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Friction mediated by transient elastic linkages : extension to loads of bounded variation

Samar Allouch^{*†} Vuk Milišić^{*‡}

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Abstract

In this work, we are interested in the convergence of a system of integro-differential equations with respect to an asymptotic parameter ε . It appears in the context of cell adhesion modelling [16, 15]. We extend the framework from [12, 13], strongly depending on the hypothesis that the external load f is in $\text{Lip}([0, T])$ to the case where $f \in \text{BV}(0, T)$ only. We show how results presented in [13] naturally extend to this new setting, while only partial results can be obtained following the comparison principle introduced in [12].

1 Introduction

Cell motility plays a central role in several important phenomena in biology : cancer cell migration, neutrophils' extravasation, chemotaxis, etc. The present paper fits in the modelling framework presented in [17, 19, 10]. The adhesive dynamics of actin filaments are at the heart of the project : they contribute to lamellipodium's stabilization and allow the cell to attach to the substrate or the surrounding tissue. This paper contributes to the better mathematical understanding of a minimal model introduced first in [12], its aim is to extend results already obtained in [12, 13] to the case of *stiffer* external loads.

More precisely, we are interested in the motion of a single binding site, linked to a one-dimensional substrate and subjected to an external force f . As in [12, 13], the position of this binding site, denoted z_ε , solves a Volterra integral equation

$$\begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f(t), & t \geq 0, \\ z_\varepsilon(t) = z_p(t), & t < 0. \end{cases} \quad (1)$$

The kernel ρ_ε above solves a non-local age-structured problem :

$$\begin{cases} \varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta_\varepsilon \rho_\varepsilon = 0, & t > 0, a > 0, \\ \rho_\varepsilon(a = 0, t) = \beta_\varepsilon(t) \left(1 - \int_0^\infty \rho_\varepsilon(t, \tilde{a}) d\tilde{a} \right), & t > 0, \\ \rho_\varepsilon(a, t = 0) = \rho_{I, \varepsilon}(a), & a \geq 0, \end{cases} \quad (2)$$

where $\beta_\varepsilon \in \mathbb{R}_+$ (resp. $\zeta_\varepsilon \in \mathbb{R}_+$) is the kinetic on-rate (resp. off-rate) function. These possibly depend on the dimensionless parameter $\varepsilon > 0$. The past positions are stored in the Lipschitz function $z_p(t) \in \mathbb{R}$, prescribed for every $t < 0$.

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Various mathematical issues related to this system have already been investigated [12, 13, 11]. In [12], the authors have introduced a specific Lyapunov functional in order to study the convergence of (2) when ε goes to 0. Indeed, due to the saturation effect in the non-local boundary condition in (2), neither the Generalized Relative Entropy [18, 5] nor more generic comparison principles [6] do apply. Then, concerning (1), under the assumptions that the force f is Lipschitz on \mathbb{R} , and because the kernel ρ_ε in (1) is non-negative, an extension of Gronwall's Lemma to integral equations, shows convergence of z_ε towards z_0 the solution of (3), the limit equation associated to (1). These two steps show that

$$\|z_\varepsilon - z_0\|_{C^0([0,T])} + \|\rho_\varepsilon - \rho_0\|_{C^0([0,T];L^1(\mathbb{R}_+))} \rightarrow 0.$$

where z_0 is given by

$$\begin{cases} \mu_{1,0}(t)\partial_t z_0(t) = f(t), & t > 0 \\ z_0(0) = z_p(0) & t = 0 \end{cases} \quad (3)$$

where $\mu_{1,0}(t) := \int_0^\infty a\rho_0(a,t)da$, and ρ_0 solves :

$$\begin{cases} \partial_a \rho_0 + \zeta_0(a,t)\rho_0 = 0, & t > 0, a > 0, \\ \rho_0(a=0,t) = \beta_0(t) \left(1 - \int_0^\infty \rho_0(t,\tilde{a}) d\tilde{a}\right), & t > 0. \end{cases} \quad (4)$$

In [13], the authors weakened some assumptions concerning the off-rate ζ_ε , by assuming that ζ_ε is not necessarily non-decreasing passed a certain age a_0 . Then, they introduce a new variable u_ε related to z_ε which transforms (1) into a transport problem with a non-local source term :

$$\begin{cases} \varepsilon\partial_t u_\varepsilon + \partial_a u_\varepsilon = \frac{1}{\mu_{0,\varepsilon}(t)} \left(\varepsilon\partial_t f + \int_0^\infty \zeta_\varepsilon(\tilde{a},t)u_\varepsilon(\tilde{a},t)\rho_\varepsilon(\tilde{a},t) d\tilde{a} \right), & t > 0, a > 0, \\ u_\varepsilon(a=0,t) = 0, & t > 0, \\ u_\varepsilon(a,t=0) = u_{I,\varepsilon}(a) := \frac{z_\varepsilon(0) - z_p(-\varepsilon a)}{\varepsilon}, & a \geq 0, \end{cases} \quad (5)$$

where $\mu_{0,\varepsilon}(t) := \int_0^\infty \rho_\varepsilon(\tilde{a},t) d\tilde{a}$ and according to (1), it holds that

$$z_\varepsilon(0) = \frac{1}{\mu_{0,\varepsilon}(0)} \left(\int_0^\infty z_p(-\varepsilon a)\rho_{I,\varepsilon}(a) da + \varepsilon f(0) \right). \quad (6)$$

If $f \in \text{Lip}(\mathbb{R})$, systems (1) and (5) are equivalent. Nevertheless, (5) admits a stability result that allows to show a weak-* convergence of $u_\varepsilon/(1+a)$ towards $u_0/(1+a)$ in $L^\infty(\mathbb{R}_+ \times (0,T))$, where u_0 is the solution of the limit problem

$$\begin{cases} \partial_a u_0 = \frac{1}{\mu_{0,0}} \int_0^\infty \zeta_0 u_0 \rho_0 d\tilde{a}, & t > 0, a > 0, \\ u_0(a=0,t) = 0, & t > 0, \end{cases} \quad (7)$$

which in turn provides the strong convergence of z_ε in $C([0,T])$ towards z_0 solving (3).

In our analysis, however, when $f \in \text{BV}((0,T))$, the derivative of f is neither a function nor it is bounded, since it is a Radon measure. Therefore, we cannot apply directly results from [12]. Instead, defining f_δ to be a specific regularization of f [20, Section 5.3] provides a *regular* function u_ε^δ solving (5). To do this, we use the framework already established in [13]. Then we show that u_ε^δ satisfies certain *a priori* estimates that are uniform with respect to both δ and ε . These provide necessary compactness in order to pass to the limit with the regularization parameter δ and give existence and uniqueness of a weak solution u_ε associated to (5) with a load $f \in \text{BV}((0,T))$. The *a priori* estimates holding also in this weaker framework, we can consider convergence with respect to ε and prove consistency with the formal limit system. We show that, in the BV framework, the equivalence between (5) and (1) still holds. For the particular case when the kernel ρ_ε is independent

on time and on ε and under suitable hypotheses, we show error estimates to be compared with [12], the comparison principle being applied to the integral of the error's modulus.

In order to clarify the interplay between parameters ε and δ , we make the following remarks : in the previous literature [12, 13], not only existence (and uniqueness) but also convergence results were *strongly* related to the Lipschitz regularity of the load f . This motivates the present work since it is not clear that the convergence occurs with respect to ε in this weaker framework. This explains also why we first regularize the problem with the parameter δ , make δ tend to zero and then consider the convergence with respect to ε .

The outline of the paper is as follows : in Section 2, collecting various results from the literature on BV-functions in one space dimension, we introduce the framework used in the rest of the paper. We make the link with the Riemann-Stieltjes integral, through a careful analysis of different definitions of BV-functions with respect to the boundary of the time domain $(0, T)$. In Section 3, we recall some results concerning (2) already established in [12]. Then in Section 4, we establish uniform (with respect to ε) *a priori* estimates for the regularized system in u_ε . After that, in Section 5, we show the weak convergence of u_ε towards u_0 , the solution of the limit problem. This implies strong convergence of z_ε in $L^\infty(0, T)$ as stated in Theorem 8. We establish, in Section ??, a specific comparison principle for Volterra equations when the density ϱ is constant in time and does not depend on ε .

2 Notations and main assumptions

We denote $L_t^p L_a^q := L^p((0, T); L^q(\mathbb{R}_+))$ for any real $(p, q) \in [1, \infty]^2$ and

$$X_T := \left\{ g \in L_{loc}^\infty((0, T) \times \mathbb{R}_+) ; \sup_{t \in (0, T)} \|g(t, a)w(a)\|_{L_a^\infty} < \infty \right\} \quad (8)$$

where $w(a) := (1 + a)^{-1}$. The space $\text{Lip}(I)$ is the set of Lipschitz functions on the interval I .

Assumptions 1. For any $T > 0$ possibly infinite, we assume that :

i) The past condition z_p is L_{z_p} -Lipschitz on \mathbb{R}_- *i.e.* :

$$|z_p(a_2) - z_p(a_1)| \leq L_{z_p} |a_2 - a_1|, \quad \forall (a_2, a_1) \in \mathbb{R}_- \times \mathbb{R}_-.$$

ii) The function $\beta_\varepsilon(t)$ is in $L^\infty(0, T)$ and $\zeta_\varepsilon(a, t)$ is in $L^\infty(\mathbb{R}_+ \times (0, T))$.

iii) For limit functions $\beta_0 \in L_t^\infty$ and $\zeta_0 \in L_t^\infty L_a^\infty$ it holds that

$$\|\zeta_\varepsilon - \zeta_0\|_{L_{a,t}^\infty} \rightarrow 0 \quad \text{and} \quad \|\beta_\varepsilon - \beta_0\|_{L_t^\infty} \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

iv) There are upper and lower bounds such that

$$0 < \zeta_{\min} \leq \zeta_\varepsilon(a, t) \leq \zeta_{\max} \quad \text{and} \quad \beta_{\min} \leq \beta_\varepsilon(a, t) \leq \beta_{\max},$$

for all $\varepsilon > 0$, $a \geq 0$ and $t > 0$.

Assumptions 2. The initial condition $\rho_{I,\varepsilon} \in L_a^\infty(\mathbb{R}_+)$ satisfies

i) positivity

$$\rho_{I,\varepsilon}(a) \geq 0, \quad \text{a.e. in } \mathbb{R}_+,$$

moreover, one has also that the total initial population satisfies

$$0 < \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a) da < 1;$$

ii) boundedness of higher moments,

$$0 < \int_{\mathbb{R}_+} a^p \rho_{I,\varepsilon}(a) da < c_p, \quad \text{for } p = 1, 2,$$

where c_p are positive constants depending only on p ;

Next, we introduce definitions of functions with bounded variation in one dimension, as well as some related properties.

Definition 1. Let $f : (0, T) \rightarrow \mathbb{R}$ be a Lebesgue measurable function. The pointwise variation (or Jordan variation) of f on $(0, T)$ is

$$\text{pvar}(f, (0, T)) := \sup_P \text{var}(f, P) \quad (9)$$

where $\text{var}(f, P) := \sum_{k=1}^n |f(t_k) - f(t_{k-1})|$ and $P = \{0 < t_0 < \dots < t_n < T\}$ is a partition of $(0, T)$.

