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ON GROUND STATES FOR THE 2D SCHRÖDINGER EQUATION WITH COMBINED NONLINEARITIES AND HARMONIC POTENTIAL

RÉMI CARLES AND YAVDAT IL'YASOV

ABSTRACT. We consider the nonlinear Schrödinger equation with a harmonic potential in the presence of two combined energy-subcritical power nonlinearities. We assume that the larger power is defocusing, and the smaller power is focusing. Such a framework includes physical models, and ensures that finite energy solutions are global in time. We address the questions of the existence and the orbital stability of standing waves. Given the mathematical features of the equation (external potential and inhomogeneous nonlinearity), the set of parameters for which standing waves exist is unclear. In the two-dimensional case, we adapt the method of fundamental frequency solutions, introduced by the second author in the higher dimensional case without potential. This makes it possible to describe accurately the set of fundamental frequency standing waves and ground states, and to prove its orbital stability.

1. INTRODUCTION

We consider the nonlinear Schrödinger (NLS) equation with harmonic potential and combined power-type nonlinearities

$$(1.1) \quad i\partial_t \psi + \Delta \psi = |x|^2 \psi - \mu |\psi|^{p-2} \psi + |\psi|^{q-2} \psi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

where ψ is a complex-valued function of (t, x) , $p, q \in (2, +\infty)$ ($p \neq q$), $N \geq 1$ and $\mu > 0$ (homogeneity considerations show that the positive coefficient in front of the last nonlinearity can be chosen equal to one). For

$$2 < p, q < \frac{2N}{(N-2)_+} =: 2^*,$$

the Cauchy problem for (1.1) with the initial value $\psi_0 \in \Sigma$ is locally well-posed and has a unique local solution $\psi \in C([0, T(\psi_0)), \Sigma) \cap C^1([0, T(\psi_0), \Sigma^*)$, for some $T(\psi_0) > 0$, where Σ denotes the domain of the harmonic oscillator, $\Sigma = H^1 \cap \mathcal{F}(H^1)$, characterized by the norm

$$\|u\|_{\Sigma}^2 = \int_{\mathbb{R}^N} ((1 + |x|^2)|u|^2 + |\nabla u|^2) = \|u\|_{L^2}^2 + \langle (-\Delta + |x|^2) u, u \rangle.$$

See e.g. [15]. In view of the uncertainty principle (see e.g. [42]),

$$(1.2) \quad \|u\|_{L^2}^2 \leq \frac{2}{N} \|xu\|_{L^2} \|\nabla u\|_{L^2},$$

we may remove the L^2 norm of u from the definition of $\|u\|_{\Sigma}$. In physics, the potential in (1.1) corresponds to some confinement, and the nonlinearity is typically cubic-quintic: the quintic part is defocusing ($q = 6$), and the cubic part is focusing ($p = 4$ and $\mu > 0$); see e.g. [1, 37]. The heart of the present paper addresses the two-dimensional case $N = 2$.

Solutions to (1.1) obey the *energy conservation law*:

$$E \equiv H_\mu(\psi(t)) := \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla \psi|^2 + \frac{|x|^2}{2} |\psi|^2 - \frac{\mu}{p} |\psi|^p + \frac{1}{q} |\psi|^q \right) dx,$$

and the *mass (charge, particle numbers) conservation law*:

$$\alpha \equiv Q(\psi(t)) := \frac{1}{2} \int_{\mathbb{R}^N} |\psi|^2 dx.$$

The assumption $2 < p < q < 2^*$ ensures global well-posedness in Σ for (1.1), since the defocusing nonlinearity dominates the focusing one (see also Lemma 4.1 below). We shall mainly consider this assumption regarding p and q , an assumption which includes the cubic-quintic nonlinearity in 2D. We mention in passing that on the other hand, the Cauchy problem for the 3D cubic-quintic equation with harmonic potential is not straightforward, and the cubic nonlinearity is energy-critical; see [32] for the Cauchy problem when $\mu = 0$, and [46] for the 3D cubic-quintic equation without harmonic potential. On the other hand, if $2 + \frac{4}{N} < p \leq 2^*$ and $2 < q < p$, then finite time blow-up is possible, as shown in [41] (see also e.g. [17, 21]).

We study the existence and stability of standing waves $\psi_\lambda(t, x) = e^{i\lambda t} u(x)$ of (1.1), where the amplitude function u satisfies

$$(1.3) \quad -\Delta u + |x|^2 u + \lambda u - \mu |u|^{p-2} u + |u|^{q-2} u = 0, \quad x \in \mathbb{R}^N.$$

Here $\lambda \in \mathbb{R}$ is the *frequency* of the standing wave.

There are at least two motivations to study (1.3), apart from the fact that (1.1) appears in some physical models (e.g. [1, 37]). First, due to the fact that the nonlinearity in (1.3) is not homogeneous, the range of possible values of the parameters λ and μ , in order for (1.3) to have nontrivial solutions, is not clear. This aspect is explained in more details below. In particular, we could not find any reference addressing the existence of ground state solutions to (1.3). Second, in the case without potential, the set of stationary solutions was studied recently in [28] by using a generalization of the Rayleigh quotient to the nonlinear framework [27]. However, the approach presented in [28] excludes the two-dimensional case (as well as the one-dimensional case, a situation where many specific techniques are available, see e.g. [8, 26]). As we will see, the introduction of the harmonic potential removes this restriction. In principle, Equation (1.3) can be studied via the nonlinear generalized Rayleigh quotient method in any space dimension N , but computations are more explicit in the 2D case, this is why we choose to focus on that case here.

The orbital stability of the ground states of many equations can be often investigated using a Lyapunov functional, determined by the action functional $S_{\lambda, \mu}(u)$ restricted to the manifold of functions u with fixed mass integral $m = Q(u)$. Such an approach was first applied to Korteweg-de Vries (KdV) solitons by Benjamin [6], and later, to three-dimensional ion-acoustic solitons in strongly magnetized plasma by Zakharov and Kuznetsov [45]. The general theorem on orbital stability of the so-called mass-prescribed solutions of nonlinear problems based on this idea was proved by T. Cazenave and P.-L. Lions [16]. However, the orbital stability of mass-prescribed solutions does not always entail the orbital stability of ground states of nonlinear Schrödinger equations, typically when the nonlinearity is not homogeneous, or in the presence of an external potential. In other words, the notions of orbital stability of (the set of) mass-prescribed solutions and orbital stability of

(action minimizing) ground states coincide in the case of homogeneous nonlinearities without potentials, as explained in [16] (see also [15]), but apart from this case (and the very singular case of a logarithmic nonlinearity, where the role of the mass is specific, see [4, 14]), the approaches introduced to prove orbital stability differ; see [25, 26, 43, 44], and also [20], in the case of orbital stability of ground states.

We emphasize that even in the case without potential, the notion of ground state solutions to (1.3) requires some care. The notion which seems to be privileged in physics consists in minimizing the energy under the constraint of a fixed mass $M > 0$,

$$(1.4) \quad \inf\{H_\mu(u); \quad Q(u) = M\}.$$

The other approach consists in seeking a minimizer of the action

$$S_{\lambda,\mu}(u) := H_\mu(u) + \lambda Q(u),$$

for a prescribed frequency $\lambda \in \mathbb{R}$; see [7, 8]. In this article, we consider the following notion (see also [30] and references therein):

Definition 1.1. *For a given $\lambda \in \mathbb{R}$, a solution \hat{u}_λ of (1.3) is said to be a ground state if $S_{\lambda,\mu}(\hat{u}_\lambda) \leq S_{\lambda,\mu}(w)$, for any $w \in \Sigma \setminus \{0\}$ such that $DS_{\lambda,\mu}(w) = 0$. For $\mu > 0$, we denote by*

$$\mathcal{G}_{\lambda,\mu} := \left\{ u \in \Sigma \setminus \{0\} : S_{\lambda,\mu}(u) = \hat{S}_{\lambda,\mu}, \quad DS_{\lambda,\mu}(u) = 0 \right\}$$

the set of the ground states of (1.3), where

$$\hat{S}_{\lambda,\mu} := \min \{ S_{\lambda,\mu}(u) : DS_{\lambda,\mu}(u) = 0, \quad u \in \Sigma \setminus \{0\} \}$$

is called the ground level.

The relation between the two notions is not clear in general: if u is a minimizer for (1.4), then there exists a Lagrange multiplier so that u solves (1.3) for some $\lambda \in \mathbb{R}$. In the case of an homogeneous nonlinearity (typically $-|\psi|^{p-2}\psi$), a scaling argument shows that any $\lambda > 0$ can be obtained, and the solution is also a minimizer of the action, see [16] (see also [15]). In the case of combined power nonlinearities, that is (1.1) without potential, it is known that not every minimizer of the action is a solution to (1.4), see [11, 34]. It has been proven recently that for a large class of nonlinearities (much larger than the sum of two power nonlinearities), every solution to (1.4) minimizes the action; see [30].

We now discuss more precisely why it is not obvious to describe the set of accessible values of λ and μ to solve (1.3). Solutions to

$$(1.5) \quad -\Delta u + V(x)u + \lambda u = f(|u|^2)u,$$

were constructed in [36] for potentials V having a finite limit as $|x| \rightarrow \infty$. In [22], the case $N = 1$ with V bounded and a cubic nonlinearity ($f(y) = y$) was considered, thanks to Lyapunov-Schmidt method, allowing to invoke perturbation arguments near the ground state associated to the case $V = 0$. These results were extended in [40], again in the case of a bounded potential, going to zero at infinity, for an homogeneous nonlinearity $f(y) = y^\sigma$. By bifurcation arguments, the authors show that for λ near $\lambda_* > 0$ ground state eigenvalue of $\Delta - V$, (1.5) has a solution. The case of an harmonic potential $V(x) = |x|^2$ was considered in [31] for the case of an L^2 -critical nonlinearity $f(y) = y^{2/N}$, thanks to a mountain pass lemma. Additional results were obtained regarding the existence and stability of standing

waves for unbounded potentials V in e.g. [23,24]. The following result is very similar to [31, Theorems 1.3 and 1.4], and is proven in Appendix A.

Proposition 1.2. *Let $N \geq 1$ and $2 < p, q < 2^* = \frac{2N}{(N-2)_+}$.*

1. *If $\lambda > -N$, then there exists a solution $u \in \Sigma \setminus \{0\}$ to*

$$(1.6) \quad -\Delta u + |x|^2 u + \lambda u = |u|^{p-2} u.$$

2. *If $\lambda < -N$, then there exists a solution $u \in \Sigma \setminus \{0\}$ to*

$$(1.7) \quad -\Delta u + |x|^2 u + \lambda u = -|u|^{p-2} u.$$

3. *If $\lambda > -N$, then there exists $\mu > 0$ and a solution $u \in \Sigma \setminus \{0\}$ to (1.3).*

Apart from the existence of standing wave, and their classification, the standard question is their dynamical stability: due to the invariances of the equation, the relevant notion is the notion of orbital stability, see e.g. [15, 20]. In the present case, the harmonic potential removes the ‘‘usual’’ space-translation invariance, and, along with time-translation invariance, only the gauge invariance remains: if ψ solves (1.1), then so does $\psi e^{i\theta}$, for any constant $\theta \in \mathbb{R}$. Like for the existence of standing waves, at least two approaches are commonly used in order to study the question of orbital stability: the stability of a set of standing waves (typically, the minimizers associated to (1.4), as in [16]), and the stability of a given standing wave (as in [25, 26, 43, 44]; see also [20]). Like in [28], our main result is in the spirit of the latter notion: we prove the orbital stability of the set of fundamental frequency solutions, defined below. Note that by resuming the approach of e.g. [12] in the 2D case (the 3D case is different, as the quintic nonlinearity becomes energy-critical), based on the method introduced in [16], it is easy to prove that the set of energy minimizers under a mass constraint (the minimizers of (1.4)) is orbitally stable. But then again, we do not know for which values of μ this set is nonempty, and which are the corresponding Lagrange multipliers λ .

