selecting the categories to consider.

Given the theory we have constructed, we can state the Deligne axioms and have theorem 6.1 for any chain of simplicial (or delta) sets $X_0 \overset{\iota_0}{\hookrightarrow} X_1 \overset{\iota_1}{\hookrightarrow} \ldots \overset{\iota_n}{\hookrightarrow} X_{n+1} = X$. Just like for the topological case, we take the categories to be either $D_\Delta(X_k)$ or $Ho(\texttt{Rep}(X_k))$, the truncation as the $\tau_{p(k)}$'s, and as the morphisms we take the derivative of $(\iota_k)^\star$ and $(\iota_k)_\star$.

Notice that even in this restrictive format we still have a lot of room for possibilities, as we can play by

- Changing the topologies considered on $\texttt{Sub}(X)$ (as we will do)
- Changing the kind of simplicial structures considered: we can take simplicial sets, delta sets, simplicial complexes, or even elements of other presheaf categories.
- Changing the chain $X_0 \leq X_1 \leq \cdots \leq X_{n+1}$

And all these changes will maintain the structure just mentioned, we will be able to state the Deligne axioms with them and we will have theorem 6.1.

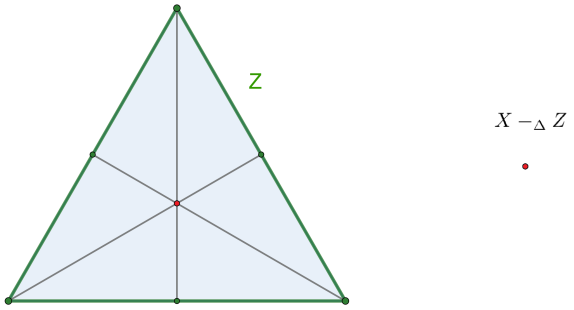
6.2 Our setting: A PL stratified pseudomanifold

We now move into the result we obtained regarding the simplicial Deligne axioms. As we said in the last section, we have three freedoms: freedom on the kind of simplicial structure, on the topologies considered and on the chain $X_0 \leq \cdots \leq X$. The key to Deligne is to get all to a balance analogous to the classical setting.

Perhaps the more puzzling of these freedoms to get is the chain $X_0 \leq \cdots \leq X$, since for topological spaces this chain is taken to be the *open sets* $U_k = X - X_{n-k}$, which are obtained by a complement of sets, and as mentioned

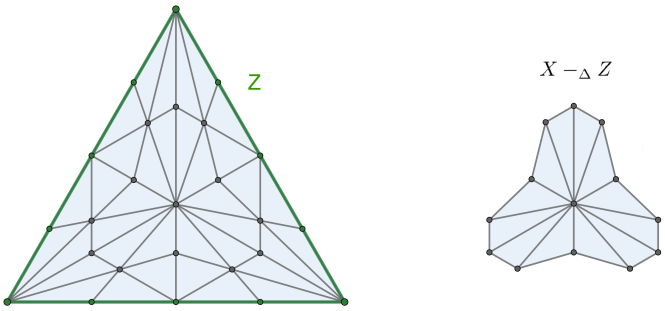
in chapter 3, the complement of simplicial structures is not simplicial and our best replacement for complement, which we have denoted as $-_\Delta$, has a bad behaviour. In particular, it does not help when working with a stratified space: If we consider for example the cone of $\partial\Delta^2$ with the cone point being X_0. If we consider $X -_\Delta X_0$ we get $\partial\Delta^2$, which is not homeomorphic to $|X| - |X_0|$.

Here is where subdivision comes into rescue. If we subdivide once we recover the three properties mentioned in section 3.2. However we do not recover the topological structure as the example in the following picture shows



Taking out Z on that picture would leave us with just a point. This is not what we want, so we subdivide again.

Now, there are many things to prove, but the following picture give us a feeling that the topology is respected when taking out Z when we take double subdivision



Since we never defined subdivision of simplicial sets, the attentive reader might guess that we are not going to work with simplicial sets. What a shame, maybe for next paper... On this document we are going to consider a simplicial complex that has been subdivided twice, particularly a simplicial complex that is the double subdivision of a fixed triangulation of a PL stratified pseudomanifold.

This means that we will not work with simplicial sets but rather with simplicial complexes. The good news on this is that our theory works wonderfully for simplicial complexes, although as said briefly in the introduction of section 5, since we will take a particular topology, we will be working with $D_\Delta(X)$ and not really looking at $Ho(\texttt{Rep}(X))$, and we will make use of the flasque condition.

This does not mean though that the theory developed lacks value, as it is useful since for instance most of the examples of sheaves come from functors from the simplex category, and furthermore might have interesting generalizations for example into considering different model categories.

We will define now and make the mathematics for all we are loosely saying.

6.2.1 Δ-topologies on $Sd^2(X)$

We start by considering X' a simplicial complex such that $|X'|$ is a PL stratified pseudomanifold with stratification given by $|X_0| \subset |X_1| \subset \cdots \subset |X_n| = |X'|$, and we subdivide it twice, so we consider $X = Sd^2(X')$ (and we call by abuse of notation $X_k = Sd^2(X_k)$).

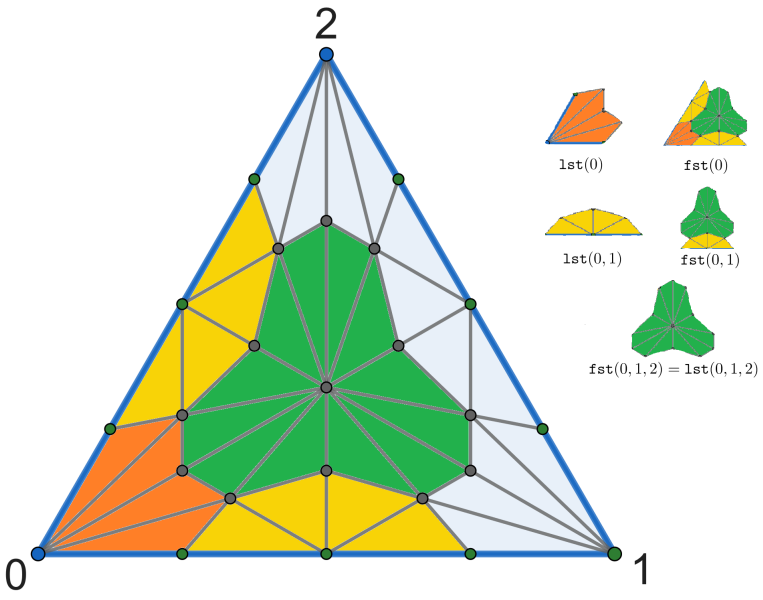
We first take a look in the topology to consider. We have that the topology given by $\mathtt{Sub}(X)$ is not very good for our purposes, and this is because if we want to make Deligne happen, we need the smallest open sets of the topology to be somewhat distinguished neighborhoods, and $\mathtt{Sub}(X)$ is generated by $\mathtt{Im}(\sigma)$'s which are far from being it.

There is one (or probably many, but we will focus on this one) topology that fulfils the task. To define it we first make a couple of definitions.

Definition. *Given X as before and $\sigma \in X'$, we define*

- $\mathtt{lst}(\sigma) = \mathtt{st}(b_\sigma)$ *the star of the barycenter*
- $\mathtt{fst}(\sigma) = \bigcup_{\sigma < \tau} \mathtt{lst}(\tau)$

We present the following picture to clarify these definitions.



And we take the topology whose base is $\{\mathtt{fst}(\sigma) | \sigma \in X'\}$ and call it

$$\mathcal{T} =< \{\mathtt{fst}(\sigma) | \sigma \in X'\} >$$

This topology is very nice, since the base is closed under intersection, and this leads the stalks to be always evaluation on $\mathtt{fst}(\sigma)$'s.

Lemma 6.1.1. *Given $F \in Sh^{\mathcal{T}}_{\Delta}(X)$ and $\sigma \in X$*

$$F_\sigma = F(\mathit{fst}(\sigma'))$$

Where $\sigma' \in X'$ is the lowest dimensional simplex with $|\mathit{Im}(\sigma)| \subset |\mathit{Im}(\sigma')|$.

Proof. This is simply the fact that

$$F_\sigma = \operatorname*{colim}_{\sigma \in Y \in \mathcal{T}} F(Y)$$

Given $\sigma' \in X'$ is the lowest dimensional simplex with $|\mathtt{Im}(\sigma)| \subset |\mathtt{Im}(\sigma')|$, $\mathtt{fst}(\sigma') = \bigcap_{\sigma \in Y \in \mathcal{T}} Y$ and since $\mathtt{fst}(\sigma') \in \mathcal{T}$ then $\operatorname*{colim}_{\sigma \in Y \in \mathcal{T}} F(Y) = F(\mathtt{fst}(\sigma'))$ □

And now we need to establish a chain of subcomplexes to apply the Deligne axioms, we consider

$$U^{\Delta}_k = X -_{\Delta} X_{n-k}$$

And we define $U^{\Delta}_k \overset{\iota_k}{\hookrightarrow} U^{\Delta}_{k+1}$ the inclusions. We have that this subcomplexes are part of the topology, as the following proposition shows.

Proposition 6.1.1. *For all $k \in \{0, \dots, n\}$*

$$U^{\Delta}_k = \bigcup_{b_\sigma \in |X| - |X_{n-k}|} \mathit{fst}(\sigma)$$

In particular $U^{\Delta}_k \in \mathcal{T}$

Proof. Consider first

$$\eta = \{\{\eta^0_0, \dots, \eta^0_{l_0}\}, \dots, \{\eta^m_0, \dots, \eta^m_{l_m}\}\} \in X -_{\Delta} X_{n-k}$$

this means that $\mathtt{Im}(\eta) \cap Sd^2(X'_{n-k}) = \emptyset$, which implies in particular that

$$\{\eta^j_0, \dots, \eta^j_{l_j}\} \notin Sd(X'_{n-k}) \text{ for all } j \in \{0, \dots, m\}$$

In particular for $j = 0$, there is $\eta^0_i \notin X'_{n-k}$. For this simplex $b_{\eta^0_i} \in |X| - |X_{n-k}|$, and we have that $\eta \in \mathtt{lst}(\eta^0_i) \leq \mathtt{fst}(\eta^0_i)$ (recall here that $\{\eta^j_0, \dots, \eta^j_{l_j}\} \subset \{\eta^{j+1}_0, \dots, \eta^{j+1}_{l_{j+1}}\}$ for all j)

On the other direction, consider

$$\eta = \{\{\eta^0_0, \dots, \eta^0_{l_0}\}, \dots, \{\eta^m_0, \dots, \eta^m_{l_m}\}\} \in \mathtt{lst}(\tau)$$

with $\sigma, \tau \in X'$ such that $b_\sigma \in |X| - |X_{n-k}|$ and $\sigma < \tau$.

$\eta \in \mathtt{lst}(\tau)$ implies that $\tau = \eta^0_i$ for some i. Now, since $\sigma < \eta^0_i$ and $b_\sigma \in |X| - |X_{n-k}|$ we have that $\eta^0_i \notin X_{n-k}$ which implies that $\{\eta^j_0, \dots, \eta^j_{l_j}\} \notin Sd(X'_{n-k})$ for all j, which implies that $\mathtt{Im}(\eta) \cap X_{n-k} = \emptyset$

□

This implies that $(\iota_k)_\star F(Z) = F(U_k^\Delta \cap Z)$. Having this at hand we can state the Deligne axioms as shown in section 6.1. We will make the actual statement. Consider $D_\Delta(\mathcal{T})$ the derived category for $\mathcal{T}$, with τ_k the truncation functor and let p be a GM-perversity.

Definition. *$F \in D_\Delta(\mathcal{T})$ is said to satisfy the Δ-Deligne axioms if*

(AX 1) $F|_{U_0^\Delta} \simeq \mathbb{R}$

(AX 2) $F|_{U_{k+1}^\Delta} \simeq \tau_{p(k)} R(\iota_k)_\star F|_{U_k^\Delta}$ $\forall k \in \{0, \dots, n\}$

For a chain of functors satisfying these axioms, we say $F \in Del_\Delta$.

And as shown in theorem 6.1, we have that the chains of functors that satisfy Δ-Deligne are given up to quasi-isomorphism by $\tau_{\bar{p}(n)} R(\iota_n)_\star \dots \tau_{\bar{p}(2)} R(\iota_2)_\star C_\Delta$

We are now after sheaves that would satisfy the Deligne axioms, and with this in mind we develop the following functor.

6.2.2 A good functor $\Phi : Sh(|X|) \to Sh_\Delta(X)$

We define a functor

$$\Phi : Sh(|X|) \to Sh_\Delta^{\mathcal{T}}(X)$$

Given by $\Phi(F)(Y) = \Gamma(|Y|, F) = \texttt{colim}_{|Y| \subseteq U} F(U)$. This is actually the pre-composition with the realization functor as we saw in 4.4. We will abuse notation and call $\Gamma(|Y|, F) = F(|Y|)$.

We need to prove that $\Phi(F)$ is a sheaf, which we state in a proposition together with another useful fact

Proposition 6.1.2. *Given the setting as before we have*

1. *For all sheaves $F \in \mathcal{Sh}(|X|)$, we have that $\Phi(F) \in \mathcal{Sh}_\Delta^T(X)$.*
2. *If F is soft then $\Phi(F)$ is flasque.*

Proof. We work presenting $\Gamma(|Y|, F)$ as $\{(s, U) : |Y| \subseteq U, s \in F(U)\}/\sim$, where $(s, U) \sim (s', U')$ if there is an open set V in $|X|$ with $|Y| \subseteq V \subseteq U \cap U'$ such that $s|_V = s'|_V$.

Now consider $Y = \bigcup_{i \in I} Y_i$

For the first gluing condition, we take $(s, U), (t, U') \in \Gamma(|Y|, F)$ with $(s, U)|_{Y_i} = (t, U')|_{Y_i}$, that is, for each i there exists V_i with $|Y_i| \subseteq V_i \subseteq U \cap U'$ and $s|_{V_i} = t|_{V_i}$. Consider $V = \bigcup_{i \in I} V_i$ which satisfies $|Y| \subseteq V \subseteq U \cap U'$, and $s|_V = t|_V$ by the gluing property of F, which is to say that $(s, U) \sim (t, U')$ in $\Gamma(|Y|, F)$.

For the second gluing condition, take $(s_i, U_i) \in \Gamma(|Y_i|, F)$ with $(s_i, U_i)|_{Y_i \cap Y_j} = (s_j, U_j)|_{Y_i \cap Y_j}$ for each i and j, which means that there are open sets V_{ij} with $|Y_i \cap Y_j| \subseteq V_{ij} \subseteq U_i \cap U_j$ such that $s_i|_{V_{ij}} = s_j|_{V_{ij}}$.

By topological properties we take open sets V_i with $|Y_i| \subseteq V_i \subseteq U_i$ and $V_i \cap V_j \subseteq V_{ij}$. To see this, first give a metric d to the (realization of the) simplicial complex. We may assume that each Y_i belongs to the base, which implies that they are compact and that each Y_i intersects only a finite amount of other Y_j's. For each $i, j \in I$ consider $d(|Y_i|, U_i^c) = \epsilon_i$ and $d(|Y_i \cap Y_j|, V_{ij}^c) = \delta_{ij}$, and then

$$V_i = \{x \in |X| : d(x, |Y_i|) < \texttt{min}\{\epsilon_i, \epsilon_j, \delta_{ij} || Y_i \cap Y_j| \neq \emptyset\}\}$$

Now, by the gluing property of F we obtain $s \in F(\bigcup_{i\in I} V_i)$ such that $s|_{V_i} = s_i|_{V_i}$. Since $|Y| \subseteq \bigcup_{i\in I} V_i$, this says that $(s, \bigcup_{i\in I} V_i)|_{Y_i} = (s_i, U_i)$.

The second statement follows from the fact that (realization of) subcomplexes are closed subsets of $|X|$. □

Some remarks of this proposition are in order.

Remarks:

- The functor Φ has a left adjoint at the level of presheaves given by $\Psi(F)(U) = F(\bigcup_{|Y|\subseteq U} Y)$ which sadly does not rise to the level of sheaves. However, a sheafification of Ψ might open an interesting line of work.

- The second statement of the proposition tells us that by considering a simplicial model $X \to |X|$, we go up a level, since flasque is a stronger condition that soft. This means that in general *soft sheaves are easier to find than flasque sheaves.* To illustrate this we present a result for which we will not develop the hole theory but it can be instinctively understood.

  If $\mathcal{R}$ is a *soft* sheaf of algebras, any sheaf of $\mathcal{R}$-modules is *soft.*

  That is, softness is a hereditary condition. In ([CSAT20] Proposition 2.6), the authors use this result to prove that $\mathbf{N}^\star_{\bar{p}}$ (described at the end of section 2.2.1) is soft, by using that it is a sheaf of $\mathbf{N}^\star_0$-modules (using the product $\mathbf{N}^\star_0 \otimes \mathbf{N}^\star_{\bar{p}} \xrightarrow{-\cup-} \mathbf{N}^\star_{\bar{p}}$), and that $\mathbf{N}^\star_0$ is soft, which is rather easy to prove.

- Since $IC^{\bar{p}}_\cdot$ is a chain of soft sheaves, we have that $\Phi(IC^{\bar{p}}_\cdot)$ is flasque. This in particular means that

  $$\mathbb{H}^\star(X, \Phi(IC^{\bar{p}}_\cdot)) \simeq H^\star(\Phi(IC^{\bar{p}}_\cdot)(X)) \simeq H^\star(IC^{\bar{p}}_\cdot(|X|)) \simeq IH^{\bar{p}}_\star(|X|)$$

  That is, the hyperhomology of $\Phi(IC^{\bar{p}}_\cdot)$ on X corresponds to the intersection homology of $|X|$. Compare this with the example(s) given at the end of section 5.2. We will develop this further in corollary 6.3.

6.3 Our main results

We have arrived now to the definitive moment for which all the previous theory was made for (if one is a goal oriented person at least), since if we do not find meaningful sheaves that do not satisfy the Δ-Deligne axioms, then the axioms fall into emptiness, in the sense that there is no point in stating them in the first place. Luckily for us ("us" being the non-empty set of anybody who cares), there are plenty of sheaves that satisfy the Δ-Deligne axioms, since our good functor Φ acts as a machine of producing Δ-Deligne sheaves.

During this section we will say "In the setting given on last section". Even though we just described this setting, given the importance of this results and in consideration of the lazy reader, we will recall the setting given on last section.

- We take X' a simplicial complex subdivided twice with a chain of subcomplexes $X_0 \leq \cdots \leq X_n = X'$ such that $|X'|$ is a PL stratified pseudomanifold with stratification $|X_0| \subset |X_1| \subset \cdots \subset |X_n| = X$. We subdivide it twice $X = Sd^2(X')$ and call by abuse of notation $X_k = Sd^2(X_k)$.

- We consider the topology $T \subset \mathtt{Sub}(X)$ with base $\{\mathtt{fst}(\sigma) | \sigma \in X'\}$. We take $U_k^\Delta = X -_\Delta X_k \in T$ and $U_k^\Delta \overset{\iota_k}{\hookrightarrow} U_{k+1}^\Delta$.
- We call $D_\Delta(T)$ the derived category of $\mathtt{Sh}_\Delta^T(X)$. On this category we say that $F \in Del_\Delta$ if

(AX 1) $F|_{U_0^\Delta} \simeq \mathbb{R}$

(AX 2) $F|_{U_{k+1}^\Delta} \simeq \tau_{p(k)} R(\iota_k)_\star F|_{U_k^\Delta}$ $\forall k \in \{0, \dots, n\}$

Theorem 6.1 says that $F \in Del_\Delta \Leftrightarrow F \simeq \tau_{\bar{p}(n)} R(\iota_n)_\star \dots \tau_{\bar{p}(2)} R(\iota_2)_\star C_\Delta$

- We have a functor $\Phi : \mathtt{Sh}(|X|) \to \mathtt{Sh}_\Delta^T(X)$ with formula $\Phi(F)(Y) = \Gamma(|Y|, F) = \underset{|Y| \subset U}{\mathtt{colim}} F(U)$, which takes soft sheaves into flasque ones.

As said, Φ is a machine of producing Δ-Deligne sheaves. One important example of this is that it transforms the sheaf $IC_\cdot^{\bar{p}}$ into a Δ-Deligne sheaf.