Moreover, we denote $\text{BPV}((0, T)) := \{f \in \mathcal{L}((0, T)), \text{ s.t. } \text{pvar}(f, (0, T)) < +\infty\}$, the space of measurable functions with pointwise bounded variation, see for example, [1, section 3.2], [9, chapter 2] and [7, section 2.2, 2.3]. The pointwise variation of f is clearly dependent on the value of f at each point of the domain, and it differs from one a.e.-representative of f to another. For this reason, for every measurable function f , one defines the essential pointwise variation :

$$\text{epvar}(f, (0, T)) := \inf \{\text{pvar}(g, (0, T)) : f(t) = g(t) \text{ a.e. } t \in (0, T)\} \quad (10)$$

In [9, Chapter 6], another functional space is defined :

Definition 2. Given an open interval $(0, T) \subset \mathbb{R}$, the space of functions with bounded variation $\text{BV}((0, T))$ is defined as the space of all functions $f \in L^1((0, T))$ for which there exists a signed Radon measure μ_f such that

$$\int_{(0, T)} f \phi' dt = - \int_{(0, T)} \phi d\mu_f, \text{ for every } \phi \in C_c^1((0, T)) \quad (11)$$

for all $\phi \in C_c^1((0, T))$. The measure μ_f is called the weak or distributional derivative of f .

Remark 1.

i) We define the total variation of $f \in L^1((0, T))$ by

$$\|Df\|((0, T)) = \sup \left\{ - \int_{(0, T)} f \phi' dt, \phi \in C_c^1((0, T)), |\phi|_\infty \leq 1 \right\}. \quad (12)$$

Moreover, $f \in \text{BV}((0, T))$ if $\|Df\|((0, T)) < +\infty$.

ii) Definitions (9) and (12) are not equivalent. For instance, the Dirichlet indicatrix function $\chi_{\mathbb{Q} \cap [0, 1]}$ is not of pointwise bounded variation in $(0, 1)$ in the sense of Definition 1 but is well defined in the sense of Definition 2. The equivalence between the two definitions holds up to a.e. equality. Moreover every integrable function $f : (0, T) \rightarrow \mathbb{R}$ such that $\text{pvar}(f, (0, T)) < +\infty$, is in $\text{BV}((0, T))$ and $\|Df\|((0, T)) \leq \text{pvar}(f, (0, T))$. On the other hand, if f belongs to $\text{BV}((0, T))$, then f admits a right continuous representative \bar{f} with bounded pointwise variation such that

$$\text{pvar}(\bar{f}, (0, T)) = \|Df\|((0, T)).$$

For more details, see, e.g., [9, theorem 7.3] and [7].

iii) Under the norm

$$\|f\|_{\text{BV}} := \|f\|_{L^1} + \text{epvar}(f, (0, T)) < \infty$$

$\text{BV}((0, T))$ is a Banach space.

Next, we provide existence of the left and right limits of functions with bounded variation [7, Proposition 2.2].

Lemma 1. Let $f \in \text{BV}((0, T))$, Then both the limits

$$f(0^+) = \lim_{s \rightarrow 0, s > 0} f(s) \quad \text{and} \quad f(T^-) = \lim_{s \rightarrow T, s < T} f(s) \quad \text{exist.}$$

Additionally, if f is integrable, the left and right limits are as follows:

Lemma 2. Suppose that $f \in \text{BV}((0, T))$, then

$$f(0^+) = \lim_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_0^\rho f(t) dt, \quad f(T^-) = \lim_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_{T-\rho}^T f(t) dt.$$

Next, we present a result used in the proof of Proposition 3, which relates the pointwise variation to the Lebesgue measure:

$$\lambda(f, h, \Omega) := \int_{\{t \in \Omega : t+h \in \Omega\}} |f(t+h) - f(t)| dt,$$

Lemma 3. If f is in $\text{BPV}((0, T))$, then $\lambda(f, h, (0, T))/|h|$ is bounded. Moreover,

$$\lambda(f, h, (0, T)) \leq |h| \text{pvar}(f, (0, T)).$$

For the proof we can see [9, Theorem 2.20].

Finally, in [7], the authors add a new notion of variation containing the boundary value in order to expand the total variation of f to $[0, T]$. This variation is defined as

$$\text{varw}(f) := \sup_{\substack{\phi \in C_c^1((0, T)) \\ |\phi|_\infty \leq 1}} \left\{ \phi(T)f(T^-) - \phi(0)f(0^+) - \int_{(0, T)} f \phi' dt \right\} \quad (13)$$

Moreover, by summarizing the results of [7, Proposition 2.3, 2.6 and 2.7] all notions of variations coincide:

$$\text{epvar}(f, (0, T)) = \|Df\|((0, T)) = \text{varw}(f)$$

The previous result allows to extend Lemma 3 to $\text{BV}((0, T))$ functions :

Lemma 4. If f is in $\text{BV}((0, T))$, then $\lambda(f, h, (0, T))/|h|$ is bounded. Moreover,

$$\lambda(f, h, (0, T)) \leq |h| \|Df\|((0, T))$$

PROOF. By taking the infimum over almost every equal measurable functions, one has

$$\inf_{f=\tilde{f} \text{ a.e.}} \lambda(\tilde{f}, h, (0, T)) \leq |h| \inf_{f=\tilde{f} \text{ a.e.}} \text{pvar}(\tilde{f}, (0, T)) = |h| \text{epvar}(f, (0, T)) = |h| \|Df\|((0, T))$$

Since the left hand side is a Lebesgue integral one has :

$$\inf_{f=\tilde{f} \text{ a.e.}} \lambda(\tilde{f}, h, (0, T)) = \lambda(f, h, (0, T))$$

which ends the proof. □

2.1 Data regularization

Theorem 1. For every $f \in \text{BV}((0, T))$, there exists a sequence of smooth functions $(f_\delta)_\delta$ in $C^\infty((0, T))$ such that

$$\lim_{\delta \rightarrow 0} \int_{(0, T)} |f_\delta - f| dt = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \int_{(0, T)} |f'_\delta| dt = \|Df\|((0, T)).$$

Although the proof is classical (see for instance [20, Theorem 5.3.3 p.225]), we need the explicit form of f_δ in the rest of the paper. For this reason, we present in Section A.1 the proof of Theorem 1.

Lemma 5. Let $f \in \text{BV}((0, T)) \cap L^\infty((0, T))$. Then the regularization function f_δ defined as (60) is bounded in $(0, T)$.

Next, we compare the left and right limits of f and its' approximation f_δ on the boundary:

Lemma 6. Let $f \in \text{BV}((0, T))$ and f_δ defined as (60), then

$$f_\delta(0^+) = f(0^+) \quad \text{and} \quad f_\delta(T^-) = f(T^-).$$

First we need the following result :

Proposition 1. Let $f \in \text{BV}((0, T))$. For every $\delta > 0$, and $t_0 \in \{0, T\}$,

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt = 0, \quad (14)$$

where $I_\tau = \{t \in \mathbb{R} : |t - t_0| < \tau\}$.

PROOF. For a fixed $t_0 \in \{0, T\}$ and $t \in I_\tau \in (0, T)$, we have

$$f_\delta(t) - f(t) = \sum_{i=0}^{\infty} [\chi_{\delta_i} * (\phi_i f) - \phi_i f]$$

by the definition of $\text{supp } \phi_i$ (see (56)), we have $1/(j_0 + i + 1) < \tau < 1/(j_0 + i - 1)$ then $i > 1/\tau - j_0 - 1$. Since, \mathbb{R} is an archimedean space then

$$\forall \tau > 0, \exists! i_0 := \left\lfloor \frac{1}{\tau} \right\rfloor - j_0 \text{ s.t } i_0 \leq \frac{1}{\tau} < i_0 + 1 \quad (15)$$

which implies by using (58) that

$$\begin{aligned} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt &= \sum_{i=i_0}^{\infty} \int_{I_\tau \cap (0, T)} [\chi_{\delta_i} * (\phi_i f) - \phi_i f] dt \leq \sum_{i=i_0}^{\infty} \delta 2^{-i} \\ &\leq \delta 2^{-i_0} \sum_{i=0}^{\infty} 2^{-i} = 2^{j_0+1} \delta 2^{-\lfloor \frac{1}{\tau} \rfloor} \quad \square \end{aligned}$$

then

$$\frac{1}{\tau} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt \leq C \delta \frac{2^{-\lfloor \frac{1}{\tau} \rfloor}}{\tau}$$

using again (15), we have

$$\frac{2^{-\lfloor \frac{1}{\tau} \rfloor}}{\tau} = \frac{\exp(-\lfloor \frac{1}{\tau} \rfloor \ln 2)}{\tau} \leq \frac{2 \exp(-\frac{1}{\tau} \ln 2)}{\tau}$$

Finally, we conclude that

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt = \lim_{\tau \rightarrow 0^+} \frac{2 \exp(-\frac{1}{\tau} \ln 2)}{\tau} = 0.$$

PROOF OF LEMMA 6. According to the Lemma 2,

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_0^\tau |f_\delta(t) - f_\delta(0^+)| dt = 0 \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_0^\tau |f(t) - f(0^+)| dt = 0.$$

Moreover, we have, thanks to Proposition 1

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_0^\tau |f_\delta - f| dt = 0.$$

Thus, for all $\varepsilon' > 0$, there exist $\delta' > 0$ such that $0 < \tau < \delta'$ implies

$$\begin{aligned} |f_\delta(0^+) - f(0^+)| &= \frac{1}{\tau} \int_0^\tau |f_\delta(0^+) - f(0^+)| dt \\ &\leq \frac{1}{\tau} \int_0^\tau |f_\delta(0^+) - f_\delta(t)| dt + \frac{1}{\tau} \int_0^\tau |f_\delta(t) - f(t)| dt + \frac{1}{\tau} \int_0^\tau |f(t) - f(0^+)| dt \leq 3\varepsilon' \end{aligned}$$

which proves the required result. Similarly, we can prove that $f_\delta(T^-) = f(T^-)$. \square

In the previous setting, the weak derivative of $f \in \text{BV}((0, T))$ defines a linear continuous form on $C((0, T))$. In the next section, we show how to *extend* this measure on functions in $C([0, T])$.

2.2 Definition and basic properties of Stieltjes integral

The Riemann–Stieltjes integral (RS–integral) is a generalization of the Riemann integral. Let \mathcal{P} a tagged partition of $[0, T]$, defined as

$$\mathcal{P} := \{(\xi_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\} \quad (16)$$

where $0 = t_1 \leq \dots \leq t_n = T$, and on each interval $[t_{i-1}, t_i]$ we choose a single value ξ_i , for $i \in \{1, \dots, n\}$.