Let us state our main result. Introduce,

$$(1.8) \quad \hat{\mu}^0 = C_{p,q} \inf_{u \in \Sigma \setminus \{0\}} \frac{(\int |\nabla u|^2 \cdot \int |x|^2 |u|^2)^{\frac{q-p}{2(q-2)}} (\int |u|^q)^{\frac{p-2}{(q-2)}}}{\int |u|^p}.$$

for some $C_{p,q} > 0$ introduced in Section 2: $\hat{\mu}^0 > 0$ (see Proposition 2.6 below).

Theorem 1.3. *Let $N = 2$, $2 < p < q < \infty$, and $\mu > \hat{\mu}^0$. Then there exists $\hat{\lambda}_\mu^* > 0$ such that:*

(1°) *For any $\lambda \in [0, \hat{\lambda}_\mu^*]$, Equation (1.3) has a ground state $\hat{u}_\lambda \in \Sigma$.*

(2°) *$S_{\lambda,\mu}(\hat{u}_\lambda) < 0$ for $\lambda \in [0, \hat{\lambda}_\mu^*)$ and $S_{\lambda,\mu}(\hat{u}_\lambda)|_{\lambda=\hat{\lambda}_\mu^*} = 0$.*

(3°) *Up to a gauge transform (replace \hat{u}_λ by $e^{i\theta} \hat{u}_\lambda$ for some constant $\theta \in \mathbb{R}$), \hat{u}_λ is positive, radially symmetric, and non-increasing in $r = |x|$.*

(4°) *For any $\lambda \in [0, \hat{\lambda}_\mu^*]$, the set of ground states $\mathcal{G}_{\lambda,\mu}$ of (1.3) is orbitally stable: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\psi_0 \in \Sigma$, satisfies*

$$\inf_{\phi \in \mathcal{G}_{\lambda,\mu}} \|\psi_0 - \phi\|_\Sigma < \delta,$$

then

$$\sup_{t \in \mathbb{R}} \inf_{\phi \in \mathcal{G}_{\lambda,\mu}} \|\psi(t) - \phi\|_\Sigma < \varepsilon,$$

where ψ is the solution to (1.1) such that $\psi|_{t=0} = \psi_0$.

The proof of the theorem is based on development of the method of fundamental frequency solutions, introduced by the second author in [28] in the higher dimensional case without potential. We deal with the so-called *prescribed action solution* of (1.3), i.e., a function $u^S \in \Sigma$ which for a given action $S \geq 0$ satisfies

$$(1.9) \quad S_{\lambda,\mu}(u^S) = S \quad \text{and} \quad DS_{\lambda,\mu}(u^S) = 0,$$

with some $\lambda \geq 0$. As mentioned above, an alternative formulation which has also been actively investigated over the last decades consists in finding the solution u to (1.3) having *prescribed mass*, via (1.4), while λ and S are unknown. Note that the frequency λ can be also considered as a value of the following conserved quantity

$$(1.10) \quad \lambda = \Lambda_\mu^S(\psi(t)) := \frac{S - H_\mu(\psi)}{Q(\psi)}.$$

Definition 1.4. For a given $S \in \mathbb{R}$, a solution \hat{u}^S of (1.3) is said to be a *fundamental frequency solution* (respectively, $e^{i\lambda t}\hat{u}$ is said to be a *fundamental frequency standing wave* of (1.1)) with a *fundamental frequency* $\hat{\lambda}_\mu^S$ if

$$\hat{\lambda}_\mu^S = \Lambda_\mu^S(\hat{u}) \geq \Lambda_\mu^S(w)$$

for any $w \in \Sigma \setminus \{0\}$ such that $D\Lambda_\mu^S(w) = 0$. For $S \in \mathbb{R}$, $\mu > 0$, we denote by

$$\mathcal{F}_\mu^S := \left\{ u \in \Sigma \setminus \{0\} : \Lambda_\mu^S(u) = \hat{\lambda}_\mu^S, \quad D\Lambda_\mu^S(u) = 0 \right\}$$

the set of the *fundamental frequency solutions* of (1.3), where

$$\hat{\lambda}_\mu^S := \min \{ \Lambda_\mu^S(u) : D\Lambda_\mu^S(u) = 0, \quad u \in \Sigma \setminus \{0\} \}$$

is called the *fundamental frequency*.

We have the following result on the existence and orbital stability of fundamental frequency solutions:

Theorem 1.5. Let $N = 2$ and $2 < p < q < \infty$. Then for any given action value $S \in \mathbb{R}$, there exists an extremal value $\hat{\mu}^S \geq 0$ such that:

(1°) For any $\mu > \hat{\mu}^S$, Equation (1.3) has a fundamental frequency solution $\hat{u}_\mu^S \in \Sigma \setminus \{0\}$. In the case $S \leq 0$, the same is true under the relaxed assumption $\mu \geq \hat{\mu}^S$.

(2°) Up to a gauge transform (replace \hat{u}_μ^S by $e^{i\theta}\hat{u}_\mu^S$ for some constant $\theta \in \mathbb{R}$), \hat{u}_μ^S is positive, radially symmetric, and non-increasing in $r = |x|$.

(3°) If $S \leq 0$, then $\hat{\mu}^S > 0$, and for any $\mu \geq \hat{\mu}^S$, the fundamental frequency solution \hat{u}_μ^S is a ground state of (1.3) with $\lambda = \Lambda_\mu^S(\hat{u}_\mu^S)$. Moreover, $\lambda \in [0, \hat{\lambda}_\mu^*)$.

(4°) If $S \leq 0$, then for any $\mu \geq \hat{\mu}^S$, the set of fundamental frequency solutions \mathcal{F}_μ^S of (1.3) is orbitally stable: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\psi_0 \in \Sigma$, satisfies

$$\inf_{\phi \in \mathcal{F}_\mu^S} \|\psi_0 - \phi\|_\Sigma < \delta,$$

then

$$\sup_{t \in \mathbb{R}} \inf_{\phi \in \mathcal{F}_\mu^S} \|\psi(t) - \phi\|_\Sigma < \varepsilon,$$

where ψ is the solution to (1.1) such that $\psi|_{t=0} = \psi_0$.

We obtain the existence of the fundamental frequency solutions by applying the nonlinear generalized Rayleigh quotient method (NG-Rayleigh quotient method) [27–29] to (1.3) with $\Lambda_\mu^S(u)$ as the Rayleigh quotient.

Remark 1.6. We conjecture that Equation (1.3) has no ground state solution with positive action $S > 0$. Furthermore, we anticipate the fundamental frequency solutions \hat{u}_μ^S of Equation (1.3) for $S > 0$ are not orbitally stable.

Notations. Let $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$ be two families of positive real numbers.

- We write $\alpha_n \ll \beta_n$ if $\limsup_{n \rightarrow +\infty} \alpha_n / \beta_n = 0$.
- We write $\alpha_n \lesssim \beta_n$ if $\limsup_{n \rightarrow +\infty} \alpha_n / \beta_n < \infty$.
- We write $\alpha_n \approx \beta_n$ if $\alpha_n \lesssim \beta_n$ and $\beta_n \lesssim \alpha_n$.

2. THE NONLINEAR GENERALIZED RAYLEIGH QUOTIENTS

For $u \in \Sigma$ and $\sigma > 0$, we denote $u_\sigma(x) := u(x/\sigma)$, $x \in \mathbb{R}^N$, and

$$\begin{aligned} T(u) &:= \int |\nabla u|^2, \quad L(u) := \int |x|^2 |u|^2, \quad Q(u) := \frac{1}{2} \int |u|^2, \\ A(u) &:= \int |u|^p, \quad B(u) := \int |u|^q. \end{aligned}$$

With these notations we have

$$S_{\lambda, \mu}(u) := \frac{1}{2}T(u) + \frac{1}{2}L(u) + \lambda Q(u) - \mu \frac{1}{p}A(u) + \frac{1}{q}B(u).$$

and

$$\begin{aligned} T(u_\sigma) &= \sigma^{N-2}T(u), \quad L(u_\sigma) = \sigma^{N+2}L(u), \quad Q(u_\sigma) = \sigma^N Q(u), \\ A(u_\sigma) &= \sigma^N A(u), \quad B(u_\sigma) = \sigma^N B(u). \end{aligned}$$

By a weak solution of (1.3) we mean a critical point $u \in \Sigma$ of $S_{\lambda, \mu}(u)$ on Σ . Observe that if $u \in \Sigma$ is a weak solution of (1.3), then the Brézis-Kato Theorem [9] and the L^γ estimates for the elliptic problems [2] yield $u \in W_{\text{loc}}^{2, \gamma}(\mathbb{R}^N)$, for any $\gamma \in (1, \infty)$, and whence by the regularity theory of the solutions of the elliptic problems $u \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ (see e.g. [2]).

This implies that any weak solution u of (1.3) satisfies the Pohozaev identity [38]

$$(2.1) \quad P(u) := \frac{N-2}{2}T(u) + \frac{N+2}{2}L(u) + \lambda N Q(u) - \mu \frac{N}{p}A(u) + \frac{N}{q}B(u) = 0.$$

See e.g. [8, Section 2.1] for details (the assumption $u \in \Sigma$ makes it possible to adapt the proof in order to include the harmonic potential, by repeating the argument on top of p. 321 in [8]). For $S \in \mathbb{R}$, introduce the so-called *action-level Rayleigh quotient*

$$(2.2) \quad \Lambda_\mu^S(u) := \frac{S - \frac{1}{2}T(u) - \frac{1}{2}L(u) + \mu \frac{1}{p}A(u) - \frac{1}{q}B(u)}{Q(u)}.$$

Notice that for any $S \in \mathbb{R}$ and $\lambda \in \mathbb{R}$,

- $\Lambda_\mu^S(u) = \lambda \Leftrightarrow S_{\lambda, \mu}(u) = S$,
- $D\Lambda_\mu^S(u) = 0$ with $\Lambda_\mu^S(u) = \lambda \Leftrightarrow DS_{\lambda, \mu}(u) = 0$, i.e., u satisfies (1.3).

Let $u \in \Sigma \setminus \{0\}$, $S \in \mathbb{R}$, $\sigma > 0$, consider

$$(2.3) \quad \lambda_u(\sigma) := \Lambda_\mu^S(u_\sigma) = \frac{\sigma^{-N}S - \sigma^{-2} \frac{1}{2}T(u) - \sigma^2 \frac{1}{2}L(u) + \mu \frac{1}{p}A(u) - \frac{1}{q}B(u)}{Q(u)}.$$

Then for $u \in \Sigma$,

$$\frac{d}{d\sigma}\lambda_u(\sigma) = 0 \Leftrightarrow -NS + \sigma^{N-2}T(u) - \sigma^{N+2}L(u) = 0.$$

The last equation,

- For $N > 2$, has a unique solution $\sigma_S(u) > 0$ for any $u \in \Sigma$ and $S < 0$. There are two positive solutions if $S > 0$ is not too large.
- For $N = 2$, has a unique solution $\sigma_S(u) > 0$ for any $u \in \Sigma$ and $S \leq \frac{1}{2}T(u)$.
- For $N = 1$, has a unique solution $\sigma_S(u) > 0$ under suitable relations between $T(u)$, $L(u)$ and S .

