A first French episode in the renewal of nonlinear theory of economic cycles
Alain Raybaut

To cite this version:
Alain Raybaut. A first French episode in the renewal of nonlinear theory of economic cycles. History of Economic Ideas, 2021, Anno XXIX (1). hal-03375804

HAL Id: hal-03375804
https://hal.science/hal-03375804
Submitted on 13 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

250 Rue A.Einstein 06560 Valbonne, France.
E-mail : raybaut@gredeg.cnrs.fr

Abstract

This paper focusses on some relatively neglected French contributions to the revival of nonlinear business theory around 1980. Drawing on the formal distinction between continuous and discrete-time modeling, we investigate the mathematical and analytical features of these contributions. The Benassy model exemplifies the use of the Poincaré-Bendixon theorem to prove the existence of endogenous cycles in a simple non-Walrasian framework. The discretization of Kaldor’s model by Dana and Malgrange mobilizes recent advances in bifurcation theory and chaotic dynamics developed at the same time by French scholars in dynamical systems. It is shown that both contributions build on the aggregate macroeconomic framework, but differ substantially in their objectives and ambition. Benassy mobilizes the nonlinear approach to extend his non-Walrasian theory to short-term dynamics. On the contrary, Dana and Malgrange are more interested in the operational aspects of nonlinear modeling. For these reasons, this second line of research will contribute to further important developments in nonlinear dynamics in France, albeit in a different perspective.

Key-Words: Endogenous business cycle theory, Nonlinear dynamics, Non-Walrasian and Kaldorian macrodynamics
A long tradition in the theory of dynamical systems stemming from the works of Poincaré and others exists in France. Relaxation oscillations, van der Pol equations and more generally the mathematics of nonlinear motions have diffused largely among French engineers, physicists and mathematicians, fueling a sustained line of research in the theory of dynamical systems and its applications, including attempts in economics.

This context facilitated undeniably the emergence of several contributions to nonlinear economic dynamics in France around 1980. This paper explores this French episode, but does not claim to draw a systematic history on this theme. We focuss instead on the mathematical and economic underpinnings of some representative advances in nonlinear business cycle modeling.

Accordingly, nonlinear dynamics may be considered as « a collection of models with nonlinear ingredients that require the use of a particular set of (relatively new) mathematical tools »\(^2\). Traditionally, the theory of nonlinear dynamical systems makes a distinction between continuous-time systems (flows) investigated by means of differential equations and difference equations in discrete-time (mappings)\(^3\). For business cycle theorists interested in recurrent solutions, the existence of limit cycles in planar continuous-time systems is generally proved using the Poincaré-Bendixon theorem\(^4\). With mappings, new developments in bifurcation theory and discrete dynamics on the interval are mobilized to characterize periodic orbits and complex solutions like deterministic chaos.

The reminder of the paper draws on this formal distinction between flows and mappings and the way the corresponding mathematical tools were used.

In this perspective, the most salient examples emerged at Cepremap, where these two directions were simultaneously explored during the period under consideration\(^5\).

---

1. Especially by scholars trained in the Grandes Ecoles, like ENS or Polytechnique. For a systematic investigation on the history of nonlinear dynamics in France during the period 1880-1940, see Ginoux 2017. After war let mention the International Conference on Economic Dynamics held in 1955 in Paris, the contributions of Allais to endogenous business cycle in the 1950s or the 1962 special issue of the Cahiers du Séminaire d’Économétrie dedicated to business cycles with a detailed mathematical analysis of Goodwin’s nonlinear models. On these points see Raybaut (2014).
3. See e.g. Medio 2000, p.1
4. For a methodological appraisal of the different tools for proving the existence of endogenous cycles in two dimensional continuous systems, see Ragupathy (2012). Notice that the Hopf bifurcation theorem proper for two dimensional flows was not yet used in the economic literature under consideration. It was also the case for 3 dimensional flows and strange attractors.
5. The Cepremap, at that time "Centre pour la Recherche en Economie Mathématique
On the one hand, Benassy develops a simple nonlinear Keynesian framework to study the emergence of endogenous business cycles as a succession of short run non-Walrasian equilibria. In this contribution, the technique mobilized is not totally innovative. The existence of closed solutions is classically proved with the Poincaré-Bendixon theorem for two-dimensional continuous-time systems. But, it is applied in an elegant way to a specific dynamic model in the non-Walrasian perspective largely initiated by the author.

On the other hand, Dana and Malgrange starting with the well-known Kaldor’s 1940 model, produce an original analysis in discrete dynamics. In particular, they build on emerging mathematical advances in bifurcation theory and chaotic dynamics of mappings developed mainly by French scholars and not yet diffused in economic analysis.

These advances in nonlinear dynamics build on the aggregate macroeconomic framework. Indeed, Benassy and Malgrange have used this modeling framework in several previous works, albeit from a different perspective. Benassy has developed a number of macroeconomic models applying the non-Walrasian methodology. In the early 1980s he aimed notably at bridging the gap between traditional Keynesian macroeconomics and the non-Walrasian theory with the help of synthetic models. Malgrange has contributed to a wide range of issues in macroeconomic modeling. He was in particular involved in the development of a simplified version of the ‘core’ Keynesian macroeconomic model of Cepremap. Hence, he made insightful exercises in the use of small scale macroeconometric models. Analyzing the long and short run properties of these models, he showed notably how the linearization procedures and the use of numerical methods may be highly misleading with respect to stability.

