Limiting reduction of hydrodynamics, singular limits, and asymptotic expansions

Vincent Ardourel

EPSA 2021

Introduction

- Question: can fluid mechanics be reduced (in the limit) to a system of particles in classical mechanics?

- Methodology:
    - focusing on “limiting reduction”(Nickles 1973, Batterman 2020)
    - analyzing recent results in applied mathematics

- Claim: fluid mechanics fails to be reduced in the limit to classical mechanics, in the current state of science.

Outline

1. Limiting reduction

2. From classical mechanics to fluid mechanics

3. Failure of limiting reduction

Limiting reduction

Limiting reduction

- Nickles (1973):
  - reduction_1: logical consequence (Nagelian reduction)
  - reduction_2: mathematical operations (e.g., mathematical limits)

- Batterman (1995, 2002, 2020, …): limiting reduction to account for how physical theories are interconnected

- **Limiting reduction**: a physical theory T_f reduces in the limit to the theory T_c :

$$\lim_{\varepsilon \to 0} T_f = T_c$$

(Batterman 2002, p. 18; 2020, sect. 2)

Limiting reduction

- Limiting reduction **holds** when limits are **regular**, i.e., when the **behaviour of the system** approaching the limit is the **same** as the behaviour of the system at the limit

    - E.g., special relativity and classical mechanics: when $v/c \rightarrow 0$, the Einsteinian momentum $p_E = mv(1-(v/c)^2)^{-1/2}$ tends to the classical mechanics momentum $p_C = mv$

- Limiting reduction **fails** with **singular** limits, i.e., when the **behaviour of the system** approaching the limit is **qualitatively different** from the behaviour of the system at the limit

    - E.g., wave optics and geometrical optics: when $\lambda \rightarrow 0$, the intensity of an interference pattern is singular at $\lambda=0$ (Berry 1994, Batterman 1995)

Regular limit: a toy example

Consider the equation (Batterman 2020):

$$x^2 + x - 9\varepsilon = 0$$

- Take $\boldsymbol{\varepsilon = 0}$, the equation changes to: $x^2+x = 0$

Two solutions: x=0 and x=-1

- Take $\boldsymbol{\varepsilon \neq 0}$, the equation has **two solutions** (which depend on ε): x_1 and x_2.

When $\varepsilon \rightarrow 0$, $x_1 \rightarrow 0$ and $x_2 \rightarrow -1$

The behaviour of the equation approaching the limit is the **same** as the behaviour of the equation at the limit

Singular limit: a toy example

Consider the equation (Batterman 2020):

$$\varepsilon x^2 + x - 9 = 0$$

- Take $\boldsymbol{\varepsilon = 0}$, the equation changes to: x-9 =0

One solution: x=9

- Take $\boldsymbol{\varepsilon \neq 0}$, the equation has **two solutions** (which depend on ε): x_1 and x_2

When $\varepsilon \rightarrow 0$, $x_1 \rightarrow \infty$ and $x_2 \rightarrow -\infty$

The behaviour of the equation approaching the limit is **qualitatively different** from the behaviour of the equation at the limit

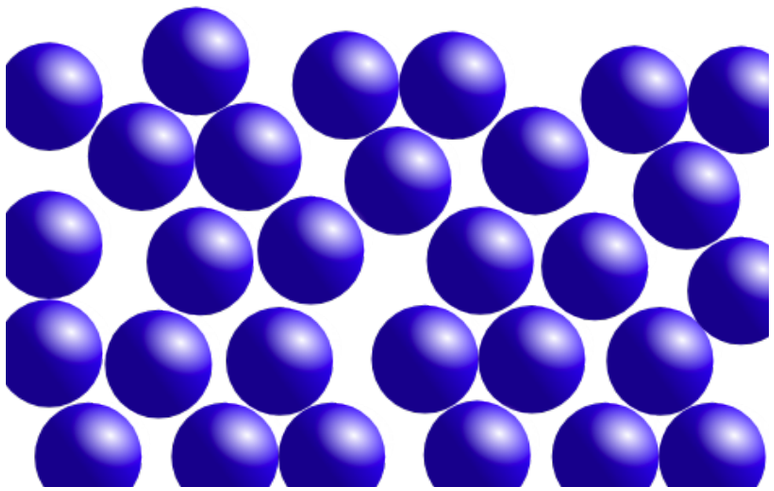
From classical mechanics to fluid mechanics