Note that for $N \neq 2$, the formula(s) for $\sigma_S(u)$ may not be easy to write, when possible.

From now on, we assume $N = 2$.

Then

$$\sigma_S(u) = \left(\frac{T(u) - 2S}{L(u)} \right)^{1/4}, \quad \forall u \in \Sigma \setminus \{0\}, \quad \text{such that } S \leq \frac{1}{2}T(u).$$

Hence, in accordance with [28], we have the following nonlinear generalized Rayleigh quotient (NG-Rayleigh quotient)

$$(2.4) \quad \lambda_\mu^S(u) := \Lambda_\mu^S(u_\sigma)|_{\sigma=\sigma_S(u)} = - \frac{\left((T(u) - 2S)^{1/2}L(u)^{1/2} - \mu \frac{1}{p}A(u) + \frac{1}{q}B(u) \right)}{Q(u)}.$$

Observe, when $T(u) > 2S$,

$$\lambda_\mu^S(u) = \max_{\sigma > 0} \Lambda_\mu^S(u_\sigma),$$

when $T(u) = 2S$,

$$\lambda_\mu^S(u) = \sup_{\sigma > 0} \Lambda_\mu^S(u_\sigma) = \frac{\frac{\mu}{p}A(u) - \frac{1}{q}B(u)}{Q(u)},$$

and when $T(u) < 2S$ (a case which may occur only for $S > 0$),

$$\sup_{\sigma > 0} \Lambda_\mu^S(u_\sigma) = \infty.$$

Denote, for $S > 0$,

$$\Sigma_S := \{u \in \Sigma; T(u) > 2S\}.$$

Corollary 2.1. *If $u \in \Sigma$ satisfies (1.3), then $\sigma_S(u) = 1$ and $\frac{d}{d\sigma}\lambda_u(\sigma)|_{\sigma=1} = 0$.*

Proof. The equalities $\frac{d}{d\sigma}\lambda_u(\sigma)|_{\sigma=1} = 0$ and $\sigma_S(u) = 1$ are consequences of Pohozaev identity (2.1). Indeed, $S_{\lambda,\mu}(u) - P(u) = \frac{1}{2}T(u) - \frac{1}{2}L(u)$ and therefore, $T(u) - 2S = L(u)$, i.e., $\sigma_S(u) = 1$. The second equality follows similarly. \square

Lemma 2.2. *Let $S \in \mathbb{R}$. Then $D\lambda_\mu^S(u) = 0$, $\lambda_\mu^S(u) = \lambda$, $\sigma_S(u) = 1$ if and only if $u \in \Sigma$ is a weak solution of (1.3) with prescribed action S .*

Proof. Since $\lambda_\mu^S(u) = \lambda$, $\sigma_S(u) = 1$, we have $\Lambda_\mu^S(u) = \lambda$, and consequently, $S_{\lambda,\mu}(u) = S$. From (2.4), by direct calculations of $D\lambda_\mu^S(u)$ it follows

$$(2.5) \quad D\lambda_\mu^S(u) = \frac{1}{Q(u)}DS_{\lambda,\mu}(u), \quad \forall u \in \Sigma \setminus \{0\} \quad \text{such that } \Lambda_\mu^S(u) = \lambda, \quad \sigma_S(u) = 1,$$

and thus, $DS_{\lambda,\mu}(u) = 0$.

Conversely, assume that u is a weak solution of (1.3) such that $S = S_{\lambda,\mu}(u)$. Then $\Lambda_\mu^S(u) = \lambda$ and $D\Lambda_\mu^S(u) = 0$. Moreover, by Corollary 2.1, $\sigma^S(u) = 1$, and hence, $\lambda_\mu^S(u) = \Lambda_\mu^S(u) = \lambda$. Consequently, (2.5) yields $D\lambda_\mu^S(u) = 0$. \square

Remark 2.3. Lemma 2.2 implies that if u is a weak solution of (1.3) with prescribed action S , then the strong inequality $T(u) > 2S$ holds, and thus, (1.1) has no standing wave $\psi_\lambda = e^{i\lambda t}u$ with action value $S := S_{\lambda,\mu}(u) > T(u)/2$.

Remark 2.4. In view of the homogeneity of $\lambda_\mu^S(u)$, we may always assume that any critical point u of $\lambda_\mu^S(u)$ satisfies $\sigma^S(u) = 1$.

From now on, we assume $2 < p < q < \infty$.

For $S \in \mathbb{R}$, we deal with the following condition:

(E_μ) : $\exists u \in \Sigma_S$ such that $\lambda_\mu^S(u) > 0$.

It is easily seen that the feasibility of this condition depends on parameter μ . The limit point of μ where the condition (E_μ) is satisfied will be called extreme value of μ . To find this value, we reapply the NG-Rayleigh quotient method to the functional $\lambda_\mu^S(u)$ (see [13]) and therefore introduce

$$\mu^S(u) := \frac{(T(u) - 2S)^{1/2} L(u)^{1/2} + \frac{1}{q}B(u)}{\frac{1}{p}A(u)}, \quad u \in \Sigma_S,$$

which is characterized by the fact that $\mu^S(u) = \mu \Leftrightarrow \lambda_\mu^S(u) = 0$, and $\mu^S(u) < \mu \Leftrightarrow \lambda_\mu^S(u) > 0$. For $S \in \mathbb{R}$, define

$$(2.6) \quad \hat{\mu}^S := \inf_{u \in \Sigma_S} \mu^S(u).$$

Obviously, $\hat{\mu}^S \geq 0$.

Proposition 2.5. *Let $N = 2$, $2 < p < q < \infty$, $S \in \mathbb{R}$. Then $\hat{\mu}^S$ is an extreme value of μ , that is for any $\mu > \hat{\mu}^S$ condition (E_μ) holds, whereas if $\mu \leq \hat{\mu}^S$, then $\lambda^S(u) \leq 0$ for any $u \in \Sigma_S$.*

Proof. Let $\mu > \hat{\mu}^S$. Then by (2.6), there exists $u \in \Sigma_S$ such that $\hat{\mu}^S < \mu^S(u) < \mu$, and therefore, $\lambda_\mu^S(u) > 0$. The converse follows immediately. \square

Observe that (2.6) can be written in an equivalent form. Indeed, consider for $S \in \mathbb{R}$,

$$(2.7) \quad M^S(u) := \frac{-S + \frac{1}{2}T(u) + \frac{1}{2}L(u) + \frac{1}{q}B(u)}{\frac{1}{p}A(u)}, \quad u \in \Sigma_S.$$

Observe that $M^S(u_\sigma) \rightarrow +\infty$ as $\sigma \rightarrow 0$ and $M^S(u_\sigma) \rightarrow +\infty$ as $\sigma \rightarrow +\infty$. An analysis similar to that for $\Lambda_\mu^S(u)$ shows that

$$(2.8) \quad \min_{\sigma > 0} M^S(u_\sigma) = \frac{(T(u) - 2S)^{1/2} L(u)^{1/2} + \frac{1}{q}B(u)}{\frac{1}{p}A(u)} = \mu^S(u).$$

Hence

$$\hat{\mu}^S = \inf_{u \in \Sigma_S} M^S(u).$$

Proposition 2.6. *Assume that $S \leq 0$. Then $\hat{\mu}^S > 0$.*

Proof. Since for $S \leq 0$, $M^S(u) \geq M^0(u)$,

$$(2.9) \quad \hat{\mu}^S \geq \inf_{u \in \Sigma \setminus \{0\}} M^0(u), \quad \forall u \in \Sigma \setminus \{0\}, \forall S \leq 0.$$

It is easily seen that for any $u \in \Sigma \setminus \{0\}$ the minimum of $\min_{t>0} M^0(tu)$ is attained at the unique minimizing point $t_m(u)$ given by

$$t_m(u) = c_{p,q} \left(\frac{T(u)^{1/2} L(u)^{1/2}}{B(u)} \right)^{1/(q-2)}, \quad c_{p,q} = \left(\frac{(p-2)q}{q-2} \right)^{1/(q-2)}.$$

Hence by (2.9) we have

$$\hat{\mu}^S \geq \inf_{u \in \Sigma \setminus \{0\}} M^0(t_m(u)u) = C_{p,q} \inf_{u \in \Sigma \setminus \{0\}} \frac{(T(u)L(u))^{\frac{q-p}{2(q-2)}} B(u)^{\frac{p-2}{(q-2)}}}{A(u)}, \quad \forall S \leq 0.$$

for

$$C_{p,q} = p(p+q-4) \left(\frac{q-2}{(p-2)q} \right)^{(p-2)/(q-2)}.$$

By (1.2), Sobolev and Hölder inequalities, we have

$$\|u\|_{L^p} \leq C_0 (\|xu\|_{L^2} \|\nabla u\|_{L^2})^{\frac{(q-p)}{p(q-2)}} \|u\|_{L^q}^{\frac{q(p-2)}{p(q-2)}}, \quad \forall u \in \Sigma \setminus \{0\},$$

where $C_0 > 0$ is a constant, and thus, $\hat{\mu}^S > 0$ for any $S \leq 0$. \square

3. EXISTENCE OF A FUNDAMENTAL FREQUENCY SOLUTION WITH PRESCRIBED ACTION

Recall that $N = 2$ and $2 < p < q < \infty$. Consider

$$(3.1) \quad \hat{\lambda}_\mu^S := \sup \{ \lambda_\mu^S(u); u \in \Sigma \setminus \{0\}, T(u) - 2S \geq 0 \},$$

with $\mu \geq \hat{\mu}^S$. By construction, we may also write

$$(3.2) \quad \hat{\lambda}_\mu^S = \hat{\Lambda}_\mu^S := \sup \{ \Lambda_\mu^S(u); u \in \Sigma \setminus \{0\}, T(u) - 2S \geq 0 \}.$$

In the sequel, we use (3.1) when we want to take advantage of the invariance of λ_μ^S with respect to spatial dilations, $\lambda_\mu^S(u_\sigma) = \lambda_\mu^S(u)$ for all $\sigma > 0$.

Lemma 3.1. *Let $S \in \mathbb{R}$. If $\mu \geq \hat{\mu}^S$, then $0 \leq \hat{\lambda}_\mu^S < \infty$. Moreover, if $\mu > \hat{\mu}^S$, then $0 < \hat{\lambda}_\mu^S < \infty$.*

Proof. By Proposition 2.5, $\hat{\lambda}_\mu^S \geq 0$. Hölder inequality yields

$$(3.3) \quad \|u\|_{L^p} \leq \|u\|_{L^2}^{1-\theta} \|u\|_{L^q}^\theta, \quad \forall u \in H^1(\mathbb{R}^2), \quad \theta = \frac{q(p-2)}{p(q-2)} \in (0, 1),$$

where we have used Sobolev embedding to ensure the finiteness of each term.

