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# Large deviations of energy transfers in nonequilibrium CFT and asymptotics of non-local Riemann-Hilbert problems

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## Abstract

A wide class of 1 + 1 dimensional unitary conformal field theories allows for an explicit construction of nonequilibrium "profile states" interpolating smoothly between different equilibria on the left and on the right. It has been recently established that the generating function for the full counting statistics of energy transfers in such states may be expressed in terms of the solution to a non-local Riemann-Hilbert problem. Following earlier works on the statistics of energy transfers, in particular the ones of Bernard-Doyon on the "partitioning protocol" in conformal field theory, the full counting statistics of energy transfers in the profile states was conjectured to satisfy a large deviation principle in the limit of long transfer-times. The present paper establishes rigorously this conjecture by carrying out the long-time asymptotic analysis of the underlying non-local Riemann-Hilbert problem.

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## 1 Introduction

The aim of the paper is to establish a large auxiliary-parameter behaviour of certain biholomorphisms that provide the holomorphic structure on conformally welded cylinders. Such cylinders are obtained by identifying the boundaries of an infinite strip in the complex plane after the composition with a line-diffeomorphism. The original

motivation for this investigation came from specific questions, described below, related to a rigorous characterisation of certain correlation functions pertaining to non-equilibrium situations in a large class of unitary 1+1 dimensional conformal field theories (CFTs). The CFT problem was first addressed, on heuristic grounds, in [2] in a closely related but more singular framework. More recently, it was reformulated in a rigorous non-singular setup in [10] where a closed formula was proven for the correlator of interest. The non-trivial building block of the obtained expression was a biholomorphism realising the rectification of a cylinder conformally-welded from a strip. What is important for the applications is the dependence of this biholomorphism on certain auxiliary parameters describing the line-diffeomorphism used to weld together the boundaries of the strip. A precise control on that dependence constitutes the main result of this work.

## 1.1 Asymptotics of conformal maps on the welded cylinders

In order to state the results, we first need to introduce a few notations so as to make the setting explicit. Let  $\alpha > 0$  and let

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : -\alpha < \Im(z) < 0\}. \quad (1.1)$$

refer to the strip of width  $\alpha$  in  $\mathbb{C}$  located below  $\mathbb{R}$ . Endow the upper and the lower boundaries of the closure  $\overline{\mathcal{S}_\alpha}$  of  $\mathcal{S}_\alpha$  with the orientation of increasing real parts as depicted in Figure 1. These boundaries are parameterised by  $p_1(x)$  and  $p_2(x)$ , where  $p_1 : \mathbb{R} \rightarrow \mathbb{R} - i\alpha$  and  $p_2 : \mathbb{R} \rightarrow \mathbb{R}$  are smooth diffeomorphisms. The welded cylinder is then defined as the manifold obtained from  $\overline{\mathcal{S}_\alpha}$  by identifying the points of  $\partial\mathcal{S}_\alpha$  parameterised by  $p_1(x)$  and  $p_2(x)$ . It comes with the complex structure such that local holomorphic functions on it are the smooth ones that are holomorphic when restricted to  $\mathcal{S}_\alpha$  [12]. Clearly, it is enough to parameterise the boundaries of the strip by taking

$$p_1(x) = g(x) - i\alpha \quad \text{and} \quad p_2(x) = x \quad (1.2)$$

with  $g$  a smooth diffeomorphism of  $\mathbb{R}$ . The corresponding welded cylinder will be denoted as  $\mathcal{S}_{\alpha,g}$ . For example, when  $g(x) = x$  then the welded cylinder is tautologically equivalent to the standard cylinder of circumference  $\alpha$ . More generally, it is of interest to consider the case where  $g$  is smooth and such that  $g - \text{id}$  is constant on the two connected components of the complement of some large enough segment  $[-M; m]$  of  $\mathbb{R}$ , *viz.*

$$g(x) = \begin{cases} x + \kappa^- & x \leq -M \\ g(x) & -M \leq x \leq m \\ x + \kappa^+ & x \geq m \end{cases}, \quad (1.3)$$

for some constants  $\kappa^\pm$ . The welded cylinder  $\mathcal{S}_{\alpha,g}$  for any such  $g$  is biholomorphically equivalent to the standard cylinder. The biholomorphism realising this equivalence may be constructed by means of solving a scalar, non-local Riemann-Hilbert problem with a jump.

**Proposition 1.1.** *Assume that  $g$  satisfies (1.3), and consider the scalar non-local Riemann-Hilbert problem with a shift consisting in finding a holomorphic function  $z \mapsto \Omega(z | \kappa^+, \kappa^-)$  on  $\mathcal{S}_\alpha$  having smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ , such that*

- $\Omega_-$ , resp.  $\Omega_+$ , is a bijection from  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ , onto  $\Omega_-(\mathbb{R} | \kappa^+, \kappa^-)$ , resp.  $\Omega_+(\mathbb{R} - i\alpha | \kappa^+, \kappa^-)$ ;
- $\Omega_+(g(x) - i\alpha | \kappa^+, \kappa^-) = \Omega_-(x | \kappa^+, \kappa^-) - i\alpha$  for  $x \in \mathbb{R}$ ;
- $\Omega(z | \kappa^+, \kappa^-) = \tilde{\gamma}_{\pm z} + C_\Omega \delta_{\pm, -} + O\left(e^{\mp \frac{2\pi}{\alpha} \tilde{\gamma}_{\pm z}}\right)$  as  $\Re(z) \rightarrow \pm\infty$ ,

for  $\tilde{\gamma}_{\pm} = \frac{-i\alpha}{\kappa^{\pm} - i\alpha}$  and some constant  $C_{\Omega} \in \mathbb{C}$ .

Then, the above problem admits a unique solution. Moreover, the latter is a biholomorphism from  $\mathcal{S}_{\alpha}$  onto its image  $\Omega(\mathcal{S}_{\alpha} | \kappa^+, \kappa^-)$ .

When  $\kappa^+ = \kappa^- = 0$ , *viz.* when the welding diffeomorphism is such that  $g - \text{id}$  has compact support, then the above proposition may be seen, after composing with obvious biholomorphisms, as a direct consequence of the material discussed in [9]. However, for general  $\kappa^{\pm}$ , the techniques of [9] are not sufficient to establish this result and one has to rely on the setting developed in the core of this paper. The proof of the above proposition is given in Appendix A.

$\Omega(\cdot | \kappa^+, \kappa^-)$  induces a biholomorphism from  $\mathcal{S}_{\alpha, g}$  onto the standard cylinder  $\mathcal{S}_{\alpha, \text{id}}$  upon identifying the endpoints  $\Omega_-(\mathbb{R} | \kappa^+, \kappa^-) \ni z \equiv z - i\alpha \in \Omega_+(\mathbb{R} - i\alpha | \kappa^+, \kappa^-)$ .

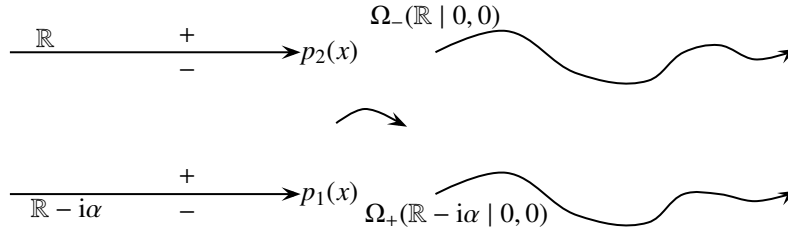


Figure 1: The strip  $\mathcal{S}_{\alpha}$ , parametrisation of its boundary along with its orientation and image thereof through the biholomorphism  $\Omega$  in the case when  $\kappa^{\pm} = 0$ .

The main interest of the present work lies in accessing the behaviour of the biholomorphism  $\Omega$  in the case when the diffeomorphism of the line  $g$  is such that  $g - \text{id}$  has compact support, *viz.*  $\kappa^{\pm} = 0$ , and is constructed from two diffeomorphisms  $g_L$  and  $g_R$  of the line in such a way that  $g$  has a non-trivial behaviour only in the neighbourhood of the points  $-w$  and  $+w$ . In order to insist on the vanishing of the constants  $\kappa^{\pm}$ , we shall henceforth denote this biholomorphism by  $\chi$ , *viz.*  $\chi(z) = \Omega(z | 0, 0)$ . Such a situation is depicted in Figures 2-3. To be more precise about the structure of  $g$ , pick  $M_R, M_L > 0$  and let  $\kappa \in \mathbb{R}$ . Then, let  $g_{L/R}$  be smooth diffeomorphisms of the real line taking the piecewise form

$$g_L(x) = \begin{cases} x & x \leq -M_L \\ g_L(x) & -M_L < x < M_L \\ x + \kappa & M_L \leq x \end{cases} \quad \text{and} \quad g_R(x) = \begin{cases} x + \kappa & x \leq -M_R \\ g_R(x) & -M_R < x < M_R \\ x & M_R \leq x \end{cases} . \quad (1.4)$$

Then, the diffeomorphism  $g$  of interest is defined as

$$g(x) = \begin{cases} x & x < -M_L - w \\ g_L(x + w) - w & -M_L - w \leq x \leq M_L - w \\ x + \kappa & M_L - w \leq x \leq w - M_R \\ g_R(x - w) + w & w - M_R < x < w + M_R \\ x & M_R + w < x \end{cases} \quad (1.5)$$

The non-local Riemann-Hilbert problem of Proposition 1.1 takes for such a  $g$  the slightly simpler form below.

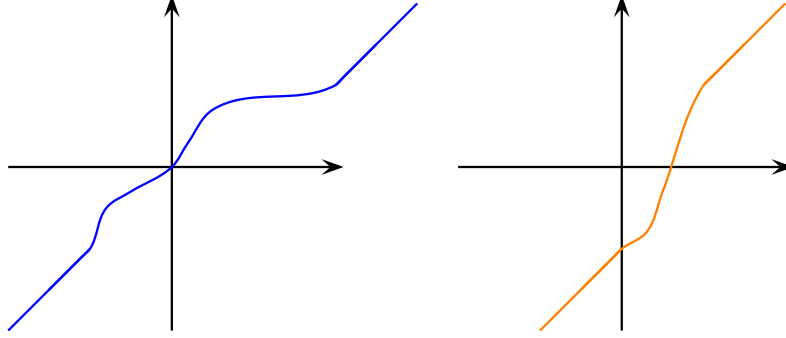


Figure 2: Diffeomorphisms  $g_L$  (on the left, blue curve) and  $g_R$  (on the right, orange curve).

**Definition 1.2.** Given a smooth diffeomorphism  $g$  of  $\mathbb{R}$  such that  $g - \text{id}$  has compact support, the non-local Riemann-Hilbert problem for  $\chi$  consists in finding a holomorphic function  $z \mapsto \chi(z)$  on  $\mathcal{S}_\alpha$  such that

- it has smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\chi_+(g(x) - i\alpha) = \chi_-(x) - i\alpha$  for  $x \in \mathbb{R}$ ;
- $\chi(z) = z + C_\chi \delta_{\pm, -} + O(e^{\mp \frac{2\pi}{\alpha} z})$  as  $\Re(z) \rightarrow \pm\infty$ ,

for some constant  $C_\chi \in \mathbb{C}$ .

As will be discussed in the following, the interest in this specific form of the diffeomorphism  $g$  and in the  $w \rightarrow +\infty$  behaviour it begets to  $\chi$  stems from certain questions occurring in 1+1 dimensional unitary conformal field theories. In order to state the main technical achievement of this work, Theorem 1.3 below, we need to introduce two auxiliary non-local Riemann-Hilbert problems with shift that are associated with the welding diffeomorphisms  $g_L$  and  $g_R$ . First, however, define

$$\gamma = \frac{-\varkappa}{\varkappa - i\alpha} \quad \text{and} \quad \tilde{\gamma} = \gamma + 1 = \frac{-i\alpha}{\varkappa - i\alpha}. \quad (1.6)$$

and denote by  $\mathcal{O}(\mathcal{S}_\alpha)$  the space of holomorphic functions on  $\mathcal{S}_\alpha$ . The left Riemann-Hilbert problem consists in finding  $\chi^{(L)} \in \mathcal{O}(\mathcal{S}_\alpha)$  that admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ , and such that

- for some constant  $C_{\chi^{(L)}}$ ,  $\chi^{(L)}(z) = C_{\chi^{(L)}} + O(e^{\frac{2\pi}{\alpha} z})$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;
- $\chi^{(L)}(z) = \gamma \cdot z + O(e^{-\frac{2\pi\tilde{\gamma}}{\alpha} z})$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- $\chi_+^{(L)}(g_L(x) - i\alpha) = \chi_-^{(L)}(x) + x - g_L(x)$ .

The right Riemann-Hilbert problem consists in finding  $\chi^{(R)} \in \mathcal{O}(\mathcal{S}_\alpha)$  that admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ , and such that

- $\chi^{(R)}(z) = O(e^{-\frac{2\pi}{\alpha} z})$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- for some constant  $C_{\chi^{(R)}}$ ,  $\chi^{(R)}(z) = \gamma z + C_{\chi^{(R)}} + O(e^{\frac{2\pi\tilde{\gamma}}{\alpha} z})$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;

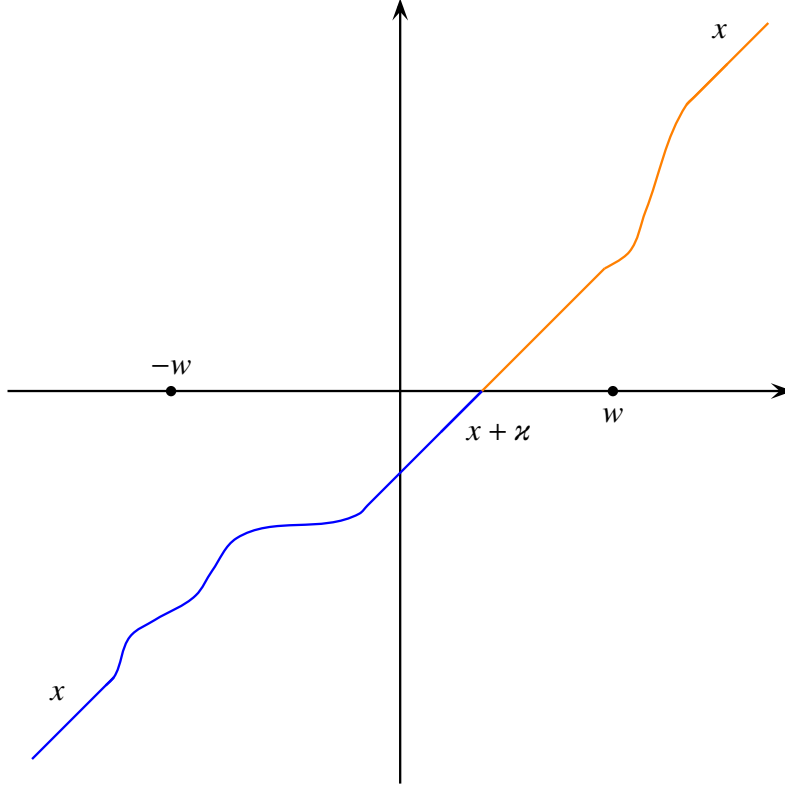


Figure 3: Diffeomorphism  $g$  constructed from the gluing of  $g_L$  and  $g_R$ .

- $\chi_+^{(R)}(g_R(x) - i\alpha) = \chi_-^{(R)}(x) + x - g_R(x)$ .