Besides the formal aspect mentioned above, the background of the protagonists in question helps to understand some substantial differences in their

---

6. It is beyond the scope of this paper to detail these advances. See e.g. Benassy (1982)


8. See e.g. Malgrange (1981). The existence of complex dynamics in the nonlinear difference equation used by Malgrange in this work was established by Uebe (1983).
contributions in nonlinear dynamics. For Benassy, the adoption of a nonlinear approach partakes of his non-Walrasian research agenda. Accordingly, he mobilizes this method to extend the scope of the non-Walrasian framework to short-term dynamics, having in mind the nonlinear Keynesian models of the business cycle. On the contrary, Dana and Malgrange are more interested in the operational aspects of nonlinear modeling. In this perspective, their goal was mainly to set out a rigorous example showing the properties and the limits of a discrete-time analysis in macrodynamics. As we will see, But it is precisely for these reasons that this second line of research will influence future developments in nonlinear dynamics in France.

The paper is organized as follows. The first section illustrates the use of the Poincaré-Bendixon theorem by Benassy. The second section is dedicated to the new advances in discrete dynamics made by Dana and Malgrange. The last section makes some concluding remarks on further developments in French nonlinear dynamics.

The Poincaré-Bendixon theorem in a Non-Walrasian framework: Benassy’s model

This section is dedicated to the non-Walrasian dynamic model of the business cycle developed by Benassy.

The model was prepared for the Conference on the dynamics of decentralized market economies held near Stockholm in August 1983, on the invitation of Richard Day. The aim of the conference was to provide a forum confronting different views on the dynamics of market economies, focusing on crucial

---

9. In this perspective, he mentions the main attempts developed since the 1950s
10. Some connections also exist between the protagonists. Benassy thanks Malgrange for stimulating discussions on his model. In the early seventies, he had collaborated directly with Malgrange on production functions and technical progress. See Benassy and Malgrange (1972), Benassy, Fouquet and Malgrange (1975). Rose-Anne Dana was at the time Associated Professor of mathematics at University Pierre and Marie Curie (Paris VI). In 1970-1973 she was Teaching assistant of mathematics and research assistant of economics at University of California in Berkeley, when Benassy was writing his Ph.D. She also made a Ph.D. in Berkeley in 1974 on optimal growth under the supervision of Roy Radner.
12. The conference was organized jointly by Gunnar Eliasson for the Industrial Institute of Economic and Social Research in Stockholm and Richard Day for the Journal of Economic Behavior and Organization, founded by Day and Sidney Winter in 1980. The model was immediately published in 1983 in the JEBO and reprinted in a book in 1986 (see below)
issues as disequilibrium, structural change and micro-macro relations in the spirit of Joseph Schumpeter.

Benassy’s interest for dynamic issues emerges naturally in his contributions on the microeconomic theory of non-Walrasian equilibria and their macroeconomic applications. But the different non-Walrasian concepts he developed in the 1970s give only some insights into how the market economies work dynamically. Indeed, the explicit derivation of dynamic findings in a disequilibrium perspective was difficult because of the added structure required which even presses downward the analytical benefit-cost ratio, especially in the microeconomic framework. This may partly explain why Benassy did not really produce a dynamic analysis until the early eighties.

The macroeconomic framework adopted by Benassy in this contribution disposes of these difficulties. In this way, as Day put it, Benassy was able “with a few deft Keynesian strokes to describe a disequilibrium macro model and derive its salient dynamic properties.”

Several growth models were developed in France during the 1980’s in the so-called disequilibrium dynamic approach. But this contribution is a rare exception in short-run dynamics. The objective is to explore this dyna-

---

13. 1983 marked the hundredth anniversary of Joseph Schumpeter’s birthday. Day has always been convinced that the dynamics of market processes works in perpetual disequilibrium. He has developed a number of adaptive and nonlinear models of growth and business cycle. (See e.g. Day 1994). His own contribution “Disequilibrium Economic Dynamics: A post-Schumpeterian Contribution” is a description of the research program he has carried out in this perspective. Let also mention among the other leading theoretical contributions, a critique of neoclassical macroeconomics by Simon, an analysis of non-tatonnement dynamics by Clower and the presentation of the Micro-Macro model by Gunnar Eliasson. The rest of the papers discussed the role of institutions, markets, technological progress, innovations and entrepreneurial activity from an historical and empirical viewpoints and the new evolutionary perspective by Winter. The main contributions, including Benassy’s paper, with supplementary pieces developed from the discussions were reprinted in the book The Dynamics of Market Economies edited by Richard Day and Gunnar Eliasson in 1986.