Suppose that $\hat{\lambda}_\mu^S = \infty$: there exists a sequence $(u_n)_n$ in $\Sigma \setminus \{0\}$, with $T(u_n) \geq 2S$, such that

$$\lambda_\mu^S(u_n) \xrightarrow{n \rightarrow \infty} \infty.$$

In particular,

$$(3.4) \quad \lambda_\mu^S(u_n) \leq \frac{1}{\|u_n\|_{L^2}^2} \left(\frac{\mu}{p} \|u_n\|_{L^p}^p - \frac{1}{q} \|u_n\|_{L^q}^q \right) \xrightarrow{n \rightarrow \infty} \infty.$$

Since $\lambda_\mu^S(u) = \lambda_\mu^S(u_\sigma)$ for any dilation parameter $\sigma > 0$, we may assume $\|u_n\|_{L^2}^2 = 1$ for all n . Then (3.3) and (3.4) yield

$$\lambda_\mu^S(u_n) \leq \frac{\mu}{p} \|u_n\|_{L^p}^p - \frac{1}{q} \|u_n\|_{L^q}^q \leq C_1 \|u_n\|_{L^q}^{\theta q} - \frac{1}{q} \|u_n\|_{L^q}^q \xrightarrow{n \rightarrow \infty} \infty,$$

which is impossible, and so $\lambda_\mu^S < \infty$.

It follows from Proposition 2.5 that for all $\mu > \hat{\mu}^S$, $\hat{\lambda}_\mu^S > 0$. \square

Lemma 3.2. *Let $S \in \mathbb{R}$. If $0 < \hat{\lambda}_\mu^S < \infty$, then there exists a maximizer \hat{u}_μ^S of (3.1), that is, $\hat{\lambda}_\mu^S = \lambda_\mu^S(\hat{u}_\mu^S)$.*

Proof. Consider a maximizing sequence $(u_n)_n$ of (3.1), $\lambda_\mu^S(u_n) \rightarrow \hat{\lambda}_\mu^S$ as $n \rightarrow +\infty$. We show that $(u_n)_n$ is bounded in Σ .

We first assume that $(u_n)_n$ is bounded in L^2 , $\|u_n\|_{L^2} \lesssim 1$ (and then show that this is necessarily the case). Suppose that $\|u_n\|_{L^q} \rightarrow +\infty$. Then, in view of (3.3),

$$\begin{aligned} -\lambda_\mu^S(u_n)Q(u_n) &= (T(u_n) - 2S)^{1/2}L(u_n)^{1/2} - \mu \frac{1}{p}A(u_n) + \frac{1}{q}B(u_n) \\ &\geq -\mu \frac{1}{p} \int |u_n|^p + \frac{1}{q} \int |u_n|^q \\ &\geq -\mu \frac{1}{p} \|u_n\|_{L^q}^{\theta p} + \frac{1}{q} \|u_n\|_{L^q}^q \rightarrow +\infty, \text{ as } \|u_n\|_{L^q} \rightarrow +\infty, \end{aligned}$$

which is absurd, since it would imply that $\lambda_\mu^S(u_n)$ becomes negative for large n . Hence by (3.3), $\|u_n\|_{L^p}$ and $\|u_n\|_{L^q}$ are bounded. In view of (3.3),

$$\lambda_\mu^S(u_n) \leq C \left(\frac{\|u_n\|_{L^q}^q}{\|u_n\|_{L^2}^2} \right)^{\frac{p-2}{q-2}} - \frac{1}{q} \frac{\|u_n\|_{L^q}^q}{\|u_n\|_{L^2}^2}.$$

This shows that $\|u_n\|_{L^q}^q / \|u_n\|_{L^2}^2$ is bounded: if we had $\|u_n\|_{L^2} \rightarrow 0$, then we would have $\|u_n\|_{L^q} \rightarrow 0$, hence $\|u_n\|_{L^p} \rightarrow 0$. Therefore,

$$\hat{\lambda}_\mu^S = \lim_{n \rightarrow +\infty} \frac{-1}{Q(u_n)} \left((T(u_n) - 2S)^{1/2}L(u_n)^{1/2} - \mu \frac{1}{p} \|u_n\|_{L^p}^p + \frac{1}{q} \|u_n\|_{L^q}^q \right) \leq 0,$$

in contradiction with Lemma 3.1, so $\|u_n\|_{L^2} \gtrsim 1$. We readily infer that $(T(u_n) - 2S)^{1/2}L(u_n)^{1/2}$ is bounded. Recall the inequality, found in e.g. [3, Lemma 4.1], in the two-dimensional case,

$$(3.5) \quad \|f\|_{L^2(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}^\theta \|xf\|_{L^2(\mathbb{R}^2)}^{1-\theta}, \quad \theta = \frac{p'}{2} \in (0, 1).$$

This implies that we cannot have $L(u_n) \rightarrow 0$, and so $T(u_n)$ is bounded: $(u_n)_n$ is bounded in $H^1(\mathbb{R}^2)$. If we had $L(u_n) \rightarrow \infty$, then consider

$$v_n(x) := u_n \left(xL(u_n)^{1/4} \right).$$

By the homogeneity of λ_μ^S with respect to dilations, $\lambda_\mu^S(v_n) = \lambda_\mu^S(u_n)$, and v_n is a maximizing sequence. It is such that $T(v_n) = T(u_n)$ is bounded, $L(v_n) = 1$, and

$$\|v_n\|_{L^2}^2 = L(u_n)^{-1/2} \|u_n\|_{L^2}^2 \xrightarrow{n \rightarrow \infty} 0,$$

while we have seen that this is impossible for a maximizing sequence. Therefore, $(u_n)_n$ is bounded in Σ . By the Banach–Alaoglu and Sobolev embedding theorems, there exists a subsequence, which we again denote by $(u_n)_n$, such that

$$\begin{aligned} u_n &\rightharpoonup \hat{u}_\mu^S \text{ in } \Sigma \\ u_n &\rightarrow \hat{u}_\mu^S \text{ in } L^\gamma(\mathbb{R}^2), \quad 2 \leq \gamma < \infty, \\ u_n &\rightarrow \hat{u}_\mu^S \text{ a.e. on } \mathbb{R}^2, \end{aligned}$$

for some $\hat{u}_\mu^S \in \Sigma$, as Σ is compactly embedded into $L^\gamma(\mathbb{R}^2)$ for such values of γ (see e.g. [39, Theorem XIII.67]). We show that $\hat{u}_\mu^S \neq 0$. To this end, it is sufficient to show that the sequence $\|u_n\|_{L^p}^p$ is separated from zero. Indeed, if $\|u_n\|_{L^p}^p \rightarrow 0$ as $n \rightarrow +\infty$, then

$$\hat{\lambda}_\mu^S = \lim_{n \rightarrow +\infty} \frac{-1}{Q(u_n)} \left((T(u_n) - 2S)^{1/2} L(u_n)^{1/2} - \mu \frac{1}{p} \|u_n\|_{L^p}^p + \frac{1}{q} \|u_n\|_{L^q}^q \right) \leq 0.$$

This is impossible, since $\hat{\lambda}_\mu^S > 0$. From this we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|u_n\|_{L^p}^p &= \|\hat{u}_\mu^S\|_{L^p}^p, \\ \lim_{n \rightarrow +\infty} \|u_n\|_{L^q}^q &= \|\hat{u}_\mu^S\|_{L^q}^q, \\ \lim_{n \rightarrow +\infty} \|u_n\|_{L^2}^2 &= \|\hat{u}_\mu^S\|_{L^2}^2. \end{aligned}$$

Recall the Brézis-Lieb lemma (see e.g. [35]):

Lemma 3.3. *Assume (v_n) is bounded in $L^\gamma(\mathbb{R}^N)$, $1 \leq \gamma < +\infty$ and $v_n \rightarrow v$ a. e. on \mathbb{R}^N , then*

$$\lim_{n \rightarrow +\infty} \|v_n\|_{L^\gamma}^\gamma = \|v\|_{L^\gamma}^\gamma + \lim_{n \rightarrow +\infty} \|v_n - v\|_{L^\gamma}^\gamma.$$

We infer

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|\nabla u_n\|_{L^2}^2 &= \|\nabla \hat{u}_\mu^S\|_{L^2}^2 + \lim_{n \rightarrow +\infty} \|\nabla(u_n - \hat{u}_\mu^S)\|_{L^2}^2 \geq \|\nabla \hat{u}_\mu^S\|_{L^2}^2, \\ \bar{L} := \lim_{n \rightarrow +\infty} \|xu_n\|_{L^2}^2 &= \|x\hat{u}_\mu^S\|_{L^2}^2 + \lim_{n \rightarrow +\infty} \|x(u_n - \hat{u}_\mu^S)\|_{L^2}^2 \geq \|x\hat{u}_\mu^S\|_{L^2}^2. \end{aligned}$$

In the case $\|\nabla \hat{u}_\mu^S\|_{L^2}^2 \geq 2S$,

$$\begin{aligned} \hat{\lambda}_\mu^S &= \lim_{n \rightarrow +\infty} \lambda_\mu^S(u_n) \leq \\ &\frac{-1}{Q(\hat{u}_\mu^S)} \left((T(\hat{u}_\mu^S) - 2S)^{1/2} L(\hat{u}_\mu^S)^{1/2} - \mu \frac{1}{p} \|\hat{u}_\mu^S\|_{L^p}^p + \frac{1}{q} \|\hat{u}_\mu^S\|_{L^q}^q \right) = \lambda_\mu^S(\hat{u}_\mu^S), \end{aligned}$$

which implies that \hat{u}_μ^S is a maximizer of (3.1).

Suppose, on the contrary, that $0 < \|\nabla \hat{u}_\mu^S\|_{L^2}^2 < 2S$ (which might occur only if $S > 0$). Then

$$\begin{aligned} \hat{\lambda}_\mu^S &= \lim_{n \rightarrow +\infty} \lambda_\mu^S(u_n) < \\ &\frac{-\left((T(\hat{u}_\mu^S) + \lim_{n \rightarrow +\infty} \|\nabla(u_n - \hat{u}_\mu^S)\|_{L^2}^2 - 2S)^{1/2} L(\hat{u}_\mu^S)^{1/2} - \mu \frac{1}{p} \|\hat{u}_\mu^S\|_{L^p}^p + \frac{1}{q} \|\hat{u}_\mu^S\|_{L^q}^q \right)}{Q(\hat{u}_\mu^S)}. \end{aligned}$$

The last term is equal to $\lambda_\mu^{S-h}(\hat{u}_\mu^S)$, where $2h = \lim_{n \rightarrow +\infty} \|\nabla(u_n - \hat{u}_\mu^S)\|_{L^2}^2$. Notice that if $S_1 < S_2$, then $\lambda_\mu^{S_1}(w) \leq \lambda_\mu^{S_2}(w)$ for any $w \in \Sigma_{S_2}$, and consequently $\hat{\lambda}_\mu^{S_1} \leq \hat{\lambda}_\mu^{S_2}$. This implies the contradiction

$$\hat{\lambda}_\mu^S < \lambda_\mu^{S-h}(\hat{u}_\mu^S) \leq \hat{\lambda}_\mu^S,$$

and we necessarily have $\|\nabla \hat{u}_\mu^S\|_{L^2}^2 \geq 2S$.