One may readily convince oneself that  $\chi^{(L)}(z) = \Omega(z | \varkappa, 0) - z$  while  $\chi^{(R)}(z) = \Omega(z | 0, \varkappa) - z$ , where in the first case,  $\Omega$  is constructed out of the diffeomorphism  $g_L$  and in the second case out of  $g_R$ .

**Theorem 1.3.** *Let  $g$  be as given above in terms of  $g_L$  and  $g_R$ . Then, the left/right non-local Riemann-problems for  $\chi^{(L/R)}$  are uniquely solvable and the unique solution to the non-local Riemann-Hilbert problem with a shift for  $\chi$  described in Definition 1.2 admits the large  $w$  asymptotic expansion which takes the patch-wise form given in Fig. 4. There,  $c = 2\gamma w - C_{\chi^{(R)}}$  and  $\delta\chi^{(R/L)}$  are holomorphic in the domains where they appear and enjoy there the uniform estimates, in  $z \in \mathcal{S}_\alpha$  and in  $w$ :*

$$\delta\chi^{(L)}(z) = \delta c + \mathcal{O}(e^{-\eta w + \eta' z}) \quad \text{and} \quad \delta\chi^{(R)}(z) = \mathcal{O}(e^{-\eta w - \eta' z}) \quad \text{with} \quad \delta c = \mathcal{O}(e^{-\eta w}). \quad (1.7)$$

Above,  $\eta, \eta' > 0$  are some constants just as  $\delta c$ . The remainder functions  $\delta\chi^{(R/L)}$  are such that  $\chi$  is indeed smooth across the separating segment  $\Gamma_0 = [0; \varkappa - i\alpha]$ . Finally, all estimates appearing above are differentiable uniformly on  $\mathcal{S}_\alpha$  and up to its boundary.

The presence of the two biholomorphisms  $\chi^{(L/R)} + \text{id}$  appearing in the leading behaviour of  $\chi$  to the left/right of  $\Gamma_0$  is certainly natural in that with  $w$  growing the two non-trivial pieces of  $g$  should cease to interact so that, locally, the overall biholomorphism should only "feel" the effect of  $g_L$  or  $g_R$ . The hardest part of the proof Theorem 1.3 consists in establishing the differentiable control on the remainder given in (1.7). It is the proof of this property, crucial for the application of Theorem 1.3 to conformal field theories, that occupies most of this work.

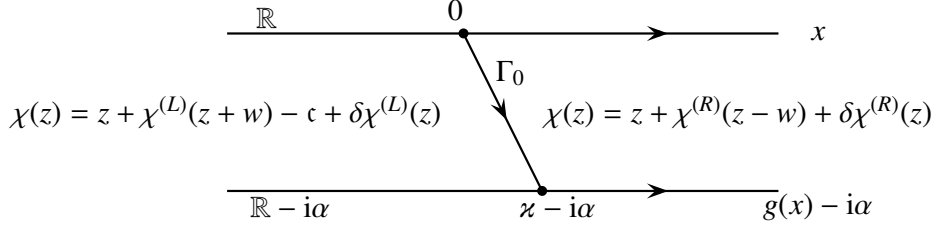


Figure 4: Piecewise expression for the solution  $\chi$ .

## 1.2 Large deviation principle for the full counting statistics of energy transfers in 1+1 dimensional conformal field theories

Let  $x \mapsto \beta(x)$  be an inverse temperature profile, *viz.* a smooth function bounded from below by a strictly positive constant such that  $\beta'(x)$  has compact support and constant sign. Furthermore, consider a 1+1 dimensional unitary conformal field theory in a finite interval  $[-L/4; L/4]$  with boundary conditions that assure no energy flux through the endpoints of the interval. Such a theory is described by a representation of the Virasoro algebra on some Hilbert space  $\mathfrak{h}$  generated by the energy-momentum tensor of the form [5, 10]

$$\mathbb{T}(x) = \frac{2\pi}{L^2} \sum_{n \in \mathbb{Z}} e^{\frac{2i\pi n}{L}(x + \frac{L}{4})} \cdot \left( L_n - \frac{c}{24} \delta_{n,0} \right), \quad (1.8)$$

in which  $L_n$  are generators of the Virasoro algebra. Out of these quantities one constructs the operator

$$\mathbb{G}_L(t) = v \int_{-\frac{L}{4}}^{\frac{L}{4}} dx \beta(x) \mathcal{E}(t, x) \quad \text{for} \quad \mathcal{E}(t, x) = \mathbb{T}(x - vt) + \mathbb{T}(-x - vt - \frac{L}{2}) \quad (1.9)$$

( $\mathcal{E}(t, x)$  is the energy density). It was proven in the work [10] that the Fourier transform of the probability measure which describes the energy transfers in time  $t$  between two baths at inverse temperatures  $\beta(-\infty)$  and  $\beta(+\infty)$  connected by the interpolating inverse temperature profile  $\beta(x)$  takes the form

$$\Psi_{t,L}(\lambda) = \frac{\text{tr}_{\mathfrak{h}} \left[ e^{\frac{i\lambda}{\Delta\beta} \mathbb{G}_L(t)} e^{-\left(\frac{i\lambda}{\Delta\beta} + 1\right) \mathbb{G}_L(0)} \right]}{\text{tr}_{\mathfrak{h}} \left[ e^{-\mathbb{G}_L(0)} \right]} \quad \text{with} \quad \Delta\beta = \int_{\mathbb{R}} \beta'(x) dx. \quad (1.10)$$

More details on the well-definiteness of the above expression can be found in the above mentioned work. Ref. [10] studied the thermodynamic limit of  $\Psi_{t,L}(\lambda)$  and it was rigorously proven there that

$$\lim_{L \rightarrow +\infty} \Psi_{t,L}(\lambda) = \prod_{\varepsilon = \pm} \Psi_t^{(\varepsilon)}(\lambda), \quad (1.11)$$

uniformly in  $\lambda$  belonging to compact subsets of  $\mathbb{R}$ , where

$$\Psi_t^{(\varepsilon)}(\lambda) = \exp \left\{ \phi^{(\varepsilon)}(t) - \frac{ic}{24\pi} \int_0^{\frac{\lambda}{\Delta\beta}} ds \int_{\mathbb{R}} dx \xi_t^{(\varepsilon)}(x) \cdot \left\{ \mathcal{S}[\chi_{s,t;-}^{(\varepsilon)}](x) - \frac{2\pi^2}{\alpha^2} (\partial_x \chi_{s,t;-}^{(\varepsilon)}(x))^2 \right\} \right\}. \quad (1.12)$$



The last formula contains several ingredients. First of all,

$$\mathcal{S}[f] = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 \quad (1.13)$$

is the Schwartzian derivative,  $\alpha$  is a constant built up from the  $\pm\infty$  limits of the inverse-temperature profile and  $\phi^{(\varepsilon)}(t)$  is an explicit, smooth and bounded function of  $t$ . Furthermore,  $\xi_t^{(\varepsilon)}$  is a smooth compactly supported function depending on the auxiliary time parameter  $t$ . One is also given two smooth diffeomorphism of the line  $g_{s,t}^{(\varepsilon)}$  such that  $g_{s,t}^{(\varepsilon)} - \text{id}$  has compact support. These depend smoothly on two auxiliary parameters: the time  $t$  and a real variable  $s$ . To each diffeomorphism  $g_{s,t}^{(\varepsilon)}$  one then associates the corresponding solution  $\chi_{s,t}^{(\varepsilon)}$  to the non-local Riemann-Hilbert problem of the strip  $\mathcal{S}_\alpha$ , as introduced in Definition 1.2. Then,  $\chi_{s,t;-}^{(\varepsilon)}$  stand for its  $-$  boundary values on  $\mathbb{R}$ . This concludes the description of the building blocks of the thermodynamic limit  $\Psi_t^{(\varepsilon)}(\lambda)$ .

The explicit construction of the functions  $\xi_t^{(\varepsilon)}$  and  $g_{s,t}^{(\varepsilon)}$  can be found in [10]. Here, we only remind the properties which are crucial for establishing the results given in Theorem 1.4 below. Namely, there exist real parameters  $\varkappa^\varepsilon$  and segments  $\mathcal{I}_L^{(\varepsilon)}$ ,  $\mathcal{I}_{\text{bk}}^{(\varepsilon)}$  and  $\mathcal{I}_R^{(\varepsilon)}$  having disjoint interiors such that:

- $\text{supp}[\xi_t^{(\varepsilon)}] \subset \mathcal{I}_L^{(\varepsilon)} \cup \mathcal{I}_{\text{bk}}^{(\varepsilon)} \cup \mathcal{I}_R^{(\varepsilon)}$ ;
- $\text{supp}[g_{s,t}^{(\varepsilon)} - \text{id}] \subset \mathcal{I}_L^{(\varepsilon)} \cup \mathcal{I}_{\text{bk}}^{(\varepsilon)} \cup \mathcal{I}_R^{(\varepsilon)}$ ;
- $\text{diam}(\mathcal{I}_L^{(\varepsilon)})$  and  $\text{diam}(\mathcal{I}_R^{(\varepsilon)})$  are  $t$  independent;
- $\text{diam}(\mathcal{I}_{\text{bk}}^{(\varepsilon)}) = \ell^\varepsilon t - C$  for some  $\ell^\varepsilon, C > 0$ ;
- $\xi_t^{(\varepsilon)}|_{\mathcal{I}_{\text{bk}}^{(\varepsilon)}} = -\varkappa^\varepsilon$  and  $(g_{s,t}^{(\varepsilon)} - \text{id})|_{\mathcal{I}_{\text{bk}}^{(\varepsilon)}} = \varkappa^\varepsilon s$ ;
- $\xi_t^{(\varepsilon)}|_{\mathcal{I}_{L,R}^{(\varepsilon)}}$  and  $(g_{s,t}^{(\varepsilon)} - \text{id})|_{\mathcal{I}_{L,R}^{(\varepsilon)}}$  have  $t$ -independent shape on those intervals.

As it is apparent from (1.12), the thermodynamic limit of  $\Psi_{t,L}(\lambda)$  depends on time  $t$ . One is interested in obtaining a large deviation principle, when  $t \rightarrow +\infty$ , for the thermodynamic limit of the associated probability measure. The rate function governing this large deviation principle may be deduced from the Legendre transform in  $i\lambda$  of the limiting functions  $\lim_{t \rightarrow +\infty} \{t^{-1} \ln \Psi_t^{(\varepsilon)}(\lambda)\}$ ,  $\varepsilon = \pm$ . In order to control this limit and compute it, one needs all the information that have been established in Theorem 1.3 given above. In fact, a direct application of this theorem shows that the Schwarzian derivative term contributes as  $O(1)$  when  $t \rightarrow +\infty$  while the only linear in  $t$  behaviour of the integral giving rise to  $\ln \Psi_t^{(\varepsilon)}(\lambda)$  is generated from the constant term in the behaviour of  $\partial_x \chi_{s,t;-}^{(\varepsilon)}(x)$  for  $x \in \mathcal{I}_{\text{bk}}^{(\varepsilon)}$  sufficiently far away from the endpoints of that segment. After straightforward calculations, one gets

**Theorem 1.4.**

$$\lim_{t \rightarrow +\infty} \{t^{-1} \ln \Psi_t^{(\varepsilon)}(\lambda)\} = -\frac{c\pi}{12\alpha} \cdot \frac{\varkappa^\varepsilon \ell^\varepsilon \lambda}{\varkappa^\varepsilon \lambda - i\alpha \Delta\beta}. \quad (1.14)$$

This theorem concludes the proof of the large deviation principle stated in [10]. The above form of large deviations for the energy transfers coincides with the one anticipated in [2], see also [3].

**Remark.** An examination of the arguments leading to Theorem 1.3 shows that the convergence in (1.14) is uniformly differentiable in  $\lambda$  belonging to compact subsets of  $\mathbb{R}$ .

### 1.3 Outline of the paper

The paper is organised as follows. Section 2 establishes the unique solvability of a class of non-local Riemann-Hilbert problems on welded cylinders. Subsection 2.1 provides the definition of a class of Riemann-Hilbert problems that will be considered there. Then, various technical results relative to the original setting of the Riemann-Hilbert problems are established, in particular an improvement of the decay at  $\infty$ , the correspondence with solutions to linear integral equations and the existence of smooth boundary values for the solution. The non-local Riemann-Hilbert problem in the optimal setting is then outlined in Subsection 2.2. Finally, Subsection 2.3 establishes the unique solvability of the non-local Riemann-Hilbert problems of interest. This is done by proving the invertibility of the operator  $\text{id} - \mathbf{K}$  which drives the linear integral equations that are satisfied by the boundary values of the solution. The preliminary notations for this result are established in Subsubsection 2.3.1. The reduction of the operator  $\text{id} - \mathbf{K}$  to  $\text{id} - \mathbf{M}$  with compact  $\mathbf{M}$  is carried out in Subsubsection 2.3.2 and, finally, the sought invertibility is established in Subsubsection 2.3.3.

Section 3 studies three auxiliary special non-local Riemann-Hilbert problems which play a role in the large- $w$  asymptotic analysis of the solution to the Riemann-Hilbert problem stated in Proposition 1.1 in the presence of the  $w$ -dependent welding diffeomorphism  $g$  as described above. Subsection 3.1 discusses properties of the Cauchy transform on a welded cylinder, Subsection 3.2 establishes the existence of the solution to the Riemann-Hilbert problem for the function  $\chi^{(L)}$  described above while Subsection 3.3 does it for the one associated with  $\chi^{(R)}$ .

Section 4 establishes Theorem 1.3. The proof given there heavily relies on technical results, relative to the uniform in large  $w$  invertibility of the integral operator  $\text{id} - \mathbf{K}_{\text{tot}}$  which drives the integral equations satisfied by the boundary values of the solution  $\chi$ . Those are established throughout Section 5. Subsection 5.1 provides a convenient decomposition of the integral kernel of  $\mathbf{K}_{\text{tot}}$ . Various technical properties issuing from this decomposition are then established throughout Subsections 5.2, 5.3, 5.4, 5.5 and 5.6. Finally, the uniform in  $w$  invertibility of  $\text{id} - \mathbf{K}_{\text{tot}}$  is established in Subsection 5.7.

Several auxiliary results are postponed to the appendices. Appendix A briefly outlines the proof of Proposition 1.1. Appendix B provides details on the inversion of certain Wiener-Hopf equations on the half-line while Appendix C discusses the inversion of a truncated Wiener-Hopf operator arising in the analysis of Section 4. This last result is achieved by solving a local  $2 \times 2$  matrix Riemann-Hilbert problem. Finally, Appendix D establishes a technical Lemma useful for certain estimates obtained in Section 4.

### 1.4 Notations

- Given an open subset  $U \subset \mathbb{C}$ ,  $\mathcal{O}(U)$  stands for the ring of holomorphic functions on  $U$ .
- Given an open subset  $U \subset \mathbb{C}$ , and a function  $f$  defined on  $U \setminus \gamma$ , with  $\gamma$  an oriented curve in  $U$ , we denote by  $f_{\pm}(s)$  the boundary values - if these exists in an appropriate sense - of  $f(z)$  on  $\gamma$  when the argument  $z$  approaches the point  $s \in \Sigma$  non-tangentially and from the left (+) or the right (-) side of the curve. Furthermore, if one deals with vector or matrix-valued function, then this notation is to be understood entry-wise.
- $\mathbb{H}^{\pm} = \{z \in \mathbb{C} : \text{Im}(\pm z) > 0\}$  is the upper/lower half-plane, and  $\mathbb{R}^{\pm} = \{x \in \mathbb{R} : \pm x \geq 0\}$  is the positive/negative real axis.
- Given a set  $A$ ,  $\bar{A}$  stands for its closure and  $\mathbf{1}_A$  stands for the indicator function of  $A$ .
- Given a ring  $\mathcal{R}$ ,  $\mathcal{M}_n(\mathcal{R})$  stands for the space of  $n \times n$  matrices over this ring.
- Given two functions  $f, g$  defined in an open neighbourhood  $U$  of a point  $\mathbf{y} = (y_1, \dots, y_n)$ , the relation  $f(\mathbf{x}) = \mathcal{O}(g(\mathbf{x}))$  means that there exists  $M > 0$  such that  $|f(\mathbf{x})| \leq M|g(\mathbf{x})|$  holds in a neighbourhood of  $\mathbf{y}$ .

