



HAL
open science

ASYMPTOTIC SMOOTHNESS IN BANACH SPACES, THREE SPACE PROPERTIES AND APPLICATIONS

R M Causey, A Fovelle, Gilles Lancien

► **To cite this version:**

R M Causey, A Fovelle, Gilles Lancien. ASYMPTOTIC SMOOTHNESS IN BANACH SPACES, THREE SPACE PROPERTIES AND APPLICATIONS. Transactions of the American Mathematical Society, 2023, 376 (3), pp.1895-1928. 10.1090/tran/8818 . hal-03377459v2

HAL Id: hal-03377459

<https://hal.science/hal-03377459v2>

Submitted on 27 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ASYMPTOTIC SMOOTHNESS IN BANACH SPACES, THREE-SPACE PROPERTIES, AND APPLICATIONS

R.M. CAUSEY, A. FOVELLE, AND G. LANCIEN

ABSTRACT. We study four asymptotic smoothness properties of Banach spaces, denoted T_p, A_p, N_p , and P_p . We complete their description by proving the missing renorming characterization for A_p . We show that asymptotic uniform flattenability (property T_∞) and summable Szlenk index (property A_∞) are three-space properties. Combined with the positive results of the first-named author, Draga, and Kochanek, and with the counterexamples we provide, this completely solves the three-space problem for this family of properties. We also derive from our characterizations of A_p and N_p in terms of equivalent renormings, new coarse Lipschitz rigidity results for these classes.

1. INTRODUCTION

A coarse Lipschitz embedding from a metric space M into a metric space N is a map which is bi-Lipschitz for large enough distances (see precise definitions in Section 6). In the seminal paper [38], M. Ribe proved that if a Banach space X coarse Lipschitz embeds into a Banach space Y , then X is finitely crudely representable into Y , which means that all finite-dimensional subspaces of X are linearly isomorphic, with a uniform distortion, to subspaces of Y . In other words, the local properties of Banach spaces, that are isomorphic properties of their finite dimensional subspaces (such as type, cotype, super-reflexivity, ...) are preserved under coarse Lipschitz embeddings. This initiated what is now called the Ribe program, which aims at finding purely metric characterizations of local properties of Banach spaces. We refer to [37] and [2] for a discussion of its origins and motivations, and for a presentation of the most striking results in this direction. In the last twenty years, the asymptotic structure of Banach spaces, which, very vaguely speaking, deals with the structure of their finite-codimensional subspaces or with the properties of weakly null sequences and trees, also proved to be central in the non-linear geometry of Banach spaces. We refer the reader to the seminal works of N. Kalton ([29] and [28] for instance) and to the survey

2020 *Mathematics Subject Classification*. Primary: 46B20, 46B80; Secondary: 46B03, 46B10, 46B26.

Key words and phrases. Asymptotic smoothness in Banach spaces, Szlenk index, upper tree estimates, three space problem, coarse Lipschitz equivalences.

The second-named and third-named authors received support from the EIPHI Graduate School (contract ANR-17-EURE-0002).

[21] and references therein. However, despite the accumulation of quite a few important stability results, there is no general analogue of Ribe's rigidity theorem in the setting of the asymptotic geometry of Banach spaces. Each new result requires an ad hoc argument. One of the main applications of this work is to exhibit two new properties related to the asymptotic smoothness of Banach spaces, that are stable under coarse Lipschitz equivalences.

In this article, we describe in details four different properties dealing with the asymptotic uniform smoothness of Banach spaces, that we shall denote T_p , A_p , N_p , and P_p . In Section 2, we give their definitions in terms of two-players games on a Banach space X . Let us just mention for this introduction that each of them coincides with the existence of a special form of so-called upper ℓ_p estimates for weakly null trees in X .

In Section 3, we give the main characterizations of these properties, which involve, upper ℓ_p estimates for weakly null trees, the existence of quantitatively good equivalent asymptotically uniformly smooth norms and, dually, the behaviour of the Szlenk index. We refer the reader to Section 3 for the precise statements and definitions. However, for the purpose of this introduction, we anticipate two definitions.

Let us first recall the definition of the Szlenk derivation. For a Banach space X , $K \subset X^*$ weak*-compact, and $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote the set of $x^* \in K$ such that for each weak*-neighborhood V of x^* , $\text{diam}(V \cap K) \geq \varepsilon$. For $1 \leq q < \infty$, we say X has *q-summable Szlenk index* provided there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon_1, \dots, \varepsilon_n \geq 0$ such that $s_{\varepsilon_1} \dots s_{\varepsilon_n}(B_{X^*}) \neq \emptyset$, $\sum_{i=1}^n \varepsilon_i^q \leq c^q$. In the $q = 1$ case, we refer to this as *summable Szlenk index* rather than 1-summable Szlenk index.

We also define the *modulus of asymptotic uniform smoothness* of X . If X is infinite-dimensional, for $\sigma \geq 0$, we define

$$\bar{\rho}_X(\sigma) = \sup_{y \in B_X} \inf_{E \in \text{cof}(X)} \sup_{x \in B_E} \|y + \sigma x\| - 1,$$

where $\text{cof}(X)$ denotes the set of finite codimensional subspaces of X .

We assume, for this introduction, that what we mean by weakly null trees is understandable and refer to Section 3 for a detailed discussion. We can now state our main new result from Section 3. It describes various characterizations of our property A_p . The renorming characterization is completely new and will be crucial for our non-linear stability results.

Theorem A. *Fix $1 < p < \infty$ and let q be conjugate to p . Let X be a Banach space. The following are equivalent*

- (i) $X \in A_p$.
- (ii) *There exists a constant $c > 0$ such that for any weak neighborhood base D at 0 in X , any $n \in \mathbb{N}$, and any weakly null tree $(x_t)_{t \in D^{\leq n}}$ in the unit ball of X , there exists $t \in D^n$ such that*

$$\forall a = (a_i)_{i=1}^n \in \ell_p^n, \quad \left\| \sum_{i=1}^n a_i x_{t_i} \right\| \leq c \|a\|_p.$$

(iii) *There exists a constant $M \geq 1$ and a constant $C > 0$, such that for any $\tau \in (0, 1]$ there exists a norm $|\cdot|$ on X satisfying $M^{-1}\|x\|_X \leq |x| \leq M\|x\|_X$ for all $x \in X$ and*

$$\forall \sigma \geq \tau, \quad \bar{\rho}_{|\cdot|}(s) \leq Cs^p.$$

(iv) *X has q -summable Szlenk index.*

Let us recall that a property (P) of Banach spaces is separably determined if a Banach space X has (P) if and only if all its separable subspaces have (P) . In Section 4, we provide a short and unified proof of the fact that all these properties are separably determined.

Section 5 is devoted to the study of the three-space problem for $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ and P_p . A property (P) of Banach spaces is a *Three-Space Property* (3SP in short) if it passes to quotients and subspaces and a Banach space X has (P) whenever it admits a subspace Y such that Y and X/Y have (P) . The properties $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ and P_p pass quite simply to subspaces, quotients or isomorphisms and it was proved in [10] that P_p is a 3SP. First, we take the opportunity of this paper to provide a more direct argument for this. Then, with a single example, we show

Theorem B. *Let $p \in (1, \infty)$. Then $\mathsf{T}_p, \mathsf{A}_p$, and N_p are not three-space properties.*

Finally, and this is the main result of this section, we show

Theorem C. *Asymptotic uniform flattenability (property T_∞) and summable Szlenk index (property A_∞) are three-space properties.*

A net in a metric space (M, d) is a subset \mathcal{M} of M such that there exist $0 < a < b$ so that for every $z \neq z'$ in \mathcal{M} , $d(z, z') \geq a$ and for every x in M , $d(x, \mathcal{M}) < b$. Let us use, for the simplicity of this introduction, that two infinite dimensional Banach spaces X and Y are coarse Lipschitz equivalent if and only if there exist two nets in X and Y that are Lipschitz equivalent. The precise definition of a coarse Lipschitz equivalence is given in Section 6 and can be roughly described as a Lipschitz equivalence at large distances. In [19] and [20], it is proved that T_p is stable under Lipschitz equivalences, P_p is stable under coarse Lipschitz equivalences and $\mathsf{A}_\infty = \mathsf{N}_\infty$ is stable under coarse Lipschitz equivalences. In [30], N. Kalton proved that for $1 < p < \infty$, the class T_p is not stable under coarse Lipschitz equivalences. Thanks to our renorming theorems from Section 3, we can almost complete this set of results.

Theorem D. *Let $p \in (1, \infty)$. Then, the class A_p and the class N_p are stable under coarse Lipschitz equivalences.*

Let us point that for \mathbf{N}_p , this is deduced from already existing results, but was unnoticed, while for \mathbf{A}_p it relies on our new renorming characterization.

In Section 7 we conclude this paper by gathering a few known examples of \mathbf{T}_∞ or \mathbf{A}_∞ spaces and related questions.

2. THE PROPERTIES

All Banach spaces are over the field \mathbb{K} , which is either \mathbb{R} or \mathbb{C} . We let B_X (resp. S_X) denote the closed unit ball (resp. sphere) of X . By *subspace*, we shall always mean closed subspace. Unless otherwise specified, all spaces are assumed to be infinite-dimensional. Throughout, we let \mathbf{Ban} denote the class of all Banach spaces over \mathbb{K} . We let \mathbf{Sep} denote the class of separable members of \mathbf{Ban} .

We now recall the definition of the Szlenk index, based on the Szlenk derivation introduced in Section 1. For a Banach space X , $K \subset X^*$ weak*-compact, and $\varepsilon > 0$, we define the transfinite derivations

$$s_\varepsilon^0(K) = K,$$

$$s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K)),$$

and if ξ is a limit ordinal,

$$s_\varepsilon^\xi(K) = \bigcap_{\zeta < \xi} s_\varepsilon^\zeta(K).$$

For convenience, we let $s_0(K) = K$. If there exists an ordinal ξ such that $s_\varepsilon^\xi(K) = \emptyset$, we let $Sz(K, \varepsilon)$ denote the minimum such ordinal, and otherwise we write $Sz(K, \varepsilon) = \infty$. We let $Sz(K) = \sup_{\varepsilon > 0} Sz(K, \varepsilon)$, where $Sz(K) = \infty$ if $Sz(K, \varepsilon) = \infty$ for some $\varepsilon > 0$. We let $Sz(X, \varepsilon) = Sz(B_{X^*}, \varepsilon)$ and $Sz(X) = Sz(B_{X^*})$. In this work, we will exclusively be concerned with Banach spaces X such that $Sz(X) \leq \omega$, where ω is the first infinite ordinal. Since $Sz(X) = 1$ if and only if X has finite dimension, and otherwise $Sz(X) \geq \omega$, we will actually only be concerned with the case $Sz(X) = \omega$. By compactness, $Sz(X) \leq \omega$ if and only if $Sz(X, \varepsilon)$ is a natural number for each $\varepsilon > 0$. We note that $Sz(X) < \infty$ if and only if X is *Asplund*. One characterization of Asplund spaces is that every separable subspace has a separable dual.

We recall that for any Banach space X and $0 < \varepsilon, \delta < 1$,

$$Sz(X, \varepsilon\delta) \leq Sz(X, \varepsilon)Sz(X, \delta).$$

From this it follows that if $Sz(X, \varepsilon)$ is a natural number for each $\varepsilon > 0$, the Szlenk power type

$$p(X) := \lim_{\varepsilon \rightarrow 0^+} \frac{\log Sz(X, \varepsilon)}{|\log(\varepsilon)|}$$

is finite. It also holds that for any ordinal ξ , any $\varepsilon > 0$, and any natural number n , if $Sz(X, \varepsilon) > \xi$, then $Sz(X, \frac{\varepsilon}{n}) > \xi n$. Indeed, this follows from

realizing

$$B_{X^*} = \frac{1}{n}B_{X^*} + \dots + \frac{1}{n}B_{X^*}$$

and noting that the $\frac{\varepsilon}{n}$ -derivations act on one summand at a time in the same way that the ε -derivations act on B_{X^*} . Therefore the ε -Szlenk index grows subgeometrically but superarithmetically. The superarithmetic growth implies that for any infinite dimensional Banach space, $\mathfrak{p}(X) \geq 1$.

We define the following modulus. For $\varepsilon > 0$,

$$\bar{\theta}_X^*(\varepsilon) = \sup\{\delta \geq 0, s_\varepsilon(B_{X^*}) \subset (1 - \delta)B_{X^*}\}.$$

We recall again the definition of the *modulus of asymptotic uniform smoothness* of X . If X is infinite-dimensional, for $\sigma \geq 0$, we define

$$\bar{\rho}_X(\sigma) = \sup_{y \in B_X} \inf_{E \in \text{cof}(X)} \sup_{x \in B_E} \|y + \sigma x\| - 1,$$

where $\text{cof}(X)$ denotes the set of finite codimensional subspaces of X . For the sake of completeness, we define $\bar{\rho}_X(\sigma) = 0$ for all $\sigma \geq 0$, when X is finite-dimensional. We note that

$$\bar{\rho}_X(\sigma) = \sup\{\limsup_\lambda \|y + \sigma x_\lambda\| - 1 : (x_\lambda) \subset B_X \text{ is a weakly null net}\}.$$

It follows easily from the triangle inequality that $\bar{\rho}_X$ is a convex function. Since $\bar{\rho}_X(0) = 0$, we deduce that $\sigma \mapsto \frac{\bar{\rho}_X(\sigma)}{\sigma}$ is non-decreasing on $(0, \infty)$. Therefore

$$\inf_{\sigma > 0} \frac{\bar{\rho}_X(\sigma)}{\sigma} = \limsup_{\sigma \rightarrow 0^+} \frac{\bar{\rho}_X(\sigma)}{\sigma}.$$

We say X is *asymptotically uniformly smooth* (in short *AUS*) if

$$\inf_{\sigma > 0} \frac{\bar{\rho}_X(\sigma)}{\sigma} = 0.$$

We say X is *asymptotically uniformly smoothable* (*AUS-able*) if X admits an equivalent AUS norm. For $1 < p < \infty$, we say X is *p-asymptotically uniformly smooth* (in short *p-AUS*) if

$$\sup_{\sigma > 0} \frac{\bar{\rho}_X(\sigma)}{\sigma^p} < \infty.$$

We say X is *p-asymptotically uniformly smoothable* (*p-AUS-able*) if X admits an equivalent *p-AUS* norm. We say X is *asymptotically uniformly flat* (*AUF*) if there exists $\sigma_0 > 0$ such that $\bar{\rho}_X(\sigma_0) = 0$. We say X is *asymptotically uniformly flattenable* if X admits an equivalent AUF norm. Of course, *p-AUS* spaces and AUF spaces are AUS spaces.

It is well known that the dual Young function of the modulus of asymptotic uniform smoothness is equivalent to the so-called modulus of weak* asymptotic uniform convexity $\bar{\delta}_X^*$ (see Proposition 2.1 and Corollary 2.3 in [13]). It is also clear that $\bar{\delta}_X^*$ is equivalent to $\bar{\theta}_X^*$. It will be more convenient for us to work with $\bar{\theta}_X^*$. We shall only need the following version of Proposition 2.1 in [13].

Proposition 2.1. *There exists a universal constant $C \geq 1$ such that for any Banach space X and any $0 < \sigma, \tau < 1$,*

- (1) *If $\bar{\rho}_X(\sigma) < \sigma\tau$, then $\bar{\theta}_X^*(C\tau) \geq \sigma\tau$.*
- (2) *If $\bar{\theta}_X^*(\tau) > \sigma\tau$, then $\bar{\rho}_X(\frac{\sigma}{C}) \leq \sigma\tau$.*

For $1 \leq q < \infty$, a Banach space X , and a sequence $(x_i)_{i=1}^\infty \subset X$, we define the (possibly infinite) quantity

$$\|(x_i)_{i=1}^\infty\|_q^w = \sup\{\|(x^*(x_i))_{i=1}^\infty\|_{\ell_q} : x^* \in B_{X^*}\}.$$

We also define this quantity for finite sequences,

$$\|(x_i)_{i=1}^n\|_q^w = \sup\{\|(x^*(x_i))_{i=1}^n\|_{\ell_q^n} : x^* \in B_{X^*}\}.$$

Note that, if $p \in (1, \infty]$ is the conjugate exponent of q , we have that

$$\|(x_i)_{i=1}^\infty\|_q^w = \inf\{c \in (0, \infty], \forall a = (a_i)_{i=1}^\infty \in \ell_p \|\sum_{i=1}^\infty a_i x_i\| \leq c \|a\|_p\}.$$

A similar formula is valid for $\|(x_i)_{i=1}^n\|_q^w$.