14. Béraud (2017) mentions that Benassy began to study dynamics in his 1973 unpublished thesis with the behavior of the simplified Keynesian model. But, this passage was not repeated in his 1976 article, “which suggests that he was not fully satisfied with the way he had dealt with this issue in his PhD. dissertation.” (“Les économistes francophones et les équilibres non-walrasiens 1975-1985”, Communication to the Gide Conference Fabricating Modern Economics, Nice December 2017, pp.24-25

15. Notably in his 1982 book. I am indebted to an anonymous referee for reminding me of this consideration.

16. The model largely builds on Benassy 1982

17. Richard Day (1983), comments on Benassy in Day and Eliasson (1986), p.148. Acknowledging Benassy for this contribution he adds that “it is thus especially noteworthy that a mathematical economist of the general equilibrium school should venture into this macroeconomic realm where much serious analytical effort needs to be allocated.” Ibid.

18. In the non-Walrasian perspective, the contribution of Somonovits (1982) is ano-
The model

The starting point is a simple IS-LM framework with fixed money supply, no public spending, flexible price and fixed wage.

The Demand Side is thus described by the equations $y = C(y, p) + I(x, r)$ and $L(y, r, p) = M$,

where $y$ is current output, $p$ current price, $r$ the interest rate and $M$ the fixed quantity of money. As mentioned by Benassy, here, the sole novelty is the explicit introduction of expected demand $x$ in the investment function, usually captured by some function of past or current income$^{20}$.

Next, consider the Supply-side of the model. Labor supply is inelastic and equal to $l_0$. The firm operates on a competitive market with a decreasing returns technology $F(l)$. She is not quantity constrained, thus for a given stock of capital, the demand for labor is given by $F[F^{-1}(y/\omega)]$.

The wage $w$ evolves according to a Phillips curve with the traditional decreasing shape:

$$\dot{w} = H(u)$$

where $u = l_0 - l$ is the level of unemployment. Using the fact that $l = F^{-1}(y)$, this relation can be rewritten as an increasing function of the level of output $y$:

$$\dot{w} = G(y)$$

To complete the model, Benassy considers that expected demand $x$ adjusts...
to actual demand $y$ with a speed of adjustment $\mu > 0$:

$$\dot{x} = \mu(y - x)$$

The level of income $y$ is a temporary equilibrium solution between aggregate demand and supply. For any level of expected demand $x$, this solution can be written as $y = Z(w, x)$, where $Z_w < 0$ and $Z_w > 0$. Consequently, the dynamics of the economy is fully described by the planar dynamical system in $w$ and $x$.

**Dynamics and existence of limit cycles**

Before investigating the dynamics of this nonlinear system, Benassy recalls that the assumptions made above on the IS-LM structure, the Phillips curve and the adaptive revision of expected demand, ensure the existence of a unique stationary solution $w = w^*$ and $x = x^* = \overline{\gamma}$, where $\overline{\gamma} = F(l_0 - \overline{\gamma})$ and $\overline{\gamma}$ is the non inflationary rate of unemployment. Then, he considers classically the linearisation of the system about this stationary solution. The determinant of the Jacobian matrix $-\mu G \cdot Z_x$ is always strictly positive, which discards saddle path dynamics. Thus, the long run equilibrium $(w^*, \overline{\gamma})$ is locally stable or unstable according to the sign of the trace of this Jacobian. Thus, the economy always converges to its long run equilibrium position when $Z_x > \bar{Z}$, where $\bar{Z} = 1 - \frac{GZ_w}{\mu}$. On the contrary, the singular point is locally unstable when $Z_x > \bar{Z} > 1$. From an economic point of view, local instability will arise when the accelerator coefficient is large enough. In such case, it is shown that there exists at least a limit cycle surrounding the long run equilibrium.

Next, Benassy proceeds to show rigourously how this instability condition eventually results in an endogenous cyclical dynamics.

21. In this non-Walrasian framework with inelastic labor supply, the level of output is normally given by the relation

$$y = \min[F(F^{-1}(\frac{I_p}{p}), l_0]$$

The type of short-run equilibrium, with full employment when $y = y_b = F(0)$ or with underemployment when $y = F(F^{-1}(\frac{I_p}{p}))$, depends on which part of the supply curve the system is. However, the assumption made by Benassy for the shape of the Phillips curve implies that the full employment equilibrium will never be reached, as wage increases become infinite when $y$ approaches $y_b$. As a consequence, $y = F(F^{-1}(\frac{I_p}{p}))$, with $y'_{\mu} > 0$ and $y'_{w} < 0$.

22. It is easy to check that the isokine $\dot{x} = 0$ is strictly above the horizontal axis with a positive slope if $Z_x > 1$ and a negative slope if $Z_x < 1$, which is a sufficient condition for local stability. In addition, the locus $\dot{w} = 0$ is always increasing.

23. Indeed, $Z_x$ the sensitivity of production to expected demand is positively related to a kind of accelerator coefficient $I_x$, by the relation $Z_x = I_x \Omega$, where $\Omega = \frac{y_{\mu} L_p + (1 - C_p) I_p + L_p I_p - C_p I_r}{y_{\mu} L_p + (1 - C_p) I_p + L_p I_p - C_p I_r} > 0$. 