The O-relation is said to be differentiable if, for all  $k_a \geq 0$ ,

$$\partial_{x_1}^{k_1} \dots \partial_{x_n}^{k_n} f(\mathbf{x}) = O\left(\max\{|\partial_{x_1}^{m_1} \dots \partial_{x_n}^{m_n} g(\mathbf{x})| : 0 \leq m_a \leq k_a\}\right) \quad (1.15)$$

holds in a neighbourhood of  $\mathbf{y}$ .

- For matrix valued functions, a relation  $M(\mathbf{x}) = O(N(\mathbf{x}))$  is to be understood entrywise, *viz.*  $M_{ab}(\mathbf{x}) = O(N_{ab}(\mathbf{x}))$ .
- Let  $\Gamma$  be Euler's  $\Gamma$ -function. We use the shorthand convention

$$\Gamma\left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_m \end{matrix}\right) = \frac{\prod_{k=1}^n \Gamma(a_k)}{\prod_{k=1}^m \Gamma(b_k)}. \quad (1.16)$$

## 2 Non-local Riemann–Hilbert problems on welded cylinders

### 2.1 General definitions and considerations

Throughout this section, we shall focus on the "affine at  $\infty$ " setting, *viz.* a situation where there exist reals  $\kappa^\pm$  such that the diffeomorphism  $g$  appearing in the welding of the strip  $\mathcal{S}_\alpha$  (1.1), *c.f.* Fig. 1, behaves as  $g(x) = x + \kappa^\pm$  when  $\pm x > M$ , for  $M$  large enough, namely

$$g(x) = \begin{cases} x + \kappa^- & x \leq -M \\ g(x) & |x| \leq M \\ x + \kappa^+ & x \geq M \end{cases}. \quad (2.1)$$

The main purpose of this section is to establish the unique solvability, along with certain other properties, of a class of non-local Riemann-Hilbert problems that arise in later subsections. Thus, we start by introducing the class of problems of interest.

The diffeomorphisms  $p_1, p_2$  realising the welding of the strip  $\mathcal{S}_\alpha$  as in Fig. 1 are denoted as in (1.2). Next, we assume being given a smooth function  $G_\Xi$  on  $\mathbb{R}$  which has the structure:

$$G_\Xi(x) = G_\Xi^{(c)}(x) + \mathcal{G}_\Xi(x) - \mathcal{G}_\Xi(g(x) - i\alpha). \quad (2.2)$$

There  $G_\Xi^{(c)}$  is smooth with compact support and  $\text{supp}[G_\Xi^{(c)}] \subset [-M; M]$ . Furthermore,

$$\mathcal{G}_\Xi \text{ is smooth on } \overline{\mathcal{S}_\alpha} \text{ and analytic on } \mathcal{S}_\alpha \cap \{z \in \mathbb{C} : |\Re(z)| > M/2\} \quad (2.3)$$

and vanishes exponentially fast at  $\Re(z) \rightarrow \pm\infty$ , *viz.* there exists  $\varrho > 0$  such that  $\mathcal{G}_\Xi(z) = O(e^{\mp\varrho z})$  uniformly on  $\overline{\mathcal{S}_\alpha} \cap \{z \in \mathbb{C} : |\Re(z)| > M/2\}$ .

One may associate with this setting the following non-local Riemann-Hilbert problem on the strip  $\mathcal{S}_\alpha$ . Find  $\Xi \in O(\mathcal{S}_\alpha)$  such that

- $\Xi_+ \circ p_1 \in L_{\text{loc}}^2(\mathbb{R})$ ,  $\Xi_- \circ p_2 \in L_{\text{loc}}^2(\mathbb{R})$ ;
- $\Xi_+(p_1(x)) = \Xi_-(p_2(x)) + G_\Xi(x)$  for  $x \in \mathbb{R}$ ;
- there exist constants  $C_\Xi, C_{-1}$  such that

$$\Xi(z) = C_\Xi \cdot \delta_{\pm;-} + \frac{C_{-1}}{z} + O(z^{-2}) \quad \text{when} \quad \Re(z) \rightarrow \pm\infty \quad (2.4)$$

uniformly up to the boundary of  $\mathcal{S}_\alpha$ .

• Improved asymptotic decay at infinity

**Lemma 2.1.** Any solution to the above Riemann-Hilbert problem decays exponentially fast at infinity as

$$\Xi(z) = C_{\Xi} \cdot \delta_{\pm; -} - \mathcal{G}_{\Xi}(z) + \mathcal{O}\left(e^{\mp \frac{2\pi}{\alpha} \tilde{\gamma}_{\pm} z}\right) \quad \text{when } \Re(z) \rightarrow \pm\infty, \quad \text{with } \tilde{\gamma}_{\pm} = \frac{-i\alpha}{-i\alpha + \kappa^{\pm}} \quad (2.5)$$

this uniformly up to the boundary.

*Proof* — To improve the bounds on the asymptotic behaviour, we first introduce the curve built out of oriented segments

$$\mathcal{C}_{r,\alpha} = [-r + \kappa^{-} - i\alpha; r + \kappa^{+} - i\alpha] \cup [r + \kappa^{+} - i\alpha; r] \cup [r; -r] \cup [-r; -r + \kappa^{-} - i\alpha]. \quad (2.6)$$

For any  $z \in \mathcal{S}_{\alpha}$ , it holds

$$\Xi(z) = \lim_{r \rightarrow +\infty} \int_{\mathcal{C}_{r,\alpha}} \frac{\tilde{\gamma}_{+} ds}{i\alpha} \frac{\Xi(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-s)}}. \quad (2.7)$$

Writing explicitly the various integrations and using that  $g([-r; r]) = [-r + \kappa^{-}; r + \kappa^{+}]$  provided that  $r$  is large enough, leads to

$$\begin{aligned} \Xi(z) = \lim_{r \rightarrow +\infty} \left[ \int_{-r}^r \frac{\tilde{\gamma}_{+} ds}{i\alpha} \left[ \frac{\Xi_{+}(g(s) - i\alpha)g'(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-g(s)+i\alpha)}} - \frac{\Xi_{-}(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-s)}} \right] \right. \\ \left. + \left\{ \int_{-r}^{-r+\kappa^{-}-i\alpha} + \int_{r+\kappa^{+}-i\alpha}^r \right\} \frac{\tilde{\gamma}_{+} ds}{i\alpha} \frac{\Xi(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-s)}} \right]. \quad (2.8) \end{aligned}$$

One may now take the  $r \rightarrow +\infty$  limit. The contribution from  $[-r; -r + \kappa^{-} - i\alpha]$  goes to zero because the numerator is bounded while the denominator blows up exponentially fast in  $r$ . The contribution from  $[r + \kappa^{+} - i\alpha; r]$  goes to zero because the denominator approaches 1 while the numerator goes to zero uniformly on this bounded segment. Finally, the integral over  $[-r; r]$  converges in the limit since, for  $s \rightarrow \pm\infty$ , the integrand is a  $\mathcal{O}(s^{-2})$  owing to the form of the uniform up to the boundary asymptotic expansion of  $\Xi$ , *c.f.* (2.4). This thus yields

$$\Xi(z) = \int_{\mathbb{R}} \frac{\tilde{\gamma}_{+} ds}{i\alpha} \left\{ \frac{\Xi_{+}(g(s) - i\alpha)g'(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-g(s)+i\alpha)}} - \frac{\Xi_{-}(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-s)}} \right\}. \quad (2.9)$$

Then, by using the jump condition, the fast decay of  $G_{\Xi}$  at infinity and the fact that, for  $x > M$

$$g(x) = x + \kappa^{+} \quad \text{while} \quad \tilde{\gamma}_{+}(-i\alpha + \kappa^{+}) = -i\alpha, \quad (2.10)$$

one gets the representation

$$\Xi(z) = \int_{-\infty}^M \frac{\tilde{\gamma}_{+} ds}{i\alpha} \Xi_{+}(g(s) - i\alpha) \left\{ \frac{g'(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-g(s)+i\alpha)}} - \frac{1}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-s)}} \right\} + \int_{\mathbb{R}} \frac{\tilde{\gamma}_{+} ds}{i\alpha} \frac{G_{\Xi}(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_{+}}{\alpha}(z-s)}}. \quad (2.11)$$

The last integral may be recast in a form which allows one to readily extract the asymptotic behaviour at  $\Re(z) \rightarrow +\infty$ . For that purpose, one observes that

$$\int_{\mathbb{R}} \frac{\tilde{\gamma}_+ ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s) - \mathcal{G}_{\Xi}(g(s) - i\alpha)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} = \int_{-\infty}^M \frac{\tilde{\gamma}_+ ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s) - \mathcal{G}_{\Xi}(g(s) - i\alpha)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} - \int_{M+\kappa^+-i\alpha}^M \frac{\tilde{\gamma}_+ ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} + \int_{\mathcal{C}'_M} \frac{\tilde{\gamma}_+ ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} \quad (2.12)$$

where

$$\mathcal{C}'_M = ] + \infty - i\alpha ; M + \kappa^+ - i\alpha ] \cup [M + \kappa^+ - i\alpha ; M] \cup [M ; +\infty[ . \quad (2.13)$$

The last integral can be taken by residues, hence leading to

$$\begin{aligned} \Xi(z) = & -\mathcal{G}_{\Xi}(z) + \int_{-\infty}^M \frac{\tilde{\gamma}_+ ds}{i\alpha} \Xi_+(g(s) - i\alpha) \left\{ \frac{g'(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-g(s)+i\alpha)}} - \frac{1}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} \right\} + \int_{\mathbb{R}} \frac{\tilde{\gamma}_+ ds}{i\alpha} \frac{G_{\Xi}^{(c)}(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} \\ & + \int_{-\infty}^M \frac{\tilde{\gamma}_+ ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s) - \mathcal{G}_{\Xi}(g(s) - i\alpha)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} - \int_{M+\kappa^+-i\alpha}^M \frac{\tilde{\gamma}_+ ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s)}{1 - e^{\frac{2\pi\tilde{\gamma}_+}{\alpha}(z-s)}} . \end{aligned} \quad (2.14)$$

The form of the asymptotic expansion at  $\Re(z) \rightarrow +\infty$  is readily deduced from this representation.

Quite similarly to the previous case, one infers the integral representation

$$\begin{aligned} \Xi(z) = & \lim_{r \rightarrow +\infty} \int_{\mathcal{C}_{r,\alpha}} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\Xi(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \\ = & \lim_{r \rightarrow +\infty} \left[ \int_{-r}^r \frac{\tilde{\gamma}_- ds}{i\alpha} \left[ \frac{\Xi_+(g(s) - i\alpha)g'(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(g(s)-z-i\alpha)} - 1} - \frac{\Xi_-(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \right] + \left\{ \int_{-r}^{-r+\kappa^- - i\alpha} + \int_{r+\kappa^+ - i\alpha}^r \right\} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\Xi(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \right]. \end{aligned} \quad (2.15)$$

The integral over the segment  $[-r ; -r + \kappa_- - i\alpha]$  produces  $C_{\Xi}$  plus terms vanishing when  $r \rightarrow +\infty$  due to the form of the  $\Re(z) \rightarrow -\infty$  asymptotics of  $\Xi$ , *c.f.* (2.4). The integral over  $[r + \kappa_+ - i\alpha ; r]$  vanishes since  $\Xi$  is bounded in that direction and the denominator blows up exponentially fast. Finally, the integrand of the first integral appearing in (2.15) is in  $L^1(\mathbb{R})$  due to the asymptotic behaviour of  $\Xi$  at infinity.

Then, proceeding analogously as in the  $\Re(z) \rightarrow +\infty$  case one gets

$$\Xi(z) = C_{\Xi} + \int_{-M}^{+\infty} \frac{\tilde{\gamma}_- ds}{i\alpha} \Xi_+(g(s) - i\alpha) \cdot \left\{ \frac{g'(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(g(s)-z-i\alpha)} - 1} - \frac{1}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \right\} + \int_{\mathbb{R}} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{G_{\Xi}(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} . \quad (2.16)$$

Furthermore, one has

$$\begin{aligned} \int_{\mathbb{R}} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s) - \mathcal{G}_{\Xi}(g(s) - i\alpha)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} &= \int_{-M}^{+\infty} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s) - \mathcal{G}_{\Xi}(g(s) - i\alpha)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \\ &- \int_{-M}^{-M+\kappa^- - i\alpha} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} + \int_{\mathcal{E}_M''} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \end{aligned} \quad (2.17)$$

where

$$\mathcal{E}_M'' = [-\infty; -M[\cup] - M; -M + \kappa^- - i\alpha] \cup [-M + \kappa^- - i\alpha; -\infty - i\alpha]. \quad (2.18)$$

This yields

$$\begin{aligned} \Xi(z) &= C_{\Xi} - \mathcal{G}_{\Xi}(z) + \int_{-M}^{+\infty} \frac{\tilde{\gamma}_- ds}{i\alpha} \Xi_+(g(s) - i\alpha) \cdot \left\{ \frac{g'(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(g(s)-z-i\alpha)} - 1} - \frac{1}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \right\} + \int_{\mathbb{R}} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{G_{\Xi}^{(c)}(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} \\ &+ \int_{-M}^{+\infty} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s) - \mathcal{G}_{\Xi}(g(s) - i\alpha)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1} - \int_{-M}^{-M+\kappa^- - i\alpha} \frac{\tilde{\gamma}_- ds}{i\alpha} \frac{\mathcal{G}_{\Xi}(s)}{e^{\frac{2\pi\tilde{\gamma}_-}{\alpha}(s-z)} - 1}. \end{aligned} \quad (2.19)$$

Thus, the asymptotic behaviour at  $\Re(z) \rightarrow -\infty$  follows, along with its uniformness up to the boundary.  $\blacksquare$

One should note that the representation (2.14) clearly indicates that the  $\pm$  boundary values  $\Xi_{\pm}(x)$  for  $x \geq M$  only depend on the boundary values  $\Xi_+(g(y) - i\alpha)$ , with  $y < M$ . The fact that the boundary values  $\Xi_+(x - i\alpha)$  and  $\Xi_-(x)$  are smooth when  $x \geq M$  is also clear from this representation. A similar property can be inferred from (2.19) relatively to the properties of the boundary values when  $x < -M$ .

