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INSTABILITY OF ALGEBRAIC STANDING WAVES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH TRIPLE POWER NONLINEARITIES

PHAN VAN TIN

ABSTRACT. We consider the following triple power nonlinear Schrödinger equation:

$$iu_t + \Delta u + a_1|u|u + a_2|u|^2u + a_3|u|^3u = 0.$$

We are interested in *algebraic standing waves* i.e standing waves with algebraic decay above equation in dimensions n ($n = 1, 2, 3$). We prove the instability of these solutions in the cases DDF (we use abbreviation D: defocusing ($a_i < 0$), F: focusing ($a_i > 0$)) and DFF when $n = 2, 3$ and in the case DFF with $a_2 < \frac{32}{15\sqrt{6}}$ when $n = 1$. In the case DFD, we only obtain an existence result of algebraic solution in dimension one.

CONTENTS

1. Introduction	1
2. Existence of solution of the elliptic equation	3
2.1. In dimension one	3
2.2. In higher dimensions	5
3. Variational characterization	6
4. Instability of algebraic standing waves	9
Acknowledgement	13
References	13

1. INTRODUCTION

In this paper, we are interested in the following triple power nonlinear Schrödinger equation:

$$iu_t + \Delta u + a_1|u|u + a_2|u|^2u + a_3|u|^3u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad (1.1)$$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $n \in \{1, 2, 3\}$.

The standing waves of (1.1) are solutions of the form $u_\omega(t, x) = e^{i\omega t}\phi_\omega(x)$, where ϕ_ω solves:

$$-\omega\phi_\omega + \Delta\phi_\omega + a_1|\phi_\omega|\phi_\omega + a_2|\phi_\omega|^2\phi_\omega + a_3|\phi_\omega|^3\phi_\omega = 0. \quad (1.2)$$

In [13], the authors study existence and stability of standing waves of (1.1) in one dimension. Existence of standing waves is obtained by ODE arguments. By studying the properties of the nonlinearity, the authors give domains of parameters for existence and nonexistence of standing waves. Stability results are obtained by studying the sign of an integral found by Iliev and Kirchev [9], based on the criteria of stability of Grillakis, Shatah and Strauss [7, 8, 16].

In the special case $\omega = 0$, the profile ϕ_0 , which for convenience we denote by ϕ , satisfies:

$$\Delta\phi + a_1|\phi|\phi + a_2|\phi|^2\phi + a_3|\phi|^3\phi = 0. \quad (1.3)$$

The equation (1.3) can be rewritten as $S'(\phi) = 0$ where S is defined by

$$S(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{a_1}{3}\|v\|_{L^3}^3 - \frac{a_2}{4}\|v\|_{L^4}^4 - \frac{a_3}{5}\|v\|_{L^5}^5. \quad (1.4)$$

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Define

$$X := \dot{H}^1(\mathbb{R}^n) \cap L^3(\mathbb{R}^n), \quad \text{and } \|u\|_X := \|\nabla u\|_{L^2} + \|u\|_{L^3}, \quad (1.5)$$

$$d := \inf\{S(v) : v \in X \setminus \{0\}, S'(v) = 0\}. \quad (1.6)$$

The algebraic standing waves are standing waves with algebraic decay. In this paper, we are only interested in a special kind of algebraic standing waves which are minimizers of the problem (1.6). Throughout this paper, for convenience, we define an algebraic standing wave as a solution of (1.3) solving problem (1.6). Thus, the function ϕ is an algebraic standing wave of (1.1) if $\phi \in \mathcal{G}$, where \mathcal{G} is defined by

$$\mathcal{G} := \{v \in X \setminus \{0\}, S'(v) = 0, S(v) = d\}. \quad (1.7)$$

The instability of algebraic standing waves was studied in [6] for double power nonlinearities. Using similar arguments as in [6], we study existence and instability of algebraic standing waves for the nonlinear Schrödinger equation with triple power nonlinearities (1.1).

First, we study the existence of algebraic standing waves of (1.1). As in [13], we will use the abbreviation D: defocusing when $a_i < 0$ and F: focusing when $a_i > 0$. In Section 2, we prove the following result.

Proposition 1.1. *Let $n = 1$. The equation (1.3) has an unique even positive solution ϕ in the space $H^1(\mathbb{R})$ in the following cases: DFF, DDF, DFD and $a_2 > \frac{8}{\sqrt{15}}$. Moreover, all solutions of (1.3) are of the form $e^{i\theta}\phi(x - x_0)$ for some $\theta, x_0 \in \mathbb{R}$. They are all algebraic standing waves of (1.1).*

In high dimensions, the situation is more complex than in the one dimension. The solutions of (1.3) are very diverse. It is not easy to describe all such solutions as in the dimension one. Thus, classifying the algebraic standing waves of (1.1) is not easy problem. It turns out that a radial positive solutions of (1.3) is also an algebraic standing wave of (1.1). To study the positive radial solutions of (1.3), we prove the following result in Section 2.

Proposition 1.2. *Let $n = 2, 3$ and DDF or DFF. Then there exists a unique radial positive solution of (1.3).*

Before stating the next results, we need some definitions. Firstly, we define the *Nehari functional* as follows:

$$K(v) := \langle S'(v), v \rangle = \|\nabla v\|_{L^2}^2 - a_1 \|v\|_{L^3}^3 - a_2 \|v\|_{L^4}^4 - a_3 \|v\|_{L^5}^5. \quad (1.8)$$

The *rescaled function* is defined by:

$$v^\lambda(x) := \lambda^{\frac{n}{2}} v(\lambda x). \quad (1.9)$$

The following is *Pohozaev functional*:

$$P(v) := \partial_\lambda S(v^\lambda)|_{\lambda=1} = \|\nabla v\|_{L^2}^2 - \frac{na_1}{6} \|v\|_{L^3}^3 - \frac{na_2}{4} \|v\|_{L^4}^4 - \frac{3na_3}{10} \|v\|_{L^5}^5. \quad (1.10)$$

The *Nehari manifold* is defined by:

$$\mathcal{K} := \{v \in X \setminus \{0\}, K(v) = 0\}.$$

Moreover, we consider the following minimization problem:

$$\mu := \inf\{S(v) : v \in \mathcal{K}\}. \quad (1.11)$$

The following is the set of minimizers of problem (1.11):

$$\mathcal{M} := \{v \in \mathcal{K}, S(v) = \mu\}. \quad (1.12)$$

Finally, we define a specific set which uses in our proof:

$$\mathcal{B} := \{v \in H^1(\mathbb{R}^n) : S(v) < \mu, P(v) < 0\}. \quad (1.13)$$

It turns out that the solution of (1.3) given by Proposition 1.2 satisfies a variational characterization and each algebraic standing wave of (1.1) is up to phase shift and translation of this special solution. More precise, in Section 3, we prove the following result.

Proposition 1.3. *Let $n = 2, 3$ and DDF or DFF. Then the radial positive solution ϕ of (1.3) given by Proposition 1.1 and Proposition 1.2 satisfies*

$$S(\phi) = \mu.$$

where S and μ are defined as in (1.4), (1.11) respectively. Moreover, all algebraic standing waves of equation (1.1) are of the form

$$e^{i\theta_0} \phi(\cdot - x_0),$$

for some $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$.

