Wave blocking in a bistable system by local introduction of a population: application to sterile insect techniques on mosquito populations

Luís Almeida, Jorge Estrada, Nicolas Vauchelet

Wave blocking in a bistable system by local introduction of a population: application to sterile insect techniques on mosquito populations

Luís Almeida [*] Jorge Estrada [†] Nicolas Vauchelet[‡]

October 13, 2021

Abstract

The Sterile Insect Technique (SIT) is a classic vector control method that has been successfully applied to fight against diverse insect plagues since the 1950s. In recent years, this strategy has been used to control mosquito populations, in order to limit the spread of the diseases they transmit. In this paper, we consider a system of reaction-diffusion equations to model the mosquito population and study the effect of the release of sterile mosquito males. Then we propose to analyze the release in a limited area inside a wider area containing natural mosquito population. We are interested in protecting a mosquito free region from invasion by mosquitoes from an exterior domain by controlling the population in a release band at the border between the two regions: we construct a barrier blocking the invasion of mosquitoes from the exterior. We adapt the geometric method of Lewis and Keener (see [15]) in this framework and extend their main result to find relations on the size of the release region and the density of the released sterile males that allow us to block the invasion. Numerical simulations are also performed to illustrate our results.

1 Introduction

Aedes mosquitoes are vectors for multiple infectious diseases, most notably dengue, yellow fever, chikungunya and zika. Currently, these diseases do not have an efficient cure or vaccine. Therefore, an usual method for preventing and controlling outbreaks is to reduce or eliminate the mosquito population since they are the vectors for the transmission of these diseases. The use of insecticides over a prolonged period of time is costly and has multiple adverse consequences such as environmental pollution, ecological impact on many species and the development of increased resistance to the insecticide. This shows the need to consider alternative control methods, such as the Sterile Insect Technique (SIT). The SIT was first proposed by Knipling [12] in the 1950s. This technique has been employed to eliminate and control different agricultural pests and disease vectors, most notably screw worms and fruit flies (see [9] for a detailed list of SIT trials and programs). The classical SIT consists in the release of a large number of sterile males (often sterilized by ionizing radiation or chemically), which results in a progressive reduction of the total population. Another closely related technique is the Incompatible Insect Technique (IIT), where released mosquitoes are not sterilized but rather infected with the bacterium *Wolbachia*, which shortens their lifespan and reduces their vector capacity (for instance for transmitting dengue). Indeed, when only *Wolbachia*-bearing males are released, the

[*]Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions UMR7598, F-75005 Paris, France. luis.almeida@sorbonne-universite.fr

[†]Laboratoire Analyse, Géométrie et Applications CNRS UMR 7539, Université Sorbonne Paris Nord, Villetaneuse, France.

[‡]Laboratoire Analyse, Géométrie et Applications CNRS UMR 7539, Université Sorbonne Paris Nord, Villetaneuse, France (vauchelet@math.univ-paris13.fr).

IIT is equivalent to the classical SIT; when both males and females are released, the expected result is that the existing population will be replaced by the *Wolbachia* infected population. In particular, for mosquito control, the SIT/IIT have been employed in multiple locations, such as Italy [4], Reunion Island [18], Polynesia[8] and China [25]. Aedes mosquitoes and the arboviroses for which they are transmission vectors have been a major problem in tropical regions since long ago. However, they are also becoming a major concern for public health authorities in temperate regions (including Europe) where, due to climate change and global exchanges, Aedes mosquitoes have become a successful invasive species..

Classical SIT and IIT have been modeled and studied in a large number of papers, from diverse mathematical viewpoints, such as discrete difference equations [16], as ODE systems by themselves [24] or linked with SIR systems of ODEs [10]. The spatial dynamics has been also studied thanks to PDE models to describe the spatial invasion of mosquitoes : for instance the recent work [21] proposes a study of the invasion of Aedes Albopictus in France. In [13], the authors propose a simple scalar reaction-diffusion equation to analyze the influence of releases of sterilized males on the traveling waves of invasion of mosquitoes.

In this paper, we are interested in the use of SIT/IIT locally as a way to block invasion or infestation of mosquitoes in a mosquito-free area. More precisely, we study the possibility of blocking a propagating front of mosquitoes by performing a local release of sterile mosquitoes in a band of width L. Such technique may be used as a sanitary cordon to avoid infestation of a mosquito-free area or re-infestation of a previously treated area. A numerical study of such strategy has been performed in [23] and in [1]. It is expected that when releasing enough sterilized mosquitoes in a wide enough area, it should be possible to block invading waves of mosquitoes. The aim of this paper is to justify rigorously these numerical results. From a mathematical point of view, it boils down to show the existence of stationary solutions for a system of reaction-diffusion equations.

More generally, let $v(x,t)$ denote the density of a species, e.g. mosquitoes, at position $x \in \mathbb{R}$ and time $t > 0$. Let $w(x,t)$ denote the density of an introduced species, e.g. sterilized males, which is released on a domain $[0,L]$ at a constant density M_0; we denote μ the death rate of this species. Then, the population dynamics is governed by the following system of reaction-diffusion equations :

$$\partial_t v - \partial_{xx} v = g(v,w), \qquad \text{on } \mathbb{R}\times\mathbb{R}^+, \tag{1a}$$

$$\partial_t w - \partial_{xx} w = M_0 \cdot \mathbf{1}_{[0,L]} - \mu w. \tag{1b}$$

In this system, the interaction between both species is modeled thanks to the reaction term g. In the absence of the species w, i.e. when $w = 0$, we denote $f(v) := g(v,0)$ the reaction term for the first species. It is supposed to be such that f is a $C^1(\mathbb{R})$ bistable function, that is

$$\exists\, \theta \in (0,1),\ f(0) = f(1) = f(\theta) = 0, \quad f < 0 \text{ on } (0,\theta), \quad f > 0 \text{ on } (\theta,1). \tag{2}$$

Moreover, we assume that

$$\int_0^1 f(v)\,dv > 0. \tag{3}$$

It is well-known (see e.g. [3, 14]) that under this condition, there exist invading traveling waves for species v in absence of species w.

The aim of this paper is to prove that, assuming appropriate conditions on the interaction function g, for $L > 0$ and M_0 large enough it is possible to block the invasion. In other words, we study the possibility to block the invasion by acting on a release function which is chosen here to be the piece-wise constant function $M_0\mathbf{1}_{[0,L]}$.

Wave-blocking in reaction-diffusion equations has been studied by several authors. For instance, the influence of the geometry on the propagation of a potential along a nerve axon has been considered in [19]. In the seminal paper [15], the authors show the existence of a wave-block due to heterogeneities in the media of propagation. To obtain their result, they introduce a geometrical technique by reasoning on the phase

plan. This technique will be recalled below. In the present paper, we adapt and extend their technique to the problem at hand. In [7], the authors propose to study the absorption by the white matter of propagating waves in the brain. In [5], wave-block in a model for criminality is proved. Wave-block in bistable reaction-diffusion systems with a drift term, usually called gene-flow models, has been investigated in [17] and [11]. Finally, we mention that control strategies on reaction-diffusion equations with bistable nonlinearity has been recently investigated by several authors e.g. [20, 22].

The outline of the paper is as follows. In section 2, we state our main results and we describe our mathematical model. Section 3 is devoted to the proof of our main result for general bistable systems, i.e. the existence of a barrier when acting on a wide enough region with large enough intensity. We use a geometric method based on a phase plane analysis to construct such barrier. In section 4 we apply this technique and result to our practical problem : the use of the sterile insect technique to avoid invasion of a species like mosquitoes. Then, we illustrate our results with some numerical applications in section 5. We end this article with some conclusion in section 6.

2 Modelling and main results

In this section we state the main results of this paper for system (1), that is the blocking of the propagation under appropriate conditions. Then, we explain how to use such results to obtain in the application we have in mind, that is the local use of the sterile insect technique to control invasion by mosquito populations.

2.1 Main results for general bistable systems

We first start by defining a barrier for system (1):

Definition 1. *Let $L \geq 0$ and $M_0 \geq 0$. We call **barrier** (of width L and height M_0) for system* (1)*, any couple $(\bar{v}, \bar{w}) \in (C^2(\mathbb{R}))^2$ which is a nonnegative solution of the following differential inequalities :*

$$\begin{cases} \bar{v}'' + g(\bar{v}, \bar{w}) \leq 0, & \text{on } \mathbb{R}, \\ \bar{w}'' + M_0 \cdot \mathbf{1}_{[0,L]} - \mu \bar{w} \geq 0, \\ \bar{v}(-\infty) = 0, \quad \bar{v}(+\infty) = 1, \quad \bar{w}(\pm\infty) = 0. \end{cases} \tag{4}$$

We say that there exists a barrier to system (1) *for $L \geq 0$ and $M_0 \geq 0$ if system* (4) *admits a nonnegative solution.*

Let us consider system (1) with initial conditions, $v(x, 0) = v_0(x)$ and $w(x, 0) = w_0(x)$, with $v_0, w_0 \in L^1 \cap L^\infty(\mathbb{R})$, $0 \leq v_0(x) \leq 1$. The following Lemma explains why we call solutions to (4) a barrier : indeed, the existence of such solutions may block the propagation.

