Modeling of bird leg

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1 Introduction

Birds have a large body mass but very thin legs with mostly ligaments. It has been shown in [2, 3] that ligaments reduce the bones to work mainly under pressure, with links between bones in rotation only without transmitting flexion. The result is a tensegrity system and this allows the pressure in the bones to be reduced. In the current study we are interested in the fact that the ligaments could ensure static equilibrium without action of the muscles on the one hand and on the other hand to ensure stability of balance. Ligaments can be modeled by a long cable guided by pulleys. The position, size and orientation of the pulleys depend on the species. Legs evolve in 3D space, and the long-term objective is to study the possibility of passive walking of such structures. The short term objective is to study equilibrium positions at rest. First in plan, then in 3D. In the medium term it will be useful to study the coordination of leg movements via the ligament for an actuation due to gravity and thigh movement.

2 Position d'équilibre au repos, modèle dans le plan sagittal

2.1 Geometric Models

In a standing equilibrium position, the leg of a human seen in the sagittal plane is straight. The alignment of the ankle, knee, hip and CoM allows for zero torque.

For a bird, the leg is not straight and the CoM is a little in front of the leg and at knee height, staying of course in the support zone.

As we limit ourselves to the study in the sagittal plane, the model with one leg is sufficient. The presence of two legs to ensure balance in the frontal plane is not discussed here.

The model of a quail is used as an inspiration. The detail of the model is given in the file: Points digitalisés sur le sq de la caille.

Points of interest are identified :

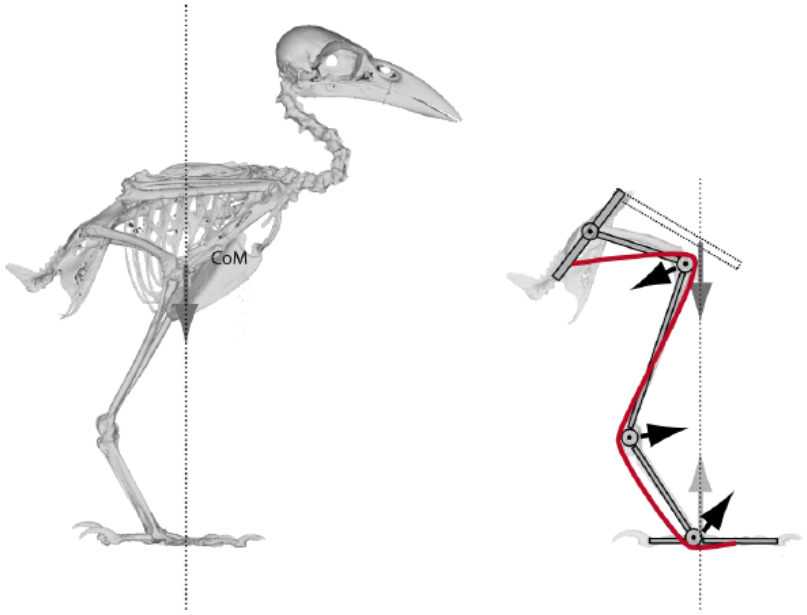

Figure 1: Bird's foot in resting position. Does the ligament shown in red ensure balance?

- S00 : CoM
- S03 : Hip pivot
- S25 : Knee CoR
- S28 : CoR ankle
- S06 : Foot CoR
- S32: point of attachment of the ligament to the body

We check that the sagittal plane defined by points S00, S01, and S02 is indeed the x,z plane of the data to project the coordinates of all the points in this plane. (x is the abscissa axis, z is the ordinate axis). The coordinates of the points of are not normalized, to give them a unit, we use the fact that the average dimensions of the bones are given in the file: $l_p = 0.03$, $l_j = 0.052$, $l_c = 0.042$. In fact we use the sum of these dimensions to make the normalization.

We model the set with 4 joints: finger, ankle, knee and hip. The segments of the legs are assumed mass-less. The mass is concentrated in the center of mass.

Finger data is selected from the illustrations. The ligament passes in front of the center of rotation of the nails. The pulley guiding the ligament is modeled as being very close to the ground. Finger dimensions are not very accurate and

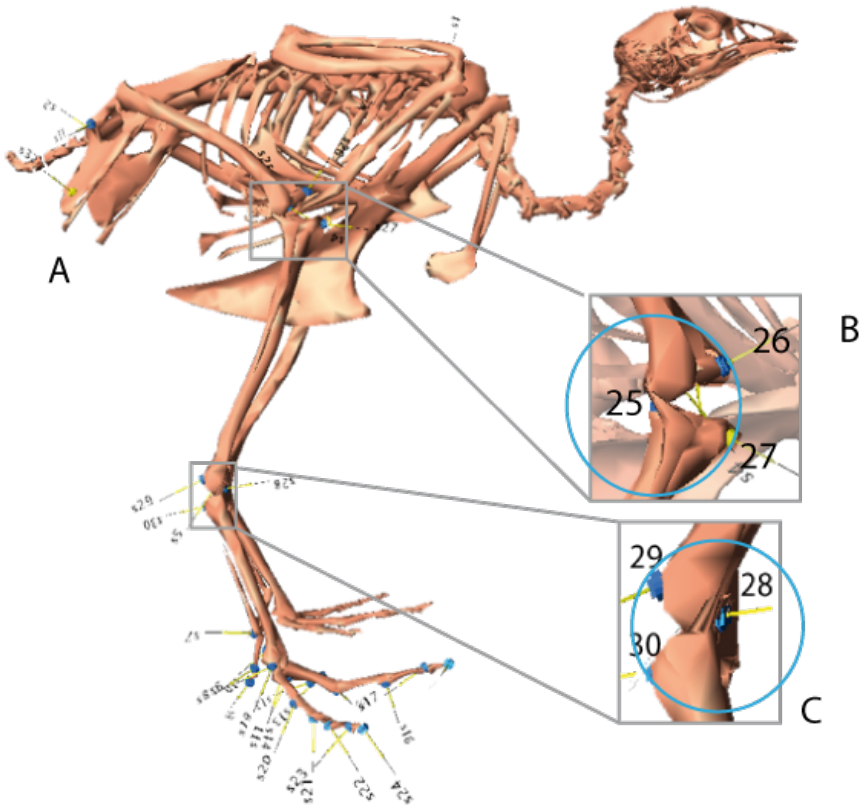

Figure 2: From measurement points on the quail skeleton we try to build a simplified model.

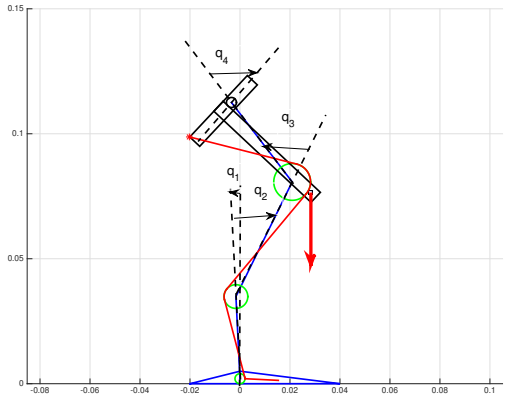

Figure 3: The modeling of the bird's foot with 4 joints q_1, at q_4 and a concentrated mass.

have no impact on the modeling except for the equilibrium condition concerning the CoM projection.

All calculations made we obtain a model, illustrated on figure 4 with the following characteristics:

The dimensions of the leg bodies $l_p = 0.0329$, $l_j = 0.0511$, $l_c = 0.04$.

The dimensions of the pulleys $r_d = 0.002$,$r_c = 0.0047$,$r_g = 0.0074$.

The dimensions of the body : The position of the CoM in relation to the attachment point in the body frame: $L_{CoM} = 0.0505, l_{CoM} = 0.0168$.

The position of the center of rotation of the hips in the body frame: $L_{at} = 0.0027, l_{at} = 0.0215$.

The chosen illustrations will be in general :$l_{dy} = 0.005$ (height of the triangle of fingers), for the length of the fingers: $l_{x1} = -0.02$, $l_{x2} = 0.04$.

2.1.1 La position du CoM

The coordinates of the CoM are:

$$\begin{aligned} x_{CoM} = \quad & -l_p sin(q_1) - l_j sin(q_1 + q_2) - l_c sin(q_1 + q_2 + q_3) \\ & -(l_{CoM} - l_{at}) sin(q_1 + q_2 + q_3 + q_4) \\ & +(L_{Com} - L_{at}) cos(q_1 + q_2 + q_3 + q_4) \end{aligned} \tag{1}$$

$$\begin{aligned} y_{CoM} = \quad & r_d + l_p cos(q_1) + l_j cos(q_1 + q_2) + l_c cos(q_1 + q_2 + q_3) \\ & +(l_{Com} - l_{at}) cos(q_1 + q_2 + q_3 + q_4) \\ & +(L_{CoM} - L_{at}) sin(q_1 + q_2 + q_3 + q_4) \end{aligned} \tag{2}$$

These equations can be written in a more compact way by introducing the parameters l_G and θ_G to write the position of the CoM in a reference frame linked

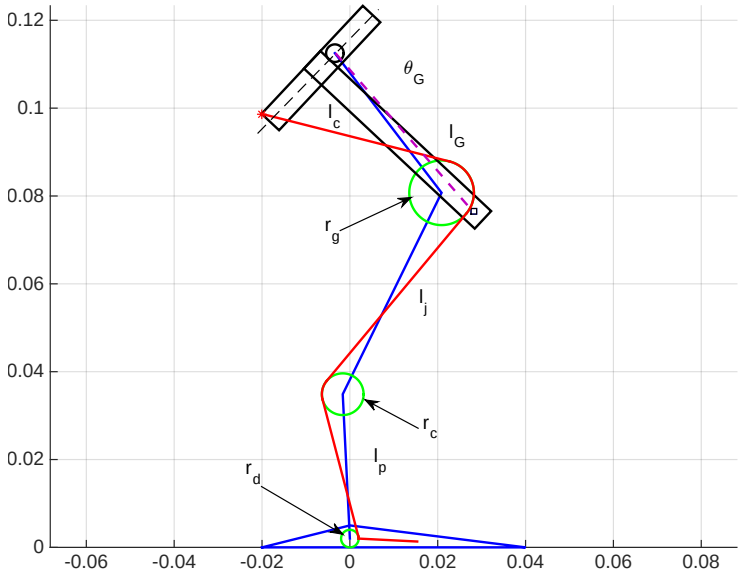

Figure 4: The different dimensions of the bodies to represent the quail.

to the body and centered on the hip joint : $l_G = \sqrt{(L_{com} - L_{at})^2 + (l_{com} - l_{at})^2} = 0.0480$, $\theta_G = -arctan(\frac{(L_{com}-L_{at})}{(l_{com}-l_{at})}) = -1.6675rd$:

$$\begin{aligned} x_{CoM} = \quad & -l_p sin(q_1) - l_j sin(q_1 + q_2) - l_c sin(q_1 + q_2 + q_3) \\ & -l_G sin(q_1 + q_2 + q_3 + q_4 + \theta_G) \end{aligned} \tag{3}$$

$$\begin{aligned} y_{CoM} = \quad & r_d + l_p cos(q_1) + l_j cos(q_1 + q_2) + l_c cos(q_1 + q_2 + q_3) \\ & +l_G cos(q_1 + q_2 + q_3 + q_4 + \theta_G) \end{aligned} \tag{4}$$

To check the stability of the balance, only the horizontal position of the CoM is important, it must be on the projection of the nails.

