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Rayleigh limit of generalized Lorenz-Mie theory for on-axis beams and its relationship with the dipole theory of forces. Part II: Non-dark axisymmetric beams of the second kind and dark axisymmetric beams, including a review.

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Abstract

A recent work has been devoted to the study of the Rayleigh limit of generalized Lorenz-Mie theory for on-axis beams in the case of non dark axisymmetric beams of the first kind. The present work complements this previous work by studying the case of non-dark axisymmetric beams of the second kind and of dark axisymmetric beams. This paper being presumably the last one of a series devoted to the Rayleigh limit of the generalized Lorenz-Mie theory in the case of lossless particles, it is complemented by a mini-review allowing one to gain an overview of the issue.

Keywords: Optical forces; Rayleigh limit; Generalized Lorenz-Mie theory; Dipole theory of forces; Poynting vector and symmetries; Beam shape coefficients and symmetries.

1 Introduction.

This paper pertains to a series of papers devoted to the study of the Rayleigh limit of generalized Lorenz-Mie theory (GLMT) and to its relationship with the dipole theory of forces, and more particularly to the case of on-axis beams when the details of the formulation are simplified with respect to the more complicated case of off-axis beams. We furthermore more specifically consider the case of on-axis axisymmetric beams which allows one to deal with more simplifications. In a Part I-paper, we paid a particular attention to the case of non dark axisymmetric beams of the first kind, with the example of Gaussian beams. The present Part II-paper is devoted to another class of on-axis axisymmetric beams encompassing both non dark axisymmetric beams of the second kind and dark beams. This work is therefore to be viewed as being located at the confluence of three rivers (i) generalized Lorenz-Mie theory, (ii) symmetries in the framework of generalized Lorenz-Mie theories, and (iii) optical forces. We now summarize a few features characterizing each of these rivers, for the sake of a better understanding of the sequel.

The generalized Lorenz-Mie theory describes the interaction between an illuminating arbitrarily shaped electromagnetic beam and a homogeneous sphere defined by its diameter and its complex refractive index, e.g. [1], [2], [3] with [4] for a recent review. In this framework, the illuminating beam is encoded in a set of beam shape coefficients (BSCs) traditionally denoted as $g_{n, TM}^m$ and $g_{n, TE}^m$, with TM and TE standing for "Transverse Magnetic" and "Transverse Electric" respectively, and n ranging from 1 to infinity, with $-n \leq m \leq +n$ (more generally, there exists a class of GLMTs describing the interaction between arbitrarily shaped beams and scattering particles possessing enough degree of symmetry to allow one to use the method of separation of variables, e.g. [5]). BSCs are useful as well when using the extended boundary condition method (EBCM) which describes scattering phenomena in the case of arbitrary shaped particles [6], [7], usually for plane wave illumination (in which case BSCs are trivial) or more generally for arbitrarily shaped illumination, e.g. subsection 8.1 in [8].

Concerning the study of symmetries in the framework of GLMT, a first attack has been published in the general framework of off-axis configurations [9], with BSCs evaluated using a technique called a quadrature technique [10], [11], leading to a series of symmetry relations making various quantities relevant to the scattering properties, such as cross-sections, invariant with respect to some coordinate changes, associated with similar invariant relations related to the BSCs. Such symmetry relations could be useful to simplify or check analytical works, or to speed up numerical evaluations by avoiding useless repetitive calculations.

These symmetries, however, were mainly related to the configurations describing the interaction between the beam and scattering particle, rather than to the beam itself. The search for symmetries of BSCs, specifically related to the beam itself, in connection with symmetry properties of the Poynting vector, started in 1996 [12]. This paper introduced the concept of axisymmetric beams according to the following definition: an axisymmetric beam is a beam for which the component of the Poynting vector in the direction of propagation does not

depend on the azimuthal angle in a suitably chosen coordinate system. In practice, the suitably chosen coordinate system is a coordinate system O_{xyz} , with the z -axis being the direction of propagation of the beam. The component of the Poynting vector \mathbf{S} in the direction of propagation is then the component S_z which, by definition, does not depend on φ , with (r, θ, φ) being the usual spherical coordinates attached to the Cartesian coordinates (x, y, z) . The description of the beam in such a configuration is called an on-axis configuration.

We then exhibited a class of BSCs symmetries characterizing an on-axis axisymmetric beam in which all BSCs are zero but the ones for which $m = \pm 1$, satisfying:

$$g_{n,TM}^m = g_{n,TE}^m = 0, m \neq \pm 1 \quad (1)$$

$$g_n/2 = g_{n,TM}^1 = g_{n,TM}^{-1}/K = -i\varepsilon g_{n,TE}^1 = i\varepsilon g_{n,TE}^{-1}/K \quad (2)$$

in which K describes the state of polarization of the beam (for instance, $K = +1$ for an on-axis Gaussian beam polarized in the x -direction at its focal waist, e.g. [13], [14]) and $\varepsilon = \pm 1$ defines the beam-propagation direction. The existence of such axisymmetric beams has been revisited in 2017 [15] in an enlarged context, including a discussion of vortex and non-vortex beams, of dark and non-dark beams, and of BSC-symmetries in spheroidal and cylindrical coordinates.

Using a refined terminology, such beams have been called on-axisymmetric beams of the first kind. Furthermore, axisymmetry, just as defined above, could also be better called longitudinal axisymmetry to remind us that the definition relies on the property of the longitudinal component of the Poynting vector. This would also allow us to introduce a complementary definition, the one of transverse axisymmetry in which the transverse component $(S_x^2 + S_y^2)^{1/2}$ would not depend on the azimuthal angle φ . A beam which is both longitudinally axisymmetric and transversely axisymmetric beam is a circularly symmetric beam. Examples will be provided in the sequel. The relationships, in terms of BSCs symmetries, between longitudinal axisymmetry, transverse axisymmetry and circularly symmetry have not yet been worked out.

Let us now introduce the concept of dark and non-dark beams as mentioned above. Dark beams (more specifically on-axis dark beams, i.e. dark beams in an on-axis configuration which constitutes the framework of the present paper) are beams in which the longitudinal component S_z of the Poynting vector is 0 on the axis (i.e. for $\theta = 0$). According to a darkness theorem [16], [15], these beams satisfy, whether they are axisymmetric or not:

$$g_{n,TM}^{\pm 1} = g_{n,TE}^{\pm 1} = 0 \quad (3)$$

Therefore, axisymmetric beams of the first kind are non-dark. However, axisymmetric beams of the first kind do not exhaust the list of non-dark axisymmetric beams, as shortly commented between Eqs.(65) and (66) of [15]. Indeed, beside non-dark axisymmetric beams of the first kind when all BSCs of the form $g_{n,X}^{\pm 1}$ are different from 0 as shown in Eq.2 and dark beams (not necessarily axisymmetric) when all of them are 0 as shown in Eq.3, there is some room for a possible case when some of them, but not all of them, are 0. Such beams, which are non-dark, are called axisymmetric beams of the second kind. The BSCs of axisymmetric beams of the second kind and of dark beams (now with the selection of axisymmetric dark beams) satisfy specific symmetry properties which will be recalled below, according to the results obtained in [17].

Concerning the confluent third river, the one of optical forces, we recall that GLMT is a perfect and rigorous tool to evaluate optical forces (and torques) as reviewed in [18]. The specific case of Rayleigh particles is however particularly interesting because it leads to simple expressions which can be easily manipulated and interpreted. An extensive study of optical forces in the Rayleigh regime of GLMT has then been recently achieved, as reviewed in Section 10. For the time being, it is sufficient to know that optical forces on Rayleigh particles by (non-dark) axisymmetric beams of the first kind have been studied in a Part-I paper [19]. The present Part II-paper deals with the cases of (non-dark) axisymmetric on-axis beams of the second kind and of axisymmetric on-axis dark beams which share the BSC-symmetries of Eqs.26 and 27 which will be presented below in their context.

The paper is organized as follows. For the sake of convenience of the reader, Section 2 recalls various basic expressions which have been published in different places of the literature and which are fundamental basic expressions for use in the sequel. Section 3 summarizes the framework in which the study of axisymmetric beams of the second kind and axisymmetric dark beams have been carried out in [17], and makes accordingly a list of different cases to be studied separately. Section 4 deals with the expression of transverse radiation cross-sections (which may be viewed as a way of expressing optical forces) for the different cases considered in Section 3 and introduces subcases. Similarly, section 5 deals with longitudinal pressure radiation cross-sections specified for the different subcases. Section 6 then considers the case of Rayleigh particles, leading to simplifications in the expressions of the transverse and of the longitudinal cross-sections. Section 7 deals with the decomposition of optical forces catalogued as gradient, scattering and non-standard forces. Sections 8 deals with a trivial case. Section 9 provides examples while section 10 provides a mini-review of the whole work carried out to the study of optical forces in the framework of the Rayleigh regime of GLMT. Section 11 is a conclusion.

2 Preliminaries.

All expressions in this section (and in all the paper) use the normalization condition $E_0 H_0^*/2 = 1$. The validity of such a normalization is ensured by the fact that beams considered in GLMT propagate in vacuum (so that the electric E_0 and magnetic H_0 strengths are proportional with a coefficient of proportionality pertaining to the set of real numbers). The transverse components of the Poynting vector then read as, e.g. [12], [15]:

$$S_x = \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \quad (4)$$

$$\left[\frac{k \sin \varphi}{r} (\psi_n'' + \psi_n) A_{nmpq} + \frac{ik \cos \theta \cos \varphi}{r} (\psi_n'' + \psi_n) B_{nmpq} + \frac{i \sin \theta \cos \varphi}{r^2} C_{nmpq} \right]$$

$$S_y = \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \quad (5)$$

$$\left[-\frac{k \cos \varphi}{r} (\psi_n'' + \psi_n) A_{nmpq} + \frac{ik \cos \theta \sin \varphi}{r} (\psi_n'' + \psi_n) B_{nmpq} + \frac{i \sin \theta \sin \varphi}{r^2} C_{nmpq} \right]$$

in which :

$$A_{nmpq} = [\alpha_{nmpq} \psi_p' \tau_p^{|q|} + q \beta_{nmpq} \psi_p \pi_p^{|q|}] P_n^{|m|} \quad (6)$$

$$B_{nmpq} = [-q \alpha_{nmpq} \psi_p' \pi_p^{|q|} - \beta_{nmpq} \psi_p \tau_p^{|q|}] P_n^{|m|} \quad (7)$$

$$C_{nmpq} = \alpha_{nmpq} (m \psi_n' \psi_p' \pi_n^{|m|} \tau_p^{|q|} + q \psi_n \psi_p \tau_n^{|m|} \pi_p^{|q|}) \quad (8)$$

$$+ \beta_{nmpq} (mq \psi_n' \psi_p \pi_n^{|m|} \pi_p^{|q|} - \psi_n \psi_p' \tau_n^{|m|} \tau_p^{|q|})$$

in which :

$$\alpha_{nmpq} = g_{p, TM}^{q*} g_{n, TE}^m - g_{n, TM}^m g_{p, TE}^{q*} \quad (9)$$

$$\beta_{nmpq} = g_{n, TM}^m g_{p, TM}^{q*} + g_{p, TE}^{q*} g_{n, TE}^m \quad (10)$$

The longitudinal component reads as:

$$S_z = \frac{-1}{r^2} \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} i c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} (\sin \theta S_{np}^{mq} + \cos \theta C_{np}^{mq}) \quad (11)$$

in which:

$$\begin{aligned} S_{np}^{mq} = & kr [-g_{n, TM}^m g_{p, TM}^{q*} \psi_p(\psi_n + \psi_n'') P_n^{|m|} \tau_p^{|q|} \\ & + g_{n, TE}^m g_{p, TE}^{q*} \psi_n(\psi_p + \psi_p'') P_p^{|q|} \tau_n^{|m|} \\ & + q g_{n, TM}^m g_{p, TE}^{q*} \psi_p'(\psi_n + \psi_n'') P_n^{|m|} \pi_p^{|q|} \\ & + m g_{n, TM}^m g_{p, TE}^{q*} \psi_n'(\psi_p + \psi_p'') P_p^{|q|} \pi_n^{|m|}] \end{aligned} \quad (12)$$

$$\begin{aligned} C_{np}^{mq} = & -g_{n, TM}^m g_{p, TM}^{q*} \psi_p \psi_n' (\tau_n^{|m|} \tau_p^{|q|} + m q \pi_n^{|m|} \pi_p^{|q|}) \\ & + g_{n, TM}^m g_{p, TE}^{q*} \psi_n \psi_p' (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|}) \\ & - g_{n, TE}^m g_{p, TM}^{q*} \psi_p \psi_n (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|}) \\ & + g_{n, TE}^m g_{p, TE}^{q*} \psi_n \psi_p' (m q \pi_n^{|m|} \pi_p^{|q|} + \tau_n^{|m|} \tau_p^{|q|}) \end{aligned} \quad (13)$$

In these equations, ψ_n denotes Riccati-Bessel functions with the argument kr (k the wavenumber) omitted for convenience, a prime denotes a derivative of a function with respect to its argument (and a double prime to a second derivative), the coefficients c_n^{pw} ("pw" standing for "plane wave") are coefficients which occur in a natural way in the Bromwich formulation of the usual Lorenz-Mie theory, and read as [14] :

$$c_n^{pw} = \frac{1}{ik} (-i)^n \frac{2n+1}{n(n+1)} \quad (14)$$

Furthermore, τ_n^m and π_n^m , with argument $\cos \theta$ omitted for convenience, are generalized Legendre functions defined according to :

$$\pi_n^m(\cos \theta) = \frac{P_n^m(\cos \theta)}{\sin \theta} \quad (15)$$

$$\tau_n^m(\cos \theta) = \frac{dP_n^m(\cos \theta)}{d\theta} \quad (16)$$

in which $P_n^m(\cos \theta)$ are the associated Legendre functions defined according to Hobson's convention :

$$P_n^m(\cos \theta) = (-1)^m (\sin \theta)^m \frac{d^m P_n(\cos \theta)}{(d \cos \theta)^m} \quad (17)$$

in which $P_n(\cos \theta)$ are the Legendre polynomials.