7
The proof of the existence of at least one limit cycle resorts to the Poincaré-Bendixon theorem. From a formal point of view, this theorem ensures that a non empty compact limit set of a continuously differentiable dynamical system in $\mathbb{R}^2$ which contains no fixed point, is a closed orbit (Hirsch and Smale 1974, p.252). As Medio put it, « the basic condition required by the theorem is the existence of a 'trapping region' which contains no fixed point, such that the orbits of the system can enter but cannot leave it. » (Medio 2000, p. 120). Without entering into the details of the proof, the argument is here to show that when the unique fixed point is locally unstable, the assumptions made above for the presentation of the model are sufficient to ensure that the trajectories remain in a bounded set.

As shown on the figure above, this set is given by the compact formed by the bold line $ABCDE$ which contains at least one limit cycle. 24 From an economic point of view, the emergence of these bounded trajectories is explained by the interplay between a destabilizing tendency on the quantity side and a stabilizing dynamics due to prices. Indeed, the destabilizing effect on the quantity side is the consequence of the adaptive revision of expected demand, the level of investment being determined by an unstable accelerator mechanism. But this instability is contained by the damping direct effect exercised by prices on aggregate demand and the adjustment of wages. The latter is captured by a traditional Phillips curve which tend to

24. The sole ambiguity concerns the segment $AB$. In order to be sure that the dynamics points inwards on $AB$, that is $\dot{y} < 0$, Benassy needs to assume that the location of point $C$ defined by $\hat{y}$ satisfies the condition $G(\hat{y}) > -\frac{\gamma}{\mu y_0 z_x}$ . This sufficient condition is not restricting, but arbitrary.
bring back the system to its long run equilibrium position.

As noticed by Day in his comments, Benassy introduces price adjustment explicitly, "the absence of which is perhaps the most important restriction in the standard IS-LM framework. The results illustrate the important point that the mere existence of price adjustments can not guarantee the emergence of equilibrium, a mistaken presumption at the root of much discussion of the market system." 25. More generally, this type of nonlinear models with low order equations "permits the derivation of sharp and interesting results that go well beyond the question of existence and comparative statics". But this payoff "necessitates difficult global analysis of dynamic behavior" 26.

In this perspective, Benassy recalls that the Poincaré-Bendixon theorem has already been applied to various Keynesian nonlinear models. However, the implementation of this technique in a dynamic non-Walrasian framework is totally original. In addition, the elegance of the qualitative proof should be emphasized. Contrary to the other contributions, no specific ad-hoc assumptions on the functional forms of the model need to be assumed 27. But this type of qualitative and global analysis of economic dynamic presents several difficulties.

First, as mentioned by Benassy himself in his concluding remarks, this simple IS-LM framework could be judiciously enriched by additional ingredients, like expectations in an augmented Phillips curve or capital accumulation and growth. But the use of the Poincaré-Bendixon theorem limits the analysis to two dimensional systems. Second, the theorem establishes the existence of at least one limit cycle and it remains in general impossible to elicit the exact number of closed trajectories. Finally, this framework deals with the qualitative and global properties of the dynamics. Thus, it is not possible to say much on the effective trajectories of the economy in the short run as in traditional business cycle modeling.

These difficulties, reinforced by the rapid decline of the aggregate perspective in mainstream macroeconomic analysis, may explain the limited influence of the model.

26. ibid.
27. As it was the case for example, on a nonlinear Phillips in the pioneering contribution of Rose (1969) or the well-known sigmoid Kaldorian investment function in Chang and Smyth (1971) and other reformulations. Notice that simultaneously to Benassy’s model, the contribution of Schinasi (1982) is also based on the qualitative dynamics of an IS-LM model, but in a classic Walrasian framework augmented of a budget constraint and a nonlinear investment function.
The model was naturally discussed in the specific Keynesian literature on nonlinear dynamics. A larger audience could be expected with the publication in 1986 of Benassy’s book *Macroeconomics An Introduction to the Non-Walrasian Approach*. The book gives an overview of the non-Walrasian approach and discussed its applications to traditional macroeconomic issues, emphasizing in particular their policy implications. Dealing with the problem of business cycles Benassy resorts to his non-Walrasian business cycle model to discuss how the dynamic evolution of a short-run non-Walrasian equilibrium of the IS-LM type may generate cycles. But in the meantime, Benassy has rapidly abandoned this IS-LM framework, moving to more sophisticated optimizing models with non-Walrasian features, notably imperfect competition. However, these contributions still remind us of the advantages of relatively simple benchmarks with analytical solutions, privileging the basic properties of the mechanisms at work in the dynamics.

New advances in a well-known model: the pioneering analysis in discrete dynamics by Dana and Malgrange

The second type of contribution adopts a different perspective, building on Kaldor’s 1940 classic business cycle model. The reformulation into a nonlinear dynamical framework of Kaldor’s initial insights is not new in the early 1980s. Let us simply recall among others, the

28. See notably Semmler (1986). In the early 1990s, the model was still mentioned in the main syntheses on nonlinear economic dynamics. For example Gabisch and Lorenz (1989) discuss lengthily Benassy’s “complicated compact set”. The model is also referred in Lorenz (1993) or Dohtani (1996) for a mathematical perspective. For a specific presentation and economic discussion of the model see Dore (1993) and the Appendix 1 in Chiarella and Flaschel (2000).