2.2 Ligament length as a function of leg position

For a ligament attached from the main body to the foot, the length of the cable depends on the values of the angles and the size of the pulleys. It is assumed that the ligament is attached at the base of the fingers. The cable is assumed to be taut, it is straight between the support areas and tangent to the pulley and then wraps along the pulley to change direction.

The cable will wrap around 3 pulleys associated with the fingers, foot, knee. These winding distances depend on the angles of the joints. We note the radii of the pulleys of the "fingers", the ankle, and the knee respectively r_d, r_c, r_g. The cable also runs along the foot and the tibia. The lengths of cable that tangent the pulleys are constant. In the same way, the distance from the point of attachment on the fingers to the first pulley is constant, we assume that the cable is guided in the trochlea and joins the pulley at a zero angle (see figure 4). The attachment length of the last pulley at the attachment point on the body depends on the angle of the body. The different dimensions used are shown on figure 4.

The detail of the cable length calculation is as follows:

- For the finger attachment, we assume that the cable attaches to the pulley at a point Q of the pulley such that the radius through Q makes a zero angle with the horizontal. Along the fingers the cable has a fixed length noted l_{0d}.
- Along the foot, the cable changes side to wind on the pulley, the length of the cable is:
$$\sqrt{(l_j)^2 - (r_d + r_c)^2}.$$
- The cable leaves the finger pulley at an angle to the horizontal $q_1 - asin((r_c + r_d)/l_p)$. The length of the cable wound along the finger pulley is therefore:
$$r_d * (q_1 - asin((r_c + r_d)/l_p)).$$
- The hook point on the pulleys of the ankle makes an angle $\pi + q_1 - asin((r_c + r_d)/l_p)$ with the horizontal.

- Along the leg, the cable changes side to wind on the pulley, the length of the cable is :
$$\sqrt{(l_j)^2 - (r_c + r_g)^2}.$$

- The cable leaves the ankle pulley at an angle to the horizontal $\pi + q_1 + q_2 - asin((r_c + r_g)/l_j)$. So the cable was wrapped around this pulley for a length:
$$r_c(q_2 - asin((r_c + r_g)/l_j + asin((r_c - r_d)/l_p)).$$

- The cable arrives on the knee pulley at an angle to the horizontal $q_1 + q_2 - asin((r_c + r_g)/l_j)$.

- The distance from the center of the knee pulley to the attachment point depends on q_4:
$$l_a(q_4) = \sqrt{l_c^2 + L_{at}^2 + l_{at}^2 - 2l_c l_{at} cos(q_4) - 2l_c L_{at} sin(q_4)} \tag{5}$$

- The length of the cable to go from the knee pulley to the attachment point depends on q_4:
$$\sqrt{l_a(q_4)^2 - r_g^2}$$

- The line reling the center of the knee pulley at the point of attachment has an orientation
$$atan(\frac{l_c - l_{at}cos(q_4) - L_{at}sin(q_4)}{-l_{at}sin(q_4) + L_{at}cos(q_4)})$$
relative to the thigh.

- the starting angle of the knee pulley cable to the horizontal is therefore:
$$\frac{\pi}{2} + q_1 + q_2 + q_3 + a_4(q_4)$$
with
$$a_4(q_4) = atan(\frac{l_c - l_{at}cos(q_4) - L_{at}sin(q_4)}{-l_{at}sin(q_4) + L_{at}cos(q_4)}) - acos(r_g/l_a(q_4)). \tag{6}$$

- The cable therefore wraps around the knee pulley for length:
$$r_g(\frac{\pi}{2} + q(3) + a_4(q_4) + asin((r_c + r_g)/l_j))$$

- In conclusion the length of the cable is written:
$$l(q) = l_0 + r_d q_1 - r_c q_2 + r_g q_3 + l_4(q_4) \tag{7}$$

with

$$\begin{aligned} l_0 = \quad & l_{0d} + \sqrt{l_p^2 - (r_d + r_c)^2} + \sqrt{(l_j)^2 - (r_c + r_g)^2} \\ & + r_d asin(r_d / l_{dy}) - (r_c + r_d) asin((r_c - r_d)/l_p) \\ & + (r_c + r_g) asin((r_c + r_g)/l_j) + r_g \tfrac{\pi}{2} \end{aligned} \tag{8}$$

and

$$l_4(q_4) = \sqrt{l_a(q_4)^2 - r_g^2} + r_g a_4(q_4) \tag{9}$$

The pulleys are essentially guides for the cables but not really pulleys in the mechanical sense. Ligaments are often clamped in guides and therefore do not detach spontaneously from the pulleys.

In summary, for the calculation of the length of the tensioned cable according to the robot configuration, we have the following conclusions:

- The cable lengths between pulleys or between the finger attachment point and the finger pulley are constant.
- Variations in the angles q_1, q_2, q_3 change the cable length in proportion to their variation and the pulley radii associated with the joint.
- The cable length decreases for a positive variation of the angle if the winding is on the left side of the pulley (ankle) and increases for a winding on the right side of the pulley (finger, knee).
- cable length is changed for a body angle variation q_4.
- A variation in the point of attachment of the cable on the finger will not affect the behavior of the system, except on the joint limits to avoid pulley detachment.
- The size of the pulleys slightly affects the behavior, but the case of pulleys of the same size is interesting because the equations are simplified (see bird model), in quail, the pulleys have significantly different radii.
- The winding direction of the cable on the sheave is an essential parameter. For the majority of pulleys, the winding direction is obvious to avoid pulley separation. For finger attachment, the 2 cases can be considered (see bird model)

2.3 Static model

The static model is used to search for equilibrium positions. The system configurations are searched for such that the equilibrium of forces are satisfied. The modeling used assumes that the foot is flat on the ground. The forces acting on the system are its weight and the force in the cable. The writing of the dynamic model via Lagrange's formalism, then the assumption of zero acceleration and zero speed of the system allow us to write the static model in the form:

$$\frac{\partial U}{\partial q} = 0 \tag{10}$$

where U is the potential energy of the system, here $U = mgy_{COM}(q) + Fl(q)$.

The static model is therefore written:

$$mg\frac{\partial y_{CoM}(q)}{\partial q} + F\frac{\partial l(q)}{\partial q} = 0 \tag{11}$$

or:

$$\begin{bmatrix} \frac{\partial y_{CoM}}{\partial q_1} \\ \frac{\partial y_{CoM}}{\partial q_2} \\ \frac{\partial y_{CoM}}{\partial q_3} \\ \frac{\partial y_{CoM}}{\partial q_4} \end{bmatrix} + \begin{bmatrix} \frac{\partial l}{\partial q_1} \\ \frac{\partial l}{\partial q_2} \\ \frac{\partial l}{\partial q_3} \\ \frac{\partial l}{\partial q_4} \end{bmatrix} \frac{F}{mg} = 0 \tag{12}$$

$$\begin{bmatrix} -l_pS1 - l_jS12 - l_cS123 - l_GS1234 \\ -l_jS12 - l_cS123 - l_GS1234 \\ -l_cS123 - l_GS1234 \\ -l_GS1234 \end{bmatrix} + \begin{bmatrix} r_d \\ -r_c \\ r_g \\ \frac{\partial l_4(q_4)}{\partial q_4} \end{bmatrix} \frac{F}{mg} = 0 \tag{13}$$

where $S1 = sin(q_1)$, $S12 = sin(q_1 + q_2)$, $S123 = sin(q_1 + q_2 + q_3)$, $S1234 = sin(q_1 + q_2 + q_3 + q_4 + \theta_G)$, $C1234 = cos(q_1 + q_2 + q_3 + q_4 + \theta_G)$

Using (9), $\frac{\partial l_4(q_4)}{\partial q_4}$ can be written:

$$\frac{\partial l_4(q_4)}{\partial q_4} = \frac{l_a(q_4)}{\sqrt{l_a(q_4)^2 - r_g^2}}\frac{\partial l_a(q_4)}{\partial q_4} + r_g\frac{\partial a_4(q_4)}{\partial q_4} \tag{14}$$

$$\begin{array}{rcl} a_4(q_4) & = & atan(\frac{U(q_4)}{V(q_4)}) - acos(r_g/l_a(q_4)), \\ U(q_4) & = & l_c - l_{at}cos(q_4) - L_{at}sin(q_4), \\ V(q_4) & = & -l_{at}sin(q_4) + L_{at}cos(q_4). \end{array} \tag{15}$$

Using (5), $\frac{\partial l_a(q_4)}{\partial q_4}$ can be written:

$$\frac{\partial l_a(q_4)}{\partial q_4} = \frac{2l_cl_{at}sin(q_4) - 2l_cL_{at}cos(q_4)}{2l_a} \tag{16}$$

Using (15), $\frac{\partial a_4(q_4)}{\partial q_4}$ can be written:

$$\begin{array}{rcl} \frac{\partial a_4(q_4)}{\partial q_4} & = & \frac{\frac{\partial V}{\partial q_4}U - \frac{\partial U}{\partial q_4}V}{U(q_4)^2 + V(q_4)^2} - \frac{r_g}{l_a(q_4)\sqrt{l_a(q_4)^2 - r_g^2}}\frac{\partial l_a(q_4)}{\partial q_4}, \\ \frac{\partial U(q_4)}{\partial q_4} & = & l_{at}sin(q_4) - L_{at}cos(q_4), \\ \frac{\partial V(q_4)}{\partial q_4} & = & -l_{at}cos(q_4) - L_{at}sin(q_4). \end{array} \tag{17}$$