Lemma 1. *Let g be Lipschitz-continuous and such that $\partial_w g(v, w) \leq 0$ for all v, w. Let us assume that there exists a barrier for system* (1) *(as in Definition 1). If the initial data (v_0, w_0) is such that $v_0 \leq \bar{v}$ and $w_0 \geq \bar{w}$, then the solution to* (1) *is such that for all $t > 0$, $v(\cdot, t) \leq \bar{v}$.*

Proof. It is straightforward to obtain this result using a comparison principle. Indeed, from (1) and (4), we deduce that

$$\begin{aligned} \partial_t(v - \bar{v}) - \partial_{xx}(v - \bar{v}) &\leq g(v, w) - g(\bar{v}, \bar{w}) \\ \partial_t(\bar{w} - w) - \partial_{xx}(\bar{w} - w) &\leq \mu(w - \bar{w}). \end{aligned}$$

We multiply the first equation by $(v - \bar{v})_+$ and the second equation by $(\bar{w} - w)_+$ and integrate over $\mathbb{R}$, where $(\cdot)_+$ denotes the positive part. Summing the resulting equalities, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left((v - \bar{v})_+^2 + (\bar{w} - w)_+^2 \right) dx \leq \int_{\mathbb{R}} (g(v, w) - g(\bar{v}, \bar{w}))(v - \bar{v})_+ \, dx - \int_{\mathbb{R}} \mu(\bar{w} - w)_+^2 \, dx.$$

From the assumptions on g, we deduce

$$\begin{aligned}(g(v,w)-g(\bar v,\bar w))(v-\bar v)_+ &= (g(v,w)-g(\bar v,w))(v-\bar v)_+ + (g(\bar v,w)-g(\bar v,\bar w))(v-\bar v)_+\\ &\le C_g(v-\bar v)_+^2 + C_g(\bar w-w)_+(v-\bar v)_+,\end{aligned}$$

where C_g is a Lipschitz constant for g. We conclude as usual by using a Gronwall type inequality and the fact that $(v_0-\bar v)_+ = 0 = (\bar w - w_0)_+$. □

As a consequence of Lemma 1, if the species w is present in the environment in such a way that $w(0,\cdot)\ge \bar w$, then any invasion front is blocked by the barrier, provided such a barrier exists. Under assumption (2) and (3), there exists a traveling wave with positive velocity for the equation for the species v alone, i.e. there exist a constant $c>0$ and a solution v_c of the system

$$cv_c' - v_c'' = f(v_c),\quad v_c(-\infty)=0,\ v_c(+\infty)=1.$$

Hence, when $L=0$, there is no barrier. The following Theorem gives conditions for existence of barriers :

Theorem 1. *Let us assume that g satisfies the following properties :*

- $g\in C^1(\mathbb{R}\times\mathbb{R})$*, and* $f(v):=g(v,0)$ *is bistable, i.e. verifies* (2) *and* (3)*;*
- $g(0,w)=0$ *and* $\dfrac{\partial g}{\partial w}(v,w)<0$*, for all* $w>0$*;*
- *There exist* $w^*>0$ *and* $\alpha_2>0$ *such that* $g(v,w)\le -\alpha_2 v$ *for all* $w>w^*$ *and* $v>0$.

Then for any $L>0$ *there exists* $M^*(L)$ *such that for all* $M_0\ge M^*(L)$*, there exists a barrier for system* (1).
Moreover, we have

(i) $L\mapsto M^*(L)$ *is decreasing and continuous;*

(ii) $\lim\limits_{L\to 0^+} M^*(L)=+\infty$*;*

(iii) $\lim\limits_{L\to+\infty} M^*(L)=M_\infty$ *where* M_∞ *is such that the function* $v\mapsto g\left(v,\frac{M_\infty}{\mu}\right)$ *admits three nonnegative roots* $0<v^-<v^+$ *and* $\int_0^{v^+} g\left(v,\frac{M_\infty}{\mu}\right)dv=0$.

Remark 1. *Let us mention that, from an experimental point of view, it is not difficult to guarantee the condition* $w_0\ge\bar w$ *to build a barrier. Indeed, it is enough to make the release with* $M\ge M_0$ *or on a domain large enough.*

2.2 Modelling for the sterile insect technique

The situation to be modeled is the effect that a release of sterilized males in a limited region in space will have on a mosquito population that is already present in part of the domain. For the sake of simplicity, we will also consider only one dimension in space and consider all biological parameters to be constant over time, thus disregarding the effects of field heterogeneity and seasonal variations.

Following ideas in e.g. [24, 6], we model the population of mosquitoes with four compartements : eggs, adult males, adult females, and sterilized mosquitoes. The dynamics is governed by the following system for $t>0$ and $x\in\mathbb{R}$:

$$\begin{cases}\dfrac{dE}{dt} = b\left(1-\dfrac{E}{K}\right)F\left(1-e^{-\beta(M+\gamma M_s)}\right)\dfrac{M}{M+\gamma M_s} - (\nu_E+\mu_E)E\\ \partial_t M-\partial_{xx}M = (1-r)\nu_E E-\mu_M M\\ \partial_t F-\partial_{xx}F = r\nu_E E-\mu_F F\\ \partial_t M_s-\partial_{xx}M_s = u-\mu_S M_s\end{cases}\tag{5}$$

The variables and parameters included in our model are listed below:

- E, M and F denote respectively the spatial density of mosquitoes in aquatic phase, adult males, and adult females;
- $M_s(x,t)$ is the density of sterile mosquitoes which are released at point x and time t, the release function is denoted u; from now on, we will consider that the release function is $u(x) = M_0 \cdot \mathbf{1}_{[0,L]}(x)$;
- the term $(1 - e^{-\beta(M+\gamma M_s)})$ is an Allee effect to model the fact that it may be difficult for a female to find a partner when the density of mosquitoes is small, similar terms have been introduced in [24];
- the fraction $\frac{M}{M+\gamma M_s}$ corresponds to the probability that a female mate with a noninfected mosquito;
- $b > 0$ is a birth rate; $\mu_E > 0$, $\mu_M > 0$, $\mu_F > 0$, and $\mu_s > 0$ denote the death rates for the mosquitoes in the aquatic phase, for adult males, for adult females, and for sterile males respectively;
- K is an environmental capacity for the aquatic phase, accounting also for the intraspecific competition;
- $\nu_E > 0$ is the rate of emergence;
- $r \in (0,1)$ is the probability that a female emerges, then $(1-r)$ is the probability that a male emerges.

Since mathematical analysis for systems like (5) is quite complicated to handle, we will make some modelling assumptions to reduce (5) to a system such as (1) for which we will perform a full analysis. To do so, we assume that $r = \frac{1}{2}$, $\mu_F < \mu_M$, and $M(x,0) \leq F(x,0)$, so that the system rewrites

$$\begin{cases} \dfrac{dE}{dt} = b\left(1 - \dfrac{E}{K}\right) F\left(1 - e^{-\beta(M+\gamma M_s)}\right) \dfrac{M}{M+\gamma M_s} - (\nu_E + \mu_E)E \\ \partial_t M - \partial_{xx} M = \dfrac{1}{2}\nu_E E - \mu_M M \\ \partial_t F - \partial_{xx} F = \dfrac{1}{2}\nu_E E - \mu_F F \\ \partial_t M_s - \partial_{xx} M_s = M_0 \cdot \mathbf{1}_{[0,L]} - \mu_S M_s. \end{cases} \tag{6}$$

Notice that the above assumptions are reasonable since the sex ratio is indeed expected to be close to $\frac{1}{2}$ and the male mortality is higher than the female mortality (see e.g. [24]). Our second assumption is to consider that the dynamics of the egg compartment is fast, which boils down to assume that the first equation is at equilibrium :

$$0 = b\left(1 - \frac{E}{K}\right) F\left(1 - e^{-\beta(M+\gamma M_s)}\right) \frac{M}{M+\gamma M_s} - (\nu_E + \mu_E)E.$$

From here we can determine E:

$$E = \mathcal{E}(F,M) = \frac{KbMF\left(1 - e^{-\beta(M+\gamma M_s)}\right)}{bMF\left(1 - e^{-\beta(M+\gamma M_s)}\right) + K(\nu_E + \mu_E)(M+\gamma M_s)}.$$

We may verify easily that $\frac{\partial}{\partial M}\mathcal{E}(F,M) \geq 0$ and $\frac{\partial}{\partial F}\mathcal{E}(F,M) \geq 0$. The second and third equations of (6) simplify into

$$\partial_t M - \partial_{xx} M = \frac{\nu_E}{2}\mathcal{E}(F,M) - \mu_M M, \tag{7a}$$

$$\partial_t F - \partial_{xx} F = \frac{\nu_E}{2}\mathcal{E}(F,M) - \mu_F F. \tag{7b}$$

We mention that a similar minimalist model for the sterile insect technique, without the space dependency, has been introduced and studied in [6]. Notice that the main difference is that the system in [6] is not cooperative,

whereas $\mathcal{E}$ being increasing with respect to F and M, system (7) is cooperative. Moreover, since $\mu_F < \mu_M$, we get

$$\partial_t M - \partial_{xx} M + \mu_F M \leq \frac{\nu_E}{2}\mathcal{E} = \partial_t F - \partial_{xx} F + \mu_F F.$$

Hence we deduce that, if initially we have $M(0,x) \leq F(0,x)$, then by the comparison principle for reaction-diffusion equations, we have $M \leq F$. Then, from (7b) and the fact that $\frac{\partial}{\partial M}\mathcal{E}(F,M) \geq 0$, we have

$$\partial_t F - \partial_{xx} F \leq \frac{\nu_E}{2}\mathcal{E}(F,F) - \mu_F F.$$

Finally, we will consider the following system

$$\partial_t F - \partial_{xx} F = \frac{\nu_E}{2}\mathcal{E}(F,F) - \mu_F F =: g(F, M_s), \tag{8}$$

$$\partial_t M_s - \partial_{xx} M_s = M_0 \cdot \mathbf{1}_{[0,L]} - \mu_s M_s. \tag{9}$$

This latter system has the same form as system (1), and any solution to (8)–(9) is a super-solution to (7)–(9).