The transverse pressure radiation cross-sections, from Eqs.(3.181) and (3.185) of [3], in which we conveniently change m to p , and p to k , read as:

$$\begin{aligned} \begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} &= \frac{\lambda^2}{2\pi} \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \sum_{p=k-1 \neq 0}^{\infty} \frac{(n+k)!}{(n-k)!} \\ &\times \left[\begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{pn}^{k-1} + S_{np}^{-k} - 2U_{pn}^{k-1} - 2U_{np}^{-k}) \left(\frac{\delta_{p,n+1}}{p^2} - \frac{\delta_{n,p+1}}{n^2} \right) \right. \\ &\left. + \frac{2n+1}{n^2(n+1)^2} \delta_{np} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (T_{pn}^{k-1} - T_{np}^{-k} - 2V_{pn}^{k-1} + 2V_{np}^{-k}) \right] \end{aligned} \quad (18)$$

in which:

$$S_{np}^m = (a_n + a_p^*) g_{n, TM}^m g_{p, TM}^{m+1*} + (b_n + b_p^*) g_{n, TE}^m g_{p, TE}^{m+1*} \quad (19)$$

$$T_{np}^m = -i(a_n + b_p^*) g_{n, TM}^m g_{p, TE}^{m+1*} + i(b_n + a_p^*) g_{n, TE}^m g_{p, TM}^{m+1*} \quad (20)$$

$$U_{np}^m = a_n a_p^* g_{n, TM}^m g_{p, TM}^{m+1*} + b_n b_p^* g_{n, TE}^m g_{p, TE}^{m+1*} \quad (21)$$

$$V_{np}^m = i b_n a_p^* g_{n, TE}^m g_{p, TM}^{m+1*} - i a_n b_p^* g_{n, TM}^m g_{p, TE}^{m+1*} \quad (22)$$

in which a_n and b_n are Mie coefficients (anticipating, the reader may refer to Eqs.110-111 below) and δ_{ij} denotes the Kronecker symbol.

The longitudinal cross-section reads as, e.g. Eq.3.159 of [3]:

$$\begin{aligned}
C_{pr,z} &= \frac{\lambda^2}{\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|m|)!}{(n-|m|)!} \right. & (23) \\
&\times \text{Re}[(a_n + a_{n+1}^* - 2a_n a_{n+1}^*)g_{n,TM}^m g_{n+1,TM}^{m*} \\
&+ (b_n + b_{n+1}^* - 2b_n b_{n+1}^*)g_{n,TE}^m g_{n+1,TE}^{m*}] \\
&+ m \frac{2n+1}{n^2(n+1)^2} \frac{(n+|m|)!}{(n-|m|)!} \\
&\times \text{Re}[i(2a_n b_n^* - a_n - b_n^*)g_{n,TM}^m g_{n,TE}^{m*}] \left. \right\}
\end{aligned}$$

which, for further use in the sequel, may conveniently be rewritten as:

$$C_{pr,z} = \frac{\lambda^2}{\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^n C_n^m \quad (24)$$

3 Restricted framework of the study and cases to be considered separately.

The study of axisymmetric beams of the second kind and of axisymmetric dark beams in [17] has been carried out in a restricted framework in which it was assumed that all BSCs are zero excepted $g_{n,TM}^M$, $g_{n,TE}^M$, $g_{n,TM}^Q$ and $g_{n,TE}^Q$, in which M and Q are specified values of the azimuthal mode index m . Under such circumstances, it has been found from Eq.11 that the longitudinal component S_z of the Poynting vector, in which these restrictions are implemented, denoted S_z^{MQ} , reads as:

$$\begin{aligned}
S_z^{MQ} &= \frac{-1}{r^2} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*} & (25) \\
&\{ e^{i(M-Q)\varphi} (\sin \theta S_{np}^{MQ} + \cos \theta C_{np}^{MQ}) + e^{i(Q-M)\varphi} (\sin \theta S_{np}^{QM} + \cos \theta C_{np}^{QM}) \} \\
&= 0
\end{aligned}$$

We now recall that we are interested in axisymmetric beams, i.e. beams for which the longitudinal component S_z does not depend on φ . It is seen from

Eq.25 that, if $M = Q$, then the beam is trivially axisymmetric. Because this case is trivial, it is excluded for the time being and will be considered separately in section 8. This case being excluded, the conditions for axisymmetry have been found to read as:

$$g_{n, TM}^M g_{p, TM}^{Q*} + g_{p, TE}^{Q*} g_{n, TE}^M = 0 \quad (26)$$

$$g_{n, TM}^M g_{p, TE}^{Q*} - g_{p, TM}^{Q*} g_{n, TE}^M = 0 \quad (27)$$

Before listing the different cases to be studied, let us point out that M and Q play equivalent roles. First, this is obvious from the fact that the list $g_{n, TM}^M, g_{n, TE}^M, g_{n, TM}^Q$ and $g_{n, TE}^Q$ of non-zero BSCs does not give any privilege to one or to the other of the superscripts M or Q . Second, formally, it can be readily deduced from Eqs.26 and 27. Then, the list of the cases to be studied is as follows:

(i) $M = 1$ and $Q \neq \pm 1$. The fact that $Q \neq 1$ comes from the fact that $M \neq Q$ (otherwise, we would be facing the trivial case whose study has been postponed). Furthermore, assume $Q = -1$. Then the list of non-zero BSCs is $g_{n, TM}^1, g_{n, TE}^1, g_{n, TM}^{-1}$ and $g_{n, TE}^{-1}$ which is solved by the first kind of axisymmetry. The beams of the present case (i) are non-dark. This case is named the case SK1 (in which "SK" stands for "second kind").

(ii) $M = -1$ and $Q \neq \pm 1$ with comments similar to those of case (i). This case is named the case SK2.

(ii) $M \neq \pm 1$ and $Q \neq \pm 1$ with $M \neq Q$. This is the case of dark beams. It is called case AD (in which "AD" stands for "Axisymmetric Dark").

We shall later see that these cases imply subcases, namely case SK1 will be separated in subcases SK1A, SK1B, SK1C and case SK2 will be separated in subcases SK2A, SK2B, SK2C.

4 Transverse pressure radiation cross-sections specified for the different cases.

4.1 Case SK1.

Eqs.19-22 show that, if $M = 1$, i.e. if $g_{n, TM}^1$ and $g_{n, TE}^1$ are non-zero, then the transverse cross-sections are zero but for $Q = 2$ and 0. This is because the BSCs in Eqs.19-22 occur in the form of products reading as $g_n^m g_p^{m+1*}$. Furthermore, the case $(m, m + 1)$ corresponding to $M = 1, Q = 2$ must be completed

by the case $(m-1, m)$ which is a variant of $(m, m+1)$ but which corresponds to $Q = 0, M = 1$. Therefore, we have:

$$C_{pr,x} = C_{pr,y} = 0 \text{ for } M = 1, Q \neq 0, 2, \pm 1 \quad (28)$$

This is the subcase SK1A. The case $M = 1, Q = 2$ is then a subcase, denoted as SK1B to be studied separately, while $M = 1, Q = 0$, to be studied separately as well is the case SK1C. For SK1B, we start from Eqs.18-22. Eq.18 involves a summation from $k = 1$ to ∞ . For $k = 1$, the coefficients S, U, T and V (subscripts and superscripts omitted) and of Eqs.19-22 occur with the superscripts 0 and -1 , therefore involving BSCs with superscripts 0 and -1 which are zero. Hence the term with $k = 1$ in Eq.18 is 0 as well. For $k = 3, S, U, T$ and V occur with the superscripts 2 and -3 . The superscript 2 implies the occurrence of BSCs g_n^2 which are not zero since $Q = 2$ is allowed, but they occur in products of the form $g_n^2 g_n^3$ which are zero since the BSCs g_n^3 are 0. The superscript -3 in S, U, T and V implies the occurrence of BSCs g_n^{-3} which are zero. Therefore, the contribution of $k = 3$ to Eq.18 is zero. The same is true for $k = 4, 5, \dots$ because the coefficients S, U, T and V occur with the superscripts 3 and $-4, 4$ and $-5, \dots$ corresponding to BSCs which are zero. It is then found that the only non-zero contribution to Eq.18 is for $k = 2$.

For $k = 2$, some coefficients S, U, T and V occur with a superscript -2 and are then zero, while the others occur with the superscript 1 and are different from 0, leading to:

$$\begin{aligned} \begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} &= \frac{\lambda^2}{2\pi} \sum_{n=2}^{\infty} \frac{(n+2)!}{(n-2)!} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} \left(\frac{S_{n+1,n}^1 - 2U_{n+1,n}^1}{(n+1)^2} - \frac{S_{n-1,n}^1 - 2U_{n-1,n}^1}{n^2} \right) \\ &+ \frac{2n+1}{n^2(n+1)^2} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (T_{nn}^1 - 2V_{nn}^1) \end{aligned} \quad (29)$$

in which:

$$S_{n+1,n}^1 = (a_{n+1} + a_n^*) g_{n+1, TM}^1 g_{n, TM}^{2*} + (b_{n+1} + b_n^*) g_{n+1, TE}^1 g_{n, TE}^{2*} \quad (30)$$

$$S_{n-1,n}^1 = (a_{n-1} + a_n^*) g_{n-1, TM}^1 g_{n, TM}^{2*} + (b_{n-1} + b_n^*) g_{n-1, TE}^1 g_{n, TE}^{2*} \quad (31)$$

$$U_{n+1,n}^1 = a_{n+1} a_n^* g_{n+1, TM}^1 g_{n, TM}^{2*} + b_{n+1} b_n^* g_{n+1, TE}^1 g_{n, TE}^{2*} \quad (32)$$

$$U_{n-1,n}^1 = a_{n-1}a_n^*g_{n-1,TM}^1g_{n,TM}^{2*} + b_{n-1}b_n^*g_{n-1,TE}^1g_{n,TE}^{2*} \quad (33)$$

$$T_{nn}^1 = -i(a_n + b_n^*)g_{n,TM}^1g_{n,TE}^{2*} + i(b_n + a_n^*)g_{n,TE}^1g_{n,TM}^{2*} \quad (34)$$

$$V_{nn}^1 = ib_na_n^*g_{n,TE}^1g_{n,TM}^{2*} - ia_nb_n^*g_{n,TM}^1g_{n,TE}^{2*} \quad (35)$$

Using Eqs.26-27, Eqs.30-35 simplify to:

$$S_{n+1,n}^1 = (a_{n+1} + a_n^* - b_{n+1} - b_n^*)g_{n+1,TM}^1g_{n,TM}^{2*} \quad (36)$$

$$S_{n-1,n}^1 = (a_{n-1} + a_n^* - b_{n-1} - b_n^*)g_{n-1,TM}^1g_{n,TM}^{2*} \quad (37)$$

$$U_{n+1,n}^1 = (a_{n+1}a_n^* - b_{n+1}b_n^*)g_{n+1,TM}^1g_{n,TM}^{2*} \quad (38)$$

$$U_{n-1,n}^1 = (a_{n-1}a_n^* - b_{n-1}b_n^*)g_{n-1,TM}^1g_{n,TM}^{2*} \quad (39)$$

$$T_{nn}^1 = i(b_n + a_n^* - a_n - b_n^*)g_{n,TM}^1g_{n,TE}^{2*} \quad (40)$$

$$V_{nn}^1 = i(b_na_n^* - a_nb_n^*)g_{n,TM}^1g_{n,TE}^{2*} \quad (41)$$

As a remark, the reader might find surprising that we have here a case of axisymmetric beams for which the transverse forces are not equal to zero (other cases will later occur as well). The explanation is that we are dealing with longitudinal axisymmetry beams, not with circularly symmetric beams which are both longitudinally symmetric and transversely symmetric. It is easy to conjecture that transversely symmetry would add supplementary conditions which would make these transverse forces becoming equal to zero.

Similarly, for SK1C, but remarking that only the term $k = 1$ contributes in the summation, we obtain:

$$\begin{aligned}
\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} &= \frac{\lambda^2}{2\pi} \sum_{n=1}^{\infty} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} \left[\frac{(n+1)!}{(n-1)!} \frac{S_{n+1,n}^0 - 2U_{n+1,n}^0}{(n+1)^2} \right. \\
&\quad - \frac{(n+2)!}{n!} \frac{S_{n,n+1}^0 - 2U_{n,n+1}^0}{(n+1)^2} \\
&\quad \left. + \frac{(n+1)!}{(n-1)!} \frac{2n+1}{n^2(n+1)^2} (T_{nn}^0 - 2V_{nn}^0) \right]
\end{aligned} \tag{42}$$

in which:

$$S_{n+1,n}^0 = (a_{n+1} + a_n^*)g_{n+1, TM}^0 g_{n, TM}^{1*} + (b_{n+1} + b_n^*)g_{n+1, TE}^0 g_{n, TE}^{1*} \tag{43}$$

$$S_{n,n+1}^0 = (a_n + a_{n+1}^*)g_{n, TM}^0 g_{n+1, TM}^{1*} + (b_n + b_{n+1}^*)g_{n, TE}^0 g_{n+1, TE}^{1*} \tag{44}$$

$$U_{n+1,n}^0 = a_{n+1}a_n^*g_{n+1, TM}^0 g_{n, TM}^{1*} + b_{n+1}b_n^*g_{n+1, TE}^0 g_{n, TE}^{1*} \tag{45}$$

$$U_{n,n+1}^0 = a_n a_{n+1}^* g_{n, TM}^0 g_{n+1, TM}^{1*} + b_n b_{n+1}^* g_{n, TE}^0 g_{n+1, TE}^{1*} \tag{46}$$

$$T_{nn}^0 = -i(a_n + b_n^*)g_{n, TM}^0 g_{n, TE}^{1*} + i(b_n + a_n^*)g_{n, TE}^0 g_{n, TM}^{1*} \tag{47}$$

$$V_{nn}^0 = ib_n a_n^* g_{n, TE}^0 g_{n, TM}^{1*} - ia_n b_n^* g_{n, TM}^0 g_{n, TE}^{1*} \tag{48}$$