### • Correspondence with integral equations

We now establish a one-to-one correspondence between solutions to the non-local Riemann-Hilbert problem for  $\Xi$  and solutions to certain linear integral equations on the space

$$\mathcal{E}(\mathbb{R}) = \left\{ f \in L_{\text{loc}}^2(\mathbb{R}); \exists C_f \text{ and } \eta > 0 \quad f(x) = C_f \delta_{\pm; -} + \mathcal{O}(e^{\mp\eta x}) \quad \text{for } x \rightarrow \pm\infty \right\}. \quad (2.20)$$

**Lemma 2.2.** *Let  $\Xi$  be a solution to the non-local Riemann-Hilbert problem for  $\Xi$ . Then  $\Xi$  has smooth boundary values on  $\partial\mathcal{S}_{\alpha}$ . Moreover, given any  $\tau > \alpha$ , the function  $\theta(x) = \Xi_+(g(x) - i\alpha)$  belongs to  $\mathcal{E}(\mathbb{R})$  and solves the linear integral equation on  $\mathcal{E}(\mathbb{R})$*

$$(\text{id} - \mathcal{K})[\theta](x) = \frac{1}{2} \left\{ G_{\Xi}(x) + \mathcal{H}_{\mathbb{R}}[G_{\Xi}](x) \right\} - \mathcal{K}_{12}[G_{\Xi}](x), \quad (2.21)$$

where

$$\mathcal{H}_{\mathbb{R}}[f](x) = \int_{\mathbb{R}} \frac{dy}{i\tau} \frac{f(y)}{\sinh\left[\frac{\pi}{\tau}(y-x)\right]}, \quad (2.22)$$

with the principal value prescription for the integral, is the sinh-Hilbert transform on  $L^2(\mathbb{R})$  and

$$K = K_{12} + K_{21} + K_{11} \quad (2.23)$$

is built up in terms of the three integral operators

$$K_{12}[h](x) = - \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{h(y)}{\sinh\left[\frac{\pi}{\tau}(y - g(x) + i\alpha)\right]}, \quad K_{21}[h](x) = \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{h(y)g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y) - x - i\alpha)\right]} \quad (2.24)$$

and

$$K_{11}[h](x) = \int_{\mathbb{R}} \frac{dy}{2i\tau} h(y) \left\{ \frac{g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y) - g(x))\right]} - \frac{1}{\sinh\left[\frac{\pi}{\tau}(y - x)\right]} \right\}. \quad (2.25)$$

Reciprocally, any solution  $\theta \in \mathcal{E}(\mathbb{R})$  to the linear integral equation (2.21) gives rise to a solution to the non-local Riemann–Hilbert problem for  $\Xi$ .

*Proof—*

The asymptotic behaviour of  $\Xi$  at infinity ensures that, for any  $z \in \mathcal{S}_\alpha$ , one has:

$$\Xi(z) = \int_{\partial\mathcal{S}_\alpha} \frac{dy}{2i\tau} \frac{\Xi(y)}{\sinh\left[\frac{\pi}{\tau}(y - z)\right]}. \quad (2.26)$$

Then, by setting  $\theta_1(x) = \Xi_+(p_1(x))$  and  $\theta_2(x) = \Xi_-(p_2(x))$ , one gets

$$\Xi(z) = - \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta_2(y)}{\sinh\left[\frac{\pi}{\tau}(y - z)\right]} + \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta_1(y)g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y) - z - i\alpha)\right]}. \quad (2.27)$$

Furthermore, the above integral representation leads to the following relations for the  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ :

$$\begin{cases} \frac{1}{2}\theta_2(x) = K_{21}[\theta_1](x) - \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta_2(y)}{\sinh\left[\frac{\pi}{\tau}(y - x)\right]} \\ \frac{1}{2}\theta_1(x) = K_{12}[\theta_2](x) + \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta_1(y)g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y) - g(x))\right]} \end{cases}. \quad (2.28)$$

Then, adding up the two above equations and using explicitly the form of the jump conditions  $\theta_1(x) = \theta_2(x) + G_\Xi(x)$ , one gets the linear integral equation (2.21). The latter allows one to represent  $\theta_1$  as

$$\theta_1(x) = K[\theta_1](x) + \frac{1}{2} \left\{ G_\Xi(x) + \mathcal{H}_{\mathbb{R}}[G_\Xi - G_\Xi(x)](x) \right\} - K_{12}[G_\Xi](x) \quad (2.29)$$

from which the smoothness of  $\Xi_+(g(x) - i\alpha)$  on  $\mathbb{R}$  is manifest. The latter ensures that  $\Xi_+(x - i\alpha)$  is smooth as well. The smoothness of  $\Xi_-$  follows from  $\Xi_-(x) = \theta_1(x) - G_\Xi(x)$ . Finally, one has that  $\theta_1 \in \mathcal{E}(\mathbb{R})$  as can be inferred from the asymptotic behaviour (2.5) established in Lemma 2.1 and the hypotheses on  $\mathcal{G}_\Xi$  that are outlined after (2.3).

Reciprocally, let  $\theta$  be a solution to the linear integral equation (2.21) on  $\mathcal{E}(\mathbb{R})$ . Then, since the integral kernels  $K_{ab}$  and  $G_{\Xi}$  are all smooth, so is  $\theta$ . Next, one defines a holomorphic function  $\tilde{\Xi}$  on  $\mathcal{S}_{\tau-\alpha}$  as

$$\tilde{\Xi}(z) = - \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta(y)g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y)-z)\right]} + \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta(y) - G_{\Xi}(y)}{\sinh\left[\frac{\pi}{\tau}(y-z+i\alpha)\right]}. \quad (2.30)$$

It is direct to check that  $\tilde{\Xi}$  admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i(\tau - \alpha)$ . Furthermore, the asymptotic behaviour of  $\theta$  at infinity ensures that

$$\tilde{\Xi}(z) = C_{\tilde{\Xi}}\delta_{\pm;-} + O(e^{\mp\eta z}) \quad \text{as } \Re(z) \rightarrow \pm\infty \quad (2.31)$$

for some  $\eta > 0$  and some constant  $C_{\tilde{\Xi}}$ . Moreover, direct calculations using the linear integral equation satisfied by  $\theta$  ensure that  $\tilde{\Xi}_-(g(x)) = \tilde{\Xi}_+(x - i(\tau - \alpha))$ . Hence,  $\tilde{\Xi}$  satisfies the non-local Riemann-Hilbert problem outlined in the beginning of Subsection 2.1 under the replacement  $g \leftrightarrow g^{-1}$  and corresponding to a vanishing shift function  $G_{\tilde{\Xi}} = 0$ . As established below, such homogeneous non-local Riemann-Hilbert problems admit only zero solutions. In particular, this entails that  $\tilde{\Xi}_-(g(x)) = \tilde{\Xi}_+(x - i(\tau - \alpha)) = 0$ . These two equations provide one with an additional set of two singular linear integral equations satisfied by  $\theta$ , namely

$$\frac{1}{2}\theta(x) = \int \frac{dy}{2i\tau} \frac{\theta(y)g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y)-g(x))\right]} + \mathcal{K}_{12}[\theta - G_{\Xi}](x) \quad (2.32)$$

$$\frac{1}{2}\theta(x) - \frac{1}{2}G_{\Xi}(x) = - \int \frac{dy}{2i\tau} \frac{\theta(y) - G_{\Xi}(y)}{\sinh\left[\frac{\pi}{\tau}(g(y)-g(x))\right]} + \mathcal{K}_{21}[\theta](x). \quad (2.33)$$

This being settled, one introduces the holomorphic function  $\Xi$  on  $\mathcal{S}_{\alpha}$  such that

$$\Xi(z) = - \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta(y) - G_{\Xi}(y)}{\sinh\left[\frac{\pi}{\tau}(y-z)\right]} + \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{\theta(y)g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y)-z-i\alpha)\right]}. \quad (2.34)$$

As before,  $\Xi$  admits smooth  $\pm$  boundary values on  $\mathcal{S}_{\alpha}$  and enjoys the asymptotic behaviour at infinity

$$\Xi(z) = C_{\Xi}\delta_{\pm;-} + O(e^{\mp\eta z}) \quad \text{as } \Re(z) \rightarrow \pm\infty \quad (2.35)$$

for some  $\eta > 0$  and some constant  $C_{\Xi}$ . The two equations (2.32)-(2.33) ensure that  $\theta(x) = \Xi_+(g(x) - i\alpha)$  and  $\Xi_-(x) = \theta(x) - G_{\Xi}(x)$ . All in all, this ensures that  $\Xi$  is indeed a solution of the non-local Riemann-Hilbert problem for  $\Xi$ .  $\blacksquare$

## 2.2 Non-local Riemann-Hilbert problem in a smooth setting

The results obtained in Lemmata 2.1 and 2.2 ensure that the original Riemann-Hilbert problem for  $\Xi$  may be equivalently formulated in a setting which involves much better behaved function.

The Riemann-Hilbert problem for  $\Xi$  of interest consists in finding

- $\Xi \in \mathcal{O}(\mathcal{S}_{\alpha})$  having smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ , in particular,  $\Xi_+(p_1(x))$  and  $\Xi_-(p_2(x))$  are both smooth on  $\mathbb{R}$ .
- $\Xi_+(p_1(x)) = \Xi_-(p_2(x)) + G_{\Xi}(x)$ , with  $x \in \mathbb{R}$ ;
- there exists a constants  $C_{\Xi}$  and  $\eta > 0$  such that

$$\Xi(z) = C_{\Xi}\delta_{\pm;-} + O(e^{-\eta|\Re(z)|}) \quad \text{when } \Re(z) \rightarrow \pm\infty \quad (2.36)$$

with an asymptotic expansion that is valid uniformly up to the boundary.



### 2.3 Unique solvability of the homogeneous non-local Riemann-Hilbert problem and invertibility of $\text{id} - \mathbf{K}$

Lemma 2.2 established that any solution to the non-local Riemann-Hilbert problem for  $\Xi$  gives rise to a solution to the linear integral equation driven by  $\text{id} - \mathbf{K}$  on  $\mathcal{E}(\mathbb{R})$ . In fact, given two solutions  $\Xi_1, \Xi_2$  to the Riemann-Hilbert problem for  $\Xi$ , their difference  $\delta\Xi = \Xi_1 - \Xi_2$  satisfies the Riemann-Hilbert problem for  $\Xi$  associated with a vanishing shift function and thus  $\delta\theta(x) = \delta\Xi_-(x) = \delta\Xi_+(g(x) - i\alpha)$  gives rise to a solution to the homogeneous integral equation

$$(\text{id} - \mathbf{K})[\delta\theta] = 0. \quad (2.37)$$

Thus, the unique solvability of the non-local Riemann-Hilbert problem for  $\Xi$ , or, equivalently, the fact that only 0 solves the non-local Riemann-Hilbert problem associated with the zero shift function, is ensured once that invertibility<sup>†</sup> of  $\text{id} - \mathbf{K}$  on  $\mathcal{E}(\mathbb{R})$  is satisfied. The arguments developed in the literature which allow to establish this property, *see* [9] for the details, build strongly on the compactness of the operator  $\mathbf{K}$ , which, however, does not hold in the present setting. Consequently, one has to recourse to a more sophisticated reasoning in order to establish the invertibility of  $\text{id} - \mathbf{K}$ .

The main idea in the present case consists in splitting the operator  $\mathbf{K}$  introduced in (2.23) into three pieces

$$\mathbf{K} = \mathbf{L}^{++} + \mathbf{L}^{--} + \mathbf{B}. \quad (2.38)$$

This splitting is such that the operators  $\mathbf{L}^{\pm\pm}$  have a purely continuous spectrum while  $\mathbf{B}$  is compact and hence has a pointwise spectrum. Upon establishing the invertibility of the operators  $\text{id} - \mathbf{L}^{\pm\pm}$ , the decomposition (2.38) reduces the question of invertibility of  $\text{id} - \mathbf{K}$  to the invertibility of an auxiliary operator  $\text{id} - \mathbf{M}$ , where  $\mathbf{M}$  is compact. Once this stage of the analysis is reached, the remainder of the reasoning will be carried out within the standard techniques outlined in [9].

In the decomposition (2.38), the operator  $\mathbf{B}$  is defined as

$$\mathbf{B} = \mathbf{K}_{11} + \delta\mathbf{K}_{12} + \delta\mathbf{K}_{21}. \quad (2.39)$$

The integral kernels of the operators  $\delta\mathbf{K}_{12}$  and  $\delta\mathbf{K}_{21}$  are given by

$$\delta\mathbf{K}_{12}(x, y) = -\frac{1}{2i\tau} \left\{ \frac{1}{\sinh\left[\frac{\pi}{\tau}(y - g(x) + i\alpha)\right]} - \sum_{\nu=\pm} \frac{\mathbf{1}_{\mathbb{R}^\nu \times \mathbb{R}^\nu}(x, y)}{\sinh\left[\frac{\pi}{\tau}(y - x - \kappa^\nu + i\alpha)\right]} \right\} \quad (2.40)$$

and

$$\delta\mathbf{K}_{21}(x, y) = \frac{1}{2i\tau} \left\{ \frac{g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y) - x - i\alpha)\right]} - \sum_{\nu=\pm} \frac{\mathbf{1}_{\mathbb{R}^\nu \times \mathbb{R}^\nu}(x, y)}{\sinh\left[\frac{\pi}{\tau}(y + \kappa^\nu - x - i\alpha)\right]} \right\}. \quad (2.41)$$

Finally, the operators  $\mathbf{L}^{\pm\pm}$  appearing in (2.38) are integral operators on  $L^2(\mathbb{R}^\pm)$  with integral kernels  $L^{\nu\nu}(x, y) = L^\nu(x - y) \cdot \mathbf{1}_{\mathbb{R}^\nu \times \mathbb{R}^\nu}(x, y)$  for the difference-dependent

$$L^\nu(x - y) = \frac{1}{2i\tau} \left\{ \frac{1}{\sinh\left[\frac{\pi}{\tau}(y + \kappa^\nu - x - i\alpha)\right]} - \frac{1}{\sinh\left[\frac{\pi}{\tau}(y - \kappa^\nu - x + i\alpha)\right]} \right\}. \quad (2.42)$$

We now establish that  $\mathbf{B}$  is a compact, Hilbert-Schmidt operator.

---

<sup>†</sup>In fact, it is enough that (2.37) does not admit solutions in the class of functions corresponding to boundary values of holomorphic functions solving the zero shift non-local Riemann-Hilbert problem for  $\Xi$ . However, as shown below, this is in fact equivalent to the invertibility.

**Proposition 2.3.** *B is a Hilbert-Schmidt operator on  $L^2(\mathbb{R})$ . Its integral kernel  $B(x, y)$  is smooth on  $\mathbb{R}^u \times \mathbb{R}^{u'}$ ,  $u, u' \in \{\pm\}$  and enjoys the bounds*

$$B(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right). \quad (2.43)$$

*The remainder appearing above is differentiable in the sense that  $\partial_x^k \partial_y^\ell B(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right)$  on  $\mathbb{R}^u \times \mathbb{R}^{u'}$  for integers  $k, \ell \in \mathbb{N}$ . The control is however not uniform with respect to the order of the derivatives.*

*Proof—*

Smoothness of  $B(x, y)$  on  $\mathbb{R}^u \times \mathbb{R}^{u'}$  is evident from (2.39)-(2.41). The Hilbert-Schmidt nature of  $B$  is a direct consequence of the bounds (2.43). Finally, in order to establish (2.43) one bounds each of the three building blocks of the operator  $B$  separately.