We next define four different two-players games on a Banach space X . Fix $1 < p \leq \infty$ and let $1/p + 1/q = 1$. For $c > 0$ and $n \in \mathbb{N}$, we define the $T(c, p)$ game on X , the $A(c, p, n)$ game, and the $N(c, p, n)$ game. Let D be a weak neighborhood base at 0 in X . In the $T(c, p)$ game, Player I chooses a weak neighborhood $U_1 \in D$, and Player II chooses $x_1 \in U_1 \cap B_X$. Player I chooses $U_2 \in D$, and Player II chooses $x_2 \in U_2 \cap B_X$. Play continues in this way until $(x_i)_{i=1}^\infty$ has been chosen. Player I wins if $\|(x_i)_{i=1}^\infty\|_q^w \leq c$, and Player II wins otherwise.

The $A(c, p, n)$ game is similar, except the game terminates after the n^{th} turn. Player I wins if $\|(x_i)_{i=1}^n\|_q^w \leq c$, and Player II wins otherwise.

In the $N(c, p, n)$ game, as in the $A(c, p, n)$ game, the game terminates after the n^{th} turn. Player I wins if $\|\sum_{i=1}^n x_i\| \leq cn^{1/p}$, and Player II wins otherwise.

Finally, in the $\Theta(c, n)$ game, Player I wins if $\|\sum_{i=1}^n x_i\| \leq c$, and Player II wins otherwise.

It is known (see [6], Section 3) that each of these games is determined. That is, in each game, either Player I or Player II has a winning strategy. We let $\mathfrak{t}_p(X)$ denote the infimum of $c > 0$ such that Player I has a winning strategy in the $T(c, p)$ game, provided such a c exists, and we let $\mathfrak{t}_p(X) = \infty$ otherwise. We let $\mathfrak{a}_{p,n}(X)$ denote the infimum of $c > 0$ such that Player II has a winning strategy in the $A(c, p, n)$ game, and we let $\mathfrak{a}_p(X) = \sup_n \mathfrak{a}_{p,n}(X)$. We note that $\mathfrak{a}_p(X)$ is the infimum of $c > 0$ such that for each $n \in \mathbb{N}$, Player I has a winning strategy in the $A(c, p, n)$ game if such a c exists, and $\mathfrak{a}_p(X) = \infty$ otherwise. We let $\theta_n(X)$ denote the infimum of $c > 0$ such that Player I has a winning strategy in the $\Theta(c, n)$ game, noting that $\theta_n(X) \leq n$. Finally, we let $\mathfrak{n}_{p,n}(X) = \theta_n(X)/n^{1/p}$ and $\mathfrak{n}_p(X) = \sup_n \mathfrak{n}_{p,n}(X)$, noting that $\mathfrak{n}_p(X)$ is the infimum of $c > 0$ such that

for each $n \in \mathbb{N}$, Player I has a winning strategy in the $N(c, p, n)$ game, provided such a c exists, and $\mathfrak{n}_p(X) = \infty$ otherwise.

Remark 2.2. The existence of winning strategies, and therefore the constants associated with these games, do not depend upon the particular choice of the weak neighborhood base D . Therefore in the case that X has a separable dual, these constants are sequentially determined. Let us indicate the argument.

Proof. If D_1, D_2 are two weak neighborhood bases at 0 in X , and Player I has a winning strategy in any of the games above when Player I is required to choose from D_1 , then this winning strategy can be used to construct a winning strategy choosing from D_2 by choosing at each stage of the game any member of D_2 which is a subset of the member of D_1 indicated by the winning strategy. From this it follows that the values of the associated constants also do not depend on D . In particular, in the case that X^* is separable, we can use a fixed countable, linearly ordered weak neighborhood base D . \square

Let $D^{\leq n} = \cup_{i=1}^n D^i$. Let $D^{<\omega} = \cup_{i=1}^{\infty} D^i$, and let D^ω denote the set of all infinite sequences whose members lie in D . Let $D^{\leq\omega} = D^{<\omega} \cup D^\omega$. For $s, t \in D^{<\omega}$, we let $s \frown t$ denote the concatenation of s with t . We let $|t|$ denote the length of t . For $0 \leq i \leq |t|$, we let $t|_i$ denote the initial segment of t having length i , where $t|_0 = \emptyset$ is the empty sequence. If $s \in \{\emptyset\} \cup D^{<\omega}$, we let $s \prec t$ denote the relation that s is a proper initial segment of t .

We say a function $\varphi : D^{<\omega} \rightarrow D^{<\omega}$ is a *pruning* provided that

- (i) $|\varphi(t)| = |t|$ for all $t \in D^{\leq n}$,
- (ii) if $s \prec t$, then $\varphi(s) \prec \varphi(t)$,
- (iii) if $\varphi((U_1, \dots, U_k)) = (V_1, \dots, V_k)$, then $V_k \subset U_k$.

We define prunings $\varphi : D^{\leq n} \rightarrow D^{\leq n}$ similarly.

Given D a weak neighborhood base of 0 in X and $(x_t)_{t \in D^{<\omega}} \subset X$, we say $(x_t)_{t \in D^{<\omega}}$ is

- (i) *weakly null of type I* provided that for each $t = (U_1, \dots, U_k)$, $x_t \in U_k$,
- (ii) *weakly null of type II* provided that for each $t \in \{\emptyset\} \cup D^{<\omega}$, $(x_{t \frown (U)})_{U \in D}$ is a weakly null net. Here D is directed by reverse inclusion.

The notions of weakly null of types I and II for collections indexed by $D^{\leq n}$ are defined similarly. Note that a weakly null collection of type I is weakly null of type II. We now link these notions with our various games.

Proposition 2.3. *Let X be a Banach space, let $p \in (1, \infty]$, and $c > 0$. Then, Player II has a winning strategy in the $T(c, p)$ game on X if and only if there exists a collection $(x_t)_{t \in D^{<\omega}} \subset B_X$ such that*

- (a) $(x_t)_{t \in D^{<\omega}}$ is weakly null of type I, and
- (b) for each $\tau \in D^\omega$, $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$.

Proof. First, if such a collection exists, we can use it to define a winning strategy for Player II in the $T(c, p)$ game. When Player I chooses $U_1 \in D$, then Player II chooses $x_{(U_1)}$. Player I chooses $U_2 \in D$, to which Player II's response is $x_{(U_1, U_2)}$. Play continues in this way, and the result is $(x_{\tau|_i})_{i=1}^\infty$ for some $\tau \in D^\omega$, which satisfies $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$. On the other hand, if Player II has a winning strategy in the $T(c, p)$ game, we define by induction on k the vector $x_{(U_1, \dots, U_k)}$ to be Player II's response according to this winning strategy following the choices $U_1, x_{(U_1)}, U_2, x_{(U_1, U_2)}, \dots, U_k$ in the $T(c, p)$ game. It follows from the rules of the game that (a) is satisfied, and it follows from the fact that Player II plays according to a winning strategy that (b) is satisfied. \square

Analogous statements about $(x_t)_{t \in D^{\leq n}} \subset B_X$ can be made for the $A(c, p, n)$, $N(c, p, n)$, and $\Theta(c, n)$ games. We also have

Proposition 2.4. *Let X be a Banach space, let $p \in (1, \infty]$, and $c > 0$. Then, Player II has a winning strategy in the $T(c, p)$ game if and only if there exists a collection $(x_t)_{t \in D^{< \omega}} \subset B_X$ such that*

- (a) $(x_t)_{t \in D^{< \omega}}$ is weakly null of type II, and
- (b) for each $\tau \in D^\omega$, $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$.

Proof. Since any collection which is weakly null of type I is also weakly null of type II, by the previous proposition, it is sufficient to note that if $(x_t)_{t \in D^{< \omega}} \subset B_X$ is weakly null of type II, then there exists a pruning $\varphi : D^{< \omega} \rightarrow D^{< \omega}$ such that $(x_{\varphi(t)})_{t \in D^{< \omega}} \subset B_X$ is weakly null of type I. Moreover, property (b) is retained by the collection $(x_{\varphi(t)})_{t \in D^{< \omega}}$. \square

Again, analogous statements hold for collections indexed by $D^{\leq n}$ and the games $A(c, p, n)$, $N(c, p, n)$, and $\Theta(c, n)$. Unless otherwise specified, by a weakly null collection $(x_t)_{t \in D^{< \omega}}$ in X , we shall mean weakly null of type II. However, it might be convenient to use that we may assume it to be weakly null of type I.

Remark 2.5. As we already mentioned, in the case that X^* is separable, we can use a fixed countable, linearly ordered weak neighborhood base D and, by identifying D with \mathbb{N} , characterize the constants $\mathfrak{t}_p(X)$, $\mathfrak{a}_{p,n}(X)$, $\mathfrak{a}_p(X)$, $\theta_n(X)$, $\mathfrak{n}_{p,n}(X)$, $\mathfrak{n}_p(X)$ using trees indexed by $\mathbb{N}^{< \omega}$ or $\mathbb{N}^{\leq n}$ rather than $D^{< \omega}$ or $D^{\leq n}$. Now, if X is separable and there exists some $1 < p \leq \infty$ such that any of the constants $\mathfrak{t}_p(X)$, $\mathfrak{a}_p(X)$, $\mathfrak{n}_p(X)$, then $Sz(X) \leq \omega$, X is Asplund, and X^* is separable. However, we can only use $\mathbb{N}^{< \omega}$ (resp. $\mathbb{N}^{\leq n}$) in place of $D^{< \omega}$ (resp. $D^{\leq n}$) as index sets to compute the values of these constants once we already know that X^* is separable, because of examples like ℓ_1 with the Schur property. So, for example, once we know $\mathfrak{p}(X)$ is finite, we can characterize its value using trees indexed by $\mathbb{N}^{< \omega}$, but we cannot use trees indexed by $\mathbb{N}^{< \omega}$ to determine whether $\mathfrak{t}_p(X)$ is finite.

We conclude this section with elementary statements that we shall use to stabilize weakly null trees. Note that the operation described below is actually a pruning.

Proposition 2.6. *Let (D, \leq_D) be any directed set, F a finite set, n a natural number, and $f : D^n \rightarrow F$ a function. There exists $\theta : D^{\leq n} \rightarrow D^{\leq n}$ preserving lengths and initial segments such that*

- (i) *if $\theta((U_1, \dots, U_k)) = (V_1, \dots, V_k)$, then $U_k \leq_D V_k$,*
- (ii) *$f \circ \theta|_{D^n}$ is constant.*

Proof. We work by induction. For $x \in F$, let $I_x = \{U \in D : f((U)) = x\}$. Since $\cup_{x \in F} I_x = D$ and F is finite, there exists some $x \in F$ such that I_x is cofinal in D . This means that for any $U \in D$, there exists $V_U \in I_x$ such that $U \leq_D V_U$. Define $\theta((U)) = (V_U)$ and note that $f \circ \theta|_{D^1} \equiv x$.

Next, assume the result holds for some n and fix $f : D^{n+1} \rightarrow F$. For each $U \in D$, define $f_U : D^n \rightarrow F$ by $f_U((U_1, \dots, U_n)) = f((U, U_1, \dots, U_n))$. By the inductive hypothesis, there exist $\theta_U : D^{\leq n} \rightarrow D^{\leq n}$ which preserves lengths and initial segments and satisfying (i) and (ii). Fix $x_U \in F$ such that $f_U \circ \theta_U|_{D^n} \equiv x_U$. Define $g : D^1 \rightarrow F$ by $g((U)) = x_U$. By the base case, there exists $\phi : D^1 \rightarrow D^1$ satisfying (i) and (ii) with f replaced by g . Define $\theta : D^{\leq n+1} \rightarrow D^{\leq n+1}$ by $\theta((U)) = \phi(U)$ and $\theta((U, U_1, \dots, U_k)) = \phi(U) \frown \theta_{\phi(U)}(U_1, \dots, U_k)$. □

Corollary 2.7. *Let (D, \leq_D) be any directed set, (K, d) a totally bounded metric space, n a natural number, and $f : D^n \rightarrow K$ a function. For any $\varepsilon > 0$, there exist $\theta : D^{\leq n} \rightarrow D^{\leq n}$ preserving lengths and initial segments and a subset B of K of diameter less than ε such that*

- (i) *if $\theta((U_1, \dots, U_k)) = (V_1, \dots, V_k)$, then $U_k \leq_D V_k$,*
- (ii) *$f(\theta(t)) \in B$ for all $t \in D^n$.*

Proof. Let B_1, \dots, B_m be a cover of K by sets of diameter less than ε . Define $g : D^n \rightarrow \{1, \dots, m\}$ by letting $g(t) = \min\{i \leq m : f(t) \in B_i\}$. Apply Proposition 2.6 to g . □

3. THE PROPERTIES AND THEIR RELATIONS

For $1 < p \leq \infty$, we let T_p denote the class of Banach spaces X such that $\mathsf{t}_p(X) < \infty$. The classes A_p and N_p are defined similarly using a_p and n_p . We let $\mathsf{P}_p = \bigcap_{1 < r < p} \mathsf{T}_r$. We let D_1 denote the class of all Banach spaces the Szlenk index of which does not exceed ω . We now present the following alternative descriptions of each class. We have chosen to quickly indicate the easy arguments, to give precise references for others and to detail the new ones. We give this overview, insisting on the renorming characterizations, as they are crucial for our non-linear applications.

We start with the description of T_p . The next theorem is the main result from [6]. We briefly explain the easy implications and emphasize the key part of the proof.

Theorem 3.1. *Fix $1 < p \leq \infty$ and let q be conjugate to p . Let X be a Banach space. The following are equivalent*

- (i) $X \in \mathsf{T}_p$.
- (ii) *There exists a constant $c > 0$ such that for any weak neighborhood base D at 0 in X and any weakly null $(x_t)_{t \in D^{<\omega}} \subset B_X$, there exists $\tau \in D^\omega$ such that $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w \leq c$.*
- (iii) X is p -AUS-able (resp. AUF-able if $p = \infty$).
- (iv) *There exist an equivalent norm $|\cdot|$ on X and $c > 0$ such that for each $\varepsilon \in [0, 1]$, $s_\varepsilon(B_{X^*}^{|\cdot|}) \subset (1 - c\varepsilon^q)B_{X^*}^{|\cdot|}$. In other words, $\bar{\theta}_{|\cdot|}^*(\varepsilon) \geq c\varepsilon^q$*

Proof. The equivalence between (i) and (ii) follows immediately from our discussions on winning strategies in the $T(c, p)$ game. More precisely, $\mathsf{t}_p(X)$ is the infimum of those c for which (ii) holds.

The equivalence between (iii) and (iv) is an immediate consequence of the duality Proposition 2.1.