Now we prove that any maximizing sequence is indeed bounded in L^2 . If we had $\|u_n\|_{L^2} \rightarrow \infty$, then by the positivity of $\hat{\lambda}_\mu^S$,

$$\|u_n\|_{L^p}^p \gtrsim \|u_n\|_{L^2}^2,$$

and together with (3.3), raised to the power p , this yields

$$\|u_n\|_{L^2}^{2\frac{q-p}{q-2}} \|u_n\|_{L^q}^{q\frac{p-2}{q-2}} \gtrsim \|u_n\|_{L^2}^2, \quad \text{hence} \quad \|u_n\|_{L^q}^q \gtrsim \|u_n\|_{L^2}^2 \xrightarrow[n \rightarrow \infty]{} \infty.$$

Again due to the positivity of $\hat{\lambda}_\mu^S$, for sufficiently large n ,

$$\|u_n\|_{L^p}^p \gtrsim \|u_n\|_{L^q}^q,$$

and (3.3) now yields

$$\|u_n\|_{L^2}^2 \gtrsim \|u_n\|_{L^q}^q,$$

so we come up with

$$\|u_n\|_{L^2}^2 \approx \|u_n\|_{L^p}^p \approx \|u_n\|_{L^q}^q.$$

Gagliardo-Nirenberg inequality entails $T(u_n) \rightarrow \infty$. Consider the dilation,

$$v_n(x) = u_n \left(\frac{x}{\|u_n\|_{L^2}} \right),$$

and recall that $\lambda_\mu^S(u_n) = \lambda_\mu^S(v_n)$. The sequence $(v_n)_n$ is bounded in $L^2 \cap L^q$: from the previous arguments (where we had considered a maximizing sequence, bounded in L^2), we see that $(v_n)_n$ is bounded in H^1 . But $T(u_n) = T(v_n) \rightarrow \infty$, hence a contradiction. \square

From Lemmas 3.1 and 3.2, we get:

Lemma 3.4. *Let $S \in \mathbb{R}$. If $\mu > \hat{\mu}^S$, then there exists a maximizer \hat{u}_μ^S of (3.1), that is, $\hat{\lambda}_\mu^S = \lambda_\mu^S(\hat{u}_\mu^S)$.*

Remark 3.5. For the existence of a maximizer of (3.1) in the case $\mu = \hat{\mu}^S$ see Lemma 5.2 below.

Lemma 3.6. *Assume that \hat{u}_μ^S is a maximizer of (3.2), i.e., $\hat{\lambda}_\mu^S = \Lambda_\mu^S(\hat{u}_\mu^S)$. Then \hat{u}_μ^S is a fundamental frequency solution of (1.3) with fundamental frequency $\hat{\lambda}_\mu^S$.*

Proof. Let \hat{u} be a maximizer of (3.2). By the Lagrange multiple rule, there exist μ_0, μ_1 such that

- (1) $|\mu_0| + |\mu_1| \neq 0$,
- (2) $\mu_0 D\Lambda_\mu^S(\hat{u}) + \mu_1 DT(\hat{u}) = 0$,
- (3) $\mu_1(T(\hat{u}) - 2S) = 0, \mu_0 \geq 0, \mu_1 \leq 0$.

Evidently, we may assume that $\mu_0 = 1$. If $T(\hat{u}) - 2S > 0$, then by (3), we have $\mu_1 = 0$, and consequently, $D\Lambda_\mu^S(\hat{u}) = 0$, and thus \hat{u} satisfies (1.3).

To obtain a contradiction, suppose that $T(\hat{u}) - 2S = 0$. By Corollary 2.1 we have

$$\frac{d\Lambda_\mu^S(\hat{u}_\sigma)}{d\sigma}\Big|_{\sigma=1} + \mu_1 \frac{dT(\hat{u}_\sigma)}{d\sigma}\Big|_{\sigma=1} = 0.$$

But $\frac{d\Lambda_\mu^S(\hat{u}_\sigma)}{d\sigma}\Big|_{\sigma=1} < 0$, $\frac{dT(\hat{u}_\sigma)}{d\sigma} = 0$. Thus, $T(\hat{u}) - 2S > 0$ and \hat{u} satisfies (1.3). The equality $\hat{\lambda}_\mu^S = \Lambda_\mu^S(\hat{u})$ entails that it is a fundamental frequency solution. \square

We can now infer:

Corollary 3.7. *The maximizers \hat{u}_μ^S of (3.2) satisfy $T(\hat{u}_\mu^S) - 2S > 0$. Up to a gauge transform, \hat{u}_μ^S is radially symmetric, nonnegative, and a non-increasing function of $|x|$.*

Proof. We explain the last point of the statement. In view of [18, Lemma 2.3], up to replacing \hat{u}_μ^S by $e^{i\theta}\hat{u}_\mu^S$ for some $\theta \in \mathbb{R}$, we may assume that \hat{u}_μ^S is real-valued. Using Schwarz symmetrization (see e.g. [35, Chapter 3]), we see that maximizers can be chosen as radially symmetric, nonnegative, and non-increasing as a function of $|x|$. It is indeed standard that symmetric rearrangements leave Lebesgue norms unchanged, and do not increase $T(u)$. They do not increase $L(u)$ either, as shown in [5, Appendix A], and more precisely, in the present case of a harmonic potential $|x|^2$, [5, Theorem 4] asserts that if u is not equal to its Schwarz symmetrization u^* , then $L(u^*) < L(u)$. Since maximizers satisfy $T(\hat{u}_\mu^S) > 2S$, we see that if they do not satisfy the announced properties, then considering their Schwarz symmetrization (strictly) increases the value of λ_μ^S , hence a contradiction. \square

4. LYAPUNOV STABILITY OF THE SET OF FUNDAMENTAL FREQUENCY SOLUTIONS

We first show why local Σ solutions of (1.1) are global. In view of e.g. [15, Theorem 9.2.6], it suffices to prove that any solutions is bounded in Σ . For the following result, we consider the general case $N \geq 1$, with combined energy-subcritical nonlinearities.

Lemma 4.1. *Let $N \geq 1$. Assume $2 < p < q < 2^*$ and $\mu > 0$. Let $\psi \in C([0, T]; \Sigma) \cap C^1([0, T]; \Sigma^*)$ be a solution of (1.1). Then ψ is bounded in Σ : there exists C depending on $\|\psi|_{t=0}\|_\Sigma$ such that for all $t \in [0, T]$, $\|\nabla\psi(t)\|_{L^2}^2 + \|x\psi(t)\|_{L^2}^2 \leq C$, and therefore we can take $T = \infty$.*

Proof. The conservations for (1.1) yield $Q(\psi(t)) = Q(\psi(0))$, $H_\mu(\psi(t)) = H_\mu(\psi(0))$ for all $t \in [0, T]$. Recall that from (3.3), there exists $\theta \in]0, 1[$ such that

$$\|\psi(t)\|_{L^p} \leq \|\psi(t)\|_{L^2}^{1-\theta} \|\psi(t)\|_{L^q}^\theta \lesssim \|\psi(t)\|_{L^q}^\theta,$$

since the L^2 -norm of ψ is independent of time. We infer

$$H_\mu(\psi(0)) = H_\mu(\psi(t)) \geq \frac{1}{2} \|\nabla\psi(t)\|_{L^2}^2 + \frac{1}{2} \|x\psi(t)\|_{L^2}^2 - C \|\psi(t)\|_{L^q}^{\theta p} + \frac{1}{q} \|\psi(t)\|_{L^q}^q.$$

Now from (3.3), $\theta p < q$: this entails that all terms in the above inequality are bounded. \square

We now prove (recall that \mathcal{F}_μ^S is defined in Definition 1.4):

Lemma 4.2. *Let $N = 2$, $2 < p < q < \infty$, $S \leq 0$, and $\mu > 0$ such that $\mathcal{F}_\mu^S \neq \emptyset$. Then the set of fundamental frequency solutions \mathcal{F}_μ^S is orbitally stable.*

Proof. Suppose that \mathcal{F}_μ^S is not orbitally stable: there exist $\varepsilon > 0$, a sequence of initial data $\psi_n(0) \in \Sigma$ such that

$$(4.1) \quad \inf_{\phi \in \mathcal{F}_\mu^S} \|\psi_n(0) - \phi\|_\Sigma \xrightarrow{n \rightarrow \infty} 0,$$

and a sequence of times $t_n \geq 0$ along which the solution to (1.1) emanating from $\psi_n(0)$ satisfies

$$(4.2) \quad \inf_{\phi \in \mathcal{F}_\mu^S} \|\psi_n(t_n) - \phi\|_\Sigma > \varepsilon, \quad \forall n \geq 1.$$

The convergence (4.1) implies that

$$\Lambda_\mu^S(\psi_n(0)) \xrightarrow{n \rightarrow \infty} \hat{\lambda}_\mu^S,$$

and therefore the conservation for Λ_μ^S yields

$$(4.3) \quad \Lambda_\mu^S(\psi_n(t_n)) \xrightarrow{n \rightarrow \infty} \hat{\lambda}_\mu^S.$$

Lemma 4.1 shows that $(\psi_n(t_n))_n$ is bounded in Σ . Consequently, there exists a subsequence which we again denote by $(\psi_n(t_n))_n$ such that

$$\psi_n(t_n) \rightharpoonup u \text{ weakly in } \Sigma, \quad \psi_n(t_n) \rightarrow u, \text{ strongly in } L^p(\mathbb{R}^2), \quad 2 \leq p < \infty.$$

for some $u \in \Sigma$. Observe that $T(u) - 2S \geq 0$ since $S \leq 0$. This and (4.3) imply that we can proceed like in the proof of Lemma 3.1, to obtain that we actually have $\psi_n(t_n) \rightarrow u$ strongly in Σ , and

$$\lambda_\mu^S(\psi_n(t_n)) \xrightarrow{n \rightarrow \infty} \hat{\lambda}_\mu^S.$$

Hence we get $\Lambda_\mu^S(u) = \hat{\lambda}_\mu^S$, that is $u \in \mathcal{F}_\mu^S$. But this contradicts (4.2). \square

5. EXISTENCE OF A MINIMIZER OF $\mu^S(u)$

Assume that $S \leq 0$. Notice that in this case we always have $\Sigma_S = \Sigma \setminus \{0\}$. Consider the minimization problem (2.6), i.e.

$$\hat{\mu}^S := \inf_{u \in \Sigma_S} \mu^S(u).$$

Lemma 5.1. *If $S \leq 0$, then there exists a minimizer \bar{v}^S of (2.6).*

Proof. By Proposition 2.6, $0 < \hat{\mu}_S < \infty$. Let $(v_n)_n$ be a minimizing sequence of (2.6), i.e., $\mu^S(v_n) \rightarrow \hat{\mu}_S$ as $n \rightarrow \infty$ and $v_n \in \Sigma_S$, $n \geq 1$. We show that $(v_n)_n$ is bounded in Σ .

Since $\mu^S(v) = \mu^S(v_\sigma)$ for any dilation parameter $\sigma > 0$, we may assume $L(v_n) = 1$. Since $\mu^S(v_n)$ is bounded, we readily have

$$(5.1) \quad 0 \leq \|\nabla v_n\|_{L^2}^2 - 2S \lesssim \|v_n\|_{L^p}^p,$$

$$(5.2) \quad \|v_n\|_{L^q}^q \lesssim \|v_n\|_{L^p}^p.$$

In view of (3.3) and (3.5),

$$\|v_n\|_{L^p}^p \lesssim \left(\|v_n\|_{L^p}^{p'} \right)^{\frac{q-p}{q-2}} \left(\|v_n\|_{L^q}^q \right)^{\frac{p-2}{q-2}},$$

hence

$$(\|v_n\|_{L^p}^p)^{\frac{q-1}{p-1}} \lesssim \|v_n\|_{L^q}^q \lesssim \|v_n\|_{L^p}^p,$$

where we have used (5.2) for the last inequality. As $2 < p < q$, we infer that $(v_n)_n$ is bounded in $L^p(\mathbb{R}^2)$, and in $L^q(\mathbb{R}^2)$, again from (5.2). Now, (3.5) implies that $(v_n)_n$ is bounded in $L^2(\mathbb{R}^2)$, and (5.1) that $(\nabla v_n)_n$ is bounded in $L^2(\mathbb{R}^2)$. Therefore, $(v_n)_n$ is bounded in Σ .