Hence, it is enough to prove the existence of a barrier for (8)–(9) to show that a blocking exists. We will show that we may apply Theorem 1 and obtain the existence of a barrier for L and M_0 large enough. Moreover, with the particular form of the reaction term g in this setting, we may obtain more information. More precisely, the main results are summarized in the following

Theorem 2. *Let us assume that*

$$\exp\left(\frac{K\beta r \nu_E}{\mu_F} - \delta\right) > 1 + \delta, \qquad \text{with } \delta = \frac{\beta K(\mu_E + \nu_E)}{2b}\left(1 + \sqrt{1 + \frac{4b}{\beta K(\mu_E + \nu_E)}}\right).$$

Then, for all $L > 0$, there exists $M^*(L)$ such that for all $M_0 > M^*(L)$, there exists a barrier for system (8)–(9). *And the function $L \mapsto M^*(L)$ has the properties (i), (ii), and (iii) in Theorem 1.*

Remark 2. *We will see in Lemma 4 that the condition on the coefficients in the statement of this Theorem guarantees that the function $F \mapsto g(F,0)$ is bistable and verifies* (3). *If this condition is not satisfied then there is no invasion by the population of mosquitoes and thus there is no need to perform a sterile insect technique.*

3 Existence of barrier

3.1 A geometric construction of steady states

In order to build a barrier we will use the geometrical construction method which has been proposed by Lewis and Keener in [15]. Let us consider the system, for $x \in \mathbb{R}$,

$$0 = u'' + h(x,u), \text{ where } h(x,u) = \begin{cases} k(x,u), & x \in [0,L], \\ f(u), & x \notin [0,L], \end{cases} \tag{10}$$

and $f(u)$ is bistable (2) and verifies (3), $k \in C^{0,1}([0,L] \times [0,+\infty))$. The idea is to work on the phase plane (u, u_x). We know (see e.g. [14]) that the equation $u_{xx} + f(u) = 0$ has a family of translated solutions $u(x) = U(x - \xi)$ with $\lim_{x\to\pm\infty} u(x) = \lim_{x\to\pm\infty} u'(x) = 0$ and $u'(x) < 0$ (resp. > 0) when $x > \xi$ (resp. $x < \xi$) for any $\xi \in \mathbb{R}$. This family traces out a homoclinic orbit associated with $(u, u_x) = (0,0)$, which we call curve $\mathcal{A}$. In $u_{xx} + f(u) = 0$, the steady state $u \equiv 1$ is stable, so there also exists a curve $\mathcal{B}$ which is its stable manifold,

with points on it tending to $(u, u_x) = (1, 0)$ when $x \to +\infty$.

We can derive an analytical expression for these two curves: Multiplying $u_{xx} + f(u) = 0$ by u_x and integrating we get

$$\frac{1}{2}u_x^2 + \int_0^u f(s)ds = C,$$

where C is a suitable constant. Taking $u = 1$, $u_x = 0$ gives us $C = 2\int_0^1 f(s)ds$ and thus the expression for curve $\mathcal{B}$ is

$$u_x = \sqrt{2\int_u^1 f(s)ds}, \quad 0 \le u \le 1.$$

Likewise taking $u = u_x = 0$ gives us the expression for curve $\mathcal{A}$:

$$u_x = \sqrt{-2\int_0^u f(s)ds}.$$

To summarize, we have

$$\mathcal{A} := \left\{ \left(u, \sqrt{-2\int_0^u f(s)\,ds} \right), 0 \le u \le \theta_c \right\}, \quad \mathcal{B} := \left\{ \left(u, \sqrt{2\int_u^1 f(s)\,ds} \right), 0 \le u \le 1 \right\}, \tag{11}$$

where θ_c is such that $\int_0^{\theta_c} f(s)\,ds = 0$ and from (3) we have $0 < \theta_c < 1$.

We take a point on curve $\mathcal{A}$ which lies on the first quadrant and consider it as an initial condition $(u(0), u_x(0))$, with $u(0) > 0$ and $u_x(0) > 0$ (we can choose $x = 0$ as the starting point because ξ is arbitrary). We solve the second-order differential equation $u''(x) = -k(x, u)$ on $(0, L)$ with this initial condition $(u(0), u_x(0))$ to get $(u(L), u_x(L))$ (assuming u does not blow up before $x = L$). If $(u(L), u_x(L))$ falls on curve $\mathcal{B}$, then for $x > L$ we follow curve $\mathcal{B}$. By doing so, we have constructed a stationary solution of (10) such that $u(-\infty) = 0$ and $u(+\infty) = 1$ (see Figure 1).

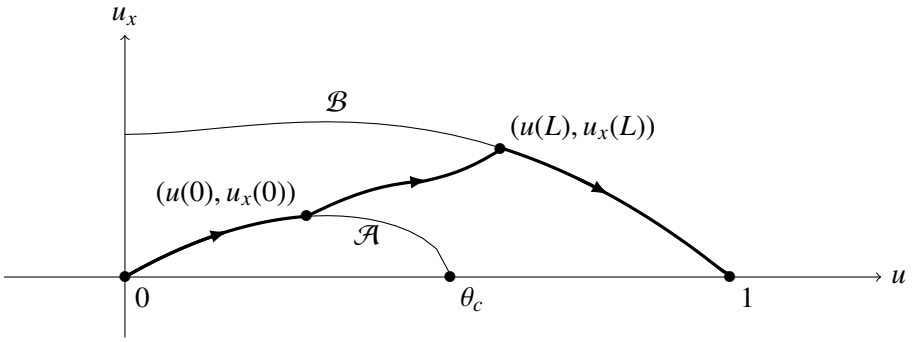

Figure 1: Schematic representation of the technique based on the phase portrait. We construct a stationary solution by following the curve in bold which belongs to $\mathcal{A}$ on $(-\infty, 0)$ and to $\mathcal{B}$ on $(L, +\infty)$.

In order to determine when this happens, for a given $L > 0$, we define the following mapping ψ_L by

$$\begin{gathered} \psi_L : (a, b) \to (u(L), u'(L)), \\ \text{where } u \text{ is solution on } [0, L] \text{ of the Cauchy problem} \\ u''(x) = -k(x, u(x)), \quad (u(0), u'(0)) = (a, b). \end{gathered} \tag{12}$$

Therefore, for a given $L > 0$, we can construct the required steady state if and only if $\psi_L(\mathcal{A})$ intersects curve $\mathcal{B}$, since then we can choose $(u(0), u_x(0))$ as the preimage ψ_L^{-1} of the intersection point.

Thanks to this approach, we may prove the existence of solutions to (10) :

Proposition 1. *Consider the system* (10) *where f is bistable, i.e. satisfies* (2)*, and verifies* (3)*, and $k \in C^{0,1}([0,L]\times[0,+\infty))$. Let us assume that there exists $\alpha > 0$ such that $k(x,u) \le -\alpha u$, for all $x \in [0,L]$. Then, there exists L^* such that for any $L \ge L^*$, there exist a solution $\bar{u}$ for* (10) *such that $\bar{u}(-\infty) = 0$ and $\bar{u}(+\infty) = 1$.*

Proof. The idea of the proof is to show that for L large enough the curves $\psi_L(\mathcal{A})$ and $\mathcal{B}$ will intersect. We first notice that for any $L > 0$, we have $\psi_L(0,0) = (0,0)$.

On $[0,L]$, we have $k(x,u) \le -\alpha u$. Then, solving $u''(x) - \alpha u(x) = \nu(x)$ with nonnegative initial data $(u(0), u'(0))$, we get on $[0,L]$,

$$u(x) = c_1 e^{\sqrt{\alpha}x} + c_2 e^{-\sqrt{\alpha}x} + \frac{1}{\sqrt{\alpha}} \int_0^x \sinh\big(\sqrt{\alpha}(x-y)\big)\nu(y)\,dy,$$

where $c_1 := \frac{u(0)}{2} + \frac{u'(0)}{2\sqrt{\alpha}}$ and $c_2 := \frac{u(0)}{2} - \frac{u'(0)}{2\sqrt{\alpha}}$. We have $c_1 > |c_2| \ge 0$. Then, since $\nu \ge 0$, the last term of the right hand side is clearly nonnegative. We deduce

$$u(L) - u(0) > \frac{u(0)}{2}\left(e^{\sqrt{\alpha}L} + e^{-\sqrt{\alpha}L} - 2\right) + \frac{u'(0)}{2\sqrt{\alpha}}\left(e^{\sqrt{\alpha}L} - e^{-\sqrt{\alpha}L}\right)$$

Then, since $u'(0) > 0$, we have

$$u(L) - u(0) > \frac{u(0)}{2}\left(e^{\sqrt{\alpha}L/2} - e^{-\sqrt{\alpha}L/2}\right)^2 = 2u(0)\sinh\left(\frac{\sqrt{\alpha}L}{2}\right)^2.$$

We also have

$$u'(x) = \sqrt{\alpha}\left(c_1 e^{\sqrt{\alpha}x} - c_2 e^{-\sqrt{\alpha}x}\right) + \int_0^x \cosh\big(\sqrt{\alpha}(x-y)\big)\,v(y)\,dy.$$

Therefore, $u'(x) \ge \sqrt{\alpha}\left(c_1 e^{\sqrt{\alpha}x} - c_2 e^{-\sqrt{\alpha}x}\right)$ and for $x = L$,

$$u'(L) > e^{\sqrt{\alpha}L}\left(\frac{\sqrt{\alpha}u(0) + u'(0)}{2}\right) - e^{-\sqrt{\alpha}L}\left(\frac{\sqrt{\alpha}u(0) - u'(0)}{2}\right) > u'(0).$$

Moreover, as above, we have

$$u'(L) - u'(0) > \frac{u'(0)}{2}\left(e^{\sqrt{\alpha}L/2} - e^{-\sqrt{\alpha}L/2}\right)^2 = 2u'(0)\sinh\left(\frac{\sqrt{\alpha}L}{2}\right)^2.$$

Since $u(0), u'(0) > 0$ and $\sqrt{\alpha}L \to \sinh\left(\frac{\sqrt{\alpha}L}{2}\right)^2$ is increasing and tends to $+\infty$ when $\sqrt{\alpha}L \to +\infty$, then not only does the mapping ψ_L increase both coordinates, but $u(L), u'(L) \to +\infty$ when $L \to +\infty$ for any $u(0), u'(0) > 0$. This guarantees that there exists L^* large enough such that, for any $L \ge L^*$, the curves $\psi_L(\mathcal{A})$ and $\mathcal{B}$ intersect, and we have existence of a solution of (10) with limit 0 at $-\infty$ and with limit 1 at $+\infty$. □

3.2 Proof of Theorem 1

Before proving our first main result, we provide some useful informations.