Using Eqs.26-27, Eqs.43-48 simplify to:

$$S_{n+1,n}^0 = (a_{n+1} + a_n^* - b_{n+1} - b_n^*)g_{n+1, TM}^0 g_{n, TM}^{1*} \tag{49}$$

$$S_{n,n+1}^0 = (a_n + a_{n+1}^* - b_n - b_{n+1}^*)g_{n, TM}^0 g_{n+1, TM}^{1*} \tag{50}$$

$$U_{n+1,n}^0 = (a_{n+1}a_n^* - b_{n+1}b_n^*)g_{n+1, TM}^0 g_{n, TM}^{1*} \tag{51}$$

$$U_{n,n+1}^0 = (a_n a_{n+1}^* - b_n b_{n+1}^*) g_{n, TM}^0 g_{n+1, TM}^{1*} \quad (52)$$

$$T_{nn}^0 = i(b_n + a_n^* - a_n - b_n^*) g_{n, TM}^0 g_{n, TE}^{1*} \quad (53)$$

$$V_{nn}^0 = i(b_n a_n^* - a_n b_n^*) g_{n, TM}^0 g_{n, TE}^{1*} \quad (54)$$

4.2 Case SK2.

We proceed similarly as for the case SK1 but, as a variant convenient for the sequel, we shall isolate terms corresponding to $n > 1$ for the sake of conciseness, while they have been preserved in the case SK1 for the sake of pedagogic skills. We then readily find that:

$$C_{pr,x} = C_{pr,y} = 0 \text{ for } M = -1, Q \neq 0, \pm 1, -2 \quad (55)$$

This is the case SK2A. The case SK2B is then for $M = -1, Q = 0$, and we also have to consider separately the case SK2C for which $M = -1, Q = -2$. For the case SK2B, we start from Eq.18, find that all terms with $k = 2, \dots, \infty$ are 0 because they involve coefficients S, T, U and V which are zero, due to the SK2B conditions, i.e. $M = -1, Q = 0$. We then express the remaining term for $k = 1$ as a 3-term summation according to :

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = C_1 + C_2 + C_3 \quad (56)$$

in which, cancelling some other coefficients S, U, T and V which are again 0 still due to the SK2B conditions:

$$C_1 = \frac{\lambda^2}{2\pi} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{(n+1)!}{(n-1)!} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{np}^{-1} - 2U_{np}^{-1}) \frac{\delta_{p,n+1}}{p^2} \quad (57)$$

$$C_2 = \frac{-\lambda^2}{2\pi} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{(n+1)!}{(n-1)!} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{np}^{-1} - 2U_{np}^{-1}) \frac{\delta_{n,p+1}}{n^2} \quad (58)$$

$$C_3 = \frac{\lambda^2}{2\pi} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{(n+1)!}{(n-1)!} \frac{2n+1}{n^2(n+1)^2} \delta_{np} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (2V_{np}^{-1} - T_{np}^{-1}) \quad (59)$$

We then find:

$$C_1 = \frac{\lambda^2}{4\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{12}^{-1} - 2U_{12}^{-1}) + (C_1)_{n>1} \quad (60)$$

in which $(C_1)_{n>1}$ only contains Mie coefficients a_n and b_n with $n > 1$. Similarly:

$$C_2 = \frac{-3\lambda^2}{4\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{21}^{-1} - 2U_{21}^{-1}) + (C_2)_{n>1} \quad (61)$$

$$C_3 = \frac{3\lambda^2}{4\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (2V_{11}^{-1} - T_{11}^{-1}) + (C_3)_{n>1} \quad (62)$$

so that we have:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = \frac{\lambda^2}{4\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{12}^{-1} - 2U_{12}^{-1} - 3S_{21}^{-1} + 6U_{21}^{-1} + 6V_{11}^{-1} - 3T_{11}^{-1}) + \begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix}_{n>1}^{SK2B} \quad (63)$$

in which the first collection of terms on the r.h.s. of Eq.63 are the the $(n = 1)$ -terms while the last term corresponds to $n > 1$, and:

$$S_{12}^{-1} = (a_1 + a_2^*)g_{1,TM}^{-1}g_{2,TM}^{0*} + (b_1 + b_2^*)g_{1,TE}^{-1}g_{2,TE}^{0*} \quad (64)$$

$$S_{21}^{-1} = (a_2 + a_1^*)g_{2,TM}^{-1}g_{1,TM}^{0*} + (b_2 + b_1^*)g_{2,TE}^{-1}g_{1,TE}^{0*} \quad (65)$$

$$T_{11}^{-1} = -i(a_1 + b_1^*)g_{1,TM}^{-1}g_{1,TE}^{0*} + i(b_1 + a_1^*)g_{1,TE}^{-1}g_{1,TM}^{0*} \quad (66)$$

$$U_{12}^{-1} = a_1a_2^*g_{1,TM}^{-1}g_{2,TM}^{0*} + b_1b_2^*g_{1,TE}^{-1}g_{2,TE}^{0*} \quad (67)$$

$$U_{21}^{-1} = a_2 a_1^* g_{2, TM}^{-1} g_{1, TM}^{0*} + b_2 b_1^* g_{2, TE}^{-1} g_{1, TE}^{0*} \quad (68)$$

$$V_{11}^{-1} = i b_1 a_1^* g_{1, TE}^{-1} g_{1, TM}^{0*} - i a_1 b_1^* g_{1, TM}^{-1} g_{1, TE}^{0*} \quad (69)$$

Using Eqs.26-27, Eqs.64-69 simplify to:

$$S_{12}^{-1} = (a_1 + a_2^* - b_1 - b_2^*) g_{1, TM}^{-1} g_{2, TM}^{0*} \quad (70)$$

$$S_{21}^{-1} = (a_2 + a_1^* - b_2 - b_1^*) g_{2, TM}^{-1} g_{1, TM}^{0*} \quad (71)$$

$$T_{11}^{-1} = i(b_1 + a_1^* - a_1 - b_1^*) g_{1, TM}^{-1} g_{1, TE}^{0*} \quad (72)$$

$$U_{12}^{-1} = (a_1 a_2^* - b_1 b_2^*) g_{1, TM}^{-1} g_{2, TM}^{0*} \quad (73)$$

$$U_{21}^{-1} = (a_2 a_1^* - b_2 b_1^*) g_{2, TM}^{-1} g_{1, TM}^{0*} \quad (74)$$

$$V_{11}^{-1} = i(b_1 a_1^* - a_1 b_1^*) g_{1, TM}^{-1} g_{1, TE}^{0*} \quad (75)$$

For case SK2C ($M = -1, Q = -2$), we proceed similarly and, noticing that all the coefficients S, T, U, V are zero, but for $k = 2$, we obtain an expression which may be simplified to:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = C_1 + C_2 + C_3 \quad (76)$$

in which:

$$C_1 = \frac{\lambda^2}{2\pi} \sum_{n=2}^{\infty} \sum_{p=1}^{\infty} \frac{(n+2)!}{(n-2)!} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{np}^{-2} - 2U_{np}^{-2}) \frac{\delta_{p,n+1}}{p^2} \quad (77)$$

$$C_2 = \frac{-\lambda^2}{2\pi} \sum_{n=2}^{\infty} \sum_{p=1}^{\infty} \frac{(n+2)!}{(n-2)!} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{np}^{-2} - 2U_{np}^{-2}) \frac{\delta_{n,p+1}}{n^2} \quad (78)$$

$$C_3 = \frac{\lambda^2}{2\pi} \sum_{n=2}^{\infty} \sum_{p=1}^{\infty} \frac{(n+2)!}{(n-2)!} \frac{2n+1}{n^2(n+1)^2} \delta_{np} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (2V_{np}^{-2} - T_{np}^{-2}) \quad (79)$$

These quantities C_i are afterward evaluated explicitly by isolating relevant terms in the summations leading to:

$$C_i = (C_i)_{n>1}, \quad i = 1, 3 \quad (80)$$

$$C_2 = \frac{-3\lambda^2}{\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{21}^{-2} - 2U_{21}^{-2}) + (C_2)_{n>1} \quad (81)$$

so that, as a whole, we obtain:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = \frac{-3\lambda^2}{\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{21}^{-2} - 2U_{21}^{-2}) + \begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix}_{n>1}^{SK2C} \quad (82)$$

in which, from Eqs.19 and 21:

$$S_{21}^{-2} = (a_2 + a_1^*)g_{2,TM}^{-2}g_{1,TM}^{-1*} + (b_2 + b_1^*)g_{2,TE}^{-2}g_{1,TE}^{-1*} \quad (83)$$

$$U_{21}^{-2} = a_2a_1^*g_{2,TM}^{-2}g_{1,TM}^{-1*} + b_2b_1^*g_{2,TE}^{-2}g_{1,TE}^{-1*} \quad (84)$$

Using axisymmetry conditions of Eqs.26-27, these last expressions simplify to:

$$S_{21}^{-2} = (a_2 + a_1^* - b_2 - b_1^*)g_{2,TM}^{-2}g_{1,TM}^{-1*} \quad (85)$$

$$U_{21}^{-2} = (a_2a_1^* - b_2b_1^*)g_{2,TM}^{-2}g_{1,TM}^{-1*} \quad (86)$$

4.3 Case AD.

For this case, we have $M \neq \pm 1$, $Q \neq \pm 1$ and $M \neq Q$. Then, examining Eq.18, it is readily found that the coefficients S , U , T and V are zero for $k = 1$ and 2. Therefore, we obtain:

$$\begin{aligned} \begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} &= \frac{\lambda^2}{2\pi} \sum_{k=3}^{\infty} \sum_{n=k}^{\infty} \sum_{p=k-1}^{\infty} \frac{(n+k)!}{(n-k)!} \\ &\times \left[\begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (S_{pn}^{k-1} + S_{np}^{-k} - 2U_{pn}^{k-1} - 2U_{np}^{-k}) \left(\frac{\delta_{p,n+1}}{p^2} - \frac{\delta_{n,p+1}}{n^2} \right) \right. \\ &\left. + \frac{2n+1}{n^2(n+1)^2} \delta_{np} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} (T_{pn}^{k-1} - T_{np}^{-k} - 2V_{pn}^{k-1} + 2V_{np}^{-k}) \right] \end{aligned} \quad (87)$$

As we shall see in the sequel, we do not need to elaborate more on this equation. We just need to remark that the coefficients a_n involved in this equation have $n > 1$.

Finally, for convenience in the sequel, we now summarize below the conditions defining the different cases:

$$\left[\begin{array}{ll} SK1A & M = 1, Q \neq 0, 2, \pm 1 \\ SK1B & M = 1, Q = 2 \\ SK1C & M = 1, Q = 0 \\ SK2A & M = -1, Q \neq 0, \pm 1, -2 \\ SK2B & M = -1, Q = 0 \\ SK2C & M = -1, Q = -2 \\ AD & M \neq \pm 1, Q \neq \pm 1, M \neq Q \\ Trivial & M = Q \end{array} \right]$$

5 Longitudinal pressure radiation cross-sections specified for the different cases.

5.1 Case SK1A.

For this case, non-zero BSCs are those which are of the form g_n^1 and g_n^Q with $Q \neq 0, 2, \pm 1$. In particular, let us consider the summations of Eq.23, and rewrite it as follows:

$$\begin{aligned}
C_{pr,z} = & \frac{\lambda^2}{\pi} \sum_{m=-1}^1 \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|m|)!}{(n-|m|)!} \right. \\
& \times \text{Re}[(a_n + a_{n+1}^* - 2a_n a_{n+1}^*)g_{n,TM}^m g_{n+1,TM}^{m*} \\
& + (b_n + b_{n+1}^* - 2b_n b_{n+1}^*)g_{n,TE}^m g_{n+1,TE}^{m*}] \\
& + m \frac{2n+1}{n^2(n+1)^2} \frac{(n+|m|)!}{(n-|m|)!} \\
& \times \text{Re}[i(2a_n b_n^* - a_n - b_n^*)g_{n,TM}^m g_{n,TE}^{m*}] \Big\}_{n=1} \\
& + \frac{\lambda^2}{\pi} \sum_{n=2}^{\infty} \sum_{m=-n}^n \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|m|)!}{(n-|m|)!} \right. \\
& \times \text{Re}[(a_n + a_{n+1}^* - 2a_n a_{n+1}^*)g_{n,TM}^m g_{n+1,TM}^{m*} \\
& + (b_n + b_{n+1}^* - 2b_n b_{n+1}^*)g_{n,TE}^m g_{n+1,TE}^{m*}] \\
& + m \frac{2n+1}{n^2(n+1)^2} \frac{(n+|m|)!}{(n-|m|)!} \\
& \times \text{Re}[i(2a_n b_n^* - a_n - b_n^*)g_{n,TM}^m g_{n,TE}^{m*}] \Big\}
\end{aligned} \tag{88}$$

In Eq.88, the second term, i.e. the summation from $n = 2$ to ∞ contains Mie coefficients a_n and b_n with $n > 1$ and will be denoted $(C_{pr,z})_{n>1}^{SK1A}$. The first term is for $n = 1$ and contains a summation over m from -1 to $+1$. The BSCs occur in the form $g_1^m g_2^{m*}$. In the case SK1A, when the non-zero BSCs are those which are of the form g_n^1 and g_n^Q with $Q \neq 0, 2, \pm 1$, we only retain $m = 1$ and Eq.88 is then found to become:

$$\begin{aligned}
C_{pr,z} = & \frac{3\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1 a_2^*)g_{1,TM}^1 g_{2,TM}^{1*} \\
& + (b_1 + b_2^* - 2b_1 b_2^*)g_{1,TE}^1 g_{2,TE}^{1*}] \\
& + \text{Re}[i(2a_1 b_1^* - a_1 - b_1^*)g_{1,TM}^1 g_{1,TE}^{1*}] \} + (C_{pr,z})_{n>1}^{SK1A}
\end{aligned} \tag{89}$$

Using the axisymmetry conditions of Eqs.26-27, this simplifies to:

$$\begin{aligned}
C_{pr,z} = & \frac{3\lambda^2}{2\pi} \{ \text{Re}(a_1 + a_2^* - 2a_1 a_2^* - b_1 - b_2^* + 2b_1 b_2^*)g_{1,TM}^1 g_{2,TM}^{1*} \\
& + \text{Re}[i(2a_1 b_1^* - a_1 - b_1^*)g_{1,TM}^1 g_{1,TE}^{1*}] \} + (C_{pr,z})_{n>1}^{SK1A}
\end{aligned} \tag{90}$$

5.2 Case SK1B.

Non-zeros BSCs are of the form g_n^1 and g_n^2 . Let us express $C_{pr,z}$ using Eq.24. For the case SK1B, when m can take only the values 1 and 2, we obtain:

$$C_{pr,z} = \frac{\lambda^2}{\pi} (C_1^1 + \sum_{n=2}^{\infty} C_n^2) \quad (91)$$

The second term contains only Mie coefficients with $n > 1$ and will be denoted $(C_{pr,z})_{n>1}^{SK1B}$ so that, working out C_1^1 , we obtain:

$$\begin{aligned} C_{pr,z} = & \frac{3\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1a_2^*)g_{1,TM}^1 g_{2,TM}^{1*}] \\ & + (b_1 + b_2^* - 2b_1b_2^*)g_{1,TE}^1 g_{2,TE}^{1*}] \\ & + \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^1 g_{1,TE}^{1*}] \} + (C_{pr,z})_{n>1}^{SK1B} \end{aligned} \quad (92)$$

which, using Eqs.26-27 simplifies to:

$$\begin{aligned} C_{pr,z} = & \frac{3\lambda^2}{2\pi} \{ \text{Re}(a_1 + a_2^* - 2a_1a_2^* - b_1 - b_2^* + 2b_1b_2^*)g_{1,TM}^1 g_{2,TM}^{1*} \\ & + \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^1 g_{1,TE}^{1*}] \} + (C_{pr,z})_{n>1}^{SK1B} \end{aligned} \quad (93)$$

Let us note that the main terms of Eqs.92 and 93 are the same than in Eqs.89 and 90 respectively.