First of all consider  $K_{11}(x, y)$ . For  $|y| \leq M$  and  $x \rightarrow \pm\infty$  one obviously gets  $K_{11}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}|x|}\right)$ . A similar bound holds for  $|x| \leq M$  and  $y \rightarrow \pm\infty$ :  $K_{11}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}|y|}\right)$ . Finally, observe that, if  $|x| \geq M$  and  $|y| \geq M$  and  $xy > 0$  then obviously  $K_{11}(x, y) = 0$ . While, for  $xy < 0$ , one gets that  $K_{11}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}|x-y|}\right) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right)$ . All in all,

$$K_{11}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right). \quad (2.44)$$

Regarding to  $\delta K_{12}(x, y)$ . Assume that  $(x, y) \in (\mathbb{R}^\pm)^2$  then, by construction  $\delta K_{12}(x, y) = 0$  if  $|x| \geq M$ , while for  $|x| \leq M$ ,  $\delta K_{12}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}|y|}\right)$ . It remains to focus on the situation when  $xy < 0$ . Then, taken the difference form of the kernel, for large arguments one gets that  $\delta K_{12}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}|x-y|}\right) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right)$ . Thus, all in all

$$\delta K_{12}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right). \quad (2.45)$$

Finally, regarding to  $\delta K_{21}(x, y)$  the reasoning is quite analogous. Assume that  $(x, y) \in (\mathbb{R}^\pm)^2$ . Then, by construction  $\delta K_{12}(x, y) = 0$  if  $|y| \geq M$ , while for  $|y| \leq M$ ,  $\delta K_{12}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}|x|}\right)$ . When  $xy < 0$  the difference form of the kernel leads to  $\delta K_{12}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right)$ . Thus

$$\delta K_{21}(x, y) = \mathcal{O}\left(e^{-\frac{\pi}{\tau}(|x|+|y|)}\right). \quad (2.46)$$

Furthermore, it is clear from the reasonings above that the remainders are differentiable on  $\mathbb{R}^u \times \mathbb{R}^{u'}$ .

Taken all together, this yields the claimed bounds on the integral kernel  $B(x, y)$ . ■

### 2.3.1 Preliminary notations

It will appear useful, at various instances, to introduce the basic building block function

$$m_\zeta(x) = \frac{1}{2i\tau \sinh\left[\frac{\pi}{\tau}(x - i\zeta)\right]} \quad \text{so that} \quad \mathcal{F}[m_{\pm\zeta}](k) = \frac{\pm e^{\mp k\zeta}}{1 + e^{\mp k\tau}}, \quad (2.47)$$

provided that  $0 < \Re(\zeta) < \tau$ . Here and in the following, we define the Fourier transform, whenever it makes sense, with the convention

$$\mathcal{F}[f](k) = \int_{\mathbb{R}} dx f(x) e^{ikx}. \quad (2.48)$$

Since one can express the difference dependent integral kernels appearing in  $K$  as

$$L^\nu(x) = m_{\alpha+i\kappa^\nu}(x) - m_{-\alpha-i\kappa^\nu}(x), \quad (2.49)$$

with  $\nu \in \{\pm\}$ , one infers that

$$\mathcal{F}[L^\nu](k) = \frac{\cosh[k(\frac{\tau}{2} - \alpha - i\kappa^\nu)]}{\cosh[\frac{k\tau}{2}]} \quad (2.50)$$

so that

$$1 - \mathcal{F}[L^\nu](k) = 2 \frac{\sinh[\frac{k}{2}(\alpha + i\kappa^\nu)] \cdot \sinh[\frac{k}{2}(\tau - \alpha - i\kappa^\nu)]}{\cosh[\frac{k\tau}{2}]} . \quad (2.51)$$

Consider the function  $k \mapsto 1 - \mathcal{F}[L^\nu](k)$  on  $\mathbb{R} + i\nu\nu$  with  $0 < \nu \ll 1$ . It is non-vanishing on this line. Furthermore, introduce

$$\mathcal{B}_\uparrow^{(\pm)} = \{z \in \mathbb{C} : \Im(z) > \pm\nu\} \quad \text{and} \quad \mathcal{B}_\downarrow^{(\pm)} = \{z \in \mathbb{C} : \Im(z) < \pm\nu\} . \quad (2.52)$$

Then,  $1 - \mathcal{F}[L^\nu]$  admits the Wiener-Hopf factorisation

$$1 - \mathcal{F}[L^\nu](k) = \frac{\alpha_\uparrow^{(\nu)}(k)}{\alpha_\downarrow^{(\nu)}(k)} \quad (2.53)$$

such that

- $\alpha_\uparrow^{(\nu)} \in \mathcal{O}(\overline{\mathcal{B}_\uparrow^{(\nu)}})$  and  $\alpha_\downarrow^{(\nu)} \in \mathcal{O}(\overline{\mathcal{B}_\downarrow^{(\nu)}} \setminus \{0\})$ ;
- $\alpha_{\uparrow/\downarrow}^{(\nu)}(k) \rightarrow 1$  when  $k \rightarrow \infty$  with  $k \in \overline{\mathcal{B}_{\uparrow/\downarrow}^{(\nu)}}$ .

The Wiener-Hopf factors are given explicitly in terms of Gamma functions as

$$\alpha_\uparrow^{(\nu)}(k) = -ik \frac{\sqrt{2\pi A^\nu B^\nu} \cdot [C^\nu]^{ikC^\nu}}{[A^\nu]^{ikA^\nu} \cdot [B^\nu]^{ikB^\nu}} \Gamma\left( \frac{1}{2} - iC^\nu k, 1 - iA^\nu k \right) \quad (2.54)$$

and

$$\alpha_\downarrow^{(\nu)}(k) = ik \sqrt{\frac{A^\nu B^\nu}{2\pi}} \frac{[C^\nu]^{ikC^\nu}}{[A^\nu]^{ikA^\nu} \cdot [B^\nu]^{ikB^\nu}} \Gamma\left( iA^\nu k, iB^\nu k, \frac{1}{2} + iC^\nu k \right) . \quad (2.55)$$

There, we adopted the conventions introduced in (1.16) and made use of the following shorthand notations:

$$A^\nu = \frac{\alpha + i\kappa^\nu}{2\pi}, \quad B^\nu = \frac{\tau - \alpha - i\kappa^\nu}{2\pi} \quad \text{and} \quad C^\nu = \frac{\tau}{2\pi} . \quad (2.56)$$

Note that  $\alpha_{\uparrow/\downarrow}^{(\nu)}$  admit meromorphic continuations to  $\mathcal{B}_{\downarrow/\uparrow}^{(\nu)}$ . Furthermore,  $\alpha_\uparrow^{(\nu)}$  has a simple zero at  $k = 0$  and this is its only zero in some open neighbourhood of  $\mathbb{R}$ . Also,  $\alpha_\downarrow^{(\nu)}$  admits a simple pole at  $k = 0$  and it is its only simple pole in some open neighbourhood of  $\mathbb{R}$ . For further convenience, we parameterise this local behaviour as

$$\alpha_\uparrow^{(\nu)}(k) \underset{k \rightarrow 0}{\sim} k \alpha_0^{(\nu)} \quad \text{and} \quad \alpha_\downarrow^{(\nu)}(k) \underset{k \rightarrow 0}{\sim} \frac{\tilde{\alpha}_0^{(\nu)}}{k} . \quad (2.57)$$

One has

$$\alpha_0^{(\nu)} \tilde{\alpha}_0^{(\nu)} = -1 . \quad (2.58)$$

### 2.3.2 Preparatory decomposition for $\text{id} - \mathbf{K}$

We are now in position to discuss the invertibility of the operator  $\text{id} - \mathbf{K}$  on the space  $\mathcal{E}(\mathbb{R})$  as defined in (2.20), with  $\mathbf{K}$  as introduced in (2.23) and rewritten in (2.38). Assume that one is given a solution  $f \in \mathcal{E}(\mathbb{R})$  to  $(\text{id} - \mathbf{K})[f] = h$  with  $h$  having an exponential fall-off at  $\pm\infty$ , viz.  $h(x) = \mathcal{O}(e^{-\eta|x|})$  for some  $\eta > 0$ . Then, one may recast the equation in a matrix form relatively to the decomposition  $\mathcal{E}(\mathbb{R}) = \mathcal{E}(\mathbb{R}^-) \oplus \mathcal{E}(\mathbb{R}^+)$  with

$$\mathcal{E}(\mathbb{R}^\sigma) = \{f \cdot \mathbf{1}_{\mathbb{R}^\sigma} : f \in \mathcal{E}(\mathbb{R})\}, \quad (2.59)$$

as

$$\begin{pmatrix} \text{id} - \mathbf{L}^{--} - \mathbf{B}^{--} & -\mathbf{B}^{-+} \\ -\mathbf{B}^{+-} & \text{id} - \mathbf{L}^{++} - \mathbf{B}^{++} \end{pmatrix} \cdot \begin{pmatrix} f^- \\ f^+ \end{pmatrix} = \begin{pmatrix} h^- \\ h^+ \end{pmatrix}. \quad (2.60)$$

Or, more explicitly

$$(\text{id} - \mathbf{L}^{--})[f^-] = \mathbf{B}^{-+}[f^+] + \mathbf{B}^{--}[f^-] + h^- \equiv H^-, \quad (2.61)$$

$$(\text{id} - \mathbf{L}^{++})[f^+] = \mathbf{B}^{++}[f^+] + \mathbf{B}^{+-}[f^-] + h^+ \equiv H^+. \quad (2.62)$$

It follows from the above and Proposition 2.3 that the functions  $f^\pm$  and  $H^\pm$  do belong to the classes considered in Subsections B.1-B.2 of the Appendix. Then, the results from these sections entail that, for  $\nu > 0$  and small enough

$$\mathcal{F}[f^-](k) = - \int_{\mathbb{R}+i\nu} \frac{ds}{2i\pi} \frac{\alpha_\downarrow^{(-)}(k) \cdot \mathcal{F}[H^-](s)}{\alpha_\uparrow^{(-)}(s) \cdot (s-k)}, \quad k \in \mathbb{R} - i\nu, \quad (2.63)$$

$$\mathcal{F}[f^+](k) = \int_{\mathbb{R}-i\nu} \frac{ds}{2i\pi} \frac{\alpha_\downarrow^{(+)}(s) \cdot \mathcal{F}[H^+](s)}{\alpha_\uparrow^{(+)}(k) \cdot (s-k)}, \quad k \in \mathbb{R} + i\nu. \quad (2.64)$$

One may recast the system of equations subordinate to (2.63)-(2.64) as a matrix integral equation on  $L^2(\mathbb{R} - i\nu) \oplus L^2(\mathbb{R} + i\nu)$  on the unknown vector

$$\mathbf{u} = \begin{pmatrix} u^- \\ u^+ \end{pmatrix} \quad \text{with} \quad u^\sigma = \mathcal{F}[f^\sigma], \quad \sigma = \pm. \quad (2.65)$$

For that purpose one observes that

$$\mathcal{F}[\mathbf{B}^{\epsilon\sigma}[f^\sigma]](k) = \int_{\mathbb{R}+i\sigma\nu} dk \widehat{\mathbf{B}}^{\epsilon\sigma}(k, s) \mathcal{F}[f^\sigma](s) \quad (2.66)$$

in which

$$\widehat{\mathbf{B}}^{\epsilon\sigma}(k, s) = \int_{\mathbb{R}^\epsilon \times \mathbb{R}^\sigma} \frac{dx dy}{2\pi} e^{ikx - isy} B(x, y). \quad (2.67)$$

The fact that the Fourier transforms  $\widehat{\mathbf{B}}^{\epsilon\sigma}(k, s)$  are well-defined for  $(k, s) \in \{\mathbb{R} - i\epsilon\nu\} \times \{\mathbb{R} + i\sigma\nu\}$  is a direct consequence of Proposition 2.3. In fact, one has that  $\widehat{\mathbf{B}}^{\epsilon\sigma}(k, s)$  is analytic in  $k, s$  belonging to a tubular neighbourhood of  $\mathbb{R}^2$ . Moreover, the asymptotics established in Proposition 2.3 entail that, for some constant  $C > 0$ ,

$$|\widehat{\mathbf{B}}^{\epsilon\sigma}(k, s)| \leq \frac{C}{(1 + |s|) \cdot (1 + |k|)} \quad (2.68)$$

uniformly throughout this tubular neighbourhood. Taken (2.63)-(2.64), it appears convenient to introduce

$$M^{+\sigma}(k, t) = \frac{1}{\alpha_{\uparrow}^{(+)}(k)} \cdot \int_{\mathbb{R}-i\nu} \frac{ds}{2i\pi} \frac{\alpha_{\downarrow}^{(+)}(s)}{s-k} \widehat{B}^{+\sigma}(s, t) \quad (2.69)$$

and

$$M^{-\sigma}(k, t) = -\alpha_{\downarrow}^{(-)}(k) \cdot \int_{\mathbb{R}+i\nu} \frac{ds}{2i\pi} \frac{\{\alpha_{\uparrow}^{(-)}(s)\}^{-1}}{s-k} \widehat{B}^{-\sigma}(s, t). \quad (2.70)$$

It is direct to check that  $M^{\epsilon\sigma}$  is smooth on  $\{\mathbb{R} + i\epsilon\nu\} \times \{\mathbb{R} + i\sigma\nu\}$ . Furthermore, it follows from Lemma D.1 that there exists a constant  $C$  such that

$$|M^{\epsilon\sigma}(k, t)| \leq \frac{C \cdot \ln(1 + |k|)}{(1 + |k|) \cdot (1 + |t|)} \quad \text{for } (k, t) \in \{\mathbb{R} + i\epsilon\nu\} \times \{\mathbb{R} + i\sigma\nu\}. \quad (2.71)$$

This entails that the operator  $\mathbf{M}$  on  $L^2(\mathbb{R} - i\nu) \oplus L^2(\mathbb{R} + i\nu)$  given in matrix form

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}^{--} & \mathbf{M}^{-+} \\ \mathbf{M}^{+-} & \mathbf{M}^{++} \end{pmatrix} \quad (2.72)$$

is Hilbert-Schmidt. Indeed, the Hilbert-Schmidt norm of interest takes the form

$$\|\mathbf{M}\|_{HS}^2 = \sum_{\epsilon, \sigma} \int_{\mathbb{R}+i\epsilon\nu} dk \int_{\mathbb{R}+i\sigma\nu} ds |M^{\epsilon\sigma}(k, s)|^2 \quad (2.73)$$

and its finiteness follows from the bounds (2.71).