Definition 3. For any function $f, g : [0, T] \rightarrow \mathbb{R}$ and a partition \mathcal{P} , we define the Riemann–Stieltjes sum by

$$S(f, dg, \mathcal{P}, [0, T]) := \sum_i f(\xi_i)[g(t_i) - g(t_{i-1})].$$

Moreover, the RS-integral of f with respect to g

$$(RS) \int_{[0, T]} f(t) dg(t)$$

exists and has a value $I \in \mathbb{R}$, if, for every $\varepsilon > 0$, there exists $\delta > 0$, such that the mesh size $\max_i(t_i - t_{i-1}) < \delta$ and for every ξ_i in $[t_i, t_{i+1}]$,

$$|S(f, dg, \mathcal{P}, [0, T]) - I| < \varepsilon.$$

Lemma 7. Suppose that f is continuous on $[0, T]$ and g is of bounded pointwise variation on $[0, T]$, then

$$\left| \int_{[0, T]} f dg \right| \leq \|f\|_\infty \text{pvar}(g, [0, T])$$

In the following Theorem we see that a Riemann–Stieltjes integral can be used to describe any bounded linear functional on $C([0, T])$ (see [3, Theorem 7.1.1] and [8, Theorem 4.4-1] for more details)

Theorem 2. Let $\Gamma_f \in (C([0, T]))'$, then there exist $g \in BPV([0, T])$ such that

$$\Gamma_f(\varphi) = \int_{[0, T]} \varphi dg, \quad \forall \varphi \in C([0, T]).$$

Theorem 3. (Integration by parts). If one of the integrals $\int_{[0, T]} f dg$ and $\int_{[0, T]} g df$ exists, then the other exists as well, and we have

$$\int_{[0, T]} f dg + \int_{[0, T]} g df = [fg(t)]_{t=0}^{t=T}.$$

Moreover, If $f \in C^1([0, T])$ and $g \in BPV([0, T])$, then $df = f' dt$ in the second term of the left hand side.

For the proof cf [14, Theorem 5.52] and [2, Lemma 2].

Lemma 8. Let $f \in \text{BV}((0, T))$. Then there exists $g \in \text{BPV}([0, T])$ s.t

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0, T)} f\varphi' dx = \int_{[0, T]} \varphi dg, \quad \forall \varphi \in C^1([0, T])$$

s.t. $f(t) = g(t)$, a.e. $t \in (0, T)$

PROOF. We regularize $f \in \text{BV}((0, T))$ by $f_\delta \in C^\infty((0, T))$ as in Theorem 1, then we have :

$$\int_{t_k}^{s_k} f'_\delta \varphi dt + \int_{t_k}^{s_k} f_\delta \varphi' dt = [f_\delta \varphi]_{t=t_k}^{t=s_k} =: L_k, \quad \forall \varphi \in C^1([0, T])$$

where $s_k \rightarrow T^-$ and $t_k \rightarrow 0^+$. We define :

$$I_k := \int_{t_k}^{s_k} f'_\delta \varphi dt, \quad J_k := \int_{t_k}^{s_k} f_\delta \varphi' dt.$$

Thanks to Lebesgue's Theorem, one has that

$$\lim_{k \rightarrow \infty} I_k = I := \int_0^T f'_\delta \varphi dt, \quad \lim_{k \rightarrow \infty} J_k = J := \int_0^T f_\delta \varphi' dt$$

and thanks to Lemma 6 and the continuity of φ ,

$$L_k = L := f(T^-)\varphi(T) - f(0^+)\varphi(0), \quad \forall k \in \mathbb{N}$$

So that we have :

$$\int_0^T f'_\delta \varphi dt + \int_0^T f_\delta \varphi' dt = [f\varphi]_{t=0^+}^{t=T^-}$$

If we set

$$\mathcal{I}_{f_\delta}(\varphi) := \int_0^T f'_\delta \varphi dt,$$

it is a linear continuous form on $C([0, T])$, since one has :

$$|\mathcal{I}_{f_\delta}(\varphi)| \leq \|f'_\delta\|_{L^1((0, T))} \|\varphi\|_{L^\infty} \leq \{\|Df\|((0, T)) + \delta\} \|\varphi\|_{L^\infty((0, T))} \quad (17)$$

where we used estimates from the proof of [20, Theorem 5.3.3]. Since it is a continuous linear form on $C([0, T])$, by Theorem 2 there exists $h_\delta \in \text{BPV}([0, T])$ s.t.

$$\mathcal{I}_{f_\delta}(\varphi) = \int_{[0, T]} \varphi dh_\delta, \quad \forall \varphi \in C([0, T])$$

in the Stieljes' sense. But by using the integration by parts from Theorem 3, we have that

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0, T)} f_\delta \varphi' dt = \int_{[0, T]} \varphi dh_\delta = [h_\delta \varphi]_{t=0}^{t=T} - \int_{[0, T]} h_\delta \varphi' dt, \quad \forall \varphi \in C^1([0, T]) \quad (18)$$

which implies that

$$\int_{(0, T)} f_\delta \varphi' dt = \int_{(0, T)} h_\delta \varphi' dt, \quad \forall \varphi \in \mathcal{D}((0, T))$$

and then we can apply [9, Lemma 7.4] and conclude that there exists $c \in \mathbb{R}$, s.t.

$$f_\delta = h_\delta + c.$$

Then setting $g_\delta := h_\delta + c$ provides a function s.t.

$$[f_\delta \varphi]_0^T - \int_0^T f_\delta \varphi' dt = \int_0^T \varphi dg_\delta, \quad \forall \varphi \in C^1([0, T])$$

and s.t.

$$f_\delta(t) = g_\delta(t), \quad \text{a.e. } t \in (0, T).$$

Thanks to (17), \mathcal{I}_{f_δ} is a linear continuous form on $C([0, T])$ uniformly bounded with respect to δ . It can be identified via the Riesz representation theorem as a Radon measure μ_δ on $[0, T]$. Therefore, there exist $\mu \in M^1([0, T])$ and a sub-sequence μ_{δ_k} such that

$$\mu_{\delta_k} \xrightarrow{*} \mu$$

in $\sigma(M^1([0, T]), C([0, T]))$ with respect to the weak-* topology. By Theorem 2, there exists $h \in \text{BPV}([0, T])$ s.t.

$$\mu(\varphi) = \int_0^T \varphi dh$$

where the left side is a Radon measure and the right hand side is the Riemann-Stieltjes integral. Because f_δ tends to f in the $L^1(0, T)$ topology, one has then that

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0, T)} f\varphi' dt = \int_{[0, T]} \varphi dh, \quad \forall \varphi \in C^1([0, T])$$

then using again integration by parts from Theorem 3, one concludes that

$$f(t) = h(t) + \tilde{c}, \quad \text{a.e. } t \in (0, T)$$

and setting $g = h + \tilde{c}$ ends the proof. \square

Corollary 1. There exists a sub-sequence $(f_{\delta_k})_{k \in \mathbb{N}}$, s.t.

$$I_{f_{\delta_k}}(\varphi) := \int_0^T \varphi f_{\delta_k}' dt \rightarrow \int_0^T \varphi dg, \quad \forall \varphi \in C([0, T])$$

when $k \rightarrow \infty$

3 Mathematical background for the linkages' density

We list here some of the results proved in [12] used in the next sections of the paper.

Theorem 4. Let Assumptions 1 and 2 hold, then for every fixed $\varepsilon > 0$ there is a unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of the problem (2). It satisfies (2) in the sense of characteristics, namely

$$\rho_\varepsilon(a, t) = \begin{cases} \beta_\varepsilon(t - \varepsilon a) \left(1 - \int_{\mathbb{R}_+} \rho_\varepsilon(\tilde{a}, t - \varepsilon a) d\tilde{a}\right) \exp\left(-\int_0^a \zeta_\varepsilon(\tilde{a}, t - \varepsilon(a - \tilde{a})) d\tilde{a}\right), & \forall a < \frac{t}{\varepsilon} \\ \rho_{I, \varepsilon}\left(a - \frac{t}{\varepsilon}\right) \exp\left(-\frac{1}{\varepsilon} \int_0^t \zeta_\varepsilon\left(\frac{\tilde{t}-t}{\varepsilon} + a, \tilde{t}\right) d\tilde{t}\right), & \forall a \geq \frac{t}{\varepsilon} \end{cases} \quad (19)$$

Moreover, it is a weak solution as well since it satisfies (cf [12, Lemma 2.1])

$$\begin{aligned} & \int_0^T \int_0^{+\infty} \rho_\varepsilon(a, t) (\varepsilon \partial_t \varphi + \partial_a \varphi - \zeta_\varepsilon \varphi) da dt - \varepsilon \int_0^{+\infty} \rho_\varepsilon(a, T) \varphi(a, T) da \\ & + \int_0^T \rho_\varepsilon(a=0, t) \varphi(a=0, t) dt + \varepsilon \int_0^{+\infty} \rho_{I, \varepsilon}(a) \varphi(a, t=0) da = 0 \end{aligned} \quad (20)$$

for every $T > 0$ and test function $\varphi \in C^\infty(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$. Now we define the moments of ρ_ε which we denote by $\mu_{p, \varepsilon}(t) := \int_{\mathbb{R}_+} a^p \rho_\varepsilon(a, t) da$, with $p = 1, 2$.

Lemma 9. Let Assumptions 1 and 2 hold, then the unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of (2) satisfies

- i) $\rho_\varepsilon(a, t) \geq 0$ for a.e (a, t) in \mathbb{R}_+^2 ,
- ii) $\mu_{0, \min} \leq \mu_{0, \varepsilon}(t) < 1, \forall t \in \mathbb{R}_+$ where $\mu_{0, \min} < \min\left(\mu_{0, \varepsilon}(0), \frac{\beta_{\min}}{\beta_{\min} + \zeta_{\max}}\right)$
- iii) $\mu_{p, \min} \leq \mu_{p, \varepsilon}(t) \leq k$, where $\mu_{p, \min} = \min\left(\mu_{p, \varepsilon}(0), \frac{\mu_{p-1, \min}}{\zeta_{\max}}\right)$

The authors provide a Liapunov functional that reads :

$$\mathcal{H}[u] := \left| \int_0^\infty u(a) da \right| + \int_0^\infty |u(a)| da ,$$

thanks to which they obtain the following convergence result for ρ_ε :

Lemma 10. Let $\zeta_{\min} > 0$ be the lower bound to $\zeta_\varepsilon(a, t)$ according to Assumptions 1, and setting $\hat{\rho}_\varepsilon := \text{one}$ has

$$\mathcal{H}[(\rho_\varepsilon - \rho_0)(\cdot, t)] \leq \mathcal{H}[\rho_{I, \varepsilon} - \rho_0(\cdot, 0)] \exp\left(\frac{-\zeta_{\min} t}{\varepsilon}\right) + \frac{2}{\zeta_{\min}} \left(\|R_\varepsilon\|_{L_a^1(\mathbb{R}_+)} + \|M_\varepsilon\|_{L_t^\infty(\mathbb{R}_+)} \right) \quad (21)$$

with $R_\varepsilon := -\varepsilon \partial_t \rho_0 - \rho_0(\zeta_\varepsilon - \zeta_0)$ and $M_\varepsilon := (\beta_\varepsilon - \beta_0)(1 - \int_0^\infty \rho_0 da)$.

This ensures the convergence of ρ_ε that reads :

Theorem 5. Let ρ_ε the solution of the system (2) and let ρ_0 given by (4), then

$$\rho_\varepsilon \rightarrow \rho_0 \text{ in } C^0((0, \infty); L^1(\mathbb{R}_+)) \text{ as } \varepsilon \rightarrow 0 ,$$

where the convergence with respect to time is meant in the sense of uniform convergence on compact subintervals.