Remark 1.4. (1) In case DFD, we only obtain the result on existence of algebraic standing waves when $n = 1$ (see Proposition 1.1). The variational characterization of algebraic standing waves and stability or instability of these solutions are open problems, even in dimension one.

(2) By using similar arguments as in [6, Proof of Proposition 3.5], we prove that the algebraic standing waves in higher dimensions ($n = 2, 3$) are also in $H^1(\mathbb{R}^n)$.

Our main result is the following.

Theorem 1.5. *Let $n = 1, 2, 3$. Assume that the parameters of (1.1) satisfy DDF or DFF when $n = 2, 3$ or DFF and $a_2 < \frac{32}{15\sqrt{6}}$ when $n = 1$. Then the algebraic standing wave ϕ given as in Proposition 1.1 and Proposition 1.3 is unstable in $H^1(\mathbb{R}^n)$.*

By scaling invariance of (1.1), we may assume $|a_1| = |a_3| = 1$ without loss of generality. This assumption will be made throughout the rest of this paper.

The rest of this paper is organized as follows. In Section 2, we find the region of parameters a_1, a_2, a_3 in which there exist solutions of the elliptic equation (1.3). Specially, in one dimension, all solution of (1.3) are algebraic standing waves. In Section 3, we establish the variational characterization of solutions given in Section 2. The existence of algebraic standing waves in high dimensions is also proved in section 3. In Section 4, we prove instability of algebraic standing waves.

2. EXISTENCE OF SOLUTION OF THE ELLIPTIC EQUATION

First, we find the region of parameters a_1, a_2, a_3 in which there exist solutions of (1.3).

2.1. In dimension one. Let $n = 1$. To study the existence of algebraic standing waves, we use the following lemma (see [2], [13, Proposition 2.1])

Lemma 2.1. *Let g be a locally Lipschitz continuous function with $g(0) = 0$ and let $G(t) = \int_0^t g(s) ds$. A necessary and sufficient condition for the existence of a solution ϕ of the problem*

$$\begin{cases} \phi \in C^2(\mathbb{R}), & \lim_{x \rightarrow \pm\infty} \phi(x) = 0, & \phi(0) > 0, \\ \phi_{xx} + g(\phi) = 0, \end{cases} \quad (2.1)$$

is that $c = \inf \{t > 0 : G(t) = 0\}$ exists, $c > 0$, $g(c) > 0$.

Using Lemma 2.1, we have the following result.

Lemma 2.2. *Let $g(u) = a_1 u^2 + a_2 u^3 + a_3 u^4$ be such that g satisfies the assumptions of Lemma 2.1 for some $a_1, a_2, a_3 \in \mathbb{R}$. Then there exists a positive solution ϕ of (2.1). Moreover, all complex valued solutions of (2.1) are of form:*

$$e^{i\theta_0} \phi(x - x_0),$$

for some $\theta_0, x_0 \in \mathbb{R}$.

Proof. By Lemma 2.1, there exists a real valued solution ϕ of (2.1). We have

$$\phi_{xx} + a_1 \phi^2 + a_2 \phi^3 + a_3 \phi^4 = 0. \quad (2.2)$$

Since $\lim_{x \rightarrow \pm\infty} \phi(x) = 0$, there exists x_0 such that $\phi_x(x_0) = 0$. Multiplying two sides of (2.2) by ϕ_x and noting that $\lim_{x \rightarrow \infty} \phi(x) = 0$ we obtain

$$\frac{1}{2} \phi_x^2 + \frac{a_1}{3} \phi^3 + \frac{a_2}{4} \phi^4 + \frac{a_3}{5} \phi^5 = 0. \quad (2.3)$$

We see that ϕ is not vanishing on \mathbb{R} . Indeed, if $\phi(x_1) = 0$ for some $x_1 \in \mathbb{R}$ then $\phi_x(x_1) = 0$ by (2.3). Thus, $\phi \equiv 0$ by uniqueness of solutions of (2.3) which is a contradiction. Then, we can assume that $\phi > 0$.

The value $\phi(x_0)$ is a positive solution of $G(u) = \frac{a_1}{3}u^3 + \frac{a_2}{4}u^4 + \frac{a_3}{5}u^5 = 0$. Since g satisfies the condition in Lemma 2.1, it follows that $G(u) = 0$ has a first positive solution c such that $g(c) > 0$. If $\phi(x_0) \neq c$ then G has another positive zero $d > c$ such that $d = \phi(x_0)$. By continuity of ϕ , there exists $x_1 > x_0$ such that $\phi(x_1) = c$ and by (2.3) $\phi_x(x_1) = 0$. This conclusion implies that every positive solution of (2.2) has a critical point such that the value of solution at this point equals to c .

Let u be a complex valued solution of (2.1). We prove that $u = e^{i\theta_0}\phi(x-x_0)$, for some $\theta_0, x_0 \in \mathbb{R}$. We use similar arguments as in [5, Theorem 8.1.4]. Multiplying the equation by \bar{u}_x and taking real part, we obtain:

$$\frac{d}{dx} \left(\frac{1}{2}|u_x|^2 + \frac{a_1}{3}|u|^3 + \frac{a_2}{4}|u|^4 + \frac{a_3}{5}|u|^5 \right) = 0.$$

Thus,

$$\frac{1}{2}|u_x|^2 + \frac{a_1}{3}|u|^3 + \frac{a_2}{4}|u|^4 + \frac{a_3}{5}|u|^5 = K.$$

Using $\lim_{x \rightarrow \pm\infty} u(x) = 0$ we have $K = 0$. In particular, $|u| > 0$. Indeed, if u vanishes then u_x vanishes at the same point, hence, $u \equiv 0$. Therefore, we may write $u = \rho e^{i\theta}$, where $\rho > 0$ and $\rho, \theta \in C^2(\mathbb{R})$. Substituting $u = \rho e^{i\theta}$ in (2.1) we have $2\rho_x\theta_x + \rho\theta_{xx} = 0$ which implies there exists $\tilde{K} \in \mathbb{R}$ such that $\rho^2\theta_x = \tilde{K}$ and so $\theta_x = \frac{\tilde{K}}{\rho^2}$. Moreover, since $|u_x|$ is bounded, it follows that $\rho^2\theta_x^2$ is bounded. Thus, $\frac{\tilde{K}^2}{\rho^2}$ is bounded. Since $\rho(x) \rightarrow 0$ as $x \rightarrow \infty$, we have $\tilde{K} = 0$. Thus, since $\rho > 0$ we have $\theta \equiv \theta_0$ for some $\theta_0 \in \mathbb{R}$. Thus $u = e^{i\theta_0}\rho$. Since ρ is a positive solution of (2.2), there exists $x_2 \in \mathbb{R}$ such that $\rho(x_2) = c$ and $\rho_x(x_2) = 0$. Thus, by uniqueness of solution of (2.2), there exists $x_3 \in \mathbb{R}$ such that $\rho(x) = \phi(x-x_3)$ and $u = e^{i\theta_0}\phi(x-x_3)$. This implies the desired result. \square

Moreover, we have the following result.