5.3 Case SK1C.

For this case in which $M = 1, Q = 0$, we obtain:

$$C_{pr,z} = \frac{\lambda^2}{\pi} (C_1^0 + C_1^1) + (C_{pr,z})_{n>1}^{SK1C} \quad (94)$$

becoming:

$$\begin{aligned}
C_{pr,z} = & \frac{\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1a_2^*)(g_{1,TM}^0 g_{2,TM}^{0*} + 3g_{1,TM}^1 g_{2,TM}^{1*})] \\
& + (b_1 + b_2^* - 2b_1b_2^*)(g_{1,TE}^0 g_{2,TE}^{0*} + 3g_{1,TE}^1 g_{2,TE}^{1*}) \} \\
& + 3 \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^1 g_{1,TE}^{1*}] + (C_{pr,z})_{n>1}^{SK1C}
\end{aligned} \quad (95)$$

which, using the axisymmetry conditions of Eqs.26-27 becomes:

$$\begin{aligned}
C_{pr,z} = & \frac{\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1a_2^* - b_1 - b_2^* + 2b_1b_2^*) \\
& \times (g_{1,TM}^0 g_{2,TM}^{0*} + 3g_{1,TM}^1 g_{2,TM}^{1*}) \\
& + 3 \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^1 g_{1,TE}^{1*}] \} + (C_{pr,z})_{n>1}^{SK1C}
\end{aligned} \quad (96)$$

5.4 Case SK2A.

For this case in which $M = -1$, $Q \neq 0, \pm 1, -2$, Eq.24 becomes:

$$C_{pr,z} = \frac{\lambda^2}{\pi} \left[\sum_{m=-1}^{+1} C_1^m + \sum_{n=2}^{\infty} \sum_{m=-n}^n C_n^m \right] \quad (97)$$

The second term involves Mie coefficients with $n > 1$ and is denoted $(C_{pr,z})_{n>1}^{SK2A}$. In the first term, in which m ranges from -1 to $+1$, the values 0 and $+1$ are excluded because $Q \neq 0, +1$ and we are left with:

$$C_{pr,z} = \frac{\lambda^2}{\pi} C_1^{-1} + (C_{pr,z})_{n>1}^{SK2A} \quad (98)$$

becoming:

$$\begin{aligned}
C_{pr,z} = & \frac{3\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1a_2^*)g_{1,TM}^{-1} g_{2,TM}^{-1*}] \\
& + (b_1 + b_2^* - 2b_1b_2^*)g_{1,TE}^{-1} g_{2,TE}^{-1*}] \\
& - \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^{-1} g_{1,TE}^{-1*}] \} + (C_{pr,z})_{n>1}^{SK2A}
\end{aligned} \quad (99)$$

Using the axisymmetry conditions of Eqs.26-27, this simplifies to:

$$C_{pr,z} = \frac{3\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1a_2^* - b_1 - b_2^* + 2b_1b_2^*)g_{1,TM}^{-1}g_{2,TM}^{-1*}] - \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^{-1}g_{1,TE}^{-1*}] \} + (C_{pr,z})_{n>1}^{SK2A} \quad (100)$$

5.5 Case SK2B.

In this case for which $M = -1$ and $Q = 0$, Eq.24 reduces to:

$$C_{pr,z} = \frac{\lambda^2}{\pi} (C_1^{-1} + C_1^0) + (C_{pr,z})_{n>1}^{SK2B} \quad (101)$$

becoming:

$$C_{pr,z} = \frac{\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1a_2^*)(3g_{1,TM}^{-1}g_{2,TM}^{-1*} + g_{1,TM}^0g_{2,TM}^{0*}) + (b_1 + b_2^* - 2b_1b_2^*)(3g_{1,TE}^{-1}g_{2,TE}^{-1*} + g_{1,TE}^0g_{2,TE}^{0*})] - 3 \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^{-1}g_{1,TE}^{-1*}] \} + (C_{pr,z})_{n>1}^{SK2B} \quad (102)$$

Using the axisymmetry conditions of Eqs.26-27, this simplifies to:

$$C_{pr,z} = \frac{\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1a_2^* - b_1 - b_2^* + 2b_1b_2^*) \times (3g_{1,TM}^{-1}g_{2,TM}^{-1*} + g_{1,TM}^0g_{2,TM}^{0*}) - 3 \text{Re}[i(2a_1b_1^* - a_1 - b_1^*)g_{1,TM}^{-1}g_{1,TE}^{-1*}] \} + (C_{pr,z})_{n>1}^{SK2B} \quad (103)$$

5.6 Case SK2C.

For this case, we readily have:

$$C_{pr,z} = \frac{\lambda^2}{\pi} C_1^{-1} + (C_{pr,z})_{n>1}^{SK2C} \quad (104)$$

becoming:

$$\begin{aligned} C_{pr,z} = & \frac{3\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1 a_2^*) g_{1, TM}^{-1} g_{2, TM}^{-1*}] \\ & + (b_1 + b_2^* - 2b_1 b_2^*) g_{1, TE}^{-1} g_{2, TE}^{-1*} \\ & - \text{Re}[i(2a_1 b_1^* - a_1 - b_1^*) g_{1, TM}^{-1} g_{1, TE}^{-1*}] \} + (C_{pr,z})_{n>1}^{SK2C} \end{aligned} \quad (105)$$

Using the axisymmetry conditions of Eqs.26-27, this simplifies to:

$$\begin{aligned} C_{pr,z} = & \frac{3\lambda^2}{2\pi} \{ \text{Re}[(a_1 + a_2^* - 2a_1 a_2^* - b_1 - b_2^* + 2b_1 b_2^*) g_{1, TM}^{-1} g_{2, TM}^{-1*}] \\ & - \text{Re}[i(2a_1 b_1^* - a_1 - b_1^*) g_{1, TM}^{-1} g_{1, TE}^{-1*}] \} + (C_{pr,z})_{n>1}^{SK2C} \end{aligned} \quad (106)$$

Let us note that the main terms of Eqs.105 and 106 are the same than the main terms of Eqs.99 and 100 respectively.

5.7 Case AD.

We now have $M \neq \pm 1$, $Q \neq \pm 1$ and $M \neq Q$ so that Eq.24 becomes:

$$C_{pr,z} = \frac{\lambda^2}{\pi} C_1^0 + (C_{pr,z})_{n>1}^{AD} \quad (107)$$

in which the second term involves Mie coefficients with $n > 1$ and, working out the first term, we obtain:

$$\begin{aligned} C_{pr,z} = & \frac{\lambda^2}{2\pi} \text{Re}[(a_1 + a_2^* - 2a_1 a_2^*) g_{1, TM}^0 g_{2, TM}^{0*}] \\ & + (b_1 + b_2^* - 2b_1 b_2^*) g_{1, TE}^0 g_{2, TE}^{0*}] + (C_{pr,z})_{n>1}^{AD} \end{aligned} \quad (108)$$

Using the axisymmetry conditions of Eqs.26-27, this simplifies to:

$$C_{pr,z} = \frac{\lambda^2}{2\pi} \text{Re}(a_1 + a_2^* - 2a_1a_2^* - b_1 - b_2^* + 2b_1b_2^*)g_{1,TM}^0g_{2,TM}^{0*} + (C_{pr,z})_{n>1}^{AD} \quad (109)$$

6 Pressure radiation cross-sections for Rayleigh particles.

In the dipole theory of forces, we only retain the Mie coefficients a_1 and b_1 [20]. These coefficients may be written as [21], and [22], pp. 143-144:

$$a_1 = \frac{2i}{3} \frac{n_p^2 - 1}{n_p^2 + 2} \alpha^3 + O(i\alpha^5) + \frac{4}{9} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 \alpha^6 \quad (110)$$

$$b_1 = O(i\alpha^5) \quad (111)$$

in which n_p is the refractive index with respect to the surrounding medium and α is the size parameter $\pi d/\lambda$. In the present paper dealing with lossless particles, the refractive index is taken to be real. The other coefficients a_i and b_i ($i > 1$) involves still higher powers of α . In the Rayleigh limit of small point-like particles, we may therefore only retain the Mie coefficient a_1 with real parts proportional to α^6 and imaginary parts proportional to α^3 while higher powers are discarded, so that we may write:

$$\text{Im}(a_1) = \frac{2}{3} \frac{n_p^2 - 1}{n_p^2 + 2} \alpha^3 \quad (112)$$

$$\text{Re}(a_1) = \frac{4}{9} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 \alpha^6 \quad (113)$$

Complementary discussions on this issue may be found in [23], [24] and [25]. Instead of assuming that n_p is real, we might have taken it as being a complex number. This would add more terms associated with optical absorption forces which are not considered in the present paper. In the sequel, Eqs.112 and 113 have to be kept in mind, even when they are not explicitly introduced in the equations. We now deal with the different cases already discussed previously but now made specific to the case of Rayleigh particles.

6.1 Transverse cross-sections.

In the case SK1A, we obtain:

$$C_{pr,x} = C_{pr,y} = 0 \quad (114)$$

which is simply Eq.28 repeated. For the case SK1B, we use Eq.29. The summation from $n = 3$ to ∞ is zero, and the only non zero terms come from the term $n = 2$, due to the existence of remaining $a_{n-1} = a_1$ -terms. We then readily obtain:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = \frac{-3\lambda^2}{\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} a_1 g_{1,TM}^1 g_{2,TM}^{2*} \quad (115)$$

For the case SK1C, we start from Eq.42 and obtain:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = \frac{\lambda^2}{4\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} [S_{21}^0 - 2U_{21}^0 - 3(S_{12}^0 - 2U_{12}^0) + 3(T_{11}^0 - 2V_{11}^0)] \quad (116)$$

in which, from Eqs.49-54:

$$S_{21}^0 = a_1^* g_{2,TM}^0 g_{1,TM}^{1*} \quad (117)$$

$$S_{12}^0 = a_1 g_{1,TM}^0 g_{2,TM}^{1*} \quad (118)$$

$$U_{21}^0 = 0 \quad (119)$$

$$U_{12}^0 = 0 \quad (120)$$

$$T_{11}^0 = i(a_1^* - a_1) g_{1,TE}^0 g_{1,TE}^{1*} \quad (121)$$

$$V_{11}^0 = 0 \quad (122)$$

so that Eq.116 simplifies to:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = \frac{\lambda^2}{4\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} [a_1^* g_{2,TM}^0 g_{1,TM}^{1*} - 3a_1 g_{1,TM}^0 g_{2,TM}^{1*} + 3i(a_1^* - a_1) g_{1,TM}^0 g_{1,TE}^{1*}] \quad (123)$$

For the case SK2A, we obtain:

$$C_{pr,x} = C_{pr,y} = 0 \quad (124)$$

which is simply Eq.55 repeated.

For the case SK2B, we use Eqs.63-69, find that the coefficients U and V vanish, while the coefficients S and T simplify, and obtain:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = \frac{\lambda^2}{4\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} [a_1 (g_{1,TM}^{-1} g_{2,TM}^{0*} + 3i g_{1,TM}^{-1} g_{1,TE}^{0*}) - 3a_1^* (g_{2,TM}^{-1} g_{1,TM}^{0*} + i g_{1,TE}^{-1} g_{1,TM}^{0*})] \quad (125)$$

For the case SK2C, we use Eqs.82, 85 and 86 to obtain:

$$\begin{pmatrix} C_{pr,x} \\ C_{pr,y} \end{pmatrix} = \frac{-3\lambda^2}{\pi} \begin{pmatrix} \text{Re} \\ \text{Im} \end{pmatrix} a_1^* g_{2,TM}^{-2} g_{1,TM}^{-1*} \quad (126)$$

For case AD, we return to Eq.87 and recall that it involves Mie coefficients a_n with $n > 1$ so that, for Rayleigh particles, we immediately obtain:

$$C_{pr,x} = C_{pr,y} = 0 \quad (127)$$

6.2 Longitudinal cross-sections.

For case SK1A, we use Eq.89 and obtain:

$$C_{pr,z} = \frac{3\lambda^2}{2\pi} \operatorname{Re}[a_1(g_{1,TM}^1 g_{2,TM}^{1*} - i g_{1,TM}^1 g_{1,TE}^{1*})] \quad (128)$$

For the case SK1B, we use Eq.92 and obtain:

$$C_{pr,z} = \frac{3\lambda^2}{2\pi} \operatorname{Re}[a_1(g_{1,TM}^1 g_{2,TM}^{1*} - i g_{1,TM}^1 g_{1,TE}^{1*})] \quad (129)$$

For the case SK1C, we use Eq.96 and obtain:

$$C_{pr,z} = \frac{\lambda^2}{2\pi} \{ \operatorname{Re}[a_1(g_{1,TM}^0 g_{2,TM}^{0*} + 3g_{1,TM}^1 g_{2,TM}^{1*})] - 3 \operatorname{Re}[i a_1 g_{1,TM}^1 g_{1,TE}^{1*}] \} \quad (130)$$