2.4 Balance condition, inextensible cable

If the cable is taut, i.e. if $l(q) = l_c$, then the cable will be able to provide the necessary force to ensure balance. The balance positions must satisfy :

$$\begin{aligned} mg\frac{\partial y_{CoM}(q_e)}{\partial q} + \frac{\partial l(q)}{\partial q}F &= 0 \\ l(q) - l_{cable} &= 0. \end{aligned} \tag{18}$$

It is a system of 5 equations with 5 unknowns q, F.

$$\begin{bmatrix} -l_pS1 - l_jS12 - l_cS123 - l_GS1234 \\ -l_jS12 - l_cS123 - l_GS1234 \\ -l_cS123 - l_GS1234 \\ -l_GS1234 \\ l_0 + r_dq_1 - r_cq_2 + r_gq_3 + l_4(q_4) \end{bmatrix} + \begin{bmatrix} r_d\frac{F}{mg} \\ -r_c\frac{F}{mg} \\ r_g\frac{F}{mg} \\ \frac{\partial l_4(q_4)}{\partial q_4}\frac{F}{mg} \\ -l_{cable} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

The third equation may allow us to express $\frac{F}{mg}$ by :

$$\frac{F}{mg} = \frac{l_c}{r_g}S123 + \frac{l_G}{r_g}S1234 \tag{20}$$

By transferring this expression into the equation (19), we obtain a system of 4 equations with 4 unknowns q_1, q_2, q_3, q_4:

$$\begin{bmatrix} -l_pS1 - l_jS12 - l_cS123 - l_GS1234 \\ -l_jS12 - l_cS123 - l_GS1234 \\ -l_GS1234 \\ l_0 + r_dq_1 - r_cq_2 + r_gq_3 + l_4(q_4) \end{bmatrix} + \begin{bmatrix} r_d(\frac{l_c}{r_g}S123 + \frac{l_G}{r_g}S1234) \\ -r_c(\frac{l_c}{r_g}S123 + \frac{l_G}{r_g}S1234) \\ \frac{\partial l_4(q_4)}{\partial q_4}(\frac{l_c}{r_g}S123 + \frac{l_G}{r_g}S1234) \\ -l_{cable} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

or:

$$\begin{bmatrix} l_pS1 + l_jS12 + l_c(1 - \frac{r_d}{r_g})S123 + l_G(1 - \frac{r_d}{r_g})S1234 \\ l_jS12 + l_c(1 + \frac{r_c}{r_g})S123 + l_G(1 + \frac{r_c}{r_g})S1234 \\ \frac{\partial l_4(q_4)}{\partial q_4}\frac{l_c}{r_g}S123 + l_G(1 + \frac{\partial l_4(q_4)}{\partial q_4}\frac{L_G}{r_g})S1234 \\ l_0 + r_dq_1 - r_cq_2 + r_gq_3 + l_4(q_4) - l_{cable} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

We note the solutions q_e. You can search for solutions numerically with fsolve for example. Then we can deduce the force to obtain the equilibrium F_e using (20).

The bird is in equilibrium if $x_{CoM}(q_e)$ is in the support area and if the force F_e is positive since a cable cannot be compressed.

In the following example, starting from the configuration corresponding to the measurements provided: $q = [0.0493, -0.5055, 1.1088, -1.4057]$ and shown in the figure 3, we calculate the length of the corresponding cable $l(q) - l_0 = 0.0523$, then we look for the appropriate balance position. We get the configuration $q_e = [0.1288, -0.2779, 0.5749, -1.6494]$ represented on the figure 5. The force in the cable is $Fe = 0.6268mg$

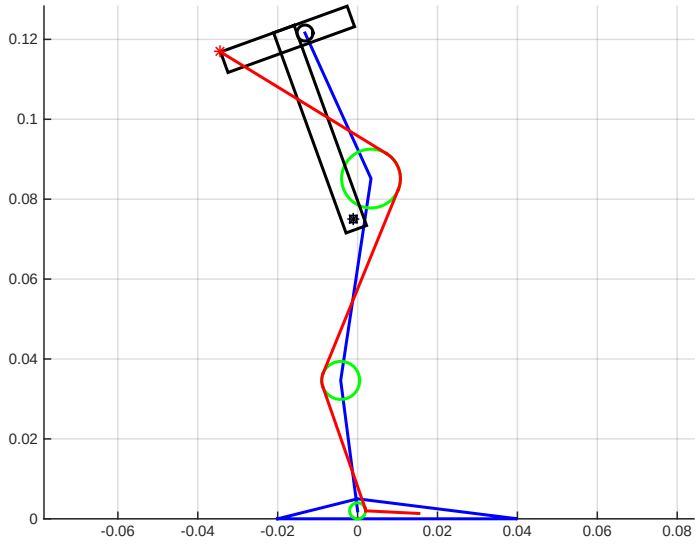

Figure 5: The equilibrium position corresponds to the length of the cable defined for the reference position measured by Anick. The position of the CoM is a little further behind the knee than expected but well in the axis of the feet.

The influence of the cable length on the equilibrium positions is studied. In figure 6, several equilibrium positions are drawn. In all cases, the center of mass projects into the support area and is very close to zero. In the figure 7, the evolution of the equilibrium configurations as a function of the length of the cable, the evolution of the CoM projection along the x-axis and the force in the cable are plotted. We note that the joint variable evolves in a quasi linear way. The angles q_1, q_2, q_3 could have negative values but remain positive when we limit the exploration to positive forces in the cables. We can have a case of stretched leg, illustrated in figure 6(a) for which the force in the cable is very low or cases of legs a little more bent 6(b,c,d) where the force in the cable increases to become higher than the weight of the quail we have normalized things by representing for the force $\frac{F}{mg}$.

2.4.1 Effect of the choice of the attachment points

The case studied previously is with point S032. If we take point S031, for the same length of cable the equilibrium position of the quail is presented on the figure 8. The same type of difference between reference position and equilibrium position is observed. The balance configuration of the leg is a bit different but in both cases the CoM is a bit lower than the knee but at the back.

It can be observed that in this case, as the point of attachment is closer to the point of the hip, variations in cable length will result in greater angular and force variations. The graph of the evolution of angles and forces for the different configurations of equilibrium in the case of an S0031 attachment point is shown in figure 9.

2.5 Effect of the size of the finger pulleys and way cable is guided

The rolling along is bones that can be assimilated to pulleys is quite visible for most joints. But for the fingers this is more complex. The ligament is essentially guided in a trochlea and the visualization of the center of rotation and thus the pulley is obtained by construction. However, the way the ligament is elongated according to the angle of the fingers is important and will change the configuration of the feet and knee in the equilibrium position. It is this point that we will explore in this section.

We will trace the evolution of the balance positions as a function of the radius of the finger pulley. A negative value of the radius corresponds to a roll in the other direction. We only consider the "variable" length of the cable, i.e. which depends on the joint configurations.

We consider finger pulley radii that vary from 0 to the size of the ankle and knee pulleys, with a positive and negative sign to indicate a different winding of the cable. The different balance configurations are shown in figure 10. The evolution of the balance joint configurations as a function of the radius of the finger "pulley" is shown in figure 11.

We can observe 3 interesting special cases.

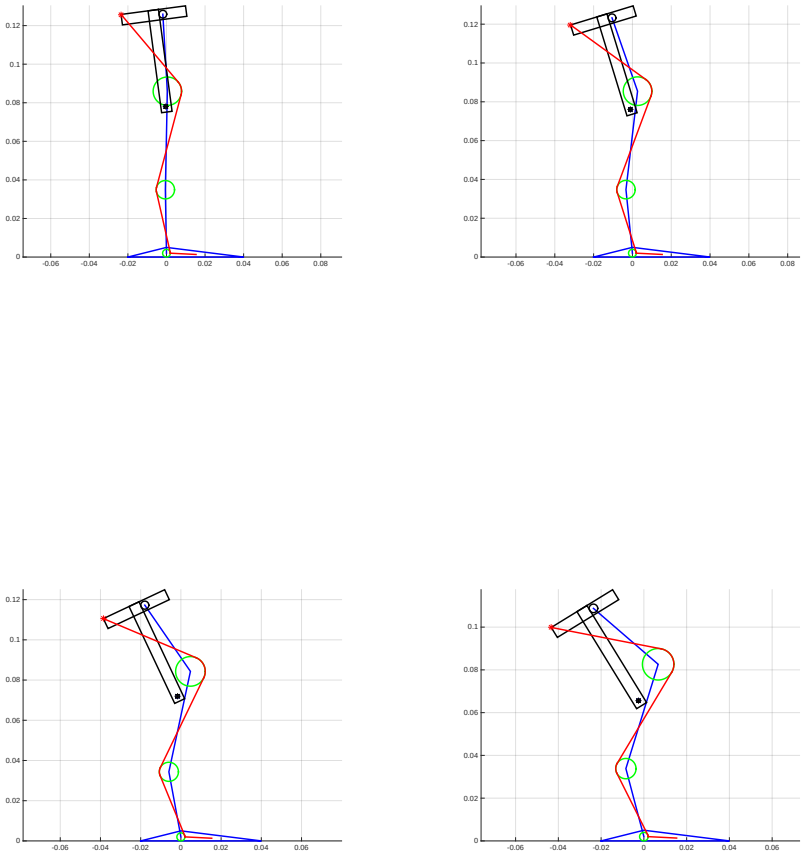

Figure 6: Some quail balance positions, for different cable lengths: (a) : the cable is shortened, to a case with a force close to 0, $Fe = 0.0852$, $l(q) - l_0 = 0.0443$, (b) : the cable is shortened, $l(q) - l_0 = 0.0.503$, $Fe = 0.4933$ (c) the cable is lengthened $l(q) - l_0 = 0.0563$, $Fe = 0.8854$ (d) the cable is lengthened $l(q) - l_0 = 0.0623$, $Fe = 1.2474$, the force in the cable is greater than the weight of the quail.