4 Existence, uniqueness and stability

Using the regularized function f_δ introduced in Theorem 1, we consider an approximation of (1) : we denote by $z_\varepsilon^\delta := z_\varepsilon^\delta(t)$ the function solving

$$\begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon^\delta(t) - z_\varepsilon^\delta(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f_\delta(t), & t \leq 0 \\ z_\varepsilon^\delta(t) = z_p(t), & t < 0 \end{cases} \quad (22)$$

We also define $u_\varepsilon^\delta(a, t)$, an approximation of the elongation variable u_ε , defined as the mild solution of

$$\begin{cases} \varepsilon \partial_t u_\varepsilon^\delta + \partial_a u_\varepsilon^\delta = \frac{1}{\mu_{0,\varepsilon}} \left(\varepsilon f'_\delta + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right), & t > 0, a > 0 \\ u_\varepsilon^\delta(a = 0, t) = 0, & t > 0 \\ u_\varepsilon^\delta(a, t = 0) = u_{I,\varepsilon}^\delta(a), & a \geq 0 \end{cases} \quad (23)$$

where

$$u_{I,\varepsilon}^\delta(a) := \frac{z_\varepsilon^\delta(0^+) - z_p(-\varepsilon a)}{\varepsilon} \quad (24)$$

and

$$z_\varepsilon^\delta(0^+) = \frac{1}{\mu_{0,\varepsilon}(0)} \left(\varepsilon f_\delta(0^+) + \int_0^\infty z_p(-\varepsilon a) \rho_{I,\varepsilon}(a) da \right). \quad (25)$$

More precisely, u_ε^δ is a solution of system (23) in the sense of characteristics, namely

$$u_\varepsilon^\delta(a, t) = \begin{cases} \int_0^a h(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t > \varepsilon a \\ \int_0^{t/\varepsilon} h(t - \varepsilon \tilde{a}) d\tilde{a} + u_{I,\varepsilon}^\delta(a - t/\varepsilon), & \text{if } t \leq \varepsilon a, \end{cases} \quad (26)$$

where

$$h(t) := \frac{1}{\mu_{0,\varepsilon}} \left(\varepsilon \partial_t f_\delta + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right).$$

By arguments similar to [12, Lemma 3], it is as well a weak solution of (23) *i.e.*

$$\begin{aligned} & - \int_0^T \int_0^\infty u_\varepsilon^\delta (\varepsilon \partial_t \varphi + \partial_a \varphi) da dt + \left[\int_0^\infty u_\varepsilon^\delta(s, a) \varphi(s, a) da \right]_{s=0}^{s=T} \\ & = \int_0^T \frac{1}{\mu_{0,\varepsilon}} \left(\varepsilon f'_\delta + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right) \left(\int_0^\infty \varphi(t, \tilde{a}) d\tilde{a} \right) dt \end{aligned} \quad (27)$$

for any function $\varphi \in C_c^\infty([0, T] \times \mathbb{R}_+)$. Although, problem (22) can be defined for weaker data (typically $L^1((0, T))$ or the space of Radon measures $M((0, T))$), the elongation problem (23), requires to give a meaning to the time derivative of f , which is more restrictive. Nevertheless, as we are mainly interested in convergence results, $f \in \text{BV}((0, T))$ seems the weakest possible regularity to our knowledge.

Theorem 6. Let Assumptions 1 hold, and let ρ_ε be the unique solution of (2) then the system (23) has a unique solution $u_\varepsilon^\delta \in X_T$.

We are in the framework of [13, Theorem 6.1], but for sake of self-containness, we recall in an abridged version the proof hereafter.

PROOF. A Banach fixed point Theorem is used to prove this result. We define the mapping $\phi(v) = u$ such that by Duhamel's principle

$$u(a, t) = \begin{cases} \int_0^a G(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t > \varepsilon a \\ \int_0^{t/\varepsilon} G(t - \varepsilon \tilde{a}) d\tilde{a} + u_{I,\varepsilon}^\delta(a - t/\varepsilon), & \text{if } t \leq \varepsilon a. \end{cases} \quad (28)$$

where

$$G(t) := \frac{1}{\mu_{0,\varepsilon}} \left(\varepsilon f'_\delta + \int_0^\infty \zeta_\varepsilon(a, t) v(a, t) \rho_\varepsilon(a, t) da \right)$$

As in [13], a simple computation shows that

$$\|u\|_{X_T} \leq \|G\|_{L^\infty((0, T))} \frac{T}{T + \varepsilon} + \left\| \frac{u_{I,\varepsilon}^\delta(\cdot)}{1 + a} \right\|_{L^\infty(\mathbb{R}_+)}$$

Moreover, since $\partial_t f_\delta \in L^\infty((0, T))$

$$\|G\|_{L^\infty((0, T))} \leq \frac{1}{\mu_{0, \min}} \varepsilon \|f'_\delta\|_{L^\infty((0, T))} + \frac{\zeta_{\max}(1+k)}{\mu_{0, \min}} \|v\|_{X_T}$$

where k is the upper bound of $\mu_{1, \varepsilon}$ proved in Lemma 9. Furthermore, by the same argument we can prove that ϕ is a contraction. Indeed, if $u_i = \phi(v_i)$ for $i \in \{1, 2\}$

$$\|u_2 - u_1\|_{X_T} \leq C \frac{T}{T + \varepsilon} \|v_2 - v_1\|_{X_T}$$

for a constant $C > 0$. Then we can choose $T < \varepsilon/C$ and we obtain the existence of a local solution in time of (23), by Banach-Picard's fixed point theorem. As the contraction time does not depend on the initial data, we shall extend the same result by continuation. This shows existence and uniqueness in X_T for any $T > 0$. \square

Lemma 11. If Assumptions 1 holds, then the solution of system (23) satisfies the uniform a priori estimates

$$\begin{aligned} \int_0^\infty \rho_\varepsilon(a, t) |u_\varepsilon^\delta(a, t)| da &\leq \int_0^t |f'_\delta| d\tilde{t} + \int_{\mathbb{R}_+} \rho_{I, \varepsilon} |u_{I, \varepsilon}^\delta| da \\ &\leq C \left(\|f\|_{\text{BV}((0, T))}, \|(1+a)\rho_I\|_{L^1(\mathbb{R}_+)}, \|z'_p\|_{L^\infty(\mathbb{R}_-)} \right) \end{aligned} \quad (29)$$

where C is independent on ε and on δ .

PROOF. Again, we proceed as in [13, Lemma 5.1], multiplying (23) by $\text{sgn}(u_\varepsilon^\delta)$, testing against ρ_ε , and integrating with respect to a gives :

$$\frac{d}{dt} \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \rho_\varepsilon da + \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \zeta_\varepsilon \rho_\varepsilon da \leq \varepsilon |f'_\delta| + \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \zeta_\varepsilon \rho_\varepsilon da$$

the rigorous proof relies on arguments exposed in [12, Lemma 3.1] and is left to the reader. Finally, after integration with respect to time, we conclude that

$$\int_0^\infty \rho_\varepsilon(a, t) |u_\varepsilon^\delta(a, t)| da \leq \int_0^t |f'_\delta| dt + \int_{\mathbb{R}_+} \rho_{I, \varepsilon} |u_{I, \varepsilon}^\delta| da$$

since

$$\begin{aligned} |u_{I, \varepsilon}^\delta(a)| &\leq \left| \frac{z_\varepsilon^\delta(0^+) - z_p(0)}{\varepsilon} \right| + \left| \frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon} \right| \\ &\leq \frac{1}{\mu_{0, \varepsilon}(0)} \left| f_\delta(0^+) + \frac{1}{\varepsilon} \int_0^\infty (z_p(-\varepsilon a) - z_p(0)) \rho_{I, \varepsilon}(a) da \right| + \left| \frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon} \right| \\ &\leq \frac{1}{\mu_{0, \min}} \left(\|z'_p\|_{L^\infty(\mathbb{R}_-)} \mu_{1, \varepsilon}(0) + f_\delta(0^+) \right) + \|z'_p\|_{L^\infty(\mathbb{R}_-)} a \\ &\leq \max \left\{ \frac{1}{\mu_{0, \min}} \|z'_p\|_{L^\infty(\mathbb{R}_-)} \mu_{1, \varepsilon}(0) + f_\delta(0^+), \|z'_p\|_{L^\infty(\mathbb{R}_-)} \right\} (1+a), \end{aligned}$$

the result follows. \square

In order to establish the convergence of u_ε^δ in X_T , for a fixed ε , we introduce an intermediate variable w defined as

$$w(a, t) := u_\varepsilon^\delta(a, t) - \frac{f_\delta(t)}{\mu_{0, \varepsilon}(t)}. \quad (30)$$

It satisfies

$$\begin{cases} \varepsilon \partial_t w + \partial_a w = \frac{1}{\mu_{0,\varepsilon}} \left(\varepsilon \frac{f_\delta \partial_t \mu_{0,\varepsilon}}{\mu_{0,\varepsilon}} + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right), & t > 0, a > 0, \\ w(a=0, t) = \frac{-f_\delta(t)}{\mu_{0,\varepsilon}(t)}, & t > 0, \\ w(a, t=0^+) = u_{I,\varepsilon}^\delta(a) - \frac{f_\delta(0^+)}{\mu_{0,\varepsilon}(0)}, & a \geq 0, \end{cases} \quad (31)$$

The following crucial result holds:

Lemma 12. For a fixed δ and ε , and under the Assumptions 1, the unknowns w and u_ε^δ , are uniformly bounded in X_T with respect to δ and ε .

PROOF. Using arguments from [12, Lemma 2.1], one can show that w defined as

$$w(a, t) := \begin{cases} w(0, t - \varepsilon a) + \int_0^a G_w(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t > \varepsilon a \\ w(a - t/\varepsilon, 0^+) + \int_0^{t/\varepsilon} G_w(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t \leq \varepsilon a. \end{cases} \quad (32)$$

is a weak solution of (31). In the latter definition $G_w(t) := \left\{ \varepsilon \frac{f_\delta \partial_t \mu_{0,\varepsilon}}{\mu_{0,\varepsilon}^2} + \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon^\delta da \right\}$. A simple computation shows that

$$\|w\|_{X_T} \leq \|G_w\|_{L^\infty((0,T))} + \|w(0, \cdot)\|_{L^\infty((0,T))} + \left\| \frac{w(\cdot, 0)}{1+a} \right\|_{L^\infty(\mathbb{R}_+)}.$$

It remains to estimate $\|G_w\|_{L^\infty((0,T))}$. For every fixed ε , $\mu_{0,\varepsilon}$ is a Lipschitz continuous function. Indeed, $\mu_{0,\varepsilon}$ satisfies

$$\varepsilon \partial_t \mu_{0,\varepsilon} - \beta_\varepsilon (1 - \mu_{0,\varepsilon}) + \int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon da = 0$$

and then

$$\|\varepsilon \partial_t \mu_{0,\varepsilon}\|_{L_t^\infty} \leq \|\beta_\varepsilon\|_{L_t^\infty} + \|\zeta_\varepsilon\|_{L_{t,a}^\infty} \|\rho_\varepsilon\|_{L_t^\infty L_a^1},$$

which shows, by using the result of Lemma 11 that G_w , and also w , are uniformly bounded in $L^\infty((0,T))$ with respect to δ and ε . Indeed,

$$\|G_w\|_{L^\infty((0,T))} \leq \frac{\|\varepsilon \partial_t \mu_{0,\varepsilon}\|_{L^\infty((0,T))} \|f_\delta\|_{L^\infty((0,T))}}{\mu_{0,\min}^2} + \frac{\zeta_{\max}}{\mu_{0,\min}} \int_{\mathbb{R}_+} \rho_\varepsilon |u_\varepsilon^\delta| da < +\infty.$$

Finally, we have that

$$\|u_\varepsilon^\delta\|_{X_T} \leq \|w\|_{X_T} + \frac{\|f_\delta\|_{L^\infty((0,T))}}{\mu_{0,\min}} < +\infty$$

which ends the proof. \square

Previous stability estimates allow to show :

Theorem 7. Under Assumption 1, one has for any fixed $\varepsilon > 0$,

$$u_\varepsilon^\delta \rightharpoonup u_\varepsilon \text{ weakly-}^* \text{ in } X_T$$

as $\delta \rightarrow 0$, where u_ε solves the weak problem

$$\begin{aligned} & - \int_0^T \int_0^\infty u_\varepsilon (\varepsilon \partial_t + \partial_a) \varphi \, da \, dt + \varepsilon \left[\int_0^\infty u_\varepsilon(a, s) \varphi(a, s) \, da \right]_{s=0}^{s=T} \\ & = \varepsilon \int_0^T \frac{\int_0^\infty \varphi(\tilde{a}, t) \, d\tilde{a}}{\mu_{0,\varepsilon}} \, dg + \int_0^T \frac{\int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \, da}{\mu_{0,\varepsilon}} \left(\int_0^\infty \varphi(\tilde{a}, t) \, d\tilde{a} \right) \end{aligned} \quad (33)$$

for any $\varphi \in C_c^1([0, T] \times \mathbb{R}_+)$.