Lemma 2. *Let $L \ge 0$ and $L^* \ge 0$. Let w be the solution of*

$$w'' + M_0 \cdot \mathbf{1}_{[0,L]} - \mu w = 0, \qquad w(\pm\infty) = 0.$$

Then, under the assumptions of Theorem 1 on g, there exists $M^* > 0$ large enough such that, for all $M_0 \ge M^*$, we have $g(v, w(x)) < -\alpha_2 v$ for all $x \in [\frac{L-L^*}{2}, \frac{L+L^*}{2}]$ and $v > 0$.

Proof. The unique solution of $-w'' + \mu w = M_0 \cdot \mathbf{1}_{[0,L]}$ that satisfies $\lim_{x\to\pm\infty} w(x) = 0$ is given by

$$w(x) = \frac{M_0}{2\sqrt{\mu}} \int_0^L e^{-\sqrt{\mu}|x-y|}\, dy.$$

After straightforward computations, we see that

$$w(x) = \frac{M_0}{\mu} \begin{cases} e^{\sqrt{\mu}(\frac{L}{2}-x)} \sinh\left(\sqrt{\mu}\frac{L}{2}\right), & x > L; \\ e^{\sqrt{\mu}(x-\frac{L}{2})} \sinh\left(\sqrt{\mu}\frac{L}{2}\right), & x < 0; \\ 1 - e^{-\sqrt{\mu}\frac{L}{2}} \cosh\left(\sqrt{\mu}(\frac{L}{2}-x)\right), & 0 \le x \le L. \end{cases} \tag{13}$$

Note that w is positive, maximal at $x = \frac{L}{2}$, and symmetric with respect to $x = \frac{L}{2}$. It is increasing on $(-\infty, \frac{L}{2}]$ and decreasing on $[\frac{L}{2}, +\infty)$. Then, on $[\frac{L-L^*}{2}, \frac{L+L^*}{2}]$, we have

$$w(x) \ge w(\frac{L-L^*}{2}) = w(\frac{L+L^*}{2}).$$

If $L^* \le L$, it gives

$$w(x) \ge \frac{M_0}{\sqrt{\mu}} \left(1 - e^{-\sqrt{\mu}\frac{L}{2}} \cosh(\sqrt{\mu}\frac{L^*}{2})\right).$$

If $L < L^*$, we obtain

$$w(x) \ge \frac{M_0}{\sqrt{\mu}} e^{-\sqrt{\mu}\frac{L^*}{2}} \sinh(\sqrt{\mu}\frac{L}{2}).$$

Thus, in both situations, we get that for M_0 large enough, $w(x) > w^*$, for $x \in [\frac{L-L^*}{2}, \frac{L+L^*}{2}]$, where w^* is defined in the assumptions of Theorem 1. Therefore, for M_0 large enough, $g(v, w(x)) < -\alpha_2 v$ for all $x \in [\frac{L-L^*}{2}, \frac{L+L^*}{2}]$ and $v > 0$.

□

We are now in position to prove our first main result.

Proof of Theorem 1. We split the proof into several steps.

Step 1. Existence of a barrier.

Let $L > 0$, $M_0 > 0$, and w be a solution of the equation

$$w'' + M_0 \mathbf{1}_{[0,L]} - \mu w = 0, \qquad w(\pm\infty) = 0.$$

Let $L^* > 0$ which will be fixed later. As a consequence of Lemma 2, we have that for $M_0 \ge M^*$ large enough, $g(v, w(x)) \le -\alpha_2 v$ for all $x \in [\frac{L-L^*}{2}, \frac{L+L^*}{2}]$ and $v \ge 0$. On $\mathbb{R} \setminus [\frac{L-L^*}{2}, \frac{L+L^*}{2}]$, we clearly have $w \ge 0$, and thus, $g(v, w) \le g(v, 0)$ since, from the assumptions on g in the statement of Theorem 1, the function g is decreasing with respect to its second variable. Let us introduce $\widetilde{v_2}$ solution of the problem

$$\begin{array}{ll} \widetilde{v_2}'' + g(\widetilde{v_2}, 0) = 0, & x \notin [\frac{L-L^*}{2}, \frac{L+L^*}{2}], \\ \widetilde{v_2}'' - \alpha_2 \widetilde{v_2} = 0, & x \in [\frac{L-L^*}{2}, \frac{L+L^*}{2}]. \end{array} \tag{14}$$

We clearly have from above remarks that $\widetilde{v_2}'' + g(\widetilde{v_2}, w) \le 0$ on $\mathbb{R}$. Since $f := g(\cdot, 0)$ is bistable, we may apply Proposition 1 for system (14) : There exists $L^* > 0$ such that system (14) admits a solution with $\widetilde{v_2}(-\infty) = 0$ and $\widetilde{v_2}(+\infty) = 1$. Then, the couple $(\widetilde{v_2}, w)$ verifies (4). Hence, we have constructed a barrier for any $L > 0$ and for M_0 large enough. Let us recall that for $L = 0$ or $M_0 = 0$, we have $w = 0$ and we know that, due to (3), there exist invading traveling wave solutions. Hence, there is no barrier when $L = 0$ or $M_0 = 0$.

We mention that if there is a barrier (v, w) for $L, M > 0$, then if we denote $\bar{w}$ the solution of

$$\bar{w}'' + M\mathbf{1}_{[0,L]} - \mu\bar{w} = 0,$$

by the comparison principle, we have $w \le \bar{w}$. Thus, $g(v, w) \ge g(v, \bar{w})$. Hence, $0 \ge v'' + g(v, w) \ge v'' + g(v, \bar{w})$, and $(v, \bar{w})$ is a barrier. Then, if there exists a barrier for $L, M > 0$, we can always choose w as

$$w_{L,M}(x) = \frac{M}{2\sqrt{\mu}} \int_0^L e^{-\sqrt{\mu}|x-y|}\, dy. \tag{15}$$

Notice that for any $L' \ge L$ and $M_0' \ge M_0$, we have $M_0'\mathbf{1}_{[0,L']} \ge M_0\mathbf{1}_{[0,L]}$. Hence, if we have a barrier $(\bar{v}, \bar{w})$ for M_0 and L, then for any $M_0' \ge M_0$ and any $L' \ge L$, $(\bar{v}, \bar{w})$ is also a barrier, i.e. verifies (4) with M_0' and L' instead of M_0 and L. For $L > 0$, we denote $M^*(L)$ the infimum of all $M_0 \ge 0$ such that there exists a barrier with M_0 and L. By definition, for any $L > 0$, there exists $M^*(L)$ such that for any $M_0 > M^*(L)$ there exists a barrier. Therefore, by continuity of g and of the function w with respect to M, we deduce that there exists a barrier also for $M_0 = M^*(L)$.

Step 2. $L \mapsto M^*(L)$ is decreasing and continuous.

It is clear from above computations that if $L > L_0$, we have $M^*(L) \le M^*(L_0)$. Let us suppose, by contradiction, that $M^*(L) = M^*(L_0)$, and denote $(v_{L_0,M^*(L_0)}, w_{L_0,M^*(L_0)})$ a barrier for $L_0, M^*(L_0)$. Then we have from (15) that $w_{L,M^*(L)} > w_{L_0,M^*(L_0)}$. We deduce that we may find $M < M^*(L) = M^*(L_0)$ such that $w_{L,M} \ge w_{L_0,M^*(L_0)}$ (for instance by taking $M = M^*(L) \max_{\mathbb{R}} \frac{w_{L_0,M^*(L_0)}}{w_{L,M^*(L)}}$ which is clearly well-defined from the expression (13)). Then, $g(v_{L_0,M^*(L_0)}, w_{L,M}) \le g(v_{L_0,M^*(L_0)}, w_{L_0,M^*(L_0)}) \le -v''_{L_0,M^*(L_0)}$. It implies that $(v_{L_0,M^*(L_0)}, w_{L,M})$ is a barrier for L, M with $M < M^*(L)$, it is a contradiction with the definition of $M^*(L)$. Hence the mapping $L \mapsto M^*(L)$ is decreasing.