Like in the proof of Lemma 3.2, there exists a subsequence, which we again denote by $(v_n)_n$, and $\bar{v}^S \in \Sigma$, such that

$$\begin{aligned} v_n &\rightharpoonup \bar{v}^S \quad \text{in } \Sigma \\ v_n &\rightarrow \bar{v}^S \quad \text{in } L^\gamma(\mathbb{R}^2), \quad 2 \leq \gamma < \infty, \\ v_n &\rightarrow \bar{v}^S \quad \text{a.e. on } \mathbb{R}^2. \end{aligned}$$

Observe that $\bar{v}^S \neq 0$. Indeed, by the proof of Proposition 2.6, we have

$$(5.3) \quad \mu^S(v_n) \geq \mu^0(t_m(v_n)v_n) = C_{p,q} \frac{(T(v_n)L(v_n))^{\frac{q-p}{2(q-2)}} B(v_n)^{\frac{p-2}{q-2}}}{A(v_n)}.$$

Since $\mu^0(t_m(su_\sigma)su_\sigma) = \mu^0(t_m(u)u)$ for any $\sigma > 0$, $s > 0$, $u \in \Sigma \setminus \{0\}$, we may assume that $T(v_n)L(v_n) = 1$, $B(v_n) = 1$, for all n . Hence if $\bar{v}^S = 0$, we get a contradiction: $\mu^S(v_n) \rightarrow +\infty$.

Now arguing as in the proof of Lemma 3.2 we obtain

$$\begin{aligned} \hat{\mu}^S &= \lim_{n \rightarrow +\infty} \mu^S(v_n) \geq \\ &\frac{1}{A(\bar{v}^S)} \left((T(\bar{v}^S) - 2S)^{1/2} L(\bar{v}^S)^{1/2} + \frac{1}{q} B(\bar{v}^S) \right) = \mu^S(\bar{v}^S), \end{aligned}$$

which implies that \bar{v}^S is a minimizer of (2.6). \square

Observing that since $\mu^S(u)$ is a homogeneous functional with respect to spatial dilations, we may always assume that the minimizer \bar{v}^S of (2.6) satisfies $\sigma_S(\bar{v}^S) = 1$.

Lemma 5.2. *Let $S \leq 0$ and \bar{v}^S be a minimizer (2.6) such that $\sigma_S(\bar{v}^S) = 1$. Then*

- (i): \bar{v}^S weakly satisfies (1.3) with $\lambda = 0$ and $\mu = \hat{\mu}^S$;
- (ii): maximizing problem (3.1) for $\mu = \hat{\mu}^S$ attains its maximizer at \bar{v}^S , i.e., $\hat{u}_{\hat{\mu}^S}^S = \bar{v}^S$, and $\hat{\lambda}_{\hat{\mu}^S}^S = \lambda_{\hat{\mu}^S}^S(\bar{v}^S) = 0$.

Proof. If \bar{v}^S is a minimizer of (2.6), then $D\mu^S(\bar{v}^S) = 0$. The assumption $\sigma_S(\bar{v}^S) = 1$ implies that $\hat{\mu}^S = \mu^S(\bar{v}^S) = M^S(\bar{v}^S)$. Using this it is easy to check by direct calculations of $D\mu^S(\bar{v}^S)$ that if $\mu^S(u) = \mu$, $\sigma_S(u) = 1$ for $u \in \Sigma_S$, then

$$D\mu^S(u) = \frac{p}{A(u)} DS_{\lambda,\mu}(u)|_{\lambda=0},$$

and thus, we get $DS_{0,\hat{\mu}^S}(\bar{v}^S) = 0$, that is \bar{v}^S weakly satisfies (1.3) with $\lambda = 0$ and $\mu = \hat{\mu}^S$. \square

Proposition 5.3. *The function $(-\infty, 0] \ni S \mapsto \hat{\mu}^S$ is continuous and monotone decreasing. Moreover, $\hat{\mu}^S \rightarrow +\infty$ as $S \rightarrow -\infty$ and $\hat{\mu}^S \rightarrow \hat{\mu}^0 := \hat{\mu}^S|_{S=0} > 0$ as $S \rightarrow 0$.*

Proof. Let $S < 0$, $\Delta S > 0$ such that $S + \Delta S \leq 0$. Proceeding like in the proof of Lemma 5.1, we see that $(\bar{v}^S)_{S \in (a, 0]}$ is bounded in Σ for any bounded $a < 0$. Using this it is not hard to show that the following Taylor expansion holds

$$(T(\bar{v}^S) - 2(S + \Delta S))^{1/2} = (T(\bar{v}^S) - 2S)^{1/2} - \Delta S (T(\bar{v}^S) - 2S)^{-1/2} + o(S, \Delta S),$$

where $o(S, \Delta S)/|\Delta S| \rightarrow 0$ as $\Delta S \rightarrow 0$ uniformly in $S \in (a, 0]$. Hence,

$$\hat{\mu}^{S+\Delta S} \leq \mu^{S+\Delta S}(\bar{v}^S) = \hat{\mu}^S - \Delta S \frac{p}{A(\bar{v}^S)} (T(\bar{v}^S) - 2S)^{-1/2} + o(\Delta S),$$

and consequently,

$$\hat{\mu}^{S+\Delta S} - \hat{\mu}^S \leq -\Delta S \frac{p}{A(\bar{v}^S)} (T(\bar{v}^S) - 2S)^{-1/2} + o(S, \Delta S).$$

Similarly we have

$$-\Delta S \frac{p}{A(\bar{v}^{S+\Delta S})} (T(\bar{v}^{S+\Delta S}) - 2(S + \Delta S))^{-1/2} + o(S + \Delta S, \Delta S) \leq \hat{\mu}^{S+\Delta S} - \hat{\mu}^S.$$

Thus, $\hat{\mu}^{S+\Delta S} - \hat{\mu}^S \rightarrow 0$ as $\Delta S \rightarrow 0$.

Observe $\hat{\mu}^{S_2} = \mu^{S_2}(\bar{v}^{S_2}) < \mu^{S_2}(\bar{v}^{S_1}) < \mu^{S_1}(\bar{v}^{S_1}) = \hat{\mu}^{S_1}$ for any $S_1 < S_2 \leq 0$. Thus, $\hat{\mu}^S$ is monotone decreasing. Moreover, it is clear that $\hat{\mu}^S \rightarrow \hat{\mu}^0 := \hat{\mu}^S|_{S=0}$ as $S \rightarrow 0$. By Proposition 2.6, $\hat{\mu}^S|_{S=0} > 0$.

Let us show that $\hat{\mu}^S \rightarrow +\infty$ as $S \rightarrow -\infty$. From the monotonicity of $\mu^S(\bar{v}^S)$ it follows $\mu^S(\bar{v}^S) \rightarrow C$ as $S \rightarrow -\infty$ for some $C \in (0, +\infty]$. Suppose, contrary to our claim, that $C < +\infty$. As above, we may assume $L(\bar{v}^S) = 1$, $S < 0$. Then

$$(5.4) \quad \mu^S(\bar{v}^S) := \frac{(T(\bar{v}^S) - 2S)^{1/2} + \frac{1}{q} \|\bar{v}^S\|_{L^q}^q}{\frac{1}{p} \|\bar{v}^S\|_{L^p}^p} \rightarrow C < +\infty \quad \text{as } S \rightarrow -\infty.$$

Then

$$(5.5) \quad \|\bar{v}^S\|_{L^q}^q \lesssim \|\bar{v}^S\|_{L^p}^p.$$

In view of (3.3) and (3.5),

$$\|\bar{v}^S\|_{L^p}^p \lesssim \left(\|\bar{v}^S\|_{L^p}^{p'} \right)^{\frac{q-p}{q-2}} \left(\|\bar{v}^S\|_{L^q}^q \right)^{\frac{p-2}{q-2}},$$

hence by (5.5),

$$\left(\|\bar{v}^S\|_{L^p}^p \right)^{\frac{q-1}{p-1}} \lesssim \|\bar{v}^S\|_{L^q}^q \lesssim \|\bar{v}^S\|_{L^p}^p.$$

Since $2 < p < q$, we infer that (\bar{v}^S) is bounded in $L^p(\mathbb{R}^2)$. However, $(T(\bar{v}^S) - 2S) \rightarrow +\infty$ as $S \rightarrow -\infty$ and thus, (5.4) implies a contradiction. \square

Proposition 5.3 yields that the range of the function $(-\infty, 0] \ni S \mapsto \hat{\mu}^S$ coincides with $[\hat{\mu}^0, +\infty)$ and thus we have:

Corollary 5.4. *There exists an inverse function $S(\mu)$ of $\hat{\mu}^S$ so that $\hat{\mu}^{S(\mu)} = \mu$ for any $\mu \in [\hat{\mu}^0, +\infty)$.*

6. EXISTENCE OF A GROUND STATE

Proposition 6.1. *Assume $\mu > 0$, $|S_2 - S_1|$ is sufficiently small, and $S_1 < S_2$ such that $\mathcal{F}_\mu^{S_1} \neq \emptyset$, $\mathcal{F}_\mu^{S_2} \neq \emptyset$. Then*

$$(6.1) \quad \frac{S_2 - S_1}{Q(\hat{u}_\mu^{S_1})} \leq \hat{\lambda}_\mu^{S_2} - \hat{\lambda}_\mu^{S_1} \leq \frac{S_2 - S_1}{Q(\hat{u}_\mu^{S_2})}, \quad \forall \hat{u}_\mu^{S_i} \in \mathcal{F}_\mu^{S_i}, \quad i = 1, 2.$$

Proof. Since $\mathcal{F}_\mu^{S_j} \neq \emptyset$, $\hat{\lambda}_\mu^{S_j} = \Lambda_\mu^{S_j}(\hat{u}_\mu^{S_j})$, $\forall \hat{u}_\mu^{S_j} \in \mathcal{F}_\mu^{S_j}$, $j = 1, 2$. Furthermore, $T(\hat{u}_\mu^{S_2}) > 2S_2 > 2S_1$. Hence,

$$\hat{\lambda}_\mu^{S_1} = \Lambda_\mu^{S_1}(\hat{u}_\mu^{S_1}) \geq \Lambda_\mu^{S_1}(\hat{u}_\mu^{S_2}) = \Lambda_\mu^{S_2}(\hat{u}_\mu^{S_2}) - \frac{S_2 - S_1}{Q(\hat{u}_\mu^{S_2})} = \hat{\lambda}_\mu^{S_2} - \frac{S_2 - S_1}{Q(\hat{u}_\mu^{S_2})},$$

and we get the second inequality in (6.1). The proof of the first one may be handled in the same way. We only have to notice that Corollary 3.7 implies that $T(\hat{u}_\mu^{S_1}) > 2S_2$ if $|S_2 - S_1|$ is sufficiently small. \square

Lemma 6.2. *Let $\mu > \hat{\mu}^0$. Then $\mathcal{F}_\mu^S \neq \emptyset$ for any $S \in (S(\mu), 0]$.*