Proposition 6.1.3. *In the setting given on last section,*

$$\Phi(IC_\cdot^{\bar{p}}) \in Del_\Delta$$

This proposition is not easy to prove, and we will spend the last section of the chapter proving it. Observe that the sheaf $\Phi(IC_\cdot^{\bar{p}})$ is very *meaningful* since, as remarked at the end of last section, its hyperhomology corresponds to intersection homology.

$$\mathbb{H}^\star(X, \Phi(IC_\cdot^{\bar{p}})) \overset{(1)}{\simeq} H^\star(\Phi(IC_\cdot^{\bar{p}})(X)) \overset{(2)}{\simeq} H^\star(IC_\cdot^{\bar{p}}(|X|)) \overset{(3)}{\simeq} IH_\star^{\bar{p}}(|X|) \qquad (8)$$

This equivalence is easy to see, however we are going to remark its the importance that it does have by explaining each isomorphism.

(1) comes from the fact that $\Phi(IC_\cdot^{\bar{p}})$ is flasque, so

$$\mathbb{H}^i(X, \Phi(IC_\cdot^{\bar{p}}) = R^i\Gamma(X, \Phi(IC_\cdot^{\bar{p}})) = H^i(\Gamma(X, \Phi(IC_\cdot^{\bar{p}}))) = H^i(\Phi(IC_\cdot^{\bar{p}}(X)))$$

(2) is because $\Phi(IC_\cdot^{\bar{p}}(X))) = \Gamma(|X|, IC_\cdot^{\bar{p}}) = IC_\cdot^{\bar{p}}(|X|)$.

(3) Corresponds to the definition of (Borel-Moore) intersection homology.

The proposition just presented has two important corollaries, that we state as theorems. The first of the corollaries tells us that the characterization up to quasi isomorphism of the Deligne axioms expresed in theorem 6.1, implies that not only $\Phi(IC_\cdot^{\bar{p}})$ satisfies the Δ-Deligne axioms, but also any other chain of sheaves in its quasi isomorphism class does.

Theorem 6.2. *In the setting given in last section we have*

$$F \in Del \Longrightarrow \Phi(F) \in Del_\Delta$$

Proof. Using theorem 6.1, if we have $F \in Del \Rightarrow F \simeq IC_\cdot^{\bar{p}} \Rightarrow \Phi(F) \simeq \Phi(IC_\cdot^{\bar{p}})$, then by the last proposition, $\Phi(IC_\cdot^{\bar{p}}) \in Del_\Delta$, and then $\Phi(F) \in Del_\Delta$ (once again using theorem 6.1). □

The second corollary expands on equation (8). Putting together proposition 6.1.3, equation (8) and theorem 6.1 we obtain a powerful corollary.

Theorem 6.3. *In the setting given in last section, we have that if $F \in Del_\Delta$ then*

$$\mathbb{H}^\star(X, F) \simeq IH^p_\star(|X|)$$

So the Δ-Deligne axioms provide a characterization of chains of sheaves, up to quasi isomorphism, that compute the classical intersection homology. This last theorem can be understood as a procedure that reads as follows

1. Consider X a PL stratified pseudomanifold
2. Take $|K| \to X$ a compatible triangulation (such that each strata is a subcomplex), and subdivide it twice.
3. If $F \in Del_\Delta$ then
$$\mathbb{H}^\star(K, F) \simeq IH^{\bar{p}}_\star(X)$$

6.4 Proof of the main result

We will advocate us into the proof of the proposition now. The first Δ-Deligne axiom is obvious. As for the second one, it is far from being so, and it will take us the rest of the document to see it.

6.4.1 Simplifying (AX 2)

We are going to prove the second axiom using the stalks

$$(\Phi(IC^{\bar{p}}_\cdot)|_{U^\Delta_{k+1}})_\sigma \simeq (\tau_{p(k)}(\iota_k)_\star \Phi(IC^{\bar{p}}_\cdot)|_{U^\Delta_k})_\sigma \text{ for all } \sigma \in X$$

And we can use the fact that the subsimplicial complexes U^Δ_k are open to simplify this a little

$$(\Phi(IC^{\bar{p}}_\cdot))_\sigma \simeq (\tau_{p(k)}(\iota_k)_\star \Phi(IC^{\bar{p}}_\cdot)|_{U^\Delta_k})_\sigma \text{ for all } \sigma \in U^\Delta_{k+1} \tag{9}$$

We recall from lemma 6.1.1 that in $\mathcal{T}$ stalks are simple evaluations.

$$F_\sigma = F(\mathtt{fst}(\sigma'))$$

So now (9) becomes

$$\Phi(IC^{\bar{p}}_\cdot)(\mathtt{fst}(\sigma')) \simeq (\tau_{p(k)}(\iota_k)_\star \Phi(IC^{\bar{p}}_\cdot)|_{U^\Delta_k})(\mathtt{fst}(\sigma')) \text{ for all } \sigma \in U^\Delta_{k+1}$$

Now, following the formulas for $(f^\star, f_\star)$ of section 4.2.1, we have that for the second part of the equation

$$(\tau_{p(k)}(\iota_k)_\star \Phi(IC^{\bar{p}}_\cdot)|_{U^\Delta_k})(\mathtt{fst}(\sigma')) \simeq \tau_{p(k)} IC^{\bar{p}}_\cdot(|U^\Delta_k \cap \mathtt{fst}(\sigma')|)$$

So we are left to prove that

$$IC^{\bar{p}}_\cdot(|\mathtt{fst}(\sigma')|) \simeq \tau_{p(k)} IC^{\bar{p}}_\cdot(|U^\Delta_k \cap \mathtt{fst}(\sigma')|)$$

For all $k \in \{0, \dots, n\}$ and $\sigma \in U^\Delta_k$. And this will be satisfied if the following lemma is true

Lemma 6.3.1. *Given $\sigma : \Delta^k \to X'$ with $b_\sigma \in |X_m| - |X_{m-1}|$, then we have that there exists a PL stratified pseudomanifold $L \subset |X|$ such that*

1. $|\mathtt{fst}(\sigma)| \simeq |\mathtt{lst}(\sigma)| \simeq \mathbb{R}^m \times cL$
2. $|\mathtt{fst}(\sigma) -_\Delta X_m| \simeq \mathbb{R}^m \times L$

(Where the isomorphism $\simeq$ here corresponds to stratified homotopy equivalence)

Then the results follows from the cone formula.

6.4.2 $\mathtt{fst}(\sigma)$ is a distinguish neighborhood

We advocate this section to prove lemma 6.3.1. So during this section, we will suppose that X' is a triangulation of a PL-stratified pseudomanifold with stratification $X'_0 \leq \cdots \leq X'_n$. We take $X = Sd^2(X')$ and call $X_k = Sd^2(X'_k)$. Consider then $\sigma : \Delta^k \to X'$ with its barycenter $b_\sigma = \{\{\sigma\}\} \in |X_m| - |X_{m-1}|$.

We use abuse of notation and call for a simplex $\eta \in X'$ the subcomplex $Sd^2(\mathtt{Im}(\eta)) = \eta$.

The proof of lemma 6.3.1 will be divided in five parts

1. $\mathtt{lst}(\sigma) \simeq (\mathtt{lst}(\sigma) \cap \sigma) \times cL^\sigma$ for some L^σ.
2. $|\mathtt{lst}(\sigma) \cap \sigma| \simeq B^k$ corresponds to a ball of dimension k.
3. $|\mathtt{fst}(\sigma)| \simeq |\mathtt{lst}(\sigma)|$
4. $|\mathtt{lst}(\sigma)| \simeq B^m \times cL$
5. $|\mathtt{lst}(\sigma) -_\Delta X_m| \simeq B^m \times L$

1. $\mathtt{lst}(\sigma) \simeq (\mathtt{lst}(\sigma) \cap \sigma) \times L^\sigma$

We start by noticing that for a simplex $\eta \in X'$, given that (the double subdivision of) $\mathtt{Im}(\eta)$ is fat, then

$$\mathtt{st}_{\mathtt{Im}(\eta)}(b_\eta) = \eta \cap \mathtt{lst}(\eta)$$

We will call $\mathtt{st}_{\mathtt{Im}(\eta)}(b_\eta) = \mathtt{st}_\eta(b_\eta)$, let us call

- $S(\eta) = \{A \in Sd(X') | \eta \in A\}$ (that is, vertices of the shape $\{\sigma_0, \ldots, \sigma_n, \eta, \tau_0, \ldots, \tau_l\}$)
- $S_-(\eta) = \{A \in S(\eta) | \eta = \max(A)\}$ (that is, vertices of the shape $\{\sigma_0, \ldots, \sigma_n, \eta\}$)
- For $B \in S_-(\eta)$ we call

  $$S_+(B) = \{A \in S(\eta) | B \subset A \text{ and } \eta < \tau \ \forall \eta \in B \ \forall \tau \in A - B\}$$

  Again this will consist on vertices of the shape $\{\sigma_0, \ldots, \sigma_n, \eta, \tau_0, \ldots, \tau_l\}$, but now with $\{\sigma_0, \ldots, \sigma_n, \eta\}$ fixed. We call $S_+(\eta) = S_+(\{\eta\})$.

Observe that $S(\eta) = \{A \cup B | A \in S_-(\eta) \ B \in S_+(\eta)\}$.

We have that $\mathtt{lst}(\eta)$ and $\mathtt{st}_\eta(b_\eta)$ take their shape according to this sets of vertices.

Lemma 6.3.2. *For a simplex $\eta \in X'$ we have that*

- $\mathtt{lst}(\eta) = \mathcal{G}(S(\eta))$
- $\mathtt{st}_\eta(b_\eta) = \mathcal{G}(S_-(\eta))$

Proof. We show first that

$$V(\mathtt{st}(b_\eta)) = S(\eta)$$

Given $A \in S(\eta)$, we have that $A \in \{\{\eta\}, A\}$ which is a 1-simplex of $\mathtt{st}(b_\eta)$, so $A \in V(\mathtt{st}(b_\eta))$. And given $\{\sigma_0, \dots, \sigma_n\} \in \mathtt{st}(b_\sigma)$ we have that there is $\tau \in X$ such that $\{\eta_0, \dots, \eta_n\} \in \tau$ and $\{\eta\} \in \tau$. Since $\{\eta\}$ is minimal, we must have that $\{\eta\} \subset \{\eta_0, \dots, \eta_n\}$.

Now we proove that $\mathtt{lst}(\eta)$ is fat. Consider $\{\Psi_0, \dots, \Psi_n\} \in X$ (so each Ψ_k has a shape $\{\eta_0, \dots, \eta_n\}$ and $\Psi_k \subset \Psi_{k+1}$ for all k) with $\Psi_k \in V(\mathtt{lst}(\eta))$ for all k, so $\Psi_0 \in S(\eta)$, in particular $\eta \in \Psi_0$, and then for $B = \{\{\eta\}, \Psi_0, \dots, \Psi_n\}$ we have that $\{\Psi_0, \dots, \Psi_n\} \subset B$ and $\{\eta\} \in B$ and therefore $\{\Psi_0, \dots, \Psi_n\} \in \mathtt{st}(b_\eta)$

For the second statement, first consider that $\mathtt{st}_\eta(b_\eta) = \mathtt{st}(\eta) \cap Sd^2(\mathtt{Im}(\tau)) = \mathcal{G}(S(\eta)) \cap \mathcal{G}(Sd(\mathtt{Im}(\tau))) = \mathcal{G}(S(\eta) \cap Sd(\mathtt{Im}(\tau)))$ by 3.1.5, so it will be enough to prove that

$$S(\eta) \cap Sd(\mathtt{Im}(\tau)) = S_-(\eta)$$

which comes from the fact that for all $\{\eta_0, \dots, \eta_n\} \in Sd(\mathtt{Im}(\tau))$ we have that $\eta_k \subset \eta$ for all k. □

Now, given $A = \{\sigma_0, \dots, \sigma_m, \sigma\} \in S_-(\sigma)$ we define

- $L^\sigma = \mathcal{G}(S_+(\sigma) - \{\{\sigma\}\})$
- $L^A = \mathcal{G}(S_+(A) - \{A\})$

We have the following lemma.

Lemma 6.3.3. *In the situation above*

- $L^\sigma \simeq L^A$ *for all* $A \in S_-(\sigma)$
- $\mathcal{G}(S_+(\sigma)) = cL^\sigma$ *and* $\mathcal{G}(S_+(A)) = cL^A$

Proof. For the first statement we use a morphism that in vertices will take $\{\sigma, \tau_0, \dots, \tau_l\} \mapsto \{\sigma_0, \dots, \sigma_m, \sigma, \tau_0, \dots, \tau_l\}$. It is easy to see that this defines a morphism and furthermore an isomorphism with inverse given in vertices by $\{\sigma_0, \dots, \sigma_m, \sigma, \tau_0, \dots, \tau_l\} \mapsto \{\sigma, \tau_0, \dots, \tau_l\}$.

For the cone shape, observe that every simplex of $\mathcal{G}(S_+(A))$ has the shape $\{\Psi_0, \dots, \Psi_n\}$ with $A \subset \Psi_0$, so it is contained in $\{A\} \cup \{\Psi_0, \dots, \Psi_n\}$. The case for cL^σ is analogous. □

So we want to prove that

$$\mathcal{G}(S(\sigma)) \simeq \mathcal{G}(S_-(\sigma)) \times \mathcal{G}(S_+(\sigma)) = \mathcal{G}(S_-(\sigma) \times S_+(\sigma))$$

(With the second equality coming from 3.1.5), and this is true because the bijection

$$f : S_-(\sigma) \times S_+(\sigma) \to S(\sigma)$$

defined by $(A, B) \mapsto A \cup B$ defines an isomorphism

$$f : \mathcal{G}(S_-(\sigma) \times S_+(\sigma)) \to \mathcal{G}(S(\sigma))$$

We use 3.1.2 to prove this

Lemma 6.3.4. *In the situation above, the function* $f : S_-(\sigma) \times S_+(\sigma) \to S(\sigma)$ *defines an isomorphism*

$$f : \mathcal{G}(S_-(\sigma) \times S_+(\sigma)) \to \mathcal{G}(S(\sigma))$$

Proof. We need to prove that f and its inverse, say g, define morphisms, that is, they send simplices to simplices.

So first consider a simplex in $\mathcal{G}(S_-(\sigma) \times S_+(\sigma))$. This simplex will have the shape

$$\{(\{\sigma_0^0, \ldots, \sigma_{k_0}^0, \sigma\}, \{\sigma, \tau_0^0, \ldots, \tau_{l_0}^0\}), \ldots, (\{\sigma_0^m, \ldots, \sigma_{k_m}^m, \sigma\}, \{\sigma, \tau_0^m, \ldots, \tau_{l_m}^m\})\}$$

With the simplex $\{\{\sigma_0^0, \ldots, \sigma_{k_0}^0, \sigma\}, \ldots, \{\sigma_0^m, \ldots, \sigma_{k_m}^m, \sigma\}\}$ being in $\mathcal{G}(S_-(\sigma))$, and $\{\{\sigma, \tau_0^0, \ldots, \tau_{l_0}^0\}, \ldots, \{\sigma, \tau_0^m, \ldots, \tau_{l_m}^m\}\}$ being in $\mathcal{G}(S_+(\sigma))$, that is

$$\{\sigma_0^i, \ldots, \sigma_{k_i}^i, \sigma\} \subset \{\sigma_0^{i+1}, \ldots, \sigma_{k_{i+1}}^{i+1}, \sigma\}$$

and

$$\{\sigma, \tau_0^i, \ldots, \tau_{l_i}^i\} \subset \{\sigma, \tau_0^{i+1}, \ldots, \tau_{l_{i+1}}^{i+1}\}$$

for all i. When applying f to this simplex we obtain

$$\{\{\sigma_0^0, \ldots, \sigma_{k_0}^0, \sigma, \tau_0^0, \ldots, \tau_{l_0}^0\}, \ldots, \{\sigma_0^m, \ldots, \sigma_{k_m}^m, \sigma, \tau_0^m, \ldots, \tau_{l_m}^m\}\}$$

Given the two inclusions before we obtain that

$$\{\sigma_0^i, \ldots, \sigma_{k_i}^i, \sigma, \tau_0^i, \ldots, \tau_{l_i}^i\} \subset \{\sigma_0^{i+1}, \ldots, \sigma_{k_{i+1}}^{i+1}, \sigma, \tau_0^{i+1}, \ldots, \tau_{l_{i+1}}^{i+1}\}$$

for all i, and therefore it is a simplex of $\mathcal{G}(S(\sigma))$

On the opposite direction, consider a simplex on $\mathcal{G}(S(\sigma))$

$$\{\{\sigma_0^0, \ldots, \sigma_{k_0}^0, \sigma, \tau_0^0, \ldots, \tau_{l_0}^0\}, \ldots, \{\sigma_0^m, \ldots, \sigma_{k_m}^m, \sigma, \tau_0^m, \ldots, \tau_{l_m}^m\}\}$$

Applying g will give us

$$\{(\{\sigma_0^0, \ldots, \sigma_{k_0}^0, \sigma\}, \{\sigma, \tau_0^0, \ldots, \tau_{l_0}^0\}), \ldots, (\{\sigma_0^m, \ldots, \sigma_{k_m}^m, \sigma\}, \{\sigma, \tau_0^m, \ldots, \tau_{l_m}^m\})\}$$

Now, since $\{\sigma_0^i, \ldots, \sigma_{k_i}^i, \sigma, \tau_0^i, \ldots, \tau_{l_i}^i\} \subset \{\sigma_0^{i+1}, \ldots, \sigma_{k_{i+1}}^{i+1}, \sigma, \tau_0^{i+1}, \ldots, \tau_{l_{i+1}}^{i+1}\}$ for all i, we will have that

$$\{\sigma_0^i, \ldots, \sigma_{k_i}^i, \sigma\} \subset \{\sigma_0^{i+1}, \ldots, \sigma_{k_{i+1}}^{i+1}, \sigma\}$$

and

$$\{\sigma, \tau_0^i, \ldots, \tau_{l_i}^i\} \subset \{\sigma, \tau_0^{i+1}, \ldots, \tau_{l_{i+1}}^{i+1}\}$$

for all i, giving us that the image by g is a simplex of $\mathcal{G}(S_-(\sigma) \times S_+(\sigma))$ □

2. $|\mathtt{lst}(\sigma) \cap \sigma| \simeq B^k$

We have that $\mathtt{lst}(\sigma) \cap \mathtt{Im}(\sigma) \subset \overset{\circ}{\sigma}$, which is a manifold. Since $\mathtt{lst}(\sigma) \cap \mathtt{Im}(\sigma) = \mathtt{st}_{\mathtt{Im}(\sigma)}(b_\sigma)$, we get that it is a ball by 3.2.1.

3. $|\mathtt{fst}(\sigma)| \simeq |\mathtt{lst}(\sigma)|$

First observe that what we have proved so far for σ is also true for any simplex of X'.

We are going to inductively (on dimension) define a retraction from $\mathtt{fst}(\sigma) = \bigcup_{\sigma<\tau} \mathtt{lst}(\tau)$ onto $\mathtt{lst}(\sigma)$ y reducing each $\mathtt{lst}(\tau)$ for $\sigma < \tau$. Before starting with the retraction we make a couple of observations

Lemma 6.3.5. *In the setting above, we have*

1. *For $\eta \in X'$ with $b_\eta \in |X_j| - |X_{j-1}|$ we have that $\mathtt{lst}(\eta) \cap \eta \leq X_j -_\Delta X_{j-1}$*
2. *For $\tau, \tau' \in X'$ we have that $\mathtt{lst}(\tau) \cap \mathtt{lst}(\tau') \neq \emptyset \Leftrightarrow \tau < \tau'$ or $\tau' < \tau$*
3. *If we have $\tau, \tau' \in \beta(\sigma) = \{\tau \in X' | \sigma < \tau\}$ of the same dimension with $\tau \neq \tau'$ then $\mathtt{lst}(\tau) \cap \mathtt{lst}(\tau') = \emptyset$*

Proof. For the first statement, consider $\{\Psi_0, \dots, \Psi_k\} \in \mathtt{st}_\eta(b_\eta)$, that is, $\Psi_i \subset \Psi_{i+1}$ for all i, and they have the shape

$$\Psi_i = \{\sigma_0^i, \dots, \sigma_{l_i}^i, \eta\}$$

with $\sigma_0^i \subset \cdots \subset \sigma_{l_i}^i \subset \eta$. We want to show that $\{\Psi_0, \dots, \Psi_k\} \in X_j -_\Delta X_{j-1}$, which is equivalent to prove that

$$\{\Psi_0, \dots, \Psi_k\} \subset Sd(X'_j) - Sd(X'_{j-1})$$

So we want that $\{\sigma_0^i, \dots, \sigma_{l_i}^i, \eta\} \in Sd(X'_j) - Sd(X'_{j-1})$ for all i. And for this it will be enough to prove that $\eta \in X_j - X_{j-1}$. Observe that $b_\eta \in |X_j| - |X_{j-1}|$ means that $\{\eta\} \in Sd(X'_j) - Sd(X'_{j-1})$ which is equivalent to what we want.