Moreover, let $\varepsilon > 0$ and $L_0 > 0$. Using (15), we have $w_{L_0,M^*(L_0)-\varepsilon} < w_{L_0,M^*(L_0)-\frac{\varepsilon}{2}}$. Then, there exists $\eta > 0$ such that for all $L < L_0 + \eta$, we have $w_{L,M^*(L_0)-\varepsilon} \le w_{L_0,M^*(L_0)-\frac{\varepsilon}{2}}$. Hence, if there exists a barrier for $L, M^*(L_0) - \varepsilon$, there will be a solution to

$$-v'' \ge g(v, w_{L,M^*(L_0)-\varepsilon}) \ge g(v, w_{L_0,M^*(L_0)-\frac{\varepsilon}{2}}).$$

It will imply that there is a barrier for $L_0, M^*(L_0) - \frac{\varepsilon}{2}$. However, it is a contradiction with the definition of $M^*(L_0)$ as the minimum of values M for which a barrier exists. Thus, we have $M^*(L_0) - \varepsilon < M^*(L) < M^*(L_0)$ for any $L_0 < L < L_0 + \eta$. Hence, $L \mapsto M^*(L)$ is continuous. It concludes the proof of the point (i).

Step 3. Proof of (ii).

As a consequence of (i), the limits $\lim_{L\to 0^+} M^*(L)$ and $\lim_{L\to+\infty} M^*(L)$ exist. Let us assume by contradiction that the point (ii) does not hold, that is $\lim_{L\to 0^+} M^*(L) = \overline{M} < +\infty$. Then, for all $\varepsilon > 0$, we may find a barrier solution to (4) with $M_0 = \overline{M}$ and $L = \varepsilon$; we call $(v_\varepsilon, w_\varepsilon)$ this barrier solution. By comparison principle and from the expression in (13), we deduce that for all $x \in \mathbb{R}$

$$w_\varepsilon(x) \le \frac{\overline{M}}{\mu}\left(1 - e^{-\sqrt{\mu}\frac{\varepsilon}{2}}\right) \to 0, \qquad \text{when } \varepsilon \to 0. \tag{16}$$

From the first two assumptions on g in the statement of Theorem 1, there exists $\eta > 0$ small enough such that $v \mapsto g(v, \eta)$ is bistable, admits 3 roots $0 < v_{1,\eta} < v_{2,\eta}$ and verifies $\int_0^{v_{2,\eta}} g(v, \eta)\, dv > 0$. Hence there exists a traveling wave solution to the parabolic equation

$$\partial_t v - \partial_{xx} v = g(v, \eta). \tag{17}$$

However, from (16), we deduce that for ε small enough, we have $g(v, w_\varepsilon) \geq g(v, \eta)$. Hence v_ε is a super-solution for the parabolic equation (17) (provided the initial conditions are chosen correctly). This is not possible since it is a stationary solution and (17) admits traveling waves solutions with positive velocity. Thus, we obtain a contradiction. We deduce that $\lim_{L \to L^*} M^*(L) = +\infty$.

Step 4. Proof of (iii).

For point (iii), let us consider M_∞ as defined in the statement of Theorem 1. Then, for any $L > 0$, the solution, denoted w_∞, of the second equation in (4) with M_∞ verifies (see (13))

$$\max_{\mathbb{R}} w_\infty = w_\infty(\tfrac{L}{2}) = \frac{M_\infty}{\mu}\left(1 - e^{-\frac{\sqrt{\mu}L}{2}}\right) < \frac{M_\infty}{\mu}.$$

Thus, since g is decreasing with respect to its second variable, for any v, $g\left(v, \frac{M_\infty}{\mu}\right) < g(v, \max_{\mathbb{R}} w_\infty) \leq f(v) = g(v, 0)$. Hence, from the definition of M_∞ and (2)–(3), $v \mapsto g(v, \max_{\mathbb{R}} w_\infty)$ is bistable, i.e. it admits three nonnegative roots denoted $0 < v_{1,\infty} < v_{2,\infty}$, and verifies $\int_0^{v_{2,\infty}} g(v, \max_{\mathbb{R}} w_\infty)\, dv > 0$. It is well-known that there exists an invading traveling wave solution, denoted v_∞, of the equation

$$\partial_t v_\infty - \partial_{xx} v_\infty = g(v_\infty, \max_{\mathbb{R}} w_\infty), \qquad v_\infty(-\infty) = 0,\ v_\infty(+\infty) = v_{2,\infty}.$$

Since, by the comparison principle, for any $M \leq M_\infty$ and any $L > 0$, the solution of (1), with the same initial data, is such that $v_\infty \leq v$, we deduce that there does not exist a barrier for any $M \leq M_\infty$. Thus for any $L > 0$, we have $M^*(L) > M_\infty$.

Let $M > M_\infty$. Let W be the unique solution of $-W'' + \mu W = M \cdot \mathbf{1}_{[0,L]}$ that satisfies $\lim_{x \to \pm\infty} W(x) = 0$. We have seen in Lemma 2 (see (13)) that, on $(0, L)$, $W(x) = \frac{M}{2\mu}(2 - e^{-\sqrt{\mu}x} - e^{-\sqrt{\mu}(L-x)})$. Then, on the interval $(\frac{L}{4}, \frac{3L}{4})$ we have $W(x) \geq \frac{M}{2\mu}(2 - e^{-\sqrt{\mu}\frac{L}{4}} - e^{-\sqrt{\mu}\frac{3L}{4}})$. As a consequence, for L large enough, we have $W(x) \geq \frac{M_\infty}{\mu}$ on $(\frac{L}{4}, \frac{3L}{4})$. Let us introduce the system

$$\begin{aligned} &\widetilde{v}'' + g(\widetilde{v}, 0) = 0, && x \notin [\tfrac{L}{4}, \tfrac{3L}{4}], \\ &\widetilde{v}'' + g(\widetilde{v}, \tfrac{M_\infty}{\mu}) = 0, && x \in [\tfrac{L}{4}, \tfrac{3L}{4}], \\ &\widetilde{v}(-\infty) = 0, \quad \widetilde{v}(+\infty) = 1. \end{aligned} \tag{18}$$

By conservation of energy, the solutions of this system are such that

$$\begin{aligned} &\frac{1}{2}(v'(x))^2 + \int_0^{v(x)} g(s, 0)\, ds = 0, \quad \text{for } 0 \leq v(x) \leq \theta_c, \quad x \in (-\infty, \frac{L}{4}), \\ &\frac{1}{2}(v'(x))^2 - \int_{v(x)}^1 g(s, 0)\, ds = 0, \quad \text{for } x \in (\frac{3L}{4}, +\infty), \\ &\frac{1}{2}(v'(x))^2 - \frac{1}{2}(v'(y))^2 = \int_{v(x)}^{v(y)} g(s, \frac{M_\infty}{\mu})\, ds, \quad \text{for } \frac{L}{4} \leq x \leq y \leq \frac{3L}{4}. \end{aligned}$$

Hence, there exists a solution to (18) if there exists $\nu_1 < \nu_2$ such that $0 < \nu_1 < \theta_c < \nu_2 < 1$, $\nu_1 = v(\frac{L}{4})$, $\nu_2 = v(\frac{3L}{4})$, and

$$\int_0^{\nu_1} g(s, 0)\, ds + \int_{\nu_1}^{\nu_2} g(s, \frac{M_\infty}{\mu})\, ds + \int_{\nu_2}^1 g(s, 0)\, ds = 0. \tag{19}$$

However, we have from (3) that $\int_0^1 g(s, 0)\, ds > 0$. Moreover, $g(s, \frac{M_\infty}{\mu}) < g(s, 0)$, and by definition of M_∞, $\int_0^1 g(s, \frac{M_\infty}{\mu})\, ds < 0$. Hence, we may always find $\nu_1 < \theta_c < \nu_2$ such that (19) is satisfied.

Thus, there exists a solution to (18). It satisfies for L large enough $\widetilde{v}'' + g(\widetilde{v}, W) \leq 0$, since $W \geq \mathbf{1}_{(\frac{L}{4}, \frac{3L}{4})}$ and g is decreasing with respect to its second variable. Finally, $(\widetilde{v}, W)$ is a barrier and we have constructed a barrier for any $M > M_\infty$.

□

4 Application to the sterile insect technique

In this section, we will apply previous results to system (8)-(9) which has been introduced to model the sterile insect technique. We first show that the reaction term fulfills the hypotheses of Theorem 1. Then, we describe the barriers.

4.1 Bistability

Let us denote g the right hand side of (8), we have

$$g(F, M_s) = F \cdot \frac{(Kr\nu_E - \mu_F F)bF\left(1 - e^{-\beta(F+\gamma M_s)}\right) - K\mu_F(\nu_E + \mu_E)(F + \gamma M_s)}{bF^2\left(1 - e^{-\beta(F+\gamma M_s)}\right) + K(\nu_E + \mu_E)(F + \gamma M_s)}. \tag{20}$$

We display in Figure 2 the function $F \to g(F, M_s)$ for some values $0 < M_s^1 < M_s^2$. In this case, it is bistable for 0 and $M_s^1 1$ and mono-stable for M_s^2. The parameter values we will use in this work are given in Table 1 below and are extracted from [24]. Clearly, with these parameters values, the function $g(\cdot, 0)$ is bistable.

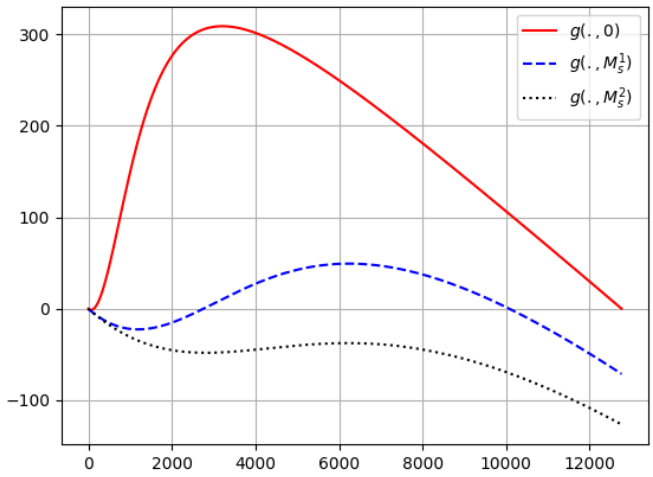

Figure 2: Plot of the functions $F \to g(F, M_s^i)$ for $0 < M_s^1 < M_s^2$.