29. Accordingly, the Web of Science data bases mention only 11 citing references (with a sum of times cited equal to 71) for the 1986 *JEBO* article. As to *Macroeconomics An Introduction to the Non-Walrasian Approach*, the data bases mention 43 citing references (with a sum of times cited equal to 251).


32. Several Kaldorian studies where developed in *Cepremap* in the 1980s, dedicated to the empirical and theoretical analysis of the relation between technical progress, productivity and growth. On this issue, see the different contributions of Boyer and Petit, notably the references in Boyer and Petit (1980). In addition, let us recall that the French translation of some selected contributions of Kaldor, *Economie et Instabilité*, including his 1940’s business cycle model, was set about under notably the direction of Robert Boyer and Pascal Petit.
Japanese attempts in the 1950s with the contribution of Ichimura (1955)\textsuperscript{33}, the application of catastrophe theory by Varian (1979) and the seminal re-formulation into a nonlinear planar differential system resorting to the Poincaré Bendixon theorem by Chang and Smyth (1971). Dana and Malgrange emphasize that most contributions deal with autonomous planar systems in continuous time. This class of nonlinear systems can only give birth to stable or unstable fixed points, saddle path dynamics or periodic orbits (limit cycles). Their aim is to develop a discrete version of the model. Building on the new mathematical literature on bifurcation and discrete dynamics, they show that more complex patterns, like deterministic chaos, can also emerge in this framework.

Thus, the contribution gives an insightful comparison of the dynamics of discrete and continuous time models. Indeed, an important methodological issue in macroeconomic modeling is whether to work in continuous or discrete time. Starting from a theoretical model in continuous time, many contributions opt directly for discrete time dynamics. Dana and Malgrange show that the transformation from continuous to discrete time dynamics is far from neutral, particularly in a nonlinear framework\textsuperscript{34}.

The discrete reformulation

This discrete version is based on a slightly modified version of Chang and Smyth (1971). Indeed, one difficulty with this reformulation is still the absence of long term growth. Cycles take place around stationary equilibrium values of the capital stock and output. Dana and Malgrange generalize the model to the case of cyclical fluctuations around an exogenous growth trend. This trend is classically explained by public expenditures $G_0$ growing at a constant rate $g$.\textsuperscript{35, 36} Then, the dynamics of the model in reduced variables $(y, k)$ is described by a system of two first order difference equations:

$$y_{t+1} = \frac{1}{1+g}[yt + \alpha(I(y_t, k_t) - S(y_t, k_t) + G_0)]$$ (5)

$$k_{t+1} = \frac{1}{1+g}[(1-\delta)k_t + I(y_t, k_t)]$$ (6)

\textsuperscript{33} On the different Japanese attempts see Velupillai 2008

\textsuperscript{34} This choice is often implicitly due to econometric considerations. Indeed, the contribution was presented at a workshop held at Erasmus University in 1982, on how the structure of complex theoretical and econometric models might be tricked out. Thus, a version of the paper was published in the book edited by J.P. Ancot (1984) \textit{Analysing the Structure of Econometric Models} that contains a collection of the conference papers.

\textsuperscript{35} The authors assume moreover that the investment $I(Y_t, K_t)$ and saving function $S(Y_t, K_t)$ are homogenous of degree 1 in global income $Y_t$ and capital $K_t$.

\textsuperscript{36} In the initial Cepremap working paper and the published version, Dana and Malgrange, start by considering the version in continuous time, give an existence theorem of at least a limit cycle using the Poincaré Bendixon theorem, and find a numerical example calibrated on French data 1960-74.
where, \( \delta \) is the rate of capital depreciation and \( \alpha \) is the speed of adjustment between global supply and demand.

On this basis, the contribution builds on several mathematical advances in discrete dynamics, not yet diffused in economic dynamics analysis at that time.\(^{37}\)

The first point is to consider that if the state of the system in period \( t \) is described by \((y_t, k_t)\), then in period \( t + 1 \), this state will be given by \((y_{t+1}, k_{t+1}) = T_\alpha(y_t, k_t)\), where, \( T_\alpha \) is a mapping from \( \mathbb{R}^2^+ \) into \( \mathbb{R}^2^+ \), with

\[
T_\alpha(y_t, k_t) = \frac{1}{1 + g} [y_t + \alpha(I(y_t, k_t) - S(y_t, k_t) + G_0), (1 - \delta)k_t + I(y_t, k_t)] \quad (7)
\]

Notice that this mapping is defined for a given value of \( \alpha \), where this speed of adjustment is a bifurcation parameter. Quite interestingly, this mapping is continuous and derivable, but not always invertible for large values of \( \alpha \), thus very complex trajectories (deterministic chaos) may exist.

From this standpoint, this model gives a clear illustration of Ruelle’s (1980) conjecture on the role that strange attractors and chaotic dynamics may play in economic dynamics, « where periodic processes (economic cycles) are well-known ».\(^{38}\)

The author began their investigations with numerical simulations of an example calibrated on French quarterly data for the period 1960-74. Different configurations are then directly observed when the parameter \( \alpha \) is increased between 0.5 to 20. Convergence to the steady state is obtained for small \( \alpha \). For intermediary values a "periodic regime" is observed, then quasi periodic solutions and eventually transition to turbulence for high \( \alpha \) are obtained.