For the second statement observe that $\mathtt{lst}(\tau) \cap \mathtt{lst}(\tau') = \mathcal{G}(S(\tau) \cap S(\tau'))$, so

$$\mathtt{lst}(\tau) \cap \mathtt{lst}(\tau') = \emptyset \Leftrightarrow S(\tau) \cap S(\tau') = \emptyset$$

Now, if we have $A \in S(\tau) \cap S(\tau')$, we would have that both $\tau, \tau' \in A$, which means that $\tau < \tau'$ or $\tau' < \tau$. On the other direction, if $\tau \subset \tau'$ then we have $\{\tau, \tau'\} \in S(\tau) \cap S(\tau')$, and if $\tau' \subset \tau$ then $\{\tau', \tau\} \in S(\tau) \cap S(\tau')$.

The third statement is a direct consequence of the second, since τ, τ' of the same dimension with $\tau \neq \tau'$ means that neither is face of the other. □

We need one more lemma before proceeding, we call for $\tau \in X'$ with $\sigma < \tau$

$$\mathtt{fst}_\tau(\sigma) = \bigcup_{\sigma \subset \eta \subsetneq \tau} \mathtt{lst}(\eta)$$

Lemma 6.3.6. *In the setting before, for $\tau \in X'$ with $\sigma < X'$ we have that*

- *$\mathtt{fst}_\tau(\sigma) \cap \mathtt{lst}(\tau) \leq \partial \mathtt{lst}(\tau)$.*
- *$|\mathtt{fst}_\tau(\sigma) \cap \mathtt{lst}(\tau) \cap \tau|$ is simply connected.*

Proof. First note that we can state everything in terms of fat subcomplexes, since

- $\mathtt{lst}(\tau) = \mathcal{G}(S(\tau))$
- $\partial \mathtt{lst}(\tau) = \mathcal{G}(S(\tau) - \{\{\tau\}\})$

- $\mathtt{lst}(\eta) = \mathcal{G}(S(\eta))$

Now, $\mathtt{lst}(\tau) \cap \bigcup_{\sigma \subset \eta \subsetneq \tau} \mathtt{lst}(\eta) = \mathcal{G}(S(\tau)) \cap \bigcup_{\sigma \subset \eta \subsetneq \tau} \mathcal{G}(S(\eta)) \leq \mathcal{G}(S(\tau)) \cap \mathcal{G}(\bigcup_{\sigma \subset \eta \subsetneq \tau} S(\eta)) = \mathcal{G}(S(\tau) \cap \bigcup_{\sigma \subset \eta \subsetneq \tau} S(\eta))$, so to prove the first statement, it will suffice to show that

$$S(\tau) \cap \bigcup_{\sigma \subset \eta \subsetneq \tau} S(\eta) \subset S(\tau) - \{\{\tau\}\}$$

And this is true since $\{\tau\} \notin \bigcup_{\sigma \subset \eta \subsetneq \tau} S(\eta)$ (because the only set of size one of $S(\eta)$ is $\{\eta\}$, and all η's in the union are different from τ)

Now we prove that in fact

$$\bigcup_{\sigma \subset \eta \subsetneq \tau} \mathcal{G}(S(\eta)) = \mathcal{G}(\bigcup_{\sigma \subset \eta \subsetneq \tau} S(\eta))$$

To see this consider $\{\Psi_0, \dots, \Psi_n\} \in \mathcal{G}(\bigcup_{\sigma \subset \eta \subsetneq \tau} S(\eta))$, that is $\Psi_i \in \bigcup_{\sigma \subset \eta \subsetneq \tau} S(\eta)$ for all i.

So for all i there is η_i such that $\Psi_i \in S(\eta_i)$, that is $\eta_i \in \Psi_i$. Notice that since $\Psi_0 \subset \cdots \subset \Psi_n$, we have that $\eta_0 \in \Psi_i$ for all i, which means that $\Psi_i \in S(\eta_0)$ for all i, so $\{\Psi_0, \dots, \Psi_n\} \in \mathcal{G}(S(\eta_0)) \subset \bigcup_{\sigma \subset \eta \subsetneq \tau} \mathcal{G}(S(\eta))$.

This means that $\mathtt{fst}_\tau(\sigma) \cap \mathtt{lst}(\tau) \cap \tau$ is a fat subcomplex of the simply conected simplicial complex $Sd^2(\mathtt{Im}(\tau))$, which gives us that $|\mathtt{fst}_\tau(\sigma) \cap \mathtt{lst}(\tau) \cap \tau|$ is simply conected. □

We are now ready to make our proof

Proposition 6.3.1. *In the setting given above, $|\mathtt{fst}(\sigma)|$ is homotopically stratified equivalent to $|\mathtt{lst}(\sigma)|$.*

Proof. We will contract each $\mathtt{lst}(\tau)$ with $\sigma < \tau$, inductively on the dimension of τ starting on dimension n down to dimension $k+1$ (recall that σ has dimension k)

First, for a n-simplex τ, we have that $\mathtt{lst}(\tau) = \mathtt{st}_\tau(b_\tau)$ is an n-ball contained in $|X_n| - |X_{n-1}|$, so we can contract it to a point in a way that respects the stratification. In particular it is homotopically stratified equivalent to $\mathtt{lst}(\tau) \cap \mathtt{fst}_\tau(\sigma)$ (which is simply connected by lemma 6.3.6).

By lemma 6.3.5, we can apply the corresponding homotopy independently on each simplex of dimension n containing σ.

Now, suppose we have reduced every simplex containing σ of dimension grater than j, and consider τ of dimension j. Recall that $\mathtt{lst}(\tau) \simeq (\mathtt{lst}(\tau) \cap \tau) \times cL^\tau$. Since $\mathtt{lst}(\tau) \cap \tau$ is a ball, we can contract it to a simply connected subset of its border, as for example $\mathtt{lst}(\tau) \cap \mathtt{fst}_\tau(\sigma) \cap \tau$ (by lemma 6.3.6). We name this homotopy h_τ. Again by lemma 6.3.5, this homotopy respects stratification. Consider now

$$h_\tau \times Id : |\mathtt{lst}(\tau) \cap \tau| \times |cL^\tau| \times [0,1] \to |\mathtt{lst}(\tau) \cap \tau| \times |cL^\tau|$$

This will give an homotopy that contracts $\mathtt{lst}(\tau)$ to its border. By lemma 6.3.5, we can apply this homotopy independently in simplices of dimension j, and it will respect the stratification since h_τ does. □

4. $|\mathtt{lst}(\sigma)| \simeq B^m \times cL$

Now consider for the point $b_\sigma \in |X_m| - |X_{m-1}|$, a distinguish neighborhood $N_\sigma = B^m \times cL$, with L a PL-stratified pseudomanifold of dimension $n-m$ compatible with the stratification. We can consider N_σ small enough so that

$$|N_\sigma| \subset |\mathtt{lst}(\sigma)|$$

We consider the simplicial complex $\mathtt{lst}(\sigma) \cap N_\sigma$. We can do a pseudoradial projection, and then make it into a simplicial complex (see the techniques throughout section 3.2.3), obtaining a subdivision K' of $\mathtt{lst}(\sigma)$ in which $\mathtt{st}_{K'}(b_\sigma) = N_\sigma$, then by 3.2.1 we have that $lk(b_\sigma, \mathtt{lst}(\sigma))$ is homeomorphic to $lk(b_s, K')$, and now we have that

$$N_\sigma = \mathtt{st}_{K'}(b_\sigma) = b_\sigma lk(b_s, K') \simeq b_\sigma lk(b_\sigma, \mathtt{lst}(\sigma)) = \mathtt{lst}(\sigma)$$

Observe that since the pseudoradial projection goes along the simplices, we have that this homeomorphism respects stratification.

5. $|\mathtt{lst}(\sigma) -_\Delta X_m| \simeq B^m \times L$

Observe that the same homotopy that contracts $\mathtt{fst}(\sigma)$ into $\mathtt{lst}(\sigma)$ will contract $\mathtt{fst}(\sigma) -_\Delta X_m$ into $\mathtt{lst}(\sigma) -_\Delta X_m$, and therefore if we prove that $|\mathtt{lst}(\sigma) -_\Delta X_m| \simeq B^m \times L$, then we will be done proving lemma 6.3.1.

We will prove that $|\mathtt{lst}(\sigma) -_\Delta X_m| \simeq B^m \times L$ by showing that

$$|\mathtt{lst}(\sigma) -_\Delta X_m| \simeq |\mathtt{lst}(\sigma)| - |X_m|$$

and then using that the latest is equivalent to $B^m \times L$, as it is a distinguished neighborhood. We make this proof in inductive steps.

Observe first that $\mathtt{lst}(\sigma) -_\Delta X_m = \mathtt{lst}(\sigma) -_\Delta (\bigcup_{\tau \in X'_m} \mathtt{Im}(\tau)) = \bigcap_{\tau \in X'_m} (\mathtt{lst}(\sigma) -_\Delta \mathtt{Im}(\tau)) = \bigcap_{\sigma < \tau \in X'_m} (\mathtt{lst}(\sigma) -_\Delta \mathtt{Im}(\tau)) = \mathtt{lst}(\sigma) -_\Delta (\bigcup_{\sigma < \tau \in X'_m} \mathtt{Im}(\tau))$. Furthermore, we can suppose that all these τ's in the union are m dimensional. Observe that for X locally finite, there are finitely many elements in $\{\tau \in X' | \sigma < \tau \in X'_m\}$. We will enumerate them

$$\{\tau \in X' | \sigma < \tau \in X'_m\} = \{\tau_1, \dots, \tau_l\}$$

For later convenience, we take τ_l to be σ itself. Let us define the following.

Definition. *Given $\{\tau_1, \dots, \tau_l\}$ as before, we call*

$$\mathtt{mst}_k = \mathtt{lst}(\sigma) -_\Delta (\mathtt{Im}(\tau_1) \cup \cdots \cup \mathtt{Im}(\tau_k))$$

And we set $\mathtt{mst}_0 = \mathtt{lst}(\sigma)$

Observe that by the ninth statement of 3.0.10, we have that

$$\mathtt{mst}_{k+1} = \mathtt{mst}_k -_\Delta \mathtt{Im}(\tau_{k+1})$$

And now we will prove that we can make the proof inductively.

Lemma 6.3.7. *In the setting given before, if we have that for all $k \geq 0$*

$$|\mathtt{mst}_k -_\Delta \mathtt{Im}(\tau_{k+1})| \simeq |\mathtt{mst}_k| - |\mathtt{Im}(\tau_{k+1})|$$

Then $|\mathtt{lst}(\sigma) -_\Delta X_m| \simeq |\mathtt{lst}(\sigma)| - |X_m|$

Proof. The proof goes inductively as follows

$$\begin{aligned}|\mathtt{lst}(\sigma) -_\Delta X_m| &= |\mathtt{mst}_l| \\ &\simeq |\mathtt{mst}_{l-1}| - |\mathtt{Im}(\tau_l)| \\ &\simeq (|\mathtt{mst}_{l-2}| - |\mathtt{Im}(\tau_{l-1})|) - |\mathtt{Im}(\tau_l)| \\ &= |\mathtt{mst}_{l-2}| - (|\mathtt{Im}(\tau_{l-1})| \cup |\mathtt{Im}(\tau_l)|) \\ &\simeq \cdots \simeq |\mathtt{lst}(\sigma)| - \bigcup_{\sigma<\tau\in X'_m} |\mathtt{Im}(\tau)|\end{aligned}$$

And $|\mathtt{lst}(\sigma)| - \bigcup_{\sigma<\tau\in X'_m} |\mathtt{Im}(\tau)| = |\mathtt{lst}(\sigma)| - |X_m|$. The only tricky step here is to see that

$$|\mathtt{mst}_{k+1}| - \bigcup_{j\geq k} |\mathtt{Im}(\tau_j)| \simeq (|\mathtt{mst}_k| - |\mathtt{Im}(\tau_{k+1})|) - \bigcup_{j\geq k} |\mathtt{Im}(\tau_j)|$$

The homotopy equivalence here is constructed by restricting the corresponding homotopy of $|\mathtt{mst}_k -_\Delta \mathtt{Im}(\tau_{k+1})| \simeq |\mathtt{mst}_k| - |\mathtt{Im}(\tau_{k+1})|$. This can be done since the inclusions of subspaces

$$|\mathtt{mst}_{k+1}| - \bigcup_{j\geq k} |\mathtt{Im}(\tau_j)| \hookrightarrow |\mathtt{mst}_{k+1}|$$

Are homotopy equivalences. This is because subtracting $|\mathtt{Im}(\tau_j)|$ corresponds to subtracting simplices of dimension $< n$ out of $\mathtt{lst}(\sigma)$, hence we can retract $|\mathtt{mst}_{k+1}|$ into $|\mathtt{mst}_{k+1}| - \bigcup_{j\geq k} |\mathtt{Im}(\tau_j)|$. And if the homotopy equivalence of $|\mathtt{mst}_k -_\Delta \mathtt{Im}(\tau_{k+1})| \simeq |\mathtt{mst}_k| - |\mathtt{Im}(\tau_{k+1})|$ respects the stratification, so will its restriction. □

We will treat σ differently from the other simplices of $\{\tau_0, \ldots, \tau_l\}$. We will also assume that each of these other simplices has dimension m. We will make use of the following lemma

Lemma 6.3.8. *In the setting given before*

$$\mathtt{lst}(\sigma) -_\Delta \mathtt{Im}(\sigma) \simeq (\mathtt{lst}(\sigma) \cap \sigma) \times L^\sigma$$

Proof. Recall by lemma 6.3.4 that

$$\mathtt{lst}(\sigma) \simeq \mathcal{G}(S_-(\sigma) \times S_+(\sigma))$$

via the identification

$$\{\sigma_0, \ldots, \sigma_m, \sigma, \tau_0, \ldots, \tau_l\} \mapsto (\{\sigma_0, \ldots, \sigma_m, \sigma\}, \{\sigma, \tau_0, \ldots, \tau_l\})$$

When removing $\mathtt{Im}(\sigma)$, we are subtracting all vertices of the shape $\{\sigma_0, \ldots, \sigma_m, \sigma\}$, that is, all $(\{\sigma_0, \ldots, \sigma_m, \sigma\}, \{\sigma\})$, and then

$$\mathtt{lst}(\sigma) -_\Delta \mathtt{Im}(\sigma) \simeq \mathcal{G}(S_-(\sigma) \times (S_+(\sigma) - \{\{\sigma\}\})) = B_\sigma \times L^\sigma$$

□

And then we have

Lemma 6.3.9. *In the setting given before*

$$|\mathtt{lst}(\sigma)| - |\mathtt{Im}(\sigma)| \simeq |\mathtt{lst}(\sigma) -_{\Delta} \mathtt{Im}(\sigma)|$$

Proof. The proof goes as follows

$$\begin{aligned}|\mathtt{lst}(\sigma)| - |\mathtt{Im}(\sigma)| &= |B_\sigma \times cL^\sigma| - |\mathtt{Im}(\sigma)| \\ &= (|B_\sigma| \times |cL^\sigma|) - |\mathtt{Im}(\sigma)| \\ &= |B_\sigma| \times (0,1) \times |L^\sigma| \\ &\simeq |B_\sigma| \times |L^\sigma| = |B_\sigma \times L^\sigma| = |\mathtt{lst}(\sigma) -_{\Delta} \mathtt{Im}(\sigma)|\end{aligned}$$

□

We will now subtract the rest of the simplices of X_m, but before start doing it we will take a closer look on L^σ, suppose first that we are working in a building block, that is, $X = \Delta^n$

Lemma 6.3.10. *Suppose that $X = \Delta^n$ and $\sigma = (0, \ldots, k)$, then*

$$L^\sigma \simeq Sd^2(\Delta^{n-k-1})$$

Where Δ^{n-k-1} is formed by the vertices of the simplex $(k+1, \ldots, k+(n-k))$

Proof. Recall that $L^\sigma = \mathcal{G}(S(\sigma) - \{\{\sigma\}\})$

We establish a bijection between $S(\sigma) - \{\{\sigma\}\}$ and $Sd(Im((k+1, \ldots, n)))$. To a face $(a_{j_0}, \ldots, a_{j_m}) < (k+1, \ldots, n)$ we assign

$$f((a_{j_0}, \ldots, a_{j_m})) = (0, \ldots, k, a_{j_0}, \ldots, a_{j_m})$$

Which naturally defines a function

$$f : Sd(Im((k+1, \ldots, n))) \to S(\sigma) - \{\{\sigma\}\}$$

Given by $f(\{\eta_0, \ldots, \eta_j\}) = \{f(\eta_0), \ldots, f(\eta_j)\}$. This is easily seen to be a function and define a morphism

$$f : \mathcal{G}(Sd(Im((k+1, \ldots, n)))) \to \mathcal{G}(S(\sigma) - \{\{\sigma\}\})$$

Furthermore, it has an inverse defined in the same fashion

$$g((0, \ldots, k, a_{j_0}, \ldots, a_{j_m})) = (a_{j_0}, \ldots, a_{j_m})$$

□

And then, in the general case, L^σ is the union of these pieces.

Lemma 6.3.11. *In the setting given before, let $\{\eta_i\}_{i \in I}$ be the set of n-simplices of X, then*

$$L^\sigma = \bigcup_{i \in I} L^\sigma_{\eta_i}$$

where $L^\sigma_{\eta_i}$ is the construction of L^σ assign to the subspace $|\mathtt{Im}(\eta_i)|$.

Proof. The proof is simply the fact that

$$L^\sigma = \bigcup_{i\in I} L^\sigma \cap \eta_i = \bigcup_{i\in I} \mathcal{G}(\{A \in S_+(\sigma) | max(A) < \eta_i\} - \{\{\sigma\}\})$$

□

Now observe that in the double subdivision, when taking off faces of some Δ^m with either $-$ or $-_\Delta$, the homotopy type does not change

Lemma 6.3.12. *Consider $Z \leq \partial\Delta^m$ a simplicial complex formed by proper faces of an m-simplex. Then*

$$|\Delta^m| = |Sd^2(\Delta^m)| \simeq |Sd^2(\Delta^m) -_\Delta Z| \simeq |Sd^2(\Delta^m)| - |Z|$$

Furthermore, if X is a union of m-simplices and $Z \leq \partial X$, then

$$|X| = |Sd^2(X)| \simeq |Sd^2(X) -_\Delta Z| \simeq |Sd^2(X)| - |Z|$$

Proof. We use that in general, for $\eta \in \Delta^m$ we have

$$|\mathtt{lst}(\eta) -_\Delta \mathtt{Im}(\eta)| \simeq |\mathtt{lst}(\eta)| - |\mathtt{Im}(\eta)|$$

which is proven in the same fashion as lemma 6.3.9

$$\begin{aligned} |\mathtt{lst}(\eta)| - |\mathtt{Im}(\eta)| &= |B_\eta \times cL^\eta| - |\mathtt{Im}(\eta)| \\ &= |B_\eta| \times (0,1) \times |L^\eta| \\ &\simeq |B_\eta| \times |L^\eta| = |B_\eta \times L^\eta| = |\mathtt{lst}(\eta) -_\Delta \mathtt{Im}(\eta)| \end{aligned}$$

We use this homotopy inductively on all simplices of Z, starting from dimension 0 up to dimension $\dim(Z)$, and in this way we obtain

$$|Sd^2(\Delta^m) -_\Delta Z| \simeq |Sd^2(\Delta^m)| - |Z|$$

The remainder equivalence $|Sd^2(\Delta^m)| \simeq |Sd^2(\Delta^m)| - |Z|$ is obvious since we have $|Z| \subset \partial|\Delta^m|$.

The same process will work for X being union of m-simplices, although it might not be obvious that we can iterate the homotopies to get to $|X| - |Z|$, for this consider the following

$$\begin{aligned} |Sd^2(X) -_\Delta Z| &= |\bigcup_{\tau\in X-Z} \mathtt{lst}(\tau) \cup \bigcup_{\tau\in Z} \mathtt{lst}(\tau) -_\Delta \bigcup_{\tau\in Z} \mathtt{Im}(\tau)| \\ &= \bigcup_{\tau\in X-Z} |\mathtt{lst}(\tau)| \cup \bigcup_{\tau\in Z} |\mathtt{lst}(\tau) -_\Delta \mathtt{Im}(\tau)| \\ &\simeq \bigcup_{\tau\in X-Z} |\mathtt{lst}(\tau)| \cup \bigcup_{\tau\in Z} |\mathtt{lst}(\tau)| - |\mathtt{Im}(\tau)| = |X| - |Z| \end{aligned}$$

□

We can now prove the last part of the result.