Lemma 3. *Let g be defined by* (20) *for $F > 0$ and $M_s \geq 0$.*

1. *We have $\frac{\partial g}{\partial M_s}(F, M_s) < 0$.*
2. *There exists $\overline{M} > 0$ such that $g(F, M_s) < 0$ for all $M_s > \overline{M}$ and $F > 0$.*
3. *We have $-\mu_F F \leq g(F, M_s) < -\mu_F F + Kr\nu_E$. Furthermore, for every fixed $M_s \geq 0$, $g(F, M_s) \sim -\mu_F F$ when $F \to 0^+$ and $g(F, M_s) \sim -\mu_F F + Kr\nu_E$ when $F \to +\infty$.*
4. *For any fixed $M_s \geq 0$, $\frac{\partial g}{\partial F}(F, M_s)$ is bounded for $F \geq 0$.*

Proof. 1. Differentiating (20) with respect to M_s and disregarding all positive factors, we have that the sign of $\frac{\partial g}{\partial M_s}$ depends on the sign of $(\beta(F + \gamma M_s) + 1)e^{-\beta(F+\gamma M_s)} - 1$. Putting $x = \beta(F + \gamma M_s)$, $h(x) = (x + 1)e^{-x} - 1$, we have that $h(x) < 0$ for all $x > 0$, which proves that $\frac{\partial g}{\partial M_s}(F, M_s) < 0$ for all $F > 0, M_s \geq 0$.

2. The sign of $g(F, M_s)$ depends on the numerator

$$(Kr\nu_E - \mu_F F)bF\left(1 - e^{-\beta(F+\gamma M_s)}\right) - K\mu_F(\nu_E + \mu_E)(F + \gamma M_s)$$

This numerator is negative for all $M_s \geq 0, F > \frac{Kr\nu_E}{\mu_F}$. For $0 \leq F < \frac{Kr\nu_E}{\mu_F}$, note that

$$\begin{aligned}
&(Kr\nu_E - \mu_F F)bF\left(1 - e^{-\beta(F+\gamma M_s)}\right) - K\mu_F(\nu_E + \mu_E)(F + \gamma M_s) \\
&< (Kbr\nu_E - K\mu_F(\nu_E + \mu_E) - b\mu_F F)F - K\mu_F(\nu_E + \mu_E)\gamma M_s \\
&\leq \frac{K^2(br\nu_E - \mu_F(\nu_E + \mu_E))^2}{4b\mu_F} - K\mu_F(\nu_E + \mu_E)\gamma M_s,
\end{aligned}$$

because $0 \leq \left(1 - e^{-\beta(F+\gamma M_s)}\right) < 1$ and $(Kbr\nu_E - K\mu_F(\nu_E + \mu_E) - b\mu_F F)F$ is a downward parabola in F. Therefore if $M_s > \bar{M} = \frac{K}{4b\gamma\mu_F^2(\nu_E+\mu_E)}(br\nu_E - \mu_F(\nu_E + \mu_E))^2$, the numerator is negative for all $F > 0$.

3. Let $h(F, M_s) = \frac{g(F,M_s)}{F}$, so

$$\begin{aligned}
h(F, M_s) &= \frac{(Kr\nu_E - \mu_F F)bF\left(1 - e^{-\beta(F+\gamma M_s)}\right) - K\mu_F(\nu_E + \mu_E)(F + \gamma M_s)}{bF^2\left(1 - e^{-\beta(F+\gamma M_s)}\right) + K(\nu_E + \mu_E)(F + \gamma M_s)} \\
&> \frac{-\mu_F bF^2\left(1 - e^{-\beta(F+\gamma M_s)}\right) - K\mu_F(\nu_E + \mu_E)(F + \gamma M_s)}{bF^2\left(1 - e^{-\beta(F+\gamma M_s)}\right) + K(\nu_E + \mu_E)(F + \gamma M_s)} = -\mu_F.
\end{aligned}$$

We compute also

$$0 < g(F, M_s) + \mu_F F = \frac{Kr\nu_E bF^2\left(1 - e^{-\beta(F+\gamma M_s)}\right)}{bF^2\left(1 - e^{-\beta(F+\gamma M_s)}\right) + K(\nu_E + \mu_E)(F + \gamma M_s)} < Kr\nu_E.$$

Since for all $M_s \geq 0$, $\lim_{F\to+\infty} g(F, M_s) + \mu_F F = Kr\nu_E$ and $\lim_{F\to 0^+} h(F, M_s) = -\mu_F$, this proves the asymptotic behavior of $g(F, M_s)$ when $F \to 0^+, F \to +\infty$ respectively.

4. By L'Hôpital's rule, we have for all $M_s \geq 0$,

$$\frac{\partial g}{\partial F}(0, M_s) = \lim_{F\to 0^+} h(F, M_s) = -\mu_F, \quad \lim_{F\to+\infty} \frac{\partial g}{\partial F}(F, M_s) = \lim_{F\to+\infty} h(F, M_s) = -\mu_F.$$

Since $\frac{\partial g}{\partial F}$ is continuous and has finite limit when $F \to 0^+$ and $F \to +\infty$, then $\frac{\partial g}{\partial F}$ is bounded for every $F \geq 0$ and $M_s \geq 0$.

□

The following lemma provides the equilibria for the dynamical system when there are no sterile mosquitoes.

Lemma 4. *Let g be defined by* (20). *Let us denote* $\delta = \frac{\beta K(\mu_E + \nu_E)}{2b}\left(1 + \sqrt{1 + \frac{4b}{\beta K(\mu_E + \nu_E)}}\right)$. *Then, when $M_s = 0$, the equation $g(F, 0) = 0$ admits*

(i) three nonnegative roots $0 < F^- < F^+$ if

$$\exp\left(\frac{K\beta r\nu_E}{\mu_F} - \delta\right) > 1 + \delta; \tag{21}$$

in this case, $g(F, 0) < 0$ on $(0, F^-) \cup (F^+, +\infty)$ and $g(F, 0) > 0$ on (F^-, F^+);

(ii) two nonnegative roots if we replace the inequality in (21) *by an equality;*

(iii) only 0 *as a root if we have the reverse inequality in* (21)*.*

Proof. From (20),

$$g(F,0) = F\frac{(Kr\nu_E - \mu_F F)b(1-e^{-\beta F}) - K\mu_F(\mu_E+\nu_E)}{bF(1-e^{-\beta F}) + K(\mu_E+\nu_E)}. \tag{22}$$

The denominator is positive for $F \geq 0$, so we concentrate on the numerator. Let us denote $a = Kbr\nu_E$, $c = b\mu_F$, $d = K\mu_F(\mu_E+\nu_E)$. Then, $g(F,0) = 0$ iff $F = 0$ or $(a-cF)(1-e^{-\beta F}) = d$. This latter equation is equivalent to

$$g_1(F) = g_2(F), \quad \text{where } g_1(F) = a - d - cF, \quad g_2(F) = (a-cF)e^{-\beta F}. \tag{23}$$

We have $g_2'(F) = (-c-\beta a + c\beta F)e^{-\beta F}$, $g_2''(F) = (a\beta + 2c - \beta cF)\beta e^{-\beta F}$ and g_1 is an affine function. On $(-\infty, 0]$, g_2 is convex, $g_2(0) = a > a - b = g_1(0)$, and $g_2'(0) = -c-\beta a < -c = g_1'(0)$. Therefore, there is no root of (23) on $(-\infty, 0]$. On $[\frac{a}{c} + \frac{1}{\beta}, +\infty)$, g_2 is nondecreasing and $g_2(\frac{a}{c}+\frac{1}{\beta}) > g_1(\frac{a}{c}+\frac{1}{\beta})$. Thus, there is no root of (23) on $[\frac{a}{c}+\frac{1}{\beta}, +\infty)$.

On $(0, \frac{a}{c}+\frac{1}{\beta})$, g_2 is decreasing and convex, whereas g_1 is affine and decreasing with a slope $-c$. Thus, there are 0, 1, or 2 roots of (23) on $(0, \frac{a}{c}+\frac{1}{\beta})$; and when there are two roots $F^- < F^+$, we have $g_2(F) < g_1(F)$ on (F^-, F^+) and $g_2(F) \geq g_1(F)$ anywhere else.

In order to find a necessary and sufficient condition for having two positive roots, we first notice that since g_2' is increasing, $g_2'(0) = -c-\beta a$, and $g_2'(\frac{a}{c}+\frac{1}{\beta}) = 0$, there exists an unique $\xi \in (0, \frac{a}{c}+\frac{1}{\beta})$ such that $g_2'(\xi) = -c$. It is equivalent to

$$ce^{\beta\xi} = c + \beta a - c\beta\xi. \tag{24}$$

Then, there are two solutions for (23) iff $g_2(\xi) < g_1(\xi)$; there is one solution for (23) iff $g_2(\xi) = g_1(\xi)$; there are no roots iff $g_2(\xi) > g_1(\xi)$.