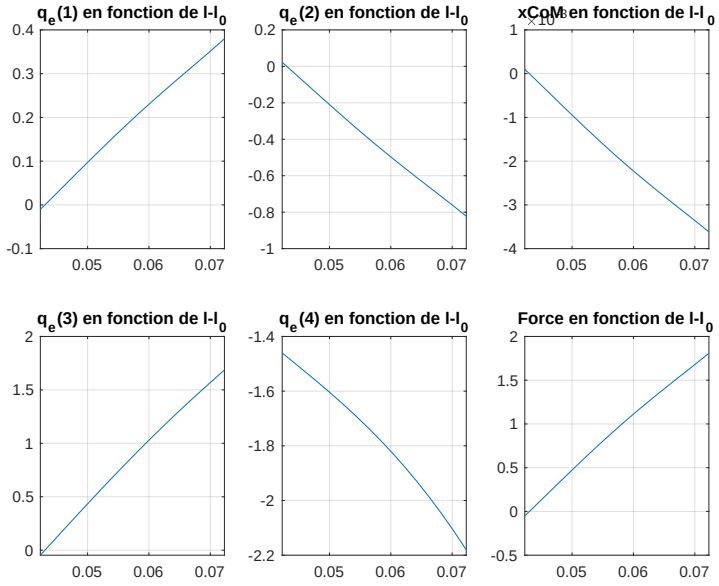

Figure 7: Evolution of the equilibrium position, CoM projection, and force in the cable as a function of the cable length.

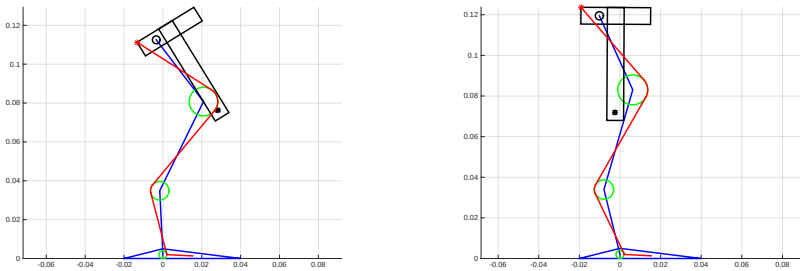

Figure 8: The quail with the S0031 quail attachment point and the equilibrium position associated with the cable length corresponding to the measuring points.

- If $r_d = r_g$, and the cable passes to the right of the center of the fingers then the 1st and 3rd line of the equation (13) tells us that the knee axis is vertical to the finger axis.
- If $r_d = r_c$, (r_a in the figure) and the cable passes to the right of the center of the fingers then, the 1st and 2nd line of the equation (13) tells us that $q_1 = 0$, the foot is vertical.
- If $r_d = 0$, then q_1 doesn't change the length of the cable and the only possible equilibrium position is when the CoM projection is on the finger joint.

2.6 Balance condition, extendable cable

It is assumed that the cable is extensible. The cable has an empty length l_v and an assumed constant stiffness K.

The potential energy of the system is then written :

$$U_e = mgy_{COM} + \frac{1}{2}K(l(q) - l_v)^2$$

We thus obtain as a static model $\frac{\partial U_e(q)}{\partial q} = 0$:

$$mg\frac{\partial y_{COM}(q)}{\partial q} + \frac{\partial l(q)}{\partial q}K(l(q) - l_v) = 0.$$

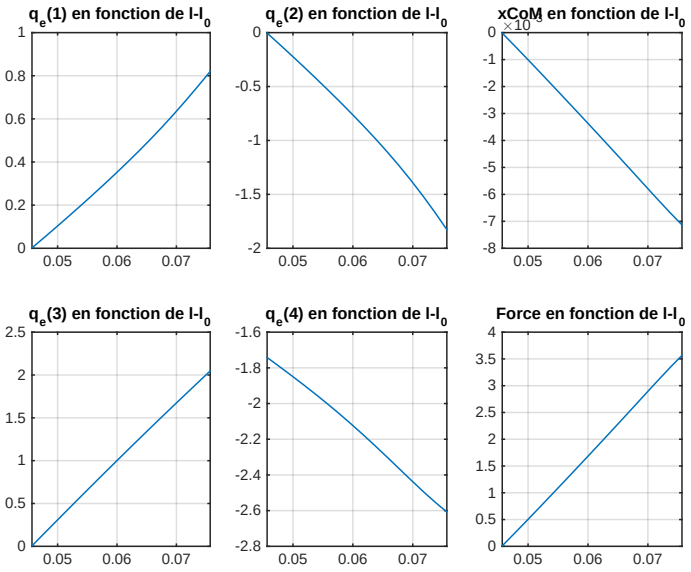

Figure 9: Evolution of the equilibrium position, the CoM projection, and the force in the cable as a function of the length of the cable for an S031 attachment point.

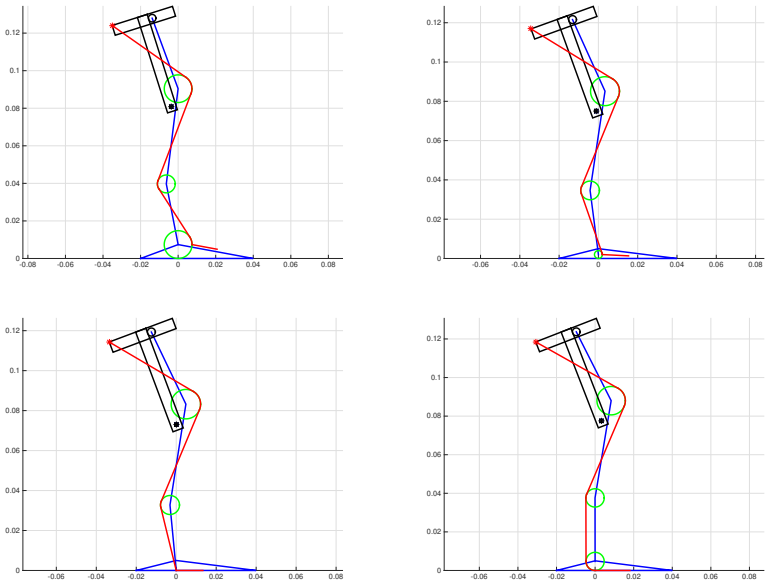

Figure 10: Several equilibrium configuration for differents value of the dimension of the finger pulley and differents way the cable passe around the pulley. In the upper left case, the pulley of the knee and finger have the same size, the knee is at the vertical of the finger joint. In the upper right case, the case studied. In the lower left case, the pulley has 0 diametre, the angle of the foot as no influence on the cable lenght, the CoM is projected at the finger joint. The lower right case, the foot is vertical.

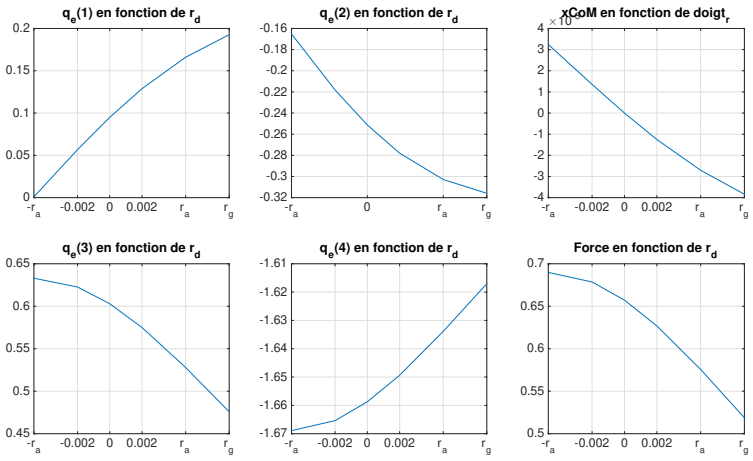

Figure 11: Evolution of the equilibrium position, the CoM projection, and the force in the cable as a function of the diametre of the finger pulley. Positive and negative value correspond to different ways the cable passe around the pulley.

A q position is therefore an equilibrium position if we have:

$$mg\frac{\partial y_{COM}(q_e)}{\partial q} + \frac{\partial l(q_e)}{\partial q}K(l(q_e) - l_v) = 0. \tag{23}$$

Or:

$$\begin{bmatrix} -l_pS1 - l_jS12 - l_cS123 - l_GS1234 \\ -l_jS12 - l_cS123 - l_GS1234 \\ -l_cS123 - l_GS1234 \\ -l_GS1234 \end{bmatrix} + \begin{bmatrix} r_d \\ -r_c \\ r_g \\ \frac{\partial l_4(q_4)}{\partial q_4} \end{bmatrix} \frac{K(l(q_e) - l_v)}{mg} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

It is a system of 4 equations with 4 unknowns q. One can search for solutions numerically with fsolve for example.

A solution obtained with the inextensible cable model q_e, F_e can be found with the extensible cable model of stiffness K, by choosing $l_v = l_{cable} - \frac{F_e}{K}$.

With the flexible cable model, one can test the stability of the q_e installation in the sense of returning to the equilibrium position in case of disturbance in addition to the mechanical stability of the contact: $x_{CoM}(q_e)$ is in the support zone.

2.7 Stability of equilibrium

With the elastic cable model, we want to know if when we move the bird away from an equilibrium position it will return to that position. We have an equilibrium position q_e that satisfies :

$$\frac{\partial U_e(q)}{\partial q} = 0 \tag{25}$$

This equilibrium is said to be stable if when the robot configuration is moved away from its equilibrium configuration it returns to it. No matter what notion of stability is adopted, the question always consists in evaluating the behavior of a system that is disturbed with respect to its equilibrium configuration by external forces and/or torques. Generally speaking, small variations in forces applied to the nodes of a system will cause changes in its configuration. If the system returns to its initial equilibrium configuration when the forces are reset to zero, it is considered stable.

For a system to return to its equilibrium configuration following an external perturbation, the deformation of the system δq under the effect of a perturbation must require a positive work input [1]. That is to say that the stiffness matrix (i.e. the Hessian matrix of the potential energy of the mechanism)

$$\mathbf{K} = \frac{\partial^2 U_e(q)}{\partial^2 q} \tag{26}$$

must be set positive. (This finding is based on the hypothesis that the system cannot move freely in translation or rotation in its space, which corresponds to the case studied in this work. For a case where the system is not constrained in its space, the stiffness matrix must rather be semi positively defined).