Then, introducing

$$\mathfrak{d}^+[h](k) = \int_{\mathbb{R}-i\nu} \frac{ds}{2i\pi} \frac{\alpha_{\downarrow}^{(+)}(s) \cdot \mathcal{F}[h^+](s)}{\alpha_{\uparrow}^{(+)}(k) \cdot (s-k)}, \quad (2.74)$$

$$\mathfrak{d}^-[h](k) = - \int_{\mathbb{R}+i\nu} \frac{ds}{2i\pi} \frac{\alpha_{\downarrow}^{(-)}(s) \cdot \mathcal{F}[h^-](s)}{\alpha_{\uparrow}^{(-)}(k) \cdot (s-k)}, \quad (2.75)$$

one ends up with the linear integral equation

$$(\text{id} - \mathbf{M})[\mathbf{u}] = \mathfrak{d}[h] \quad \text{with} \quad \mathbf{u} = \begin{pmatrix} u^- \\ u^+ \end{pmatrix} \quad \text{and} \quad \mathfrak{d}[h] = \begin{pmatrix} \mathfrak{d}^-[h] \\ \mathfrak{d}^+[h] \end{pmatrix}. \quad (2.76)$$

Reciprocally, given any solution  $\mathbf{u}$  to the above equation, by using the analyticity properties of the functions  $\mathfrak{d}^{\sigma}[h]$  and the integral kernels  $M^{\sigma\nu}(k, s)$ , it is direct to infer from the representation

$$u^{\sigma}(k) = \mathbf{M}^{\sigma+}[u^+](k) + \mathbf{M}^{\sigma-}[u^-](k) + \mathfrak{d}^{\sigma}[h](k) \quad (2.77)$$

that  $u^{\sigma} \in \mathcal{O}(\mathbb{H}^{\sigma})$  and that  $u^{\sigma}(k) = C^{\sigma}/k + \mathcal{O}(k^{-2})$  on  $\mathbb{H}^{\sigma}$ . This entails that the functions

$$\psi^{(\sigma)}(x) = \int_{\mathbb{R}+i\sigma\nu} \frac{dk}{2\pi} e^{-ikx} u^{\sigma}(k) \quad (2.78)$$

are supported on  $\mathbb{R}^\sigma$  and enjoy the asymptotic behaviour when  $x \rightarrow \sigma\infty$ :

$$\psi^{(\sigma)}(x) = C_{\psi^{(\sigma)}} + \mathcal{O}(e^{-\sigma vx}), \quad (2.79)$$

for some constants  $C_{\psi^{(\sigma)}}$ . Then, by carrying backwards the reasonings described in Subsections B.1-B.2, one infers that

$$(\text{id} - L^{\sigma\sigma})[\psi^{(\sigma)}](x) = \mathbf{B}^{\sigma+}[\psi^{(+)}](x) + \mathbf{B}^{\sigma-}[\psi^{(-)}](x) + h^\sigma(x) \quad (2.80)$$

for  $\sigma = \pm$  and  $x \in \mathbb{R}^\sigma$ . Upon setting  $\psi = \psi^{(+)} + \psi^{(-)}$ , one gets that equation (2.80) can be recast as  $(\text{id} - \mathbf{K})[\psi] = h$ . As a consequence, since constants are in the kernel of  $\text{id} - \mathbf{K}$ , it follows that the function  $\theta = \psi - C_{\psi^{(+)}}$  solves

$$(\text{id} - \mathbf{K})[\theta] = h \quad \text{with} \quad \theta(x) = C_\theta \delta_{\pm,-} + \mathcal{O}(e^{-v|x|}) \quad \text{when} \quad x \rightarrow \pm\infty. \quad (2.81)$$

### 2.3.3 Invertibility of $\text{id} - \mathbf{K}$ and unique solvability of the Riemann-Hilbert problem

**Proposition 2.4.** *The operator  $\mathbf{M}$  on  $L^2(\mathbb{R} - iv) \oplus L^2(\mathbb{R} + iv)$  defined through (2.69),(2.70) and (2.72) satisfies  $\det[\text{id} - \mathbf{M}] \neq 0$ . Moreover, the non-local Riemann-Hilbert problem for  $\Xi$  is uniquely solvable and the operator  $\text{id} - \mathbf{K}$  on  $\mathcal{E}(\mathbb{R})$  is invertible.*

Once that developments of Sub-Section 2.3.2 have been laid down, the proof closely follows the reasoning outlined in [9].

*Proof* —

We first establish that the non-local Riemann-Hilbert problem for  $\Xi$  associated with a zero shift function, *viz.* corresponding to  $G_\Xi = 0$ , has only the trivial solution  $\Xi = 0$ . Let  $\Xi$  be a non-vanishing solution to this zero shift problem. Thus,  $\Xi^n$  also solves this problem for any  $n \in \mathbb{N}$ . Setting  $\theta = \Xi_+ \circ p_2$ , one infers from Lemma 2.2 that  $\theta^n$  has the asymptotic behaviour when  $x \rightarrow \pm\infty$ :  $\theta(x) = C_\theta^n \delta_{\pm,-} + \mathcal{O}(e^{-v|x|})$  for some  $v > 0$  and solves  $(\text{id} - \mathbf{K})[\theta^n] = 0$ . Since  $\Xi$  is non-identically vanishing,  $\theta$  is non-identically vanishing as well. By building on the earlier considerations, one infers that

$$\mathbf{u}_n = \begin{pmatrix} \mathcal{F}[(\theta^n)^-] \\ \mathcal{F}[(\theta^n)^+] \end{pmatrix} \in \ker(\text{id} - \mathbf{M}). \quad (2.82)$$

We now establish that the  $\mathbf{u}_1, \dots, \mathbf{u}_k$  are linearly independent for any  $k$ . Let  $c_a$  be such that  $\sum_{n=1}^k c_n \mathbf{u}_n = 0$ . Component-wise this yields  $\sum_{n=1}^k c_n \mathcal{F}[(\theta^n)^\pm] = 0$ . Hence, by taking the inverse Fourier transform of the sum of these two relations, one gets that  $\sum_{n=1}^k c_n \theta^n = 0$ . Since the function  $\theta$  is non-zero, it also cannot be constant owing to its asymptotics at  $+\infty$ . Since  $\theta$  is non-constant, there exists  $x_0$  such that  $\theta'(x_0) \neq 0$ . Thus,  $\theta$  is a diffeomorphism in the neighbourhood of  $x_0$ . This entails that there exist a sequence  $x_1, x_2, \dots$  of pairwise distinct reals such that  $\theta(x_a) \neq \theta(x_b)$  for any  $a \neq b$ . One then infers from the relation  $\sum_{n=1}^k c_n \theta^n = 0$  the system of equations

$$\sum_{n=1}^k c_n \theta^n(x_s) = 0, \quad s = 1, \dots, k. \quad (2.83)$$

However, the latter has only trivial solutions owing to the invertibility of the associated Vandermonde matrix which stems from the condition  $\theta(x_a) \neq \theta(x_b)$ .

The linear independence of  $\mathbf{u}_1, \dots, \mathbf{u}_k$  for any  $k$  thus entails that  $\ker[\text{id} - \mathbf{M}]$  cannot be finite dimensional contradicting the compactness of  $\mathbf{M}$ . Thus the non-local Riemann-Hilbert problem for  $\Xi$  associated with a zero shift has only the trivial solution  $\Xi = 0$ .

We now establish that,  $\ker[\text{id} - \mathbb{M}] = 0$ . If not, then let  $\mathbf{u} \in \ker(\text{id} - \mathbb{M})$ ,  $\mathbf{u} \neq \mathbf{0}$ . Since  $\mathbf{u} \in L^2(\mathbb{R} - i\nu) \oplus L^2(\mathbb{R} + i\nu)$ , the large- $k$  asymptotic expansion of  $\widehat{B}^{\sigma\epsilon}(k, s)$  which follows from integration by parts and differentiability of the remainder in Proposition 2.3

$$\widehat{B}^{\sigma\epsilon}(k, s) \simeq \sum_{\ell \geq 0} \frac{\varphi_\ell(s)}{k^\ell} \quad \text{with} \quad |\varphi_\ell(s)| \leq \frac{C_\ell}{1 + |s|}, \quad (2.84)$$

entails that  $u^\sigma(k)$  admits the asymptotic expansion

$$u^\sigma(k) \simeq \sum_{\ell \geq 0} k^{-\ell} c_\ell^{(\sigma)} \quad \text{for} \quad \Re(k) \rightarrow \pm\infty. \quad (2.85)$$

Furthermore, the very structure of the integral kernels of the operator  $\mathbb{M}$  ensures that  $u^+ \in \mathcal{O}(\mathcal{B}_\uparrow^{(-)} \setminus \{0\})$ , resp.  $u^- \in \mathcal{O}(\mathcal{B}_\downarrow^{(+)} \setminus \{0\})$ . Also,  $u^\sigma$  admits a simple pole at 0. Thus, by taking the inverse Fourier transform, one gets a solution  $\theta$  to (2.81) with  $h = 0$ . By following the reasoning outlined in the proof of Lemma 2.2, this solution gives rise to a solution  $\widetilde{\Xi}$  to the non-local Riemann-Hilbert problem for  $\Xi$  having zero shift and subordinate to the replacement  $g \mapsto g^{-1}$  in the welding diffeomorphism. But then, by the above, this Riemann-Hilbert problem has only trivial solutions. This allows one to introduce, following the proof of Lemma 2.2, a holomorphic function  $\Xi$  on  $\mathcal{S}_\alpha$  solving the non-local Riemann-Hilbert problem for  $\Xi$  with a zero shift and such that  $\theta(x) = \Xi_+(g(x) - i\alpha)$ . Since this problem has only trivial solutions,  $\theta = 0$  and thus, by going backwards,  $\mathbf{u} = \mathbf{0}$  as well, which is a contradiction.

We have just established that  $\ker[\text{id} - \mathbb{M}] = 0$ . Thus, since  $\mathbb{M}$  is compact,  $\text{id} - \mathbb{M}$  is invertible and in particular  $\det[\text{id} - \mathbb{M}] \neq 0$ . The latter, by virtue of the construction described earlier on, ensures that  $\text{id} - \mathbb{K}$  is invertible as well.  $\blacksquare$

### 3 Special non-local Riemann-Hilbert problems

In the following, we shall consider two smooth diffeomorphisms of  $\mathbb{R}$ ,  $g_L$  and  $g_R$  which both satisfy  $g'_{L/R} > 0$  and such that

$$g_L(x) = \begin{cases} x & x < -M_L \\ g_L(x) & -M_L < x < M_L \\ x + \varkappa & M_L < x \end{cases} \quad \text{and} \quad g_R(x) = \begin{cases} x + \varkappa & x < -M_R \\ g_R(x) & -M_R < x < M_R \\ x & M_R < x \end{cases}, \quad (3.1)$$

for some  $M_L, M_R > 0$ . The purpose of this section is to establish the unique solvability of two non-local Riemann-Hilbert problems with shifts associated with the diffeomorphisms  $g_{L/R}$ . Prior to that, however, we shall establish some properties of the Cauchy transform on a welded strip which will play some role in later steps of the analysis.

#### 3.1 Cauchy transform on a welded strip

In the following, we shall set

$$\gamma = \frac{-\varkappa}{\varkappa - i\alpha} \quad \text{and} \quad \widetilde{\gamma} = \gamma + 1 = \frac{-i\alpha}{\varkappa - i\alpha} \quad (3.2)$$

where  $\varkappa \in \mathbb{R}$ . Observe that the constant  $\gamma$  is such that  $f(z) = \gamma z$  satisfies to the jump condition

$$f_+(x + \varkappa - i\alpha) = f_-(x) - \varkappa \quad x \in \mathbb{R}. \quad (3.3)$$

In this subsection, we establish the main properties of a Cauchy transform which has the appropriate symmetry to deal properly with the welding diffeomorphisms  $p_{1;\text{sev}}(x) = x + \varkappa - i\alpha$  from  $\mathbb{R}$  onto  $\mathbb{R} - i\alpha$ .

**Lemma 3.1.** *Let  $\Upsilon$  be a holomorphic function on  $\{z \in \mathbb{C} : -\alpha < \Im(z) < 0, y - 2|\kappa| < \Re(z) < y + \kappa + 2|\kappa|\}$  for some  $y \in \mathbb{R}$ , having continuous  $-$ , resp.  $+$ , boundary values on the upper, resp. lower, pieces of this domain and satisfying  $\Upsilon_+(x + \kappa - i\alpha) = \Upsilon_-(x)$ . Then, the Cauchy transform*

$$C_\Gamma[\Upsilon](z) = \int_\Gamma \frac{\tilde{\gamma} ds}{i\alpha} \frac{\Upsilon(s)}{e^{\frac{2\pi\tilde{\gamma}}{\alpha}(s-z)} - 1}, \quad (3.4)$$

with  $\Gamma = [y; y - i\alpha + \kappa]$  satisfies the following non-local Riemann-Hilbert problem on  $\mathcal{S}_\alpha$ :

- $C_\Gamma[\Upsilon] \in O(\mathcal{S}_\alpha \setminus \Gamma)$  and has holomorphic  $-$ , resp.  $+$ , boundary values on  $\Gamma$ ;
- $C_{\Gamma,+}[\Upsilon](s) - C_{\Gamma,-}[\Upsilon](s) = \Upsilon(s)$  for  $s \in \Gamma$ ;
- $C_\Gamma[\Upsilon](x + \kappa - i\alpha) = C_\Gamma[\Upsilon](x)$  for  $x \in \mathbb{R}$ ;
- up to the boundary  $\partial\mathcal{S}_\alpha$ , it holds

$$C_\Gamma[\Upsilon](z) = -\delta_{\pm,+} \int_\Gamma \frac{\tilde{\gamma} ds}{i\alpha} \Upsilon(s) + O\left(e^{\mp \frac{2\pi\tilde{\gamma}}{\alpha} z}\right) \quad \text{when} \quad \Re(z) \rightarrow \pm\infty. \quad (3.5)$$

*Proof*—

Most of the statements are rather evident, the non-local jump condition on the boundary following from

$$C_\Gamma[\Upsilon](x + \kappa - i\alpha) - C_\Gamma[\Upsilon](x) = \int_\Gamma \frac{\tilde{\gamma} ds}{i\alpha} \Upsilon(s) \cdot \left\{ \frac{1}{e^{\frac{2\pi}{\alpha}[\tilde{\gamma}(s-x) - \tilde{\gamma}(\kappa - i\alpha)]} - 1} - \frac{1}{e^{\frac{2\pi\tilde{\gamma}}{\alpha}(s-x)} - 1} \right\} = 0, \quad (3.6)$$

owing to  $\tilde{\gamma}(\kappa - i\alpha) = -i\alpha$ .

Furthermore, observe that  $C_\Gamma[\Upsilon]$  has cuts on  $\mathbb{C}$  along the curves  $\Gamma + i\frac{\alpha}{\tilde{\gamma}}\mathbb{Z}$ , viz. at the points

$$z = y + (-i\alpha + \kappa)t + i\alpha - n\kappa \quad \text{with} \quad t \in [0; 1], \quad n \in \mathbb{Z}, \quad (3.7)$$

which form the line in  $\mathbb{C}$  passing through  $\Gamma$ . For  $n \neq 0$ , none of these points is contained in  $\mathcal{S}_\alpha$ , thus  $\Gamma$  is indeed the sole discontinuity curve for  $C_\Gamma[\Upsilon]$  in the strip  $\mathcal{S}_\alpha$ .

The fact that  $z \mapsto C_{\Gamma,\pm}[\Upsilon](z)$  are holomorphic in a neighbourhood of  $\Gamma$  follows, for  $z \in \text{Int}(\Gamma)$ , from a contour deformation in (3.4) made possible by the fact that  $\Upsilon$  is analytic in the neighbourhood of  $\Gamma$ . Holomorphicity in the vicinity of the endpoints of  $\Gamma$  needs an extra care.

In the domain depicted in Figure 5, one defines, in the neighbourhood of the curve  $\Gamma_{\text{ext}}$  an analytic function  $\widehat{\Upsilon}$ . The fact that it is analytic follows from the jump conditions on  $\partial\mathcal{S}_\alpha$  satisfied by  $\Upsilon$ .

The curve  $\Gamma$  may be parameterised as

$$y + (-i\alpha + \kappa)t \quad \text{with} \quad t \in [0; 1], \quad \text{namely} \quad y - i\frac{\alpha t}{\tilde{\gamma}}, \quad \text{with} \quad t \in [0; 1]. \quad (3.8)$$



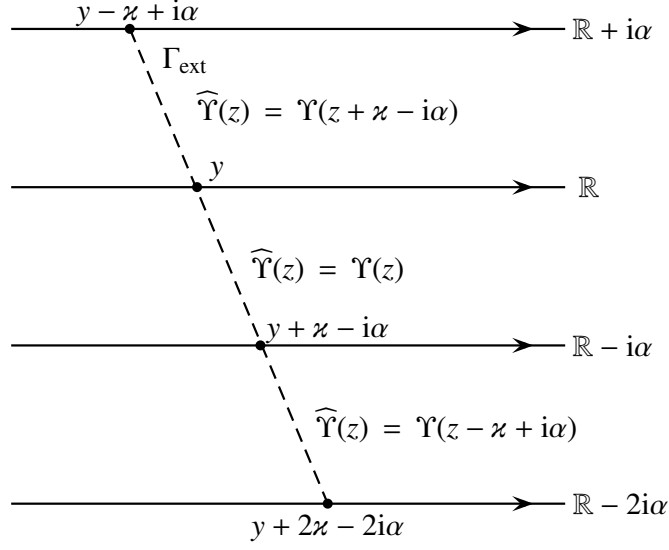


Figure 5: The extended strip with the extended curve  $\Gamma_{\text{ext}}$  such that  $\Gamma = \Gamma_{\text{ext}} \cap \mathcal{S}_\alpha$ .