The condition $g_2(\xi) < g_1(\xi)$ reads

$$(a-c\xi)e^{-\beta\xi} < a - d - c\xi.$$

Using (24), we obtain

$$\frac{a}{c} - \xi < (a-d-c\xi)(c+\beta a - c\beta\xi),$$

which is equivalent to

$$0 < c^2\beta\xi^2 - c\beta(2a-d)\xi + a^2\beta - cd - a\beta d = c^2\beta(\xi-\xi_1)(\xi-\xi_2), \tag{25}$$

where

$$\xi_1 = \frac{a}{c} - \frac{\delta}{\beta}, \qquad \xi_2 = \frac{a-d}{c} + \frac{\delta}{\beta},$$

and δ has been defined in the statement of the Lemma: with our notation we have $\delta = \frac{\beta}{2c}\Big(d + \sqrt{d^2 + \frac{4cd}{\beta}}\Big)$. Clearly, we have $\delta > \frac{\beta d}{c}$. Hence, condition (25) is satisfied iff $\xi \notin [\xi_1, \xi_2]$.

Since $\xi \mapsto e^{\beta\xi}$ is increasing and $\xi \mapsto c + \beta a - c\beta\xi$ is decreasing, we deduce from (24) that $\xi < \xi_1$ if and only if $ce^{\beta\xi_1} > c + \beta a - c\beta\xi_1$; and $\xi > \xi_2$ if and only if $ce^{\beta\xi_2} < c + \beta a - c\beta\xi_2$. However, recalling that $\delta > \frac{\beta d}{c}$, we have

$$\begin{aligned} ce^{\beta\xi_2} &= e^{\beta a/c}ce^{\delta - \beta d/c} \geq e^{\beta a/c}c(1+\delta-\beta d/c) > c \\ &\geq c + \beta d - c\delta = c + \beta a - c\beta\xi_2. \end{aligned}$$

As a consequence, we always have $\xi < \xi_2$.

Hence, we have $g_2(\xi) < g_1(\xi)$ iff $\xi < \xi_1$ and $ce^{\beta\xi_1} > c + \beta a - c\beta\xi_1$. Replacing a, c, and ξ_1 by their respective expressions, we obtain (21). By the same token, $g_2(\xi) = g_1(\xi)$ iff $ce^{\beta\xi_1} = c + \beta a - c\beta\xi_1$. And $g_2(\xi) > g_1(\xi)$ iff $ce^{\beta\xi_1} < c + \beta a - c\beta\xi_1$. □

Corollary 1. *Under assumption* (21)*, the function* $F \mapsto g(\frac{F}{F^+}, 0)$ *is bistable in the sense of* (2)*.*

Proof. Clearly, we see from (22) that $F \mapsto g(F, 0)$ belongs to $C^1(\mathbb{R})$. From Lemma 3, we have seen that we have $g(F, 0) \sim -\mu_F F$ in a neighborhood of zero. Moreover, with Lemma 4 (i) the equation $g(F, 0) = 0$ has three simple roots $0 < F^- < F^+$, and $g(F, 0)$ is negative on $(0, F^-) \cup (F^+, +\infty)$ and positive on (F^-, F^+). □

Lemma 5. *Let* g *be defined as in* (20)*.*

For any $M_s > \overline{M}$*, we define*

$$\alpha(M_s) = -\max_{F>0} \frac{g(F, M_s)}{F}.$$

Then, $\alpha(M_s) \in (0, \mu_F)$*,* $M_s \mapsto \alpha(M_s)$ *is increasing, and* $g(F, M_s) \leq -\alpha(M_s)F$*, for all* $F > 0$*.*

Proof. Recall that we have proved in Lemma 3 that for $M_s > \overline{M}$ we have $g(F, M_s) < 0$, for all $F > 0$ and that $h(F, M_s) = \frac{g(F,M_s)}{F}$ is bounded, continuous, $h(F, M_s) > -\mu_F$, and $\lim_{F\to 0^+} h(F, M_s) = \lim_{F\to+\infty} h(F, M_s) = -\mu_F$.

Therefore, $h(F, M_s)$ has a maximum for $F \in (0, +\infty)$, and $\alpha(M_s)$ is well-defined. By definition $g(F, M_s) \leq -\alpha(M_s)F$. Moreover, since $0 > g(F, M_s) > -\mu_F F$ for $F \in (0, +\infty)$ (see Lemma 3), then $\alpha(M_s) \in (0, \mu_F)$. Also, since $g(F, M_s)$ is decreasing in M_s, $\alpha(M_s)$ is increasing.

□

4.2 Wave-blocking region

This part is devoted to the proof of Theorem 2.

Proof of Theorem 2. Under assumption (21) the function $F \mapsto g(F, 0)$ is bistable (see Lemma 4) and thus, it admits three nonnegative roots $0 < F^- < F^+$. If $\int_0^{F^+} g(F, 0)\, dF \leq 0$, then there exists a traveling waves v with negative velocity $c \leq 0$ which is a solution of the system : $cv' - v'' = g(v, 0)$, $v(-\infty) = 0$, $v(+\infty) = F^+$, and v is increasing. Therefore, $-v'' = g(v, 0) - cv' \geq g(v, 0)$, so v verifies the barrier equation in (4). Thus, we have a blocking for any $L \geq 0$ or $M \geq 0$.

Let us assume now that (21) holds and that $\int_0^{F^+} g(F, 0)\, dF > 0$. From Lemma 3, Corollary 1, and Lemma 5, we have that assumptions in Theorem 1 are satisfied for the function g defined in (20). Hence, we may apply Theorem 1 and deduce that for any $L > 0$ there exists $M^*(L)$ such that for any $M_0 \geq M^*(L)$ system (8)–(9) admits a barrier. □

Remark 3. *Since we always have the lower bound* $g(F, M_s) \geq -\mu_F F$*, which means that* F *is bounded from below by the solution of* $\partial_t F_0 - \partial_{xx} F_0 + \mu_F F_0 = 0$*, the solution is* $O(e^{-\mu_F t})$ *in* $[0, L]$*. This means that no matter how many sterile males we introduce in this interval, we cannot make* F *tend to zero asymptotically faster than* $e^{-\mu_F t}$*. This result can be seen as a consequence of the fact that introducing the sterile males only perturbs the reproduction and does not impact the mortality of the females. Therefore, the females cannot decrease with a mortality rate larger than* μ_F *which is, in this simple model, the mortality rate they would have in isolation. However, when we introduce a large number of sterile males, we can choose our linear bound* $\alpha \approx \mu_F$*, which reflects that the probability of females mating with fertile males, and laying viable eggs, decreases, and therefore the female population decreases at a rate close to* μ_F*.*

5 Numerical simulations

To illustrate our theoretical results, we provide some numerical simulations for the mathematical model used for the sterile insect technique (8)–(9). The function g defining the right hand side is given in (20). The values of the numerical parameters are taken from [24] and are given in Table 1. The carrying capacity is computed

similarly as in [24] by taking the value of the number of males at equilibrium (without performing the SIT). We may assume that when no control technique is performed, the system (5) is at equilibrium with value $(\overline{E}, \overline{M}, \overline{F})$ which verifies (see system (5))

$$\overline{E} = \frac{\mu_M}{(1-r)\nu_E}\overline{M}, \qquad \overline{F} = \frac{r\nu_E}{\mu_F}\overline{E}.$$

Once the value of the equilibrium is determined. It is easy to compute the carrying capacity since it should be such that $g(\overline{F}, 0) = 0$. Therefore, with the expression of g in (20) we get

$$K = \frac{b\mu_F \overline{F}(1 - e^{-\beta\overline{F}})}{r\nu_E b(1 - e^{-\beta\overline{F}}) - \mu_F(\nu_E + \mu_E)}.$$

For instance, for a value $\overline{M} = 5106\ \text{km}^{-2}$ (see [24]) we find $K = 20793\ \text{km}^{-2}$.

Parameter	β	b	r	μ_E	ν_E	μ_F	μ_M	γ	μ_s	D_u	K
Value	10^{-4}	10	0.5	0.03	0.05	0.04	0.1	1	0.12	0.0125	20688.8

Table 1: Table of the numerical values used in the numerical simulations. These values are taken from [24]

With these parameter values, we first verify easily that the conditions on the parameters in Theorem 2 are satisfied. Then, we perform numerical simulations of system (8)–(9) in one space dimension. More precisely, we consider a domain of width 50 km discretized by 1000 points. We discretize this system with a finite difference scheme where the right hand side is treated explicitly. Since, for numerical purposes we are in a bounded domain, we impose Neumann boundary conditions at the boundary of the numerical domain.

We show in Figure 3 the density of the females mosquitoes F computed by this discetization of system (8)–(9) for $L = 10$ km, and for the two values $M_0 = 10\,000\ \text{km}^{-2}$ (fig.3-left) and $M_0 = 30\,000\ \text{km}^{-2}$ (fig.3-right). We observe that there is an invading wave of mosquitoes coming from the left of the domain. When performing the SIT with the value $M_0 = 10\,000\ \text{km}^{-2}$ in the domain $[0, L]$, the wave is slowing down but can cross this region and continue to invade the whole domain. Increasing M_0 sufficiently, we notice that the invasion seems to be blocked, as illustrated in fig.3-right for $M_0 = 30\,000\ \text{km}^{-2}$.