The term $K_{i,j}$ of the stiffness matrix is :

$$K_{i,j} = \frac{\partial^2 U_e(q)}{\partial q_i \partial q_j}$$

soit

$$K_{i,j} = \frac{\partial}{\partial q_i}(mg\frac{\partial y_{COM}(q_e)}{\partial q_j} + \frac{\partial l(q_e)}{\partial q_j}K(l(q_e) - l_v))$$

$$K_{i,j} = mg\frac{\partial^2 y_{COM}(q_e)}{\partial q_i \partial q_j} + \frac{\partial^2 l(q_e)}{\partial q_i \partial q_j}K(l(q_e) - l_v) + K\frac{\partial l(q_e)}{\partial q_j}\frac{\partial l(q_e)}{\partial q_i} \tag{27}$$

This stiffness matrix can be written :

$$\mathbf{K} = \mathbf{K_y} + K\mathbf{K_{ll}} + K\mathbf{K_l} \tag{28}$$

with

$$\mathbf{K_y} = -\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & y_2 & y_3 & y_4 \\ y_3 & y_3 & y_3 & y_4 \\ y_4 & y_4 & y_4 & y_4 \end{bmatrix},$$

$$\mathbf{K_{ll}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{ll4} \end{bmatrix}$$

$$\mathbf{K_{l}} = \begin{bmatrix} r_d^2 & -r_d r_c & r_d r_g & r_d dl4 \\ -r_d r_c & r_c^2 & -r_c r_g & -r_c dl4 \\ r_d r_g & -r_c r_g & r_g^2 & r_g dl4 \\ r_d dl4 & -r_c dl4 & r_g dl4 & dl4^2 \end{bmatrix},$$

and $y_1 = l_p C1 + l_j C12 + l_c C123 + l_G C1234$, $y_2 = l_j C12 + l_c C123 + l_G C1234$, $y_3 = l_c C123 + l_G C1234$, $y_4 = l_G C1234$, $dl4 = \frac{\partial l_4}{q_4}$, $K_{ll4} = \frac{\partial^2 l(q_e)}{\partial q_4 \partial q_4}(l(q_e) - l_v)$.

The balance positions studied are positions in which the legs are erect but with the body pointing downwards, and a center of mass lower than the knee so we have: $y_4 < y_3 < 0 < y_2 < y_1$.

In order to have a positive defined stiffness matrix, it is necessary and sufficient that the 4 main miners of this matrix have positive determinants. This condition can be established for with a set of variables written in any order.

A first necessary condition is therefore deduced: the terms of the diagonal must be positive. We note that $dl4 < 0$ and that K_{ll4} is very small in front of $dl4^2$. We have only 2 values of y_i which are positive, so we have only 2 conditions on K :

$$\begin{array}{ll} K > & \frac{y_1}{r_d^2} \\ K > & \frac{y_2}{r_c^2} \end{array} \tag{29}$$

Ceci nous donne la valeur minimum de K :

$$K > max(\frac{y_1}{r_d^2}, \frac{y_2}{r_c^2}) \tag{30}$$

In our case study, like $y_1 > y_2$ and $r_d < r_c$, the first term is most penalizing term. We note that the smaller r_d is, the greater the stiffness of the cable must be to ensure balance. We must have $K > 1.8262e+04$, we choose $K = 20000$. It can be noted that for a higher pose of the CoM, e.g. if the CoM is higher that the hip has it exist for human (and not for bird), we will have more condition to satisfy. If the radius of the finger pulley is the smallest one, the most penalizing condition will not changed.

Order 2 minors must have positive determinants. We consider the minor of $\mathbf{K}(1:2, 1:2)$:

$$det = (Kr_d^2 - y_1)(Kr_c^2 - y_2) - (Kr_d r_c - y_2)^2 \tag{31}$$

$$det = -Kr_d^2 y_2 + y_1 y_2 - Kr_c^2 y_1 + 2Kr_d r_c y_2 - y_2^2 \tag{32}$$

We write r_c in the form $r_c = r_d \alpha$.We can then write the equation in the form:

$$det = -Kr_d^2 y_2 + y_1 y_2 - K\alpha^2 r_d^2 y_1 + 2K\alpha r_d^2 y_2 - y_2^2 \tag{33}$$

We're looking for the value of α that maximizes the value of the determinant. α is such that $\frac{\partial det}{\partial \alpha}$:

$$\frac{\partial det}{\partial \alpha} = -2K\alpha r_d^2 y_1 + 2Kr_d^2 y_2 = 0 \tag{34}$$

or

$$\alpha = \frac{2Kr_d^2 y_2}{2Kr_d^2 y_1} = \frac{y_2}{y_1} \tag{35}$$

If we transfer this value in the expression of the determinant we obtain:

$$det = -Kr_d^2 y_2 + Kr_d^2 \frac{y_2^2}{y_1} + y_1 y_2 - y_2^2 \tag{36}$$

$$det = -\frac{Kr_d^2}{y_1}(y_2 y_1 - y_2^2) + y_1 y_2 - y_2^2 \tag{37}$$

.

$$det = (1 - \frac{Kr_d^2}{y_1}) y_2 (y_1 - y_2) \tag{38}$$

It is known that $y_1 > y_2$, and that $y_2 > 0$ so the last 2 terms of this expression are positive. To have the first positive minor, we must have $Krd^2 > y_1$ so the first term $(1 - \frac{Kr_d^2}{y_1})$ is negative. We deduce therefore that an extremum of the determinant is negative by scanning the values of α, (ratio between the radii of the pulleys of the fingers and ankle).

To determine whether this extremum is a minimum or a maximum, we determine the second derivative of the determinant with respect to α. By deriving the equation (34) with respect to α we obtain:

$$\frac{\partial^2 det}{\partial \alpha^2} = -2Kr_d^2 y_1 < 0. \tag{39}$$

The optimum obtained is therefore a maximum of the *det* function, so we can deduce that there is no set of parameters that correspond to a stable equilibrium of the structure.

It can be noted that the difficulty in obtaining a stable equilibrium is mainly reflected by the difficulty in obtaining a positive term for this determinant. For the hip and knee joints, stable equilibrium is obtained simply by the fact that the CoM is lower than the axis of the joint.

We have defined 3 conditions necessary to have stability: $\mathbf{K}(1,1) > 0$, $\mathbf{K}(2,2) > 0$, $det(\mathbf{K}(1:2,1:2)) > 0$. The first 2 conditions can be satisfied by the choice of the ligament stiffness K but this is not the case for the third condition. To satisfy this condition, it is proposed to add a ligament around the fingers and/or ankle.

3 Ligament addition

To ensure balance, we propose to add in our model ligaments that will act around each joint independently with a pulley of radius R_d, R_c, and a stiffness cable K_d, K_c.

3.1 Modeling

The idea is to add a cable that roll around a finger pullay of radius R_d and that is attached to the connected bones, and a cable that roll around a ankle pulley of radius R_c and that is attached to the connected bones. Depending of the side of the pulley were the cable is, we will defined a coefficient s_d and s_c for each cable, the lenght of the cable will be :

$$\begin{array}{ll} l_a & = l_{d0} + s_d R_d q_1 \\ l_c & = l_{c0} + s_c R_c q_2 \end{array} \tag{40}$$

where $s_i = 1$, for $i = a, c$ if the cable is on the rigt side and $s_i = -1$ if the cable is on the left side, l_{d0} and l_{c0} is a constant value that do not depend on the configuration of the leg.

A treatment similar to what was done for the long cable will lead us to define a new potential energy for the system:

$$U_{e2} = mgy_{COM} + \frac{1}{2}K(l(q) - l_v)^2 + \frac{1}{2}K_d(s_d R_d q_1 + l_{d0} - l_{vd})^2$$

$$+\frac{1}{2}K_c(s_c R_c q_2 + l_{c0} - l_{vc})^2.$$

where l_{vd} and l_{vc} are the no-load length of the springs. We can introduce new variables : $l_{dv} = l_{d0} - l_{vd}$, $l_{cv} = l_{c0} - l_{cd}$ to simplify the equation.

We thus obtain as a static model $\frac{\partial U_e(q)}{\partial q} = 0$ that can be developped as :

$$mg\frac{\partial y_{COM}(q)}{\partial q} + \begin{bmatrix} r_d \\ -r_c \\ r_g \\ \frac{\partial l_4(q_4)}{\partial q_4} \end{bmatrix} K(l(q_e) - l_v) + \begin{bmatrix} s_d R_d K_d(s_d R_d q_1 - l_{dv}) \\ s_c R_c K_c(s_c R_c q_2 - l_{cv}) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

or if we want to explicitly write the forces in the cables :

$$mg\frac{\partial y_{COM}(q)}{\partial q} + \begin{bmatrix} r_d \\ -r_c \\ r_g \\ \frac{\partial l_4(q_4)}{\partial q_4} \end{bmatrix} F + \begin{bmatrix} s_d R_d \\ 0 \\ 0 \\ 0 \end{bmatrix} F_d + \begin{bmatrix} 0 \\ s_c R_c \\ 0 \\ 0 \end{bmatrix} F_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

with positive forces into the 3 cables : $F = K(l(q) - l_v)$ as previously, and 2 new forces $F_d = K_d(s_d R_d q_1 - l_{dv})$ et $F_c = K_c(s_c R_c q_2 - l_{cv})$.

From the static model (41), the equilibrium position can be calculated. The no-load lengths of the extra ligaments cannot be chosen so as not to change the equilibrium position q_e, otherwise the force in the ligament would be zero and since the forces exerted must be positive, the ligament could only stabilize the whole in one direction. Depending on the no-load position for the additional cables added, the equilibrium position will be modified. This additional degree of freedom can be used to have an equilibrium position close to the reference

position chosen in figure 4, we will note this configuration q_n (nominal configuration) but in the nominal configuration the CoM projection is very far from the finger joint and the force in the cable will be very hight.

Stability is studied by writing the second derivative of potential energy, the stiffness matrix is then written :

$$\mathbf{K}_+ = \mathbf{K_y} + \mathbf{K_l} + \mathbf{K_{ll}} + \mathbf{K_N} \tag{43}$$

with $\mathbf{K_y}$, $\mathbf{K_l}$, et $\mathbf{K_{ll}}$ définis dans l'équation (62), et

$$\mathbf{K_N} = \begin{bmatrix} K_d R_d^2 & 0 & 0 & 0 \\ 0 & K_c R_c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the same stiffness matrice is obtained independently of the side where the cables passe, for any value of s_c, s_a. It can be noticed that only the matrix $\mathbf{K_y}$, the terms $dl4$ and K_{ll4} depends on the equilibrium configuration while the other coefficient depend only of the stifness ceofficient and the dimension of the pulleys.