Then, one has that

$$C_\Gamma[\Upsilon](z) = - \int_0^1 dt \frac{\Upsilon(y + (-i\alpha + \kappa)t)}{e^{\frac{2\pi\bar{\gamma}}{\alpha}(y-z)} e^{-2i\pi t} - 1} \quad (3.9)$$

$$= - \int_0^{\frac{1}{2}} dt \frac{\Upsilon(y + (-i\alpha + \kappa)t)}{e^{\frac{2\pi\bar{\gamma}}{\alpha}(y-z)} e^{-2i\pi t} - 1} - \int_{-\frac{1}{2}}^0 dt \frac{\Upsilon(y + (-i\alpha + \kappa)t + (-i\alpha + \kappa))}{e^{\frac{2\pi\bar{\gamma}}{\alpha}(y-z)} e^{-2i\pi t} - 1} \quad (3.10)$$

$$= \int_{\widehat{\Gamma}} \frac{\bar{\gamma} ds}{i\alpha} \frac{\widehat{\Upsilon}(s)}{e^{\frac{2\pi\bar{\gamma}}{\alpha}(s-z)} - 1} \quad (3.11)$$

in which  $\widehat{\Gamma} = [y + (i\alpha - \kappa)/2; y - (i\alpha - \kappa)/2]$ . The above representation produces manifestly holomorphic  $\pm$  boundary values around  $z = 0$ . A similar analysis allows one to conclude relatively to the point  $y + \kappa - i\alpha$ . ■

### 3.2 Left Riemann-Hilbert problem

The problem consists in finding  $\chi^{(L)} \in \mathcal{O}(\mathcal{S}_\alpha)$  such that  $\chi^{(L)}$  admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ , such that

- $\chi^{(L)}(z) = C_{\chi^{(L)}} + \mathcal{O}(e^{\frac{2\pi}{\alpha}z})$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;
- $\chi^{(L)}(z) = \gamma \cdot z + \mathcal{O}(e^{-\frac{2\pi\bar{\gamma}}{\alpha}z})$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- $\chi_+^{(L)}(g_L(x) - i\alpha) = \chi_-^{(L)}(x) + x - g_L(x)$ .

Above,  $\gamma$  and  $\tilde{\gamma}$  are as introduced in (3.2) while  $g_L$  is as given by (3.1).

**Proposition 3.2.** *The left non-local Riemann-Hilbert problem stated above admits a unique solution.*

*Proof—*

First, introduce the holomorphic function on an open neighbourhood  $S_\alpha$

$$\omega^{(L)}(z) = \frac{\gamma z}{e^{-\frac{2\pi}{\tau}z} + 1}, \quad (3.12)$$

where  $\tau > 2\alpha$ . The function  $\omega^{(L)}(z)$  may be decomposed as

$$\omega^{(L)}(z) = \gamma z + \omega_R^{(L)}(z) \quad \text{with} \quad \omega_R^{(L)}(z) = -\frac{\gamma z}{e^{\frac{2\pi}{\tau}z} + 1}. \quad (3.13)$$

As a consequence, the following estimates hold

$$\omega^{(L)}(z) = \begin{cases} O\left(ze^{\frac{2\pi}{\tau}z}\right) & \Re(z) \rightarrow -\infty \\ \gamma z + O\left(ze^{-\frac{2\pi}{\tau}z}\right) & \Re(z) \rightarrow +\infty \end{cases}. \quad (3.14)$$

The decomposition (3.13) entails that  $\omega^{(L)}$  satisfies

$$\omega^{(L)}(x + \varkappa - i\alpha) - \omega^{(L)}(x) = -\varkappa + \omega_R^{(L)}(x + \varkappa - i\alpha) - \omega_R^{(L)}(x) \quad \text{for } x \in \mathbb{R}. \quad (3.15)$$

Then, one makes the substitution in the Riemann-Hilbert problem for  $\chi^{(L)}$  as described in Figure 6.

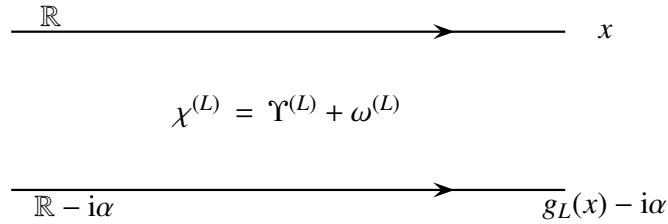


Figure 6: The substitution for the Riemann-Hilbert problem for  $\chi^{(L)}$ .

What results is the following Riemann-Hilbert problem for  $\Upsilon^{(L)}$ :

- $\Upsilon^{(L)} \in \mathcal{O}(S_\alpha)$ ;
- $\Upsilon^{(L)}(z) = -\omega_R^{(L)}(z) + O\left(e^{-\frac{2\pi\tilde{\gamma}}{\alpha}z}\right)$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- $\Upsilon^{(L)}(z) = C_{\chi^{(L)}} - \omega^{(L)}(z) + O\left(e^{\frac{2\pi}{\alpha}z}\right)$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;
- $\Upsilon^{(L)}$  admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\Upsilon_+^{(L)}(g_L(x) - i\alpha) = \Upsilon_-^{(L)}(x) + G_{\Upsilon^{(L)}}(x)$ , where the jump function takes the form

$$G_{\Upsilon^{(L)}}(x) = \begin{cases} \omega^{(L)}(x) - \omega^{(L)}(x - i\alpha) & x < -M_L \\ x - g_L(x) + \omega^{(L)}(x) - \omega^{(L)}(g_L(x) - i\alpha) & -M_L \leq x \leq M_L \\ \omega_R^{(L)}(x) - \omega_R^{(L)}(x + \varkappa - i\alpha) & M_L < x \end{cases}. \quad (3.16)$$

It follows from the stated properties of  $\omega^{(L)}$  that  $G_{\Upsilon^{(L)}}$  has the form (2.2) for  $g = g_L$ ,  $M = M_L$  and

$$\mathcal{G}_{\Upsilon^{(L)}}(z) = \omega_R^{(L)}(z) + \varphi_L(\Re(z)) \gamma z, \quad (3.17)$$

where  $\varphi_L(x)$  is a smooth interpolating function equal to 1 for  $x < -M_L/2$  and to 0 for  $x > M_L/2$  so that  $\mathcal{G}_{\Upsilon^{(L)}}$  is analytic on  $\mathcal{S}_\alpha$  for  $|\Re(z)| > M_L/2$  with exponential falloff at infinity. By virtue of Proposition 2.4 the non-local Riemann-Hilbert problem for  $\Upsilon^{(L)}$  is uniquely solvable, and hence, so is the one for  $\chi^{(L)}$ .

We point out that  $\chi^{(L)}$  readily allows one to build the solution to the non-local Riemann-Hilbert problem associated with a shifted function. Namely, for any  $w \in \mathbb{R}$ , let

$$G_L(x) = g_L(x+w) - w = \begin{cases} x & x < -M_L - w \\ g_L(x+w) - w & -M_L - w < x < M_L - w \\ x + w & M_L - w < x \end{cases}. \quad (3.18)$$

Then  $\Xi^{(L)}(z) = \chi^{(L)}(z+w)$  solves the Riemann-Hilbert problem:

- $\Xi^{(L)} \in \mathcal{O}(\mathcal{S}_\alpha)$  and admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\Xi^{(L)}(z) = C_{\Xi^{(L)}} + \mathcal{O}\left(e^{\frac{2\pi}{\alpha}(z+w)}\right)$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;
- $\Xi^{(L)}(z) = \gamma \cdot (z+w) + \mathcal{O}\left(e^{-\frac{2\pi\tilde{\gamma}}{\alpha}(z+w)}\right)$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- $\Xi_+^{(L)}(G_L(x) - i\alpha) = \Xi_-^{(L)}(x) + x - G_L(x)$ .

We stress that the remainders at  $\Re(z) \rightarrow \pm\infty$  appearing above are uniform in  $w$ .

### 3.3 Right Riemann-Hilbert problem

The right Riemann-Hilbert problem consists in finding  $\chi^{(R)} \in \mathcal{O}(\mathcal{S}_\alpha)$  such that

- $\chi^{(R)}$  admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\chi^{(R)}(z) = \mathcal{O}\left(e^{-\frac{2\pi}{\alpha}z}\right)$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- $\chi^{(R)}(z) = \gamma \cdot z + C_{\chi^{(R)}} + \mathcal{O}\left(e^{\frac{2\pi\tilde{\gamma}}{\alpha}z}\right)$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;
- $\chi_+^{(R)}(g_R(x) - i\alpha) = \chi_-^{(R)}(x) + x - g_R(x)$ .

We remind that  $\gamma$  and  $\tilde{\gamma}$  have been introduced in (3.2) while  $g_R$  is given by (3.1).

**Proposition 3.3.** *The Riemann-Hilbert problem for  $\chi^{(R)}$  admits a unique solution.*

*Proof* — Let

$$\omega^{(R)}(z) = \frac{\gamma z}{1 + e^{\frac{2\pi}{\tau}z}}, \quad (3.19)$$

where  $\tau > 2\alpha$ . The function  $\omega^{(R)}(z)$  may be decomposed as

$$\omega^{(R)}(z) = \gamma z + \omega_L^{(R)}(z) \quad \text{with} \quad \omega_L^{(R)}(z) = -\frac{\gamma z}{e^{-\frac{2\pi}{\tau}z} + 1}. \quad (3.20)$$

As a consequence, one has

$$\omega^{(R)}(z) = \begin{cases} \gamma z + O\left(ze^{\frac{2\pi}{\tau}z}\right) & \Re(z) \rightarrow -\infty \\ O\left(ze^{-\frac{2\pi}{\tau}z}\right) & \Re(z) \rightarrow +\infty \end{cases}. \quad (3.21)$$

The decomposition (3.13) entails that  $\omega^{(R)}$  satisfies

$$\omega^{(R)}(x + \varkappa - i\alpha) - \omega^{(R)}(x) = -\varkappa + \omega_L^{(R)}(x + \varkappa - i\alpha) - \omega_L^{(R)}(x). \quad (3.22)$$

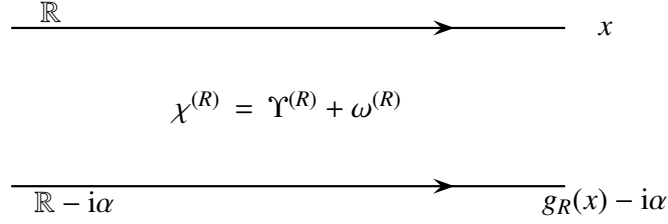


Figure 7: The substitution for the Riemann-Hilbert problem for  $\chi^{(R)}$ .

Upon implementing the substitution in the Riemann-Hilbert problem for  $\chi^{(R)}$  as described in Figure 7, one gets that  $\Upsilon^{(R)} \in \mathcal{O}(\mathcal{S}_\alpha)$  solves the Riemann-Hilbert problem

- $\Upsilon^{(R)}(z) = -\omega^{(R)}(z) + O\left(e^{-\frac{2\pi}{\alpha}z}\right)$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- $\Upsilon^{(R)}(z) = C_{\chi^{(R)}} - \omega_L^{(R)}(z) + O\left(e^{\frac{2\pi\tilde{\gamma}}{\alpha}z}\right)$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;
- $\Upsilon^{(R)}$  admits smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\Upsilon_+^{(R)}(g_R(x) - i\alpha) = \Upsilon_-^{(R)}(x) + G_{\Upsilon^{(R)}}(x)$ , where the jump function takes the form

$$G_{\Upsilon^{(R)}}(x) = \begin{cases} \omega_L^{(R)}(x) - \omega_L^{(R)}(x + \varkappa - i\alpha) & x < -M_R \\ x - g_R(x) + \omega^{(R)}(x) - \omega^{(R)}(g_R(x) - i\alpha) & -M_R \leq x \leq M_R \\ \omega^{(R)}(x) - \omega^{(R)}(x - i\alpha) & M_R < x \end{cases}. \quad (3.23)$$

It follows from the stated properties of  $\omega^{(R)}$  that  $G_{\Upsilon^{(R)}}$  has the form (2.2) for  $g = g_R$ ,  $M = M_R$  and

$$\mathcal{G}_{\Upsilon^{(R)}}(z) = \omega_L^{(R)}(z) + \varphi_R(\Re(z))\gamma z, \quad (3.24)$$

where  $\varphi_R(x)$  is a smooth interpolating function equal to 0 for  $x < -M_R/2$  and to 1 for  $x > M_R/2$  so that  $\mathcal{G}_{\Upsilon^{(R)}}$  is analytic on  $\mathcal{S}_\alpha$  for  $|\Re(z)| > M_R/2$  with exponential falloff at infinity. Again, these properties entail by virtue of Proposition 2.4 that the non-local Riemann-Hilbert problem for  $\Upsilon^{(R)}$  is uniquely solvable, and hence, so is the one for  $\chi^{(R)}$ . ■

The solution  $\chi^{(R)}$  gives rise to the solution of the non-local Riemann-Hilbert problem associated with a shifted function. Namely, for any  $w \in \mathbb{R}$ , let

$$G_R(x) = g_R(x - w) + w = \begin{cases} x + \varkappa & x < -M_R + w \\ g_R(x - w) + w & -M_R + w < x < M_R + w \\ x & M_R + w < x \end{cases} \quad (3.25)$$

Then  $\Xi^{(R)}(z) = \chi^{(R)}(z - w)$  solves the Riemann-Hilbert problem:

- $\Xi^{(R)} \in \mathcal{O}(\mathcal{S}_\alpha)$  and admits  $L^2(\mathbb{R})$   $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ,
- $\Xi^{(R)}(z) = \mathcal{O}\left(e^{-\frac{2\pi}{\alpha}(z-w)}\right)$  when  $\Re(z) \rightarrow +\infty$  and up to the boundary;
- $\Xi^{(R)}(z) = \gamma \cdot (z - w) + C_{\Xi^{(R)}} + \mathcal{O}\left(e^{\frac{2\pi\bar{\gamma}}{\alpha}(z-w)}\right)$  when  $\Re(z) \rightarrow -\infty$  and up to the boundary;
- $\Xi_+^{(R)}(G_R(x) - i\alpha) = \Xi_-^{(R)}(x) + x - G_R(x)$ .

## 4 Asymptotic behaviour of the global non-local Riemann-Hilbert problem

From now on we fix two positive reals  $M_L, M_R > 0$  and consider the function  $g$  defined as

$$g(x) = \begin{cases} x & x < -M_L - w \\ g_L(x + w) - w & -M_L - w \leq x \leq M_L - w \\ x + \varkappa & M_L - w \leq x \leq w - M_R \\ g_R(x - w) + w & w - M_R < x < w + M_R \\ x & M_R + w < x \end{cases} \quad (4.1)$$

where  $g_L$  and  $g_R$  correspond to the functions introduced in (3.1).

Below, we shall establish the main theorem of this work, Theorem 1.3.