Lemma 2.3. *Let g and ϕ be as in Lemma 2.2. Then $\phi \in H^1(\mathbb{R})$.*

Proof. Firstly, since g satisfies the assumption of Lemma 2.1, we have $a_1 < 0$ (see the arguments in the proof of Proposition 1.1). As in the proof of Lemma 2.2, up to a translation, we may assume that $\phi_x(0) = 0$ and let $c = \phi(0)$. Then ϕ is an even function of x . Furthermore, ϕ satisfies

$$\frac{1}{2}\phi_x^2 + G(\phi) = 0. \quad (2.4)$$

Moreover, $\phi_{xx}(0) = -g(\phi(0)) = -g(c) < 0$. Therefore, there exists $a > 0$ such that $\phi_x < 0$ on $(0, a)$. We claim that $a = \infty$. Otherwise, there would exist $b > 0$ such that $\phi_x < 0$ on $(0, b)$ and $\phi_x(b) = 0$. Thus, $\phi(b) < c$ is a positive zero of G . This is a contradiction since c is the first positive solution of G . Hence, $\phi_x < 0$ on $(0, \infty)$. Thus, there exists $0 \leq l < c$ such that $\lim_{x \rightarrow \infty} \phi(x) = l$. In particular, there exists $x_m \rightarrow \infty$ such that $\phi_x(x_m) \rightarrow 0$ as $m \rightarrow \infty$. Passing to the limit in (2.4) we have $G(l) = 0$ and hence $l = 0$ by definition of c . Therefore ϕ decreases to 0, as $x \rightarrow \infty$. Thus, from (2.4), for $|x|$ large enough, we have

$$\phi_x^2 \approx -\frac{a_1}{3}\phi^3.$$

Then

$$-\phi_x \approx c\phi^{\frac{3}{2}}, \text{ for some } c > 0.$$

Thus, for $|x|$ large enough, we have

$$0 \geq \phi_x + c\phi^{\frac{3}{2}}.$$

It follows that $\phi \leq \frac{1}{(cx+d)^2}$ for some $c, d > 0$. Hence $\phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, especially $\phi \in L^2(\mathbb{R})$. Combining this and (2.4), we obtain that $\phi_x \in L^2(\mathbb{R})$. Thus, $\phi \in H^1(\mathbb{R})$, this completes the proof of Lemma 2.3. \square

Now, we come back to the proof of Proposition 1.1.

Proof of Proposition 1.1. A solution of (1.3) in the space X satisfies

$$u_{xx} + g(u) = 0, \quad u \in C^2(\mathbb{R}), \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} u(x) = 0, \quad (2.5)$$

From Lemma 2.1, the necessary condition for existence of solutions of (2.5) is $a_1 < 0$. Indeed, let c is the first positive root of $G(u)$ then $G'(c) = g(c) > 0$. Thus, G do not change sign on $(0, c)$ and is increasing in a neighborhood of c . It follows that $G < 0$ on $(0, c)$ and hence $a_1 < 0$.

To conclude the existence of solution of (2.5), we consider the three cases DDF, DFF, DFD. In the case DDD we have $G < 0$ on $(0, \infty)$, therefore there is no solution of (2.5).

In the case DDF (i.e $a_1 = -1$, $a_2 < 0$, $a_3 = 1$), we have

$$\begin{aligned} g(s) &= -s^2 + a_2 s^3 + s^4, \\ G(s) &= -\frac{1}{3}s^3 + \frac{a_2}{4}s^4 + \frac{1}{5}s^5. \end{aligned}$$

Thus ,

$$c = \frac{-\frac{a_2}{4} + \sqrt{\frac{a_2^2}{16} + \frac{4}{15}}}{\frac{2}{5}},$$

and $g(c) = c^2(c^2 + a_2c - 1)$. It easy to check that c is larger than the largest root of $x^2 + a_2x - 1$. Thus, $g(c) > 0$. It follows that in case DDF, there exists a solution of (2.5).

By similar arguments, in the case DFF, (2.5) has a solution. In the case DFD, (2.5) has a solution if and only if $a_2 > \frac{8}{\sqrt{15}}$.

Let ϕ be a solution of (2.5). From Lemma 2.2 all solution of (2.5) are of the form $e^{i\theta}\phi(x - x_0)$, and belong to $H^1(\mathbb{R})$ by Lemma 2.3. Thus, they are all algebraic standing waves of (1.1). This completes the proof of Proposition 1.1. \square

2.2. In higher dimensions. In this section, we prove existence and uniqueness of a radial positive solution of (1.3) when $a_1 = -1$, $a_3 = 1$ and $n = 2, 3$. The existence result is a consequence of the following theorem.

Theorem 2.4 ([3], Theorem I.1). *Let g be a locally Lipschitz continuous function from \mathbb{R}^+ to \mathbb{R} with $g(0) = 0$, satisfying*

- (1) $\alpha = \inf\{\zeta > 0, g(\zeta) \geq 0\}$ exists, and $\alpha > 0$.
- (2) There exists a number $\zeta > 0$ such that $G(\zeta) > 0$, where

$$G(t) = \int_0^t g(s) ds.$$

Define $\zeta_0 = \inf\{\zeta > 0, G(\zeta) > 0\}$. Then, ζ_0 exists, and $\zeta_0 > \alpha$.

- (3) $\lim_{s \downarrow \alpha} \frac{g(s)}{s - \alpha} > 0$.
- (4) $g(s) > 0$ for $s \in (\alpha, \zeta_0]$. Let $\beta = \inf\{\zeta > \zeta_0, g(\zeta) = 0\}$. Then, $\zeta_0 < \beta \leq \infty$.
- (5) If $\beta = \infty$ then $\frac{g(s)}{s^l} = 0$, with $l < \frac{n+2}{n-2}$, (If $n = 2$, we may choose for l just any finite real number).

Then there exists a number $\zeta \in (\zeta_0, \beta)$ such that the solution $u \in C^2(\mathbb{R}^+)$ of the Initial Value problem

$$\begin{cases} -u'' - \frac{n-1}{r}u' = g(u), & \text{for } r > 0, \\ u(0) = \zeta, \quad u'(0) = 0 \end{cases}$$

has the properties: $u > 0$ on \mathbb{R}^+ , $u' < 0$ on \mathbb{R}^+ and

$$\lim_{r \rightarrow \infty} u(r) = 0.$$

In our case, we have

$$g(s) = -s^2 + a_2 s^3 + s^4, \quad (2.6)$$

$$G(s) = -\frac{1}{3}s^3 + \frac{a_2}{4}s^4 + \frac{1}{5}s^5. \quad (2.7)$$

It is easy to check that the function g and G satisfy the conditions of Theorem 2.4 when $n = 2, 3$ with $\alpha = \frac{-a_2 + \sqrt{a_2^2 + 4}}{2}$ (the positive zero of g), $\zeta_0 = \frac{-a_2 + \sqrt{a_2^2 + \frac{64}{15}}}{\frac{2}{5}}$ (the positive zero of G), $\beta = \infty$

and $4 < l < 5$ when $n = 3$ and $l > 4$ when $n = 2$. Thus, in high dimensions ($n = 2, 3$), there exists a decreasing radial positive solution of (1.3).

The uniqueness of a radial positive solution is obtained by following result.