We now detail the rather simple implication (iii) \Rightarrow (i) and assume, as we may, that X is p -AUS. We look at $1 < p < \infty$ and $p = \infty$ separately. First consider the case $1 < p < \infty$. We note that $\sup_{\sigma > 0} \bar{\rho}_X(\sigma)/\sigma^p < \infty$ if and only if there exists a constant $C \geq 1$ such that for any $x \in X$ and $\sigma \geq 0$, there exists a weak neighborhood U of 0 in X such that for any $y \in U \cap B_X$, $\|x + \sigma y\|^p \leq \|x\|^p + C^p \sigma^p + \varepsilon$. A finite net argument yields that for any compact $G \subset X$ and $\varepsilon > 0$, there exists a weak neighborhood U of 0 such that for any $x \in G$, any scalar b with $|b| \leq 1$, and any $y \in U \cap B_X$, $\|x + by\|^p \leq \|x\|^p + C^p |b|^p$. Using this fact, for $\varepsilon > 0$, we can define a winning strategy for Player I in the $T(C + \varepsilon, p)$ game by fixing $(\varepsilon_i)_{i=1}^\infty \subset (0, 1)$. Player I's initial choice U_1 is arbitrary. Once $U_1, x_1, \dots, U_n, x_n$ have been chosen, let

$$G = \left\{ \sum_{i=1}^n b_i x_i : (b_i)_{i=1}^n \in B_{\ell_p^n} \right\}$$

and choose U_{n+1} such that for any $x \in G$, $y \in U_{n+1} \cap B_X$, and any b with $|b| \leq 1$, $\|x + by\|^p \leq \|x\|^p + C^p |b|^p + \varepsilon_{n+1}$. This completes the recursive construction. For any $m \in \mathbb{N}$ and $(b_i)_{i=1}^m \in B_{\ell_p^m}$,

$$\begin{aligned} \left\| \sum_{i=1}^m b_i x_i \right\|^p &\leq \left\| \sum_{i=1}^{m-1} b_i x_i \right\|^p + C^p |b_m|^p + \varepsilon_m \\ &\leq \left\| \sum_{i=1}^{m-2} b_i x_i \right\|^p + C^p |b_{m-1}|^p + C^p |b_m|^p + \varepsilon_{m-1} + \varepsilon_m \\ &\leq C^p \sum_{i=1}^m |b_i|^p + \sum_{i=1}^m \varepsilon_i \leq C^p + \sum_{i=1}^\infty \varepsilon_i. \end{aligned}$$

If $\sum_{i=1}^{\infty} \varepsilon_i$ was chosen small enough, depending on the modulus of continuity of the function $t \mapsto t^p$ on $[0, C + 1]$, this strategy is a winning strategy for Player I in the $T(C + \varepsilon, p)$ game. Therefore X has T_p . For the $p = \infty$ case, the argument is similar, except there exists a constant C such that for any $x \in X$ and $\sigma > 0$, there exists a weak neighborhood U of 0 such that for any $y \in U \cap B_X$, $\|x + \sigma y\| \leq \max\{\|x\|, C\sigma\}$.

Finally, we refer the reader to [6] for the difficult implication (i) \Rightarrow (iv). \square

We now turn to the characterizations of A_p . Note that item (iii) is a completely new characterization. For that reason we recall the old arguments and detail the new ones. As we will see later, $\mathsf{A}_\infty = \mathsf{N}_\infty$, so we limit ourselves to $p \in (1, \infty)$ in the next statement.

Theorem 3.2. *Fix $1 < p < \infty$ and let q be conjugate to p . Let X be a Banach space. The following are equivalent*

- (i) $X \in \mathsf{A}_p$.
- (ii) *There exists a constant $c > 0$ such that for any weak neighborhood base D at 0 in X , any $n \in \mathbb{N}$, and any weakly null $(x_t)_{t \in D^{\leq n}} \subset B_X$, there exists $t \in D^n$ such that $\|(x_{t_i})_{i=1}^n\|_q^w \leq c$.*
- (iii) *There exist a constant $M \geq 1$ and a constant $C > 0$, such that for any $\tau \in (0, 1]$ there exists a norm $|\cdot|$ on X satisfying $M^{-1}\|x\|_X \leq |x| \leq M\|x\|_X$ for all $x \in X$ and*

$$\forall \sigma \geq \tau, \quad \bar{\rho}_{|\cdot|}(s) \leq Cs^p.$$

- (iv) X has q -summable Szlenk index.

Proof. The equivalence between (i) and (ii) follows again from our initial discussion on games.

The implication (ii) \Rightarrow (iii) is new. Let us prove it. Fix $1 < p < \infty$. Suppose that X is a Banach space and $a \geq 1$ is such that for each $n \in \mathbb{N}$ and $(x_t)_{t \in D^{\leq n}} \subset B_X$ weakly null, there exists $t \in D^n$ such that for all scalar sequences $(a_i)_{i=1}^n$,

$$\left\| \sum_{i=1}^n a_i x_{t_i} \right\|^p \leq a^p \sum_{i=1}^n |a_i|^p.$$

We first note that for any $x \in X$, $n \in \mathbb{N}$, and $(x_t)_{t \in D^{\leq n}} \subset B_X$ weakly null, there exists $t \in D^n$ such that for all scalar sequences $(a_i)_{i=1}^n$,

$$(1) \quad \left\| x + \sum_{i=1}^n a_i x_{t_i} \right\|^p \leq (2a)^p \left[\|x\|^p + \sum_{i=1}^n |a_i|^p \right].$$

Indeed, for an appropriate branch t , it holds that

$$\begin{aligned} \left\| x + \sum_{i=1}^n a_i x_{t|i} \right\|^p &\leq 2^p \max \left\{ \|x\|^p, \left\| \sum_{i=1}^n a_i x_{t|i} \right\|^p \right\} \\ &\leq (2a)^p \max \left\{ \|x\|^p, \sum_{i=1}^n |a_i|^p \right\} \leq (2a)^p \left[\|x\|^p + \sum_{i=1}^n |a_i|^p \right]. \end{aligned}$$

Let now $A = 2a$. Set $f_0(x) = \frac{\|x\|}{A}$ and for $n \in \mathbb{N}$, define

$$f_n(x) = \left[\sup_{(x_t)} \inf_t \sup_{(a_i)} \frac{1}{A^p} \left\| x + \sum_{i=1}^n a_i x_{t|i} \right\|^p - \sum_{i=1}^n |a_i|^p \right]^{1/p},$$

where the outer supremum is taken over all weakly null collections $(x_t)_{t \in D^{\leq n}}$ in B_X , the infimum is taken over $t \in D^n$, and the inner supremum is taken over all scalar sequences $(a_i)_{i=1}^n$. It follows from taking $x_t = 0$ for all t that $f_n(x) \geq \frac{\|x\|}{A}$ for all $x \in X$ and $n \in \mathbb{N}$. On the other hand, it follows from (1) that $f_n(x) \leq \|x\|$ for all $n \in \mathbb{N}$. We also have that $f_n(cx) = |c|f_n(x)$, for each $n \in \mathbb{N} \cup \{0\}$, each $x \in X$, and each scalar c . Let us detail this last fact. It is clear that $f_n(0) = 0$, so assume $c \neq 0$. It is also clear that $f_0(cx) = |c|f_0(x)$. Then we fix $n \in \mathbb{N}$, $x \in X$ and $n \in \mathbb{N}$. For an arbitrary $(x_t)_{t \in D^{\leq n}} \subset B_X$ weakly null and $b > f_n(x)$, there exists $t \in D^n$ such that for all $(a_i)_{i=1}^n$,

$$\frac{1}{A^p} \left\| x + \sum_{i=1}^n a_i x_{t|i} \right\|^p - \sum_{i=1}^n |a_i|^p \leq b^p.$$

Then

$$\begin{aligned} &\frac{1}{A^p} \left\| cx + \sum_{i=1}^n a_i x_{t|i} \right\|^p - \sum_{i=1}^n |a_i|^p \\ &= |c|^p \left[\frac{1}{A^p} \left\| x + \sum_{i=1}^n c^{-1} a_i x_{t|i} \right\|^p - \sum_{i=1}^n |c^{-1} a_i|^p \right] \leq |c|^p b^p. \end{aligned}$$

Since this holds for any $(x_t)_{t \in D^{\leq n}} \subset B_X$ weakly null, it holds that $f_n(cx) \leq |c|f_n(x)$. Repeating the argument, we deduce that $f_n(x) = f_n(c^{-1}cx) \leq |c|^{-1}f_n(cx)$, which gives the reverse inequality.

The key step will be to show the following: for each $n \in \mathbb{N} \cup \{0\}$, each weakly null net $(x_U)_{U \in D} \subset B_X$, and each $\sigma > 0$,

$$(2) \quad \limsup_U f_n(x + \sigma x_U)^p \leq f_{n+1}(x)^p + \sigma^p.$$

So assume $\eta < \limsup_U f_n(x + \sigma x_U)^p$. By passing to a subnet and relabeling, we can assume $\eta < f_n(x + \sigma x_U)^p$ for all U . For each U , we find $(x_t^U)_{t \in D^{\leq n}} \subset B_X$ weakly null such that for each $t \in D^n$, there exists $(a_i)_{i=1}^n$ such that

$$\frac{1}{A^p} \left\| x + \sigma x_U + \sum_{i=1}^n a_i x_{t|i}^U \right\|^p - \sum_{i=1}^n |a_i|^p > \eta.$$

We define the weakly null collection $(x_t)_{t \in D^{\leq n+1}} \subset B_X$ by letting $x_U = x_U$ and $x_{(U, U_1, \dots, U_k)} = x_{(U_1, \dots, U_k)}^U$ for $1 \leq k \leq n$. By the definition of $f_{n+1}(x)^p$, for any $\varepsilon > 0$, there exists $s \in D^{n+1}$ such that for all $(b_i)_{i=1}^{n+1}$,

$$\frac{1}{A^p} \left\| x + \sum_{i=1}^{n+1} b_i x_{s|i} \right\|^p - \sum_{i=1}^{n+1} |b_i|^p \leq f_{n+1}(x)^p + \varepsilon.$$

Write $s = (U, U_1, \dots, U_n)$ and let $t = (U_1, \dots, U_n)$. Then there exists $(a_i)_{i=1}^n$ such that, combining this paragraph with the previous and letting $b_1 = \sigma$ and $b_{i+1} = a_i$ for $1 \leq i \leq n$, it holds that

$$\begin{aligned} \eta &< \frac{1}{A^p} \left\| x + \sigma x_U + \sum_{i=1}^n a_i x_{s|i}^U \right\|^p - \sum_{i=1}^n |a_i|^p \\ &= \frac{1}{A^p} \left\| x + \sum_{i=1}^{n+1} b_i x_{s|i} \right\|^p - \sum_{i=1}^{n+1} |b_i|^p + \sigma^p \leq f_{n+1}(x)^p + \varepsilon + \sigma^p. \end{aligned}$$

Therefore $\eta < f_{n+1}(x)^p + \varepsilon + \sigma^p$. Since $\eta < \limsup_U f_n(x + \sigma x_U)^p$ and $\varepsilon > 0$ were arbitrary, we have proved (2).

The next step is to average the f_n^p 's. So, fix $N \in \mathbb{N}$ and define

$$g_N(x)^p = \frac{1}{N} \sum_{n=0}^{N-1} f_n(x)^p.$$

Clearly, we still have that for all $x \in X$ and $N \in \mathbb{N}$, $\frac{\|x\|}{A} \leq g_N(x) \leq \|x\|$ and $g_N(cx) = |c|g_N(x)$ for all scalars c . Then, applying (2) for each $n \in \{0, \dots, N-1\}$, we obtain that for any weakly null net $(x_U)_{U \in D} \subset B_X$, any $N \in \mathbb{N}$, and $x \in AB_X$,

$$(3) \quad \limsup_U g_N(x + \sigma x_U)^p \leq g_N(x)^p + \sigma^p + \frac{A^p}{N}.$$

The last stage of the proof is to “convexify” our function g_N . For that purpose, we set

$$|x|_N = \inf \left\{ \sum_{i=1}^n g_N(x_i) : n \in \mathbb{N}, x = \sum_{i=1}^n x_i \right\},$$

which defines an equivalent norm on X satisfying $\frac{\|x\|}{A} \leq |x|_N \leq \|x\|$. Moreover, $B_X^{| \cdot |_N}$ is the closed, convex hull of $\{x \in X : g_N(x) < 1\}$. We shall now prove that

$$(4) \quad \forall \sigma > 0, \quad \bar{\rho}_{(X, | \cdot |_N)}(\sigma) \leq \frac{A^p}{p} \left(\sigma^p + \frac{1}{N} \right).$$

First we fix $y \in X$ such that $g_N(y) < 1$. From this it follows that $\|y\| \leq A$. Fix $\sigma > 0$ and $(y_U)_{U \in D} \subset B_X^{| \cdot |_N}$ weakly null, define $x_U = A^{-1}y_U \in B_X$, so

$(x_U)_{U \in D} \subset B_X$ is weakly null. Then we apply (3) to get

$$\begin{aligned} \limsup_U |y + \sigma y_U|_N^p &\leq \limsup_U g_N(y + \sigma A x_U)^p \\ &\leq g_N(y) + \sigma^p A^p + \frac{A^p}{N} < 1 + \sigma^p A^p + \frac{A^p}{N}. \end{aligned}$$

Therefore, by concavity of the function $h(t) = (1+t)^{1/p}$,

$$\limsup_U |y + \sigma y_U|_N - 1 \leq \left(1 + \sigma^p A^p + \frac{A^p}{N}\right)^{1/p} - 1 \leq \frac{A^p}{p} \left(\sigma^p + \frac{1}{N}\right).$$

Next fix $x \in B_X^{|\cdot|_N}$. As noted above, $B_X^{|\cdot|_N}$ is the closed, convex hull of $\{y \in X : g_N(y) < 1\}$. Therefore for each $\varepsilon > 0$, we can find $y_1, \dots, y_k \in X$ with $g_N(y_i) < 1$ and convex coefficients w_1, \dots, w_k such that $|x - \sum_{i=1}^k w_i y_i| < \varepsilon$. Then

$$\begin{aligned} \limsup_U |x + \sigma y_U| - 1 &\leq \varepsilon + \limsup_U \sum_{i=1}^k w_i (|y_i - \sigma y_U|_N - 1) \\ &\leq \varepsilon + \sum_{i=1}^k w_i \left(\frac{A^p}{p} \left(\sigma^p + \frac{1}{N}\right)\right) = \varepsilon + \frac{A^p}{p} \left(\sigma^p + \frac{1}{N}\right). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this finishes the proof of (4).

Finally, it is clear, by taking N large enough in (4), that for any $\tau > 0$ there exists an equivalent norm $|\cdot|$ on X such that $\frac{\|x\|}{A} \leq |x| \leq \|x\|$ and for any $\sigma \geq \tau$, $\bar{\rho}_{(X, |\cdot|)}(\sigma) \leq \frac{A^p}{p} \sigma^p$. We have proved that X satisfies (iii).

Next we prove (iii) \Rightarrow (iv), which is also new. So assume (iii) is satisfied. Then, it follows from Proposition 2.1 that there exists $\gamma \in (0, 1]$ so that for any $t_0 \in (0, 1]$ there exists a norm $|\cdot|$ on X satisfying

$$\forall x \in X, M^{-1} \|x\|_X \leq |x| \leq M \|x\|_X \quad \text{and} \quad \forall t \in [t_0, 1], \bar{\theta}_{|\cdot|}^*(t) \geq \gamma t^q.$$

Fix now $\varepsilon_1, \dots, \varepsilon_n \in (0, 1]$ and pick an equivalent norm $|\cdot|$ as above for $t_0 = \min\{\frac{\varepsilon_1}{4M^2}, \dots, \frac{\varepsilon_n}{4M^2}\}$. Assume that $s_{\varepsilon_1} \dots s_{\varepsilon_n} B_{X^*}$ is not empty. Then $s_{\varepsilon_1} \dots s_{\varepsilon_n} (M B_{|\cdot|} |^*)$ is non empty and by homogeneity, so is $s_{\frac{\varepsilon_1}{M}} \dots s_{\frac{\varepsilon_n}{M}} (B_{|\cdot|} |^*)$. Thus, if we denote σ_ε the Szlenk derivation on X^* where the diameter is taken with respect to the norm $|\cdot|$, we have that $\sigma_{\frac{\varepsilon_1}{M^2}} \dots \sigma_{\frac{\varepsilon_n}{M^2}} B_{|\cdot|} |^*$ is non empty. Then classical manipulations on the Szlenk derivation imply that

$$\frac{1}{2} B_{|\cdot|} |^* \subset \sigma_{\frac{\varepsilon_1}{4M^2}} \dots \sigma_{\frac{\varepsilon_n}{4M^2}} B_{|\cdot|} |^* \subset \prod_{k=1}^n \left(1 - \frac{\gamma \varepsilon_k^q}{4^q M^{2q}}\right) B_{|\cdot|} |^*.$$

The argument for the first inclusion can be found in [35] (proof of Proposition 3.3) and the second inclusion follows from the definition of $\bar{\theta}_{|\cdot|}^*$ and homogeneity. Finally we use the fact that $t \leq -\log(1-t)$, for $t \in [0, 1)$ and elementary calculus to deduce that $\sum_{k=1}^n \varepsilon_k^q \leq \frac{4^q M^{2q}}{\gamma} \log 2$. This finishes the proof.

