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Direct proof of Fermat's Last Theorem based on Induction on z not on n

The French mathematician Pierre de Fermat (1601-1665), conjectured that the equation $x^n + y^n = z^n$ has no solution in positive integers x , y and z if n is a positive integer ≥ 3 . He wrote in the margin of his personal copy of Brachet's translation of Diophantus' *Arithmetica*: "I have discovered a truly marvellous demonstration of this proposition that this margin is too narrow to contain".

Many researchers believe that Fermat does not find a demonstration of his proposition but some others think there is a proof and Fermat's claim seems right.

The search of a solution of equation $x^n + y^n = z^n$ are splitted in two directions.

The first one is oriented to search a solution for a specific value of the exponent n and the second is more general, oriented to find a solution for any value of the exponent n .

- Babylonian (570,495 BC) studied the equation $x^2 + y^2 = z^2$ and found the solution (3,4,5).
- Arabic mathematician Al-Khazin studied the equation $x^3 + y^3 = z^3$ in the X century and his work mentioned in a philosophic book by Avicenne in the XI century.
- A defective proof of FLT was given before 972 by the Arab Alkhodjandi
- The Arab Mohamed Beha Eddin ben Alhossain (1547-1622) listed among the problems remaining unsolved from former times that to divide a cube into two cubes.(refer Image of Arabic manuscript from British Museum. Problem N4 Red color at line 8 from top).
- Fermat (1601, 1665), Euler (1707, 1783) and Dirichlet (around 1825) solved the equation for $n=3, 4$ and 5 .
- In 1753, Leonhard Euler presented a proof for $x^3 + y^3 = z^3$
- Fermat found a proof of $x^4 + y^4 = z^4$ using his famous "infinite descent". This method combines proof by contradiction and proof by backward induction.
- Dirichlet (in 1825) solved the equation $x^5 + y^5 = z^5$.
- Sophie Germain (in 1823) generalized the result of Dirichlet for prime p if $2p+1$ is prime.. Let p prime, $x^p + y^p = z^p$ has no solution in positive integers if $2p+1$ is prime.
- In XIX century E.Kummer continued the work of Gauss and innovated by using numbers of cyclotomic field and introduced the concept of "prime factor ideal".
- Andrew Wiles, a professor at Princeton University, provided an indirect proof of Fermat's Conjecture in two articles published in the May 1995 issue of *Annals of Mathematics*. Andrew Wiles solved a high level problem in modular forms about elliptic curves and **the consequence is a solution for FLT. Thanks to the results of Andrew Wiles, we know that Fermat's Last Theorem is true.**

I think he opens a space for mathematicians to search proofs for FLT comprehensible by a normal student in mathematics and may be to find new concepts or ideas. **This result should imply a direct proof of FLT.**

In this paper, I would like to suggest a direct proof using mathematical concepts (Forward Induction and Backward Induction) and tools of the Fermat's era; valid for whatever value $n > 2$. This direct proof is comprehensible for a normal student and mathematical lovers.

I- Proof by Forward Induction :

The induction proof is on z not on n.

The induction proof starts from $z = 2$ and $z = 3, z = 4 \dots$ until $z = p$. with $n > 2$

Observe that for $z=2$, the equation $x^n + y^n = 2^n$ has no solutions in integers x and y if $n > 2$.

There is no equality between the sum (x^n+y^n) and 2^n . if $n > 2$.

The case $x=y$ gives $2x^n=z^n$ which leads to $z=\sqrt[n]{2} * x$ which is not an integer.

Hence x is different from y .

A. Basis step / anchoring

Let's assume $z=3$ with $1 \leq x < y < z$ and $x^n+y^n = 3^n$

The cells of the table below show the different values of the sum (x^n+y^n) when x and y varies from 1 to 3 .

The cells of the last line show the value of $z^n=(\text{Max}(x) + 1)^n=(2+1)^n=3^n$.

x^n	1	2^n	3^n
y^n			
1	1+1	$2^n + 1$	$1 + 3^n$
2^n		$2^n + 2^n$	$2^n + 3^n$
3^n			$3^n + 3^n$
z^n	3^n	3^n	

It is obvious that x^n+y^n is never equal to $z^n=(\text{Max}(x)+1)^n=(2+1)^n=3^n$.