Lemma 6.3.13. *In the setting given before, we have that for all $k \geq 0$*

$$|\mathtt{mst}_k -_\Delta \mathtt{Im}(\tau_{k+1})| \simeq |\mathtt{mst}_k| - |\mathtt{Im}(\tau_{k+1})|$$

Recalling lemma 6.3.7, this will give us the final result.

Proof. Recall by lemma 6.3.4 that

$$\texttt{lst}(\sigma) \simeq \mathcal{G}(S_-(\sigma) \times S_+(\sigma))$$

via the identification

$$\{\sigma_0, \ldots, \sigma_m, \sigma, \tau_0, \ldots, \tau_l\} \mapsto (\{\sigma_0, \ldots, \sigma_m, \sigma\}, \{\sigma, \tau_0, \ldots, \tau_l\})$$

Now, when we perform $-_\Delta \texttt{Im}(\tau)$ we are subtracting vertices of the form

$$(\{\sigma_0, \ldots, \sigma\}, \{\sigma, \tau_0, \ldots, \tau_l\})$$

With $\tau_l < \tau$, so we have that

$$\begin{aligned}\texttt{lst}(\sigma) -_\Delta \texttt{Im}(\tau) &= \mathcal{G}(S_-(\sigma)) \times \mathcal{G}(S_+(\sigma) - \{A \in S_+(\sigma) | max(A) < \tau\}) \\ &\simeq B_\sigma \times (L^\sigma -_\Delta \mathcal{G}(\{A \in S_+(\sigma) | max(A) < \tau\}))\end{aligned}$$

Now, since $\dim(\tau) < n$ then $\mathcal{G}(\{A \in S_+(\sigma) | max(A) < \tau\}) \leq \partial L^\sigma$, and since L^σ is a union of m-simplices, we can use the previous lemma to conclude (Observe here that taking off all the simplices finally will take a subcomplex contained in the frontier of L^σ, which does not change the homotopy type (which was the hole point of last lemma). We also take $Id \times -$ to create the homotopy). □

7 Epilogue: final comments and further research

Dear reader, we have come to the end of today's journey. The author hopes that this four years of his life amounted into a great reading experience, and of course useful results that can expand the human knowledge of mathematics.

Many adventures await us for the future of Δ-Deligne axioms. For starters, a seemingly simple generalization of the results presented is to use more general perversities instead of the GM perversities that were used.

The reader might have noticed also that up to the statement of the axioms we worked with simplicial sets, to then turn into simplicial complexes at the end. In fact, the challenge of finding Δ-Deligne sheaves for simplicial sets is open.

Furthermore, following that same line, the Δ-Deligne axioms can be stated for both $D_\Delta(X)$ and $Ho(X)$. The homotopy category holds the interesting fact that the image category can be any *model category*, expanding to unexpected grounds, if we can find the proper replacement for the truncation functors (which satisfy the properties expressed in section 6.1). This is very interesting, as it expands the reach of Deligne axioms beyond their original intends, expanding the theory of intersection homology to unknown realms. One possible example of this is taking the category of DGA, which have a natural model category structure, and can add *products* into the hole theory.

And if the reader loves to generalize (as the author does), the reader might have notice the generalization that we made for presheaf categories in chapter 5, for which the simplest presheaf category **Set** recovers the classical sheaf theory. Could this particular presentation of sheaves on other presheaf categories be holding a key to unlock great mysteries of mathematics? We might someday know.

Finally, we mention that we used a functor $\Phi : \mathtt{Sh}(|X|) \to Sh^T_\Delta(X)$ naturally defined to prove that $F \in Del \Rightarrow \Phi(F) \in Del_\Delta$. One can naturally wonder about the question for the other way around, that is, to find a functor $\Psi : Sh^T_\Delta(X) \to \mathtt{Sh}(|X|)$ that transforms Δ-Deligne into Deligne sheaves. One possible line of work for this is to follow the construction made in [CSAT20] for the blown-up cochain sheaf.

As we part ways, the author wishes that the reader has a great future full of happiness to come, and that all that has been here written serves in one way or another to the well being of us all.

8 Appendix

8.1 Kan Extensions

As the reader most likely knows, through the years category theory has proven its strength in the development of mathematical theories, mainly by the use of universal concepts (such as limits, adjoints, initial objects, etc), which provide optimal and coherent ways to make constructions that fill important places on many, if not most, branches of mathematics. There is a way to place what is called **Kan extensions** at the heart of universal constructions, considering as it was famously expressed by MacLane that "the concept of Kan extensions subsumes all the other fundamental concepts on category theory". He in fact presents in [ML98] a way to construct the fundamental universals using Kan extensions.

In this document, our main reason to invoke the presence of Kan extensions is to construct adjoints of precomposition functors. That is, given a functor $K : \mathbf{C} \to \mathbf{D}$ and a category $\mathbf{A}$, we want an adjoints of

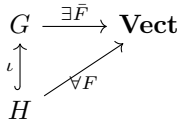
$$- \circ K : \mathtt{Func}(\mathbf{D}, \mathbf{A}) \to \mathtt{Func}(\mathbf{C}, \mathbf{A})$$

Thanks to Kan extensions we have not only the existence of a left and right adjoints, but also a (co)limit formula for them, provided that $\mathbf{C}$ is small and $\mathbf{A}$ is (co)complete.

During this appendix we go through the concept of Kan extensions, focusing on the uses that we are give them in this document. The classical reference to study Kan extensions is [ML98]. We will refer also to [Leh14], as it presents a pleasant and very complete exposition of the subject, drawing on and enlarging MacLane.

Perhaps the best starting point to visualize Kan extensions (which is the second way in which we use them), is to extend functors from a subcategory. Take as an illustrative example the case of group representations, which are functors from some group G seen as a one object groupoid to the category **Vect**. Imagine that we have a subgroup $H \leq G$, then it would be natural to ask whether there are ways to obtain representations of H from the representations of G and vice-versa.

There is an obvious way to obtain representations of H, which is by restriction of representations of G. Observe that restricting to H is the same as precomposing with the inclusion $H \overset{\iota}{\hookrightarrow} G$, and by adopting this perspective we might suspect that the most natural way to do the vice-versa is to find an adjoint of $- \circ \iota$, which would in turn fill the following diagram

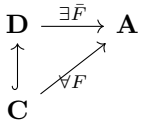


Extending the representations of the subgroup H in the most general way possible. As the reader might (or might not) expect this is a very desirable construction to perform, and in fact it plays an important roll on representation theory, important enough to have a famous name, *induced representations*.

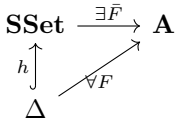
Now, what can be the formula of the induced representation? The answer is not obvious at all, and in fact it is not obvious that such a criteria (having an adjoint of the restriction functor) can be satisfied in the first place.

Here is where Kan extensions come to rescue, not only guaranteeing the existence of the induction functor, but also providing a formula for it. All this for free, we just need to say that H is a small category and that **Vect** is (co)complete.

We can generalize this idea by considering any subcategory $\mathbf{C} \hookrightarrow \mathbf{D}$



which for example is what we do in section 3.1



Using the Yoneda embedding to consider Δ as a subcategory of **SSet**.

Once again, Kan extensions provide not only the possibility to do this naturally (that is, giving left and right adjoints of the restriction functor $- \circ \iota$), but it also provides a formula for this construction. We only ask that $\mathbf{C}$ is a small category and that $\mathbf{A}$ is (co)complete.

Now, this idea can be generalized even further, and in fact it will be useful to do so, by considering instead of a fully faithful functor ι, any functor. That is, to find left and right adjoints of a precomposition functor $- \circ K$.

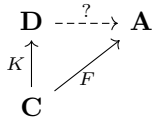
As an example of how this will be useful, consider (pre)sheaves over topological spaces. These are functors $F : Op(X)^{op} \to R\text{-mod}$ from the category of open sets of X into a familiar category such as **Set** or R-mod. It is desirable to compare (pre)sheaves from related topological spaces. In more specific words, given $f : X \to Y$ a continuous function, are there functors back and foward relating $\texttt{Func}(Op(X), R\text{-mod})$ and $\texttt{Func}(Op(Y), R\text{-mod})$ in a way compatible with f?

There is an obvious functor

$$f_\star : Func(Op(X)^{op}, R\text{-mod}) \to Func(Op(Y)^{op}, R\text{-mod})$$

which corresponds to the precomposition with $f^{-1} : Op(Y)^{op} \to Op(X)^{op}$. A compatible way to go in the other direction is not obvious at first glance, but it can be performed using Kan extensions.

So in general, what we expect is that given a functor $K : \mathbf{C} \to \mathbf{D}$ and $F : \mathbf{C} \to \mathbf{A}$ we can fill a commutative diagram



That is, we want a functor Kan_K such that $F = Kan_K \circ K$. This is possible when K is fully faithful, $\mathbf{C}$ is small and $\mathbf{A}$ is (co)complete, but in general it

does not have to be true. The best possible answer we can hope for is to take universals between natural transformations $F \to G \circ K$ and $H \circ K \to F$, and this is how we define Kan extensions. We first define a left Kan extension.

Definition. *Given $F : \boldsymbol{C} \to \boldsymbol{A}$ and $K : \boldsymbol{C} \to \boldsymbol{D}$, a **left Kan extension** is the information of*

- *A functor $Lan_K F : \boldsymbol{D} \to \boldsymbol{A}$*
- *A natural transformation $\eta : F \to Lan_K F \circ K$*

Such that for each other pair $(G : \boldsymbol{D} \to \boldsymbol{A}, \gamma : F \to G \circ K)$ there exist a unique $\alpha : Lan_K F \to G$ such that $\gamma = \alpha_K \circ Lan_K F$ (where $\alpha_K : Lan_K F \circ K \to G \circ K$ is defined in components as $(\alpha_K)_c = \alpha_{K(c)}$)

A right Kan extension has the dual definition.

Definition. *Given $F : \boldsymbol{C} \to \boldsymbol{A}$ and $K : \boldsymbol{C} \to \boldsymbol{D}$, a **right Kan extension** is the information of*

- *A functor $Ran_K F : \boldsymbol{D} \to \boldsymbol{A}$*
- *A natural transformation $\epsilon : Ran_K F \circ K \to F$*

Such that for each other pair $(H : \boldsymbol{D} \to \boldsymbol{A}, \delta : H \circ K \to F)$ there exist a unique $\beta : Lan_K F \to G$ such that $\delta = Ran_K F \circ \beta_K$ (where $\beta_K : Lan_K F \circ K \to G \circ K$ is defined in components as $(\beta_K)_c = \beta_{K(c)}$)

The Kan extensions do not necessarily exist in all cases, but as we will see it is not hard to have conditions for their existence. For example if $\mathbf{C}$ is small and $\mathbf{A}$ is (co)complete, then the left and right Kan extensions exist for all $F : \mathbf{C} \to \mathbf{A}$. These two conditions are satisfied by all the cases we consider in this document, hence we will never encounter any existential crises here.

Kan extensions define functors

Proposition 8.0.1. *Given $K : \boldsymbol{C} \to \boldsymbol{D}$, if the Kan extensions exist for all $F : \boldsymbol{C} \to \boldsymbol{A}$, we have that*

$$Lan_K, Ran_K : \texttt{Func}(\boldsymbol{C}, \boldsymbol{A}) \to \texttt{Func}(\boldsymbol{D}, \boldsymbol{A})$$

Define functors.

Proof. We define of course $Lan_K(F) = Lan_K F$ and $Ran_K(F) = Ran_K F$. We will see what Ran_K does to morphisms, and the action of Lan_K is analogous. Recall that each $Ran_K F$ comes along with a natural transformation $\epsilon^F : Ran_K F \to F$.

Consider $\alpha : F \to G$ on $\texttt{Func}(\mathbf{C}, \mathbf{A})$. We have that

$$\alpha \circ \epsilon^G : Ran_K G \circ K \xrightarrow{\epsilon^G} G \xrightarrow{\alpha} F$$

factorizes through ϵ^F. That is, there exists an unique

$$\bar{\alpha} : Ran_K G \to Ran_K F$$

such that $\alpha \circ \epsilon^G = \epsilon^F \circ \bar{\alpha}_K$. It is an easy exercise to prove that if we have $F, G, H : \mathbf{C} \to \mathbf{A}$ with $H \xrightarrow{\beta} G \xrightarrow{\alpha} F$ then $\alpha \bar{\circ} \beta = \bar{\alpha} \circ \bar{\beta}$, that is, Ran_K is functorial. □

Furthermore, the Kan extensions correspond to the left and right adjoints of the precomposition functor

$$- \circ K : \mathtt{Func}(\mathbf{D}, \mathbf{A}) \to \mathtt{Func}(\mathbf{C}, \mathbf{A})$$

As the following theorem shows

Theorem 8.1. *Given $K : \boldsymbol{C} \to \boldsymbol{D}$, if the Kan extensions exist for all $F : \boldsymbol{C} \to \boldsymbol{A}$, we have that*

$$(Lan_K \dashv - \circ K \dashv Ran_K)$$

That is, they form an adjoint triple. The unit and counit are given by $\eta^F : F \to Lan_K F \circ K$ and $\epsilon^F : Ran_K F \circ K \to F$ respectively.

We usually visualize this by saying that for a functor $K : \mathbf{C} \to \mathbf{D}$ we have a diagram of adjoints

$$\mathtt{Func}(\mathbf{D}, \mathbf{A}) \underset{Lan_K}{\overset{Ran_K}{\underset{\longleftarrow}{\overset{\longleftarrow}{-\circ K}}}} \mathtt{Func}(\mathbf{C}, \mathbf{A})$$

Proof. We will make the proof now for Lan_K, and then the case for the right Kan extension is analogous.

By Yoneda lemma, each $\gamma : F \to GK$ defines a natural transformation

$$\gamma^\star : \hom(G, -) \to \hom(F, - \circ K)$$

with $\gamma^\star_H(\alpha) = \alpha_K \circ \gamma$. We take $G = Lan_K F$. The universal property gives us that $\eta^\star$ is a natural isomorphism, and by its representability (given by Yoneda) we have that this is natural on G. □

What makes Kan extensions particularly useful is the fact that when $\mathbf{C}$ is small and $\mathbf{A}$ is (co)complete, they have a formula given by (co)limits. To establish these we give some notation.

For $K : \mathbf{C} \to \mathbf{D}$ as before and $d \in \mathbf{D}$ an object, recall that the comma category $d \downarrow \mathbf{D}$ consists of the category whose

- Objects are pairs $(f : d \to K(c), c \in \mathbf{C})$
- Morphisms are given by $\hom(d \xrightarrow{f} K(c), d \xrightarrow{f'} K(c')) = \{h : c \to c' | f' = K(h) \circ f\}$, that is, commutative diagrams

$$\begin{array}{ccc} K(c) & \xrightarrow{\quad K(h) \quad} & K(c') \\ & {\scriptstyle f} \nwarrow \quad \nearrow {\scriptstyle f'} & \\ & d & \end{array}$$

And we define a projection functor

$$Q^d : d \downarrow K \to \mathbf{C}$$

That acts in objects as $Q^d(d \xrightarrow{f} K(c)) = c$ and on morphisms as $Q^d(h) = h$. We now construct a formula for the right Kan extension

Construction of $Ran_K F$:

Consider as before $K : \mathbf{C} \to \mathbf{D}$ and $F : \mathbf{C} \to \mathbf{A}$, and suppose that the following limits exist for all $d \in \mathbf{D}$

$$R(d) = \lim_{d \downarrow K} F \circ Q^d$$

We have that any $g : d \to d'$ defines a functor $g : d' \downarrow K \to d \downarrow K$ by precomposition (that is $d' \xrightarrow{f} K(c)$ is sent to $d \xrightarrow{g} d' \xrightarrow{f} K(c)$), and since $Q^d \circ g = Q^{d'}$, this defines a morphism

$$R(g) : \lim_{d \downarrow K} F \circ Q^d \to \lim_{d' \downarrow K} F \circ Q^{d'}$$

We have that $R(gg') = R(g)R(g')$, making $R : \mathbf{D} \to \mathbf{A}$ into a functor.

Furthermore, for any $c \in \mathbf{C}$ we have a limit morphism

$$R(K(c)) = \lim_{K(c) \downarrow K} F \circ Q^{K(c)} \xrightarrow{l_{1_{K(c)}}} F(c)$$

Defining a natural transformation $\epsilon : R \circ K \to F$ given by $\epsilon_c = l_{1_{K(c)}}$.

And then we have the following

Theorem 8.2. *In the context of the previous construction, (R, ϵ) is a right Kan extension of F along K.*

We refer to ([Leh14] Theorem 3.1) for a proof of this theorem. The reader will have no complications following the proof since we use the same notation as the author of the reference.

On a dual way, the left Kan extension has a formula given by

$$Lan_K F(d) = \operatorname*{\texttt{colim}}_{K \downarrow d} F \circ Q^d$$

when those colimits exist. We now establish the condition for the existence of Kan extensions that we were mentioned all along

Corollary 8.2.1. *Given $K : \boldsymbol{C} \to \boldsymbol{D}$ and $F : \boldsymbol{C} \to \boldsymbol{A}$ as before. If $\boldsymbol{C}$ is small and $\boldsymbol{A}$ is complete and cocomplete, then $Lan_K F$ and $Ran_K F$ exist.*

Proof. This is simply because in this case the limits and colimits of the constructions before exist. □

It is common to left the comma categories and projection functors be obvious from the context and write

$$Lan_K F(d) = \operatorname*{\texttt{colim}}_{K(c) \to d} F(c)$$

for the left Kan extension, and

$$Ran_K F(d) = \lim_{d \to K(c)} F(c)$$

for the right Kan extensions.

We will summarize all this discussion in one theorem, since it is a result used all over this document.

Theorem 8.3. *Let $\boldsymbol{D}$ be a category, $\boldsymbol{C}$ be a small category and $\boldsymbol{A}$ be a complete and cocomplete category. For any functor $K : \boldsymbol{C} \to \boldsymbol{D}$ there is an adjoint triple*

$$\boldsymbol{Func}(\boldsymbol{D}, \boldsymbol{A}) \underset{\overleftarrow{Lan_K}}{\overset{\overleftarrow{Ran_K}}{\overleftarrow{-\circ K}}} \boldsymbol{Func}(\boldsymbol{C}, \boldsymbol{A})$$

Where Lan_K and Ran_K are defined on $F : \boldsymbol{C} \to \boldsymbol{A}$ and $d \in \boldsymbol{D}$ by

$$Lan_K F(d) = \underset{K(c) \to d}{\boldsymbol{colim}} F(c) \quad Ran_K F(d) = \lim_{d \to K(c)} F(c)$$

We now present one other useful corollary of the (co)limit formula.

Corollary 8.3.1. *In the context as last corollary, if K is fully faithful, then $Ran_K F \circ K \to F$ and $F \to Lan_K F \circ K$ are isomorphisms*

Proof. If K is fully faithful, any $f : K(c) \to K(c')$ equals $K(h)$ for some unique $h : c \to c'$, which implies that $(1_{K(c)}, K(c))$ is initial in $K(c) \downarrow K$ and final in $K \downarrow K(c)$, and this implies that the limit morphism

$$\lim_{K(c) \downarrow K} F \circ Q^{K(c)} \overset{l_{1_{K(c)}}}{\longrightarrow} F(c)$$

is an isomorphism. The same goes for the corresponding colimit. □

This last corollary gives us the commutative diagrams that extend functors presented at the introductory part of the section.

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{\exists \bar{F}} & \mathbf{A} \\ {\scriptstyle K}\uparrow & \nearrow_{\forall F} & \\ \mathbf{C} & & \end{array}$$

As one final point to mention, in many occasions during this document the categories $\mathbf{C}$ and $\mathbf{D}$ are preorders, and this makes the Kan extensions take a particularly simple shape, that we will see in the following example.