Theorem 2.5 ([15], Theorem 1). *Let us consider, for $n \geq 2$, the following equation*

$$\Delta u + g(u) = 0, \quad (2.8)$$

where g satisfies the following conditions:

- (a) g is continuous on $[0, \infty)$ and $g(0) = 0$,
- (b) g is a C^1 -function on $(0, \infty)$,
- (c) There exists $a > 0$ such that $g(a) = 0$ and

$$g(u) < 0 \text{ for } 0 < u < a,$$

$$g(u) > 0 \text{ for } u > a.$$

- (d) $\frac{d}{du} \left[\frac{G(u)}{g(u)} \right] \geq \frac{n-2}{2n}$, for $u > 0, u \neq a$, where $G(s) = \int_0^s f(\tau) d\tau$.

Then (2.8) admits at most one radial positive solution.

The function g given in (2.6) satisfies conditions (a), (b), (c) of Theorem 2.5 for a the positive root of g . When $n = 2, 3$, the condition (d) is satisfied if only if

$$\frac{d}{ds} \left[\frac{\frac{1}{5}s^3 + \frac{a_2}{4}s^2 - \frac{1}{3}s}{s^2 + a_2s - 1} \right] \geq \frac{n-2}{2n}, \text{ for } s > 0, s \neq a. \quad (2.9)$$

We prove that (2.9) holds. We only need to show that

$$\frac{d}{ds} \left[\frac{\frac{1}{5}s^3 + \frac{a_2}{4}s^2 - \frac{1}{3}s}{s^2 + a_2s - 1} \right] \geq \frac{1}{6}, \text{ for } s \neq a.$$

This is equivalent to

$$\frac{1}{5}s^4 + \frac{2a_2}{5}s^3 + \left(\frac{a_2^2}{2} + \frac{2}{5} \right) - a_2s + 1 \geq 0,$$

which is true for all $s > 0, a_2 \in \mathbb{R}$ by the fact that

$$\frac{1}{5}s^4 + \frac{2a_2}{5}s^3 + \left(\frac{a_2^2}{2} + \frac{2}{5} \right) - a_2s + 1 = \frac{1}{5}(s^2 + a_2s)^2 + \frac{3}{10} \left(a_2 - \frac{5}{3} \right)^2 + \frac{2}{5}s^2 + \frac{1}{6} > 0.$$

Thus, there exists a unique radial positive solution of (1.3) by Theorem 2.5. This completes the proof of Proposition 1.2.

3. VARIATIONAL CHARACTERIZATION

Let $n = 1, 2, 3$. In this section, we prove Proposition 1.3. By the assumption of Proposition 1.3, we may pick $a_1 = -1$ and $a_3 = 1$. We recall that S, K, P are defined in (1.4), (1.8) and (1.10).

Let \mathcal{M} and \mathcal{K} be defined as (1.12) and (1.8). First, as in [6], we prove that \mathcal{M} is not empty. We set

$$J(v) = \frac{1}{4} \|\nabla v\|_{L^2}^2 + \frac{1}{12} \|v\|_{L^3}^3 + \frac{1}{20} \|v\|_{L^5}^5,$$

which is well defined on X . The functional S is rewritten as

$$\begin{aligned} S(v) &= \frac{1}{2} K(v) - \frac{1}{6} \|v\|_{L^3}^3 + \frac{a_2}{4} \|v\|_{L^4}^4 + \frac{3}{10} \|v\|_{L^5}^5, \\ S(v) &= \frac{1}{4} K(v) + J(v). \end{aligned}$$

We can rewrite μ as

$$\mu = \inf \{ J(v) : v \in \mathcal{K} \}. \quad (3.1)$$

Lemma 3.1. *Let $v \in H^1(\mathbb{R}^n)$. If $K(v) < 0$ then $\mu < J(v)$. In particular,*

$$\mu = \inf \{ J(v) : v \in X \setminus \{0\}, K(v) \leq 0 \}. \quad (3.2)$$

Proof. Since $K(v) < 0$ and $K(\lambda v) > 0$ if $\lambda > 0$ small enough, there exists $\lambda_1 \in (0, 1)$ such that $K(\lambda_1 v) = 0$. Therefore, by (3.1) and since the function $\lambda \mapsto J(\lambda v)$ on $(0, \infty)$ is increasing, we have

$$\mu \leq J(\lambda_1 v) < J(v).$$

This completes the proof. \square

Lemma 3.2. *The following is true:*

$$\mu > 0.$$

Proof. Let $v \in \mathcal{K}$. By using the Gagliardo-Nirenberg inequalities, for some $\theta \in (0, 5)$ and $\tilde{\theta} \in (0, 4)$, we have

$$\begin{aligned} \|v\|_{L^5}^5 &\lesssim \|\nabla v\|_{L^2}^\theta \|v\|_{L^3}^{5-\theta} \leq C_1 \|\nabla v\|_{L^2}^5 + C_2 \|v\|_{L^3}^5, \\ \|v\|_{L^4}^4 &\lesssim \|\nabla v\|_{L^2}^{\tilde{\theta}} \|v\|_{L^3}^{4-\tilde{\theta}} \leq C_3 \|\nabla v\|_{L^2}^4 + C_4 \|v\|_{L^3}^4, \end{aligned}$$

we have

$$0 = K(v) \geq (1 - C_1 \|\nabla v\|_{L^2}^3 - |a_2| C_3 \|\nabla v\|_{L^2}^2) \|\nabla v\|_{L^2}^2 + (1 - C_2 \|v\|_{L^3}^2 - |a_2| C_4 \|v\|_{L^3}) \|v\|_{L^3}^3,$$

It follows that $1 \leq C_1 \|\nabla v\|_{L^2}^3 + |a_2| C_3 \|\nabla v\|_{L^2}^2 \leq C \|\nabla v\|_{L^2}^3 + \frac{1}{2}$ or $1 \leq C_2 \|v\|_{L^3}^2 + |a_2| C_4 \|v\|_{L^3} \leq \tilde{C} \|v\|_{L^3}^2 + \frac{1}{2}$, for some $C, \tilde{C} > 0$. Hence, $\|\nabla v\|_{L^2}$ or $\|v\|_{L^3}^3$ bounded below by some constant. In two cases, $J(v)$ is bounded below by some constant. Combining with (3.1) we have the conclusion. \square

We need the following results.

Lemma 3.3 ([1, 11]). *Let $p \geq 1$. Let (f_n) be a bounded sequence in $\dot{H}^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. Assume that there exists $q \in (p, 2^* - 1)$ such that $\limsup_{n \rightarrow \infty} \|f_n\|_{L^{q+1}} > 0$. Then there exist $(y_n) \subset \mathbb{R}^n$ and $f \in \dot{H}^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n) \setminus \{0\}$ such that $(f_n(\cdot - y_n))$ has a subsequence that converges to f weakly in $\dot{H}^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$.*

Lemma 3.4 ([4]). *Let $1 \leq r < \infty$. Let (f_n) be a bounded sequence in $L^r(\mathbb{R}^n)$ and $f_n \rightarrow f$ a.e in \mathbb{R}^n as $n \rightarrow \infty$. Then*

$$\|f_n\|_{L^r}^r - \|f_n - f\|_{L^r}^r - \|f\|_{L^r}^r \rightarrow 0,$$

as $n \rightarrow \infty$.

Now, we comeback to prove the set \mathcal{M} is not empty.