**Bifurcation theory and dynamics for mappings : the "Nice School"**

Next, in order to investigate more closely these findings, the authors refer to recent developments in the theory of bifurcation and dynamics of nonlinear mappings. Contrary to continuous system, in mathematics and physics

\(^{37}\) Let us recall that the fully-fledged working paper is recorded March 1981, but the study began in the end of the 1970s, notably 1978 for the example. In the same perspective, the sole economic exception are the yet unpublished papers of Day and Benhabib and Day, mentioned by the authors.

\(^{38}\) Ruelle (1980), quoted by Dana and Malgrange (1981) p.10. Ruelle supposes that the dynamics of the macroeconomic system may depend on a parameter \( \mu \). Then, like in physics, « we would guess that for small \( \mu \) the economy is a steady state and that as \( \mu \) increases periodic or quasi periodic cycles may develop. For high \( \mu \) chaotic behavior with sensitive dependence on initial condition would be present » (ibid.).
the systematic investigation in the theory of discrete dynamics develop only in the 1970s. In this perspective, the contribution of Dana and Malgrange (1981) provides one of the first synthetic presentation in an economic paper of some main findings.\footnote{Simultaneously with Day’s papers, dedicated more specifically to chaotic dynamics, notably on Li and Yorke (1975) and May (1976) results, and not to bifurcation theory for mappings.}

It is obviously outside the scope of this paper to survey this growing and not yet completely coordinated technical literature.\footnote{For surveys of the period, see e.g. Marsden and Mac-Cracken (1976), Ruelle and Takens (1971), Ruelle (1977) and the seminal synthesis of Guckenheimer and Holmes (1983).} We simply focus on the major advances made by different scholars in Nice at that time.

Indeed, a local culture of nonlinearity and chaos existed in Nice since the early 1970s. The French mathematician and astronomer Michel Hénon a leading figure in the field of stellar dynamics moved from the Institut d’Astrophysique de Paris to the Nice Observatory in 1968 where he spent the rest of his career. Hénon inspired a very active group of scholars in nonlinear dynamics in Nice. His name is in particular linked to the discovery in 1976 of the so-called Hénon map which is one of the most studied chaotic systems.\footnote{Coullet and Pommeau (2016) explain that Hénon found out this mapping during lunch time after a seminar at the Nice Observatory where Yves Pommeau and Jean-Louis Ibanez have presented some new findings on Lorenz’s attractor.} But he also contributed with other astrophysics of the Nice Observatory, notably Claude Froeschlé, to the implementation of performing numerical techniques using Monte Carlo methods.\footnote{See Chazottes and Monticelli (2015).} In the meantime, two French mathematicians Gérard Iooss and Alain Chenciner, have joined the University of Nice, making breakthroughs in the analysis of nonlinear dynamics and bifurcation theory.\footnote{Both have studied at the École Polytechnique during the same period, Chenciner from 1963 to 1965 and Iooss from 1964 to 1966. Iooss joined in 1974 the University of Nice where he retired in 2007; Chenciner arrived in 1975 moved back to the University of Paris VII in 1978. Dana and Malgrange in their initial Cepremap working paper acknowledge Alain Chenciner for his help during the preparation of their paper. Notice that Jean-Michel Grandmont mentions that he also benefited from the expertise of Alain Chenciner, in particular from a course he gave on the Hopf bifurcation in Marseille in 1986.} In 1977 an important international meeting between physicists and mathematicians about non-linear problems and their applications supported by the French Physical Society was held in Nice.\footnote{Or Société Française de Physique (SFP) the main professional society of French physicists founded in 1873. The proceedings, with notably contributions of Michel Hénon, Gérard Iooss, Alain Chenciner, Pierre Coullet and Charles Tresser, were published in Journal de Physique C5, Vol.39, 1978 August. For an assessment of this meeting see Petitgirard (2004). Petitgirard describes how this conference shaped the research in these fields with an increasing implication of the French National Center for Scientific Research.
In this context, Pierre Coullet and Charles Tresser discovered at the same time as Feigenbaum in the US the scenario of transition to turbulence by accumulation of period doubling by iteration of simple maps.\(^{45}\)

In addition, Iooss (1979) gives one of the first systematic discussion and new results on Hopf bifurcation theory for maps, directly mobilized in Dana and Malgrange.\(^ {46}\)

Indeed, Iooss’ (1979) theorem is illustrated with a detailed application to the discrete Kaldor model.\(^ {47}\) The existence of an invariant orbit is proved theoretically. The period and the amplitude of the cycled are also investigated.

As shown in Iooss (1979), this analysis needs to consider the normal form in polar coordinates and is rather intricate, mobilizing the concept of rotation number of an associated homeomorphism \(f_\alpha\) which captures the dynamics of the initial map \(T_\alpha\) restricted to the invariant curve \(\Gamma_\alpha\).\(^ {48}\) The authors refer to an algorithm recently developed by Arneodo, Coullet, Iooss and Tresser (1980), showing that the amplitude can be approximated by a square root function depending on the parameter \(\alpha\) and coefficients of the normal form of the model. But computations were not so easy at today, and they conclude that it is impossible to derive economic practical estimations on theses theoretical advances.\(^ {49}\)

However, a main limitation with these findings is the local property of the...
discrete dynamics. The initial conditions should belong to a neighborhood of the stationary state and the bifurcation parameter closed to its critical value. This limitation contrasts with the continuous systems where global results can be obtained, notably on the existence of limit cycles with the Poincaré-Bendixon theorem.