Example:
Suppose that $\mathbf{C}$ and $\mathbf{D}$ are preorders (that is, the hom sets have one or zero elements), and we have $K : \mathbf{C} \to \mathbf{D}$ a functor between them (that is, an order preserving function). Let $\mathbf{A}$ be a (co)complete category, then we have the adjoint triple

$$\mathtt{Func}(\mathbf{D}, \mathbf{A}) \underset{\overleftarrow{Lan_K}}{\overset{\overleftarrow{Ran_K}}{\overleftarrow{-\circ K}}} \mathtt{Func}(\mathbf{C}, \mathbf{A})$$

Where for any $F : \mathbf{C} \to \mathbf{A}$ and $d \in \mathbf{D}$

$$\mathrm{Lan}_K F(d) = \underset{K(c) \leq d}{\mathtt{colim}} F(c) \quad \mathrm{Ran}_K F(d) = \lim_{d \leq K(c)} F(c)$$

*We need to make the observation here that in many cases we take a preorders $(\mathbf{C}, \leq)$, $(\mathbf{D}, \leq)$ and then work with their opposite categories, considering functors $F : \mathbf{C}^{op} \to \mathbf{A}$. In this case $K(c) \to d$ means $d \leq K(c)$ and this is why the formulas that we present take the following shape

$$\mathrm{Lan}_K F(d) = \operatorname*{colim}_{d \leq K(c)} F(c) \quad \mathrm{Ran}_K F(d) = \lim_{K(c) \leq d} F(c)$$

These formulas can be read as follows, since we cannot calculate precisely "$F(d)$" when d does not belong to the image of K, we approximate it using (co)limits: $Lan_K F(d)$ will approximate "$F(d)$" from above and $Ran_K F(d)$ will approximate it from below. Then the fact that these are Kan extensions assures us that this process will yield a well behaved construction (in this case, harmonic with K, or more precisely, adjoint of $- \circ K$).

8.2 Model categories and Reedy model structure

Model categories present a way to generalize the homotopical theory to environments different from **Top** in a way in which we can give a structure on simplicial sets that makes it equivalent to the topological counterpart. The theory of model categories allows us to consider multiple useful examples outside this setting and it has evolved into being considered an important subject.

In this section of the appendix we will go through the definition of a model category and then see a particular class of examples of model categories on functor categories called Reedy model category structures, which is used in chapter 5 as we endow a model structure on $\mathtt{Rep}(X)$.

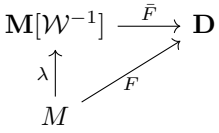
We will not in general give full proves of well established facts if it does not provide important insights, providing references when necessary.

8.2.1 Model categories

The main example of model categories is **Top**. To reflect the homotopical structure what we do is to create (optimally) a category in which weak homotopy equivalences are made to be isomorphisms, which will make topological spaces with the same homotopy groups actually isomorphic.

So in general the hole idea of model categories (just like for derived categories) is to *localize* a category $\mathbf{M}$ with respect to certain class of morphisms, which in this case we will call $\mathcal{W}$, which means that we want to make the morphisms of $\mathcal{W}$ be isomorphisms in an optimal way.

In mathematical terms, we are looking for a category $\mathbf{M}[\mathcal{W}^{-1}]$ with a functor $\lambda : \mathbf{M} \to \mathbf{M}[\mathcal{W}^{-1}]$ such that for all $F : \mathbf{M} \to \mathbf{D}$ that sends elements of $\mathcal{W}$ to isomorphisms, there is an unique $\bar{F} : \mathbf{M}[\mathcal{W}^{-1}] \to \mathbf{D}$ such that the following commutes



It is a classical result that

Proposition 8.3.1. *For a category $\boldsymbol{C}$ and a class of morphisms $\mathcal{W}$, we have that $\boldsymbol{C}[\mathcal{W}^{-1}]$ exists and it is unique up to a unique equivalence that preserves structure.*

However in general the construction of this category can be very cumbersome and even have problematic size issues. The idea of model categories (as well

as for derived categories) is that it presents a rather simple model to present $\mathbf{M}[\mathcal{W}^{-1}]$ for a large class of categories which we call model categories.

Definition. *A **model category** is a category $\boldsymbol{M}$ equipped with 3 classes of morphisms*

- *$\mathcal{W}$ of **weak equivalences**, drawn in diagrams as $\xrightarrow{\sim}$*
- *$\mathcal{F}$ of **fibrations**, drawn in diagrams as $\twoheadrightarrow$*
- *$\mathcal{C}$ of **cofibrations**, drawn in diagrams as $\hookrightarrow$*

satisfying the following five axioms

MC 1 $\boldsymbol{M}$ *is complete and cocomplete*

MC 2 *The class $\mathcal{W}$ satisfies the "2 out of 3" property, that is, for any diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ if two elements of $\{\{f, g, gf\}$ belong to $\mathcal{W}$ then $\{f, g, gf\} \subset \mathcal{W}$*

MC 3 *The classes $\mathcal{W}$, $\mathcal{F}$ and $\mathcal{C}$ are stable under retract. That is, if we have f, g, i, i', r and r' with $ri = Id$, $r'i' = Id$ such that the following diagram commutes*

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{r} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{i'} & Y & \xrightarrow{r'} & B \end{array}$$

Then $f \in \mathcal{W} \Rightarrow g \in \mathcal{W}$, $f \in \mathcal{F} \Rightarrow g \in \mathcal{F}$ and $f \in \mathcal{C} \Rightarrow g \in \mathcal{C}$

MC 4 *For any commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

(with $i \in \mathcal{C}$ and $p \in \mathcal{F}$), if either $i \in \mathcal{W}$ or $p \in \mathcal{W}$, then there exists a lift $h : B \to X$ making the following diagram commute

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

MC 5 *Every morphism $f : X \to Y$ admits factorizations $X \xrightarrow{\sim} \cdot \twoheadrightarrow Y$ and $X \hookrightarrow \cdot \xrightarrow{\sim} Y$, in a way functorial with respect to f.*

Observe that (MC1) implies that a model category $\mathbf{M}$ has an initial $\emptyset$ and a final $\star$ object.

Definition. *Given $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ a model category*

- *$X \in \boldsymbol{M}$ is called **cofibrant** if $\emptyset \to X$ is a cofibration.*
- *$X \in \boldsymbol{M}$ is called **fibrant** if $X \to \star$ is a fibration.*

- *The class $\mathcal{F} \cap \mathcal{W}$ is called* ***acyclic fibrations*** *and $\mathcal{C} \cap \mathcal{W}$ is called* ***acyclic cofibrations****.*

All objects $X \in \mathbf{M}$ in a model category can be represented by a fibrant of cofibrant object in the following way

Proposition 8.3.2. *For all $X \in \boldsymbol{M}$, there exist*

- $Q_X \xrightarrow{\sim} X$*, with Q_X cofibrant.*
- $X \xhookrightarrow{\sim} R_X$*, with R_X fibrant.*

Functorially with respect to X.

Proof. Apply (MC 5) to $\emptyset \to X$ and $X \to \star$. □

In this last proposition, R_X is called **fibrant replacement** and Q_X is called **cofibrant replacement**.

Examples of model categories include topological spaces, simplicial sets and chain complexes.

Examples:

- The most famous model structure on **Top** is what is called Quillen model structure, where
  - The class $\mathcal{W}$ is given by weak homotopy equivalences.
  - The class $\mathcal{F}$ is given by serre fibrations.
  - The class $\mathcal{C}$ is given by retracts of generalized relative CW-complexes.
- There is also a model structure on **SSet** called Quillen model structure, where
  - The class $\mathcal{W}$ is given by morphisms whose realization are weak homotopy equivalences
  - The class $\mathcal{F}$ is given by Kan fibrations.
  - The class $\mathcal{C}$ is given by monomorphisms.
- For R a commutative ring, there is a model structure, called projective model structure, on $Ch_{\geq 0}(R)$ in which
  - The class $\mathcal{W}$ is given by quasi isomorphisms.
  - The class $\mathcal{F}$ is given by maps that are epimorphisms in degree ≥ 1.
  - The class $\mathcal{C}$ is given by injections with projective cokernel.
- For R a commutative ring, there is a model model structure, called injective model structure, on $Ch_{\geq 0}(R)$ in which
  - The class $\mathcal{W}$ is given by quasi isomorphisms
  - The class $\mathcal{F}$ is given by surjective maps with injective kernel.
  - The class $\mathcal{C}$ is given by maps injective in degree ≥ 1.

The homotopy category of a model category corresponds to its localization with respect to weak equivalences

Definition. *Given $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ a model category, we call the* ***homotopy category*** *of $\boldsymbol{M}$ to*

$$Ho(\boldsymbol{M}) = \boldsymbol{M}[\mathcal{W}^{-1}]$$

In this category, the objects are the same as in $\mathbf{M}$, and weak equivalences are now isomorphisms.

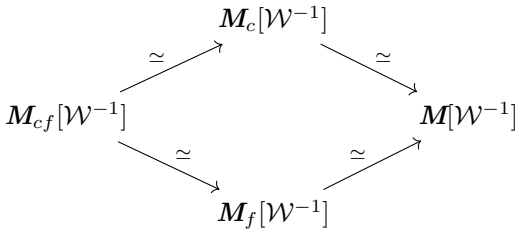
There are different ways to construct the homotopy category. For starters, it might be nice to know that we can restrict our attention to fibrant and/or cofibrant objects

Definition. *Given $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ a model category, we define three full subcategories*

- *M_f of fibrant objects.*
- *M_c of cofibrant objects.*
- *M_{cf} of objects that are both fibrant and cofibrant.*

And we have the following proposition, whose detailed proof can be found in ([Hov99] Proposition 1.2.3).

Proposition 8.3.3. *Given $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ a model category, there are equivalences of categories*



Proof. Since the inclusion $\mathbf{M}_c \to \mathbf{M} \to \mathbf{M}[\mathcal{W}^{-1}]$ sends $\mathcal{W}$ to isomorphisms, it defines $\mathbf{M}_c[\mathcal{W}^{-1}] \to \mathbf{M}[\mathcal{W}^{-1}]$. The inverse is defined by

$$\begin{aligned} Q \colon \mathbf{M} &\longrightarrow \mathbf{M}_c \longrightarrow \mathbf{M}_c[\mathcal{W}^{-1}] \\ X &\longmapsto Q_X \end{aligned}$$

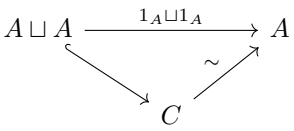
where $Q_X \xrightarrow{\sim} X$ is the cofibrant replacement of X. The other equivalences are analogous. □

There is a particularly classical way to construct $Ho(\mathbf{M})$ using a generalization of homotopies. We will see this in what follows.

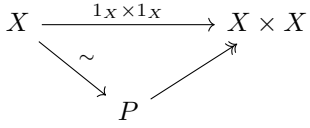
We begin with a definition

Definition. *Given $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ a model category and $A, X \in \boldsymbol{M}$ objects.*

- *A* ***cylinder*** *of A is a factorization of $A \sqcup A \xrightarrow{1_A \sqcup 1_A} A$ as*



- *A **path object** of X is a factorization of $X \xrightarrow{1_X \times 1_X} X \times X$ as*



Observe how these two definitions are dual. Observe also that from (MC 5) we obtain that every object has a cylinder and a path object.

As an example, in **Top** with Quillen structure a cylinder is given by

$$A \times \{0,1\} \xrightarrow{(\iota_0,\iota_1)} A \times [0,1] \xrightarrow{\sim} A$$

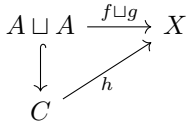
and a path object is given by

$$X \xrightarrow{\sim} X^{[0,1]} \xrightarrow{(ev_0,ev_1)} X$$

With these objects we can abstractly define the concept of homotopy. It turns out that two maps can be homotopic in two different (and dual) ways, which under good conditions will coincide.

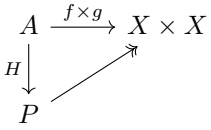
Definition. *Given $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ a model category, $A, X \in \boldsymbol{M}$ objects and $f, g : A \to X$ morphisms*

- *A **left homotopy** between f and g is a cylinder $A \sqcup A \hookrightarrow C \xrightarrow{\sim} A$ togueter with a morphism $h : C \to X$ such that*



  commutes. We say that f and g are left homotopic and write it as $f \overset{l}{\simeq} g$.

- *A **right homotopy** between f and g is a path object $X \xrightarrow{\sim} P \twoheadrightarrow X \times X$ togueter with a morphism $H : A \to P$ such that*



  commutes. We say that f and g are left homotopic and write it as $f \overset{r}{\simeq} g$.

The proof of the following proposition and theorem can be found in ([Hov99] Chapter 1.2).

Proposition 8.3.4. *Given $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ a model category, $A, X \in \boldsymbol{M}$ objects and $f, g : A \to X$ morphisms*

1. - *For all $h : X \to Y$ we have $f \overset{l}{\simeq} g \Rightarrow hf \overset{l}{\simeq} hg$*

- *For all* $h : B \to A$ *we have* $f \overset{r}{\simeq} g \Rightarrow fh \overset{r}{\simeq} gh$

2. - *If* A *is cofibrant then* $\overset{l}{\simeq}$ *is an equivalence relation on* $\hom_{\boldsymbol{M}}(A, X)$
   - *If* X *is fibrant then* $\overset{r}{\simeq}$ *is an equivalence relation on* $\hom_{\boldsymbol{M}}(A, X)$

3. *If* A *is cofibrant and* X *is fibrant, then* $f \overset{l}{\simeq} g \Leftrightarrow f \overset{r}{\simeq} g$. *We call the relation in this case simply by* $\simeq$.

Given $A \in \mathbf{M}$ cofibrant and $X \in \mathbf{M}$ fibrant, we have that $\simeq$ is an equivalence relation on $\hom_{\mathbf{M}}(A, X)$. We call in this case

$$[A, X] = \hom_{\mathbf{M}}(A, X)/\simeq$$

Now we define the category $\pi\mathbf{M}_{cf}$ to be the category in which

- Objects are given by $Obj(\pi\mathbf{M}_{cf}) = Obj(\mathbf{M}_{cf})$, objects of $\mathbf{M}$ which are fibrant and cofibrant.
- Morphisms are given by $\hom_{\pi\mathbf{M}_{cf}}(A, X) = [A, X]$

We arrive then to the theorem that gives shape to the homotopy category

Theorem 8.4. *Given* $(\boldsymbol{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ *a model category, we have*

$$Ho(\boldsymbol{M}) \simeq \pi\boldsymbol{M}_{cf}$$

And for $A, X \in \boldsymbol{M}$

$$\hom_{Ho(\boldsymbol{M})}(A, X) = [Q_A, R_X]$$

where $Q_A \overset{\sim}{\twoheadrightarrow} A$ *is a cofibrant replacement of* A *and* $X \overset{\sim}{\hookrightarrow} R_X$ *is a fibrant replacement of* X.

We end this section defining what work as *morphisms* between model categories.

Definition. *Given an adjunction between model categories*

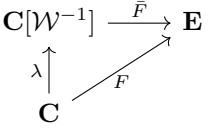
$$F : \boldsymbol{M} \leftrightarrows \boldsymbol{N} : G$$

We say that the pair (F, G) *is a* ***Quillen adjunction*** *if one of the following equivalent properties is satisfied.*

1. F *preserves cofibrations and acyclic cofibrations.*
2. G *preserves fibrations and acyclic fibrations.*
3. F *preserves cofibrations and* G *preserves fibrations.*
4. F *preserves acyclic cofibrations and* G *preserves acyclic fibrations.*

If a Quillen adjunction (F, G) is an equivalence of categories, we call it **Quillen equivalence**. The idea of having this kind of adjunctions is to assure the existence of *derived functors*.

We motivate a discussion about derived functors by considering the general setting of localization. Given a category $\mathbf{C}$ and a class of morphisms $\mathcal{W}$, when presented the diagram



defining the localization of $\mathbf{C}$ with respect to $\mathcal{W}$, we said that every functor that *sends $\mathcal{W}$ into isomorphisms* can be extended along λ. One might wonder what happens to the rest of the functors, is there a way or *a best way possible* to extend any $F : \mathbf{C} \to \mathbf{E}$ along λ?

Observe that this question is the same one asked for *Kan extensions* (see appendix 8.1).

Definition. *Given $\lambda : \boldsymbol{C} \to \boldsymbol{C}[\mathcal{W}^{-1}]$ a localization. A **right localization** of a functor $F : \boldsymbol{C} \to \boldsymbol{E}$ (if it exists) is a left Kan extension of F along λ*

$$RL(F) = Lan_{\lambda}(F)$$

*If the right localization exists we say that F is **right localizable**.*

Similarly left localization is a right Kan extension of F along λ. As remarked in the definition, these right and left localizations of functors need not to exist. However, their existence is guaranteed under certain conditions.

Proposition 8.4.1. *Given $F : \boldsymbol{M} \leftrightarrows \boldsymbol{N} : G$ a Quillen adjunction, then*

- *The functor $\boldsymbol{M} \xrightarrow{F} \boldsymbol{N} \to Ho(\boldsymbol{N})$ is left localizable. We call LF its left localization and name it the **left derived functor** of F.*
- *The functor $\boldsymbol{N} \xrightarrow{G} \boldsymbol{M} \to Ho(\boldsymbol{M})$ is right localizable. We call RG its right localization and name it the **right derived functor** of G.*

Furthermore, we have an adjunction

$$LF : Ho(\boldsymbol{M}) \leftrightarrows Ho(\boldsymbol{N}) : RG$$

And we have formulas to calculate the derived functors

$$LF(A) = F(Q_A) \quad RG(X) = G(R_X)$$

Where $Q_A \xrightarrow{\simeq} A$ is a cofibrant replacement of A, and $X \xrightarrow{\simeq} R_X$ is a fibrant replacement of X.

This is proven throughout ([Hov99] Section 1.3).

8.2.2 Reedy model category structure

When working with structures of homotopical nature one arrives to the necessity of insert a model structure into a functor category $\texttt{Func}(\mathbf{C}, \mathbf{M})$. This happens in cases such as the one in this paper, that is, sheaves or representations of certain kind, but also when working with colimits and limits. If we wish to make (co)limits that are compatible with the homotopical structure we

arrive to the necessity of establishing a model structure in the functors from the corresponding diagrams.

In both these examples, the codomain category is a model category. In general, we wish that the structure of the model category inserted in $\mathtt{Func}(\mathbf{C}, \mathbf{M})$ inherits in some way the model structure of $\mathbf{M}$. In particular it is desirable that for any $F, G \in \mathtt{Func}(\mathbf{C}, \mathbf{M})$

$$f : F \to G \text{ is a weak equivalence } \Leftrightarrow f_c : F(c) \to G(c) \text{ is a weak equivalence } \forall c \in \mathbf{C}$$

There are two "obvious" ways to do this which are called projective and injective model structures, although existence of these structures depends in general on conditions for $\mathbf{M}$. We are going to focus our attention on a third way to do this, which imposes conditions on the domain category $\mathbf{C}$. It has to be what is called a **Reedy category**.

Definition. *A Reedy category is a small category $\boldsymbol{C}$, together with two full subcategories $\overrightarrow{\boldsymbol{C}}$ and $\overleftarrow{\boldsymbol{C}}$, in which every object of $\boldsymbol{C}$ is asigned a non-negative integer called the degree, such that*

- *Every non-identity morphism in $\overrightarrow{\boldsymbol{C}}$ raises the degree.*
- *Every non-identity morphism in $\overleftarrow{\boldsymbol{C}}$ lowers the degree.*
- *Every morphism g in $\boldsymbol{C}$ has a unique factorization $g = \overrightarrow{g}\overleftarrow{g}$ with $\overrightarrow{g}$ in $\overrightarrow{\boldsymbol{C}}$ and $\overleftarrow{g}$ in $\overleftarrow{\boldsymbol{C}}$*

We are going to work over the following Reedy category.

Example:
Let Δ be the skeleton category of finite linearly order sets and order preserving maps between them. As it is usual, we call $[n] = \{0 < \cdots < n\}$. This is a Reedy category for which

i) Arrows of $\overleftarrow{\Delta}$ correspond to surjective functions.

ii) Arrows of $\overrightarrow{\Delta}$ correspond to injective functions.

We have that Δ^{op} is a Reedy category as well, in view of the observation that if $\mathbf{C}$ is a Reedy category, then $\mathbf{C}^{op}$ is a Reedy category with $\overleftarrow{\mathbf{C}^{op}} = \overrightarrow{\mathbf{C}}^{op}$ and $\overrightarrow{\mathbf{C}^{op}} = \overleftarrow{\mathbf{C}}^{op}$ (using the same degree function).

Given $\mathbf{C}$ a Reedy category and $\mathbf{M}$ a model category, we can induce a model category structure on the functor category $\mathbf{M}^{\mathbf{C}}$. This will allow us to have a model category structure on any category of simplicial objects $\mathbf{M}^{\Delta^{op}}$.