Lemma 3.5. *If $(v_n) \in X$ is a minimizing sequence for μ , that is,*

$$K(v_n) \rightarrow 0, \quad S(v_n) \rightarrow \mu,$$

then there exist $(y_n) \subset \mathbb{R}^n$, a subsequence (v_{n_j}) , and $v_0 \in X \setminus \{0\}$ such that $v_{n_j}(\cdot - y_{n_j}) \rightarrow v_0$ in X . In particular, $v_0 \in \mathcal{M}$.

Proof. Since $K(v_n) \rightarrow 0$ and $S(v_n) \rightarrow \mu$, we have

$$J(v_n) \rightarrow \mu, \tag{3.3}$$

$$\frac{-1}{6} \|v\|_{L^3}^3 + \frac{a_2}{4} \|v\|_{L^4}^4 + \frac{3}{10} \|v\|_{L^5}^5 \rightarrow \mu. \tag{3.4}$$

From (3.3), we infer that (v_n) is bounded in X . Also, since $\mu > 0$ by Lemma 3.2 and the Gagliardo-Nirenberg inequality $\|v\|_{L^5}^5 \lesssim \|\nabla v\|_{L^2}^5 + \|v\|_{L^4}^5$, we have $\limsup_{n \rightarrow \infty} \|v_n\|_{L^4} > 0$. Then, by Lemma 3.3 there exist $(y_n) \subset \mathbb{R}^n$ and $v_0 \in X \setminus \{0\}$ and a subsequence of $(v_n(\cdot - y_n))$, which we still denote by the same notation, such that $v_n(\cdot - y_n) \rightharpoonup v_0$ weakly in X . we put $w_n := v_n(\cdot - y_n)$.

We can assume that $w_n \rightarrow v_0$ a.e in \mathbb{R}^n and we prove that $w_n \rightarrow v_0$ strongly in X . By Lemma 3.4, we have

$$J(w_n) - J(w_n - v_0) \rightarrow J(v_0), \tag{3.5}$$

$$K(w_n) - K(w_n - v_0) \rightarrow K(v_0). \tag{3.6}$$

Since $J(v_0) > 0$ by $v_0 \neq 0$, it follows from (3.5) and (3.3) that

$$\lim_{n \rightarrow \infty} J(w_n - v_0) = \lim_{n \rightarrow \infty} J(w_n) - J(v_0) < \lim_{n \rightarrow \infty} J(w_n) = \mu.$$

From this and (3.2) we have $K(w_n - v_0) > 0$ for n large. Thus, since $K(v_n) \rightarrow 0$ and (3.6) we obtain $K(v_n) \leq 0$. By (3.2) and weak lower semicontinuity of the norms, we have

$$\mu \leq J(v_0) \leq \lim_{n \rightarrow \infty} J(w_n) = \mu.$$

Combining with (3.5) imply that $J(w_n - v_0) \rightarrow 0$ thus, $w_n \rightarrow v_0$ strongly in X . This completes the proof. \square

Proof of Proposition 1.3. Firstly, we prove the variational characterization of ϕ as follows

$$S(\phi) = \mu.$$

This means that ϕ is a minimizer of (1.11). From Lemma 3.5, we have $\mathcal{M} \neq \emptyset$. Let $\varphi \in \mathcal{M}$. We divide the proof of this to three steps.

Step 1. There exists $\theta \in \mathbb{R}$ such that $e^{i\theta}\varphi$ is a positive function.

We use similar arguments as in [6, Lemma 2.10]. Put $v := |\operatorname{Re}\varphi|$, $w := |\operatorname{Im}\varphi|$ and $\psi := v + iw$. By a phase modulation, we may assume that $v \neq 0$.

Since $|\psi| = |\varphi|$ and $|\nabla\psi| = |\nabla\varphi|$, we have $K(\psi) = K(\varphi)$ and $S(\psi) = S(\varphi)$. Thus, $\psi \in \mathcal{M}$. Then, there exists $\gamma \in \mathbb{R}$ such that

$$S'(\psi) = \gamma K'(\psi).$$

Hence,

$$\gamma \langle K'(\psi), \psi \rangle = \langle S'(\psi), \psi \rangle = K(\psi) = 0. \quad (3.7)$$

Moreover, using $K(\psi) = 0$ we have

$$\begin{aligned} \langle K'(\psi), \psi \rangle &= \partial_\lambda K(\lambda\psi)|_{\lambda=1} \\ &= \partial_\lambda K(\lambda\psi)|_{\lambda=1} - 4K(\psi) \\ &= (2\|\nabla\psi\|_{L^2}^2 + 3\|\psi\|_{L^3}^3 - 4a_2\|\psi\|_{L^4}^4 - 5\|\psi\|_{L^5}^5) - 4(\|\nabla\psi\|_{L^2}^2 + \|\psi\|_{L^3}^3 - a_2\|\psi\|_{L^4}^4 - \|\psi\|_{L^5}^5) \\ &= -2\|\nabla\psi\|_{L^2}^2 - \|\psi\|_{L^3}^3 - \|\psi\|_{L^5}^5 < 0. \end{aligned}$$

Combining with (3.7), we deduce $\gamma = 0$. Thus, $S'(\psi) = 0$. Hence, v solves the following equation

$$(-\Delta + |\varphi| - a_2|\varphi|^2 - |\varphi|^3)v = 0.$$

Since v is nonnegative and not identically equal to zero, using [12, Theorem 9.10], we infer that v is positive function. Furthermore, since $K(|\psi|) \leq K(\psi)$ and $S(|\psi|) \leq S(\psi)$, it follows from Lemma 3.1 we have $K(|\psi|) = K(\psi)$ and $S(|\psi|) = S(\psi)$. Then, $\|\nabla|\psi|\|_{L^2} = \|\nabla\psi\|_{L^2}$. By [12, Theorem 7.8], there exists a constant c such that $w = cv$ for some $c \geq 0$.

Since v is continuous and positive, $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ do not change sign. Then, there exist constants $\lambda = \pm 1$ and $\eta \in \mathbb{R}$ such that $\operatorname{Re}\varphi = \lambda v$ and $\operatorname{Im}\varphi = \eta v$. Taking $\theta \in \mathbb{R}$ such that $e^{-i\theta} = \frac{\lambda + i\eta}{|\lambda + i\eta|}$, we have $e^{i\theta}\varphi = e^{i\theta}(\lambda + i\eta)v = |\lambda + i\eta|v$. This completes the step 1.

Step 2. Radial symmetry of minimizer.

Since [10, Theorem 1], there exists $y \in \mathbb{R}^n$ such that $e^{i\theta}\varphi(\cdot - y)$ is a radial and decreasing function.

Step 3. Conclusion.

Since ϕ and $e^{i\theta}\varphi(\cdot - y)$ are positive radial solutions of (1.3), using Proposition 1.2, we obtain

$$\phi = e^{i\theta}\varphi(\cdot - y),$$

Thus, $S(\phi) = S(\varphi) = \mu$, $\phi \in \mathcal{M}$ and each element of \mathcal{M} is of form $e^{i\theta}\phi(\cdot - x_0)$ for some $\theta, x_0 \in \mathbb{R}$.

It remains to classify all algebraic standing waves of (1.1). We only need to prove that $\mathcal{G} = \mathcal{M} \neq \emptyset$, where \mathcal{G} and \mathcal{M} are defined in (1.7) and (1.12), respectively. We use similar arguments as in [6, Proof of Theorem 2.1]. We divide the proof of this in two steps.

Step 1. $\mathcal{M} \subset \mathcal{G}$.