For the case SK2A, we use Eq.99 and obtain:

$$C_{pr,z} = \frac{3\lambda^2}{2\pi} \operatorname{Re}[a_1(g_{1,TM}^{-1} g_{2,TM}^{-1*} + i g_{1,TM}^{-1} g_{1,TE}^{-1*})] \quad (131)$$

For the case SK2B, we use Eq.102 and obtain:

$$C_{pr,z} = \frac{\lambda^2}{2\pi} \operatorname{Re}[a_1(3g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^0 g_{2,TM}^{0*} + 3i g_{1,TM}^{-1} g_{1,TE}^{-1*})] \quad (132)$$

For the case SK2C, we use Eq.106 and obtain:

$$C_{pr,z} = \frac{3\lambda^2}{2\pi} \{ \operatorname{Re}[a_1 g_{1,TM}^{-1} g_{2,TM}^{-1*}] + \operatorname{Re}[i a_1 g_{1,TM}^{-1} g_{1,TE}^{-1*}] \} \quad (133)$$

For the case AD, we use Eq.108 and obtain:

$$C_{pr,z} = \frac{\lambda^2}{2\pi} \operatorname{Re}(a_1 g_{1,TM}^0 g_{2,TM}^{0*}) \quad (134)$$

7 Decomposition of optical forces.

7.1 General expressions.

The gradient transverse forces in the x -direction are given by, e.g. Eqs.(42) and (38) of [26]:

$$C_{pr,x}^G = \frac{-\lambda^2}{4\pi} \text{Im}(a_1) \text{Im}(H) = \frac{-\lambda^2}{6\pi} \frac{n_p^2 - 1}{n_p^2 + 2} \alpha^3 \text{Im}(H) \quad (135)$$

in which:

$$\begin{aligned} H = & g_{2,TM}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\ & - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*}) \\ & + 3i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1)] \end{aligned} \quad (136)$$

The scattering transverse forces in the x -direction are given by, e.g. Eqs.(77) and (78) of [26], and Eqs.(22) and (23) of [27]:

$$\begin{aligned} C_{pr,x}^S &= \frac{3\lambda^2}{4\pi} \text{Re}(a_1) \text{Re} i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1)] \\ &= \frac{\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \text{Re} i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1)] \end{aligned} \quad (137)$$

which, by using $\text{Re}(iz) = -\text{Re}(iz^*)$ may be rewritten as:

$$\begin{aligned} C_{pr,x}^S &= \frac{3\lambda^2}{4\pi} \text{Re}(a_1) \text{Re} i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^0(g_{1,TM}^{1*} - g_{1,TM}^{-1*})] \\ &= \frac{\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \text{Re} i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^0(g_{1,TM}^{1*} - g_{1,TM}^{-1*})] \end{aligned} \quad (138)$$

The non-standard transverse forces in the x -direction are given by, e.g. Eq.(76) of [26] and (21) of [27]:

$$\begin{aligned}
C_{pr,x}^{NS} &= \frac{\lambda^2}{4\pi} \text{Re}(a_1) \quad (139) \\
&\times \text{Re}[g_{2,TM}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\
&\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*})] \\
&= \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \\
&\times \text{Re}[g_{2,TM}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\
&\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*})]
\end{aligned}$$

The gradient transverse forces in the y -direction are given by, e.g. Eq.(43) and (38) of [26], corrected by Eqs.(19) and (15) of [27]:

$$C_{pr,y}^G = \frac{\lambda^2}{4\pi} \text{Im}(a_1) \text{Re}(H') = \frac{\lambda^2}{6\pi} \frac{n_p^2 - 1}{n_p^2 + 2} \alpha^3 \text{Re}(H') \quad (140)$$

in which:

$$\begin{aligned}
H' &= g_{2,TM}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) - 3g_{1,TM}^0(g_{2,TM}^{1*} - g_{2,TM}^{-1*}) \quad (141) \\
&\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} - g_{1,TM}^{-1} g_{2,TM}^{-2*}) \\
&\quad - 3i[g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*}) - g_{1,TE}^{0*}(g_{1,TM}^{-1} + g_{1,TM}^1)]
\end{aligned}$$

The scattering transverse forces in the y -direction are given by, e.g. Eqs.(86)-(87) of [26], corrected by Eqs.(25) and (26) of [27]:

$$\begin{aligned}
C_{pr,y}^S &= \frac{3\lambda^2}{4\pi} \text{Re}(a_1) \text{Im} i[g_{1,TE}^{0*}(g_{1,TM}^{-1} + g_{1,TM}^1) - g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*})] \quad (142) \\
&= \frac{\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \text{Im} i[g_{1,TE}^{0*}(g_{1,TM}^{-1} + g_{1,TM}^1) - g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*})]
\end{aligned}$$

The non-standard transverse forces in the y -direction are given by, e.g. Eq.(85) of [26]; corrected by Eq.(24) of [27]:

$$\begin{aligned}
C_{pr,y}^{NS} &= \frac{\lambda^2}{4\pi} \text{Re}(a_1) \\
&\times \text{Im}[g_{2,TM}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) - 3g_{1,TM}^0(g_{2,TM}^{1*} - g_{2,TM}^{-1*}) \\
&\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} - g_{1,TM}^{-1} g_{2,TM}^{-2*})] \\
&= \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \\
&\times \text{Im}[g_{2,TM}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) - 3g_{1,TM}^0(g_{2,TM}^{1*} - g_{2,TM}^{-1*}) \\
&\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} - g_{1,TM}^{-1} g_{2,TM}^{-2*})]
\end{aligned} \tag{143}$$

The longitudinal gradient force reads as, e.g. Eq.(27) of [28] and Eqs.(2) and (64) of [29]:

$$C_{pr,z}^G = \frac{-3\lambda^2}{2\pi} \text{Im}(a_1) \text{Im}(G) = \frac{-\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 2} \alpha^3 \text{Im}(G) \tag{144}$$

in which:

$$G = g_{1,TM}^{-1}(g_{2,TM}^{-1*} + i g_{1,TE}^{-1*}) + g_{1,TM}^1(g_{2,TM}^{1*} - i g_{1,TE}^{1*}) + \frac{1}{3} g_{1,TM}^0 g_{2,TM}^{0*} \tag{145}$$

From Eq.(26) of [28], and Eqs.(2) and (97) of [29], we introduce forces proportional to the sixth-power of α and to $\text{Re}(G)$ according to:

$$C_{pr,z}^s = \frac{3\lambda^2}{2\pi} \text{Re}(a_1) \text{Re}(G) = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \text{Re}(G) \tag{146}$$

Considering separately in G terms which do not contain any coupling between $(n = 1)$ - and $(n = 2)$ -partial waves and those which do contain such couplings, we may express $C_{pr,z}^s$ as the sum of two terms, one corresponding to standard scattering forces denoted as $C_{pr,z}^S$ and the other corresponding to non-standard forces denoted as $C_{pr,z}^{NS}$, according to:

$$C_{pr,z}^s = C_{pr,z}^S + C_{pr,z}^{NS} \tag{147}$$

in which:

$$\begin{aligned}
C_{pr,z}^S &= \frac{3\lambda^2}{2\pi} \text{Re}(a_1) \text{Re}[i(g_{1,TM}^{-1}g_{1,TE}^{-1*} - g_{1,TM}^1g_{1,TE}^{1*})] \quad (148) \\
&= \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \text{Re}[i(g_{1,TM}^{-1}g_{1,TE}^{-1*} - g_{1,TM}^1g_{1,TE}^{1*})]
\end{aligned}$$

$$\begin{aligned}
C_{pr,z}^{NS} &= \frac{3\lambda^2}{2\pi} \text{Re}(a_1) \text{Re}(g_{1,TM}^{-1}g_{2,TM}^{-1*} + g_{1,TM}^1g_{2,TM}^{1*} + \frac{1}{3}g_{1,TM}^0g_{2,TM}^{0*}) \quad (149) \\
&= \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2}\right)^2 \alpha^6 \text{Re}(g_{1,TM}^{-1}g_{2,TM}^{-1*} + g_{1,TM}^1g_{2,TM}^{1*} + \frac{1}{3}g_{1,TM}^0g_{2,TM}^{0*})
\end{aligned}$$

7.2 Simplified expressions for SK- and AD-beams.

We are now rewriting the general expressions of the previous subsection after implementation of Eqs.26-27. The gradient transverse forces in the x -direction of Eq.135 can then be rewritten as:

$$C_{pr,x}^G = \frac{-\lambda^2 n_p^2 - 1}{6\pi n_p^2 + 2} \alpha^3 \text{Im}(H_1 + H_2) \quad (150)$$

in which:

$$\begin{aligned}
\text{Im}(H_1) &= \text{Im}[g_{2,TM}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\
&\quad - 12(g_{1,TM}^1g_{2,TM}^{2*} + g_{1,TM}^{-1}g_{2,TM}^{-2*})] \quad (151)
\end{aligned}$$

while $\text{Im}(H_2)$, using Eqs.26-27, can be simplified to:

$$\begin{aligned}
\text{Im}(H_2) &= 6 \text{Re}(g_{1,TM}^0g_{1,TE}^{-1*} - g_{1,TM}^{1*}g_{1,TE}^0) \quad (152) \\
&= 6 \text{Re}(g_{1,TE}^0g_{1,TM}^{-1*} - g_{1,TE}^{1*}g_{1,TM}^0)
\end{aligned}$$

Concerning products of BSCs which are products of TE -coefficients, they can be expressed in terms of products of TM -coefficients using Eq.26. We choose this convention – expressing TE -coefficients in terms of TM -coefficients –

instead of the reverse one. Therefore, $\text{Im}(H_1)$ is not affected by the axisymmetry conditions.

The scattering transverse forces in the x -direction of Eq.138 similarly simplifies to:

$$C_{pr,x}^S = 0 \quad (153)$$

The non-standard transverse forces in the x -direction of Eq.139 only contains products of TM -coefficients and is therefore not affected. So, we still have:

$$\begin{aligned} C_{pr,x}^{NS} &= \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 \alpha^6 \quad (154) \\ &\quad \text{Re}[g_{2,TM}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\ &\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*})] \end{aligned}$$

The gradient transverse forces in the y -direction of Eq.140 can be rewritten as:

$$C_{pr,y}^G = \frac{\lambda^2}{6\pi} \frac{n_p^2 - 1}{n_p^2 + 2} \alpha^3 \text{Re}(H'_1 + H'_2) \quad (155)$$

in which:

$$\begin{aligned} \text{Re}(H'_1) &= \text{Re}[g_{2,TM}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) - 3g_{1,TM}^0(g_{2,TM}^{1*} - g_{2,TM}^{-1*}) \\ &\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} - g_{1,TM}^{-1} g_{2,TM}^{-2*})] \quad (156) \end{aligned}$$

is not affected while $\text{Re}(H'_2)$ can be rewritten as:

$$\begin{aligned} \text{Re}(H'_2) &= 6 \text{Im}(g_{1,TM}^0 g_{1,TE}^{-1*} + g_{1,TE}^0 g_{1,TM}^{1*}) \quad (157) \\ &= 6 \text{Im}(g_{1,TM}^0 g_{1,TE}^{1*} + g_{1,TE}^0 g_{1,TM}^{-1*}) \end{aligned}$$

The scattering transverse forces in the y -direction of Eq.142 simplifies to:

$$C_{pr,y}^S = 0 \quad (158)$$

The non-standard transverse forces in the y -direction of Eq.143 is not affected:

$$C_{pr,y}^{NS} = \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 \alpha^6 \quad (159)$$

$$\text{Im}[g_{2,TM}^{0*} (g_{1,TM}^{-1} - g_{1,TM}^1) - 3g_{1,TM}^0 (g_{2,TM}^{1*} - g_{2,TM}^{-1*}) - 12(g_{1,TM}^1 g_{2,TM}^{2*} - g_{1,TM}^{-1} g_{2,TM}^{-2*})]$$

The longitudinal gradient force of Eq.144-145 is not affected. We still have:

$$C_{pr,z}^G = \frac{-\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 2} \alpha^3 \text{Im}(G) \quad (160)$$

in which:

$$G = g_{1,TM}^{-1} (g_{2,TM}^{-1*} + i g_{1,TE}^{-1*}) + g_{1,TM}^1 (g_{2,TM}^{1*} - i g_{1,TE}^{1*}) + \frac{1}{3} g_{1,TM}^0 g_{2,TM}^{0*} \quad (161)$$

Similarly, the longitudinal standard scattering forces and the non-standard forces of Eqs.148-149 are not affected and are still given by:

$$C_{pr,z}^S = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 \alpha^6 \text{Re}[i(g_{1,TM}^{-1} g_{1,TE}^{-1*} - g_{1,TM}^1 g_{1,TE}^{1*})] \quad (162)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^1 g_{2,TM}^{1*} + \frac{1}{3} g_{1,TM}^0 g_{2,TM}^{0*}) \quad (163)$$

7.3 Simplified expressions for the special cases. Transverse cross-sections.

For the case SK1A, we use Eqs.150-159 of subsection 7.2 and obtain:

$$C_{pr,x}^G = C_{pr,y}^G = C_{pr,x}^S = C_{pr,y}^S = C_{pr,x}^{NS} = C_{pr,y}^{NS} = 0 \quad (164)$$

agreeing with Eq.114 of subsection 6.1 which tells us that $C_{pr,x} = C_{pr,y} = 0$, and with Eqs.135-143 of subsection 7.1. Concerning the use of equations of subsection 6.1, there is however a remark to be done, useful in the sequel. Namely, in the equations to be examined, a separation has to be done between terms proportional to α^3 which are gradient terms and terms proportional to α^6 which correspond to a summation of scattering and non-standard terms. The same remark is valid for the use of the equations of subsection 6.2 to be used in the next subsection devoted to longitudinal cross-sections.

Similarly for the case SK1B, using the results of subsection 7.2, we obtain:

$$C_{pr,x}^G = \frac{2\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 \text{Im}(g_{1,TM}^1 g_{2,TM}^{2*}) \quad (165)$$

$$C_{pr,x}^{NS} = \frac{-4\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \text{Re}(g_{1,TM}^1 g_{2,TM}^{2*}) \quad (166)$$

$$C_{pr,y}^G = \frac{-2\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 \text{Re}(g_{1,TM}^1 g_{2,TM}^{2*}) \quad (167)$$

$$C_{pr,y}^{NS} = \frac{-4\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \text{Im}(g_{1,TM}^1 g_{2,TM}^{2*}) \quad (168)$$

$$C_{pr,x}^S = C_{pr,y}^S = 0 \quad (169)$$

which agree with the results we would have obtained from subsection 7.1, and from Eq.115 of subsection 6.1.