Some Notation: For a functor $F : \mathbf{C} \to \mathbf{D}$, and a object in $\mathbf{D}$, we call $F \downarrow a$ the comma category whose objects are morphisms $F(b) \to a$ (with b object of $\mathbf{C}$), and whose morphisms are

$$\mathrm{Hom}_{\mathbf{C}\downarrow a}(F(b_1) \xrightarrow{\alpha_1} a, F(b_2) \xrightarrow{\alpha_2} a) = \{F(b_1) \xrightarrow{F(f)} F(b_2) \mid \alpha_2 F(f) = \alpha_1\}$$

For the identity functor $I_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ we call $I_{\mathbf{C}} \downarrow a$ simply by $\mathbf{C} \downarrow a$. We call the oposite category $a \downarrow \mathbf{C}$.

Definition. *Given $\boldsymbol{C}$ a Reedy category, $\boldsymbol{M}$ a model category, X a functor in $\boldsymbol{M}^{\boldsymbol{C}}$ and a an object in $\boldsymbol{C}$, we define the following two objects of $\boldsymbol{M}$*

1. *The latching object of X at a is defined as $L_aX = \underset{\partial(\overrightarrow{C}\downarrow a)}{colim} X$.*
2. *The matching object of X at a is defined as $M_aX = \underset{\partial(a\downarrow\overleftarrow{C})}{lim} X$.*

Where $\partial(\overrightarrow{C} \downarrow a)$ is the full subcategory of $\overrightarrow{C} \downarrow a$ with all objects except for the identity 1_a (similarly for $\partial(a \downarrow \overleftarrow{C})$). We are calling also by X the induced functors $X(b \to a) = X(b)$ in $\partial(\overrightarrow{C} \downarrow a)$ and $X(a \to b) = X(b)$ in $\partial(a \downarrow \overleftarrow{C})$.

We call *latching map* and *matching map* the natural morphisms $L_aX \to X(a)$ and $X(a) \to M_aX$.

Using these maps, for a natural transformation $f : X \to Y$ in $\mathbf{M^C}$ and an object a in $\mathbf{C}$, we have the conmutative squares

$$\begin{array}{ccc} L_aX & \longrightarrow & X(a) \\ {\scriptstyle L_a(f)}\downarrow & & \downarrow{\scriptstyle f_a} \\ L_aY & \longrightarrow & Y(a) \end{array}$$

and

$$\begin{array}{ccc} X(a) & \longrightarrow & M_aX \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle M_a(f)} \\ Y(a) & \longrightarrow & M_aY \end{array}$$

Which define morphisms $X(a) \sqcup_{L_aX} L_aY \to Y(a)$ and $X(a) \to Y(a) \times_{M_aY} M_aX$, called the relative latching and relative matching maps.

Now we are ready to define a model category structure on $\mathbf{M^C}$. The proof of the following can be found in ([Hir03], Chapter 15, 15.3.1).

Theorem 8.5. *Given $\boldsymbol{C}$ a Reedy category and $\boldsymbol{M}$ a model category, there is a model category structure (called Reedy model category structure) on $\boldsymbol{M}^{\boldsymbol{C}}$, in which*

1. *A map $f : X \to Y$ is a weak equivalence if for every object a in $\boldsymbol{C}$ the map*

$$f_a : X(a) \to Y(a)$$

   is a weak equivalence in $\boldsymbol{M}$.

2. *A map $f : X \to Y$ is a cofibration if for every object a in $\boldsymbol{C}$ the relative latching map*

$$X(a) \sqcup_{L_aX} L_aY \to Y(a)$$

   is a cofibration in $\boldsymbol{M}$.

3. *A map $f : X \to Y$ is a fibration if for every object a in $\boldsymbol{C}$ the relative matching map*

$$X(a) \to Y(a) \times_{M_a Y} M_a X$$

is a fibration in $\boldsymbol{M}$.

Observe that an object X in $\mathbf{M}^{\mathbf{C}}$ is cofibrant in this model structure if the latching map $L_a X \to X(a)$ is a cofibration for all a in $\mathbf{C}$. Similarly, X is fibrant if the matching map $X(a) \to M_a X$ is a fibration for all a in $\mathbf{C}$.

Under certain conditions for some reedy categories there will be a pleasant way of characterize fibrant objects. We will explain it in what follows

For a Reedy category , call $' = [R^{op}, Set]$ the asociated presheaf category. To objects a, b, c in , we call A, B, C the image under the Yoneda embedding (or simply by the same leters a, b, c when there's no risk of confusion).

For functors L, K in $'$, if the piecewhise inclusion $i_a : L(a) \to K(a)$ is a morphism, we say that L is subobject of K. We write $L \subset K$.

Consider now the Reedy model structure of $\mathbf{M}^{^{op}}$. The existence of the following object guaranties a simple shape for the matching maps.

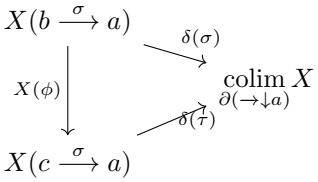
Condition: For $a \in$, supose that there is an object $\partial A \subset A$ in $'$ such that

i) For $b \in$ with $d(b) < d(a)$, $Hom_{'}(B, \partial A) = Hom_{'}(B, A)$ (equivalently $\partial A(b) = A(b)$)

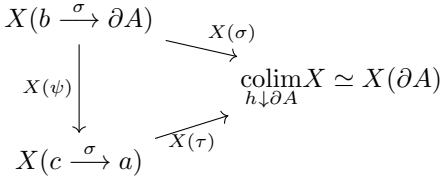
ii) For $b \in$ with $d(b) \geq d(a)$, and $\sigma \in \partial A(b)$, there exist a map $\overleftarrow{f} : b \to c$ in $\leftarrow$, with $d(c) < d(a)$, such that $\sigma = \sigma' \circ \overleftarrow{f}$.

Proposition 8.5.1. *Supose that satisfies the condition above for $a \in$, then for a functor $X \in \boldsymbol{M}^{^{op}}$ we have that $M_a X \simeq X(\partial A)$ (where X is being extended as in Prop1). Moreover, the matching map corresponds to the projection $X(A) \xrightarrow{X(i)} X(\partial A)$.*

Proof. We are going to make the proof for the oposites. Observe that for $Y \in \mathbf{M}^{R^{op}}$, $\partial(a \downarrow \overleftarrow{^{op}}) = \partial(a \downarrow \overrightarrow{R}^{op}) = \partial(\to \downarrow a)^{op}$, and then $\lim_{\partial(a \downarrow \overleftarrow{^{op}})} Y = \operatorname{colim}_{\partial(\to \downarrow a)} Y^{op}$. So we just need to prove that the colimit of a functor $X \in \mathbf{M}$ over the category of representables of ∂A is the same as the colimit over $\partial(\to \downarrow a)$. We have the two corresponding cones



and



From part i) of the condition, we easily deduce the existence of an arrow from the first colimit to the second. In fact, since for elements $b \xrightarrow{\sigma} a$ of $\partial(\to \downarrow a)$ we have that $d(b) < d(a)$, and $\mathrm{Hom}(B, \partial A) = \mathrm{Hom}(B, A)$ in such case, we get that $X(\partial A)$ toguether with the maps $\{X(\sigma)\}_{\sigma \in \partial(\to \downarrow a)}$ form a cone for the first diagram. We call

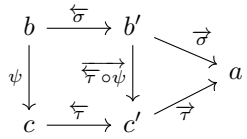
$$f : \operatorname*{colim}_{\partial(\to \downarrow a)} X \to X(\partial A)$$

the unique map such that

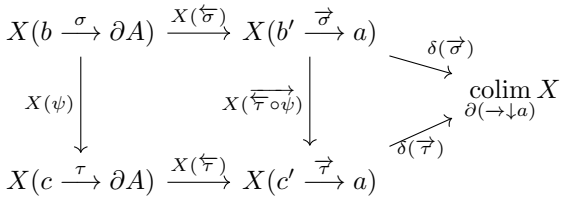
$$f \circ \delta(\sigma) = X(\sigma) \text{ for all } \sigma \in \partial(\to \downarrow a) \tag{10}$$

To get an arrow in the other direction, consider a map $B \xrightarrow{\sigma} \partial A$. If $d(b) < d(a)$, σ is an arrow in , and we consider its descomposition $b \xrightarrow{\sigma} a = b \xrightarrow{\overleftarrow{\sigma}} b' \xrightarrow{\overrightarrow{\sigma}} a$. If $d(b) \geq d(a)$, by condition ii), $\sigma = \sigma' \circ \overleftarrow{f}$, so σ is in fact a morphism in , we consider its descomposition $b \xrightarrow{\sigma} a = b \xrightarrow{\overleftarrow{\sigma}} b' \xrightarrow{\overrightarrow{\sigma}} a$.

Now, for a morphism ψ from $\sigma : b \to a$ to $\tau : c \to a$, observe that $\overrightarrow{\sigma} \circ \overleftarrow{\sigma} = \overrightarrow{\tau} \circ \overleftarrow{\tau} \circ \psi = (\overrightarrow{\tau} \circ \overrightarrow{\overleftarrow{\tau} \circ \psi}) \circ \overleftarrow{\overleftarrow{\tau} \circ \psi}$, and so $\overleftarrow{\sigma} = \overleftarrow{\overleftarrow{\tau} \circ \psi}$. This means that we have the following commutative diagram



This allows us to consider the following natural transformation



From here, we get a morphism

$$g : X(\partial A) \to \operatorname*{colim}_{\partial(\to \downarrow a)} X$$

which is the unique morphism such that

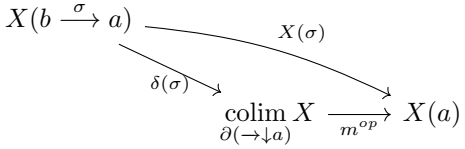
$$g \circ X(\sigma) = \delta(\overrightarrow{\sigma}) \circ X(\overleftarrow{\sigma}) \text{ for all } \sigma : b \to \partial A \tag{11}$$

Now, f and g are inverse of each other. To see this we need to check that $f \circ g \circ X(\sigma) = X(\sigma)$ for all $\sigma \in h_R \downarrow \partial A$, and $g \circ f \circ \delta(\sigma) = \delta(\sigma)$ for all $\sigma \in \partial(\to \downarrow a)$. This two come easily using the equations (1) and (2).

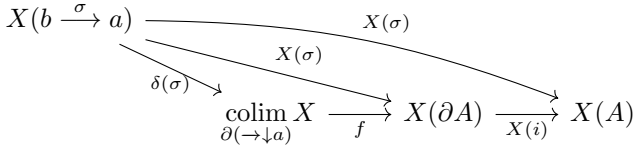
And so we have

$$X(\partial A) \simeq \operatorname*{colim}_{\partial(\to \downarrow a)} X$$

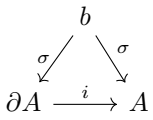
Call m the matching map, this is the only map that makes the diagrams



commute. But $X(i) \circ f$ also makes the diagrams



whose left part commute by (1), and whose right part commute because the diagrams



commute. □

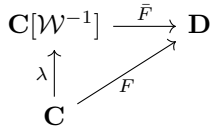
Following this, if the object ∂A exists for all $a \in$, we can conclude that an object $X \in \mathbf{M}^{op}$ is fibrant if the projections $X(A) \to X(\partial A)$ are fibrations for all $a \in$.

In the case of $= \Delta$, for each $n \in \mathbb{N}$, an object that satisfies the condition of the last proposition is $\partial\Delta^n$, hence $X \in \mathbf{M}^{\Delta^{op}}$ is fibrant if the projections $X(\Delta^n) \to X(\partial\Delta^n)$ are fibrations in $\mathbf{M}$.

8.3 Derived Categories of abelian categories with enough injectives

Just as with model categories, the idea of derived categories is born as a way to consider *quasi isomorphisms* as isomorphisms, so that we can make theoretical distentions by regarding the *homology*.

So once again, just like with model categories, we are looking how to *localize* a category



As we recall this usually a hard problem. Model categories gives us one setting in which this is possible, and now *derived categories* gives us another one. We are going to focus in a particular kind of category, an *abelian category with enough injectives*, in which the construction of the localized category with respect to *quasi isomorphisms* can be described without too much pain. For a detail exposition at a basic level the reader can check ([Alu09] Chapter IX.). Less basic level expositions can be found in [Sch11], [Ive86] Chapter I and in [Ban07].

8.3.1 Abelian categories

Abelian categories can be described as *good places to do homology theory.* They are environments in which we can work with (co)kernels, direct sums, exact sequences, etc. and pleasant features such as description of mono and epiness based on (co)kernels and isomorphism being both mono and epimorphism. These features emulate the category R-mod, and in a very real sense, this emulation is backed up. Furthermore, abelian categories are a environment in which we can perform homology theory, expanding this subject far beyond its original land of topology.

We start by defining additive categories

Definition. *A category $\boldsymbol{C}$ is said to be **additive** if it satisfies the following*

1. $\hom_{\boldsymbol{C}}(X, Y)$ *is an abelian group for all $X, Y \in \boldsymbol{C}$.*
2. *The composition is bilinear with respect to the abelian group operation.*
3. *$\boldsymbol{C}$ has a zero object (an object that is initial and final).*
4. *$\boldsymbol{C}$ admits products and coproducts.*

Observe that this means that $\hom_{\mathbf{C}}(X, Y)$ is empty for all $X, Y \in \mathbf{C}$.

Lemma 8.5.1. *Given a category satisfying 1-3 in the definition above. Then the forth condition is equivalent to*

- *For any $X, Y \in \boldsymbol{C}$, there exists $Z \in \boldsymbol{C}$ and morphisms $i_1 : X \to Z$, $i_2 : Y \to Z$, $p_1 : Z \to X$ and $p_2 : Z \to Y$ satisfying*

$$\begin{aligned} p_1 \circ i_1 = 1_X, \qquad & p_1 \circ i_2 = 0 \\ p_2 \circ i_2 = 1_Y, \qquad & p_2 \circ i_1 = 0 \\ i_1 \circ p_1 + i_2 \circ p_2 = 1_Z & \end{aligned}$$

And we have that $X \sqcup Y \simeq X \times Y$ for all $X, Y \in \boldsymbol{C}$

The proof of the lemma can be found in ([Sch11] Lemma 3.1.3). This lemma means that the finite product and coproduct are equivalent in an additive category. We denote the coproduct by $\oplus$ and call it **direct sum**.

Definition. *A functor $F : \boldsymbol{C} \to \boldsymbol{C'}$ between additive categories is called **additive** if for all $X, Y \in \boldsymbol{C}$*

$$F : \hom_{\boldsymbol{C}}(X, Y) \to \hom_{\boldsymbol{C}}(F(X), F(Y))$$

is a morphism of groups.

We move right into defining abelian categories.

Definition. *An additive category $\boldsymbol{A}$ is said to be **abelian** if*

1. *Every morphism has a Kernel and a Cokernel.*
2. *Every monomorphism is a Kernel and every epimorphism is a Cokernel.*

We have a condition which is equivalent, and sometimes more useful than the second condition just given. For this consider an additive category with (co)kernels. For a morphism $f : X \to Y$ in $\mathbf{C}$ we define

$$\texttt{Coim}(f) = \texttt{Coker}(\texttt{Ker}(f) \to X) \quad \texttt{im}(f) = \texttt{Ker}(Y \to \texttt{Coker}(f))$$

The properties universal properties of (co)kernels assures the existence of a morphism $u : \texttt{Coim}(f) \to \texttt{im}(f)$, that can be visualized in the following diagram

$$\begin{array}{ccccccc} \texttt{Ker}(f) & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & \texttt{Coker}(f) \\ & & \downarrow & & \uparrow & & \\ & & \texttt{Coim}(f) & \xrightarrow{u} & \texttt{im}(f) & & \end{array}$$

It is proven in ([Alu09] Chapter X.1.5) that condition 2. of the last definition is equivalent to $u : \texttt{Coim}(f) \to \texttt{im}(f)$ being an isomorphism. So we may define an abelian category as follows.

Definition. *An additive category $\boldsymbol{A}$ is said to be **abelian** if*

1. *Every morphism has a Kernel and a Cokernel.*
2. *For every morphism $f : X \to Y$ in $\boldsymbol{A}$, the canonical morphism $u : \texttt{Coim}(f) \to \texttt{im}(f)$ is an isomorphism.*

We also mention that all abelian categories considered in this document are complete and cocomplete.

Working in abelian categories is very pleasant in regards to the mono and epiness of morphisms.

Lemma 8.5.2. *In an abelian category $\boldsymbol{A}$ we have that*

- *$f : X \to Y$ is a monomorphism iff $\texttt{Ker}(f) \simeq 0$ and it is a epimorphism iff $\texttt{Coker}(f) \simeq 0$.*
- *$f : X \to Y$ is an isomorphism iff $\texttt{Ker}(f) \simeq 0$ and $\texttt{Coker}(f) \simeq 0$.*
- *If $f : X \to Y$ is an isomorphism, then $X \simeq \texttt{Coim}(f)$ and $Y \simeq \texttt{im}(f)$.*

The proof of this can be found in ([Alu09] Chapter X 1.5). Let us see some examples now.

Examples:

1. The cannonical example of an abelian category is R-mod. In fact, as mentioned, all abelian categories *feel like* working on R-mod, and this is backed up by what is called the Freyd-Mitchell theorem, which says that for any small abelian category $\mathbf{A}$, there is an exact fully faithful functor $\mathbf{A} \hookrightarrow R$-mod, for some ring R (see below the definition of exact functor). This theorem allows to work with many constructions using objects as in R-mod.
2. If $\mathbf{A}$ is abelian, then $\mathbf{A}^{op}$ is abelian.

3. If $\mathbf{A}$ is abelian, then for any category $\mathbf{C}$, the functor category

$$\texttt{Func}(\mathbf{C}, \mathbf{A})$$

Is an abelian category, since (Co)limits (in particular (co)kernels) are computed componentwise, as well as the morphism $\texttt{Coim}(f) \to \texttt{im}(f)$.

4. The previous example directly implies that for a topological space X and an abelian category $\mathbf{A}$, the category of *presheaves* $\texttt{PSh}(X) = \texttt{Func}(Op(X)^{op}, \mathbf{A})$ is an abelian category.

Furthermore the category *sheaves* $\texttt{Sh}(X)$, defined in 2.1.2, over an abelian category form an abelian category.

5. For an abelian category $\mathbf{A}$, the category

$$Diff(\mathbf{A}) = \texttt{Func}(\mathbb{Z}, \mathbf{A})$$

And the category

$$Diff_{\geq 0}(\mathbf{A}) = \texttt{Func}(\mathbb{N}, \mathbf{A})$$

Are abelian categories as the third example shows.

6. We denote by $Ch(\mathbf{A})$ and $Ch_{\geq 0}(\mathbf{A})$ the full subcategories of $Diff(\mathbf{A})$ (and $Diff_{\geq 0}(\mathbf{A})$ respectively) whose objects are functors F such that $F(n \to n+2) = 0$ for all $n \in \mathbb{Z}$ (and for all $n \in \mathbb{N}$ respectively). It is easy to see that these are abelian categories as well with (co)limits defined componentwise.

The category $Ch(\mathbf{A})$ can be better visualized as follows

- Objects are sequences

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \dots$$

Such that $d^k \circ d^{k-1} = 0$ for all $k \in \mathbb{Z}$

These sequences are denoted as $(X^\cdot, d^\cdot_X)$, or simply as $(X^\cdot, d^\cdot_X)$ or even as $X^\cdot$ if things are clear for context. The morphisms $d^\cdot$ are called **differencials**.

- Morphisms $f^\cdot : X^\cdot \to Y^\cdot$ are commutative diagrams

$$\begin{array}{ccccccc} \dots & \longrightarrow & X^n & \xrightarrow{d^n_X} & X^{n+1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \dots & \longrightarrow & Y^n & \xrightarrow{d^n_Y} & Y^{n+1} & \longrightarrow & \dots \end{array}$$

This last example is very important, and we will dedicate it some lines right after defining what *exact sequences and functors* are.

Consider a complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{A}$, that is $g \circ f = 0$. Since $g \circ f = 0$, there is a morphism $\texttt{Coim}(f) \to \texttt{Ker}(g)$, and since $\mathbf{A}$ is abelian, there is a morphism $\texttt{im}(f) \to \texttt{Ker}(g)$.

Definition. *We say that a given sequence in an abelian category* $\boldsymbol{A}$

$$X^k \xrightarrow{d^k} X^{k+1} \xrightarrow{d^{k+1}} \cdots \to X^n$$

such that $d^{j+1} \circ d^j = 0$ *for all* $j \in \{k, \ldots, n-1\}$ *is* ***exact*** *if* $\mathtt{im}(d^j) \xrightarrow{\simeq} \mathtt{Ker}(d^{j+1})$ *for all* $j \in \{k, \ldots, n-1\}$.