We now turn to (iv) implies (ii). This was already proved in [7] in a more general setting. We include the simpler proof in our situation for the sake of clarity. So, let M be such that if $\varepsilon_1, \dots, \varepsilon_n \geq 0$ are such that $s_{\varepsilon_1} \dots s_{\varepsilon_n}(B_{X^*}) \neq 0$, then $\sum_{i=1}^n \varepsilon_i^q \leq M^q$. Let D be a weak neighborhood base of 0 in X and assume that $c > 0$ is such that, for some $n \in \mathbb{N}$ and $(x_t)_{t \in D^{\leq n}}$ weakly null in B_X we have that for each $t \in D^n$, there exists $(a_i)_{i=1}^n \in B_{\ell_p^n}$ satisfying $\|\sum_{i=1}^n a_i x_{t|_i}\| > c$. For each $t \in D^n$, fix $x_t^* \in B_{X^*}$ and $(a_i^t)_{i=1}^n \in B_{\ell_p^n}$ such that

$$\operatorname{Re} x_t^* \left(\sum_{i=1}^n a_i^t x_{t|_i} \right) = \left\| \sum_{i=1}^n a_i^t x_{t|_i} \right\| > c.$$

Define $f : D^n \rightarrow B_{\ell_\infty^n}$ by letting $f(t) = (x_t^*(x_{t|_1}), \dots, x_t^*(x_{t|_n}))$. Fix $\delta > 0$ arbitrary. By Corollary 2.7, there exist $(b_i)_{i=1}^n \in B_{\ell_\infty^n}$ and $\theta : D^n \rightarrow D^n$ preserving lengths and initial segments such that for all $t \in D^n$ and $1 \leq i \leq n$,

- (i) if $\theta((U_1, \dots, U_k)) = (V_1, \dots, V_k)$, then $V_k \subset U_k$, and
- (ii) $|x_{\theta(t)}^*(x_{\theta(t)|_i}) - b_i| < \delta$.

By replacing x_s with $x_{\theta(s)}$ and x_t^* with $x_{\theta(t)}^*$ for each $s \in D^{\leq n}$ and $t \in D^n$, we can relabel and assume that the original collections $(x_t)_{t \in D^{\leq n}}$ and $(x_t^*)_{t \in D^n} \subset B_{X^*}$ satisfy this property.

Define $\varepsilon_i = \max\{0, |b_i| - 2\delta\}$ for each $1 \leq i \leq n$. We will prove that for each $0 \leq j \leq n$ and $t \in D^{n-j}$, there exists $x^* \in s_{\varepsilon_{n-j+1}} \dots s_{\varepsilon_n}(B_{X^*})$, and if $j < n$, this x^* can be chosen such that for each $1 \leq i \leq n-j$, $|x^*(x_{t|_i}) - b_i| \leq \delta$. We prove this claim by induction on j . By convention, in the $j = 0$ case, $s_{\varepsilon_{n+1}} s_{\varepsilon_n}(B_{X^*}) = B_{X^*}$ and we just take $x^* = x_t^* \in B_{X^*}$. Next, assume the result holds for some $0 \leq j < n$. By the inductive hypothesis, for each $t \in D^{n-j-1}$ and $U \in D$, since $t \frown (U) \in D^{n-j}$, there exists $x_U^* \in s_{\varepsilon_{n-j+1}} \dots s_{\varepsilon_n}(B_{X^*})$ such that for each $1 \leq i \leq n-j-1$, $|x_U^*(x_{t|_i}) - b_i| \leq \delta$ and $|x_U^*(x_{t \frown (U)}) - b_{n-j}| \leq \delta$. If $\varepsilon_{n-j} = 0$, we pick any U in D and set $x^* = x_U^*$. Note that the conclusions are satisfied, since, by convention

$$x_U^* \in s_{\varepsilon_{n-j+1}} \dots s_{\varepsilon_n}(B_{X^*}) = s_{\varepsilon_{n-j}} s_{\varepsilon_{n-j+1}} \dots s_{\varepsilon_n}(B_{X^*}).$$

Consider now the case $\varepsilon_{n-j} > 0$. If x^* is any weak*-cluster point of $(x_U^*)_{U \in D}$, then clearly $|x^*(x_{t|_i}) - b_i| \leq \delta$ for each $1 \leq i \leq n-j-1$. Note also that, since $(x_{t \frown (U)})_{U \in D}$ is weakly null, there exists $U_0 \in D$ such that for all $U \subset U_0$, $|x^*(x_{t \frown (U)})| < \delta$. This implies that

$$\forall U \subset U_0, \|x_U^* - x^*\| \geq |(x_U^* - x^*)(x_{t \frown (U)})| > |b_{n-j}| - 2\delta = \varepsilon_{n-j}.$$

We now use that x^* is a weak*-cluster point of $(x_U^*)_{U \subset U_0} \subset s_{\varepsilon_{n-j+1}} \dots s_{\varepsilon_n}(B_{X^*})$ to deduce that $x^* \in s_{\varepsilon_{n-j}} \dots s_{\varepsilon_n}(B_{X^*})$. This finishes the inductive proof of our claim. Applying this claim for $j = n$ yields the existence of some $x^* \in s_{\varepsilon_1} \dots s_{\varepsilon_n}(B_{X^*})$, from which it follows that $\sum_{i=1}^n \varepsilon_i^q \leq M^q$. We can now use this information to estimate the constant c . We define $I = \{i \leq n :$

$|b_i| > 2\delta\}$. Then, for any $t \in D^n$,

$$\begin{aligned} c &< \operatorname{Re} x_t^* \left(\sum_{i=1}^n a_i^t x_{t|_i} \right) \leq \delta n + \sum_{i=1}^n |a_i^t| |b_i| \leq 3\delta n + \sum_{i \in I} |a_i^t| |b_i| \\ &\leq 5\delta n + \sum_{j \in I} |a_j^t| \varepsilon_j \leq 5\delta n + \|(a_i^t)_{i \in I}\|_{\ell_p^n} \|(\varepsilon_i)_{i \in I}\|_{\ell_q^n} \leq 5\delta n + M. \end{aligned}$$

Since $\delta > 0$ was arbitrary, we conclude that $c \leq M$. This finishes the proof of this last implication. \square

Next we describe the class \mathbf{N}_p . A more general version of the following result is proved in [9].

Theorem 3.3. *Fix $1 < p \leq \infty$ and let q be conjugate to p . Let X be a Banach space. The following are equivalent.*

- (i) $X \in \mathbf{N}_p$.
- (ii) *There exists a constant $K > 0$ such that for any $n \in \mathbb{N}$ and any weakly null collection $(x_t)_{t \in D^{\leq n}}$ in B_X , there exists $t \in D^n$ such that $\|\sum_{i=1}^n x_{t|_i}\| \leq Kn^{1/p}$.*
- (iii) *There exists a constant $M \geq 1$ and a constant $c > 0$ such that for each $\sigma \in (0, 1]$, there exists a norm $|\cdot|$ on X such that $M^{-1}|x| \leq \|x\|_X \leq M|x|$ for all $x \in X$ and*
 - (a) *if $1 < p < \infty$, $\bar{\rho}_1|\cdot|(\sigma) \leq c\sigma^p$*
 - (b) *if $p = \infty$, $\bar{\rho}_1|\cdot|(c) \leq \sigma$.*

Proof. Again, the equivalence between (i) and (ii) follows from our initial discussion on games.

The argument for (ii) \Rightarrow (iii) is an adaptation of the proof of Theorem 4.2 in [20] to the non-separable case. We refer the reader to this paper or to [9].

Let us briefly explain the simple implication (iii) \Rightarrow (ii). Let us assume, as we may, that $|\cdot|$ satisfies (iii) for $\sigma = \frac{1}{2}$. We shall show the existence of a constant $C \geq 2$ such that (ii) is satisfied. Let $(x_t)_{t \in D^{\leq n}}$ be a weakly null tree in B_X . Pick $t \in D^{\leq n}$ such that $2 < \|\sum_{i=1}^k x_{t|_i}\| \leq 3$ (if this is not possible we are done). Now we pick recursively U_{k+1}, \dots, U_n so that for all $k < l \leq n$, we have, if we denote $s = t \frown (U_{k+1}, \dots, U_n)$, $\|\sum_{i=1}^l x_{s|_i}\| > 2$ and $\|\sum_{i=1}^l x_{s|_i}\| \leq \|\sum_{i=1}^{l-1} x_{s|_i}\|(1 + 2c2^{-p})$. It now follows from classical use of Orlicz functions (see for instance the proof of Theorem 6.1 in [31]) that there exist a constant $K > 0$ so that $\|\sum_{i=1}^n x_{t|_i}\| \leq Kn^{1/p}$. \square

We recall that \mathbf{P}_p is defined to be $\bigcap_{1 < r < p} \mathbf{T}_r$. Then we have.

Theorem 3.4. *Fix $1 < p \leq \infty$ and let q be conjugate to p . Let X be a Banach space. The following are equivalent*

- (i) $X \in \mathbf{P}_p$.

- (ii) For each $1 < r < p$, X is r -AUS-able.
- (iii) There exists an equivalent norm $|\cdot|$ on X such that for all $1 < r < p$, X is r -AUS.
- (iv) For each $1 < r < p$, $\theta_n(X) = o(n^{1/r})$.
- (v) $\mathfrak{p}(X) \leq q$.

Proof. The equivalence between (i) and (ii) follows from Theorem 3.1. The fact that (ii) implies (v) follows from Proposition 2.1. The implication (v) \Rightarrow (iii) is proved in [8] in a very general setting (non-separable, for higher ordinals and operators). Obviously (iii) implies (ii). The implication (ii) \Rightarrow (iv) also follows from Theorem 3.1. Finally (iv) \Rightarrow (i) relies on an averaging of the norms provided by (iii) in Theorem 3.3. \square

We finally summarize what is known about the inclusions between these classes.

Theorem 3.5. *Recall that D_1 denotes the class the of all Banach spaces with Szlenk index at most ω . Then*

- (i) $D_1 = \bigcup_{1 < p \leq \infty} T_p = \bigcup_{1 < p \leq \infty} A_p = \bigcup_{1 < p \leq \infty} N_p = \bigcup_{1 < p \leq \infty} P_p$.
- (ii) For $1 < p < \infty$, $T_p \subsetneq A_p \subsetneq N_p \subsetneq P_p$.
- (iii) $T_\infty \subsetneq A_\infty = N_\infty \subsetneq P_\infty$.

Proof. Let $1 < p \leq \infty$. We clearly have that $T_p \subset A_p \subset N_p \subset P_p$. It follows from (iv) in Theorem 3.1 and (ii) in Theorem 3.4 that $P_p \subset D_1$. We have already explained that if $X \in D_1$, then $\mathfrak{p}(X) < \infty$, so Theorem 3.4 implies that $X \in T_r$, for some $1 < r < p$. Our statement (i) follows from gathering all these pieces of information.

The fact that the inclusions are strict in (ii), as well as $T_\infty \neq A_\infty = N_\infty \neq P_\infty$ are proved in [9]. \square

4. SEPARABLE DETERMINATION

We start with a simple but fundamental statement about selecting weakly null sequences from weakly null nets in AUS-able Banach spaces.

Proposition 4.1. *Let X be a Banach space with $Sz(X) \leq \omega$. Let D be a weak neighborhood base at 0 in X . For any $(x_U)_{U \in D} \subset B_X$ such that $x_U \in U$ for all $U \in D$, there exists a function $f : \mathbb{N} \rightarrow D$ such that $(x_{f(n)})_{n=1}^\infty$ is a weakly null sequence.*

Proof. Since $Sz(X) \leq \omega$, $X \in T_r$ for some $1 < r < \infty$. Let $1/r + 1/s = 1$ and $c > \mathfrak{t}_r(X)$. Let ϕ be a winning strategy in the $T(c, r)$ game. Let V_1 be determined by ϕ and fix $U_1 \in D$ such that $U_1 \subset V_1$. Let Player II choose $x_{U_1} \in U_1 \cap B_X$. Let V_2 be determined by ϕ and fix $U_2 \in D$ such that $U_2 \subset V_2$. Let Player II choose $x_{U_2} \in U_2 \cap B_X$. Continue in this way until U_1, U_2, \dots have been chosen. Define $f(n) = U_n$ and note that $\|(x_{f(n)})_{n=1}^\infty\|_s^w = \|(x_{U_n})_{n=1}^\infty\|_s^w \leq c < \infty$. Therefore $(x_{f(n)})_{n=1}^\infty$ is weakly null. \square

We are now ready to give a unified proof of the separable determination of all the properties considered in this paper. Before to state it, let us mention that summable Szlenk index and having power type Szlenk index was proved to be separably determined by Draga and Kochanek in [14].

Theorem 4.2. *If X is a Banach space with $Sz(X) \leq \omega$, then for each $1 < p \leq \infty$,*

$$\mathfrak{t}_p(X) = \sup\{\mathfrak{t}_p(E) : E \leq X \text{ is separable}\},$$

and this supremum is attained, although possibly infinite. The same is true of $\mathfrak{a}_p(X)$, $\mathfrak{n}_p(X)$, and $\theta_n(X)$. In particular, if X is a Banach space all of whose separable subspaces lie in \mathbb{T}_p , then X lies in \mathbb{T}_p . The same conclusion holds for \mathbb{A}_p , \mathbb{N}_p , \mathbb{P}_p and \mathbb{D}_1

Proof. It is clear that $\mathfrak{t}_p(X) \geq \sup\{\mathfrak{t}_p(E) : E \leq X \text{ is separable}\}$. If $c > \mathfrak{t}(X)$, then there exists a collection $(x_t)_{t \in D^{<\omega}}$ such that for each $\tau \in D^\omega$, $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$.

First, we build $\varphi : \mathbb{N}^{<\omega} \rightarrow D^{<\omega}$ which preserves lengths and immediate predecessors such that $(x_{\varphi(t)})_{t \in \mathbb{N}^{<\omega}}$ is weakly null. We define $\varphi(t)$ by induction on $|t|$. By Proposition 4.1 applied to $(x_{(U)})_{U \in D}$, there exists $f : \mathbb{N} \rightarrow D$ such that $(x_{(f(n))})_{n=1}^\infty$ is weakly null. Define $\varphi((n)) = (f(n))$. Next, if $\varphi(t)$ has been defined, apply Proposition 4.1 to $(x_{\varphi(t) \frown (U)})_{U \in D}$ to select $g : \mathbb{N} \rightarrow D$ such that $(x_{\varphi(t) \frown (g(n))})_{n=1}^\infty$ is weakly null. Define $\varphi(t \frown (n)) = \varphi(t) \frown (g(n))$. This completes the construction.

Define $y_t = x_{\varphi(t)}$. It follows that for any $\tau_1 \in \mathbb{N}^\omega$, there exists a unique $\tau \in D^\omega$ such that $\varphi(\tau_1|_i) = \tau|_i$ for all $i \in \mathbb{N}$, so that

$$\|(y_{\tau_1|_i})_{i=1}^\infty\|_q^w = \|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c.$$

Therefore if F is the closed linear span of $(y_t)_{t \in \mathbb{N}^{<\omega}}$, then $\mathfrak{t}_p(F) > c$. This shows that $\mathfrak{t}_p(X) \leq \sup\{\mathfrak{t}_p(E) : E \leq X \text{ is separable}\}$. Next, let R denote the set of rational numbers r such that $\mathfrak{t}_p(X) > r$. For each $r \in R$, let F_r be a separable subspace of X such that $\mathfrak{t}_p(F_r) > r$, and let E be the closed span of F_r , $r \in R$. Then $\mathfrak{t}_p(E) = \mathfrak{t}_p(X)$, and the supremum is attained. The arguments for $\mathfrak{a}_p(X)$, $\mathfrak{n}_p(X)$, $\theta_n(X)$ are similar.