PROOF. The uniform bound on u_ε^δ in X_T , proved in Lemma 12, implies that

$$\frac{u_\varepsilon^\delta}{1+a} \xrightarrow{*} \frac{u_\varepsilon}{1+a}$$

in $L^\infty((0, T) \times \mathbb{R}_+)$ in the weak- $*$ sense and the limit function u_ε belongs X_T . For every $\psi \in L^\infty((0, T) \times \mathbb{R}_+)$, we have $\zeta_\varepsilon(1+a)\rho_\varepsilon\psi \in L^1((0, T) \times \mathbb{R}_+)$ and then

$$\int_0^T \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon^\delta \psi \, da \, dt \rightarrow \int_0^T \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \psi \, da \, dt.$$

By Corollary 1, the first term of right-hand in (27) tends to $\int_0^T \int_0^\infty \varphi(t, \tilde{a}) \, d\tilde{a} / \mu_{0,\varepsilon} \, dg$ as $\delta \rightarrow 0$, for any $\varphi \in C_c^1([0, T] \times \mathbb{R}_+)$. Regarding the second term of the right hand side in (27), one has that $\zeta_\varepsilon \rho_\varepsilon / \mu_{0,\varepsilon} \in L^1((0, T) \times \mathbb{R}_+)$ and this leads, thanks again to the weak- $*$ convergence above to write :

$$\int_0^t \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon^\delta \, da \, d\tilde{t} \longrightarrow \int_0^t \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \, da \, d\tilde{t} \quad \text{as } \delta \rightarrow 0, \quad (34)$$

which ends the proof. \square

The latter theorem allows to prove a convergence result when returning to the z_ε variable :

Proposition 2. Under the same assumptions as above, it holds that

$$z_\varepsilon^\delta \rightarrow z_\varepsilon \text{ strongly in } L^\infty((0, T)) \text{ as } \delta \rightarrow 0,$$

where z_ε satisfies

$$z_\varepsilon(t) = z_\varepsilon(0^+) + \int_0^t \frac{\varepsilon}{\mu_{0,\varepsilon}} \, dg + \int_0^t \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon u_\varepsilon \rho_\varepsilon \, da \, d\tilde{t} \quad (35)$$

which is also a solution of (1).

Before showing this result, we make some comments : if u_ε^δ is a solution of (23) then z_ε^δ defined as

$$z_\varepsilon^\delta(t) := z_\varepsilon^\delta(0^+) + \int_0^t \frac{1}{\mu_{0,\varepsilon}(\tilde{t})} \left(\varepsilon f'_\delta(\tilde{t}) + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon \, da \right) \, d\tilde{t} \quad (36)$$

solves (22). Conversely, if z_ε^δ solves (22) then u_ε^δ , given by

$$u_\varepsilon^\delta(a, t) = \begin{cases} \frac{z_\varepsilon^\delta(t) - z_\varepsilon^\delta(t - \varepsilon a)}{\varepsilon}, & \text{if } t > \varepsilon a \\ \frac{z_\varepsilon^\delta(t) - z_p(t - \varepsilon a)}{\varepsilon}, & \text{if } t \leq \varepsilon a \end{cases} \quad (37)$$

is a solution of (23). For more details see [13, Lemma 4.1 and Lemma 4.2].

PROOF. First, by using (34) in the proof of Theorem 7, and Lemma 6, we have that $f_\delta(0^+) = f(0^+)$ and z_ε^δ given by (36) converge strongly in $L^\infty((0, T))$ to z_ε which verifies (35). Using [13, Lemma 4.2], if z_ε^δ is defined as (36) it solves (22). Multiplying (22) by a test function $\varphi \in L^1(0, T)$ gives :

$$\frac{1}{\varepsilon} \int_0^T \int_0^{+\infty} (z_\varepsilon^{\delta k}(t) - z_\varepsilon^{\delta k}(t - \varepsilon a)) \rho_\varepsilon(a, t) \varphi(t) da dt = \int_0^T f_{\delta k}(t) \varphi(t) dt \quad (38)$$

As $z_\varepsilon^{\delta k}$ converges strongly in $L^\infty(0, T)$ to z_ε , the difference $z_\varepsilon^{\delta k}(t) - z_\varepsilon^{\delta k}(t - \varepsilon a)$ converges almost everywhere for any fixed (a, t) in $\mathbb{R}_+ \times (0, T)$ towards $z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)$. Thanks to the L^∞ bounds on $u_\varepsilon^{\delta k}/(1+a)$, and the bounds in $L^1(\mathbb{R}_+ \times (0, T))$ on the first moment of ρ_ε , there exists an integrable majorizing function $g(a, t)$ on $\mathbb{R}_+ \times (0, T)$ s.t.

$$|z_\varepsilon^{\delta k}(t) - z_\varepsilon^{\delta k}(t - \varepsilon a)| |\varphi(t)| \rho_\varepsilon(a, t) \leq g(a, t)$$

uniformly for every k . Thus one can apply the Lebesgue's Theorem in the right hand side of (38). Since f_δ converges in $L^1(0, T)$ the convergence occurs in (38) for every $\varphi \in \mathcal{D}(0, T)$ and thus almost everywhere in $(0, T)$ and thus z_ε solves (1). \square

5 Weak convergence when ε goes to zero

Next, we prove the weak convergence of u_ε from which we deduce the strong convergence of z_ε .

Theorem 8. Under the same assumptions as above, one has

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly-}^* \text{ in } X_T$$

as $\varepsilon \rightarrow 0$, where u_0 satisfies (7) and

$$\int_0^\infty u_0(a, t) \rho_0(a, t) da = f(t) \text{ a.e } t \in (0, T).$$

Furthermore, it also holds that

$$z_\varepsilon \rightarrow z_0 \text{ strongly in } L^\infty((0, T)) \text{ as } \varepsilon \rightarrow 0.$$

PROOF. The proof follows the same steps as in [13, Theorem 6.2]. First, by Lemma 12, u_ε^δ is uniformly bounded in X_T with respect to δ and ε , and therefore u_ε is uniformly bounded in X_T with respect to ε , then u_ε is weakly convergent to u_0 in X_T . On the other hand, Theorem 3.2 and Lemma 3.4 imply that

$$(1+a)\rho_\varepsilon \rightarrow (1+a)\rho_0$$

strongly in $L^1((0, T) \times \mathbb{R}_+)$. These arguments justify that for every $\psi \in L^\infty((0, T) \times \mathbb{R}_+)$ one has

$$\int_0^T \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \psi da dt \rightarrow \int_0^T \int_0^\infty \zeta_0 \rho_0 u_0 \psi da dt.$$

Indeed, one has

$$\begin{aligned} & \int_0^T \int_0^\infty \{\zeta_\varepsilon \rho_\varepsilon u_\varepsilon - \zeta_0 \rho_0 u_0\} \psi da dt = \int_0^T \int_0^\infty (\zeta_\varepsilon - \zeta_0) \rho_\varepsilon u_\varepsilon \psi da dt \\ & + \int_0^T \int_0^\infty \zeta_0 (\rho_\varepsilon - \rho_0) u_\varepsilon \psi da dt + \int_0^T \int_0^\infty \zeta_0 \rho_0 (u_\varepsilon - u_0) \psi da dt \end{aligned}$$

As $\zeta_\varepsilon \rightarrow \zeta_0$ by Assumptions 1, and thanks to the weak convergence of u_ε , both terms on the right-hand side tend to zero as $\varepsilon \rightarrow 0$. Note that this implies the weak convergence of $\int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon u_\varepsilon da$ in $L^1((0, T))$, since we can choose $\psi \in L^\infty((0, T))$. Moreover, thanks to Lemma 7 we obtain that

$$\left| \int_{[0, T]} \frac{\int_0^{+\infty} \varphi da}{\mu_{0, \varepsilon}} dg \right| \leq \frac{C \|\varphi\|_{L_{t, a}^\infty}}{\mu_{0, \min}} \text{pvar}(g, [0, T]) \leq C \|f\|_{\text{BV}((0, T))}.$$

As in [13, Theorem 6.2], passing to the limit in the weak formulation (33) we obtain

$$- \int_0^T \int_0^\infty u_0 \partial_a \psi da dt = \int_0^T \int_0^\infty \frac{\zeta_0 \rho_0 u_0 \psi}{\mu_{0, 0}} da dt$$

which implies that u_0 satisfies

$$\begin{cases} \partial_a u_0 = \frac{1}{\mu_{0, 0}} \int_0^\infty \zeta_0(\tilde{a}, t) u_0(\tilde{a}, t) \rho_0(\tilde{a}, t) d\tilde{a}, & t > 0, a > 0, \\ u_0(a = 0, t) = 0, & t > 0, \end{cases} \quad (39)$$

Similarly, we have the weak convergence of $\int_0^\infty u_\varepsilon(t, a) \rho_\varepsilon(t, a) da$ towards $\int_0^\infty u_0(t, a) \rho_0(t, a) da$ in $L^1((0, T))$. Hence, one concludes that u_0 satisfies also

$$\int_0^\infty u_0(t, a) \rho_0(t, a) da = f(t), \quad \text{a.e. } t \in (0, T).$$

As the the right hand side of (39) does not depend on age, one has that $u_0 = \gamma(t) a$, where in order to satisfy the last compatibility condition implies that

$$\gamma(t) \int_{\mathbb{R}_+} a \rho_0(a, t) da = f(t), \quad \text{a.e. } t \in (0, T).$$

Thus $u_0(a, t) = f(t)/\mu_{1, 0}(t)a$ for almost every $t \in (0, T)$ and every $a \in \mathbb{R}_+$. Using again the weak convergence of $\int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon u_\varepsilon da$ in $L^1((0, T))$ combined with the strong convergence of $\mu_{0, \varepsilon}$ allows to pass to the limit in the third term of (35).