All the sums are:

$$1+1=2 < 3^n; \quad 1+2^n < 3^n; \quad \text{and} \quad 2^n + 2^n < 3^n$$

The basis case with $z = 3$ does not need any proof, because you can calculate that regardless of what you choose for x, y , with $n > 2$. **The sum $(x^n + y^n)$ is either less than z^n or greater than z^n but never equal to z^n .**

Let $f(t) = 3^n - (2^n + t^n)$ with $1 \leq t \leq 3$. $f(t)$ is continuous on the interval $[1,3]$

$$f(1) = 3^n - (2^n + 1^n) > 0$$

$$f(2) = 3^n - (2^n + 2^n) > 0$$

$$f(3) = 3^n - (2^n + 3^n) < 0$$

So, as the value of t moves between 2 and 3, the value $2^n + t^n$ goes from being smaller than 3^n to being greater than 3^n . There is a change of sign of $f(t)$ in the interval $[2,3]$. Thus, there must be some value of t in the interval $[2, 3]$ for which $2^n + t^n = 3^n$. The Intermediate Value Theorem (IVT) which states that « if a function is continuous on $[a, b]$, and if L is any number between $f(a)$ and $f(b)$, then there must be a value, $x = c$, where $a < c < b$, such that $f(c) = L$ », confirms our conclusion and we get

$t^n = 3^n - 2^n$ if $n=3$ we get $3 \log t = \log (27 - 8) = \log 19$ so $t=2,668$

$t = 2,668$ in the interval $[2,3]$: $2 < 2,668 < 3$

$n=3$ $2^3 + (2,668)^3 = 3^3$ or $8 + 19 = 27$

We can repeat the same process for any value of n and compute the right value of t (or y) in the interval $[2, 3]$ in order to get $x^n + y^n = z^n$ but y is a real value not integer. x and z remain integer.

$n=2$ $2^2 + (2,236)^2 = 3^2$ or $4 + 5 = 9$

$n=4$ $2^4 + (2,839)^4 = 3^4$ or $16 + 65 = 81$

and so on

This method will be used later in Inductive step to show that **the sum ($x^n + y^n$) is either less than z^n or greater than z^n but never equal to z^n .**

B. inductive step:

Now assume the Fermat's conjecture is true until $z=p$.

We know that every equation $x^n + y^n = z^n$ from $z=2$ and $z=3$ until $z=p$ has no solutions in integers x and y if $n > 2$.

We want to prove that FLT is true for $z=p+1$.

The cells of the table below show the different values of the sum ($x^n + y^n$) when x and y varies from 1 to p .

The last line shows the value of z^n .

x^n	1	2^n	3^n	4^n			$(p-1)^n$	p^n
y^n								
1	1+1	2^n+1	3^n+1	4^n+1				p^n+1
2^n		2^n+2^n	3^n+2^n	4^n+2^n				p^n+2^n
3^n			3^n+3^n	4^n+3^n				p^n+3^n
4^n				4^n+4^n				p^n+4^n
					...			
						$(p-2)^n+(p-2)^n$		$p^n+(p-2)^n$
$(p-1)^n$...	$p^n+(p-1)^n$
p^n								p^n+p^n
z^n	p^n	p^n	p^n	p^n				p^n

In the above table (**first row**=nth-powers of integer x , **first column**=nth-powers of integer y).

Each cell contains the sum ($x^n + y^n$).

The last row contains the value of $z^n = p^n$.

Assume Fermat's conjecture is true until $z=p$.

All the sum (x^n+y^n) are not equal to p^n . For all integers values of x and y if $n > 2$ the sum $(x^n + y^n)$ is either less than p^n or greater than p^n :

If $x^n+y^n < p^n$ then $(x^n + y^n) < p^n < (p+1)^n$.

It is OK for FLT with $z=p+1$

If $x^n+y^n > p^n$ then there are two cases:

- $(x^n + y^n) < (p+1)^n$ or $(x^n+y^n) > (p+1)^n$.
- It is OK for FLT with $z=p+1$.

We have to prove that $(x^n + y^n)$ is never equal to $(p+1)^n$, because we already know that all the sum (x^n+y^n) are not equal to p^n .

In the last column $\text{Max}(x)=p$ all the sum $(p^n + y^n)$ with $1 < y < p$ are either less than $(p+1)^n$ or greater than $(p+1)^n$. We have to show that there is no equality between the sum (x^n+y^n) and $(p+1)^n$.