**Dynamics on the interval : The Kaldor’s map**

Finally, Dana and Malgrange consider Kaldor’s conjecture that firms adjust production much faster than the capital stock in the very short run. It is then possible to analyze the adjustment of global income for a given level of capital. Consequently, the dynamics of \( y \) can be represented by a one dimensional mapping, \( T_k(y) \), the « Kaldor map ». We have

\[
y_{t+1} = T_k(y_t) = \frac{1}{1 + g}[y_t + \alpha\{I(y_t, k_0) - S(y_t, k_0) + G_0\}] \quad (8)
\]

For a given value of the parameter \( \alpha \), it can be shown that a typical « Kaldor map » admits an invariant interval \([y_0, y_1] \in \mathbb{R}^+\), such that \( T_k(y) \) is unimodal.

The dynamics of this class of maps on the interval was quite well understood in the early 1980’s. Thus, Dana and Malgrange apply directly these advances on iterated maps into the interval to their model.\(^{50}\) In particular, the concept of negative Scharwtzen derivative in unimodal functions on the interval is mobilized.\(^{51}\)

The interesting point is to focus on mappings with high values of the parameter \( \alpha \) for which complex dynamical patterns emerge\(^{52}\). In this perspective, the authors consider the iterations of the critical points of their mapping \( T_k(y) \), for different values of the capital level and draw the bifurcation diagram, showing the transition to turbulence\(^{53}\). We reproduced below

\[^{50}\] They refer notably to the contributions of May (1976), Collet and Eckmann (1980)\(^{51}\). The Scharwtzen derivative of a map \( T \) is defined by

\[
S(T) = T'' / T' - 3/2(T'' / T')^2
\]

For unimodal maps on the interval with negative Scharwtzen derivative, several important and generic results on the existence of periodic or non periodic orbits have been established (see e.g Medio 2000 p. 215 for details). These properties will be extensively mobilized in the economic literature, notably by Grandmont (1985) in OLG models. See notably Barnett, Geweke and Shell (1989) and Michel and Wignolle (1996).\(^{52}\) The authors refer here to the work of Arneodo, Coullet and Tresser (1980) and the advances made in Nice on the doubling period scenario. See above\(^{53}\).

\[^{53}\] A point is said to be a critical point if \( T'_k(y) = 0 \)
this bifurcation diagram for $\alpha = 20$ and $120 \leq k \leq 220$, computed by Tresser from the research team in nonlinear dynamics of Nice.$^{54}$

The diagram clearly shows the successive branch splitting and the period doubling scenario typical of the transition to turbulence within the black areas. Thus, the discrete reformulation of Kaldor’s model may exhibit a richer dynamics than the mere limit cycles obtained in the continuous time frameworks.

This pioneering contribution exemplify the benefits of a discrete approach in nonlinear economic dynamics.$^{55}$ But Dana and Malgrange also perceive the methodological difficulties raised by this approach. They mention the potential bias in calculation induced by complex numerical computations. In addition, the time paths generated by such dynamical systems were extremely sensitive to initial conditions and changes in the parameters. That would make it hard to formulate robust macroeconometric estimates and reliable representations of reality. In this perspective, it became also clear that formulating reliable forecasts would become extremely difficult in this type of nonlinear framework.$^{56}$

$^{54}$ Dana and Malgrange 1983, f.n. 8 p. 129. These elements are added in the published version, with the new results on the dynamics on the interval. On the contrary, the developments of the working paper inspired by Varian (1979) in terms of catastrophe theory disappear.

$^{55}$ The initial 1981 Cepremap working paper and its published version in the Cahiers du Séminaire d’Économétrie (1983) is most likely the first economic application of these advances in France, with the comment of Uebe (1983) on Malgrange (1981). The international economic literature on chaotic dynamics emerges at the same time with Day and Benhabib, but the bifurcation diagram was introduced latter. See notably Day (1994).

$^{56}$ Theses remarks anticipate further discussions notably by Baumol and Quandt (1985) in “Models and their implications for forecasting.”, Eastern Economic Journal, Jan-Mar, 11, pp.3-15. The point was particularly sensitive for Malgrange involved in applied macroeconomic modeling.
Concluding remarks

This paper focussed on some relatively neglected French contributions to the revival of nonlinear business theory. Drawing on the formal distinction between continuous and discrete-time modeling, we investigated the mathematical and analytical features of these contributions. The Benassy model exemplifies the use of the Poincaré-Bendixon theorem to prove the existence of endogenous cycles in a simple non-Walrasian framework. The discretization of Kaldor’s model by Dana and Malgrange mobilizes some recent advances in bifurcation theory and chaotic dynamics.