Hard-spheres gas

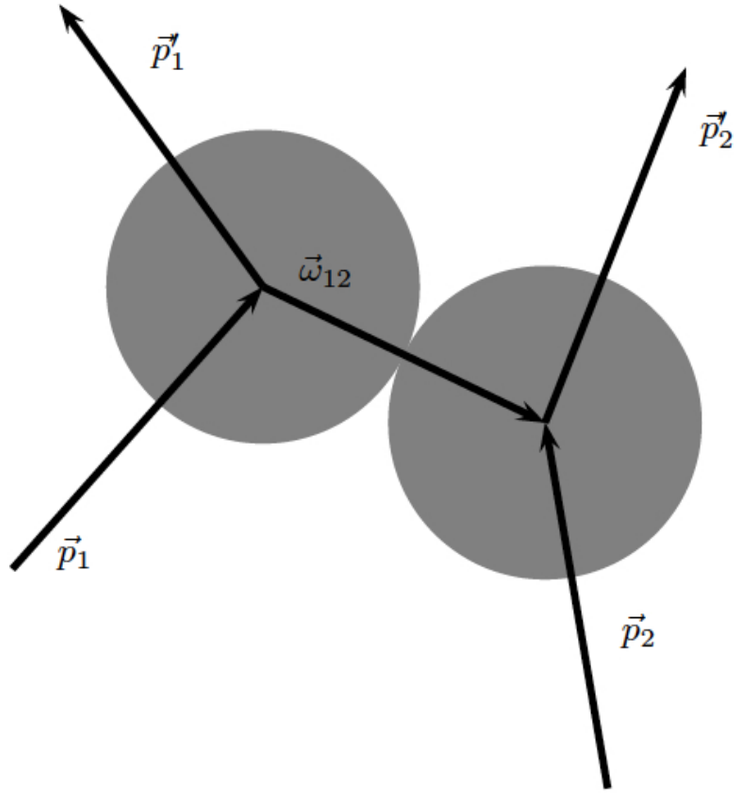
Classical mechanics
Microscopic scale

$$\frac{\partial \mu}{\partial t} = \{H, \mu\}$$

Liouville equation



- *N* hard spheres: mass *m*, diameter *a*
- motion equations of classical mechanics
- elastic collisions



Hard-spheres gas

Classical mechanics
Microscopic scale

$$\frac{\partial \mu}{\partial t} = \{H, \mu\}$$

Liouville equation

Lanford's theorem (Boltzmann-Grad limit: $N \to \infty$, $a \to 0$, $Na^2 \sim 1$)

Kinetic theory of gases
Mesoscopic scale

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F)$$

Boltzmann equation

Hard-spheres gas

Classical mechanics
Microscopic scale

$$\frac{\partial \mu}{\partial t} = \{H, \mu\}$$

Liouville equation

Lanford's theorem ↓ (Boltzmann-Grad limit: $N \to \infty$, $a \to 0$, $Na^2 \sim 1$)

Kinetic theory of gases
Mesoscopic scale

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F)$$

Boltzmann equation

Asymptotic expansions ↓ (hydrodynamic limit: $Na^2 \to \infty$)

Fluid mechanics
Macroscopic scale

$$\partial_t u + \nabla_x u - \nu \Delta_x u = 0 :$$

Navier-Stokes equations

A mathematical derivation

Due to: Lanford, Cercignani, Spohn, Gallagher, Golse, Saint-Raymond…

Hilbert's sixth problem (International Congress of Mathematicians):