Similarly, for the other cases, using the results of subsection 7.2 which are found to be agreement with those of subsection 7.1, and with the corresponding equations of subsection 6.1, we obtain, keeping in mind, from Eqs.153 and 158 that $C_{pr,x}^S = C_{pr,y}^S = 0$:

$$C_{pr,x}^G = \frac{\lambda^2 n_p^2 - 1}{6\pi n_p^2 + 1} \alpha^3 [\text{Im}(g_{2,TM}^0 g_{1,TM}^{1*} + 3g_{1,TM}^0 g_{2,TM}^{1*}) + 6 \text{Re}(g_{1,TM}^0 g_{1,TE}^{1*})], \text{ SK1C} \quad (170)$$

$$C_{pr,x}^{S+NS} = \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{2,TM}^0 g_{1,TM}^{1*} - 3g_{1,TM}^0 g_{2,TM}^{1*}), \text{ SK1C} \quad (171)$$

$$C_{pr,y}^G = \frac{-\lambda^2 n_p^2 - 1}{6\pi n_p^2 + 1} \alpha^3 [\text{Re}(g_{2,TM}^0 g_{1,TM}^{1*} + 3g_{1,TM}^0 g_{2,TM}^{1*}) - 6 \text{Im}(g_{1,TM}^0 g_{1,TE}^{1*})], \text{ SK1C} \quad (172)$$

$$C_{pr,y}^{S+NS} = \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Im}(g_{2,TM}^0 g_{1,TM}^{1*} - 3g_{1,TM}^0 g_{2,TM}^{1*}), \text{ SK1C} \quad (173)$$

$$C_{pr,x}^G = C_{pr,x}^S = C_{pr,x}^{NS} = C_{pr,y}^G = C_{pr,y}^S = C_{pr,y}^{NS} = 0, \text{ SK2A} \quad (174)$$

$$C_{pr,x}^G = \frac{-\lambda^2 n_p^2 - 1}{6\pi n_p^2 + 1} \alpha^3 [\text{Im}(g_{1,TM}^{-1} g_{2,TM}^{0*} + 3g_{2,TM}^{-1} g_{1,TM}^{0*}) + 6 \text{Re}(g_{1,TM}^{-1} g_{1,TE}^{0*})], \text{ SK2B} \quad (175)$$

$$C_{pr,x}^{S+NS} = \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^{-1} g_{2,TM}^{0*} - 3g_{2,TM}^{-1} g_{1,TM}^{0*}), \text{ SK2B} \quad (176)$$

$$C_{pr,y}^G = \frac{\lambda^2 n_p^2 - 1}{6\pi n_p^2 + 1} \alpha^3 [\text{Re}(g_{1,TM}^{-1} g_{2,TM}^{0*} + 3g_{2,TM}^{-1} g_{1,TM}^{0*}) - 6 \text{Im}(g_{1,TM}^{-1} g_{1,TE}^{0*})], \text{ SK2B} \quad (177)$$

$$C_{pr,y}^{S+NS} = \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Im}(g_{1,TM}^{-1} g_{2,TM}^{0*} - 3g_{2,TM}^{-1} g_{1,TM}^{0*}), \text{ SK2B} \quad (178)$$

$$C_{pr,x}^G = \frac{-2\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 \operatorname{Im}(g_{2,TM}^{-2} g_{1,TM}^{-1*}), \text{ SK2C} \quad (179)$$

$$C_{pr,x}^{S+NS} = \frac{-4\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \operatorname{Re}(g_{2,TM}^{-2} g_{1,TM}^{-1*}), \text{ SK2C} \quad (180)$$

$$C_{pr,y}^G = \frac{2\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 \operatorname{Re}(g_{2,TM}^{-2} g_{1,TM}^{-1*}), \text{ SK2C} \quad (181)$$

$$C_{pr,y}^{S+NS} = \frac{-4\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \operatorname{Im}(g_{2,TM}^{-2} g_{1,TM}^{-1*}), \text{ SK2C} \quad (182)$$

$$C_{pr,x}^G = C_{pr,x}^S = C_{pr,x}^{NS} = C_{pr,y}^G = C_{pr,y}^S = C_{pr,y}^{NS} = 0, \text{ AD} \quad (183)$$

Let us note that the agreement between the results of subsections 6.1, 7.1 and 7.2 provide a very satisfactory checking of the results displayed.

7.4 Simplified expressions for the special cases. Longitudinal cross-sections.

Similarly, we obtain the results for the longitudinal cross-sections from subsection 7.2 in agreement with those of subsection 7.1 and of subsection 6.2 (instead of subsection 6.1). These results read as:

$$C_{pr,z}^G = \frac{\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 [\operatorname{Re}(g_{1,TM}^1 g_{1,TE}^{1*}) - \operatorname{Im}(g_{1,TM}^1 g_{2,TM}^{1*})], \text{ SK1A and SK1B} \quad (184)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 [\operatorname{Re}(g_{1,TM}^1 g_{2,TM}^{1*}) + \operatorname{Im}(g_{1,TM}^1 g_{1,TE}^{1*})], \text{ SK1A and SK1B} \quad (185)$$

$$C_{pr,z}^S = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \operatorname{Im}(g_{1,TM}^1 g_{1,TE}^{1*}), \text{ SK1A and SK1B} \quad (186)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \operatorname{Re}(g_{1,TM}^1 g_{2,TM}^{1*}), \text{ SK1A and SK1B} \quad (187)$$

$$C_{pr,z}^G = \frac{-\lambda^2}{3\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 [\operatorname{Im}(g_{1,TM}^0 g_{2,TM}^{0*} + 3g_{1,TM}^1 g_{2,TM}^{1*}) - 3 \operatorname{Re}(g_{1,TE}^1 g_{1,TE}^{1*})], \text{ SK1C} \quad (188)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 [\operatorname{Re}(g_{1,TM}^0 g_{2,TM}^{0*} + 3g_{1,TM}^1 g_{2,TM}^{1*}) + 3 \operatorname{Im}(g_{1,TE}^1 g_{1,TE}^{1*})], \text{ SK1C} \quad (189)$$

$$C_{pr,z}^S = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \operatorname{Im}(g_{1,TE}^1 g_{1,TE}^{1*}), \text{ SK1C}$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 [\operatorname{Re}(g_{1,TE}^0 g_{2,TE}^{0*} + 3g_{1,TE}^1 g_{2,TE}^{1*})], \text{ SK1C} \quad (190)$$

$$C_{pr,z}^G = \frac{-\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 [\operatorname{Re}(g_{1,TE}^{-1} g_{1,TE}^{-1*}) + \operatorname{Im}(g_{1,TE}^{-1} g_{2,TE}^{-1*})], \text{ SK2A} \quad (191)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 [\operatorname{Re}(g_{1,TE}^{-1} g_{2,TE}^{-1*}) - \operatorname{Im}(g_{1,TE}^{-1} g_{1,TE}^{-1*})], \text{ SK2A} \quad (192)$$

$$C_{pr,z}^S = \frac{-2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \operatorname{Im}(g_{1,TE}^{-1} g_{1,TE}^{-1*}), \text{ SK2A} \quad (193)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \operatorname{Re}(g_{1,TE}^{-1} g_{2,TE}^{-1*}), \text{ SK2A} \quad (194)$$

$$C_{pr,z}^G = \frac{-\lambda^2}{3\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 [\operatorname{Im}(g_{1,TE}^0 g_{2,TE}^{0*} + 3g_{1,TE}^{-1} g_{2,TE}^{-1*}) + 3 \operatorname{Re}(g_{1,TE}^{-1} g_{1,TE}^{-1*})], \text{ SK2B} \quad (195)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 [\text{Re}(g_{1,TM}^0 g_{2,TM}^{0*} + 3g_{1,TM}^{-1} g_{2,TM}^{-1*}) - 3 \text{Im}(g_{1,TM}^{-1} g_{1,TE}^{-1*})], \text{ SK2B} \quad (196)$$

$$C_{pr,z}^S = \frac{-2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Im}(g_{1,TM}^{-1} g_{1,TE}^{-1*}), \text{ SK2B} \quad (197)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^0 g_{2,TM}^{0*} + 3g_{1,TM}^{-1} g_{2,TM}^{-1*}), \text{ SK2B} \quad (198)$$

$$C_{pr,z}^G = \frac{-\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 [\text{Re}(g_{1,TM}^{-1} g_{1,TE}^{-1*}) + \text{Im}(g_{1,TM}^{-1} g_{2,TM}^{-1*})], \text{ SK2C} \quad (199)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 [\text{Re}(g_{1,TM}^{-1} g_{2,TM}^{-1*}) - \text{Im}(g_{1,TM}^{-1} g_{1,TE}^{-1*})], \text{ SK2C} \quad (200)$$

$$C_{pr,z}^S = \frac{-2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Im}(g_{1,TM}^{-1} g_{1,TE}^{-1*}), \text{ SK2C} \quad (201)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^{-1} g_{2,TM}^{-1*}), \text{ SK2C} \quad (202)$$

$$C_{pr,z}^G = \frac{-\lambda^2}{3\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 \text{Im}(g_{1,TM}^0 g_{2,TM}^{0*}), \text{ AD} \quad (203)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^0 g_{2,TM}^{0*}), \text{ AD} \quad (204)$$

$$C_{pr,z}^S = 0, \text{ AD} \quad (205)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^0 g_{2,TM}^{0*}), \text{ AD} \quad (206)$$

8 The trivial case.

We now consider the trivial case in which $M = Q$. From Eqs.18-22, due to the fact that BSCs occur in the form of $g_n^m g_n^{m+1*}$, we immediately have that, for the trivial case, whether we deal or not with Rayleigh particles, the transverse forces read as:

$$C_{pr,x} = C_{pr,y} = 0 \quad (207)$$

We then have to consider only the longitudinal case of Eq.23 which is valid for the trivial case insofar as BSCs occur in the form of $g_n^m g_n^{m*}$. In the case of Rayleigh particles, however, only the $(n = 1)$ -terms must be retained in the summation. Therefore, from Eq.23, retaining only these terms, we obtain, after rearranging and separating the α^3 -terms from the α^6 -terms:

$$C_{pr,z}^G = \frac{\lambda^2}{6\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 \sum_{m=-1}^{+1} [3m \frac{(1 + |m|)!}{(1 - |m|)!} \text{Re}(g_{1, TM}^m g_{1, TE}^{m*}) - \frac{(2 + |m|)!}{(1 - |m|)!} \text{Im}(g_{1, TM}^m g_{2, TM}^{m*})] \quad (208)$$

$$C_{pr,z}^{S+NS} = \frac{\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 \sum_{m=-1}^{+1} [\frac{(2 + |m|)!}{(1 - |m|)!} \text{Re}(g_{1, TM}^m g_{2, TM}^{m*}) + 3m \frac{(1 + |m|)!}{(1 - |m|)!} \text{Im}(g_{1, TM}^m g_{1, TE}^{m*})] \quad (209)$$

The trivial case must then be separated into three subcases, namely T1 for $M = Q = -1$, T2 for $M = Q = 0$ and T3 for $M = Q = 1$.

For case T1, we use Eqs.208-209 and obtain:

$$C_{pr,z}^G = \frac{-\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 [\text{Re}(g_{1, TM}^{-1} g_{1, TE}^{-1*}) + \text{Im}(g_{1, TM}^{-1} g_{2, TM}^{-1*})] \quad (210)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1}\right)^2 \alpha^6 [\text{Re}(g_{1, TM}^{-1} g_{2, TM}^{-1*}) - \text{Im}(g_{1, TM}^{-1} g_{1, TE}^{-1*})] \quad (211)$$

From Eqs.160-161, we then recover Eq.210 while, from Eqs.162-163, we respectively obtain:

$$C_{pr,z}^S = \frac{-2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Im}(g_{1,TM}^{-1} g_{1,TE}^{-1*}) \quad (212)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^{-1} g_{2,TM}^{-1*}) \quad (213)$$

which agree with Eq.211.

For the case T2, we use Eqs.208-209 and obtain:

$$C_{pr,z}^G = \frac{-\lambda^2}{3\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 \text{Im}(g_{1,TM}^0 g_{2,TM}^{0*}) \quad (214)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^0 g_{2,TM}^{0*}) \quad (215)$$

From Eqs.160-161, we then recover Eq.214 while, from Eqs.162-163, we respectively obtain:

$$C_{pr,z}^S = 0 \quad (216)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{9\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \text{Re}(g_{1,TM}^0 g_{2,TM}^{0*}) \quad (217)$$

which agree with Eq.215.

For case T3, we use Eqs.208-209 and obtain:

$$C_{pr,z}^G = \frac{\lambda^2}{\pi} \frac{n_p^2 - 1}{n_p^2 + 1} \alpha^3 [\text{Re}(g_{1,TM}^1 g_{1,TE}^{1*}) - \text{Im}(g_{1,TM}^1 g_{2,TM}^{1*})] \quad (218)$$

$$C_{pr,z}^{S+NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 [\text{Re}(g_{1,TM}^1 g_{2,TM}^{1*}) + \text{Im}(g_{1,TM}^1 g_{1,TE}^{1*})] \quad (219)$$

From Eqs.160-161, we then recover Eq.218 while, from Eqs.162-163, we respectively obtain:

$$C_{pr,z}^S = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \operatorname{Im}(g_{1,TM}^1 g_{1,TE}^{1*}) \quad (220)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 \operatorname{Re}(g_{1,TM}^1 g_{2,TM}^{1*}) \quad (221)$$

which agree with Eq.219.

9 Examples.

The examples considered in this section are examples which have been used in [17] when dealing with axisymmetric beams of the first kind. They are now considered again but for the cases of SK- and AD-beams.

9.1 (1,0) x-polarized circularly symmetric Bessel beams.