Beyond the formal aspects discussed in detail in the text, these two contributions differ substantially in their objectives and ambition.

Benassy’s main goal was the extension of the non-Walrasian macroeconomic model to a short-run dynamic setting. Facing the revival of business cycle theories, the non-Walrasian approach had to take part in the debates. From this standpoint, the implementation of nonlinear tools was a necessary detour for proving rigorously the existence of endogenous Keynesian cycles. In the meantime, the focus shifted to the New-Keynesian research programme and the intertemporal optimizing framework. In addition, the main critique to Lucas and the equilibrium business cycle approach had to be formulated by Grandmont in his model of endogenous cycles. This may partially explain why Benassy did not contribute later to nonlinear dynamics, privileging the introduction of non-Walrasian features into the stochastic general equilibrium framework. On the contrary, for Dana and Malgrange, the Kaldorian setting plays basically the role of a pedagogical framework in nonlinear discrete dynamics. Accordingly, their discrete version contributes to highlight the relevance as well as the methodological difficulties of this new approach in macroeconomic modeling. In this perspective, the contribution also emphasizes that the discretization of a continuous time macroeconomic model may be critical in a nonlinear context.

For these reasons, this second line of research will contribute alone to further important developments in nonlinear economic dynamics, albeit in a different perspective. Pierre Malgrange will co-sign with Grandmont in 1986 the introduction of the special issue of the Journal of Economic Theory dedicated to nonlinear dynamics. In addition, as we may recall, Grandmont (1985) thanks Dana and Malgrange for his introduction to the mathematics on nonlinear discrete dynamics. In a recent contribution based on several interviews of Jean-Michel Grandmont, Linnemer and Visser (2018) clarify the crucial role played by Dana and Malgrange. They explain that before a

---

57. On the specific case of Grandmont on Keynesianism, see Grandmont (1989).
session of the Roy seminar in 1983, Grandmont was working on Lucas' model in the seminar room, "drafting ideas about how behavior could converge to several cyclical equilibria in the same type of models used by the New Classical economists. By chance, Pierre Malgrange saw the graphs Grandmont was drawing and asked about them. (...) Jean-Michel explained that he was lost, to which Pierre retorts 'You are lost because this is chaos theory.'" 58. Pierre Malgrange was precisely working at the time with Rose-Anne Dana on their discrete version of Kaldor's model. His familiarity with the generic properties of nonlinear mappings was certainly decisive for bringing to Grandmont's attention the explanatory power of bifurcation theory and chaotic dynamics. On this basis, Grandmont was able to worked out completely his answer to Lucas on endogenous business cycles and stabilization policy 59.

Consequently, the first French episode in the renewal of nonlinear business cycle theory ends with the publication of Grandmont's article 60. Indeed, these different contributions make a breakaway in the nonlinear perspective. The 'old-type' aggregated macroeconomic framework is abandoned for the overlapping generations model or the inter-temporal framework with infinite horizon agents. In this perspective, the 1990s see a growing diffusion of the mathematics on nonlinear difference equations, bifurcations and chaos within French theoretically oriented macro-economists 61. But this is beyond the scope of this article.

59. Firstly in his 1983 Cepremap working paper "On Endogenous Competitive Business Cycles", followed in 1984 by the related Standford technical report and the lecture at the Meetings of the Econometric Society, which form the basis of his 1985 *Econometrica* article. We do not enter here into the content of this well-known contribution. A synthetic presentation is given in Linnemer and Visser (2018). They also mention that it is now Grandmont's most cited publication with more than 1,200 Google Scholar citations.
60. Among the related contributions, let mention different Cepremap working papers on endogenous cycles by Grandmont, Grandmont and Laroque and Laroque and the 1986 special issue of the *Journal of Economic Theory*. We did not consider the literature on sun spots. See *JET* (1986) and Barnett and al. (1989)
61. These advances were mainly developed by several young French scholars directly supervised or advised by Grandmont. Let mention Stefano Bosi, Jean-Pierre Drugeon, Patrick Pintus, Thomas Seegmuller or Alain Venditti. For a comprehensive list see Linnemer and Visser (2018). Phillipe Michel was another main French proponent of discrete dynamics in the overlapping generations growth model. The survey of Michel and Wignolle (1993) is a good illustration of these French developments on complex discrete dynamics during the 1990s. For the investigations on bifurcation, cycles and indeterminacy in the optimal growth model, see notably Venditti (1996)
References


Chiarella C. and Flaschel P. 2000, The Dynamics of Keynesian Monetary Growth : Macro Foundations, CUP.


Henon M., 1976, « A Two dimensional mapping with a strange attractor », Comm. in Mathematical Physics, 69-77.


Iooss G. 1979, Bifurcation of Maps and Applications, North Holland.


Medio A. and Lines M. 2000, Nonlinear dynamics : a Primer, CUP.

Petitgirard L. 2000, Le chaos : des questions théoriques aux enjeux sociaux. Philosophie, épistémologie, histoire et impact sur les institutions 1880-2000,
thèse, Université Lumière Lyon 2.

Ragupathy V. 2012, « Existence proofs in nonlinear endogenous theory of the business cycle on the plane », Social School of Sciences/ASSRU, University of Trento, June.