If X is a Banach space all of whose separable subspaces lie in $\mathbb{T}_p \subset \mathbb{D}_1$, then $\mathfrak{t}_p(X) = \sup\{\mathfrak{t}_p(E) : E \leq X \text{ is separable}\}$ must be finite. Indeed, if the supremum were infinite, then since it is attained, there would exist some separable $E \leq X$ such that $\mathfrak{t}_p(E) = \infty$, and E does not belong to \mathbb{T}_p . Similar arguments hold for \mathbb{A}_p and \mathbb{N}_p .

For \mathbb{P}_p , we note that

$$\begin{aligned} X \in \mathbb{P}_p &\Leftrightarrow (\forall 1 < r < p)(X \in \mathbb{T}_r) \\ &\Leftrightarrow (\forall 1 < r < p)(\forall E \leq X \text{ separable})(E \in \mathbb{T}_r) \\ &\Leftrightarrow (\forall E \leq X \text{ separable})(\forall 1 < r < p)(E \in \mathbb{T}_r) \\ &\Leftrightarrow (\forall E \leq X \text{ separable})(E \in \mathbb{P}_p) \end{aligned}$$

Assume now that X is not in D_1 . Then, for any $p \in \mathbb{Q} \cap (1, \infty)$, X does not belong to T_p . So for any $p \in \mathbb{Q} \cap (1, \infty)$, there exists a separable subspace E_p of X so that E_p is not in T_p . Then the closed linear span of these E_p 's is a separable subspace of X which does not belong to D_1 . \square

5. THREE-SPACE PROPERTIES

5.1. Introduction. We recall that a property (P) of Banach spaces is a *Three-Space Property* (3SP in short) if it passes to quotients and subspaces and a Banach space X has (P) whenever it admits a subspace Y such that Y and X/Y have (P) .

Note first that the properties considered in this paper pass easily to subspaces and quotients.

Proposition 5.1. *Fix $1 < p \leq \infty$ and $X \in \text{Ban}$. If X is in any of the classes T_p, A_p, N_p , or P_p , then any subspace, quotient, or isomorph of X lies in the same class.*

Proof. For subspaces and isomorphs, the result is clear. For quotients, the result follows easily from the dual characterizations of these properties, which clearly pass to weak*-closed subspaces of X^* (we recall that $(X/Y)^*$ is canonically isometric to $Y^\perp \subset X^*$ by a weak*-weak*-bi-continuous map). \square

The following lemma will allow us, when convenient, to reduce our questions to the separable setting.

Lemma 5.2. *Let \mathfrak{l} be a class of Banach spaces which contains all subspaces, quotients, and isomorphs of its members. Suppose also that membership in $\mathfrak{l} \cap \text{Sep}$ is a 3SP, and that if X is a Banach space such that every separable subspace of X lies in \mathfrak{l} , then X lies in \mathfrak{l} . Then membership in \mathfrak{l} is a 3SP.*

Proof. Let X be a Banach space and suppose that Y is a subspace of X such that $Y, X/Y \in \mathfrak{l}$. If X is not in \mathfrak{l} , then there exists a separable subspace E of X such that E is not in \mathfrak{l} . Fix a countable, dense subset S of E and for each $x \in S$, fix a countable subset R_x of Y such that

$$\|x\|_{X/Y} = \inf_{y \in R_x} \|x - y\|.$$

Let G denote the closed linear span of

$$E \cup \bigcup_{x \in S} R_x$$

and let F denote the closed linear span of $\bigcup_{x \in S} R_x$. Then F, G are separable and F , being a subspace of Y , lies in $\mathfrak{l} \cap \text{Sep}$. Moreover, it follows from the construction of G that G/F is isometric to a subspace of X/Y , which means G/F also lies in $\mathfrak{l} \cap \text{Sep}$. Therefore G lies in \mathfrak{l} , as does $E \leq G$. Therefore every separable subspace of X lies in \mathfrak{l} , as does X . \square

5.2. Past results. It was shown by Draga, Kochanek, and the first-named author in [10] that membership in \mathbf{P}_p is a 3SP, although it was not stated in this way. We isolate here a shorter and more direct argument. We will show the following.

Theorem 5.3. *Fix a Banach space X , a closed subspace Y of X , and $1 < p \leq \infty$.*

- (i) *If $\mathfrak{n}_p(Y)$ and $\mathfrak{n}_p(X/Y)$ are finite, then there exist constants C, λ such that for all $2 \leq n \in \mathbb{N}$, $\mathfrak{n}_p(X) \leq C(\log n)^\lambda$.*
- (ii) *If Y and X/Y have \mathbf{P}_p , so does X .*

The fact that \mathbf{P}_p is a 3SP was shown in Theorem 7.5 of [10]. The proof there established an inequality similar to Theorem 5.3(i), but using \mathfrak{a}_p rather than \mathfrak{n}_p . In fact, the argument there was given for asymptotic Rademacher type p , which deals with Rademacher averages of arbitrary linear combinations of the branches of weakly null trees, which added significant technicality to the proof. Because \mathfrak{n}_p deals only with flat linear combinations, we sketch the simpler proof below.

We will use the following, which is an analogue of a lemma of Enflo, Lindenstrauss, and Pisier in their solution of the Palais problem [15].

Lemma 5.4. *For any Banach space X , any closed subspace Y of X , and any $m, n \in \mathbb{N}$,*

$$\theta_{mn}(X) \leq 6(\theta_m(X/Y)\theta_n(X) + \theta_m(X)\theta_n(Y)).$$

Let us first deduce Theorem 5.3 from Lemma 5.4.

Proof of Theorem 5.3. (i) By Lemma 5.4, for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathfrak{n}_{p,n^2}(X) &= \frac{\theta_{n^2}(X)}{n^{2/p}} \leq 6 \left(\frac{\theta_n(X/Y)}{n^{1/p}} + \frac{\theta_n(Y)}{n^{1/p}} \right) \frac{\theta_n(X)}{n^{1/p}} \\ &\leq c \mathfrak{n}_{p,n}(X), \end{aligned}$$

where $c = 10(\mathfrak{n}_p(X/Y) + \mathfrak{n}_p(Y))$. We argue as in Theorem 3 of [15] to deduce the existence of the constants C and λ .

(ii) Assume $Y, X/Y$ have \mathbf{P}_p . Fix $1 < r < s < p$. Since $Y, X/Y$ have \mathbf{P}_p , they also have \mathbf{N}_s , which means there exist constants C, λ such that for all $2 \leq n$, $\theta_n(X) = \mathfrak{n}_{s,n}(X)n^{1/s} \leq C(\log n)^\lambda n^{1/s}$. Then $\mathfrak{n}_{r,n}(X) = \theta_n(X)n^{-1/r} \leq C(\log n)^\lambda n^{1/s-1/r}$, which vanishes as n tends to infinity. Therefore, for $1 < r < p$, $\mathfrak{n}_r(X) < \infty$, so $X \in \mathbf{P}_p = \bigcap_{1 < r < p} \mathbf{N}_r$. \square

We next recall an easy technical piece which we will need for the proof of Lemma 5.4.

Claim 1. *Let X be a Banach space and Y a closed subspace. For any weak neighborhood U_1 of 0 in X and $R, \delta > 0$, there exists a weak neighborhood U_2 of 0 in X such that if $x \in U_2 \cap RB_X$ with $\|x\|_{X/Y} < \delta$, then there exists $y \in U_1 \cap RB_Y$ such that $\|x - y\| < 6\delta$.*

Proof. If it were not so, then for some weak neighborhood U_1 of 0 in X and some $R, \delta > 0$, there would exist a weakly null net $(x_\lambda) \subset RB_X$ such that, for all λ , $\|x_\lambda\|_{X/Y} < \delta$ and for all $y \in U_1 \cap RB_Y$, $\|x_\lambda - y\| \geq 6\delta$. For each λ , we can fix $y_\lambda \in Y$ such that $\|x_\lambda - y_\lambda\| < \delta$. By passing to a subnet and relabeling, we can assume (y_λ) is weak*-convergent to some $y^{**} \in B_{X^{**}}$. Fix $\varepsilon > 0$ and a finite subset F of X^* such that $V := \{x \in X : (\forall x^* \in F)(|x^*(x)| < 2\varepsilon)\} \subset U_1$. Since (x_λ) is weakly null and (y_λ) is weak*-convergent to y^{**} , we can find λ_1 , a finite subset G of the index set of (x_λ) , and positive numbers $(w_\lambda)_{\lambda \in G}$ summing to 1 such that

- (i) for all $x^* \in F$ and $\lambda \in \{\lambda_1\} \cup G$, $|y^{**}(x^*) - x^*(y_\lambda)| < \varepsilon$,
- (ii) $\|\sum_{\lambda \in G} w_\lambda x_\lambda\| < \delta$.

Let $y_1 = y_{\lambda_1} - \sum_{\lambda \in G} w_\lambda y_\lambda \in V$ and note that

$$\|y_1 - x_{\lambda_1}\| \leq \|y_{\lambda_1} - x_{\lambda_1}\| + \sum_{\lambda \in G} w_\lambda \|y_\lambda - x_\lambda\| + \left\| \sum_{\lambda \in G} w_\lambda x_\lambda \right\| < 3\delta.$$

Since $\|x_{\lambda_1}\| \leq R$, $\|y_1\| \leq R + 3\delta$. If $\|y_1\| \leq R$, let $y = y_1$, and otherwise let $y = \frac{R}{\|y_1\|} y_1$, noting that $\|y - x\| \leq \|y - y_1\| + \|y_1 - x\| < 6\delta$. By convexity of V , $y \in V \subset U_1$, and we reach a contradiction. \square

Let us now sketch the proof of Lemma 5.4.

Sketch. If Y is finite dimensional, then $\theta_n(Y) = 0$ and $\theta_n(X/Y) = \theta_n(X)$ for all $n \in \mathbb{N}$. Then the inequality follows, without the factor of 6, using submultiplicativity of $\theta_n(X)$. A similar conclusion holds if X/Y is finite dimensional. We can therefore assume $Y, X/Y$ are infinite dimensional, and $\theta_n(Y), \theta_n(X/Y) \geq 1$ for all $n \in \mathbb{N}$.

Of course, the idea is to consider a weakly null tree indexed by D^{mn} to consist of inner trees of height m , and outer trees of height n . For $\psi > \theta_m(X/Y)$, $\psi_1 > \theta_m(X)$, $\phi > \theta_n(Y)$, and $\phi_1 > \theta_n(X)$, we can fix winning strategies χ, χ_1, ϖ , and ϖ_1 for Player I in each of the games $\Theta(\psi, m)$ on X/Y , $\Theta(\psi_1, m)$ on X , $\Theta(\phi, n)$ on Y , and $\Theta(\phi_1, n)$ on X , respectively. For a weakly null collection $(x_t)_{t \in D^{\leq mn}} \subset B_X$, we claim that we can recursively select $t_1 \in D^m$, $y_1 \in Y$, $t_2 \in D^{2m}$ such that $t_1 \prec t_2$, $y_2 \in Y, \dots$, such that, with $t = t_n \in D^{mn}$, for all $1 \leq i \leq n$,

- (i) $\|y_i - \sum_{j=(i-1)m+1}^{im} x_{t_j}\| \leq 6\psi$,
- (ii) $\|\sum_{j=(i-1)m+1}^{im} x_{t_j}\| \leq \psi_1$,
- (iii) $\|\sum_{i=1}^n \frac{y_i}{10\psi_1}\| \leq \phi$,
- (iv) $\|\sum_{i=1}^n \frac{y_i - \sum_{j=(i-1)m+1}^{im} x_{t_j}}{10\psi}\| \leq \phi_1$.

Then

$$\begin{aligned} \left\| \sum_{i=1}^{mn} x_{t_i} \right\| &\leq \left\| \sum_{i=1}^m [y_i - \sum_{j=(i-1)m+1}^{im} x_{t_{|j}}] \right\| + \left\| \sum_{i=1}^n y_i \right\| \\ &\leq 6\psi\phi_1 + 6\psi_1\phi. \end{aligned}$$

Since $\psi > \theta_m(X/Y)$, $\psi_1 > \theta_m(X)$, $\phi > \theta_n(Y)$, and $\phi_1 > \theta_n(X)$ were arbitrary, this will yield the inequality.

We now explain how to choose t_i and y_i . Assume that for some $k < n$, we have already chosen $t_1 \prec \dots \prec t_k$, $t_i \in D^{im}$, and y_1, \dots, y_k . Assume also that $(y_i/10\psi_1)_{i=1}^k$ and $((y_i - \sum_{j=(i-1)m+1}^{im} x_{t_{|j}})/10\psi)_{i=1}^k$ have been chosen by Player II against Player I, who is using strategies ϖ and ϖ_1 , respectively. Let U, U_1 be the weak neighborhoods chosen for the next stage of the game by strategies ϖ and ϖ_1 , respectively. By replacing these sets with subsets if necessary, we can assume they are convex.

By Claim 1, there exists a weak neighborhood W of 0 in X , which we can also assume is convex, such that if $x \in W \cap \psi_1 B_X$ satisfies $\|x\|_{X/Y} \leq \psi$, then there exists $y \in U \cap \frac{1}{2}U_1 \cap \psi_1 B_Y$ such that $\|y - x\| \leq 6\psi$. Let $Q : X \rightarrow X/Y$ denote the quotient map and, using the strategies χ and χ_1 , choose $t_k \prec s_1 \prec \dots \prec s_m = t_{k+1} \in D^{(k+1)m}$ such that for each $1 \leq j \leq m$,

$$x_{s_j} \in G_j \cap Q^{-1}(H_j) \cap \frac{1}{m}W \cap \frac{1}{2m}U_1.$$

Here, the sets H_j are determined by χ playing against Player II's choices of $x_{s_1} + X/Y, \dots, x_{s_m} + X/Y$ and the sets G_j are determined by χ_1 playing against Player II's choices of x_{s_1}, \dots, x_{s_m} . Note that $G_j \cap Q^{-1}(H_j)$ is a weak neighborhood of 0 in X . Playing according to χ and χ_1 guarantees that

$$\left\| \sum_{j=km+1}^{(k+1)m} x_{t_{k+1|j}} \right\| = \left\| \sum_{j=1}^m x_{s_j} \right\| \leq \psi_1$$

and

$$\left\| \sum_{j=km+1}^{(k+1)m} x_{t_{k+1|j}} \right\|_{X/Y} = \left\| \sum_{j=1}^m x_{s_j} \right\|_{X/Y} \leq \psi.$$

Since $\sum_{j=1}^m x_{s_j} \in \frac{1}{m}V + \dots + \frac{1}{m}V = V$, there exists $y_{k+1} \in U \cap \frac{1}{2}U_1 \cap \psi_1 B_Y$ such that

$$\left\| y_{k+1} - \sum_{j=1}^m x_{s_{|j}} \right\| \leq 6\psi.$$

Note also that $y_{k+1} - \sum_{j=1}^m x_{s_j} \in \frac{1}{2}U_1 + \frac{1}{2m}U_1 + \dots + \frac{1}{2m}U_1 = U_1$. Therefore

$$\frac{y_{k+1}}{\phi_1} \in U \cap B_Y \quad \text{and} \quad \frac{y_{k+1} - \sum_{j=km+1}^{(k+1)m} x_{t_{k+1|j}}}{10\psi} \in U_1 \cap B_X$$

have been chosen in accordance with the winning strategies ϖ and ϖ_1 , respectively. This completes the recursive choices. Items (i) and (ii) are

seen to be satisfied from the construction, while items (iii) and (iv) follow from the fact that the outer sequences were chosen according to χ and χ_1 . \square

5.3. A counterexample. For $p \in (1, \infty)$, contrary to \mathbf{P}_p , none of the properties \mathbf{T}_p , \mathbf{A}_p , \mathbf{N}_p is a three-space property. Before proving this result by giving a counterexample, let us introduce the following definition.