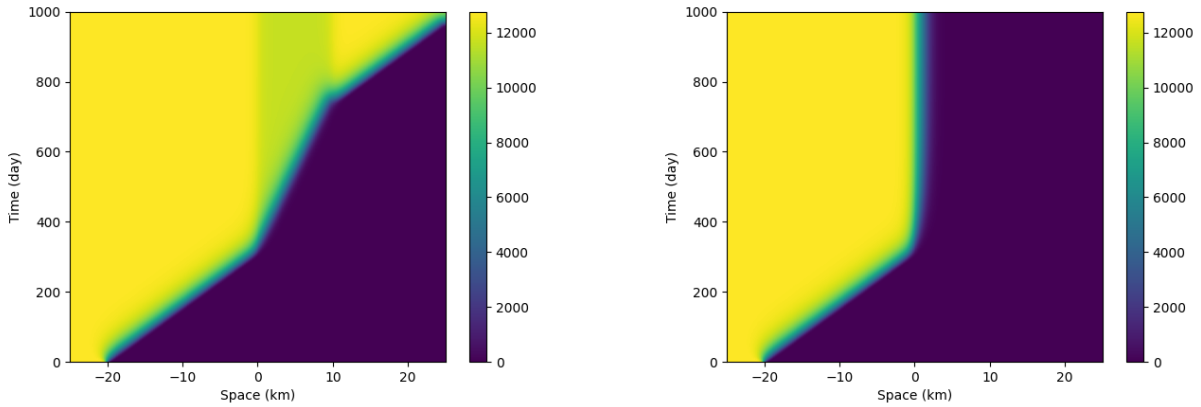

Figure 3: Numerical simulations of system (8)–(9) with $L = 10$ km and $M_0 = 10\,000\ \text{km}^{-2}$ (left), and $M_0 = 30\,000\ \text{km}^{-2}$ (right).

In order to illustrate Theorems 1 and 2, we display in Figure 4 the curve $L \to M^*(L)$ obtained for the function g used to model the sterile insect technique (see (20)) with the values of Table 1. To obtain this

curve, we test numerically, for a given value of the width L of the band, for which value of M_0 the invading wave is blocked at least numerically. We consider that the invading wave is blocked when the difference between two consecutive time iterations is smaller than a given threshold (here we choose 10^{-6}). We observe that this curve $L \to M^*(L)$ verifies the properties stated in Theorem 1.

Moreover, we also compute the limit value M_∞, defined in Theorem 1, by a dichotomy method. More precisely, for a given value of M_0, we compute the maximal root F_+ of the function $g(\cdot, \frac{M_0}{\mu_s})$ thanks to a Newton method. Then we compute the integral $\int_0^{F_+} g(F, \frac{M_0}{\mu_s})\, dF$ by a trapezoidal rule. By dichotomy, we obtain an approximation of M_∞ such that this latter integral vanishes. With the numerical values at hand, we find $M_\infty = 25083,58$ km^{-2}. We verify in Figure 4-left that the curve seems to converge to this asymptotic value M_∞.

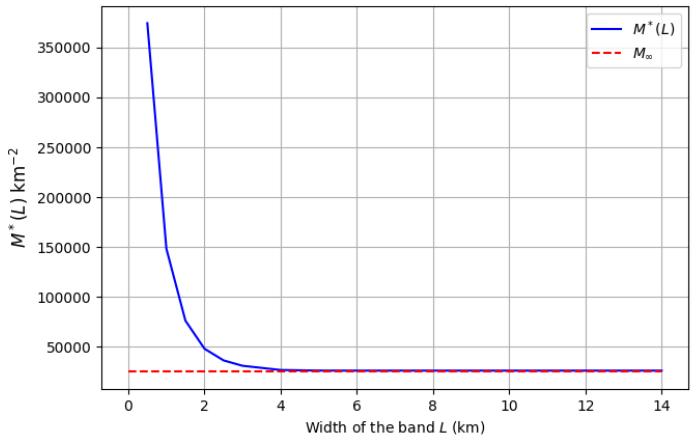

Figure 4: Left : Curve $L \to M^*(L)$ in the case of the sterile insect technique with the values of Table 1.

6 Conclusion

We provide in this paper a rigorous study of the feasability to block an invading species by performing a sterile insect technique in a band, which may be seen as a saniraty cordon to protect an area. The main result of this paper may be summarized as follows : for any width of the band, there exists a minimal value of the number of sterile insect to release continuously in this band to block the invasion of the species. We provide some numerical simulations for this strategy for the case of mosquito invasion.

This study raises several questions. One question is directly related to our observation from numerical experiments. Indeed it will be interesting to study the slowing down of the wave since for practical situations it may be interesting to only slow down the invasion to protect an area for several years. Another extension of this work is the so-called "rolling carpet" strategy which consists in moving the band where the blocking occurs in the opposite direction of the wave to eradicate the insect species in a whole area. Such a study has been performed in [2]. Moreover, we have assumed in this study that all biological parameters are constant in space and time. Obviously, this is a rough approximation and a more precise study should take into account the variation due to the spatial heterogeneity and due to the seasonality.

References

[1] L. Almeida, J. Estrada, N. Vauchelet, *The sterile insect technique used as a barrier control against reinfestation*, Radon Series 2020.

[2] L. Almeida, A. Leculier, N. Vauchelet, *Analysis of the "Rolling carpet" strategy to eradicate an invasive species*, preprint hal-03261142.

[3] D. G. Aronson, H. F. Weinberger, 1975, *Nonlinear Diffusion In Population Genetics, Combustion, And Nerve Pulse Propagation*, in: Goldstein J.A. (eds) *Partial Differential Equations and Related Topics. Lecture Notes in Mathematics*, vol 446. Springer, Berlin, Heidelberg.

[4] R. Bellini, M. Calvitti, A. Medici et al, 2007, *Use of the Sterile Insect Technique Against Aedes albopictus in Italy: First Results of a Pilot Trial*. In: Vreysen M.J.B., Robinson A.S., Hendrichs J. (eds) *Area-Wide Control of Insect Pests*, Springer, Dordrecht.

[5] H. Berestycki, N. Rodríguez, L. Ryzhik, Traveling wave solutions in a reaction-diffusion model for criminal activity, *Multiscale Model. Simul.* **11** (2013), no. 4, 1097—1126.

[6] P.-A. Bliman, D. Cardona-Salgado, Y. Dumont, O. Vasilieva, Implementation of Control Strategies for Sterile Insect Techniques, *Mathematical Biosciences*, 2019, vol. 314, pp. 43–60.

[7] G. Chapuisat, R. Joly, Asymptotic profiles for a traveling front solution of a biological equation, *Math. Mod. Methods Appl. Sci.* **21** (2011) 10, 2155–2177.

[8] E. Chambers, L.K.M. Hapairai, B.A. Peel, H. Bossin, S. Dobson, Male mating competitiveness of a Wolbachia-introgressed Aedes polynesiensis strain under semi-field conditions, PLoS Negl. Trop. Dis. 5 (2011) e1271.

[9] V. A. Dyck, J. Hendrichs, A. S. Robinson, Sterile Insect Technique Principles and Practice in Area-Wide Integrated Pest Management, (2005), Springer.

[10] Y. Dumont, J. M. Tchuenche, *Mathematical studies on the sterile insect technique for the Chikungunya disease and Aedes albopictus*, 2012, J. Math. Biol. Vol. 65, pp. 809–854, doi:10.1007/s00285-011-0477-6.

[11] S. Eberle, Front blocking versus propagation in the presence of a drift term in the direction of propagation, *Nonlinear Analysis*, 2020, vol.197.

[12] E. F Knipling, 1955, *Possibilities of insect control or eradication through the use of sexually sterile males*, Journal of Economic Entomology, Vol. 48 no. 4, pp. 459–462.

[13] M. A. Lewis, P. van den Driessche, Waves of extinction from sterile insect release, *Math Biosci.* 1993 Aug, **116** (2):221–47.

[14] T. J. Lewis, *The Effects of Nonexcitable Regions on Signal Propagation in Excitable Media: Propagation and Reflection*, 1998, Ph. D. Thesis, University of Utah.

[15] T. J. Lewis, J. P. Keener, 2000, *Wave-Block in Excitable Media due to Regions of Depressed Excitability*, SIAM Journal of Applied Mathematics, Vol. 61 no. 1, pp. 293–316.

[16] J. Li, Z. Yuan, 2015, *Modelling releases of sterile mosquitoes with different strategies*, Journal of Biological Dynamics, doi: https://doi.org/10.1080/17513758.2014.977971.

[17] G. Nadin, M. Strugarek, N. Vauchelet, *Hindrances to bistable front propagation: application to Wolbachia invasion*, J. Math. Biol. 76 (2018), no 6, 1489–1533.

[18] C. F. Oliva, M. Jacquet, J. Gilles et al, 2012, *The Sterile Insect Technique for Controlling Populations of Aedes albopictus(Diptera: Culicidae) on Reunion Island: Mating Vigour of Sterilized Males*, Journal PLOS ONE vol. 7 no. 11, doi: 10.1371/journal.pone.0049414.

[19] J. Pauwelussen, 1982, *One way traffic of pulses in a neuron*, J. Math. Biol. Vol 15, pp. 151–171, https://doi.org/10.1007/BF00275071.

[20] C. Pouchol, E. Trélat, E. Zuazua, *Phase portrait control for 1D monostable and bistable reaction-diffusion equations*, Nonlinearity, 05, 2018.

[21] L. Roques, T. Boivin, J Papaïx, S. Soubeyrand, and O. Bonnefon, *Dynamics of Aedes Albopictus invasion insights from a spatio temporal model*, submitted.

[22] D. Ruiz-Balet, E. Zuazua, *Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations*, J. Math. Pures Appl, vol. 143, (2020), pp. 345-375

[23] S. Seirin Lee, R. E. Baker, E. A. Gaffney, S. M. White, *Optimal barrier zones for stopping the invasion of Aedes aegypti mosquitoes via transgenic or sterile insect techniques*, *Theor. Ecol.* (2013) **6**: 427–442.

[24] M. Strugarek, H. Bossin, Y. Dumont, 2019, *On the use of the sterile insect release technique to reduce or eliminate mosquito populations*, Applied Mathematical Modeling, Vol. 68, pp. 443-470.

[25] X. Zheng, D. Zhang, Y. Li et al, *Incompatible and sterile insect techniques combined eliminate mosquitoes*, 2019, Nature, Vol. 572.