Let $\psi \in \mathcal{M}$. Then, $S'(\psi) = 0$. Now, we show that $\psi \in \mathcal{G}$. Let $v \in X \setminus \{0\}$ such that $S'(v) = 0$. From $K(v) = \langle S'(v), v \rangle = 0$ and by definition of \mathcal{M} , we have $S(\psi) \leq S(v)$. Thus, $\psi \in \mathcal{G}$ and $\mathcal{M} \subset \mathcal{G}$.

Step 2. $\mathcal{G} \subset \mathcal{M}$ and conclusion.

Let $\psi \in \mathcal{G}$. Then $K(\psi) = \langle S'(\psi), \psi \rangle = 0$. As the above, $\phi \in \mathcal{M}$. As in step 1, $\phi \in \mathcal{G}$. Therefore, $S(\psi) = S(\phi) = \mu$, which implies $\psi \in \mathcal{M}$. Thus $\mathcal{G} \subset \mathcal{M}$, which completes the proof of Proposition 1.3. \square

It turns out that the algebraic standing waves of (1.1) in high dimensions ($n = 2, 3$) belongs to $H^1(\mathbb{R}^n)$. To prove this, we need the following lemma (see [6, Lemma 3.4]).

Lemma 3.6. *Let $\varphi \in C^1([0, \infty))$ be a positive function. If there exist $\rho, A > 0$ such that*

$$\varphi'(r) + A\varphi(r)^{1+\rho} \leq 0, \text{ for all } r > 0,$$

then

$$\varphi(r) \leq \left(\frac{1}{\rho Ar} \right)^{\frac{1}{\rho}}.$$

Proof of Remark 1.4(2). We use similar arguments as in [6, Proof of Proposition 3.5]. Firstly, we denote $\phi(r)$ as function of ϕ respect to variable $r = |x|$. Since ϕ is positive decreasing radial function, we have

$$\|\phi\|_{L^3}^3 \geq \int_{|x| \leq R} |\phi|^3 dx \geq |\mathcal{B}(R)| |\phi(R)|^3 = CR^n |\phi(R)|^3,$$

for all $R > 0$. Hence,

$$\phi(x) \leq |x|^{-\frac{n}{3}} \|\phi\|_{L^3}, \text{ for all } x \in \mathbb{R}.$$

For $r > r_0$ large enough, we have

$$|a_2|\phi^3 + \phi^4 \leq \frac{1}{2}\phi^2,$$

Since ϕ solves (1.3) and is decreasing as a function of r , this implies

$$\phi''(r) \geq \phi''(r) + \frac{n-1}{r}\phi'(r) = \phi^2 - a_2\phi^3 - \phi^4 \geq \frac{1}{2}\phi^2, \text{ for } r > r_0.$$

Multiplying the two sides by ϕ' and integrating it on $[r, \infty)$, we get

$$\phi'(r)^2 \geq \frac{1}{3}\phi^3, \text{ for } r \geq r_0.$$

Since $\phi' < 0$ we obtain that

$$\phi'(r) + \sqrt{\frac{1}{3}}\phi^{\frac{3}{2}} \leq 0, \text{ for } r \geq r_0.$$

By Lemma 3.6, we deduce that

$$\phi(r) \leq Cr^{-2}, \text{ for } r \geq r_0.$$

Thus, $\phi \in L^2(\mathbb{R}^n)$, for $n = 2, 3$. From the proof of Proposition 1.3, we have $\phi \in \mathcal{M}$. Hence, $|\nabla\phi| \in L^2(\mathbb{R}^n)$ and $\phi \in H^1(\mathbb{R}^n)$. This completes the proof. \square

4. INSTABILITY OF ALGEBRAIC STANDING WAVES

Let $n = 1, 2, 3$. In this section, we prove Theorem 1.5. Throughout this section, we consider the case DDF or DFD and a_2 small. Then we may pick $a_1 = -1$ and $a_3 = 1$. First, we prove the following result by using similar arguments as in [14] (see also [6, Proof of Proposition 5.1]).

Proposition 4.1. *Assume that*

$$\partial_\lambda^2 \mathcal{S}(\phi^\lambda)|_{\lambda=1} < 0, \text{ where } v^\lambda(x) := \lambda^{\frac{n}{2}} v(\lambda x). \quad (4.1)$$

Then the algebraic standing wave ϕ is unstable.

We define a tube around the standing wave by

$$\mathcal{N}_\varepsilon := \left\{ v \in H^1(\mathbb{R}^n) : \inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^n} \|v - e^{i\theta}\phi(\cdot - y)\|_{H^1} < \varepsilon \right\}.$$

Lemma 4.2. *Assume (4.1) holds. Then there exist $\varepsilon_1, \delta_1 \in (0, 1)$ such that: For any $v \in \mathcal{N}_{\varepsilon_1}$ there exists $\Lambda(v) \in (1 - \delta_1, 1 + \delta_1)$ such that*

$$\mu \leq S(v) + (\Lambda(v) - 1)P(v).$$

Proof. First, we recall that S , K and P are defined as in (1.4), (1.8) and (1.10), respectively.

Since $\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0$, by the continuity of the function

$$(\lambda, v) \mapsto \partial_\lambda^2 S(v^\lambda),$$

there exist $\varepsilon_1, \delta_1 \in (0, 1)$ such that $\partial_\lambda^2 S(v^\lambda) < 0$ for any $\lambda \in (1 - \delta_1, 1 + \delta_1)$ and $v \in \mathcal{N}_{\varepsilon_1}$. Moreover, by the definition of P we have

$$S(v^\lambda) \leq S(v) + (\lambda - 1)P(v), \quad (4.2)$$

for $\lambda \in (1 - \delta_1, 1 + \delta_1)$ and $v \in \mathcal{N}_{\varepsilon_1}$.

Moreover, consider the map:

$$(\lambda, v) \mapsto K(v^\lambda) = \lambda^2 \|\nabla v\|_{L^2}^2 + \lambda^{\frac{n}{2}} \|v\|_{L^3}^3 - a_2 \lambda^n \|v\|_{L^4}^4 - \lambda^{\frac{3n}{2}} \|v\|_{L^5}^5.$$

Note that $K(\phi) = 0$ and

$$\partial_\lambda K(\phi^\lambda)|_{\lambda=1} = 2\|\nabla \phi\|_{L^2}^2 + \frac{n}{2}\|\phi\|_{L^3}^3 - na_2\|\phi\|_{L^4}^4 - \frac{3n}{2}\|\phi\|_{L^5}^5.$$

Thus,

$$\begin{aligned} \partial_\lambda K(\phi^\lambda)|_{\lambda=1} &= \partial_\lambda K(\phi^\lambda)|_{\lambda=1} - 5P(\phi) \\ &= -3\|\nabla \phi\|_{L^2}^2 - \frac{n}{3}\|\phi\|_{L^3}^3 + \frac{na_2}{4}\|\phi\|_{L^4}^4. \end{aligned}$$