Proof. Take $S \in (S(\mu), 0]$. From the above properties, $\hat{\mu}^S > 0$. Since $S > S(\mu)$, Proposition 5.3 implies $\mu = \hat{\mu}^{S(\mu)} > \hat{\mu}^S$. Hence by Lemma 3.4 we have $\mathcal{F}_\mu^S \neq \emptyset$. \square

Hence we have

Corollary 6.3. *For any $\mu > \hat{\mu}^0$, the function $S \mapsto \hat{\lambda}_\mu^S$ is continuous and strictly monotone increasing on $(S(\mu), 0]$, i.e., $\hat{\lambda}_\mu^{S_2} > \hat{\lambda}_\mu^{S_1}$, for any $S(\mu) < S_1 < S_2 \leq 0$. Furthermore, $Q(\hat{u}_\mu^{S_1}) > Q(\hat{u}_\mu^{S_2})$, $\forall \hat{u}_\mu^{S_j} \in \mathcal{F}_\mu^{S_j}$, $j = 1, 2$.*

Lemma 6.4. *Assume that $\mu > \hat{\mu}^0$. Suppose $S \in (S(\mu), 0]$ such that \hat{u}_μ^S is a fundamental frequency solution of (1.3) with $\lambda = \hat{\lambda}_\mu^S$, then $\hat{u}_{\lambda, \mu} := \hat{u}_\mu^S$ is a ground state of (1.3) with ground level S .*

Proof. Suppose the assertion of the lemma is false. Then there exists a solution w of (1.3) with $\lambda = \hat{\lambda}_\mu^S$ such that

$$S_1 := S_{\hat{\lambda}_\mu^S, \mu}(w) < S_{\hat{\lambda}_\mu^S, \mu}(\hat{u}_\mu^S) = S.$$

The equality $S_1 := S_{\hat{\lambda}_\mu^S, \mu}(w)$ implies $\Lambda_\mu^{S_1}(w) = \hat{\lambda}_\mu^S$. Moreover, since $D\Lambda_\mu^{S_1}(w) = 0$, Lemma 2.2 (see also Remark 2.3) implies that $T(w) - 2S_1 > 0$. Lemma 3.1 implies that $\hat{\lambda}_\mu^{S_1} < +\infty$. Furthermore, we have

$$(6.2) \quad \hat{\lambda}_\mu^{S_1} = \max_{u \in \Sigma_{S_1}} \Lambda_\mu^{S_1}(u) \geq \Lambda_\mu^{S_1}(w) = \hat{\lambda}_\mu^S > 0.$$

Thus $0 < \hat{\lambda}_\mu^{S_1} < +\infty$, and by Lemma 3.2, $\mathcal{F}_\mu^{S_1} \neq \emptyset$. Hence, the inequality $S > S_1$ by Proposition 6.1 implies that $\hat{\lambda}_\mu^{S_1} < \hat{\lambda}_\mu^S$, which contradicts (6.2). \square

Lemma 6.5. *Assume that $\mu > \hat{\mu}^0$. Then $\lim_{S \rightarrow S(\mu)} \hat{\lambda}_\mu^S = 0$ and $0 < \lim_{S \rightarrow 0} \hat{\lambda}_\mu^S = \hat{\lambda}_\mu^* < \infty$.*

Proof. Since $\hat{\mu}^{S(\mu)} = \mu$, Lemma 5.2 implies that $\lim_{S \rightarrow S(\mu)} \hat{\lambda}_\mu^S = \hat{\lambda}_\mu^{S(\mu)} = 0$. By the monotonicity of $\hat{\lambda}_\mu^S$ it follows that there exists $\lim_{S \rightarrow 0} \hat{\lambda}_\mu^S = \hat{\lambda}_\mu^* \leq \infty$. Suppose, contrary to our claim, that $\hat{\lambda}_\mu^* = \infty$. By (6.1) and since $Q(\hat{u}_\mu^{S_1}) > Q(\hat{u}_\mu^{S_2})$, for

$S(\mu) < S_1 < S_2 < 0$, this is possible only if $Q(\hat{u}_\mu^S) \rightarrow 0$ as $S \rightarrow 0$. An analysis similar to that in the proof of Lemma 5.1 shows that $(\hat{u}_\mu^S)_{S \in (\delta, 0)}$ is bounded in Σ for any $\delta \in (S(\mu), 0)$, and thus there exists a subsequence $(\hat{u}_\mu^{S_n})_n$, such that $S_n \rightarrow 0$ and $\hat{u}_\mu^{S_n} \rightharpoonup \bar{w}$ in Σ , $\hat{u}_\mu^{S_n} \rightarrow \bar{w}$ in $L^\gamma(\mathbb{R}^2)$, $2 \leq \gamma < \infty$, $\hat{u}_\mu^{S_n} \rightarrow \bar{w}$ a.e. on \mathbb{R}^2 , for some $\bar{w} \in \Sigma$. As in proof of Lemma 3.2, it follows that $\bar{w} \neq 0$. However, this contradicts the convergence $Q(\hat{u}_\mu^{S_n}) \rightarrow 0$. \square

Corollary 6.3 implies that the function $S \mapsto \hat{\lambda}_\mu^S$ is invertible so that for any $\lambda \in [0, \hat{\lambda}_\mu^*]$ there exists a unique $S_\lambda \in [S(\mu), 0]$ such that $\hat{\lambda}_\mu^{S_\lambda} = \lambda$. Hence, by Lemmas 6.4, 6.5 we have:

Corollary 6.6. *Assume that $\mu > \hat{\mu}^0$. Then for any $\lambda \in [0, \hat{\lambda}_\mu^*]$ equation (1.3) has a ground state \hat{u}_λ . Moreover, $S_{\lambda, \mu}(\hat{u}_\lambda) < 0$ for $\lambda \in [0, \hat{\lambda}_\mu^*)$ and $S_{\lambda, \mu}(\hat{u}_\lambda)|_{\lambda = \hat{\lambda}_\mu^*} = 0$.*

Lemma 6.7. *Assume that $\mu > \hat{\mu}^0$. Suppose $\lambda \in [0, \hat{\lambda}_\mu^*]$ and \hat{u}_λ is a ground state of (1.3) with the corresponding ground level $S = S_{\lambda, \mu}(\hat{u}_\lambda) \leq 0$. Then \hat{u}_λ is a fundamental frequency solution of (1.3) with fundamental frequency λ .*

Proof. Suppose that $\lambda \in [0, \hat{\lambda}_\mu^*]$ and \hat{u}_λ is a ground state of (1.3) with $S = S_{\lambda, \mu}(\hat{u}_\lambda)$. Then $\Lambda_\mu^S(\hat{u}_\lambda) = \lambda_\mu^S(\hat{u}_\lambda) = \lambda$, and thus $\lambda \leq \hat{\lambda}_\mu^S$.

Suppose, contrary to our claim, that $\lambda < \hat{\lambda}_\mu^S$. It follows from the above that there exists $S_\lambda \in (S(\mu), 0]$ and a fundamental frequency solution $\hat{u}_\mu^{S_\lambda}$ such that $\lambda = \hat{\lambda}_\mu^{S_\lambda} = \Lambda_\mu^{S_\lambda}(\hat{u}_\mu^{S_\lambda})$ and $S_{\lambda, \mu}(\hat{u}_\mu^{S_\lambda}) = S_\lambda$. Since $\hat{\lambda}_\mu^{S_\lambda} = \lambda < \hat{\lambda}_\mu^S$, Corollary 6.3 implies that $S_\lambda < S \leq 0$, which contradicts the assumption that \hat{u}_λ is a ground state. \square

7. PROOF OF THEOREMS 1.3 AND 1.5

Proof of Theorem 1.3. Let $N = 2$, $2 < p < q < \infty$ and $\mu > \hat{\mu}^0$. Take $\hat{\lambda}_\mu^* \in (0, +\infty)$ defined by Lemma 6.5.

Let $\lambda \in [0, \hat{\lambda}_\mu^*]$. Then Corollary 6.6 yields that Equation (1.3) has a ground state $\hat{u}_\lambda \in \Sigma$, and thus we have (1°). Moreover, $S_{\lambda, \mu}(\hat{u}_\lambda) < 0$ for $\lambda \in [0, \hat{\lambda}_\mu^*)$ and $S_{\lambda, \mu}(\hat{u}_\lambda)|_{\lambda = \hat{\lambda}_\mu^*} = 0$, and thus (2°) holds. Corollaries 3.7 and 6.6 yield (3°). Assertion (4°) follows from Lemma 4.2 and Corollary 6.6. \square

Proof of Theorem 1.5. Let $N = 2$, $2 < p < q < \infty$, $S \in \mathbb{R}$. Proposition 2.5 implies that the value $\hat{\mu}^S \geq 0$ given by (2.6) is an extremal value. By Lemmas 3.4 and 3.6, it follows that Equation (1.3) has a fundamental frequency solution $\hat{u}_\mu^S \in \Sigma \setminus \{0\}$ for any $\mu > \hat{\mu}^S$, and thus assertion (1°) is satisfied. Assertion (2°) follows from Corollary 3.7. Corollary 6.6 yields (3°). Since (1°), $\mathcal{F}_\mu^S \neq \emptyset$ for $S \leq 0$, $\mu > \hat{\mu}^S$, and thus, Lemma 4.2 yields that the set of fundamental frequency solutions \mathcal{F}_μ^S is orbitally stable. \square

APPENDIX A. PROOF OF PROPOSITION 1.2

We resume the approach introduced in [19] (see also [8]), and, more precisely, the proof of [10, Proposition 3.1]. In all cases, we consider the constraint $u \in B_p$, where

$$B_p := \left\{ \phi \in \Sigma, \quad \frac{1}{p} \|\phi\|_{L^p(\mathbb{R}^N)}^p = 1 \right\}.$$

For the homogeneous nonlinearity, we introduce the functional

$$F(u) := \frac{1}{2} \langle Hu, u \rangle + \frac{\lambda}{2} \|u\|_{L^2}^2, \quad H = -\Delta + |x|^2,$$

and

$$\delta = \inf_{u \in B_p} F(u).$$

We know the spectrum of H , $\sigma_p(H) = N + 2\mathbb{N}$, and that Hermite functions provide an L^2 -eigenbasis; see e.g. [33]. The uncertainty principle also reads

$$\langle H\phi, \phi \rangle \geq N \|\phi\|_{L^2}^2, \quad \forall \phi \in \Sigma.$$

First case: $\lambda > -N$. We show that $0 < \delta < \infty$. The finiteness is obvious, and the uncertainty principle implies, for all $u \in \Sigma$,

$$(A.1) \quad F(u) \geq \frac{1}{2} (N + \lambda) \|u\|_{L^2}^2 > 0.$$

If we had $\delta = 0$, then there would exist a minimizing sequence $u_n \in B_p$ such that $u_n \rightarrow 0$ in L^2 . The definition of F implies that $u_n \rightarrow 0$ in Σ too. Our assumption on p implies $\Sigma \subset H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, and thus u_n leaves B_p for n sufficiently large, leading to a contradiction: $\delta > 0$.

In view of (A.1), and minimizing sequence is bounded in L^2 , and again from the definition of F , it is bounded in Σ . The Banach–Alaoglu theorem implies that up to a subsequence, u_n converges weakly to some function $\tilde{u} \in \Sigma$. The presence of the harmonic potential makes the embedding $\Sigma \hookrightarrow L^r(\mathbb{R}^N)$ compact for all $2 \leq r < 2^*$ (see e.g. [39, Theorem XIII.67]): $u_n \rightarrow \tilde{u}$ strongly in $L^2 \cap L^p$, and $\tilde{u} \in B_p$ solves

$$(H + \lambda)\tilde{u} = \nu |\tilde{u}|^{p-2} \tilde{u},$$

for some Lagrange multiplier $\nu \in \mathbb{R}$. Since $\delta > 0$, taking the inner product with \tilde{u} in the above equation shows that $\nu > 0$. The function $u = \nu^{1/(p-2)} \tilde{u} \in \Sigma \setminus \{0\}$ then solves (1.6).