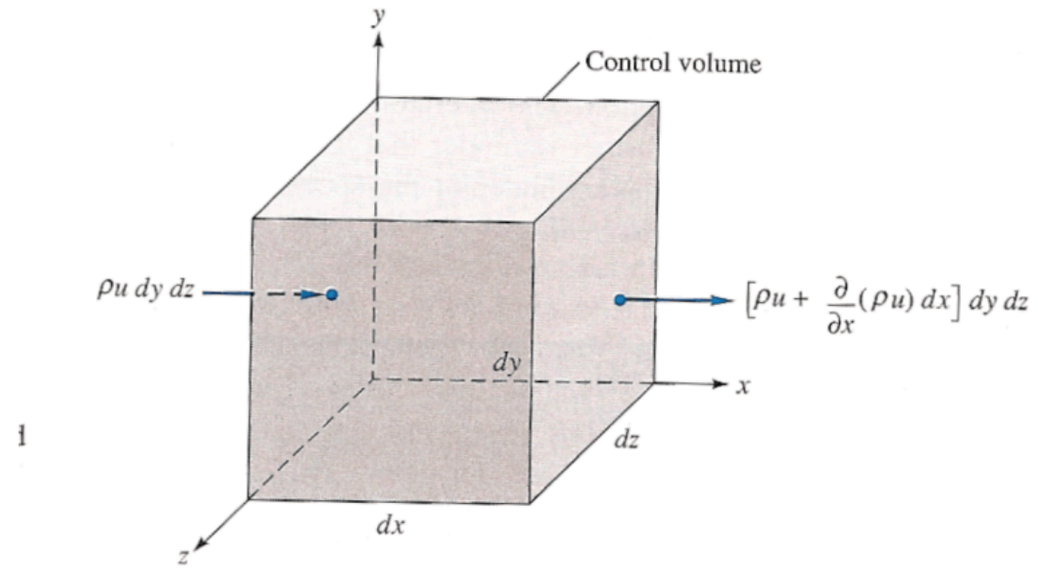
to develop "mathematically the limiting processes [. . .] which lead from the atomistic view to the laws of motion of continua"(Hilbert 1900)

It is not complete proof (yet). However, there are promising partial results (notably on linearized cases)

Other derivations in physics

- The constructions of the Navier-Stokes equations from Newton equations (in physics textbooks)

- Not based on the microdynamics: applying Newton's equations to volume elements of the fluid, not to hard spheres



- "This method fails to relate equations of state or transport coefficients to microscopic data."(Golse 2015)

From meso to macro

Kinetic theory of gases
Mesoscopic scale

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F)$$

Boltzmann equation

Asymptotic expansions ↓ (hydrodynamic limit: $Na^2 \to \infty$)

Fluid mechanics
Macroscopic scale

$$\partial_t u + \nabla_x u - \nu \Delta_x u = 0 :$$

Navier-Stokes equations

Failure of limiting reduction

Asymptotic expansions

The solution of the Boltzmann equation is expanded as infinite power series of the parameter ε (Hilbert expansion, Chapman-Enskog expansion)

$\varepsilon = 1/Na^2$ is the mean free distance between particle collisions

$$f = \sum \varepsilon^n f_n$$

If truncated at the first order, this solution is asymptotic to the solution of the Navier-Stokes equations (Grad 1963, Gallagher 2019, Slemrod 2018)

When $\varepsilon \to 0$, $f_0 + \varepsilon f_1 \to$ the solution of the Navier-Stokes equations

Singular limit

The asymptotic expansion is singular.

The Boltzmann equation used to asymptotically obtain the NS equations:

$$\partial f/\partial t + v\partial f/\partial x = \mathbf{1/\varepsilon\ Q(f,f)}$$

Harold Grad: “the parameter ε enters into the Boltzmann equation in a **singular** manner, and **the equation loses all meaning if ε =0**. **On the other hand, all solutions converge as ε → 0**” (1963, p. 148).