La matrice $\mathbf{K}_+$ est une matrice symétrique :

$$\begin{bmatrix} -y_1 + Kr_d^2 + K_d R_d^2 & -y_2 - Kr_d r_c & -y_3 + Kr_d r_g & -y_4 + Kr_d dl4 \\ ... & -y_2 + Kr_c^2 + K_c R_c^2 & -y_3 - Kr_c r_g & -y_4 - Kr_c dl4 \\ ... & ... & -y_3 + Kr_g^2 & -y_4 + Kr_g dl4 \\ ... & ... & ... & -y_4 + K(dl4^2 + K_{ll4}) \end{bmatrix}$$

3.2 The parameters choosen

An iterative method is proposed to find the properties of the extra ligaments that will ensure balance. The condition of stability concerns the different minors. We will write the successive determinants to be studied by considering the joint, q_4 then q_3, then q_2 then q_1. The first condition to be satisfied is: $det_1 = \mathbf{K}(4,4) > 0$. For our case, this condition is satisfied for any stiffness of the ligament, we have $det_1 > 0$. The second condition to be satisfied is $det_2 = \mathbf{K}(3:4,3:4) > 0$. As soon as the stiffness of the long cable K is greater than 100 N/m, this condition is satisfied. For the following we will choose $K = 20000N/m$, value for which $det_2 > 0$. We then consider $det_3 = \mathbf{K}(2:4,2:4) > 0$, we calculate the determinant without adding stiffness K_c, the determinant is then noted det_{3i}. If $det_{3i} > 0$, there is no need to introduce an ankle stiffness. Introducing K_c stiffness to the ankle will change the value of the determinant to $det_3 = det_{3i} + K_c R_c^2 det_2$, if $det_{3i} < 0$, choosing $K_c R_c^2 > -\frac{det_{3i}}{det_2}$ will ensure that det_3 is positive. The method can then be repeated for the finger. Note that the choice of the stiffness at the long cable and the ankle will influence the stiffness needed at the ankle and then at the finger .... High stiffness in the long cable will reduce the stiffness needed in the other joints.

On the example treated, a stable solution can be obtained by choosing $K_c R_c^2 = 0.1, K_d R_d^2 = 0.2$. By choosing for all the cables a stiffness of 20000, this

gives us the size of the associated pulleys: $R_c = 0.0022$, $R_d = 0.0032$. There are of course many other possibilities.

You can choose the no-load positions of the cables l_v, l_{dv}, q_{cv}, so that the equilibrium position is as close as possible to q_n. Care must be taken to ensure that the forces in the cables are positive, the choice of the side the cables roll can be chosen to satisfy this constraint.

The equilibrium configuration satisfy the four equations (41) with three forces F, F_d, F_c, thus not any configuration can be achieved and the closest configuration to q_n can be chosen.

From equation (41) it can be seen that the 2 last line are actuated only by the long cable. The two equations must be satisfyed:

$$\begin{bmatrix} -l_c S123 - l_G S1234 \\ -l_G S1234 \end{bmatrix} + \begin{bmatrix} r_g \\ \frac{\partial l_4(q_4)}{\partial q_4} \end{bmatrix} \frac{F}{mg} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{44}$$

It is possible to calculate F using the third equation and using the fourth one, the equilibrium configuration must satisfied :

$$l_G S1234 + \frac{\partial l_4(q_4)}{\partial q_4} \frac{l_c S123 + l_G S1234}{r_g} = 0 \tag{45}$$

Among the set of possible configuration, we choose the closest to the configuration mesure in the museum. We choose to minimize the norm of the equilibriuml configuration and the nominal using absolute angle to emphasize on the orientation of the body.

The desired equilibrium configuration is : $qe = [0.0493, -0.5055, 1.2325, -1.8279]$

The caracteristique of the long cable is similar to previous case. The force required in the added cable are calculated. The side of the pulley where the cable must passed are defined to have positive force. The left size must be used for the finger pulley and the ankle pulley. The values of l_{dv}, q_{cv} are deduced to have the desired force. The equilibrium obtained is shown in figure 12.

3.3 Change of equilibrium configuration with changed of the long cable

The equilibria when the length of the cable is changed will also be slightly disturbed, the evolution of the equilibrium position and stability conditions as a function of the length of the long cable is plotted in the following figures 13 to 16. Note that in order to have a balance, the forces in the three cables must be positive, which limits the possible variations in cable length. A variation in the no-load length of the long cable is chosen as the abscissa for plotting the different equilibrium positions. However, a variation in elongation also exists, due to the redistribution of forces between the different cables. The real length of the long cable is therefore different from the one used as abscissa of the curves.

It can be observed that the equilibrium in the sense of the return to the equilibrium position in case of a disturbance is well satisfied in all cases (Figure 15). The cables at the level of the bolts and pins play an important role with

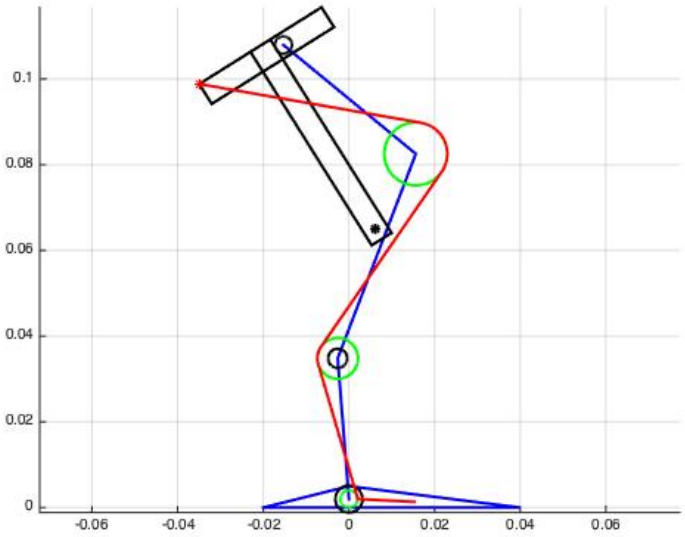

Figure 12: Stable equilibrium of a bird leg with 3 ligaments.

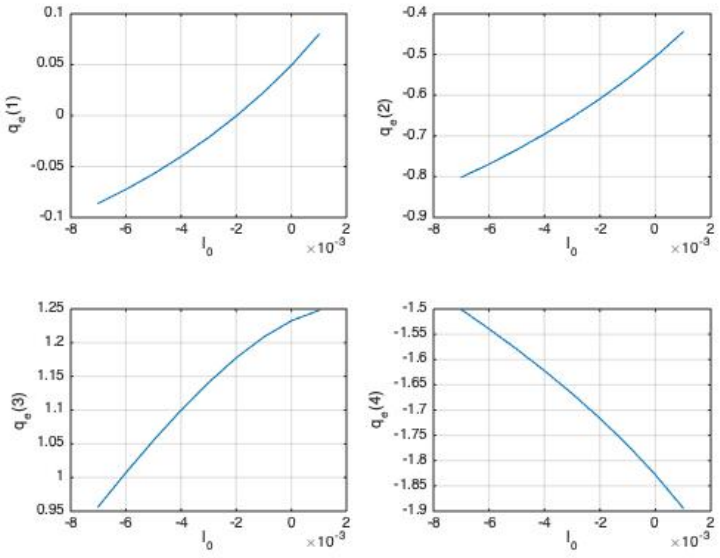

Figure 13: Equilibrium of a bird leg with 3 ligaments for various variation of the long cable.

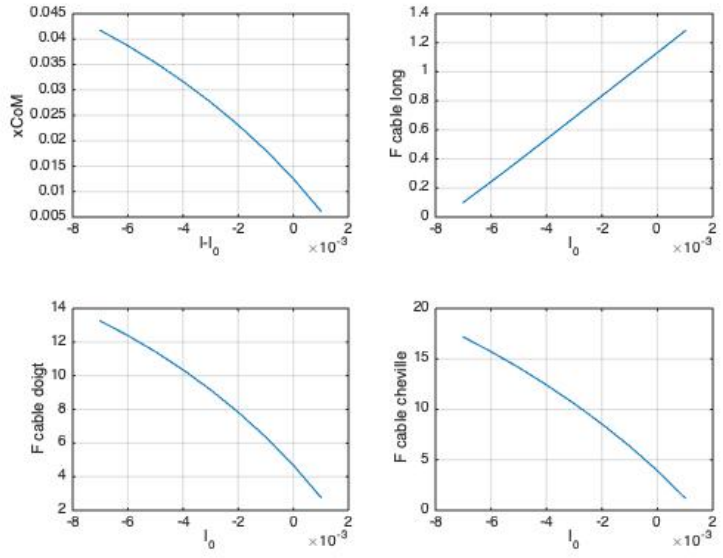

Figure 14: Forces in the cable for stables equilibrium for various variation of the long cable.

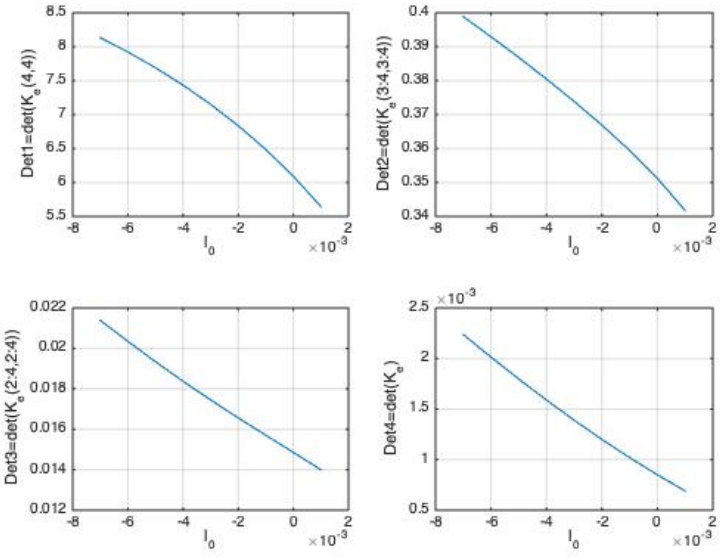

Figure 15: With 3 cables, equilibrium can be stable as shown by the determinant of the mineurs of the Hessian.