Bessel beams have been introduced by Durnin and co-workers [30], [31]. They possess an invariance property, namely the fact that the intensity of the beam is constant along the direction of propagation meaning that they are non-diffracting. They are self-healing as well, and they furthermore share a not often mentioned property, namely that their speed of propagation may be much smaller than the speed of light, being equal to $c \cos \alpha_0$ [32], [33], in which c is the speed of light and α_0 an angle, named axicon angle, or half-cone angle, which occurs in the propagation term reading $\exp(\pm ikz \cos \alpha_0)$ instead of being of the form $\exp(\pm ikz)$. There actually exists an infinity of kinds of Bessel beams (i) with different polarizations and (ii) depending on the form given to a certain function $g(\alpha_0)$, see [32], [34], [35], [36]. In particular, when $g(\alpha_0) = (1 + \cos \alpha_0)/4$, the beam reduces to a Davis circularly symmetric beam, e.g. [32], [37] while, when $g(\alpha_0) = 1/2$ it reduces to another kind of beams discussed in [38], [39], [40]. In this section, we consider more specifically (1,0) x-polarized circularly symmetric Bessel beams (see next subsection for a more general assessment of the (p_x, p_y) notation used here with $p_x = 1$ and $p_y = 0$). Using a quadrature technique and solving quadratures analytically, BSCs of circularly symmetric beams have been obtained in closed forms in an off-axis configuration in [36] and conveniently rewritten in [28]. In the case of on-axis configurations, they simplify to [41]:

$$g_{n,TM}^{l+1} = -g(\alpha_0) \exp(ik_z z_0) A_n^{l+1} \quad (222)$$

$$g_{n,TM}^{l-1} = -g(\alpha_0) \exp(ik_z z_0) B_n^{l-1} \quad (223)$$

$$g_{n,TE}^{l+1} = ig(\alpha_0) \exp(ik_z z_0) A_n^{l+1} \quad (224)$$

$$g_{n,TE}^{l-1} = -ig(\alpha_0) \exp(ik_z z_0) B_n^{l-1} \quad (225)$$

in which l denotes the order of the beam, $k_z = k \cos \alpha_0$ is the longitudinal wave number, and:

$$A_n^{l+1} = (-1)^{(l+1-|l+1|)/2} \frac{(n-l-1)!}{(n+|l+1|)!} [\tau_n^{l+1}(\cos \alpha_0) + (l+1)\pi_n^{l+1}(\cos \alpha_0)] \quad (226)$$

$$B_n^{l-1} = (-1)^{(l-1-|l-1|)/2} \frac{(n-l+1)!}{(n+|l-1|)!} [\tau_n^{l-1}(\cos \alpha_0) - (l-1)\pi_n^{l-1}(\cos \alpha_0)] \quad (227)$$

Being circularly symmetric beams, they are *a fortiori* axisymmetric beams. For $l = 0$, they are of the first kind. For $l = \pm 2$, they are of the second kind. Otherwise, they are dark. Examples will be given for $l = 1$ (dark beams) and $l = 2$ (second kind).

For $l = 1$ ($m = 0$ and 2), from Eq.127, the transverse forces are found to read as:

$$C_{pr,x} = C_{pr,y} = 0 \quad (228)$$

which was to be expected for a circularly symmetric beam.

For the longitudinal cross-sections, we use Eqs.203, 205-206 and, implementing the expressions for the BSCs given above, we obtain :

$$C_{pr,z}^G = 0 \quad (229)$$

which was to be expected for a Bessel beam, and:

$$C_{pr,z}^S = 0 \quad (230)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 |g(\alpha_0)|^2 \sin^2 \alpha_0 \cos \alpha_0 \quad (231)$$

in which we have used $\tau_1^0 = -\sin \alpha_0$ and $\tau_2^0 = -3 \cos \alpha_0 \sin \alpha_0$. We remark that the only non zero forces are longitudinal non-standard forces.

For $l = 2$, we are facing a SK1A-case. The transverse cross-sections are then given by Eq.164 and, therefore, are all zero.

For the longitudinal cross-sections, we use Eqs.184, 186-187 and, implementing the values of BSCs given above, we obtain:

$$C_{pr,z}^G = 0 \quad (232)$$

again to be expected for a Bessel beam, and:

$$C_{pr,z}^S = \frac{-\lambda^2}{6\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 |g(\alpha_0)|^2 (1 - \cos \alpha_0)^2 \quad (233)$$

$$C_{pr,z}^{NS} = \frac{\lambda^2}{6\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 |g(\alpha_0)|^2 (1 - \cos \alpha_0)(1 + \cos \alpha_0 - 2 \cos^2 \alpha_0) \quad (234)$$

in which we used $\tau_1^1 = -\cos \alpha_0$, $\pi_1^1 = -1$, $\tau_2^1 = -3(2 \cos^2 \alpha_0 - 1)$ and $\pi_2^1 = -3 \cos \alpha_0$. Note that all these forces are zero for $\alpha_0 = 0$, as expected since we are then facing the case of a plane wave interacting with a dipole.

9.2 Bessel beams. Other polarizations.

Other polarizations, which are circularly symmetric as well [34], have been discussed in [35]. BSCs for an off-axis configuration are given in [42]. For on-axis configurations, they read as:

$$\begin{aligned}
g_{n, TM}^m &= ig(\alpha_0)(-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} \exp(ik_z z_0) \{ & (235) \\
& [ip_x \tau_n^m(\cos \alpha_0) + p_y m \pi_n^m(\cos \alpha_0)] \\
& \times [i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \\
& + [ip_y \tau_n^m(\cos \alpha_0) - p_x m \pi_n^m(\cos \alpha_0)] \\
& \times i[-i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \}
\end{aligned}$$

$$\begin{aligned}
g_{n, TE}^m &= g(\alpha_0)(-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} \exp(ik_z z_0) \{ & (236) \\
& [ip_x m \pi_n^m(\cos \alpha_0) + p_y \tau_n^m(\cos \alpha_0)] \\
& \times [i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \\
& + [-p_x \tau_n^m(\cos \alpha_0) + ip_y m \pi_n^m(\cos \alpha_0)] \\
& \times i[-i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \}
\end{aligned}$$

in which ϕ_0 is an angle used to specify the rotational location of the off-axis coordinate system (soon to be cancelled in agreement with the fact that it does not matter for on-axis configurations), $J_k(\cdot)$ are cylindrical Bessel function of the first kind of order k , and p_x, p_y define the polarization of the beam in the framework of an angular spectrum decomposition according to $(p_x, p_y) = (1, 0)$ for x -polarization, $(0, 1)$ for y -polarization, $(1, i)$ for left circular polarization, $(1, -i)$ for right circular polarization, $(\cos \beta, \sin \beta)$ for radial polarization, and $(-\sin \beta, \cos \beta)$ for azimuthal polarization in which β is an azimuthal angle [40], [43].

Because $J_k(0) = \delta_{k0}$, Eqs.235-236 show that the only nonzero BSCs are those for which $m = l \pm 1$, according to:

$$\begin{aligned}
g_{n, TM}^{l+1} &= ig(\alpha_0)(-1)^{(l+1-|l+1|)/2} \frac{(n-l-1)!}{(n+|l+1|)!} \exp(ik_z z_0) & (237) \\
& \times (ip_x + p_y) [\tau_n^{l+1}(\cos \alpha_0) + (l+1) \pi_n^{l+1}(\cos \alpha_0)]
\end{aligned}$$

$$\begin{aligned}
g_{n, TM}^{l-1} &= ig(\alpha_0)(-1)^{(l-1-|l-1|)/2} \frac{(n-l+1)!}{(n+|l-1|)!} \exp(ik_z z_0) & (238) \\
& \times (ip_x - p_y) [\tau_n^{l-1}(\cos \alpha_0) - (l-1) \pi_n^{l-1}(\cos \alpha_0)]
\end{aligned}$$

$$g_{n,TE}^{l+1} = g(\alpha_0)(-1)^{(l+1-|l+1|)/2} \frac{(n-l-1)!}{(n+|l+1|)!} \exp(ik_z z_0) \quad (239)$$

$$\times (ip_x + p_y)[\tau_n^{l+1}(\cos \alpha_0) + (l+1)\pi_n^{l+1}(\cos \alpha_0)]$$

$$g_{n,TE}^{l-1} = -g(\alpha_0)(-1)^{(l-1-|l-1|)/2} \frac{(n-l+1)!}{(n+|l-1|)!} \exp(ik_z z_0) \quad (240)$$

$$\times (ip_x - p_y)[\tau_n^{l-1}(\cos \alpha_0) - (l-1)\pi_n^{l-1}(\cos \alpha_0)]$$

which satisfies $g_{nTM}^{l+1} = ig_{n,TE}^{l+1}$ and $g_{nTM}^{l-1} = -ig_{n,TE}^{l-1}$. We again restrict ourselves to $l = 1$ ($m = 0$ and 2) which is the case of a dark beam and to $l = 2$ ($m = 1$ and 3) which is the case of a second kind beam.

For $l = 1$ ($m = 0$ and 2), we have from Eq.127:

$$C_{pr,x} = C_{pr,y} = 0 \quad (241)$$

which was to be expected for a circularly symmetric beam.

For the longitudinal cross-sections, we use Eqs.203, 205-206 and, implementing the expressions for the BSCs given above, we obtain :

$$C_{pr,z}^G = 0 \quad (242)$$

which was to be expected for a Bessel beam, and:

$$C_{pr;,z}^S = 0 \quad (243)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 |g(\alpha_0)|^2 (p_x^2 + p_y^2) \sin^2 \alpha_0 \cos \alpha_0 \quad (244)$$

in which we have used $\tau_1^0 = -\sin \alpha_0$ and $\tau_2^0 = -3 \cos \alpha_0 \sin \alpha_0$. We remark that the only non zero forces are longitudinal non standard forces.

For $l = 2$, we are facing a SK1A-case. The transverse cross-sections are then given by Eq.164 and, therefore, are all zero.

For the longitudinal cross-sections, we use Eqs.184, 186-187 and, implementing the values of BSCs given above, we obtain:

$$C_{pr,z}^G = 0 \quad (245)$$

again to be expected for a Bessel beam, and:

$$C_{pr,z}^S = \frac{-\lambda^2}{6\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 |g(\alpha_0)|^2 (p_x^2 + p_y^2) (1 - \cos \alpha_0)^2 \quad (246)$$

$$C_{pr,z}^{NS} = \frac{\lambda^2}{6\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 |g(\alpha_0)|^2 (p_x^2 + p_y^2) (1 - \cos \alpha_0) (1 + \cos \alpha_0 - 2 \cos^2 \alpha_0) \quad (247)$$

in which we used $\tau_1^1 = -\cos \alpha_0$, $\pi_1^1 = -1$, $\tau_2^1 = -3(2 \cos^2 \alpha_0 - 1)$ and $\pi_2^1 = -3 \cos \alpha_0$. Note that all these forces are zero for $\alpha_0 = 0$, as expected since we are then facing the case of a plane wave interacting with a dipole.

9.3 Lommel beams.

Lommel beams are constituted by a combination of Bessel beams of various orders (i.e. topological charges) as exemplified by the expression of the basic electric modes in cylindrical coordinates (r, φ, z) reading as, e.g. Eq.(1) in [44]:

$$E_l = \exp(ik_z z) \sum_{p=0}^{\infty} (-1)^p c^{2p} \exp[i(l + 2p)\varphi] J_{\nu+2p}(k_t r) \quad (248)$$

in which E_l is the x - or y -component of the electric field which is obtained from a solution of the scalar wave equation (meaning that the beam is not Maxwellian), $k_z = k \cos \alpha_0$ and $k_t = k \sin \alpha_0$ are the longitudinal and transverse wavenumbers respectively. The beam being not Maxwellian, its expression in terms of BSCs will therefore provide a remodelling of the beam, turning it from a non-Maxwellian to a Maxwellian beam. In [44], these BSCs are evaluated using the integral localized approximation [45] which is fairly satisfactory for small enough axicon angles, e.g. [46], [47], [48], [49], although it is likely that quadratures of the quadrature techniques could be performed analytically, leading to localized BSCs for localized Lommel beams. In [17] in which the case of axisymmetric beams of the first kind has been discussed, the case $l = 1$ corresponding to a dark beam has not been worked out. The dark beam case for $l = 4$ has instead been chosen because it was displayed in a figure by the authors in their original paper. Therefore, we here consider this case again. It will

furthermore provide an interesting example with specific particularities which can be generalized. For this case in which $m = 4 \pm 1 = 5, 3$, the BSCs are given by [44], [17]:

$$g_{n, TM}^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{Z_n^3}{2} J_4(w_n) \exp(ik_z z_0) \quad (249)$$

$$g_{n, TM}^5 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{Z_n^5}{2} J_4(w_n) \exp(ik_z z_0) \quad (250)$$

$$g_{n, TE}^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} \frac{Z_n^3}{2} J_4(w_n) \exp(ik_z z_0) \quad (251)$$

$$g_{n, TE}^5 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} \frac{Z_n^5}{2} J_4(w_n) \exp(ik_z z_0) \quad (252)$$

in which x and y denote x - and y -polarizations respectively and:

$$w_n = \left(n + \frac{1}{2}\right) \sin \alpha_0 \quad (253)$$

$$Z_n^m = \left(\frac{-2i}{2n+1}\right)^{|m|-1} \text{ for } m \neq 0 \quad (254)$$

From Eq.127, we still have, for the transverse cross-sections:

$$C_{pr,x} \begin{pmatrix} x \\ y \end{pmatrix} = C_{pr,y} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (255)$$

For the longitudinal cross-sections, however, we have to rely on Eqs.203, 205-206. We then observe that the BSCs occurring in these equations are of the form $g^0 g^{0*}$ with $m = 0$ so that we may conclude that all the cross-sections (transverse and longitudinal) are actually equal to 0. It would be the same for all values of $l \geq 3$ (and of $l \leq -3$) where the values of m would be different from 0.

For $l = 2$ ($m = 3, 1$) we are facing again a SK1A-case. The BSCs read as:

$$g_{n, TM}^1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{Z_n^1}{2} J_2(w_n) \exp(ik_z z_0) \quad (256)$$

$$g_{n,TM}^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{Z_n^3}{2} J_2(w_n) \exp(ik_z z_0) \quad (257)$$

$$g_{n,TE}^1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} \frac{Z_n^1}{2} J_2(w_n) \exp(ik_z z_0) \quad (258)$$

$$g_{n,TE}^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} \frac{Z_n^3}{2} J_2(w_n) \exp(ik_z z_0) \quad (259)$$

The transverse cross-sections are then given by Eq.164 and, therefore, are all zero.