Definition 5.5. Let X be a Banach space and $p \in (1, \infty)$. We say that X has the *weak p -Banach-Saks property* if there exists a positive constant C such that for every weakly null sequence (x_n) in B_X and every $k \in \mathbb{N}$, we can find a subsequence $(x_{n_j})_j$ of (x_n) such that

$$\left\| \sum_{j=1}^k x_{n_j} \right\| \leq Ck^{1/p}$$

for all $n_1 < \dots < n_k$.

Let us notice that every Banach space with property \mathbf{N}_p , $1 < p < \infty$, has the weak p -Banach-Saks property. For instance, use item (iii) of Theorem 3.3 and mimic the argument of (iii) \Rightarrow (ii) in the proof of that statement.

Proposition 5.6. *Let $p \in (1, \infty)$. Then the properties \mathbf{T}_p , \mathbf{A}_p , and \mathbf{N}_p are not three space properties.*

Proof. Let us consider the Kalton-Peck reflexive spaces Z_p (see [32] or [5]), that satisfies the following: Z_p may be normed in such a way that it has a closed subspace M isometric to ℓ_p with Z_p/M also isometric to ℓ_p . It is known that Z_p does not have the weak p -Banach-Saks property (see [5]). Hence, since ℓ_p has property \mathbf{T}_p , we get the result by combining the previous remark and item (ii) of Theorem 3.5. \square

In fact, we can even prove that Z_p does not have any of the concentration properties indexed by p considered in [16] and therefore deduce the following.

Proposition 5.7. *Let $p \in (1, \infty)$. Then the properties \mathbf{HFC}_p , \mathbf{HIC}_p , \mathbf{HC}_p , $\mathbf{HFC}_{p,d}$, and $\mathbf{HC}_{p,d}$ introduced in [16] are not three spaces properties.*

For clarity, we will only define one of them here. First, for $k \in \mathbb{N}$, let us denote $[\mathbb{N}]^k = \{\bar{n} = (n_1, \dots, n_k) \in \mathbb{N}^k; n_1 < n_2 < \dots < n_k\}$. We endow $[\mathbb{N}]^k$ with the following distance, called the Hamming distance:

$$\forall \bar{n}, \bar{m} \in [\mathbb{N}]^k, d_{\mathbb{H}}(\bar{n}, \bar{m}) = |\{1 \leq j \leq k; n_j \neq m_j\}|.$$

We say that a Banach space X has property \mathbf{HFC}_p if there exists $\lambda > 0$ such that, for every $k \in \mathbb{N}$ and every 1-Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \rightarrow X$, one can find an infinite subset \mathbb{M} of \mathbb{N} so that

$$\forall \bar{n}, \bar{m} \in [\mathbb{M}]^k, \|f(\bar{n}) - f(\bar{m})\| \leq \lambda k^{\frac{1}{p}}.$$

We refer the reader to [16] for the definitions of the other concentration properties.

Proof of Proposition 5.7. We use the notation from Theorem 6.1 [32]. For $k \in \mathbb{N}$, the map $g : \begin{cases} [\mathbb{N}]^k & \rightarrow Z_p \\ \bar{n} & \mapsto \sum_{j=1}^k u_{n_j} \end{cases}$ is 2-Lipschitz and satisfies

$$\|g(\bar{n}) - g(\bar{m})\| = 2^{1/p} \left(\frac{\ln(2k)}{p} + 1 \right) k^{1/p}$$

for all integers $n_1 < m_1 < n_2 < \dots < n_k < m_k$. Therefore, Z_p does not have any of the properties mentioned above. The result follows from the fact that ℓ_p has them all, which is due to Kalton and Randrianarivony [31]. \square

5.4. Asymptotic uniform flatenability. In the case $p = \infty$, the situation is different. First, we easily have the following.

Theorem 5.8. *The property T_∞ is a three-space property.*

Proof. Let us first recall that a separable Banach space is T_∞ if and only if it is isomorphic to a subspace of c_0 (see [19]). It was shown by Johnson and Zippin [27], who attributed it to Lindenstrauss, that being isomorphic to a subspace of c_0 is a three-space property. So we deduce from Theorem 4.2, Proposition 5.1 and Lemma 5.2 that T_∞ is a three-space property. \square

By looking at the argument in [27], we can actually show slightly more.

Proposition 5.9. *Let $p \in (1, \infty]$ and \mathbf{B} be any one of the properties T_p , A_p and N_p . Let X be a Banach space with a closed subspace Y such that Y has T_∞ and X/Y has \mathbf{B} . Then X has \mathbf{B} .*

Proof. By Theorem 4.2, we may assume that X is separable. Let $T : Y \rightarrow c_0$ be a linear embedding. It follows from Sobczyk's theorem that c_0 has the separable extension property. Therefore T extends to a bounded linear map $S : X \rightarrow c_0$. Define now $U : X \rightarrow c_0 \oplus X/Y$ by $Ux = (Sx, Qx)$ where $Q : X \rightarrow X/Y$ is the quotient map. It then easy to check that U is a linear embedding from X into $c_0 \oplus X/Y$. Finally, since \mathbf{B} passes clearly to direct sums, we deduce that $c_0 \oplus X/Y$ and therefore X have \mathbf{B} . \square

5.5. Summable Szlenk index. This subsection contains the proof of our main result on three-space properties. We will show that A_∞ is a three-space property.

Recall that a Banach space is said to have property A_∞ provided there exists a constant $c > 0$ such that for each $n \in \mathbb{N}$, Player I has a winning strategy in the $N(c, \infty, n)$ game (since $A_\infty = N_\infty$ according to Theorem 3.5): Player I chooses a weak neighborhood U_1 of 0 in X and Player II chooses $x_1 \in U_1 \cap B_X$. Player I chooses a weak neighborhood U_2 of 0 in X and Player II chooses $x_2 \in U_2 \cap B_X$. Play continues in this way until x_1, \dots, x_n have been chosen. Player I wins if $\|\sum_{i=1}^n x_i\| \leq c$ and Player II wins otherwise.

As we are going to use a separable reduction, we will be able to use trees indexed by \mathbb{N} . For that purpose we let $T_n = \mathbb{N}^{\leq n}$.

It will be convenient for us to introduce the notions of \mathcal{U} -weakly (or weak*) null sequences or collections indexed by T_n in a Banach space, where \mathcal{U} is a given free ultrafilter on \mathbb{N} . Let us assume, more generally, that \mathcal{U} is a filter on \mathbb{N} . Given a Banach space X , we say that a sequence $(x_i)_{i=1}^\infty \subset X$ is \mathcal{U} -weakly null if it converges to 0 over \mathcal{U} in the weak topology. The notion of \mathcal{U} -weak*-null for a sequence $(x_i^*)_{i=1}^\infty \subset X^*$ is defined similarly. A collection $(x_t)_{t \in T_n} \subset X$ is \mathcal{U} -weakly null provided that for each $t \in \{\emptyset\} \cup T_{n-1}$, $(x_{t \wedge (m)})_{m=1}^\infty$ is \mathcal{U} -weakly null. We say $(x_t^*)_{t \in T_n} \subset X^*$ is \mathcal{U} -weak*-null provided that for each $t \in \{\emptyset\} \cup T_{n-1}$, $(x_{t \wedge (m)}^*)_{m=1}^\infty$ is \mathcal{U} -weak*-null. Note that for each Banach space X and each $n \in \mathbb{N}$, B_X admits a \mathcal{U} -weakly null collection and B_{X^*} admits a \mathcal{U} -weak*-null collection, namely the collections consisting entirely of zeros.

In the remainder of this section, we shall always assume that \mathcal{U} is a free ultrafilter on \mathbb{N} .

Proposition 5.10. *Let Z be a Banach space such that Z^* is separable, and $(z_m^*)_{m=1}^\infty \subset Z^*$ a \mathcal{U} -weak*-null sequence in Z^* . For any $\delta > 0$, there exists a \mathcal{U} -weakly null sequence $(z_m)_{m=1}^\infty \subset B_Z$ such that*

$$\lim_{m \in \mathcal{U}} \operatorname{Re} z_m^*(z_m) \geq \lim_{m \in \mathcal{U}} \frac{\|z_m^*\|}{2} - \delta.$$

Proof. If $\lim_{m \in \mathcal{U}} \|z_m^*\| = 0$, simply take $z_m = 0$ for all $m \in \mathbb{N}$. Suppose that $r := \lim_{m \in \mathcal{U}} \|z_m^*\| > 0$. For each $m \in \mathbb{N}$, fix $x_m \in B_X$ such that $\operatorname{Re} z_m^*(x_m) > \|z_m^*\| - \delta$. Let $x^{**} = \operatorname{weak}^* \text{-} \lim_{m \in \mathcal{U}} x_m$, where the limit is taken in $B_{X^{**}}$. Since X^* is separable, the weak*-topology on $B_{X^{**}}$ is metrizable, which means some subsequence $(u_m)_{m=1}^\infty$ of $(x_m)_{m=1}^\infty$ is weak*-convergent to x^{**} . Define $u_0 = 0$. For each $m \in \mathbb{N}$, let $C_m = \{i \in \mathbb{N}_0 : |z_m^*(u_i)| \leq \delta\}$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Note that $0 \in C_m$ for all $m \in \mathbb{N}$. Define $f : \mathbb{N} \rightarrow \mathbb{N}_0$ by letting $f(m) = \max C_m$ if $\max C_m < m$, and let $f(m) \in C_m \cap [m, \infty)$ be arbitrary if $\max C_m \geq m$. Note that $\lim_{m \in \mathcal{U}} f(m) = l \in \mathbb{N}_0 \cup \{\infty\}$, where $\mathbb{N}_0 \cup \{\infty\}$ is the one-point compactification of \mathbb{N}_0 . We claim that $l = \infty$. Indeed, if $l < \infty$, then

$$f^{-1}(\{l\}) \cap \{m \in \mathbb{N} : |z_m^*(u_{l+1})| < \delta\} \in \mathcal{U},$$

and therefore there exists some $l < m_0 \in f^{-1}(\{l\}) \cap \{m \in \mathbb{N} : |z_m^*(u_{l+1})| < \delta\}$. But this means that $l + 1 \in C_{m_0}$ and $l = \max C_{m_0}$, this is a contradiction.

Define now $z_m = \frac{1}{2}(x_m - u_{f(m)}) \in B_X$. Since $\lim_{m \in \mathcal{U}} f(m) = \infty$ and $\operatorname{weak}^* \text{-} \lim_{m \rightarrow \infty} u_m = x^{**}$, $\operatorname{weak}^* \text{-} \lim_{m \in \mathcal{U}} u_{f(m)} = x^{**}$. Therefore $\operatorname{weak}^* \text{-} \lim_{m \in \mathcal{U}} z_m = 0$. By our choice of $f(m)$, $|z_m^*(u_{f(m)})| \leq \delta$ for all $m \in \mathbb{N}$, and

$$\lim_{m \in \mathcal{U}} \operatorname{Re} z_m^*(z_m) \geq \lim_{m \in \mathcal{U}} \operatorname{Re} \frac{1}{2} z_m^*(x_m) - \lim_{m \in \mathcal{U}} \frac{1}{2} |z_m^*(u_{f(m)})| \geq \frac{r}{2} - \delta.$$

□

We define $\alpha_n^{\mathcal{U}}(X)$ to be the infimum of $a \geq 0$ such that for any \mathcal{U} -weakly null $(x_t)_{t \in T_n} \subset B_X$,

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| \leq a.$$

We define $\beta_n^{\mathcal{U}}(X)$ to be the infimum of $b > 0$ such that for any \mathcal{U} -weak*-null $(x_t^*)_{t \in T_n} \subset B_{X^*}$,

$$b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| \geq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|.$$

The next proposition details a characterization and a dual formulation of A_∞ for spaces with separable dual.

Proposition 5.11. *Let X be a Banach space such that X^* is separable.*

- (i) X has A_∞ if and only if $\sup_n \alpha_n^{\mathcal{U}}(X) < \infty$.
- (ii) For each $n \in \mathbb{N}$, $\alpha_n^{\mathcal{U}}(X) \leq 2\beta_n^{\mathcal{U}}(X)$.
- (iii) For each $n \in \mathbb{N}$, $\beta_n^{\mathcal{U}}(X) \leq 2\alpha_n^{\mathcal{U}}(X)$.

Proof. (i) Since X^* is separable, there exists a metric d on B_X that is compatible with the weak topology. For each $n \in \mathbb{N}$, let $U_n = \{x \in X : d(x, 0) < 1/n\}$.

First assume that X does not have property A_∞ . Then for each $a > 0$, there exists $n \in \mathbb{N}$ such that Player I fails to have a winning strategy in the $A(a, n)$ game. Since the $A(a, n)$ game is determined, Player II must have a winning strategy in the $A(a, n)$ game. We will choose $(x_t)_{t \in T_n}$ according to this winning strategy. First, let $x_{(m)} \in U_m \cap B_X$ be Player II's response if Player I opens the game with U_m . For $1 < k \leq n$ and $t = (m_1, \dots, m_k)$, let $x_t \in U_{m_k} \cap B_X$ be Player II's response if Players I and II have chosen $U_{m_1}, x_{(m_1)}, \dots, x_{(m_1, \dots, m_{k-1})}, U_{m_k}$. For each $t = (m_i)_{i=1}^n \in T_n$, since $U_{m_1}, x_{(m_1)}, U_{m_2}, x_{(m_1, m_2)}, \dots, U_{m_n}, x_{(m_1, \dots, m_n)}$ were chosen according to Player II's winning strategy, $\left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| > a$. Therefore

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| \geq a.$$

Since $x_{t \smallfrown (m)} \in U_m$ for each $t \in \{\emptyset\} \cup T_{n-1}$ and $m \in \mathbb{N}$, it follows that $(x_{t \smallfrown (m)})_{m=1}^\infty$ is weakly null, and is therefore a \mathcal{U} -weakly null sequence, for each $t \in \{\emptyset\} \cup T_{n-1}$. Therefore this collection $(x_t)_{t \in T_n} \subset B_X$ witnesses the fact that $\alpha_n^{\mathcal{U}}(X) \geq a$. Since $a > 0$ was arbitrary, $\sup_n \alpha_n^{\mathcal{U}}(X) = \infty$. By contraposition, if $\sup_n \alpha_n^{\mathcal{U}}(X) < \infty$, X has property A_∞ .

Next, suppose that X has property A_∞ . Fix $a_0 > 0$ such that for all $n \in \mathbb{N}$, Player I has a winning strategy in the $A(a_0, n)$ game. Suppose that for some $n \in \mathbb{N}$, $\alpha_n^{\mathcal{U}}(X) > a_0$. Fix $\alpha_n^{\mathcal{U}}(X) > a > a_0$. There exists

$(x_t)_{t \in T_n} \subset B_X$ which is \mathcal{U} -weakly null and such that

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| > a.$$

Let V_1 be Player I's first choice according to a winning strategy in the $A(a_0, n)$ game. This means we can choose

$$m_1 \in \{m \in \mathbb{N} : x_{(m)} \in V_1\} \cap \{m \in \mathbb{N} : \|x_{(m)}\| > a\} \in \mathcal{U}, \text{ if } n = 1$$

and

$$m_1 \in \{m \in \mathbb{N} : x_{(m)} \in V_1\} \\ \cap \left\{ m \in \mathbb{N} : \lim_{m_2 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| x_{(m)} + \sum_{i=2}^n x_{(m, m_2, \dots, m_i)} \right\| > a \right\} \in \mathcal{U}, \text{ if } n > 1$$

Let Player II's choice in the $A(a_0, n)$ game be $x_{(m_1)}$. Next, assume that for some $1 \leq k < n$, V_1, \dots, V_k and m_1, \dots, m_k have been chosen such that

- (a) $V_1, x_{(m_1)}, \dots, V_k, x_{(m_1, \dots, m_k)}$ have been chosen in the $A(a_0, n)$ game with Player I playing according to a winning strategy,
- (b) we have the inequality

$$\lim_{m_{k+1} \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| > a,$$

We now describe the recursive step of the construction.