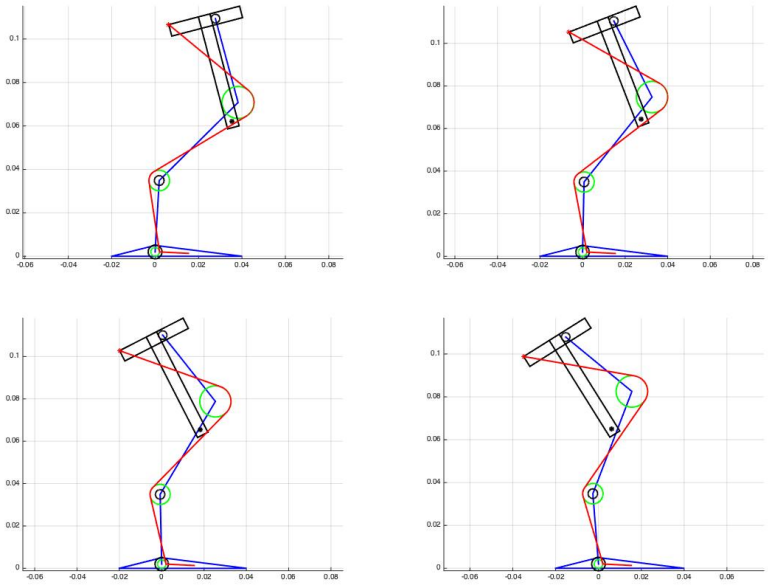

Figure 16: Several equilibrium configurations for the model with 3 cables (long, ankle and finger) with variation of the lenght cable. a) dl=-0.005, b) dl=-0.003, c) dl=-0.001, d) dl=0.001.

significant forces in the cables (Figure 14). This allows for a more limited force in the long cable, usually less than the body weight of the bird. The counterpart is that the configurations of the fingers and ankles are constrained by these forces and their configuration varies less than in the case with a single cable in proportion to the knee and hip joints (Figure 13 and Figure 16). It is then possible that the projection of the CoM tends to leave the foot support and that static equilibrium is lost (Figure 14).

This arrangement of additional ligaments was chosen based on the shape of the stiffness matrix to ensure stable balance. Alternatively, the strong ligaments of the bird's leg can be used as a model for other arrangements.

4 Choice of ligaments based on bio-inspiration

The most powerful muscles/ligaments in bird are represented in the figure 17. One ligament is bi-articular, represented in yellow, it includes finger and ankle. One ligament is bi-articular, represented in green, it includes ankle and knee. Two parallele ligaments, represented in blue are bi-articular they includes hip and knee. The contribution of the knee on their elongation differs since the ligaments because the ligaments do not pass on the same side of the knee pulley, whereas they both pass on the same side of the hip. The hypotheses of alternating or simultaneous work of the two hip-knee ligaments will be considered successively. The pulley at ankle for the two ligaments can be consider identical. This is not the case for the knee, the pulley for the blue ligaments is larger than for the green one, thus we will used two differents dimension r_g and r_{g2}.

By analogy with what was done in the calculation of the length of the ligament in section 2.2. We can write here for our three ligaments noted l_{da}, l_{ak}, l_{kh}.

$$l_{da}(q) = l_{da0} + r_d q_1 - r_c q_2 \tag{46}$$

$$l_{ak}(q) = l_{ak0} - r_c q_2 - r_{g2} q_3 \tag{47}$$

$$l_{kh}(q) = l_{kh0} + r_g q_3 + l_4(q_4) \tag{48}$$

$$l_{kh2}(q) = l_{kh20} - r_g q_3 + l_4(q_4) \tag{49}$$

A treatment similar to what was done previously will lead us to define a new potential energy for the system:

$$U_{e2} = mgy_{COM} + \frac{1}{2}K_{da}(r_d q_1 - r_c q_2 - l_{dav})^2 + \frac{1}{2}K_{ak}(-r_c q_2 - r_{g2} q_3 - l_{akv})^2$$

$$+\frac{1}{2}K_{kh}(r_g q_3 + l_4(q_4) - l_{khv})^2 + \frac{1}{2}K_{kh2}(-r_g q_3 + l_4(q_4) - l_{kh2v})^2.$$

where l_{dav}, l_{akv}, l_{khv} and l_{kh2v} are the difference between the no-load length of the springs and the lenght of the cable for $\mathbf{q=0}$.

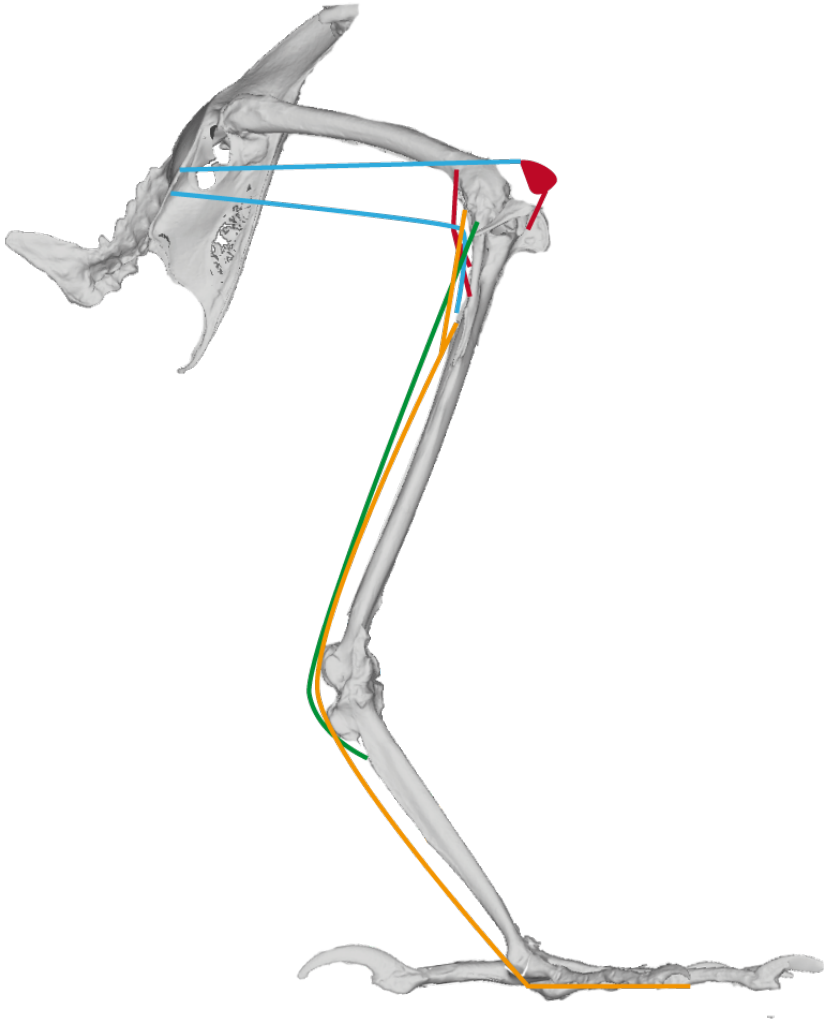

Figure 17: The main muscles implied in passive extension for bird leg.

We thus obtain as a static model $\frac{\partial U_e(q)}{\partial q} = 0$ that can be developped as function of the forces in the cables :

$$mg\frac{\partial y_{COM}(q)}{\partial q}+\begin{bmatrix} r_d \\ -r_c \\ 0 \\ 0 \end{bmatrix} F_{da}+\begin{bmatrix} 0 \\ -r_c \\ -r_{g2} \\ 0 \end{bmatrix} F_{ak}+\begin{bmatrix} 0 \\ 0 \\ r_g \\ \frac{\partial l_4(q_4)}{\partial q_4} \end{bmatrix} F_{kh}+\begin{bmatrix} 0 \\ 0 \\ -r_g \\ \frac{\partial l_4(q_4)}{\partial q_4} \end{bmatrix} F_{kh2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{50}$$

with positive forces into the 2 first cables :

$$F_{da} = K_{da}(r_d q_1 - r_c q_2 - l_{dav}) \tag{51}$$

$$F_{ak} = K_{ak}(-r_c q_2 - r_{g2} q_3 - l_{akv}) \tag{52}$$

and positive or nul forces into the 2 first cables, a negative force will be replaced by a nul one since cables can not produce a negative force. An unstressed cable cannot contribute to the stability of the balance.

$$F_{kh} = K_{kh}(r_g q_3 + l_4(q_4) - l_{khv}) \tag{53}$$

$$F_{kh2} = K_{kh2}(-r_g q_3 + l_4(q_4) - l_{kh2v}) \tag{54}$$

4.1 Research for balance positions

In the presence of 4 ligaments many parameters will affect the balance position and stability. For the size of the pulleys, bio-inspiration can help us. We keep the previous values and choose $r_{g2} = \frac{3}{4} r_c$. The stiffnesses K_{da}, K_{ak}, K_{kh}, K_{kh2} will have an important effect on the stability of the equilibrium. The lengths l_{dav}, l_{akv}, l_{khv}, l_{kh2v} will adjust the equilibrium positions. The way they are chosen is explained now.

An equilibrium configuration is a solution which satisfies the equation 50 under the constraints of positivity of the forces (here the 4). We will look for the q_e configuration closest to the equilibrium position of the figure 5 (i.e. $[0.1288, -0.2779, 0.5749, -1.6494]$) by considering the joint configurations written in an absolute reference frame, to favour the absolute orientation of the bodies, such that all the forces are greater than or equal to 5% of the bird's weight. A local minimum is the configuration $q_e = [0.1403, -0.2752, 0.5491, -1.6221]$ shown in figure 18. The corresponding forces are $F = [0.5966, 0.1217, 0.5934, 0.0640]mg$. The forces in the da and kh cables are close to the forces when a single cable is used, whereas the forces in the ak and $kh2$ cables are lower. There are many other possible equilibria with other forces.

These forces can be produced by many conditions on stiffnesses and open lengths. These choices can affect the stability of the balance.