- $p^n + 1^n < (p+1)^n$
- $p^n + 2^n < (p+1)^n$
- $p^n + 3^n < (p+1)^n$
-
- $p^n + t^n < (p+1)^n$
- $p^n + (t+1)^n > (p+1)^n$
-
- $p^n + p^n > (p+1)^n$ with the condition $\log(1+1/p) < (\log 2) / n$
-
- With n fixed we can compute p in order to get $2 \cdot p^n > (p+1)^n$

n\l	1	2	3	4	5	6
p	p=1	p=3	p=4	p=5	p=7	p=9
	$\log(1+1/2) < 2$	$\log(1+1/3) < 2/2$	$\log(1+1/4) < 2/3$	$\log(1+1/6) < 2/6$	$\log(1+1/7) < 2/7$	$\log(1+1/9) < 2/9$
	$2 \cdot 1^1 \geq (1+1)^1$	$2 \cdot 3^2 > (3+1)^2$	$2 \cdot 4^3 > (4+1)^3$	$2 \cdot 6^4 > (6+1)^4$	$2 \cdot 7^5 < (7+1)^5$	$2 \cdot 9^6 > (9+1)^6$
	$2 \geq 2$	$18 > 16$	$128 > 125$	$2592 > 2401$	$33614 > 32768$	$1062882 > 10^6$

So, as the variable t moves between t and $t+1$, the value $p^n + t^n$ goes from being lesser than $(p+1)^n$ to being greater to $(p+1)^n$. There is a change of sign. Thus, there must be some value of t in the interval $[t, t+1]$ for which $p^n + t^n = (p+1)^n$. This is the common sense but there is actually a mathematical theorem, known as the Intermediate Value Theorem, which confirms our conclusion.

Summary. The Intermediate Value Theorem (IVT) is a precise mathematical statement (theorem) concerning the properties of continuous functions. The IVT states that if a function

is continuous on $[a, b]$, and if L is any number between $f(a)$ and $f(b)$, then there must be a value, $t = c$, where $a < c < b$, such that $f(c) = L$.

The expression $p^n + t^n$ with $1 < t < p$ starts less than $(p+1)^n$. It increases when t increases and becomes close to $(p+1)^n$. When t reaches some integer value $(p^n + t^n)$ becomes greater than $(p+1)^n$. Change of value occurs between t and $t+1$. The expression $(p^n + t^n)$ may be equal to $(p+1)^n$ with t irrational but never with t integer. There is no integer value between t and $t+1$.

Thus, there must be some value of t in the interval $[t, t+1]$ for which

$$p^n + (T)^n = (p+1)^n$$

and again, since $T > 2$, this must be an irrational value.

Thus, since there are infinitely many increasing triples (x, y, z) , there are infinitely many irrational values of T for which there is a solution to $x^n + y^n = z^n$.

Let the function $G(t) = (p+1)^n - p^n - t^n$ with $1 \leq t \leq p$, p is fixed

The derivative is $G'(t) = -n t^{n-1}$. It is always negative then $G(t)$ is decreasing from a positive value $G(1) = (p+1)^n - p^n - 1$ to a negative value $G(p) = (p+1)^n - (p)^n - p^n$ because $2 * p^n > (p+1)^n$ as showed above.

Change of sign of $G(t)$ occurs between t and $t+1$. The expression $(p^n + t^n)$ may be equal to $(p+1)^n$ with t irrational but never with t integer. There is no integer value between t and $t+1$.

In particular, note that if $x=y$ then $x^n + y^n = z^n$ becomes $2x^n = z^n$ or $2 = (z/x)^n$.

If we take now log to the base (z/x) (written $\log(z/x)$ of $()$); we get

$\log(z/x)$ of $(2) = \log(z/x)$ of $((z/x)^n) = n \log(z/x)$ of $(z/x) = n$. which is not integer.

Example, we see that

$t^n + t^n = 4^n$ when $\log t = \log 4 - 1/n \log(2)$ which is real

if $n=3$ then $\log t = \log 4 - 1/3 \log 2$

$3^n + 3^n = 4^n$ when $n = \log 4 / \log(2) = 2,40942$ then Fermat is not true with exponent $n =$ real

All the sums $(x^n + y^n)$ are less than $(p+1)^n$ or greater than $(p+1)^n$ and $(x^n + y^n)$ is not equal to $(p+1)^n$ for all x, y and $n > 2$.

FLT is true until $z=p$ implies FLT is true for $z=p+1$

C. Conclusion

With the principle of strong mathematical induction, we can then conclude that the equation

$x^n + y^n = p^n$ has no solutions in positive integers x , y and p if $n > 2$.

This proof based on Induction on z not on n , is short, direct, comprehensible by any student in Mathematics and lovers of Mathematics.

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