A **short exact sequence** is an exact sequence

$$0 \to X \to Y \to Z \to 0$$

Exact sequences can be viewed as a less restrictive way of performing direct sums, since the sequence

$$0 \to X \xrightarrow{i_X} X \oplus Y \xrightarrow{p_Y} Y \to 0$$

is a short exact sequence. Situations in which the middle term can be regarded as a direct sum are referred as *split* exact sequences. The following proposition, proven in ([Sch11] Proposition 4.1.7), reflects on and makes precise this idea.

Proposition 8.5.2. *Let* $\boldsymbol{A}$ *be an abelian category and consider a short exact sequence*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

The following conditions are equivalent

1. *There exists* $h : X'' \to X$ *such that* $g \circ h = 1_{X''}$
2. *There exists* $k : X \to X'$ *such that* $k \circ f = 1_{X'}$
3. *There exist* $\phi = (k, g) : X \to X' \oplus X''$ *and* $\psi = \begin{pmatrix} f \\ h \end{pmatrix}$ *that are inverses one of the other. In particular*

$$X \simeq X' \oplus X''$$

4. *The complex* $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ *is isomorphic to* $0 \to X' \to X' \oplus X'' \to X'' \to 0$

If one of these conditions is satisfied we said that the exact sequence ***splits****.*

We define now exact functors

Definition. *Given* $F : \boldsymbol{A} \to \boldsymbol{A'}$ *a functor between abelian categories, we say that*

1. F *is* ***left exact*** *if it commutes with finite limits.*
2. F *is* ***right exact*** *if it commutes with finite colimits.*
3. F *is* ***exact*** *if it is left and right exact.*

And we have the following characterization, which corresponds to ([Sch11] Lemma 4.2.2)

Proposition 8.5.3. *The following conditions are equivalent*

- *F is left exact.*
- *F commutes with Kernels.*
- *For any exact sequence $0 \to X \to Y \to Z$ in $\boldsymbol{A}$, the sequence $0 \to F(X) \to F(Y) \to F(Z)$ is exact*

An analogous proposition is true for the right exact functors.

Examples:

1. The functor $\hom_{\mathbf{A}} : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Ab}$ is left exact with respect of both their arguments.
2. If $F : \mathbf{A} \to \mathbf{A}'$ has a left adjoint then it is left exact. If it has a right adjoint it is right exact.
3. For a category $\mathbf{I}$, the limit functor $\lim : \mathtt{Func}(\mathbf{I}^{op}, \mathbf{A}) \to \mathbf{A}$ is right exact.
4. For a category $\mathbf{I}$, the colimit functor $\mathtt{colim} : \mathtt{Func}(\mathbf{I}, \mathbf{A}) \to \mathbf{A}$ is right exact. If $\mathbf{I}$ is a *filtrant category* and $\mathbf{A} = R$-mod for some ring R, then this functor is exact.

We will now define *injective and projective objects*, which are essential for the construction of the derived categories that we attempt to perform. As we just saw, the functor $\hom_{\mathbf{A}} : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Ab}$ is left exact with respect of both their arguments. The objects that make this functor *right* exact as well in each variable are called injective and projective objects.

Definition. *Given $\boldsymbol{A}$ an abelian category, we say that*

1. *An object I is **injective** if the functor $\hom_{\boldsymbol{A}}(-, I)$ is exact.*
2. *An object P is **projective** if the functor $\hom_{\boldsymbol{A}}(P, -)$ is exact.*

A injective object I can be visualized as objects that complete a commutative diagram

$$\begin{array}{ccc} X & \xhookrightarrow{f} & Y \\ \downarrow & \swarrow & \\ I & & \end{array}$$

For all monomorphisms $X \overset{f}{\hookrightarrow} Y$ and morphisms $X \to I$. There is a analogous diagram for projective objects.

One importance of injective objects is that they make sequences split, as the properties in ([Sch11] Lemmas 4.3.3, 4.3.4, Proposition 4.3.5) show

Proposition 8.5.4. *Consider $\boldsymbol{A}$ an abelian category and $X, X', X'' \in \boldsymbol{A}$, then*

- *If $0 \to X' \to X \to X'' \to 0$ is an exact sequence and X' is injective, then the sequence splits.*
- *If $0 \to X' \to X \to X'' \to 0$ is an exact sequence and X', X are injective, then X'' is injective.*
- *X' and X'' is injective if and only if $X' \oplus X''$ is injective.*

The key property that allows to greatly simplify the construction of derived categories is the existence of monomorphisms into injective objects for any object in an abelian category.

Definition. *We say that an abelian category* ***A has enough injectives*** *if for any $X \in \boldsymbol{A}$ there exists a monomorphism $X \hookrightarrow I$ into an injective object.*

Similarly it will have enough projectives for all objects there is a epimorphism from a projective object.

Examples:

1. The category R-mod has enough injectives and projectives.
2. The category $\mathtt{Sh}(X)$ defined in 2.1.2 has enough injectives.
3. It is a famous result shown in [Gro57] that for a small category $\mathbf{C}$, if $\mathbf{A}$ has enough injectives then $\mathtt{Func}(\mathbf{C}, \mathbf{A})$ has enough injectives.

8.3.2 Chain complexes and Homology

The category $Ch(\mathbf{A})$ is called the category of **chain complexes** over $\mathbf{A}$. We can better visualize this category as follows

- Objects are sequences

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \dots$$

  Such that $d^k \circ d^{k-1} = 0$ for all $k \in \mathbb{Z}$

  These sequences are denoted as $(X^\cdot, d^\cdot_X)$, or simply as $(X^\cdot, d^\cdot_X)$ or even as $X^\cdot$ if things are clear for context. The morphisms $d^\cdot$ are called **differencials**.

- Morphisms $f^\cdot : X^\cdot \to Y^\cdot$ are commutative diagrams

$$\begin{array}{ccccccc} \dots & \longrightarrow & X^n & \xrightarrow{d^n_X} & X^{n+1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \dots & \longrightarrow & Y^n & \xrightarrow{d^n_Y} & Y^{n+1} & \longrightarrow & \dots \end{array}$$

Given R a ring, the chain complexes over R-mod recieve a special notation

$$Ch(R\text{-mod}) := Ch(R)$$

can also define full subcategories of $Ch(\mathbf{A})$.

- $Ch_{\geq 0}(\mathbf{A})$ corresponding to the chains $X^\cdot$ such that $X^k = 0$ for all $k < 0$
- $Ch_b(\mathbf{A})$ of **bounded chains**, corresponding to the chains $X^\cdot$ such that there is $N \in \mathbb{N}$ such that $X^k = 0$ for all $|k| \geq N$
- $Ch_+(\mathbf{A})$ of **bounded below** corresponding to the chains $X^\cdot$ such that there is $N \in \mathbb{N}$ such that $X^k = 0$ for all $k \leq N$

- $Ch_{-}(\mathbf{A})$ of **bounded above** corresponding to the chains $X^{\cdot}$ such that there is $N \in \mathbb{N}$ such that $X^k = 0$ for all $k \geq N$

All this categories are abelian categories. We might refer sometimes to all of them by saying "$Ch_{\star}(\mathbf{A})$ (with $\star = ub, b, \geq 0, +, -$)", where ub means "unbounded" and refers to $Ch(\mathbf{A})$.

We commonly identify $\mathbf{A}$ as a subcategory of $Ch_{\star}(\mathbf{A})$ (with $\star = ub, b, \geq 0, +, -$) by concentrating an object of $\mathbf{A}$ on the degree 0 of a complex.

$$\begin{aligned} in_0 \colon \mathbf{A} &\hookrightarrow Ch_{\star}(\mathbf{A}) \\ X &\mapsto (\cdots \to 0 \to X \to 0 \to \cdots) \end{aligned}$$

We now define *homology*. Observe that the condition

$$d^n \circ d^{n-1} = 0$$

For the differentials implies that there are maps $\mathtt{im}(d^{n-1}) \to \mathtt{Ker}(d^n)$

Definition. *Given $\boldsymbol{A}$ an abelian category and $(X^{\cdot}, d^{\cdot}) \in Ch(\boldsymbol{A})$, we define the n-**th cohomology object** of $X^{\cdot}$ as*

$$H^n(X^{\cdot}) = \mathtt{Coker}(\mathtt{im}(d^{n-1}) \to \mathtt{Ker}(d^n)) := \mathtt{Ker}(d^n)/\mathtt{im}(d^{n-1})$$

This as the reader probably knows is an object of great significance in topology and other fields. In a way, cohomology is measuring the lack of exactness of a chain complex at different degrees of the chain. Observe that in fact if a piece of a chain

$$X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1}$$

is exact, then $H^k(X) = 0$, moreover

$$X^{\cdot} \in Ch(\mathbf{A}) \text{ is quasi isomorphic to } 0 \iff \text{X}^{\cdot} \text{ is exact.}$$

All the constructions on this section revolve around (co)homology, as the derived categories are nothing but the localization of $Ch(\mathbf{A})$ over the quasi isomorphisms, which are morphisms that *turn into* isomorphisms in homology. So, yes, cohomology defines a functor.

Proposition 8.5.5. *Cohomology defines a functor for all $n \in \mathbb{Z}$*

$$H^n : Ch(\boldsymbol{A}) \to \boldsymbol{A}$$

The action on a morphism $f : X^{\cdot} \to Y^{\cdot}$ is given by the fact that this morphism induces morphisms in $f : \mathtt{Ker}(d_X^n) \to \mathtt{Ker}(d_Y^n)$ and $f : \mathtt{im}(d_X^{n-1}) \to \mathtt{im}(d_Y^{n-1})$, and hence defines a morphism

$$H^n(f) : H^n(X^{\cdot}) \to H^n(Y^{\cdot})$$

Definition. *We say that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $Ch(\boldsymbol{A})$ is a **quasi isomorphism** if*

$$H^n(f) : H^n(X^{\cdot}) \to H^n(Y^{\cdot})$$

Are isomorphisms for all $n \in \mathbb{Z}$.

We present now a couple of endofunctors in $Ch(\mathbf{A})$ that affect the cohomology of a chain in useful and aesthetic ways.

Definition. *Given $\boldsymbol{A}$ an abelian category and an integer $m \in \mathbb{Z}$, we define*

- *The **shift functor** as the endofunctor*

$$\begin{aligned}(-)[m]\colon Ch(\boldsymbol{A}) &\longrightarrow Ch(\boldsymbol{A})\\ (X^{\cdot}, d_X^{\cdot}) &\longmapsto (X[m]^{\cdot}, d_{X[m]}^{\cdot})\end{aligned}$$

  Where $(X[m])^n = X^{n+p}$ and $d^n_{X[m]} = (-1)^m d_X^{n+p}$. For a morphism $f : X^{\cdot} \to Y^{\cdot}$ we define

$$f[m] : X[m]^{\cdot} \to Y[m]^{\cdot}$$

  by $(f[m])^n = f^{n+m}$

- *The **truncation functors** as the endofunctors*

$$\tau_{\leq m}, \tau_{\geq m} : Ch(\boldsymbol{A}) \to Ch(\boldsymbol{A})$$

  Defined on a chain $X^{\cdot} = \cdots \to X^{m-1} \xrightarrow{d^{m-1}} X^m \xrightarrow{d^m} X^{m+1} \to \dots$ as

  - $\tau_{\leq m}(X^{\cdot}) = \cdots \to X^{m-2} \to X^{m-1} \to \texttt{Ker}(d^m) \to 0 \to \dots$
  - $\tau_{\geq m}(X^{\cdot}) = \cdots \to 0 \to \texttt{Coker}(d^{m-1}) \to X^{m+1} \to X^{m+2} \to \dots$

  And they are defined in morphisms in the obvious way.

For these functors, we have the following easy to check properties.

Proposition 8.5.6. *Given $\boldsymbol{A}$ an abelian category, $n, m \in \mathbb{Z}$ and $X^{\cdot}, Y^{\cdot} \in Ch(\boldsymbol{A})$, we have that*

1. *The shift functor defines a morphism of groups*

$$\begin{aligned}[-]\colon \mathbb{Z} &\longrightarrow Aut(Ch(\boldsymbol{A}))\\ m &\longmapsto (-)[m]\end{aligned}$$

   Where $Aut(Ch(\boldsymbol{A}))$ is the group of invertible endomorphisms of $Ch(\boldsymbol{A})$.

2. *$\tau_{\leq n}\tau_{\leq m}X^{\cdot} = \tau_{\leq \texttt{min}(n,m)}X^{\cdot}$ and $\tau_{\leq n}(X^{\cdot}[m]) = (\tau_{n+m}X^{\cdot})[m]$.*

3. *If $f : X^{\cdot} \to Y^{\cdot}$ is such that*

$$H^k(f) : H^k(X^{\cdot}) \to H^k(Y^{\cdot})$$

   is isomorphism for all $k \leq m$, then $\tau_{\leq m} f : \tau_{\leq m}X^{\cdot} \to \tau_{\leq m}Y^{\cdot}$ is a quasi isomorphism.

4. *The cohomology of the shifted chain is the shift of the cohomology*

$$H^k(X^{\cdot}[m]) = H^{k+m}(X^{\cdot})$$

5. *The cohomology of a truncated chain corresponds to the truncation of the cohomology*

$$H^k(\tau_{\leq m}X^\cdot) = \begin{cases} H^k(X^\cdot) & \text{if } k \leq m \\ 0 & \text{if } k > m \end{cases}$$

$$H^k(\tau_{\geq m}X^\cdot) = \begin{cases} H^k(X^\cdot) & \text{if } k \geq m \\ 0 & \text{if } k < m \end{cases}$$

We now move into the construction of the derived category. For this purpose we first need to define the *homotopy category*. Homotopy between chains is inspired directly from how the homotopy between topological spaces looks at the level of chains.

Definition. *Given $\boldsymbol{A}$ an additive category, we say that*

1. *The morphisms $f, g : X^\cdot \to Y^\cdot$ in $Ch(\boldsymbol{A})$ are **homotopic** if for all m there is $h^m : X^m \to Y^{m-1}$ such that*

$$f^m - g^m = h^{m+1} \circ d_X^m + d_Y^{m-1} \circ h^m$$

   We denote $f \simeq_h g$ when f and g are homotopic.

2. *We say that f is a **homotopy equivalence** if there is $l : Y^\cdot \to X^\cdot$ such that $f \circ l \simeq_h 1_{Y^\cdot}$ and $l \circ f \simeq_h 1_{X^\cdot}$.*

3. *$X^\cdot \in Ch(\boldsymbol{A})$ is said to be **homotopic to** 0 if $1_{X^\cdot}$ is homotopic to the zero morphism.*

We have some easy to check properties of the homotopy relation

Proposition 8.5.7. *Given $X^\cdot, Y^\cdot \in Ch(\boldsymbol{A})$ and $f, g : X^\cdot \to Y^\cdot$*

1. *$\simeq_h$ is an equivalence relation on $\hom_{Ch(\boldsymbol{A})}(X^\cdot, Y^\cdot)$.*

2. *$f \simeq_h g \Leftrightarrow f - g \simeq_h 0$.*

3. *If f or g are homotopic to 0, then $f \circ g \simeq_h 0$.*

4. *If $f \simeq_h g$ then $H^n(f) = H^n(g)$ for all n.*

The second property suggests that we have all the information of homotopic maps by considering maps *homotopic to zero*. As an example, a split short exact sequence

$$0 \to X \to X \oplus Y \to Y \to 0$$

Is homotopic to 0.

We now wish that homotopic to zero could mean to be *isomorphic* to zero, and this wish inspires the construction of the homotopy category.

Definition. *For an additive category $\boldsymbol{A}$, we define its **homotopy category** $K(\boldsymbol{A})$ to be the addivite category whose*

- *Objects are chain complexes $\mathtt{Ob}(K(\boldsymbol{A})) = \mathtt{Ob}(Ch(\boldsymbol{A}))$*
- *Morphisms are homotopy classes*

$$\hom_{K(\boldsymbol{A})}(X^\cdot, Y^\cdot) = \hom_{Ch(\boldsymbol{A})}(X^\cdot, Y^\cdot)/\{f : X^\cdot \to Y^\cdot | f \simeq_h 0\}$$

This means that in $K(\mathbf{A})$ a map homotopic to zero becomes the zero morphism, whereas an homotopy equivalence becomes an isomorphism. We define similarly the categories $K_\star(\mathbf{A})$ (with $\star = ub, b, \geq 0, +, -$).

Observe that proposition 8.5.7 is crucial for this definition, the third statement allows us to take the quotient and the second statement says that we are not loosing information by taking "up to homotopic to zero". The forth statement implies the following.

Proposition 8.5.8. *Given $\boldsymbol{A}$ an abelian category. For all n the functor $H^n : Ch(\boldsymbol{A}) \to \boldsymbol{A}$ extends to a functor*

$$H^n : K(\boldsymbol{A}) \to \boldsymbol{A}$$

Observe also that the shift functor extends to an automorphism

$$-[1] : K(\mathbf{A}) \xrightarrow{\simeq} K(\mathbf{A})$$

We also have that every additive functor takes morphisms homotopic to zero into morphisms homotopic to zero.

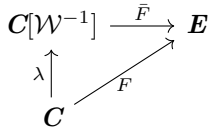
8.3.3 Construction of the derived category

Throughout this subsection, we are going to work with an abelian category $\mathbf{A}$. Call

$$\mathcal{Q} = \{f \in \texttt{Morf}(K(\mathbf{A})) | f \text{ is a quasi isomorphism}\}$$

We recall the definition of the localization from the beginning of the appendix 8.2.1.

Definition. *Given a category $\boldsymbol{C}$ and a class of morphisms $\mathcal{W}$. The **localization** of $\boldsymbol{C}$ with respect to $\mathcal{W}$ is a category $\boldsymbol{C}[\mathcal{W}^{-1}]$ together with a functor $\boldsymbol{C} \to \boldsymbol{C}[\mathcal{W}^{-1}]$ such that for any functor $F : \boldsymbol{C} \to \boldsymbol{E}$ that sends elements of $\mathcal{W}$ to isomorphisms, there is an unique (up to unique isomorphism) $\bar{F} : \boldsymbol{C}[\mathcal{W}^{-1}] \to \boldsymbol{E}$ such that the following commutes*



The localization of a category always exists and it is unique up to equivalence of categories. The derived category is the localization of $K(\mathbf{A})$ with respect to the class of quasi isomorphisms.

Definition. *We define the derived category of $\boldsymbol{A}$ to be the localization of $K(\boldsymbol{A})$ with respect to quasi isomorphisms*

$$D(\boldsymbol{A}) = K(\boldsymbol{A})[\mathcal{Q}^{-1}]$$

We similarly define $D_\star(\mathbf{A})$ (with $\star = ub, b, \geq 0, +, -$) as the localization $K_\star(\mathbf{A})[\mathcal{Q}^{-1}]$. The idea of this category is to have an environment in which quasi isomorphic means isomorphic.

Remark:

1. Observe that the truncation functors send a complex homotopic to zero to a complex homotopic to zero, they are well defined in $K(\mathbf{A})$. Moreover, since they send quasi isomorphisms to quasi isomorphisms, they are well defined on the derived category.

$$\tau_{\leq m} : D(\mathbf{A}) \to D_{-}(\mathbf{A})$$
$$\tau_{\geq m} : D(\mathbf{A}) \to D_{+}(\mathbf{A})$$

2. The cohomology functor is also well defined in the derived categories

$$H^k : D(\mathbf{A}) \to \mathbf{A}$$

We have that when $\mathbf{A}$ is an abelian category with enough injectives, there is a simple description of $D(\mathbf{A})$ based on injectives.

Proposition 8.5.9. *If $\boldsymbol{A}$ is an abelian category with enough injectives, then we have an equivalence of categories*

$$K_{+}(\mathcal{I}_{\boldsymbol{A}}) \xrightarrow{\simeq} D_{+}(\boldsymbol{A})$$

Where $\mathcal{I}_{\boldsymbol{A}}$ is the full subcategory of $\boldsymbol{A}$ consisting of injective objects.

To arrive into $K_{+}(\mathcal{I}_{\mathbf{A}})$ we perform *injective resolutions*, which are the analogous for the (co)fibrant replacements performed in model categories.