4.2 Stability of equilibrium

To test the stability of the equilibrium, the stiffness matrix (i.e. the Hessian matrix of the potential energy of the mechanism) must be defined and its positivity

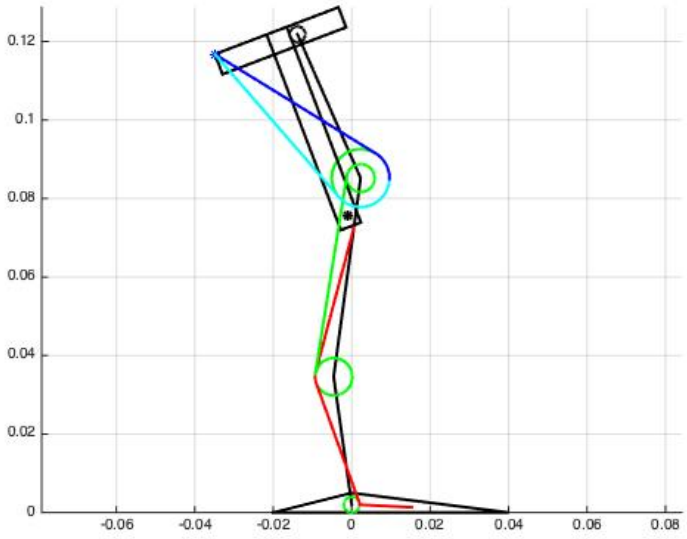

Figure 18: The closest equilibrium position to that shown in Figure 5 is with 4 bio-inspired cables and forces $> 5\%$ of the weight in each cable.

tested.

$$\mathbf{K} = \frac{\partial^2 U_e(q)}{\partial^2 q} \tag{55}$$

The term $K_{i,j}$ of the stiffness matrix is :

$$K_{i,j} = \frac{\partial^2 U_e(q)}{\partial q_i \partial q_j}$$

soit

$$\begin{aligned} K_{i,j} = \quad & mg\frac{\partial^2 y_{COM}(q_e)}{\partial q_i \partial q_j} + \frac{\partial^2 (l_4(q_4))}{\partial q_i \partial q_j} K_{kh}(r_g q_3 + l_4(q_4) - l_{khv}) + \\ & \frac{\partial^2 (l_4(q_4))}{\partial q_i \partial q_j} K_{kh2}(-r_g q_3 + l_4(q_4) - l_{khv}) + \\ & K_{da}\frac{\partial (r_d q_1 - r_c q_2)}{\partial q_i}\frac{\partial (r_d q_1 - r_c q_2)}{\partial q_j} + K_{ak}\frac{\partial (-r_c q_2 - r_{g2} q_3)}{\partial q_i}\frac{\partial (-r_c q_2 - r_{g2} q_3)}{\partial q_j} + \\ & K_{kh}\frac{\partial (r_g q_3 + l_4(q_4))}{\partial q_i}\frac{\partial (r_g q_3 + l_4(q_4))}{\partial q_j} + K_{kh2}\frac{\partial (-r_g q_3 + l_4(q_4))}{\partial q_i}\frac{\partial (-r_g q_3 + l_4(q_4))}{\partial q_j} \end{aligned} \tag{56}$$

This stiffness matrix can be written :

$$\mathbf{K} = \mathbf{K_y} + K_{kh}\mathbf{K_{ll}} + +K_{kh2}\mathbf{K_{ll2}} + K_{da}\mathbf{K_{l_{da}}} + K_{ak}\mathbf{K_{l_{ak}}} + K_{kh}\mathbf{K_{l_{kh}}} + K_{kh2}\mathbf{K_{l_{kh2}}} \tag{57}$$

with $\mathbf{K_y}$, $\mathbf{K_{ll}}$ previously defined, but with $K_{ll4} = \frac{\partial^2 l(q_e)}{\partial q_4 \partial q_4}(r_g q_3 + l_4(q_4) - l_{khv})$ and $\mathbf{K_{ll2}}$ as $\mathbf{K_{ll}}$, but with $K_{ll4} = \frac{\partial^2 l(q_e)}{\partial q_4 \partial q_4}(-r_g q_3 + l_4(q_4) - l_{kh2v})$.

$$\mathbf{K_{l_{da}}} = \begin{bmatrix} r_d^2 & -r_d r_c & 0 & 0 \\ -r_d r_c & r_c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K_{l_{ak}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & r_c^2 & r_c r_{g2} & 0 \\ 0 & r_c r_{g2} & r_{g2}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K_{l_{kh}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_g^2 & r_g dl4 \\ 0 & 0 & r_g dl4 & dl4^2 \end{bmatrix},$$

$$\mathbf{K_{l_{kh2}}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_g^2 & -r_g dl4 \\ 0 & 0 & -r_g dl4 & dl4^2 \end{bmatrix},$$

and $dl4 = \frac{\partial l_4}{q_4}$.

for the previous equilibrium configuration, we look for an identical stiffness value for all the cables which allows us to obtain a positive Hessian matrix and we find : $\mathrm{K}_{da} = K_{ak} = K_{kh} = K_{kh2} > 60000$. A 2 times lower stiffness on $K_{kh} = K_{kh2}$ is also acceptable.

We note that for the stiffness matrix, whereas for a cable, we could show that it was impossible to find a solution so that the minor $\mathbf{K}(1:2, 1:2)$ has a positive determinant, here the fact of cutting the ligament in 2 at the level of the knee makes it possible to increase the contribution of the stiffnesses for the term $\mathbf{K}(2,2)$ by two, and thus to make the equilibrium stable.

We can ask ourselves if this element is preponderant to ensure stability and therefore if it would be possible to have a balance with a bi-articular finger-ankle cable and a tri-articular ankle-knee-hip cable.

pour la configuration d'équilibre précédente, on recherche une valeur de raideur identique pour tous les câbles qui permettent d'obtenir une matrice hessienne positive et on trouve : $K_{da} = K_{ak} = K_{kh} = K_{kh2} > 60000$. Une raideur 2 fois plus faible sur $K_{kh} = K_{kh2}$ est aussi accepatble.

5 Importance of the bi-articular ligament route

A notable difference between the conceptual red line and the practical implementation via muscles is the passage of the ligaments via the joints which are on the same side for the real implementation and crossed for the conceptual red

line. The work developed in Wanda Zhao's thesis and the notion of buckling will be taken up here to compare 2 very implified bird's foot architecture. This work is also related to the master thesis of Navdeep.

We consider an inverted double pendulum that carries a mass, a bi-articular ligament encompasses the 2 joints. Only the finger-ankle-ankle joints are considered and the mass is placed at the hip. The ligament hooks onto the "thigh" and "sole". The equations become :

It is assumed that the cable is extensible. The cable has an empty length l_v and an assumed constant stiffness K.

The potential energy of the system is then written :

$$U_e = mgy_{COM} + \frac{1}{2}K(l(q) - l_v)^2$$

We thus obtain as a static model $\frac{\partial U_e(q)}{\partial q} = 0$:

$$mg\frac{\partial y_{COM}(q)}{\partial q} + \frac{\partial l(q)}{\partial q}K(l(q) - l_v) = 0.$$

A q position is therefore an equilibrium position if we have:

$$mg\frac{\partial y_{COM}(q_e)}{\partial q} + \frac{\partial l(q_e)}{\partial q}K(l(q_e) - l_v) = 0. \tag{58}$$

Or:

$$\begin{bmatrix} -l_pS1 - l_jS12 - l_cS123 \\ -l_jS12 - l_cS123 \\ -l_cS123 \end{bmatrix} + \begin{bmatrix} 0 \\ s_c * r_c \\ r_g \end{bmatrix} \frac{K(l(q_e) - l_v)}{mg} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{59}$$

It is a system of 3 equations with 3 unknowns q. One can search for solutions numerically with fsolve for example. s_c peut valoir 1 ou -1 selon la position du ligament c'est ce que nous étudions ici.

With the elastic cable model, we want to know if when we move the bird away from an equilibrium position it will return to that position. We have an equilibrium position q_e that satisfies :

$$\frac{\partial U_e(q)}{\partial q} = 0 \tag{60}$$

The term $K_{i,j}$ of the stiffness matrix is :

$$K_{i,j} = \frac{\partial^2 U_e(q)}{\partial q_i \partial q_j}$$

soit

$$K_{i,j} = mg\frac{\partial^2 y_{COM}(q_e)}{\partial q_i \partial q_j} + K\frac{\partial l(q_e)}{\partial q_j}\frac{\partial l(q_e)}{\partial q_i} \tag{61}$$

This stiffness matrix can be written :

$$\mathbf{K} = \mathbf{K_y} + K\mathbf{K_l} \tag{62}$$

with

$$\mathbf{K_y} = -\begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_2 & y_3 \\ y_3 & y_3 & y_3 \end{bmatrix},$$

$$\mathbf{K_l} = \begin{bmatrix} r_d^2 & -r_d r_c & r_d r_g \\ -r_d r_c & r_c^2 & s_c r_c r_g \\ r_d r_g & s_c r_c r_g & r_g^2 \end{bmatrix},$$

and $y_1 = l_p C1 + l_j C12 + l_c C123 + l_G C1234$, $y_2 = l_j C12 + l_c C123 + l_G C1234$, $y_3 = l_c C123 + l_G C1234$.

6 Conclusion

The representation proposed with a cable to balance the weight of the bird in resting position is interesting to define this position. It would be necessary to compare the configurations found with configurations observed in birds from the geometric characteristics of the different species.

It can be observed that for the proposed cabling, ligament passing to the left of the finger pulley, the configuration obtained corresponds to a foot close to the vertical. This result was not expected. Wiring with a ligament passing to the right of the finger pulley would make it possible to approach an equilibrium position with one knee in the axis of the fingers.

While the presence of a cable allows the weight of the bird to be balanced, it does not ensure stability of the installation. This result will be obtained whatever the geometrical parameters describing the bird.

It is possible to obtain a stable equilibrium configuration if other ligaments are taken into account. The proposal studied here takes into account an additional ligament at the finger, and ankle joints. Two ligaments have been added here, it is possible that stability can be ensured for different sets of parameters by fewer ligaments or by a set of ligaments that could be bi-articular.

Inspired by the arrangement of the main ligaments on a bird's foot, a study with 4 ligaments 1 bi-articular finger-ankle, 1 bi-articular ankle-knee and 2 bi-articular cruciate knee-hip. In this case a stable equilibrium configuration with fairly stiff ligaments could be obtained It was then shown that the grouping of the ligaments was not appropriate to ensure stability.

It is also possible that a modeling of the joints by tensegrity systems (in X) and not by pivots can avoid the use of these additional ligaments.

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