Assume first that $k+1 < n$. Let V_{k+1} be Player I's next choice according to the winning strategy, let Player II's choice be $x_{(m_1, \dots, m_{k+1})}$, where

$$m_{k+1} \in \{m \in \mathbb{N} : x_{(m_1, \dots, m_k, m)} \in V_{k+1}\} \cap \\ \left\{ m \in \mathbb{N} : \lim_{m_{k+2} \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| x_{(m_1, \dots, m_k, m)} + \sum_{i \neq k+1; 1 \leq i \leq n} x_{(m_1, \dots, m_i)} \right\| > a \right\} \in \mathcal{U}$$

Assume, for the last step, that $k+1 = n$. Let $V_{k+1} = V_n$ be Player I's next choice according to the winning strategy, let Player II's choice be $x_{(m_1, \dots, m_n)}$, where

$$m_{k+1} = m_n \in \{m \in \mathbb{N} : x_{(m_1, \dots, m_k, m)} \in V_n\} \cap \\ \left\{ m \in \mathbb{N} : \left\| x_{(m_1, \dots, m_k, m)} + \sum_{i=1}^k x_{(m_1, \dots, m_i)} \right\| > a \right\} \in \mathcal{U}$$

This completes the recursive construction, from which we it follows that $\left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| > a$. However, since $V_1, x_{(m_1)}, \dots, V_n, x_{(m_1, \dots, m_n)}$ were chosen with Player I playing according to a winning strategy in the $A(a_0, n)$ game, $\left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| \leq a_0$. Since $a_0 < a$, this is a contradiction. Hence $\sup_n \alpha_n^{\mathcal{U}}(X) \leq a_0 < \infty$. Therefore if X has property A_∞ , $\sup_n \alpha_n^{\mathcal{U}}(X) < \infty$.

(ii) Fix $a < \alpha_n^{\mathcal{U}}(X)$. Then there exists a \mathcal{U} -weakly null collection $(x_t)_{t \in T_n} \subset B_X$ such that

$$a < \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\|.$$

For each $t = (m_1, \dots, m_n) \in \mathbb{N}^n$, choose $y_t^* \in \frac{1}{2}B_{X^*}$ such that

$$\operatorname{Re} y_t^* \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)} \right) = \frac{1}{2} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\|.$$

Then

$$\frac{a}{2} < \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \operatorname{Re} y_{(m_1, \dots, m_n)}^* \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)} \right).$$

For $t \in \mathbb{N}^n$, set $z_t^* = y_t^*$ and for each $t \in \{\emptyset\} \cup T_{n-1}$, define

$$z_t^* = \lim_{m_{|t|+1} \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} y_{t \frown (m_{|t|+1}, \dots, m_n)}^*,$$

where all limits are taken with respect to the weak*-topology. For $(m_1, \dots, m_k) \in T_n$, define $x_{(m_1, \dots, m_k)}^* = z_{(m_1, \dots, m_k)}^* - z_{(m_1, \dots, m_{k-1})}^* \in B_{X^*}$. Note that $(x_t^*)_{t \in T_n} \subset B_{X^*}$ is \mathcal{U} -weak*-null, which implies that

$$\begin{aligned} \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| &\leq \beta_n^{\mathcal{U}}(X) \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| \\ &= \beta_n^{\mathcal{U}}(X) \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \|z_{(m_1, \dots, m_n)}^* - z_{\emptyset}^*\| \\ &\leq \beta_n^{\mathcal{U}}(X). \end{aligned}$$

Note that because $(x_t)_{t \in T_n}$ is \mathcal{U} -weakly null and $(x_t^*)_{t \in T_n}$ is \mathcal{U} -weak*-null, it holds that for distinct $i, j \in \{1, \dots, n\}$,

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} x_{(m_1, \dots, m_i)}^*(x_{(m_1, \dots, m_j)}^*) = 0.$$

Similarly, for each $1 \leq i \leq n$, $\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} z_{\emptyset}^*(x_{(m_1, \dots, m_i)}^*) = 0$.

Combining the facts above, we can write

$$\begin{aligned}
\frac{a}{2} &< \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \operatorname{Re} z_{(m_1, \dots, m_n)}^* \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)} \right) \\
&= \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \operatorname{Re} (z_{(m_1, \dots, m_n)}^* - z_{\emptyset}^*) \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)} \right) \\
&= \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \operatorname{Re} \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right) \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)} \right) \\
&= \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \operatorname{Re} x_{(m_1, \dots, m_i)}^* (x_{(m_1, \dots, m_i)}) \\
&= \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \operatorname{Re} x_{(m_1, \dots, m_i)}^* (x_{(m_1, \dots, m_i)}) \\
&\leq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| \leq \beta_n^{\mathcal{U}}(X).
\end{aligned}$$

Since $a < \alpha_n^{\mathcal{U}}(X)$ was arbitrary, we are done.

(iii) Fix $b < \beta_n^{\mathcal{U}}(X)$ and $\delta > 0$. Then there exists a collection $(x_t^*)_{t \in T_n} \subset B_{X^*}$ which is \mathcal{U} -weak*-null and

$$b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| < \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|.$$

Note that this implies that

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| > 0.$$

We define a collection $(x_t)_{t \in T_n} \subset B_X$ which is \mathcal{U} -weakly null and such that for each $1 \leq i \leq n$,

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \operatorname{Re} x_{(m_1, \dots, m_i)}^* (x_{(m_1, \dots, m_i)}) \geq \frac{1}{2} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| - \delta.$$

To that end, for each $t \in \{\emptyset\} \cup T_{n-1}$, choose, as it is allowed by Proposition 5.10, $(x_{t \smallfrown (m)})_{m=1}^{\infty} \subset B_X$ to be a \mathcal{U} -weakly null sequence such that

$$\lim_{m \in \mathcal{U}} \operatorname{Re} x_{t \smallfrown (m)}^* (x_{t \smallfrown (m)}) \geq \frac{1}{2} \lim_{m \in \mathcal{U}} \|x_{t \smallfrown (m)}^*\| - \delta.$$

Note that, since $(x_t)_{t \in T_n}$ is \mathcal{U} -weakly null and $(x_t^*)_{t \in T_n}$ is \mathcal{U} -weak*-null, we have again that for all $1 \leq i \neq j \leq n$

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} x_{(m_1, \dots, m_i)}^* (x_{(m_1, \dots, m_j)}) = 0.$$

Then, we can write

$$\begin{aligned}
& \frac{b}{2} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| - \delta n \\
& < \frac{1}{2} \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| - \delta n \\
& \leq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \operatorname{Re} x_{(m_1, \dots, m_i)}^*(x_{(m_1, \dots, m_i)}) \\
& = \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \operatorname{Re} \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right) \left(\sum_{i=1}^n x_{(m_1, \dots, m_i)} \right) \\
& \leq \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)} \right\| \\
& \leq \alpha_n^{\mathcal{U}}(X) \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\|.
\end{aligned}$$

Since $\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| > 0$, and since $\delta > 0$ and $b < \beta_n^{\mathcal{U}}(X)$ were arbitrary, we are done. \square

Remark 5.12. In item (i) of the preceding proof, we actually showed that if X^* is separable, then for each $n \in \mathbb{N}$, $\alpha_n^{\mathcal{U}}(X)$ is the infimum of $a > 0$ such that Player I has a winning strategy in the $A(a, n)$ game.

We can now turn to the heart of the proof.

Lemma 5.13. *For any Banach space X with X^* separable and any subspace Y of X ,*

$$\alpha_n^{\mathcal{U}}(X) \leq 40 \max\{\alpha_n^{\mathcal{U}}(Y), \alpha_n^{\mathcal{U}}(X/Y)\}^2.$$

Proof. If X is finite-dimensional, then $\alpha_n^{\mathcal{U}}(X) = \alpha_n^{\mathcal{U}}(Y) = \alpha_n(X/Y) = 0$, so assume X is infinite-dimensional. In this case, at least one of Y , X/Y must also be infinite-dimensional, which means

$$b := \max\{\beta_n^{\mathcal{U}}(Y), \beta_n^{\mathcal{U}}(X/Y)\} \geq 1.$$

Fix $(x_t^*)_{t \in T_n} \subset B_{X^*}$ \mathcal{U} -weak*-null. We will define a bounded, \mathcal{U} -weak*-null $(y_t^*)_{t \in T_n} \subset Y^\perp$ such that for each $1 \leq i \leq n$,

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^* - y_{(m_1, \dots, m_i)}^*\| \leq 2 \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp}.$$

To that end, for each $t = (m_1, \dots, m_i) \in T_n$, fix $w_t^* \in Y^\perp$ such that

$$\|x_t^* - w_t^*\| < \|x_t^*\|_{X^*/Y^\perp} + 2^{-m_i}$$

and note that $w_t^* \in 3B_{Y^\perp}$. For $t \in \{\emptyset\} \cup T_{n-1}$, let $v_t^* = \operatorname{weak}^* \lim_{m \in \mathcal{U}} w_{t \smallfrown (m)}^* \in 3B_{Y^\perp}$ and let $y_{t \smallfrown (m)}^* = w_{t \smallfrown (m)}^* - v_t^*$. It is clear that $(y_t^*)_{t \in T_n} \subset Y^\perp$ is

bounded and \mathcal{U} -weak*-null. Note that for any $t \in \{\emptyset\} \cup T_{n-1}$, weak*- $\lim_{m \in \mathcal{U}} (w_{t \wedge(m)}^* - x_{t \wedge(m)}^*) = v_t^* - 0 = v_t^*$. By weak*-lower semicontinuity of the norm, it follows that

$$\|v_t^*\| \leq \lim_{m \in \mathcal{U}} \|w_{t \wedge(m)}^* - x_{t \wedge(m)}^*\| = \lim_{m \in \mathcal{U}} \|x_{t \wedge(m)}^*\|_{X^*/Y^\perp}.$$

Therefore

$$\lim_{m \in \mathcal{U}} \|x_{t \wedge(m)}^* - y_{t \wedge(m)}^*\| \leq \lim_{m \in \mathcal{U}} \|x_{t \wedge(m)}^* - w_{t \wedge(m)}^*\| + \|v_t^*\| \leq 2 \lim_{m \in \mathcal{U}} \|x_{t \wedge(m)}^*\|_{X^*/Y^\perp}.$$

From this it follows that

$$\lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^* - y_{(m_1, \dots, m_i)}^*\| \leq 2 \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp}.$$

Since $(y_t^*)_{t \in T_n} \subset Y^\perp = (X/Y)^*$ is \mathcal{U} -weak*-null and bounded, by homogeneity, we have that

$$b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n y_{(m_1, \dots, m_i)}^* \right\| \geq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|y_{(m_1, \dots, m_i)}^*\|.$$

We start with the easy case and suppose first that

$$\sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp} \geq \frac{1}{1+4b} \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|.$$

Since $(x_t^*|_Y)_{t \in T_n} \subset B_{Y^*} = B_{X^*/Y^\perp}$ is \mathcal{U} -weak*-null,

$$\begin{aligned} b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| &\geq b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\|_{X^*/Y^\perp} \\ &\geq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp} \\ &\geq \frac{1}{1+4b} \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|. \end{aligned}$$

Next suppose that

$$\sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp} < \frac{1}{1+4b} \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|.$$

Then since $b \geq 1$,

$$\begin{aligned}
& b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n x_{(m_1, \dots, m_i)}^* \right\| \geq b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \left\| \sum_{i=1}^n y_{(m_1, \dots, m_i)}^* \right\| \\
& - b \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^* - y_{(m_1, \dots, m_i)}^*\| \\
& \geq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|y_{(m_1, \dots, m_i)}^*\| - 2b \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp} \\
& \geq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| - \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^* - y_{(m_1, \dots, m_i)}^*\| \\
& - 2b \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp} \\
& \geq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| - 4b \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|_{X^*/Y^\perp} \\
& \geq \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| - \frac{4b}{1+4b} \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\| \\
& = \frac{1}{1+4b} \sum_{i=1}^n \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \|x_{(m_1, \dots, m_i)}^*\|.
\end{aligned}$$

Combining the two previous paragraphs we get

$$\beta_n^{\mathcal{U}}(X) \leq b(1+4b) \leq 5b^2 = 5 \max\{\beta_n^{\mathcal{U}}(Y), \beta_n^{\mathcal{U}}(X/Y)\}^2.$$

Combining this inequality with items (ii) and (iii) of Proposition 5.11 yields

$$\alpha_n^{\mathcal{U}}(X) \leq 40 \max\{\alpha_n^{\mathcal{U}}(Y), \alpha_n^{\mathcal{U}}(X/Y)\}^2.$$

□

We can now state and prove our result.

Theorem 5.14. *The property A_∞ is a three-space property.*

Proof. Assume first that Y is a closed subspace of a Banach space X such that Y and X/Y are in $A_\infty \cap \text{Sep}$. Then Y and X/Y are separable Asplund spaces and $Y^* = X^*/Y^\perp$ and $(X/Y)^* = Y^\perp$ are separable. So X^* is separable and we can apply Lemma 5.13 and item (i) of Proposition 5.11 to deduce that X has A_∞ . We have shown that membership in $A_\infty \cap \text{Sep}$ is a 3SP. It then follows from Theorem 4.2 and Lemma 5.2 that A_∞ is a 3SP. □

Remark 5.15. Since reflexivity is also a three-space property (cf [33]), we can therefore deduce that property HFC_∞ of [16], which is known to be equivalent to being reflexive and asymptotic- c_0 (cf [3]), is a three-space property.

However, the question of whether or not property HC_∞ is a three-space property remains open.

6. NON-LINEAR STABILITIES

We first define the non-linear equivalences between Banach spaces that we will discuss in this section.

Definition 6.1. Let (M, d) and (N, δ) be two metric spaces. A map $f : M \rightarrow N$ is a *Lipschitz equivalence* (or *Lipschitz isomorphism*) from M to N if f is a Lipschitz bijection from M to N with Lipschitz inverse. If there exists a Lipschitz equivalence from M to N , we say that M and N are *Lipschitz equivalent* (or *Lipschitz isomorphic*) and we denote $M \stackrel{L}{\sim} N$.

Definition 6.2. Let (M, d) and (N, δ) be two unbounded metric spaces and $f : M \rightarrow N$ be a map. We say that f is *coarse Lipschitz* if there exist $A, B \geq 0$ such that

$$\forall x, y \in M, \quad \delta(f(x), f(y)) \leq Ad(x, y) + B.$$

We say that f is a *coarse Lipschitz equivalence* from M to N , if it is coarse Lipschitz and there exists a coarse Lipschitz map $g : N \rightarrow M$ and a constant $C \geq 0$ such that

$$\forall x \in M \quad d((g \circ f)(x), x) \leq C \quad \text{and} \quad \forall y \in N \quad \delta((f \circ g)(y), y) \leq C.$$

If there exists a coarse Lipschitz equivalence from M to N , we say that M and N are *coarse Lipschitz equivalent* and denote $M \stackrel{CL}{\sim} N$.

This notion of coarse Lipschitz equivalent metric spaces is the same as the notion of quasi-isometric metric spaces introduced by Gromov in [22] (see also the book [17] by E. Ghys and P. de la Harpe).

We now turn to the notion of net in a metric space.

Definition 6.3. Let $0 < a \leq b$. An (a, b) -*net* in the metric space (M, d) is a subset \mathcal{M} of M such that for every $z \neq z'$ in \mathcal{M} , $d(z, z') \geq a$ and for every x in M , $d(x, \mathcal{M}) < b$.

Then a subset \mathcal{M} of M is a *net* in M if it is an (a, b) -net for some $0 < a \leq b$.

Let us now give two technical equivalent formulations of the notion of coarse Lipschitz equivalence between Banach spaces. We refer to [11] or [21] for details.