Thus, in the case $a_2 < 0$, we have $\partial_\lambda K(\phi^\lambda)|_{\lambda=1} < 0$. In the case $a_2 \geq 0$, using $P(\phi) = 0$, we have

$$\begin{aligned} \frac{na_2}{4}\|\phi\|_{L^4}^4 &= \|\nabla \phi\|_{L^2}^2 + \frac{n}{6}\|\phi\|_{L^3}^3 - \frac{3n}{10}\|\phi\|_{L^5}^5 \\ &\leq 3\|\nabla \phi\|_{L^2}^2 + \frac{n}{3}\|\phi\|_{L^3}^3, \end{aligned}$$

hence we also have $\partial_\lambda K(\phi^\lambda)|_{\lambda=1} < 0$. In all cases, by the implicit function theorem, taking ε_1 and δ_1 small enough, for any $v \in \mathcal{N}_{\varepsilon_1}$ there exists $\Lambda(v) \in (1 - \delta_1, 1 + \delta_1)$ such that $\Lambda(\phi) = 1$ and $K(v^{\Lambda(v)}) = 0$. Therefore, by definition of μ as in (1.11) we obtain:

$$\mu \leq S(v^{\Lambda(v)}) \leq S(v) + (\Lambda(v) - 1)P(v).$$

This completes the proof. \square

Let $u_0 \in \mathcal{N}_\varepsilon$ and $u(t)$ be the associated solution of (1.1). We define the exit time from the tube \mathcal{N}_ε by

$$T_\varepsilon^\pm(u_0) := \inf\{t > 0 : u(\pm t) \notin \mathcal{N}_\varepsilon\}.$$

We set $I_\varepsilon(u_0) := (-T_\varepsilon^-(u_0), T_\varepsilon^+(u_0))$.

Lemma 4.3. *Assume (4.1) holds and let ε_1 be given by Lemma 4.2. Then for any $u_0 \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1}$, where \mathcal{B} is defined as in (1.13), there exists $m = m(u_0) > 0$ such that $P(u(t)) \leq -m$ for all $t \in I_{\varepsilon_1}(u_0)$.*

Proof. For $t \in I_{\varepsilon_1}(u_0)$, since $u(t) \in \mathcal{N}_{\varepsilon_1}$, it follows from Lemma 4.2 that

$$\mu - S(u_0) = \mu - S(u(t)) \leq -(1 - \Lambda(u(t)))P(u(t)).$$

In particular, since $\mu > S(u_0)$ by $u_0 \in \mathcal{B}$, we have $P(u(t)) \neq 0$. By continuity of the flow and $P(u_0) < 0$ we obtain

$$P(u(t)) < 0, \quad 1 - \Lambda(u(t)) > 0.$$

Therefore, we obtain

$$-P(u(t)) \geq \frac{\mu - S(u_0)}{1 - \Lambda(u(t))} \geq \frac{\mu - S(u_0)}{\delta_1} =: m(u_0) > 0.$$

This completes the proof. \square

Lemma 4.4. *Assume (4.1) holds. Then $|I_{\varepsilon_1}| < \infty$ for all $u_0 \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1} \cap \Sigma$, where*

$$\Sigma = \{v \in H^1(\mathbb{R}) : xv \in L^2(\mathbb{R})\}.$$

Proof. Let $u(t)$ be associated solution of $u_0 \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1} \cap \Sigma$. By the virial identity and Lemma 4.3 we have

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8P(u(t)) \leq -8m(u_0)$$

for all $t \in I_{\varepsilon_1}(u_0)$, which implies $|I_{\varepsilon_1}(u_0)| < \infty$. This completes the proof. \square

Let χ be a smooth cut-off function such that

$$\chi(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

and for $R > 0$ define $\chi_R(x) = \chi\left(\frac{|x|}{R}\right)$.

The following is similar as in [6, Lemma 4.5].

Lemma 4.5. *There exists a function $R : (1, \infty) \rightarrow (0, \infty)$ such that $\chi_{R(\lambda)}\phi^\lambda \in \mathcal{B} \cap \Sigma \cap \mathcal{N}_{\varepsilon_1}$ for all $\lambda > 1$ close to 1, and that $\chi_{R(\lambda)}\phi^\lambda \rightarrow \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$.*

Proof. We divide the proof in three steps.

Step 1: Prove $\phi^\lambda \rightarrow \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$.

We have

$$\begin{aligned} & \|\phi^\lambda - \phi\|_{\dot{H}^1} + \|\phi^\lambda - \phi\|_{L^2} \\ & \leq \|\lambda^{\frac{n}{2}}\phi(\lambda \cdot) - \phi(\lambda \cdot)\|_{\dot{H}^1} + \|\phi(\lambda \cdot) - \phi(\cdot)\|_{\dot{H}^1} + \|\lambda^{\frac{n}{2}}\phi(\lambda \cdot) - \phi(\lambda \cdot)\|_{L^2} + \|\phi(\lambda \cdot) - \phi(\cdot)\|_{L^2} \\ & = (\lambda^{\frac{n}{2}} - 1)(\lambda^{1-\frac{n}{2}}\|\phi\|_{\dot{H}^1} + \lambda^{-\frac{n}{2}}\|\phi\|_{L^2}) \\ & \quad + \|\phi(\lambda \cdot) - \phi(\cdot)\|_{\dot{H}^1} + \|\phi(\lambda \cdot) - \phi(\cdot)\|_{L^2}. \end{aligned} \tag{4.3}$$

$$\tag{4.4}$$

The term (4.3) converges to zero as $\lambda \rightarrow 1$. To prove the term (4.4) converges to zero as $\lambda \rightarrow 1$, we prove for all $\phi \in L^p$, $1 < p < \infty$, then the following holds

$$\|\phi(\lambda x) - \phi(x)\|_{L^p} \rightarrow 0, \text{ as } \lambda \rightarrow 1.$$

Indeed, we only need to consider ϕ is a integrable step function, by density of step function in $L^p(\mathbb{R}^n)$. It is sufficient to consider $\phi = \mathbb{1}_{\mathcal{A}}$, for some measurable set \mathcal{A} . We have $\phi(\lambda x) = \mathbb{1}_{\frac{1}{\lambda}\mathcal{A}}$ and

$$\begin{aligned} \|\phi(\lambda x) - \phi(x)\|_{L^p}^p &= \|\mathbb{1}_{\frac{1}{\lambda}\mathcal{A}} - \mathbb{1}_{\mathcal{A}}\|_{L^p}^p \\ &= \mu(\{\lambda x \in \mathcal{A}, x \notin \mathcal{A}\} \cup \{x \in \mathcal{A}, \lambda x \notin \mathcal{A}\}) \\ &\leq \mu(\mathcal{A}) + \mu\left(\frac{1}{\lambda}\mathcal{A}\right) - 2\mu\left(\mathcal{A} \cap \frac{1}{\lambda}\mathcal{A}\right), \end{aligned}$$

this converges to zero when λ converges to 1. Thus, if we consider $\nabla\phi$ as a vector function then the term (4.4) converges to zero as λ converges to 1.

Step 2: $\chi_{R(\lambda)}\phi^\lambda \rightarrow \phi$ as $\lambda \rightarrow 1$ for some function R .

Choosing $R : (1, \infty) \rightarrow (0, \infty)$ such that $R(\lambda) \rightarrow \infty$ as $\lambda \rightarrow 1$. Thus, for all $v \in H^1(\mathbb{R}^n)$, we have

$$\chi_{R(\lambda)}v \rightarrow v, \text{ as } \lambda \rightarrow 1$$

and $\chi_{R(\lambda)}\phi^\lambda \rightarrow \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$, since step 1.