Moreover,

$$\begin{aligned} |z_\varepsilon(0^+) - z_p(0)| &= \frac{1}{\mu_{0, \varepsilon}(0)} \left| \int_0^\infty (z_p(-\varepsilon a) - z_p(0)) \rho_{I, \varepsilon}(a) da + \varepsilon f(0^+) \right| \\ &\leq \frac{\varepsilon k \|z_p'\|_{L^\infty(\mathbb{R}_-)}}{\mu_{0, \min}} + \frac{\varepsilon |f(0^+)|}{\mu_{0, \min}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \end{aligned}$$

where k is the constant from Lemma 9.

All together this provides that z_0 solves :

$$z_0(t) = z_p(0) + \int_0^t \frac{f(\tau)}{\mu_{1, 0}(\tau)} \frac{\int_{\mathbb{R}_+} \zeta_0(a, \tau) \rho_0(a, \tau) da}{\mu_{0, 0}(\tau)} d\tau \quad (40)$$

but because ρ_0 solves (4), one has that $a\rho_0$ solves :

$$\partial_a(a\rho_0) - \rho_0 + a\zeta_0(a, t)\rho_0 = 0,$$

which after integration in time shows that

$$\frac{\int_{\mathbb{R}_+} a\zeta_0(a, t)\rho_0(a, t) da}{\mu_{0, 0}(t)} = 1$$

and this shows in turn that (40) reduces to

$$z_0(t) = z_0(0) + \int_0^t \frac{f(\tau)}{\mu_{1, 0}(\tau)} d\tau$$

which is the integrated version of (3). \square

6 A comparison principle

In this section, we give error estimates between z_ε and z_0 , the solution of the problem

$$\begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)) \varrho(a) da = f(t), & t \geq 0, \\ z_\varepsilon(t) = z_p(t), & t < 0, \end{cases} \quad (41)$$

where ϱ is constant in time and satisfies

$$\begin{cases} \partial_a \varrho + \zeta(a) \varrho = 0, & a > 0, \\ \varrho(0) = \beta \left(1 - \int_0^\infty \varrho(\tilde{a}) d\tilde{a} \right), \end{cases} \quad (42)$$

where the data of (41) and (42) satisfy

Assumptions 3.

- i) $f \in \text{BV}((0, T))$,
- ii) $z_p \in \text{Lip}(\mathbb{R}_-)$,
- iii) $\beta \in \mathbb{R}_+^*$,
- iv) $\zeta \in L^\infty(\mathbb{R}_+)$ such that

$$0 < \zeta_{\min} \leq \zeta(a) \leq \zeta_{\max}, \quad \text{a.e. } a \in \mathbb{R}_+.$$

Setting $\hat{z}_\varepsilon(t) := z_\varepsilon(t) - z_0(t)$, it solves :

$$\hat{z}_\varepsilon(t) = \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} \hat{z}_\varepsilon(t - \varepsilon a) \varrho(a) da + \tilde{h}_\varepsilon(t)$$

where

$$\begin{aligned} \tilde{h}_\varepsilon(t) &= \frac{\varepsilon}{\mu_0} \int_0^{t/\varepsilon} \left(\frac{z_0(t) - z_0(t - \varepsilon a)}{\varepsilon} - a \partial_t z_0(t) \right) \varrho(a) da \\ &+ \frac{\varepsilon}{\mu_0} \int_{t/\varepsilon}^{+\infty} \left(\frac{z_0(t) - z_0(0)}{\varepsilon} - a \partial_t z_0(t) \right) \varrho(a) da + \frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} (z_p(t - \varepsilon a) - z_p(0)) \varrho(a) da \end{aligned} \quad (43)$$

where $\mu_0 := \int_0^{+\infty} \varrho(a) da$ then

$$|\hat{z}_\varepsilon(t)| \leq \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} |\hat{z}_\varepsilon(t - \varepsilon a)| \varrho(a) da + |\tilde{h}_\varepsilon(t)| \quad (44)$$

Then integrating in time and setting

$$\hat{Z}_\varepsilon(t) := \int_0^t |\hat{z}_\varepsilon(\tau)| d\tau$$

one has that :

$$\hat{Z}_\varepsilon(t) = \int_0^t |\hat{z}_\varepsilon(\tau)| d\tau \leq \frac{1}{\mu_0} \int_0^t \int_0^{\frac{\tau}{\varepsilon}} |\hat{z}_\varepsilon(\tau - \varepsilon a)| \varrho(a) da d\tau + \int_0^t |\tilde{h}_\varepsilon(\tau)| d\tau$$

then, we change the order of integration and the domain of integration becomes $D' := \{(a, \tau) \in (0, t/\varepsilon) \times (\varepsilon a, t)\}$. We use the change of variable $\tilde{t} = \tau - \varepsilon a$ in order to write :

$$\begin{aligned} \int_0^t \int_0^{\frac{\tau}{\varepsilon}} |\hat{z}_\varepsilon(\tau - \varepsilon a)| \varrho(a) da d\tau &= \int_0^{\frac{t}{\varepsilon}} \int_{\varepsilon a}^t |\hat{z}_\varepsilon(\tau - \varepsilon a)| d\tau \varrho(a) da \\ &= \int_0^{\frac{t}{\varepsilon}} \int_0^{t - \varepsilon a} |\hat{z}_\varepsilon(\tilde{t})| d\tilde{t} \varrho(a) da = \int_0^{\frac{t}{\varepsilon}} \hat{Z}_\varepsilon(t - \varepsilon a) \varrho(a) da \end{aligned}$$

So that finally \hat{Z}_ε solves :

$$\hat{Z}_\varepsilon(t) \leq \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} \hat{Z}_\varepsilon(t - \varepsilon a) \varrho(a) da + \int_0^t \tilde{h}_\varepsilon(\tau) d\tau = \int_0^t \hat{Z}_\varepsilon(\tilde{a}) K_\varepsilon(t - \tilde{a}) d\tilde{a} + \int_0^t \tilde{h}_\varepsilon(\tau) d\tau$$

where $K_\varepsilon(\tilde{a}) := \frac{1}{\varepsilon\mu_0} \varrho\left(\frac{\tilde{a}}{\varepsilon}\right)$ is the kernel of the integral operator. We use a comparison principle [6, the Generalised Gronwall Lemma 8.2 p. 257] and construct a majorizing function U_ε of the form $U_\varepsilon(t) = \varepsilon(K_0 + K_1 t)$ where K_0 and K_1 are suitably chosen, such that $U_\varepsilon \geq |\hat{Z}_\varepsilon|$ and $U_\varepsilon \sim \varepsilon$. The following two lemmas are required in order to apply this comparison principle :

Lemma 13. The Volterra kernel K_ε satisfies :

$$\|K_\varepsilon\|_{L^\infty(0,T)} := \operatorname{ess\,sup}_{t \in (0,T)} \int_0^t |K_\varepsilon(\tilde{a})| d\tilde{a} < 1$$

PROOF. To prove this result, we need to show that

$$0 \leq \int_0^t |K_\varepsilon(\tilde{a})| d\tilde{a} = \frac{\int_0^{\frac{t}{\varepsilon}} \varrho(a) da}{\int_0^{+\infty} \varrho(a) da} < 1. \quad (45)$$

The kernel ϱ solves (42), thus it can be explicitly computed as

$$\varrho(a) = \frac{\beta}{1 + \beta I} \exp\left(-\int_0^a \zeta(s) ds\right)$$

one has the lower bound :

$$\varrho(a) \geq \frac{\beta}{1 + \beta I} \exp(-\zeta_{\max} a) > 0, \quad \forall a \in \mathbb{R}_+.$$

This in turn shows that

$$\int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da > 0$$

which is equivalent to the claim. \square

Lemma 14. Consider the expectation value of a given density ϱ with respect to the tail $a > t/\varepsilon$,

$$A_1[\varrho](t) := \frac{\int_0^{+\infty} a \varrho(a + \frac{t}{\varepsilon}) da}{\int_0^{+\infty} \varrho(a + \frac{t}{\varepsilon}) da} \quad (46)$$

then under Assumptions 3, one has

$$A_1[\varrho](t) \leq \frac{\zeta_{\max}}{\zeta_{\min}^2}.$$

PROOF. Setting

$$q(a, t) := \frac{\varrho(a + \frac{t}{\varepsilon})}{\varrho(\frac{t}{\varepsilon})}, \quad (47)$$

it solves

$$\partial_a q(a, t) + \zeta(a + t/\varepsilon) q(a, t) = 0, \quad q(0, t) = 1.$$

This problem admits an explicit solution of the form

$$q(a, t) = \exp\left(-\int_0^a \zeta(\bar{a} + t/\epsilon) d\bar{a}\right) = \exp\left(-\int_{t/\epsilon}^{a+t/\epsilon} \zeta(\hat{a}) d\hat{a}\right). \quad (48)$$

Then, we shall rewrite (46) as :

$$A_1[\varrho](t) := \frac{\int_0^{+\infty} a q(a, t) da}{\int_0^{+\infty} q(a, t) da}$$

by using hypothesis iv) from Assumptions 3, one has

$$\exp(-\zeta_{\max} a) \leq q(a, t) \leq \exp(-\zeta_{\min} a)$$

This gives :

$$\int_0^{+\infty} q(a, t) da \geq \int_0^{+\infty} \exp(-\zeta_{\max} a) da = \frac{1}{\zeta_{\max}}$$

and

$$\int_0^{+\infty} a q(a, t) da \leq \int_0^{+\infty} a \exp(-\zeta_{\min} a) da = \frac{1}{\zeta_{\min}^2}$$

which shows the final upper bound. \square

Proposition 3. Under Assumptions 3, for $0 < t < T$ one has the estimates :

$$\tilde{H}_\epsilon(t) := \int_0^t |\tilde{h}_\epsilon(\tau)| d\tau \leq \epsilon^2 C_1$$

where C_1 depends on $\mu_2, \mu_1, \|\partial_t z_0\|_{\text{BV}((0, T))}$ and on $\|z_p\|_{\text{Lip}(\mathbb{R}_-)}$ but not on ϵ .