Definition. *Given $\mathcal{J}$ an additive full subcategory of $\boldsymbol{A}$.*

- *We say that $\mathcal{J}$ is **cogenerating** if for all $X \in \boldsymbol{A}$ there is a monomorphism $X \hookrightarrow J$ to some object $J \in \mathcal{J}$.*
- *For an object $X \in \boldsymbol{A}$, we say that $J^{\cdot} \in Ch_{+}(\boldsymbol{A})$ is a **resolution** if there is a quasi isomorphism*

  $$X \xrightarrow{qiso} J^{\cdot}$$

  *And we say that it is a $\mathcal{J}$-**resolution** if $J^i \in \mathcal{J}$ for all $i \in \mathbb{N}$. A $\mathcal{I}_{\boldsymbol{A}}$-resolution is called an **injective resolution**.*

Observe that $J^{\cdot} = 0 \to J^0 \to J^1 \to \dots$ is a resolution of X if and only if there is a exact sequence

$$0 \to X \to J^0 \to J^1 \to \dots$$

We have that the cogeneration guarantees the existence of resolutions, as it is stated in ([Sch11] Proposition 4.5.3).

Proposition 8.5.10. *Let $\boldsymbol{A}$ be an abelian category and $\mathcal{J}$ be a cogenerating category, then*

- *For all $X \in \boldsymbol{A}$ there is an exact sequence*

  $$0 \to X \to J^0 \to J^1 \to \dots$$

  With $J^n \in \mathcal{J}$ for all $n \in \mathbb{N}$.

- *For any $X^{\cdot} \in Ch_+(\boldsymbol{A})$ there is a quasi isomorphism*

$$X^{\cdot} \xrightarrow{qiso} J^{\cdot}$$

For some $J^{\cdot} \in Ch_+(\boldsymbol{A})$.

The key property that make injectives so tuned to work for the derived categories are the properties shown in ([Sch11] Proposition 4.5.5 and Proposition 4.5.6).

Proposition 8.5.11. *Consider $\boldsymbol{A}$ an abelian category with enough injectives*

1. *Let $f : X^{\cdot} \to I^{\cdot}$ be a morphism in $Ch_+(\boldsymbol{A})$ with $I^{\cdot} \in Ch_+(\mathcal{I}_{\boldsymbol{A}})$. If $X^{\cdot}$ is exact then $f \simeq_h 0$.*
2. *If $I^{\cdot} \in Ch_+(\mathcal{I}_{\boldsymbol{A}})$ is exact then it is homotopic to 0.*
3. *Given $f : X \to Y$ a morphism in $\boldsymbol{A}$, consider $0 \to X \to X^{\cdot}$ a resolution of X and $0 \to Y \to J^{\cdot}$ a complex with all the J^k's injective. Then there exists $f^{\cdot} : X^{\cdot} \to J^{\cdot}$, unique up to homotopy, making the following diagram commute*

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & X^{\cdot} \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^{\cdot}} \\ 0 & \longrightarrow & Y & \longrightarrow & J^{\cdot} \end{array}$$

Proof. We make the construction of the third statement, leaving the rest of the proof to be followed in the reference.

This construction is made inductively as follows. f^0 is given by the property that defines injective objects

$$\begin{array}{ccc} X & \xhookrightarrow{f} & Y \\ \downarrow & \swarrow & \\ I & & \end{array}$$

Suppose now that we constructed $f^0, \ldots, f^n$, let

$$g^n = d_Y^n \circ f^n : X^n \to J^{n+1}$$

We have that g^n factorizes through $h^n : X^n/\mathtt{im}d_X^{n-1} \to J^{n+1}$. Now, since $X^{\cdot}$ is exact, the sequence $0 \to X^n/\mathtt{im}d_X^{n-1} \to X^{n+1}$ is exact. Since J^{n+1} is injective, h^n extends as

$$f^{n+1} : X^{n+1} \to J^{n+1}$$

□

This proposition implies that given $X \in \mathbf{A}$, an injective resolution

$$X \to I_X^{\cdot}$$

Is unique up to homotopy. Also, given $f : X \to Y$ a morphism in $\mathbf{A}$, by this proposition there is a morphism $f^{\cdot} : I_X^{\cdot} \to I_Y^{\cdot}$ unique up to homotopy.

This means that we can construct a *injective resolution* functor from $\mathbf{A}$ to $K_+(\mathcal{I}_{\mathbf{A}})$

Definition. *Given* $\boldsymbol{A}$ *an abelian category with enough injectives, we call*

$$\mathcal{I} : \boldsymbol{A} \to K_+(\mathcal{I}_{\boldsymbol{A}})$$

The functor taking an object to a resolution $X \mapsto I_X^{\cdot}$*, and a morphism* $f : X \to Y$ *to the morphism constructed in proposition 8.5.11.*

So we have in fact constructed a functor to the derived category

$$\mathcal{I} : \mathbf{A} \to D_+(\mathbf{A})$$

This functor is more important than it appears at first glance, since it is the key to *derive* functors in a pleasant way.

We begin a discussion about derived functors by considering the general setting of localization. Given a category $\mathbf{C}$ and a class of morphisms $\mathcal{W}$, when presented the diagram

$$\begin{array}{ccc} \mathbf{C}[\mathcal{W}^{-1}] & \xrightarrow{\bar{F}} & \mathbf{E} \\ {\scriptstyle\lambda}\uparrow & \nearrow_{F} & \\ \mathbf{C} & & \end{array}$$

defining the localization of $\mathbf{C}$ with respect to $\mathcal{W}$, we said that every functor that *sends* $\mathcal{W}$ *into isomorphisms* can be extended along λ. One might wonder what happens to the rest of the functors, is there a way or *a best way possible* to extend any $F : \mathbf{C} \to \mathbf{E}$ along λ?

Observe that this question is the same one asked for *Kan extensions* (see appendix 8.1).

Definition. *Given* $\lambda : \boldsymbol{C} \to \boldsymbol{C}[\mathcal{W}^{-1}]$ *a localization. A* ***right localization*** *of a functor* $F : \boldsymbol{C} \to \boldsymbol{E}$ *(if it exists) is a left Kan extension of* F *along* λ

$$RL(F) = Lan_{\lambda}(F)$$

If the right localization exists we say that F *is* ***right localizable****.*

Similarly left localization is a right Kan extension of F along λ. As remarked in the definition these right and left localizations of functors need not to exist, however, their existence is guaranteed under not hard conditions, specially when we give the categories involved a great amount of specifications such as *being an abelian category with enough injectives.*

In the context of abelian categories, the right localization is given a special name. Consider a functor

$$F : \mathbf{A} \to \mathbf{A}'$$

Between abelian categories. It naturally defines a functor

$$K_+(F) : K_+(\mathbf{A}) \to K_+(\mathbf{A}')$$

We call $K_+(F)$ simply as F when there is no risk of confusion.

Definition. *We say that a functor* $F : \boldsymbol{A} \to \boldsymbol{A}'$ ***has a right derived functor*** *if the functor* $F : K_+(\boldsymbol{A}) \to K_+(\boldsymbol{A}')$ *is right localizable. In such case one denotes*

$$RF : D_+(\boldsymbol{A}) \xrightarrow{Lan_{\lambda}(F)} K_+(\boldsymbol{A}') \xrightarrow{\lambda'} D_+(\boldsymbol{A}')$$

The ***right derived functor*** *of* F*.*

We present a condition for the existence of a right derived functor.

Theorem 8.6. *Given a functor $F : \boldsymbol{A} \to \boldsymbol{A}'$ between abelian categories. If $\boldsymbol{A}$ has enough injectives and F is left exact, then F has a right derived functor*

$$RF : D_+(\boldsymbol{A}) \to D_+(\boldsymbol{A}')$$

Moreover, we can assume that there is a *F-injective subcategory* (to be define now) and that F is left exact in order to guarantee the existence of RF.

We will see now a way to construct F.

Definition. *Given $F : \boldsymbol{A} \to \boldsymbol{A}'$ a left exact functor between abelian categories and $\mathcal{J}$ a full additive subcategory of $\boldsymbol{A}$. We say that $\mathcal{J}$ is F-**injective** if*

1. *$\mathcal{J}$ is cogenerating.*
2. *For any exact sequence $0 \to X \to Y \to Z \to 0$ in $\boldsymbol{A}$ we have $X, Y \in \mathcal{J} \Rightarrow Z \in \mathcal{J}$.*
3. *For any exact sequence $0 \to X \to Y \to Z \to 0$ with $X \in \mathcal{J}$, we have that $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact.*

This definition will mean that F takes an exact sequence $J^{\cdot}$ of F-injective objects into an exact sequence $F(J)^{\cdot}$ (in particular with null cohomology). We have a couple of important examples.

Examples:

1. The category of injective objects $\mathcal{I}_{\mathbf{A}}$ is F-injective for any $F : \mathbf{A} \to \mathbf{A}'$.
2. We say that $X \in \mathbf{A}$ is F**-acyclic** if $RF(X)$ has null cohomology. The full subcategory of *F-acyclic objects* is F-injective for any $F : \mathbf{A} \to \mathbf{A}'$. We have also that being injective implies being F-acyclic.
3. In sheaf theory, we have that *flasque, soft, c-soft* and *acyclic* sheaves are examples of $\Gamma(Y, -)$-injective sheaves (see 2.1.2 for the definitions).

We have the following way to compute the right derived functor.

Theorem 8.7. *Given $F : \boldsymbol{A} \to \boldsymbol{A}'$ a left exact functor between abelian categories. Suppose that $\boldsymbol{A}$ has enough injectives and we have an F-injective category $\mathcal{J}$ that contains $\mathcal{I}_{\boldsymbol{A}}$.*

Then for $X^{\cdot} \in K_+(\boldsymbol{A})$ if there is a quasi isomorphism $X^{\cdot} \xrightarrow{qiso} J^{\cdot}$ for $J^{\cdot} \in K_+(\mathcal{J})$ then

$$RF(X^{\cdot}) \simeq F(J^{\cdot}) \text{ in } D_+(\boldsymbol{A}')$$

This means that up to homology $RF(X^{\cdot})$ can be computed by finding a resolution $X^{\cdot} \xrightarrow{qiso} J^{\cdot}$ of F-injectives, and then applying F. Finally we will define

Definition. *For $F : \boldsymbol{A} \to \boldsymbol{A}'$ a left exact functor between abelian categories, where $\boldsymbol{A}$ has enough injectives, we define*

$$R^k F : \boldsymbol{A} \xrightarrow{\mathcal{I}} K_+(\mathcal{I}_{\boldsymbol{A}}) \xrightarrow{F} K_+(\boldsymbol{A}') \xrightarrow{H^k} \boldsymbol{A}'$$

Observe that in the context of the definition we have

$$R^k F = H^k RF$$

Which just means that we are computing the homology of the derived functor. Just in the case that the reader is confused at this point (as the author was on first reading on the subject), we present steps to calculate $R^n F(X)$

1. Build a resolution $X \to I_X^0 \to I_X^1 \to \dots$ with $I_X^k \in \mathcal{J}$ for all $k \in \mathbb{N}$, with $\mathcal{J}$ a F-injective resolution that contains $\mathcal{I}_{\mathbf{A}}$.
2. Apply F to $I_X^{\cdot}$.
3. Take the n-th cohomology.

We present a variety of properties of the derived functor

Proposition 8.7.1. *Given $F : \boldsymbol{A} \to \boldsymbol{A}'$ a left exact functor between abelian categories, where $\boldsymbol{A}$ has enough injectives, let $X \in \boldsymbol{A}$, then we have the following*

- *$R^n F : \boldsymbol{A} \to \boldsymbol{A}'$ is an additive functor for all n.*
- *$R^n F(X) \simeq 0$ for $n < 0$.*
- *$R^0 F(X) \simeq F(X)$.*
- *If F is exact then $R^n F(X) \simeq 0$ for $n \neq 0$.*
- *If X is injective then $R^n F(X) \simeq 0$ for $n \neq 0$.*
- *Moreover, if $X \in \mathcal{J}$, with $\mathcal{J}$ a F-injective resolution that contains $\mathcal{I}_{\boldsymbol{A}}$, then $R^n F(X) \simeq 0$ for $n \neq 0$*

But the most important property of derived functor is the transformation of short exact sequences into long exact sequences

Theorem 8.8. *Given $F : \boldsymbol{A} \to \boldsymbol{A}'$ a left exact functor between abelian categories, where $\boldsymbol{A}$ has enough injectives. Consider $0 \to X \to Y \to Z \to 0$ a short exact sequence. Then there is a long exact sequence*

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1 F(X) \to \cdots \to R^k F(X) \to R^k F(Y) \to R^k F(Z) \to \dots$$

Bibliography

[Alu09] Paolo Aluffi. *Algebra: chapter 0*. Vol. 104. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009, pp. xx+713. ISBN: 978-0-8218-4781-7. DOI: 10.1090/gsm/104. URL: https://doi.org/10.1090/gsm/104.

[Ba08] A. Borel and et al. *Intersection cohomology*. Modern Birkhäuser Classics. Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. x+234. ISBN: 978-0-8176-4764-3.

[Ban07] M. Banagl. *Topological invariants of stratified spaces*. Springer Monographs in Mathematics. Springer, Berlin, 2007, pp. xii+259. ISBN: 978-3-540-38585-1; 3-540-38585-1.

[Bre02] Francesco Brenti. "Kazhdan-Lusztig polynomials: history problems, and combinatorial invariance". In: *Sém. Lothar. Combin.* 49 (2002/04), Art. B49b, 30.

[Bre97] Glen E. Bredon. *Sheaf theory*. Second. Vol. 170. Graduate Texts in Mathematics. Springer-Verlag, New York, 1997, pp. xii+502. ISBN: 0-387-94905-4. DOI: 10.1007/978-1-4612-0647-7. URL: https://doi.org/10.1007/978-1-4612-0647-7.

[CSAT18a] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré. "Intersection cohomology, simplicial blow-up and rational homotopy". In: *Mem. Amer. Math. Soc.* 254.1214 (2018), pp. viii+108. ISSN: 0065-9266. DOI: 10.1090/memo/1214. URL: https://doi.org/10.1090/memo/1214.

[CSAT18b] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré. "Blown-up intersection cohomology". In: *An alpine bouquet of algebraic topology*. Vol. 708. Contemp. Math. Amer. Math. Soc., [Providence], RI, [2018] ©2018, pp. 45–102. DOI: 10.1090/conm/708/14263. URL: https://doi.org/10.1090/conm/708/14263.

[CSAT20] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré. "Blown-up intersection cochains and Deligne's sheaves". In: *Geom. Dedicata* 204 (2020), pp. 315–337. ISSN: 0046-5755. DOI: 10.1007/s10711-019-00458-w. URL: https://doi.org/10.1007/s10711-019-00458-w.

[CSAT21] David Chataur, Martin Saralegi-Aranguren, and Daniel Tanré. *Simlpicial intersection homology revisited*. 2021. arXiv: 2109.00261 [math.AT].

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*. Vol. 205. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001, pp. xxxiv+535. ISBN: 0-387-95068-0. DOI: 10.1007/978-1-4613-0105-9. URL: https://doi.org/10.1007/978-1-4613-0105-9.

[FP90] Rudolf Fritsch and Renzo A. Piccinini. *Cellular structures in topology*. Vol. 19. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990, pp. xii+326. ISBN: 0-521-32784-9. DOI: 10.1017/CBO9780511983948. URL: https://doi.org/10.1017/CBO9780511983948.

[Fri10] Greg Friedman. "Intersection homology with general perversities". In: *Geom. Dedicata* 148 (2010), pp. 103–135. ISSN: 0046-5755. DOI: 10.1007/s10711-010-9482-y. URL: https://doi.org/10.1007/s10711-010-9482-y.

[Fri12] Greg Friedman. "Survey article: An elementary illustrated introduction to simplicial sets". In: *Rocky Mountain J. Math.* 42.2 (2012), pp. 353–423. ISSN: 0035-7596. DOI: 10.1216/RMJ-2012-42-2-353. URL: https://doi.org/10.1216/RMJ-2012-42-2-353.

[Fri20] Greg Friedman. *Singular Intersection Homology*. New Mathematical Monographs. Cambridge University Press, 2020. DOI: 10.1017/9781316584446.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Vol. 174. Progress in Mathematics. Birkhäuser Verlag, Basel, 1999, pp. xvi+510. ISBN: 3-7643-6064-X. DOI: 10.1007/978-3-0348-8707-6. URL: https://doi.org/10.1007/978-3-0348-8707-6.

[GM80] Mark Goresky and Robert MacPherson. "Intersection homology theory". In: *Topology* 19.2 (1980), pp. 135–162. ISSN: 0040-9383. DOI: 10.1016/0040-9383(80)90003-8. URL: https://doi.org/10.1016/0040-9383(80)90003-8.

[GM83] Mark Goresky and Robert MacPherson. "Intersection homology. II". In: *Invent. Math.* 72.1 (1983), pp. 77–129. ISSN: 0020-9910. DOI: 10.1007/BF01389130. URL: https://doi.org/10.1007/BF01389130.

[Gro57] Alexander Grothendieck. "Sur quelques points d'algèbre homologique". In: *Tohoku Math. J. (2)* 9 (1957), pp. 119–221. ISSN: 0040-8735. DOI: 10.2748/tmj/1178244839. URL: https://doi.org/10.2748/tmj/1178244839.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457. ISBN: 0-8218-3279-4. DOI: 10.1090/surv/099. URL: https://doi.org/10.1090/surv/099.

[Hov99] Mark Hovey. *Model categories*. Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, pp. xii+209. ISBN: 0-8218-1359-5.

[Ive86] Birger Iversen. *Cohomology of sheaves*. Universitext. Springer-Verlag, Berlin, 1986, pp. xii+464. ISBN: 3-540-16389-1. DOI: 10.1007/978-3-642-82783-9. URL: https://doi.org/10.1007/978-3-642-82783-9.

[Joh02] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*. Vol. 44. Oxford Logic Guides. The Clarendon Press, Oxford University Press, Oxford, 2002, i–xxii, 469–1089 and I1–I71. ISBN: 0-19-851598-7.

[Kin85] Henry C. King. "Topological invariance of intersection homology without sheaves". In: *Topology Appl.* 20.2 (1985), pp. 149–160. ISSN: 0166-8641. DOI: 10.1016/0166-8641(85)90075-6. URL: https://doi.org/10.1016/0166-8641(85)90075-6.

[KL80] David Kazhdan and George Lusztig. "Schubert varieties and Poincaré duality". In: *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*. Proc. Sympos. Pure Math., XXXVI. Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203.

[Kle07] Steven L. Kleiman. "The development of intersection homology theory". In: *Pure Appl. Math. Q.* 3.1, Special Issue: In honor of Robert D. MacPherson. Part 3 (2007), pp. 225–282. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2007.v3.n1.a8. URL: https://doi.org/10.4310/PAMQ.2007.v3.n1.a8.

[KW06] Frances Kirwan and Jonathan Woolf. *An introduction to intersection homology theory*. Second. Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. xiv+229. ISBN: 978-1-58488-184-1; 1-58488-184-4. DOI: 10.1201/b15885. URL: https://doi.org/10.1201/b15885.

[Leh14] Marina Christina Lehner. *Concepts are Kan Extensions": Kan Extensions as the Most Universal of the Universal Constructions*. https://www.math.harvard.edu/media/lehner.pdf. 2014.

[Man14] Ciprian Manolescu. "Triangulations of manifolds". In: *ICCM Not.* 2.2 (2014), pp. 21–23. ISSN: 2326-4810. DOI: 10.4310/ICCM.2014.v2.n2.a2. URL: https://doi.org/10.4310/ICCM.2014.v2.n2.a2.

[McF12] P. McFaddin. *Cohomology on a Grothendieck Site*. https://mcfaddin.github.io/documents/LAAST_talk_sheaf_cohom.pdf. 2012.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314. ISBN: 0-387-98403-8.

[Poi00] H. Poincaré. "Second Complement a l'Analysis Situs". In: *Proc. Lond. Math. Soc.* 32 (1900), pp. 277–308. ISSN: 0024-6115. DOI: 10.1112/plms/s1-32.1.277. URL: https://doi.org/10.1112/plms/s1-32.1.277.

[Pol05] Guido Pollini. "Intersection differential forms". In: *Rend. Sem. Mat. Univ. Padova* 113 (2005), pp. 71–97. ISSN: 0041-8994.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967, iv+156 pp. (not consecutively paged).

[RS72] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. Springer-Verlag, New York-Heidelberg, 1972, pp. viii+123.

[Sch11] Pierre Schapira. *Algebra and Topology*. `http://www.math.jussieu.fr/~schapira/lectnotes`. 2011.

[Sul70] D. Sullivan. "A general problem". In: *Springer Lecture Notes in Math.* 197 (1970), p. 230.