Step 3: Conclusion.

We claim that $\phi^\lambda \in \mathcal{B}$ for $\lambda > 1$ close to 1. Since $\partial_\lambda S(\phi^\lambda)|_{\lambda=1} = 0$ and $\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0$, there exists $\lambda_1 > 1$ such that $\partial_\lambda S(\phi^\lambda) < 0$ and $S(\phi^\lambda) < \mu$ for $\lambda \in (1, \lambda_1)$. We see that $P(\phi^\lambda) = \lambda \partial_\lambda S(\phi^\lambda) < 0$ for $\lambda \in (1, \lambda_1)$. Moreover, taking λ_1 close to 1, we get $\phi^\lambda \in \mathcal{N}_{\varepsilon_1}$ for all $\lambda \in (1, \lambda_1)$. Since $\chi_{R(\lambda)}$ has compact support and $\|\chi_{R(\lambda)}\phi^\lambda - \phi^\lambda\|_{H^1} \rightarrow 0$ as $\lambda \rightarrow 1$, we have $\chi_{R(\lambda)}\phi^\lambda \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1} \cap \Sigma$ for λ close to 1. This completes the proof. \square

Proof of Proposition 4.1. By Lemma 4.5, there exists $R : (1, \infty) \rightarrow (0, \infty)$ such that $\chi_{R(\lambda)}\phi^\lambda \rightarrow \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$. Moreover, $\chi_{R(\lambda)}\phi^\lambda \in \mathcal{B} \cap \Sigma \cap \mathcal{N}_{\varepsilon_1}$ for $\lambda > 1$ close to 1. Thus, by Lemma 4.4, $|I_{\varepsilon_1}(\chi_{R(\lambda)}\phi^\lambda)| < \infty$ for $\lambda > 1$ close to 1 and since $\chi_{R(\lambda)}\phi^\lambda \rightarrow \phi$ as $\lambda \rightarrow 1$ in $H^1(\mathbb{R}^n)$ we have ϕ is unstable. This completes the proof. \square

Proof of Theorem 1.5. Using Proposition 4.1, we only need to check the condition (4.1). We have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\nabla\phi\|_{L^2}^2 + \frac{n(n-2)}{12}\|\phi\|_{L^3}^3 - \frac{n(n-1)a_2}{4}\|\phi\|_{L^4}^4 - \frac{3n(3n-2)}{20}\|\phi\|_{L^5}^5.$$

We divide into three cases.

Case $n = 1$:

In this case, we have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\phi'\|_{L^2}^2 - \frac{1}{12}\|\phi\|_{L^3}^3 - \frac{3}{20}\|\phi\|_{L^5}^5.$$

In the case DDF, using $K(\phi) = 0$ and $P(\phi) = 0$ we have

$$0 = P(\phi) - \frac{1}{4}K(\phi) = \frac{3}{4}\|\phi'\|_{L^2}^2 - \frac{1}{12}\|\phi\|_{L^3}^3 - \frac{1}{20}\|\phi\|_{L^5}^5.$$

Thus,

$$\|\phi'\|_{L^2}^2 = \frac{1}{9}\|\phi\|_{L^3}^3 + \frac{1}{15}\|\phi\|_{L^5}^5.$$

It follows that

$$\begin{aligned} \partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} &= \frac{1}{36}\|\phi\|_{L^3}^3 - \frac{1}{12}\|\phi\|_{L^5}^5 \\ &= \frac{1}{36}\|\phi\|_{L^3}^3 - \frac{1}{12}\frac{10}{3}\left(\|\phi'\|_{L^2}^2 + \frac{1}{6}\|\phi\|_{L^3}^3 - \frac{a_2}{4}\|\phi\|_{L^4}^4 - P(\phi)\right) \\ &= -\frac{5}{18}\|\phi'\|_{L^2}^2 - \frac{1}{54}\|\phi\|_{L^3}^3 + \frac{5a_2}{72}\|\phi\|_{L^4}^4. \end{aligned} \quad (4.5)$$

Thus,

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0.$$

This implies the instability of algebraic standing waves in the case DDF.

In the case DFF, using (4.5) and the fact that $a\|\phi\|_{L^3}^3 + b\|\phi\|_{L^5}^5 \geq 2\sqrt{ab}\|\phi\|_{L^4}^4$ for all $a, b > 0$ we have

$$\begin{aligned} \partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} &= -\frac{5}{18}\left(\frac{1}{9}\|\phi\|_{L^3}^3 + \frac{1}{15}\|\phi\|_{L^5}^5\right) - \frac{1}{54}\|\phi\|_{L^3}^3 + \frac{5a_2}{72}\|\phi\|_{L^4}^4 \\ &= -\frac{4}{81}\|\phi\|_{L^3}^3 - \frac{1}{54}\|\phi\|_{L^5}^5 + \frac{5a_2}{72}\|\phi\|_{L^4}^4 \\ &\leq -\frac{4}{27\sqrt{6}}\|\phi\|_{L^4}^4 + \frac{5a_2}{72}\|\phi\|_{L^4}^4 < 0, \end{aligned}$$

since we have assumed $a_2 < \frac{32}{15\sqrt{6}}$. Thus, in the case DFF and $a_2 < \frac{32}{15\sqrt{6}}$ we obtain the instability of algebraic standing waves.

Case $n = 2$:

In this case, we have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\nabla\phi\|_{L^2}^2 - \frac{a_2}{2}\|\phi\|_{L^4}^4 - \frac{6}{5}\|\phi\|_{L^5}^5. \quad (4.6)$$

Moreover,

$$0 = P(\phi) = \|\nabla\phi\|_{L^2}^2 + \frac{1}{3}\|\phi\|_{L^3}^3 - \frac{a_2}{2}\|\phi\|_{L^4}^4 - \frac{3}{5}\|\phi\|_{L^5}^5.$$

Replacing $\frac{a_2}{2}\|\phi\|_{L^4}^4 = \|\nabla\phi\|_{L^2}^2 + \frac{1}{3}\|\phi\|_{L^3}^3 - \frac{3}{5}\|\phi\|_{L^5}^5$ in (4.6), we obtain

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = -\frac{1}{3}\|\phi\|_{L^3}^3 - \frac{3}{5}\|\phi\|_{L^5}^5 < 0.$$

The instability of algebraic standing waves in the case $n = 2$ follows.

Case $n = 3$:

In this case, we have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\nabla\phi\|_{L^2}^2 + \frac{1}{4}\|\phi\|_{L^3}^3 - \frac{3a_2}{2}\|\phi\|_{L^4}^4 - \frac{63}{20}\|\phi\|_{L^5}^5. \quad (4.7)$$

Moreover,

$$0 = P(\phi) = \|\nabla\phi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^3}^3 - \frac{3a_2}{4}\|\phi\|_{L^4}^4 - \frac{9}{10}\|\phi\|_{L^5}^5.$$

Hence,

$$\begin{aligned}\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} &= \partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} - 2P(\phi) \\ &= -\|\nabla\phi\|_{L^2}^2 - \frac{3}{4}\|\phi\|_{L^3}^3 - \frac{27}{20}\|\phi\|_{L^5}^5 < 0.\end{aligned}$$

The instability of algebraic standing waves in case $n = 3$ follows. This completes the proof of Theorem 1.5. \square

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