PROOF. Recalling the definition of \tilde{h}_ϵ in (43), we split $\int_0^t |\tilde{h}_\epsilon(\tau)| d\tau$ into three parts. First, we define

$$I_1 := \frac{\epsilon}{\mu_0} \int_0^t \int_0^{\tau/\epsilon} \left| \frac{z_0(\tau) - z_0(\tau - \epsilon a)}{\epsilon} - a \partial_t z_0(\tau) \right| \varrho(a) da d\tau$$

Since $\partial_t z_0 \in \text{BV}((0, T))$, then I_1 can be written in the form

$$I_1 = \frac{1}{\mu_0} \int_0^t \int_0^{\tau/\epsilon} \left| \int_{\tau - \epsilon a}^{\tau} (\partial_t z_0(\tilde{t}) - \partial_t z_0(\tau)) d\tilde{t} \right| \varrho(a) da d\tau$$

by switching the integration order between τ and a , and using the change of variable $\tilde{t} = \tau + h$, we get that

$$\begin{aligned} I_1 &\leq \frac{1}{\mu_0} \int_0^{t/\epsilon} \int_{\epsilon a}^t \int_{\tau - \epsilon a}^{\tau} |\partial_t z_0(\tilde{t}) - \partial_t z_0(\tau)| d\tilde{t} d\tau \varrho(a) da \\ &\leq \frac{1}{\mu_0} \int_0^{t/\epsilon} \int_{\epsilon a}^t \int_{-\epsilon a}^0 |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| dh d\tau \varrho(a) da \\ &\leq \frac{1}{\mu_0} \int_0^{t/\epsilon} \int_{-\epsilon a}^0 \int_{\epsilon a}^t |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau dh \varrho(a) da \end{aligned} \quad (49)$$

and thus applying Lemma 3, one has the estimate of the inner integral of the latter right hand side :

$$\int_{\epsilon a}^t |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau \leq |h| \|\partial_t z_0\|_{\text{BV}}, \quad \forall h \in (-\epsilon a, 0)$$

which implies that

$$I_1 \leq \frac{\|\partial_t z_0\|_{BV}}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^0 |h| dh \varrho(a) da \leq \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a) da.$$

Next, we set:

$$\begin{aligned} I_2 &= \frac{\varepsilon}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} \left| \frac{z_0(\tau) - z_0(0)}{\varepsilon} - a \partial_t z_0(\tau) \right| \varrho(a) da d\tau \\ &\leq \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} \left| \int_0^\tau \partial_t z_0(\tilde{t}) d\tilde{t} - \tau \partial_t z_0(\tau) \right| \varrho(a) da d\tau + \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} |\tau - \varepsilon a| |\partial_t z_0(\tau)| \varrho(a) da d\tau \\ &=: I_{2,1} + I_{2,2} \end{aligned}$$

As in the estimates of I_1 , first, one switches the order of integration and then one integrates on

$$D := \{(a, \tau) \in (0, t/\varepsilon) \times (0, \varepsilon a)\} \cup \{(a, \tau) \in (t/\varepsilon, +\infty) \times (0, t)\}, \quad (50)$$

and one makes the change of variable $\tilde{t} = \tau + h$ in order to obtain

$$\begin{aligned} I_{2,1} &= \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_0^{\varepsilon a} \int_{-\tau}^0 |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| dh d\tau \varrho(a) da \\ &\quad + \frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} \int_0^t \int_{-\tau}^0 |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| dh d\tau \varrho(a) da \\ &= \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^0 \int_{-h}^{\varepsilon a} |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau dh \varrho(a) da \\ &\quad + \frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} \int_{-t}^0 \int_{-h}^t |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau dh \varrho(a) da \end{aligned}$$

also, by using Lemma 4, we get that

$$\begin{aligned} I_{2,1} &\leq \frac{\|\partial_t z_0\|_{BV}}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^0 |h| dh \varrho(a) da + \frac{\|\partial_t z_0\|_{BV}}{\mu_0} \int_{t/\varepsilon}^{+\infty} \int_{-t}^0 |h| dh \varrho(a) da \\ &\leq \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a) da + \frac{\|\partial_t z_0\|_{BV}}{2\mu_0} \int_{t/\varepsilon}^{+\infty} t^2 \varrho(a) da \\ &\leq \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a) da + \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_{t/\varepsilon}^{+\infty} a^2 \varrho(a) da \\ &\leq \frac{\varepsilon^2 \mu_2 \|\partial_t z_0\|_{BV}}{\mu_0}. \end{aligned}$$

The second term $I_{2,2}$ is estimated in the same way as $I_{2,1}$. We have

$$\begin{aligned} I_{2,2} &= \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} |\tau - \varepsilon a| |\partial_t z_0(\tau)| \varrho(a) da d\tau \\ &= \frac{1}{\mu_0} \left\{ \int_0^{t/\varepsilon} \int_0^{\varepsilon a} + \int_{t/\varepsilon}^{+\infty} \int_0^t |\tau - \varepsilon a| |\partial_t z_0(\tau)| d\tau \varrho(a) da \right\} \\ &\leq \frac{\varepsilon^2 \|\partial_t z_0\|_{\infty}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a) da + \frac{\varepsilon^2 \|\partial_t z_0\|_{\infty}}{2\mu_0} \int_{t/\varepsilon}^{+\infty} a^2 \varrho(a) da \leq \frac{\varepsilon^2 \mu_2 \|\partial_t z_0\|_{\infty}}{\mu_0}. \quad \square \end{aligned}$$

Finally, by similar computations, one has

$$\frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} |z_p(\tau - \varepsilon a) - z_p(0)| \varrho(a) da d\tau \leq \varepsilon^2 \|z_p\|_{\text{Lip}(\mathbb{R}_-)} \frac{\mu_2}{\mu_0}$$

Theorem 9. Under Assumptions 3, z_ε tends to z_0 , the solution of (3), strongly in $L^1(0, T)$ as ε goes to zero. Moreover, there exists a generic constant C depending only on the data of the problem but not on ε , such that :

$$\|z_\varepsilon - z_0\|_{L^1(0, T)} \leq \varepsilon C.$$

PROOF. We have proved in Lemma 13 that the Volterra kernel K_ε is non-positive and bounded (with a bound strictly less than one) in the sense of [6, Definition 2.2 p. 227 and Proposition 2.7 p.231] so [6, the Generalised Gronwall Lemma 8.2 p. 257] applies. First observe, by using Proposition 3, that

$$\hat{Z}_\varepsilon(t) - \int_0^t \hat{Z}_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) d\tilde{a} \leq \tilde{h}_{1,\varepsilon}(t) + \tilde{h}_{2,\varepsilon}(t)$$

We construct a function U_ε which satisfies,

$$U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) d\tilde{a} \geq \tilde{h}_{1,\varepsilon} + \tilde{h}_{2,\varepsilon} \quad (51)$$

We split the integral operator applied to U_ε in two parts

$$\begin{aligned} U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) d\tilde{a} &= U_\varepsilon(t) - \frac{1}{\mu_0} \int_0^{t/\varepsilon} U_\varepsilon(t - \varepsilon a) \varrho(a) da \\ &= \underbrace{\frac{1}{\mu_0} \int_0^{+\infty} (U_\varepsilon(t) - U_\varepsilon(t - \varepsilon a)) \varrho(a) da}_{:=H_{1,\varepsilon}} + \underbrace{\frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} U_\varepsilon(t - \varepsilon a) \varrho(a) da}_{:=H_{2,\varepsilon}} \end{aligned}$$

and we shall specify U_ε such that $H_{1,\varepsilon} \geq \tilde{H}_\varepsilon(t)$ and $H_{2,\varepsilon} \geq 0$. To this end we set

$$U_\varepsilon(t) := \varepsilon(K_0 + K_1 t), \quad \forall t \in \mathbb{R} \quad (52)$$

with constants K_0 and K_1 to be specified. One has obviously that

$$H_{1,\varepsilon}(t) = \frac{\varepsilon^2 K_1 \mu_1}{\mu_0} \geq \varepsilon^2 C_1 \geq \tilde{H}_\varepsilon(t)$$

a.e. on \mathbb{R}_+ , provided that K_1 is chosen as

$$K_1 > \frac{\mu_0}{\mu_1} C_1$$

Using (52) and the change of variable $\tilde{a} = -t/\varepsilon + a$, we obtain that

$$H_{2,\varepsilon} = \int_0^{+\infty} (K_0 - \varepsilon K_1 \tilde{a}) \varrho(\tilde{a} + t/\varepsilon) d\tilde{a} = (K_0 - \varepsilon K_1 A_\varepsilon[\varrho](t)) \int_{\mathbb{R}_+} \varrho\left(\frac{t}{\varepsilon} + a\right) da$$

We are in the hypotheses of Lemma 14 : $A_\varepsilon[\varrho](t)$, the expectation of a given density ϱ with respect to the tail $a > t/\varepsilon$ is bounded by a positive constant A_{\max}

$$A_\varepsilon[\varrho](t) := \frac{\int_{\mathbb{R}_+} a \varrho\left(\frac{t}{\varepsilon} + a\right) da}{\int_{\mathbb{R}_+} \varrho\left(\frac{t}{\varepsilon} + a\right) da} \leq A_{\max}.$$

Therefore it suffices to chose $K_0 > \varepsilon K_1 A_{\max}$ in order to obtain that $H_{2,\varepsilon} \geq 0$. These computations show that U_ε is a super-solution. Then the comparison principle implies that, for all $0 \leq t \leq T$,

$$0 \leq \hat{Z}_\varepsilon(t) = \int_0^t |z_\varepsilon(s) - z_0(s)| ds \leq U_\varepsilon(t) = \varepsilon(K_0 + K_1 t) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

hence $\hat{Z}_\varepsilon \rightarrow 0$ in $C([0, T])$, which ends the proof since $\|z_\varepsilon - z_0\|_{L^1(0, T)} \leq \|\hat{Z}_\varepsilon\|_{C([0, T])}$. \square

7 A simple example

We construct *by hand* solutions of problems (1) and (3) when the load f is explicitly defined as

$$f(t) := \begin{cases} 1/2 & \text{if } 0 < t \leq \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\ 3/2 & \text{if } \frac{2}{3} < t < 1, \end{cases} \quad (53)$$

and the kernel ϱ is a simple exponential (see more precise statements below). So defined f is of course of bounded variation on $(0, 1)$. The solution z_ε solving (1) and its limit z_0 show different regularities (see Figure 1) : the adhesive approximation is rougher than the limit solution. This is an interesting feature of our approach.

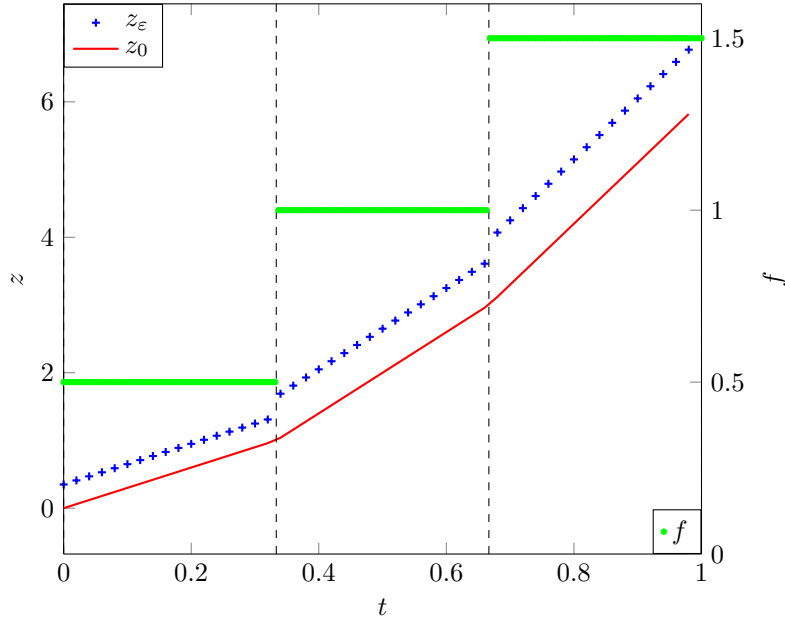


Figure 1: The solutions z_ε and z_0 as a function of time, for a fixed $\varepsilon = 10^{-1}$ on the left y-axis. The load f on the right y-axis.

Assumptions 4.