Second case: $\lambda < -N$. For any c , $u_c(x) = ce^{-|x|^2/2}$ satisfies $Hu_c = Nu_c$, so picking $c > 0$ so that $u_c \in B_p$ shows that $\delta < 0$.

Suppose $\delta = -\infty$: there would exist a sequence $(u_n)_n$ in B_p such that $F(u_n) \rightarrow -\infty$. As $F(u) \geq \frac{\lambda}{2} \|u\|_{L^2}^2$, $(u_n)_n$ is unbounded in $L^2(\mathbb{R}^N)$. We remark that $(xu_n)_{n \in \mathbb{N}}$ and $(\nabla u_n)_{n \in \mathbb{N}}$ are also unbounded in $L^2(\mathbb{R}^N)$, with norms of the same order as $\|u_n\|_{L^2}$. Indeed, if we had $\|\nabla u_n\|_{L^2} \gg \|u_n\|_{L^2}$ and/or $\|xu_n\|_{L^2} \gg \|u_n\|_{L^2}$, then we would have $F(u_n) \geq 0$ for n sufficiently large. In view of (1.2), the three terms $\|\nabla u_n\|_{L^2}$, $\|xu_n\|_{L^2}$ and $\|u_n\|_{L^2}$ go to infinity with the same order of magnitude. Set

$$\tilde{u}_n = \frac{1}{\|u_n\|_{L^2}} u_n.$$

This is a bounded sequence in Σ , whose L^2 norm is equal to one. Up to extracting a subsequence, \tilde{u}_n converges weakly in Σ , and strongly in $L^2 \cap L^p$ (as we have seen in the first case), to some $\tilde{u} \in B_p$ such that $\|\tilde{u}\|_{L^2} = 1$. We infer

$$\|u_n\|_{L^p} = \|u_n\|_{L^2} \|\tilde{u}_n\|_{L^p} \approx \|u_n\|_{L^2} \rightarrow +\infty.$$

Therefore, u_n cannot remain in B_p , hence the finiteness of δ .

Knowing that $-\infty < \delta < 0$, we infer that any minimizing sequence is bounded in Σ , and we conclude like in the first case. The only difference is that now the Lagrange multiplier $\nu < 0$, and we set $u = |\nu|^{1/(p-2)} \tilde{u}$.

Third case: combined nonlinearities, $\lambda > -N$. The proof in the inhomogeneous case follows the same lines as the first case. We now consider

$$G(u) := \frac{1}{2} \langle Hu, u \rangle - \frac{\lambda}{2} \|u\|_{L^2}^2 + \frac{1}{q} \|u\|_{L^q}^q,$$

and the minimization problem

$$\delta = \inf_{u \in B_p} G(u).$$

The uncertainty principle yields

$$G(u) \geq \frac{1}{2} (N + \lambda) \|u\|_{L^2}^2 + \frac{1}{q} \|u\|_{L^q}^q,$$

and it is easy to check that the proof of the first case can be repeated, up to the final homogeneity argument: there exists a Lagrange multiplier μ , which is positive, but we have essentially no information regarding its value.

REFERENCES

- [1] F. K. Abullaev and M. Salerno. Gap-Townes solitons and localized excitations in low-dimensional Bose-Einstein condensates in optical lattices. *Phys. Rev. A*, 72:033617, 2005. [1](#), [2](#)
- [2] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Commun. Pure Appl. Math.*, 12:623–727, 1959. [6](#)
- [3] P. Antonelli, R. Carles, and C. Sparber. On nonlinear Schrödinger type equations with nonlinear damping. *Int. Math. Res. Not.*, 2015:740–762, 2015. [10](#)
- [4] A. H. Ardila. Orbital stability of Gaussian solutions to logarithmic Schrödinger equations. *Electron. J. Differential Equations*, pages Paper No. 335, 9, 2016. [3](#)
- [5] J. Bellazzini, N. Boussaïd, L. Jeanjean, and N. Visciglia. Existence and stability of standing waves for supercritical NLS with a partial confinement. *Comm. Math. Phys.*, 353(1):229–251, 2017. [13](#)
- [6] T. B. Benjamin. The stability of solitary waves. *Proc. Roy. Soc. London Ser. A*, 328:153–183, 1972. [2](#)
- [7] H. Berestycki, T. Gallouët, and O. Kavian. Équations de champs scalaires euclidiens non linéaires dans le plan. *C. R. Acad. Sci. Paris Sér. I Math.*, 297(5):307–310, 1983. [3](#)
- [8] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.*, 82(4):313–345, 1983. [2](#), [3](#), [6](#), [18](#)
- [9] H. Brézis and T. Kato. Remarks on the Schrödinger operator with singular complex potentials. *J. Math. Pures Appl. (9)*, 58:137–151, 1979. [6](#)
- [10] R. Carles. Rotating points for the conformal NLS scattering operator. *Dyn. Partial Differ. Equ.*, 6(1):35–51, 2009. [18](#)
- [11] R. Carles, C. Klein, and C. Sparber. On soliton (in-)stability in multi-dimensional cubic-quintic nonlinear Schrödinger equations. Preprint. Archived at <https://hal.archives-ouvertes.fr/hal-03085923>, 2020. [3](#)
- [12] R. Carles and C. Sparber. Orbital stability vs. scattering in the cubic-quintic Schrödinger equation. *Rev. Math. Phys.*, 33(3):2150004, 2021. [4](#)
- [13] M. L. Carvalho, Y. Il'yasov, and C. A. Santos. Separating solutions of nonlinear problems using nonlinear generalized Rayleigh quotients. *Topol. Meth. Nonl. Anal*, 2022. to appear. Archived at <https://arxiv.org/abs/1906.07759>. [8](#)
- [14] T. Cazenave. Stable solutions of the logarithmic Schrödinger equation. *Nonlinear Anal.*, 7(10):1127–1140, 1983. [3](#)
- [15] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003. [1](#), [3](#), [4](#), [13](#)
- [16] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982. [2](#), [3](#), [4](#)

- [17] X. Cheng, C. Miao, and L. Zhao. Global well-posedness and scattering for nonlinear Schrödinger equations with combined nonlinearities in the radial case. *J. Differ. Equations*, 261(6):2881–2934, 2016. 2
- [18] S. Cingolani, L. Jeanjean, and S. Secchi. Multi-peak solutions for magnetic NLS equations without non-degeneracy conditions. *ESAIM Control Optim. Calc. Var.*, 15(3):653–675, 2009. 13
- [19] S. Coleman, V. Glaser, and A. Martin. Action minima among solutions to a class of Euclidean scalar field equations. *Comm. Math. Phys.*, 58(2):211–221, 1978. 18
- [20] S. De Bièvre, F. Genoud, and S. Rota Nodari. Orbital stability: analysis meets geometry. In *Nonlinear optical and atomic systems*, volume 2146 of *Lecture Notes in Math.*, pages 147–273. Springer, Cham, 2015. 3, 4
- [21] B. Feng. On the blow-up solutions for the fractional nonlinear Schrödinger equation with combined power-type nonlinearities. *Commun. Pure Appl. Anal.*, 17(5):1785–1804, 2018. 2
- [22] A. Floer and A. Weinstein. Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.*, 69:397–408, 1986. 3
- [23] R. Fukuizumi. Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential. *Discrete Contin. Dyn. Syst.*, 7(3):525–544, 2001. 4
- [24] R. Fukuizumi and M. Ohta. Stability of standing waves for nonlinear Schrödinger equations with potentials. *Differ. Integral Equ.*, 16(1):111–128, 2003. 4
- [25] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74(1):160–197, 1987. 3, 4
- [26] I. D. Iliev and K. P. Kirchev. Stability and instability of solitary waves for one-dimensional singular Schrödinger equations. *Differential Integral Equ.*, 6(3):685–703, 1993. 2, 3, 4
- [27] Y. Ilyasov. On extreme values of Nehari manifold method via nonlinear Rayleigh’s quotient. *Topol. Methods Nonlinear Anal.*, 49(2):683–714, 2017. 2, 5
- [28] Y. Il’yasov. On fundamental frequency solutions with prescribed action value of the NLS equations. *J. Math. Sci.*, 2021. To appear. 2, 4, 5, 7
- [29] Y. Il’yasov. Rayleigh quotients of the level set manifolds related to the nonlinear PDE. *Minimax Theory and its Applications*, The Nehari manifold: theory, applications, related topics, 2022. to appear. Archived at <https://arXiv.org/abs/2108.00891>. 5
- [30] L. Jeanjean and S.-S. Lu. On global minimizers for a mass constrained problem. Preprint. Archived at <https://arxiv.org/abs/2108.04142>, 2021. 3
- [31] O. Kavian and F. Weissler. Self-similar solutions of the pseudo-conformally invariant nonlinear Schrödinger equation. *Michigan Math. J.*, 41(1):151–173, 1994. 3, 4
- [32] R. Killip, M. Visan, and X. Zhang. Energy-critical NLS with quadratic potentials. *Comm. Partial Differential Equations*, 34(10-12):1531–1565, 2009. 2
- [33] L. D. Landau and E. M. Lifshitz. *Quantum mechanics: non-relativistic theory. Course of Theoretical Physics, Vol. 3*. Addison-Wesley Series in Advanced Physics. Pergamon Press Ltd., London-Paris; for U.S.A. and Canada: Addison-Wesley Publishing Co., Inc., Reading, Mass.;, 1958. Translated from the Russian by J. B. Sykes and J. S. Bell. 19
- [34] M. Lewin and S. Rota Nodari. The double-power nonlinear Schrödinger equations and its generalizations: uniqueness, non-degeneracy and applications. *Calc. Var. Partial Differ. Equ.*, 59(197), 2020. 3
- [35] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001. 11, 13
- [36] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984. 3
- [37] B. Malomed. Vortex solitons: Old results and new perspectives. *Physica D*, 399(1):108–137, 2019. 1, 2
- [38] S. I. Pokhozhaev. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Sov. Math., Dokl.*, 6:1408–1411, 1965. 6
- [39] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. 11, 19
- [40] H. A. Rose and M. I. Weinstein. On the bound states of the nonlinear Schrödinger equation with a linear potential. *Physica D*, 30(1-2):207–218, 1988. 3
- [41] T. Tao, M. Visan, and X. Zhang. The nonlinear Schrödinger equation with combined power-type nonlinearities. *Comm. in Partial Diff. Eq.*, 32:1281–1343, 2007. 2

- [42] M. I. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.*, 87(4):567–576, 1982/83. [1](#)
- [43] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985. [3](#), [4](#)
- [44] M. I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39(1):51–67, 1986. [3](#), [4](#)
- [45] V. S. Zakharov and S. A. Kuznetsov. On threedimensional solitons. *Zhurnal Eksp. Teoret. Fiz.*, pages 594–597, 1974. [2](#)
- [46] X. Zhang. On the Cauchy problem of 3-D energy-critical Schrödinger equations with subcritical perturbations. *J. Differential Equ.*, 230(2):422–445, 2006. [2](#)

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