- when ε =0, the Boltzmann equation is not defined
- when ε → 0, the solution of the Boltzmann equation tends to the solution of the NS equations

Failure of the limiting reduction

The **hydrodynamic limit** $\varepsilon \to 0$ that connects the Boltzmann equation to the Navier-Stokes equations is **singular.**

Another instance of a failure of the limiting reduction between two physical theories

wave optics	kinetic theory of gases
geometrical optics	fluid mechanics
$\lambda \to 0$	$\varepsilon \to 0$
shortwave limit	hydrodynamic limit

Failure of the limiting reduction

The definition of the fluid velocity requires the scale invariance of u.

“The definition of u consists in **intertwining** the evolution of the Boltzmann equation with the invariance group of the Navier-Stokes equations”(Golse 2005)

Scale invariance: if $u \equiv u(t, x)$ is a solution of the Navier-Stokes equations, then $\alpha u(\alpha^2 t, \alpha x)$ is also a solution of the Navier-Stokes equations

The challenge of the reduction

What philosophical morals?

No limiting reduction for fluid mechanics from this mathematical derivation of the Navier-Stokes equations

However, fluid mechanics is not doomed to be irreducible

- Future scientific progress might be made to avoid a singular limit and asymptotic expansions

- We might have to (re)turn to other accounts of reduction

Divergent series and approximation

Asymptotic expansion in an infinite power series $f = \sum \varepsilon^n f_n$

Truncation at the first order, the solution is asymptotic to the solution of the Navier-Stokes equations

However, the infinite series is **divergent** (Grad 1963, p. 149)

Should we interpret these asymptotic expansions as providing approximations?

> “While truncations of convergent expansions can naturally be interpreted as approximations, the same is not true of divergent expansions” (Miller forthcoming, p. 3).

Conclusion

Focus on the derivation of the Navier-Stokes equations from a hard-spheres gas, by using the Boltzmann equation as an intermediate step.

Claim: the hydrodynamic limit ($\varepsilon \to 0$) is singular, which prevents a limiting reduction

Another case for the literature on the limiting reduction and singular limits.

Further discussion is needed (divergent series)

Some references

Berry, M. V. (1994). Asymptotics, singularities and the reduction of theories. In Prawitz, D., Skyrms, B, and Westerstahl, D., (Eds.), *Logic, Methodology and Philosophy of Science, Studies in Logic and Foundations of Mathematics*, pp. 597-607.

Batterman, R. (2020), Intertheory Relations in Physics, *The Stanford Encyclopedia of Philosophy*, E. Zalta (ed.). <https://plato.stanford.edu/archives/fall2020/entries/physics-interrelate/>

Batterman B. (2002). *The Devil in the Details.* OUP, New York.

Batterman B. (2012). "The Tyranny of Scales," in Robert Batterman (ed.), *The Oxford Handbook of Philosophy of Physics*, Oxford: Oxford University Press, pp. 255–286

Batterman, R. (1995). Theories between theories: asymptotic limiting intertheoretic relations, *Synthese*, 103: 171-201

Butterfield., J. (2011). Less is Different: Emergence and Reduction Reconciled. *Foundations of Physics,* 41(6): 1065-1135.

Gallagher, I. (2019). From Newton to Navier-Stokes, or How to Connect Fluid mechanics Equations from Microscopic to Macroscopic Scales, *Bull. Amer. Math. Soc*. **56**, 65-85.

Golse, F. (2005). The Boltzmann Equation and Its Hydrodynamic Limits, HANDBOOK OF DIFFERENTIAL EQUATIONS Evolutionary Equations, volume 2

Grad, H. (1963). Asymptotic Theory of the Boltzmann Equation, *The Physics of Fluids,* 6: 147-181.

Miller, M. (forthcoming). Mathematical Structure and Empirical Content, *British Journal for the Philosophy of Science.* <http://philsci-archive.pitt.edu/12678/1/structure_and_content_archive.pdf>

Nickles, T. (1973). Two Concepts of Intertheoretic Reduction. *The Journal of Philosophy,* 70(7), 181-201.

Slemrod, M. (2018). Hilbert's sixth problem and the failure of the Boltzmann to Euler limit, *Phil. Trans. R. Soc. A* 376: 20170222.