- i) the load f is defined in (53),
- ii) the on and off rates are constants defined as :

$$\zeta_\varepsilon = \zeta_0 = \zeta, \quad \beta_\varepsilon = \beta_0 = \beta.$$

- iii) the initial condition is at equilibrium :

$$\rho_{I,\varepsilon} = \rho_0 = \frac{\beta\zeta}{\beta + \zeta} e^{-\zeta a}.$$

Lemma 15. Under Assumptions 4, one has that $\mu_{0,\varepsilon} = \mu_{0,0} = \beta/(\beta + \zeta)$, $\mu_{1,\varepsilon} = \mu_{1,0} = \beta/(\zeta(\beta + \zeta))$ and $\rho_0(a) = \mu_{0,0}\zeta e^{-\zeta a}$. Then the solution z_ε of (1) is *BPV*([0, 1]) and it is explicitly given by

$$z_\varepsilon(t) = \int_0^t \frac{f}{\mu_{1,0}} ds + \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{1}{\mu_{0,0}} \int_0^{+\infty} z_p(-\varepsilon a) \rho_0 da$$

and hence,

$$z_\varepsilon(t) - z_0(t) = \varepsilon \frac{f(t)}{\mu_{0,0}} + \int_{-\infty}^0 z'_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds$$

with $z_0(t) = z_p(0) + \int_0^t f(s) ds / \mu_{1,0}$ is a continuous functions in $[0, 1]$. Note that the last term is an ε order term according to Assumption 1, indeed it holds that

$$\left| \int_{-\infty}^0 z'_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds \right| \leq \frac{\varepsilon}{\zeta} \|z_p\|_{W^{1,\infty}(\mathbb{R}_-)}.$$

PROOF. The specific choice of data and kernel allows to rephrase (1) as

$$z_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t z_\varepsilon(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) ds = \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) ds.$$

Next, setting

$$q_\varepsilon(t) = z_\varepsilon(t) \exp\left(\frac{\zeta t}{\varepsilon}\right), \quad t \geq 0.$$

Then we can rewrite (15) for all $t \geq 0$ as

$$q_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t q_\varepsilon(s) ds = \varepsilon \exp\left(\frac{\zeta t}{\varepsilon}\right) \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds. \quad (54)$$

By differentiating (54) in time and using (6), we prove that q_ε solve the equation

$$\begin{cases} \dot{q}_\varepsilon(t) - \frac{\zeta}{\varepsilon} q_\varepsilon(t) = \frac{\exp\left(\frac{\zeta t}{\varepsilon}\right)}{\mu_{0,0}} (\zeta f(t) + \varepsilon f'(t)), & t > 0, \\ q_\varepsilon(0^+) = \varepsilon f(0) / \mu_{0,0} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds \end{cases} \quad (55)$$

and therefore q_ε is explicitly given by

$$q_\varepsilon(t) = \exp\left(\frac{\zeta t}{\varepsilon}\right) \left(\varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds + \int_0^t \frac{f(s)}{\mu_{1,0}} ds \right)$$

which gives the formula of z_ε . Moreover, it is clear that z_ε is of bounded variation since f is it and z_0 is an absolutely continuous function. \square

A Auxiliary proofs

In this appendix, the domain Ω is an open set of \mathbb{R}^n .

A.1 Proof of Theorem 1

As in the proof of [9, Theorem 14.9], the aim is to show that for every $\delta > 0$, there exists a sequence $\{f_\delta\}_\delta$ in $C^\infty(\Omega)$ such that

$$\int_\Omega |f - f_\delta| dt < \delta \quad \text{and} \quad |Df_\delta|(\Omega) < \|Df\|(\Omega) + \delta.$$

Since the total variation $\|Df\|(\Omega)$ is bounded,

$$\lim_{j \rightarrow +\infty} \|Df\|(\Omega \setminus \{t, \text{dist}(t, \partial\Omega) > 1/j, |t| < j\}) = 0$$

then for fixed $\delta > 0$, there exists a $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\|Df\|(\Omega \setminus \{t, \text{dist}(t, \partial\Omega) > 1/j, |t| < j\}) \leq \delta.$$

For $i \in \mathbb{N}$, we define the subdomain Ω_i of Ω by

$$\Omega_i := \{t \in \Omega, \text{dist}(t, \partial\Omega) > 1/j_0 + i, |t| < j_0 + i\}$$

such that $\Omega_i \subset\subset \Omega_{i+1}$ and $\bigcup_{i=0}^{\infty} \Omega_i = \Omega$. Let $W_0 = \Omega_1$ and $W_i = \Omega_{i+1} \setminus \bar{\Omega}_{i-1}$, where $\Omega_{-1} = \Omega_0 := \emptyset$, and let $\{\phi_i\}$ be a partition of the unity subordinate to the covering $\{W_i\}_{i \in \mathbb{N}}$

$$\phi_i \in C_0^\infty(W_i), \quad 0 \leq \phi_i \leq 1, \quad \sum_{i=0}^{\infty} \phi_i = 1.$$

For $i \in \mathbb{N}$, the idea is to find $\delta_i > 0$ so small that

$$\text{supp } \chi_{\delta_i} * (\phi_i f) \subset W_i \tag{56}$$

which allows to write

$$\|\chi_{\delta_i} * (\phi_i f) - \phi_i f\|_{L^1(\Omega)} = \|\chi_{\delta_i} * (\phi_i f) - \phi_i f\|_{L^1(W_i)} \tag{57}$$

and then from the local approximation Theorem (see Theorem 2 p.125 in [4]),

$$\|\chi_{\delta_i} * (\phi_i f) - \phi_i f\|_{L^1(W_i)} < C\delta$$

and also by convenience, we can take $C = \frac{1}{2^i}$. Hence we conclude that

$$\int_{\Omega} |\chi_{\delta_i} * (\phi_i f) - \phi_i f| \, dt < \delta 2^{-i} \tag{58}$$

$$\int_{\Omega} |\chi_{\delta_i} * (f \phi_i') - f \phi_i'| \, dt < \delta 2^{-i}. \tag{59}$$

for a positive mollifiers χ_δ defined as

$$\chi_\delta(t) := \frac{1}{\delta} \chi\left(\frac{t}{\delta}\right), \quad \chi(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{if } |t| > 2 \end{cases}$$

Define

$$f_\delta = \sum_{i=0}^{\infty} \chi_{\delta_i} * (\phi_i f). \tag{60}$$

By the construction of $\{W_i\}$, we have $W_i \cap W_{i+1} \neq \emptyset$ and $W_i \cap W_{i+1} \cap W_{i+2} = \emptyset$ which give that for every $t \in \Omega$,

$$\#\{i \in \mathbb{N} : \chi_{\delta_i} * (\phi_i f)(t) \neq 0\} \leq 2$$

and since the finite sum of infinitely differentiable functions is infinitely differentiable, we conclude that $f_\delta \in C^\infty(\Omega)$ and

$$\int_{\Omega} |f_\delta - f| \, dt \leq \sum_{i=1}^{\infty} \int_{\Omega} |\chi_{\delta_i} * (\phi_i f) - \phi_i f| \, dt < \sum_{i=1}^{\infty} \frac{\delta}{2^i} \leq \delta,$$

since $f = \sum_{i=0}^{\infty} f\phi_i$. Thus $f_\delta \rightarrow f$ in $L^1(\Omega)$ as $\delta \rightarrow 0$. Using the theorem of Lower semi-continuity of variation measure (see chap 5, [4]), we have

$$\|Df\|(\Omega) \leq \liminf_{\delta \rightarrow 0} |Df_\delta|(\Omega). \quad (61)$$

it remains to show that

$$\limsup_{\delta \rightarrow 0} |Df_\delta|(\Omega) \leq \|Df\|(\Omega).$$

Let $\psi \in C_c^\infty(\Omega)$ be such that $|\psi|_\infty \leq 1$. Since $\phi_i f \in \text{BV}(\Omega)$ for every $i \in \mathbb{N}$, we have by Lemma 14.10 in [9] that

$$\int_{\Omega} \chi_{\delta_i} * (\phi_i f) \psi' dt = \int_{\Omega} \phi_i f (\chi_{\delta_i} * \psi)' dt \quad (62)$$

let $\psi_{\delta_i} = \chi_{\delta_i} * \psi$, and using the fact that support of ψ is compact and that the partition of unity is locally finite, we have that

$$\begin{aligned} \int_{\Omega} f_\delta \psi' dt &= \sum_{i=1}^{+\infty} \int_{\Omega} \chi_{\delta_i} * (\phi_i f) \psi' dt \\ &= \sum_{i=1}^{+\infty} \int_{\Omega} \phi_i f \psi'_{\delta_i} dt \\ &= \sum_{i=1}^{+\infty} \int_{\Omega} f((\phi_i \psi_{\delta_i})' - \phi_i' \psi_{\delta_i}) dt := I_1 + I_2 \end{aligned}$$

since $\text{supp}(\phi_i \psi_{\delta_i}) \subset W_i$ and $|\phi_i \psi_{\delta_i}|_\infty \leq 1$, it follows that

$$\begin{aligned} I_1 &= \int_{\Omega} f (\phi_i \psi_{\delta_i})' dt + \sum_{i=2}^{+\infty} \int_{\Omega} f (\phi_i \psi_{\delta_i})' dt \\ &\leq \|Df\|(\Omega) + \sum_{i=2}^{+\infty} \|Df\|(W_i) \\ &\leq \|Df\|(\Omega) + 3\|Df\|(\Omega \setminus \Omega_1) \\ &\leq \|Df\|(\Omega) + 3\delta \end{aligned}$$

since each $t \in \Omega$ belongs at most two of the sets U_i . On the other hand, by Fubini's theorem we have

$$\begin{aligned} I_2 &= - \sum_{i=1}^{+\infty} \int_{\Omega} \chi_{\delta_i} * (f\phi_i') \psi dt \\ &= - \sum_{i=1}^{+\infty} \int_{\Omega} (\chi_{\delta_i} * (f\phi_i') - f\phi_i') \psi dt \end{aligned}$$

by using the fact that $\sum_{i=1}^{+\infty} \phi_i' = 0$. We now use (59) and the fact that $|\psi|_\infty \leq 1$, to conclude that $I_2 \leq \delta$ and then we obtain that

$$\int_{\Omega} f_\delta \psi' dt \leq \|Df\|(\Omega) + 3\delta.$$

By taking the supremum and passing to the limit when $\delta \rightarrow 0$, we obtain that

$$\limsup_{\delta \rightarrow 0} \|Df_\delta\|(\Omega) \leq \|Df\|(\Omega) \quad (63)$$

Finally, (61) and (63) together concludes the proof.

A.2 Proof of Lemma 5:

Let $f \in \text{BV}(\Omega) \cap L^\infty(\Omega)$. By using (60) and (56) we can write for all t in Ω ,

$$\begin{aligned} |f_\delta(t)| &= \left| \sum_{i=0}^{\infty} \chi_{\delta_i} * (f\phi_i)(t) \right| \\ &\leq \sum_{i=0}^{\infty} \|f\|_{L^\infty} \int_{W_i} \phi_i(t-t') \chi_{\delta_i}(t') dt' \\ &\leq \sum_{i=0}^{\infty} \|f\|_{L^\infty} \int_{W_i} \chi_{\delta_i}(t') dt' \end{aligned}$$

since $0 \leq \phi_i \leq 1$. On the other hand, we have $W_i \cap W_{i-1} = \Omega_i \setminus \Omega_{i-1}$ and $W_{i-1} \cap W_i \cap W_{i+1} = \emptyset$ which implies that

$$|f_\delta(t)| \leq \|f\|_{L^\infty} \int_{W_{i-1}} \chi_{\delta_i}(t') dt' + \|f\|_{L^\infty} \int_{W_i} \chi_{\delta_{i-1}}(t') dt' \leq 2\|f\|_{L^\infty(\Omega)} \|\chi\|_{L^1(\mathbb{R})}$$

which ends the proof.

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