For the longitudinal cross-sections, we use Eqs.184, 186-187, implement the values of BSCs given above, use Eqs.253-254, leading to:

$$C_{pr,z}^G = 0 \quad (260)$$

again to be expected due to the fact that Lommel beams are a combination of Bessel beam, and:

$$C_{pr,z}^S = \frac{-\lambda^2}{6\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 [J_2(\frac{3}{2} \sin \alpha_0)]^2 \quad (261)$$

$$C_{pr,z}^{NS} = \frac{\lambda^2}{6\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 J_2(\frac{3}{2} \sin \alpha_0) J_2(\frac{5}{2} \sin \alpha_0) \quad (262)$$

Note that all these forces are zero for $\alpha_0 = 0$, as again expected.

9.4 Bessel-Gauss beams using finite series.

From Eqs.(20) and (31) in [50], required TM-BSCs can be written as:

$$(g_{n,TM}^{l-1})_{n-m \text{ even}} = i^{n+1} G_{nl}^{even}(\alpha_0, s) \quad (263)$$

$$(g_{n,TM}^{l-1})_{n-m \text{ odd}} = i^n G_{nl}^{odd}(\alpha_0, s) \quad (264)$$

in which $G_{nl}^{even}(\alpha_0, s)$ and $G_{nl}^{odd}(\alpha_0, s)$ are real functions of the axicon angle α_0 and of the confinement factor $s = 1/(kw_0)$, depending as well on the subscripts n and l . The TE-BSCs are furthermore related to the TM BSCs by the following relation [50]:

$$g_{n, TM}^{l\pm 1} = \pm i g_{n, TE}^{l\pm 1} \quad (265)$$

For $l = 1$ ($m = 0$ and 2 , i.e. for an axisymmetric dark beam), we have from Eq.127:

$$C_{pr,x} = C_{pr,y} = 0 \quad (266)$$

For the longitudinal cross-sections, we use Eqs.203, 205-206 and, implementing the expressions for the BSCs given above, we obtain :

$$C_{pr,z}^G = C_{pr,z}^S = 0 \quad (267)$$

$$C_{pr,z}^{NS} = \frac{-2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 G_{11}^{odd}(\alpha_0, s) G_{21}^{even}(\alpha_0, s) \quad (268)$$

For $l = 2$, we are facing a SK1A-case. The transverse cross-sections are then given by Eq.164 and, therefore, are all zero.

For the longitudinal cross-sections, we use Eqs.184, 186-187 and, implementing the values of BSCs given above, we obtain:

$$C_{pr,z}^G = 0 \quad (269)$$

again to be expected for a Bessel beam, and:

$$C_{pr,z}^S = \frac{-2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 [G_{12}^{even}(\alpha_0, s)]^2 \quad (270)$$

$$C_{pr,z}^{NS} = \frac{2\lambda^2}{3\pi} \left(\frac{n_p^2 - 1}{n_p^2 + 1} \right)^2 \alpha^6 G_{12}^{even}(\alpha_0, s) G_{22}^{odd}(\alpha_0, s) \quad (271)$$

9.5 Laguerre-Gauss beams freely propagating.

For the BSCs, we refer to [51] and it is found to be sufficient to evaluate two BSCs, namely $g_{1,TM}^0$ and $g_{1,TM}^1$ using respectively Eqs.(102) and (69) in [51], leading to:

$$g_{1,TM}^0 = g_{1,TM}^1 = 0 \quad (272)$$

For $l = 1$ ($m = 0$ and 2 , i.e. again for an axisymmetric dark beam), we have from Eq.127:

$$C_{pr,x} = C_{pr,y} = 0 \quad (273)$$

For the longitudinal cross-sections, we use Eqs.203, 205-206 and, implementing the expressions for the BSCs given above, we obtain :

$$C_{pr,z}^G = C_{pr,z}^S = C_{pr,z}^{NS} = 0 \quad (274)$$

For $l = 2$ ($m = 3$ and 1), we are facing a SK1A-case.

The transverse cross-sections are then given by Eq.164 and, therefore, are all zero.

For the longitudinal cross-sections, we use Eqs.184, 186-187 and, implementing the values of BSCs given above, we obtain:

$$C_{pr,z}^G = C_{pr,z}^S = C_{pr,z}^{NS} = 0 \quad (275)$$

i.e. all kinds of optical forces are zero, both for $l = 1$ and $l = 2$.

10 Optical forces in the Rayleigh regime of GLMT for lossless particles: a mini-review.

As already mentioned, GLMTs provide rigorous frameworks to study optical forces and torques exerted on particles as reviewed in [18], not only for homogeneous spherical particles (in the framework of GLMT *stricto sensu*) but also for multilayered spherical particles [52], for assemblies of spheres and aggregates [53], for spheres with an eccentrically located spherical inclusion [54], and for spheroids [55], [56].

Beside practical interest, for instance concerning the optical levitation and optical manipulation of small particles, e.g. [57], the Rayleigh regime presents the advantage of allowing one to deal with simple mathematical expressions which then has the advantage of leading to easy physical insights and interpretations. A first mention of the use of the Rayleigh regime in the GLMT framework is to be found in [21] for on-axis Gaussian beams in the weak confinement limit. In [21], optical forces were catalogued as gradient and scattering forces (discussed below) but the fact that the weak confinement limit was used prevented the observation of non standard forces (again, discussed below), see [19] for a more extended discussion of this work.

A systematic study of the Rayleigh regime of GLMT began with [28] who dealt with longitudinal forces exerted by off-axis Bessel beams. Although there was a focus on Bessel beams, this paper however contains general expressions for longitudinal forces, independent of the specific beams under studies. These expressions are isolated and recalled in section 7 of the present paper. The prefactor of Eq. (1) of [28] was however slightly incorrect. It should read $\lambda^2/(2\pi)$ instead of $\lambda^2/(4\pi)$. This is simply equivalent to a change of the normalization factor. Expressions given in section 7 of the present paper have been accordingly corrected. The case of transverse optical forces, again in the case of off-axis Bessel beams, has been considered in [26], with the expressions for the y -component of the transverse cross-sections later slightly corrected by Eqs.(19) and (15) of [27]. These expressions are recalled in section 7 which, therefore, contains the expressions for longitudinal and transverse optical forces in the Rayleigh regime of GLMT. Although derived in papers devoted to Bessel beams, the expressions recalled in section 7 are valid for arbitrary shaped electromagnetic beams.

The optical forces in section 7 (and in the original papers) are catalogued as gradient forces proportional to the gradient of $|\mathbf{E}|^2$, scattering forces proportional to the Poynting vector and non-standard forces which do not explicitly appear in the dipole theory of forces (actually in the Rayleigh regime of the dipole theory of forces). In [28] and [26], they were given the name of axicon forces because they become equal to 0 when the axicon angle is equal to 0. It has later been recognized that these non-standard forces were not specific of Bessel beams, and have then been renamed non-standard forces. More specifically gradient forces are proportional to α^3 and involve BSCs of the form g_1^m and g_2^m (therefore associated with partial waves of order $n = 1$ and $n = 2$), while scattering forces and non-standard forces are proportional to α^6 . However, while non-standard forces again involve BSCs of the form g_1^m and g_2^m , scattering forces only involve BSCs of the form g_1^m , i.e. they depend only on partial waves of order $n = 1$. A fairly interesting conclusion is that, for certain types de beams, the only radiation force on the Rayleigh particle is non-standard.

The existence of these non-standard forces could have been a clue that the Rayleigh regime of the GLMT might not identify with the Rayleigh limit of the dipole theory of forces, an issue to be investigated with the complementary issue, if the identification were indeed valid, to determine what would be the sta-

tus of the non-standard forces. Furthermore, supposing that the identification was indeed valid, another question would be to examine whether the Rayleigh limit of the GLMT would not have specific properties with respect to the dipole theory.

In the conclusion of [28], the "axicon" forces were pointed out as being "new", making these questions more vivid (it will be revealed later that these non-standard forces were indeed new insofar as they have not been previously isolated in their own right, as far as we are aware of, or at least new in the context of GLMT). In the end of subsection 5.1 of [26], due to the occurrence of these "axicon" forces, it was furthermore stated that "the Rayleigh limit of the GLMT does not exactly identify with the Rayleigh dipole theory of optical forces". This remains true insofar as the word "exactly" has been used, pointing out to the fact that these "axicon" forces had not yet been properly isolated in the framework of the dipole theory. A similar comment has been reiterated in section 6.1 of [58]. It also points out to the fact, discussed below, that the GLMT approach may be viewed as a *reduced* approach to the dipole approach.

To advance toward the answers to the above questions, it has then been numerically demonstrated, in the case of Bessel beams, that the Rayleigh regime of GLMT for longitudinal optical forces agrees with the dipole theory within 1,000 decimal places [59], while an agreement up to 3,000 decimal places was revealed in [27] for the transverse optical forces, leading to the conjecture that indeed both approaches should eventually be found to be equivalent. After fairly heavy algebraic manipulations, such an equivalence has indeed been revealed both in the case of longitudinal forces [29] (the prefactor of the second term in the r.h.s. of Eq.(40) in this paper has been misprinted to 2π instead of 4π , as can be checked e.g. from Eq.(40) of [27] and also from Eq.(20) of [59]), and of transverse forces [60]. As a by-product, it was confirmed that the non-standard forces did not explicitly appear in the dipole theory but that they constituted an ingredient of "spin-curl" forces in the terminology used by Albaladejo [61]. Such spin-curl forces are intrinsically dependent on the beam's polarization and are associated with the existence of a spin angular momentum density for the incident light (see also [62] for a comment on the aforementioned reference, with a reply in [63]). This interpretation of the non-standard optical forces is discussed in [29] and refined in [60].

The identification between the Rayleigh regime of the GLMT and the dipole theory of forces does not mean that both approaches are exactly identical in all aspects. There is already the fact that non-standard forces were not explicitly isolated in the dipole theory. More important, the dipole theory expresses the optical forces in terms of the total electric field of the illuminating beam, e.g. [23], [24], [64]. For this reason, it has been stated that the dipole approach is somehow "inconsistent" (although exact) insofar as it involves *total* illuminating fields, i.e. it includes *all* partial waves from $n = 1$ to ∞ , e.g. [27].

Conversely, the equations of section 7 show that the Rayleigh limit of GLMT expresses the optical forces only in terms of the partial waves of order 1 and 2, that is to say using only BSCs of the form g_1^m and g_2^m , meaning that the contributions of all other partial waves, from $n = 3$ to ∞ are exactly equal

to 0, a fact that has not been explicitly stated in the framework of the dipole theory. Therefore the GLMT approach may be viewed as a reduced version of the dipole approach.

In general, if we retain only the partial waves of order $n = 1$ in the gradient terms (which should involve both $n = 1$ and $n = 2$ partial waves), we observe the occurrence of terms which have not been cancelled out in the process and which have been called axicon terms in [58]. Such terms never appear if we properly take account of both partial waves of order $n = 1$ and 2 as summarized in [65]. It may however happen indeed that partial waves of order $n = 1$ are the only ones required to exactly express the gradient optical forces in the Rayleigh regime, for instance in the case of on-axis Bessel beams [41].

The fact that only partial waves of low-order are sufficient to express optical forces may be related to the van de Hulst principle of localization, a fact which is particularly easy to explain in the case of on-axis configurations. Indeed, according to this principle, originally stated for plane waves (p. 208 of [22]) and later on extended to arbitrary shaped beams in order to develop localized beam models, e.g. [66], a beam may be thought of as being made up of separate ways of light each independently pursuing its own path. Furthermore, a partial wave of order n corresponds to a ray passing at a distance from the origin equal to $(n + 1/2)\lambda/(2\pi)$. The partial waves of order $n = 1$ and 2 therefore pass at distances roughly equal to $\lambda/4$ and $\lambda/2$ from the point-like particle. Loosely speaking, we may then state that higher-order partial waves pass too far away from the particle to interact with it. The issue of van de Hulst principle of localization in the context of the present work has been discussed in subsection 6.2 of [58] and in the conclusion of [19].

On-axis beams present significant simplifications with respect to off-axis beams. It has therefore been found interesting to examine specifically such special cases. This has been first carried out in the case of Bessel beams which has originally be examined in the case of an off-axis configuration [41]. More generally, the case of non-dark axisymmetric beams of the first kind has been discussed in a Part I-paper [19], with an extensive discussion of the case of Gaussian beams, while the present paper deals with non-dark axisymmetric beams of the second kind and axisymmetric dark beams, after a preliminary investigation of such symmetries achieved in [17].

As a final remark, the previous discussions have been carried out in the framework of the GLMT *stricto sensu*, i.e. for the case when the scatterer is a homogeneous spherical particle defined by its diameter and its complex refractive index. They may however be readily extended to other kinds of spherical particles namely multilayered spheres when the expressions of the BSCs are unchanged and when we only need to modify the expressions for the Mie coefficients, e.g. [67], [68], and to other kinds of particles which lead to expressions which are formally identical to the ones of GLMT, namely assemblies of spheres and aggregates [53], and spheres with an eccentrically located spherical inclusion [54], [69], [70].

11 Conclusion.

This paper is presumably the last one we publish concerning the Rayleigh approach of GLMT to the computations of optical forces. It complements a previous Part I-paper devoted to optical forces exerted on Rayleigh particles by axisymmetric of the first kind, and deals with axisymmetric beams of the second kind and with axisymmetric dark beams. Several examples are provided pertaining to the cases of Bessel beams, Lommel beams, Bessel-Gauss beams and Laguerre-Gauss beams. It confirms and exemplifies the interest of the GLMT approach, namely that it produces a reduced theory of the dipole approach insofar as the expressions of optical forces are obtained specifically in terms of BSCs associated with low-order partial waves, namely $n = 1$ partial waves for scattering forces, and $n = 1$ partial waves complemented with $n = 2$ partial waves for gradient and non standard forces. In order to help the reader navigating on the papers devoted to the issue, the present paper ends with a mini-review stressing the main points which have been investigated and the main results which have been obtained during the research devoted to the issue.

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