Proposition 6.4. *Let X and Y be two Banach spaces and let $f : X \rightarrow Y$ be a map. The following assertions are equivalent.*

- (i) *The map f is a coarse Lipschitz equivalence.*
- (ii) *There exist $A_0 > 0$ and $K \geq 1$ such that for all $A \geq A_0$ and all maximal A -separated subset \mathcal{M} of X , $\mathcal{N} = f(\mathcal{M})$ is a net in Y and*

$$\forall x, x' \in \mathcal{M} \quad \frac{1}{K} \|x - x'\| \leq \|f(x) - f(x')\| \leq K \|x - x'\|.$$

(iii) There exist two continuous coarse Lipschitz maps $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ and a constant $C \geq 0$ such that $\|\varphi(x) - f(x)\| \leq C$ for all x in X and

$$\forall x \in X \quad \|(\psi \circ \varphi)(x) - x\| \leq C \quad \text{and} \quad \forall y \in Y \quad \|(\varphi \circ \psi)(y) - y\| \leq C.$$

The following results were obtained by Godefroy, Kalton, and the third-named author in [19] and [20].

Theorem 6.5. *Let $p \in (1, \infty]$.*

- (1) *The class T_p is stable under Lipschitz equivalences.*
- (2) *The class P_p is stable under coarse Lipschitz equivalences.*
- (3) *The class $\mathsf{A}_\infty = \mathsf{N}_\infty$ is stable under coarse Lipschitz equivalences.*

In fact, statements (2) and (3) are only proved for uniform homeomorphisms in [20] in the separable case. The adaptation for coarse Lipschitz equivalences relies on characterization (iii) in Proposition 6.4, which allows to apply the so-called Gorelik principle (see also [11] for details). Then the non-separable case can easily be deduced by a standard separable saturation argument combined with the separable determination of these properties. It is then natural to wonder about the non-linear stability of the classes A_p and N_p for $1 < p < \infty$. The results we have detailed in Section 3 together with a careful examination of the statements in [20] or [11] will allow us to easily obtain strong new stability results. We start with the following.

Theorem 6.6. *For any $p \in (1, \infty)$, the class A_p is stable under coarse Lipschitz equivalences.*

Proof. Let $X \in \mathsf{A}_p$ and Y a Banach space such that there exists a coarse Lipschitz equivalence f from X to Y . Then, Corollary 6.7 in [11], which is an extension of results in [20], insures the existence of a universal constant $K > 0$ and a constant $M > 0$ (depending on f) so that for any $\varepsilon > 0$, there exists a norm $|\cdot|$ on Y satisfying

$$\forall y \in Y, \quad \|y\|_Y \leq |y| \leq M\|y\|_Y \quad \text{and} \quad \forall \sigma \in [0, 1], \quad \bar{\rho}_Y(KM^2\sigma) \leq \bar{\rho}_X(\sigma) + \varepsilon.$$

With this result in hands, it is clear that characterization (iii) of A_p in Theorem 3.2 is stable under coarse Lipschitz equivalences. \square

We also have.

Theorem 6.7. *For any $p \in (1, \infty)$, the class N_p is stable under coarse-Lipschitz equivalences.*

Proof. Similarly to the previous proof, this is a direct consequence of Corollary 6.7 in [11] and characterization (iii) of N_p in Theorem 3.3. \square

Obviously the above argument can also be applied to prove that A_∞ is stable under coarse Lipschitz equivalences, which, as we explained, was already known.

Problem 6.8. In [30], N. Kalton proved that for $1 < p < \infty$, the class \mathbb{T}_p is not stable under uniform homeomorphisms. It is not known however whether the class \mathbb{T}_∞ is stable under coarse Lipschitz isomorphisms (or even uniform homeomorphisms). In fact, a positive answer would imply that a Banach space coarse Lipschitz equivalent to c_0 is linearly isomorphic to c_0 , which is an important open question. Indeed, it is known that the class of all \mathcal{L}_∞ spaces is stable under coarse Lipschitz equivalences [24] and that a \mathcal{L}_∞ subspace of c_0 is isomorphic to c_0 [26].

7. EXAMPLES

We gather in this section a few known examples of \mathbb{T}_∞ or \mathbb{A}_∞ spaces and related problems.

7.1. Non-separable uniformly flatenable spaces. The first obvious examples of non-separable \mathbb{T}_∞ (or equivalently, AUF-renormable) spaces are given by $c_0(\Gamma)$ spaces, with Γ uncountable.

Proposition 7.1. *For any set Γ , the space $c_0(\Gamma)$ equipped with its natural norm is AUF.*

Proof. It follows immediately from the definition of the norm of $c_0(\Gamma)$ that

$$\forall t \in (0, 1) \quad \bar{\rho}_{c_0(\Gamma)}(t) = 0.$$

□

The next result was already known (see the remark after the proof). We present first a proof using that \mathbb{T}_∞ is a 3SP.

Theorem 7.2. *Let K be a compact scattered space such that its Cantor derived set of order ω , $K^{(\omega)}$ is empty. Then $C(K)$ is \mathbb{T}_∞ .*

Proof. We shall prove it by induction on $n \in \mathbb{N}$ such that $K^{(n)} = \emptyset$. If $n = 1$, then $K' = \emptyset$ and K is finite. Therefore $C(K)$ is finite-dimensional and thus is \mathbb{T}_∞ . Assume that the statement is true for $n \in \mathbb{N}$ and that $K^{(n+1)} = \emptyset$. The subspace of $C(K)$ defined by $Y = \{f \in C(K), f|_{K'} = 0\}$ is clearly isometric to $c_0(K \setminus K')$ and by Proposition 7.1 is \mathbb{T}_∞ . Let now Q be the restriction mapping from $C(K)$ to $C(K')$. It follows from the Tietze extension theorem that Q is onto. Since Y is the kernel of Q , we have that $C(K')$ is isomorphic to $C(K)/Y$. By induction hypothesis, $C(K')$ and thus $C(K)/Y$ are \mathbb{T}_∞ . It now follows from Theorem 5.8 that $C(K)$ is \mathbb{T}_∞ . □

Remark 7.3. As we already mentioned, this is not a new result. Let us indicate a few other ways to prove it.

- (1) Let K be a compact space such that $K^{(n)} = \emptyset$, $n \in \mathbb{N}$. The dual of $C(K)$ is isometric to $\ell_1(K)$. Define the following equivalent norm on $\ell_1(K)$:

$$\forall \mu \in \ell_1(K), \quad |\mu| = \sum_{x \in K} \alpha_x |\mu(x)|,$$

where $\alpha_x = 2^{-i}$ with $0 \leq i \leq n - 1$ such that $x \in K^{(i)} \setminus K^{(i+1)}$. This formula comes from [34], where it is proved that this norm is 1-AUC* and is the dual norm of an equivalent norm on $C(K)$. So its predual norm is AUF.

- (2) Let X be a separable subspace of $C(K)$ and denote Y the closed sub*-algebra of $C(K)$ generated by X . Then Y is isometric to a space $C(L)$, where L is a compact metrizable space such that $L^{(\omega)} = \emptyset$. It follows from [4] that Y is either finite dimensional or isomorphic to $c_0(\mathbb{N})$. So X is T_∞ and we can apply the separable determination of T_∞ (Theorem 4.2) to deduce that $C(K)$ is T_∞ .
- (3) We conclude with the most sophisticated argument. It is known that if K is a compact space such that $K^{(\omega)} = \emptyset$, then $C(K)$ is Lipschitz isomorphic to some $c_0(\Gamma)$ (see [12]). On the other hand, being AUF renormable is stable under Lipschitz isomorphisms ([19] for the separable case and [11] for the general case, or use separable determination and saturation).

It is also important to mention that Theorem 7.2 provides (only in the non-separable setting) examples of T_∞ spaces that are not isomorphic to a quotient or a subspace of a $c_0(\Gamma)$ space. Indeed we have

Theorem 7.4. *There exists a compact space K such that $K^{(3)} = \emptyset$, but $C(K)$ is not isomorphic to a quotient of a subspace of a $c_0(\Gamma)$ space.*

Let us indicate this now classical construction. There exists a scattered separable uncountable compact space K so that $K^{(3)} = \emptyset$. This space is often called the Mrówka-Isbell space and its construction is due to Mrówka [36] and Isbell (credited in [18]). We also refer to its description in [40] page 1757, where its construction is based on the Johnson-Lindenstrauss space JL_0 [25]. Since K is separable, $C(K)$ admits a countable family of separating functionals (the Dirac maps at the points of a dense countable subset of K). But $C(K)$ is not separable, as K is uncountable and scattered and therefore non metrizable. It follows that $C(K)$ is not weakly Lindelöf determined (WLD in short): see Theorem 5.37 and Proposition 5.40 in [23], or see [39]. We conclude by recalling that $c_0(\Gamma)$ is always WLD and that being WLD is stable by passing to subspaces or quotients (see also [23] and references therein).

Problem 7.5. We do not know if there exists a T_∞ space which is not isomorphic to quotient of a subspace of a $C(K)$ space with $K^{(\omega)} = \emptyset$.

7.2. An interesting \mathbf{A}_∞ space. We already explained that showing that T_∞ is stable under coarse Lipschitz equivalences would imply that a Banach space coarse Lipschitz equivalent to c_0 is linearly isomorphic to c_0 . At this point it is only known that a Banach space coarse Lipschitz equivalent to c_0 is \mathbf{A}_∞ and \mathcal{L}_∞ . Another hope was to show that a separable Banach space which is \mathbf{A}_∞ and \mathcal{L}_∞ is necessarily T_∞ (see conjecture after Theorem

5.6 in [20]). Let us mention here that this question has been solved negatively by Argyros, Gasparis, and Motakis in [1], who showed the existence of a separable Banach space X which is A_∞ and \mathcal{L}_∞ but so that every infinite dimensional subspace of X contains an infinite dimensional reflexive subspace.

Acknowledgments. We thank Petr Hájek for valuable discussions on non-separable $C(K)$ spaces and Gilles Godefroy for pointing to us the important example by Argyros, Gasparis and Motakis. We also thank the referee for her/his suggestions that helped improve the presentation of this article.

REFERENCES

- [1] S. A. Argyros, I. Gasparis, and P. Motakis, *On the structure of separable \mathcal{L}_∞ -spaces*, *Mathematika* **62** (2016), no. 3, 685–700.
- [2] K. Ball, *The Ribe programme*, Séminaire Bourbaki. Volume 2011/2012. Exposés 1043–1058, 2013, pp. 147–159, ex.
- [3] F. Baudier, G. Lancien, P. Motakis, and Th. Schlumprecht, *A new coarsely rigid class of Banach spaces*, *J. Inst. Math. Jussieu* **20** (2021), no. 5, 1729–1747.
- [4] C. Bessaga and A. Pełczyński, *Spaces of continuous functions. IV. On isomorphical classification of spaces of continuous functions*, *Studia Math.* **19** (1960), 53–62.
- [5] J. M. F. Castillo and M. González, *Three-space problems in Banach space theory*, *Lecture Notes in Mathematics*, vol. 1667, Springer-Verlag, Berlin, 1997.
- [6] R. M. Causey, *Power type asymptotically uniformly smooth and asymptotically uniformly flat norms*, *Positivity* **22** (2018), no. 5, 1197–1221.
- [7] ———, *Concerning q -summable Szlenk index*, *Illinois J. Math.* **62** (2018), no. 1-4, 381–426.
- [8] ———, *Power type ξ -asymptotically uniformly smooth norms*, *Trans. Amer. Math. Soc.* **371** (2019), no. 3, 1509–1546, DOI 10.1090/tran/7336. MR3894026
- [9] ———, *Three and a half asymptotic properties*, *Studia Math.* **257** (2021), no. 2, 155–212.
- [10] R. M. Causey, S. Draga, and T. Kochanek, *Operator ideals and three-space properties of asymptotic ideal seminorms*, *Trans. Amer. Math. Soc.* **371** (2019), no. 11, 8173–8215.
- [11] A. Dalet and G. Lancien, *Some properties of coarse Lipschitz maps between Banach spaces*, *North-West. Eur. J. Math.* **3** (2017), 41–62, i.
- [12] R. Deville, G. Godefroy, and V. Zizler, *The three space problem for smooth partitions of unity and $C(K)$ spaces*, *Math. Ann.* **288** (1990), no. 4, 613–625.
- [13] S. J. Dilworth, D. Kutzarova, G. Lancien, and N. L. Randrianarivony, *Equivalent norms with the property (β) of Rolewicz*, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **111** (2017), no. 1, 101–113.
- [14] S. Draga and T. Kochanek, *The Szlenk power type and tensor products of Banach spaces*, *Proc. Amer. Math. Soc.* **145** (2016), 1685–1698.
- [15] P. Enflo, J. Lindenstrauss, and G. Pisier, *On the “three space problem”*, *Math. Scand.* **36** (1975), no. 2, 199–210.
- [16] A. Fovelle, *Hamming graphs and concentration properties in Banach spaces*, arXiv preprint arXiv:2106.04297 (2021).
- [17] E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d’après Mikhael Gromov*, *Progress in Mathematics*, vol. 50, Birkhäuser, 1990.
- [18] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, NJ., 1960.

- [19] G. Godefroy, N. J. Kalton, and G. Lancien, *Subspaces of $c_0(\mathbf{N})$ and Lipschitz isomorphisms*, Geom. Funct. Anal. **10** (2000), no. 4, 798–820.
- [20] ———, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. **353** (2001), no. 10, 3895–3918.
- [21] G. Godefroy, G. Lancien, and V. Zizler, *The non-linear geometry of Banach spaces after Nigel Kalton*, Rocky Mountain J. of Math. **44** (2014), no. 5, 1529–1583.
- [22] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.
- [23] P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, and V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 26, Springer, New York, 2008.
- [24] S. Heinrich and P. Mankiewicz, *Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces*, Studia Math. **73** (1982), no. 3, 225–251.
- [25] W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. **17** (1974).
- [26] W. B. Johnson and M. Zippin, *On subspaces of quotients of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$* , Israel J. Math. **13** (1972), 311–316 (1973).
- [27] ———, *Subspaces and quotient spaces of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$* , Israel J. Math. **17** (1974), 50–55.
- [28] N. J. Kalton, *The uniform structure of Banach spaces*, Math. Ann. **354** (2012), 1247–1288.
- [29] ———, *Uniform homeomorphisms of Banach spaces and asymptotic structure*, Trans. Amer. Math. Soc. **365** (2013), 1247–1288.
- [30] ———, *Examples of uniformly homeomorphic Banach spaces*, Israel J. Math. **194** (2013), no. 1, 1051–1079.
- [31] N. J. Kalton and N. L. Randrianarivony, *The coarse Lipschitz geometry of $\ell_p \oplus \ell_q$* , Math. Ann. **341** (2008), no. 1, 223–237.
- [32] N. J. Kalton and N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. **255** (1979), 1–30.
- [33] M. Krein and V. Šmulian, *On regularly convex sets in the space conjugate to a Banach space*, Ann. of Math. (2) **41** (1940), 556–583.
- [34] G. Lancien, *On uniformly convex and uniformly Kadec-Klee renormings*, Serdica Math. J. **21** (1995), no. 1, 1–18.
- [35] ———, *A survey on the Szlenk index and some of its applications*, RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **100** (2006), no. 1-2, 209–235 (English, with English and Spanish summaries).
- [36] S. Mrówka, *On completely regular spaces*, Fund. Math. **41** (1954), 105–106.
- [37] A. Naor, *An introduction to the Ribe program*, Jpn. J. Math. **7** (2012), no. 2, 167–233.
- [38] M. Ribe, *On uniformly homeomorphic normed spaces*, Ark. Mat **16** (1978), 1–9.
- [39] J. Vanderwerff, J. H. M. Whitfield, and V. Zizler, *Markušević bases and Corson compacta in duality*, Canad. J. Math. **46** (1994), no. 1, 200–211.
- [40] V. Zizler, *Nonseparable Banach spaces*, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1743–1816.

R.M CAUSEY

Email address: `rmcausey1701@gmail.com`

A. FOVELLE, LABORATOIRE DE MATHÉMATIQUES DE BESANÇON, UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ, 16 ROUTE DE GRAY, 25030 BESANÇON CÉDEX, BESANÇON, FRANCE

Email address: `audrey.fovelle@univ-fcomte.fr`

G. LANCIEN, LABORATOIRE DE MATHÉMATIQUES DE BESANÇON, UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ, 16 ROUTE DE GRAY, 25030 BESANÇON CÉDEX, BESANÇON, FRANCE

Email address: `gilles.lancien@univ-fcomte.fr`