*Proof of Theorem 1.3.*

The substitution  $\chi(z) = \tilde{\chi}(z) + z$  turns the non-local Riemann-Hilbert problem with shift for  $\chi$  into the one of finding  $\tilde{\chi} \in \mathcal{O}(\mathcal{S}_\alpha)$  such that

- $\tilde{\chi}$  has smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\tilde{\chi}(z) = C_\chi \delta_{\pm, -} + \mathcal{O}\left(e^{\mp \frac{2\pi}{\alpha} z}\right)$  for  $\Re(z) \rightarrow \pm\infty$ , this up to the boundary and for some constant  $C_\chi$ ;
- $\tilde{\chi}_+(g(x) - i\alpha) = \tilde{\chi}_-(x) + x - g(x)$ , with  $x \in \mathbb{R}$ .

The unique solvability of the Riemann-Hilbert problem for  $\tilde{\chi}$  is a direct consequence of Proposition 2.4. This thus establishes the unique solvability of the non-local Riemann-Hilbert problem with shift for  $\chi$ . However, the approach that was adopted for establishing Proposition 2.4 does not allow one for a uniform in  $w$  control on the solution. To achieve it, first introduce

$$\Psi(z) = C_{\Gamma_0}[\delta\Xi](z) \quad (4.2)$$

in which  $\Gamma_0 = [0; -i\alpha + \varkappa]$  while  $\delta\Xi = \Xi^{(L)} - \Xi^{(R)}$  where  $\Xi^{(L/R)}$  are the unique solutions to the left/right shifted Riemann-Hilbert problems that were discussed in Subsections 3.2-3.3. Note that, owing to the large  $z$  asymptotics of the solutions  $\chi^{(L/R)}$ , it holds

$$\delta\Xi(s) = \gamma(s + w) + \mathcal{O}\left(e^{-\frac{2\pi\bar{\gamma}}{\alpha}(s+w)}\right) - \gamma(s - w) - C_{\Xi^{(R)}} - \mathcal{O}\left(e^{\frac{2\pi\bar{\gamma}}{\alpha}(s-w)}\right) = 2\gamma w - C_{\Xi^{(R)}} + \mathcal{O}\left(e^{-\frac{2\pi\bar{\gamma}}{\alpha}w}\right) \quad (4.3)$$

uniformly in  $s \in \Gamma_0$  and where  $\bar{\gamma}$  is as defined in (3.2). Furthermore, the jump condition  $\delta\Xi_+(x + \varkappa - i\alpha) = \delta\Xi_-(x)$ , valid in some fixed,  $w$ -independent, neighbourhood of  $x = 0$ , ensures that one can invoke Lemma 3.1 so as to conclude that  $\Psi$  satisfies

- $\Psi \in \mathcal{O}(\mathcal{S}_\alpha \setminus \Gamma)$  and has holomorphic  $-$ , resp.  $+$ , boundary values on  $\Gamma_0$ ;
- $\Psi_+(x) - \Psi_-(x) = \delta\Xi(x)$  for  $x \in \Gamma_0$ ;
- $\Psi(x + \kappa - i\alpha) = \Psi(x)$  for  $x \in \mathbb{R}$ ;
- $\Psi(z) = \delta_{\pm,+}c(w) + \mathcal{O}\left(we^{\mp\frac{2\pi\tilde{\gamma}}{\alpha}z}\right)$  when  $\Re(z) \rightarrow \pm\infty$  this up to the boundary  $\partial\mathcal{S}_\alpha$ , and uniformly in  $w$  with

$$c(w) = - \int_{\Gamma_0} \frac{\tilde{\gamma} ds}{i\alpha} \delta\Xi(s) = 2\gamma w - C_{\Xi^{(R)}} + \mathcal{O}\left(e^{-\frac{2\pi\tilde{\gamma}}{\alpha}w}\right). \quad (4.4)$$

Finally, the asymptotic expansion (4.3) for  $\delta\Xi$  which holds uniformly on  $\Gamma_0$  allows one to get uniform in  $w$  estimates for  $\Psi$ . Indeed, this entails that

$$\Psi(z) = (2\gamma w - C_{\Xi^{(R)}}) \mathbf{1}_{\mathcal{D}_R}(z) + C_{\Gamma_0}[\delta\Xi - 2\gamma w + C_{\Xi^{(R)}}](z) \quad (4.5)$$

where  $\mathcal{D}_R$  is the domain depicted in Fig. 8 and we used that  $C_{\Gamma_0}[1](z) = \mathbf{1}_{\mathcal{D}_R}(z)$ . The second term may be estimated as

$$C_{\Gamma_0}[\delta\Xi - 2\gamma w + C_{\Xi^{(R)}}](z) = \begin{cases} \mathcal{O}\left(e^{-\frac{2\pi\tilde{\gamma}}{\alpha}w(1 + e^{-\frac{2\pi\tilde{\gamma}}{\alpha}z})}\right) & z \in \mathcal{D}_R \\ \mathcal{O}\left(e^{-\frac{2\pi\tilde{\gamma}}{\alpha}(w-z)}\right) & z \in \mathcal{D}_L \end{cases}. \quad (4.6)$$

These estimates are uniform up to the boundary of  $\mathcal{S}_\alpha$  and up to  $\Gamma_0$ , as follows from the local holomorphicity of  $\delta\Xi$  around  $\Gamma_0$  and the fact that it satisfies in this neighbourhood  $\delta\Xi_+(s + \kappa - i\alpha) = \delta\Xi_-(s)$ .

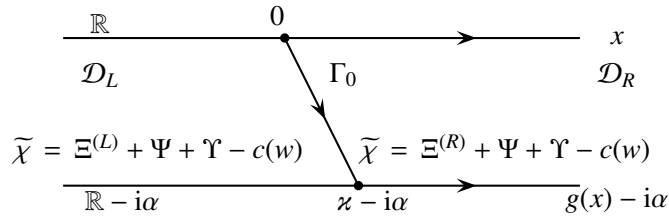


Figure 8: The substitution for the Riemann-Hilbert problem for  $\tilde{\chi}$ .

Then, one makes the substitution in the Riemann-Hilbert problem for  $\tilde{\chi}$  as described in Figure 8. One gets that, by construction,  $\Upsilon$  is continuous across  $\Gamma_0$ . It thus solves the non-local Riemann-Hilbert problem

- $\Upsilon \in \mathcal{O}(\mathcal{S}_\alpha)$  having  $L^2(\mathbb{R})$   $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\Upsilon(z) = C_\Upsilon \delta_{\pm;-} + \mathcal{O}\left(e^{-\eta|\Re(z)|}\right)$  for  $\Re(z) \rightarrow \pm\infty$ , this up to the boundary for some  $C_\Upsilon$  and  $\eta > 0$ ;
- $\Upsilon_+(g(x) - i\alpha) = \Upsilon_-(x) + G_\Upsilon(x)$ , with  $x \in \mathbb{R}$ ,

where the jump function reads  $G_\Upsilon(x) = \Psi(x) - \Psi(g(x) - i\alpha)$ . Observe further that for  $x \in [M_L - w; -M_R + w]$  it holds  $g(x) = x + \kappa$  and thus  $G_\Upsilon(x) = 0$  by virtue of the periodicity of  $\Psi$ . Furthermore, the estimates for  $\Psi$  at infinity entail that for  $x \in \mathbb{R} \setminus [M_L - w; -M_R + w]$

$$G_\Upsilon(x) = \mathcal{O}\left(we^{\mp\frac{2\pi\tilde{\gamma}}{\alpha}x}\right) \quad \pm x > 0 \quad (4.7)$$

which is uniformly exponentially small in  $w$  and has an exponential fall-off in  $x$  at infinity. This will allow to control the behaviour of the function  $\Upsilon$  both in  $z$  and in  $w$ . The argument goes as follows.

By Proposition 2.4 and Lemma 2.2, the function  $Y_1(x) = \Upsilon_+(g(x) - i\alpha)$  corresponds to the unique solution to

$$(\text{id} - \mathbb{K}_{\text{tot}})[Y_1] = \frac{1}{2}\{G_\Upsilon + \mathcal{H}[G_\Upsilon]\} - \mathbb{K}_{\text{tot};12}[G_\Upsilon] \quad (4.8)$$

where  $\mathbb{K}_{\text{tot}}$ ,  $\mathbb{K}_{\text{tot};12}$  and  $\mathcal{H}$  are the integral operators introduced in Lemma 2.2 whose integral kernels are expressed in terms of the function  $g$  given in (4.1). Due to the properties of  $G_\Upsilon$ , the results that will be established in Section 5, in particular Theorem 5.5, ensure that

$$Y_1(x) = C_{Y_1}\delta_{\pm,-} + \mathcal{O}(e^{-w\eta' - \eta|x|}) \quad \text{as } x \rightarrow \pm\infty \quad (4.9)$$

uniformly in  $w$  and for some  $\eta, \eta' > 0$  and with  $C_{Y_1} = \mathcal{O}(e^{-w\eta'})$ . Moreover, it holds  $\|Y_1\|_{L^\infty(\mathbb{R})} = \mathcal{O}(e^{-w\eta'})$ . Thus, since

$$\Upsilon(z) = - \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{Y_1(y) - G_\Upsilon(y)}{\sinh\left[\frac{\pi}{\tau}(y-z)\right]} + \int_{\mathbb{R}} \frac{dy}{2i\tau} \frac{Y_1(y)g'(y)}{\sinh\left[\frac{\pi}{\tau}(g(y) - z - i\alpha)\right]}, \quad (4.10)$$

one gets that

$$\Upsilon(z) = C_{Y_1}\delta_{\pm,-} + \mathcal{O}(e^{-w\eta' - \eta|z|}) \quad \text{as } \Re(z) \rightarrow \pm\infty, \quad (4.11)$$

uniformly throughout  $\mathcal{S}_\alpha$  and in  $w$ . The comparison of the decompositions of Figures 4 and 8 permits then to end the proof of Theorem 1.3.  $\blacksquare$

## 5 Invertibility of an auxiliary integral operator

Let  $\mathbb{K}_{\text{tot}}$  be the integral operator, introduced in (2.23), and associated with the function  $g$  given in (4.1). It follows from the analysis in Subsection 2.3 that the operator  $\text{id} - \mathbb{K}_{\text{tot}}$  is invertible on an appropriate functional space.

The goal of this section is to establish, uniformly in  $w \rightarrow +\infty$ , bounds on the inverse of  $\text{id} - \mathbb{K}_{\text{tot}}$ . This will be done by relying on the various results established in the previous sections.

### 5.1 Decomposition of $\mathbb{K}_{\text{tot}}$

To start with, it is convenient to introduce three intervals

$$I_w^- = ] - \infty; -w] \quad , \quad I_w^0 = ] - w; w[ \quad , \quad I_w^+ = [w; +\infty[ \quad . \quad (5.1)$$

Next, introduce three operators on  $L^2(\mathbb{R})$ ,  $L_w^{++}$ ,  $L_w^0$ ,  $L_w^{--}$  with integral kernels

$$L_w^{\pm\pm}(x, y) = L^{(\pm)}(x - y) \cdot \mathbf{1}_{I_w^\pm \times I_w^\pm}(x, y) \quad (5.2)$$

$$L_w^0(x, y) = L^{(0)}(x - y) \cdot \mathbf{1}_{I_w^0 \times I_w^0}(x, y) \quad , \quad (5.3)$$

where

$$L^{(\pm)}(x - y) = \frac{1}{2i\tau} \left\{ \frac{1}{\sinh\left[\frac{\pi}{\tau}(y - x - i\alpha)\right]} - \frac{1}{\sinh\left[\frac{\pi}{\tau}(y - x + i\alpha)\right]} \right\}, \quad (5.4)$$

$$L^{(0)}(x - y) = \frac{1}{2i\tau} \left\{ \frac{1}{\sinh\left[\frac{\pi}{\tau}(y + \kappa - x - i\alpha)\right]} - \frac{1}{\sinh\left[\frac{\pi}{\tau}(y - \kappa - x + i\alpha)\right]} \right\}. \quad (5.5)$$

From now on we agree to denote by  $K_R$  and  $B_R$ , resp.  $K_L$  and  $B_L$ , the operators  $K$  as introduced in (2.23) and  $B$  as introduced in (2.39) which are subordinate to the function  $g_R$ , resp.  $g_L$ , introduced in (3.1).

Define the operator  $B_{\text{tot}}$  on  $L^2(\mathbb{R})$

$$B_{\text{tot}}(x, y) = \mathbf{1}_{\mathbb{R}^+}(x)\mathbf{1}_{]w;+\infty[}(y) \cdot B_R(x-w, y-w) + \mathbf{1}_{\mathbb{R}^-}(x)\mathbf{1}_{]-\infty;w[}(y) \cdot B_L(x+w, y+w) \\ + \mathbf{1}_{\mathbb{R}^+}(x)G_R(x, y) + \mathbf{1}_{\mathbb{R}^-}(x)G_L(x, y). \quad (5.6)$$

Upon using the function  $m_\alpha$  introduced in (2.47), the functions  $G_{R/L}$  are expressed as

$$G_L(x, y) = \left\{ m_\alpha(g_R(y-w) + w - x)g'_R(y-w) + m_0(g_R(y-w) + 2w - g_L(x+w))g'_R(y-w) - m_0(y-x) \right. \\ \left. + K_{L;12}(x+w, y+w) \right\} \mathbf{1}_{]w-M_R;+\infty[}(y) - K_L(x+w, y+w)\mathbf{1}_{]w-M_R;w[}(y) \quad (5.7)$$

and

$$G_R(x, y) = \left\{ m_\alpha(g_L(y+w) - w - x)g'_L(y+w) + m_0(g_L(y+w) - 2w - g_R(x-w))g'_L(y+w) - m_0(y-x) \right. \\ \left. + K_{R;12}(x-w, y-w) \right\} \mathbf{1}_{]-\infty; -w+M_L[}(y) - K_R(x-w, y-w)\mathbf{1}_{]-w; -w+M_L[}(y). \quad (5.8)$$

It is clear that the functions  $G_{L/R}$  satisfy the bounds

$$G_L(x, y) = O\left(e^{-\frac{\pi}{\tau}|x-y|}\right) \cdot \mathbf{1}_{]w-M_R;+\infty[}(y) \quad \text{and} \quad G_R(x, y) = O\left(e^{-\frac{\pi}{\tau}|x-y|}\right) \cdot \mathbf{1}_{]-\infty; M_L-w[}(y). \quad (5.9)$$

Moreover, the remainders appearing above also hold for the derivatives, namely, for any  $(k, \ell) \in \mathbb{N}^2$ , one has

$$\partial_x^\ell \partial_y^k G_L(x, y) = O\left(e^{-\frac{\pi}{\tau}|x-y|}\right) \cdot \mathbf{1}_{]w-M_R;+\infty[}(y) \quad \text{and} \quad \partial_x^\ell \partial_y^k G_R(x, y) = O\left(e^{-\frac{\pi}{\tau}|x-y|}\right) \cdot \mathbf{1}_{]-\infty; M_L-w[}(y). \quad (5.10)$$

**Lemma 5.1.** *One has*

$$K_{\text{tot}} = L_w^{++} + L_w^0 + L_w^{--} + B_{\text{tot}} \quad (5.11)$$

*Proof—*

Recall that  $g(x) = g_L(x+w) - w$  whenever  $-\infty < x < w - M_R$ . Hence, for  $x \leq w - M_R$ ,

$$K_{\text{tot};12}(x, y) = K_{L;12}(x+w, y+w) \quad \text{for any } y \in \mathbb{R}. \quad (5.12)$$

Also, in the same range of  $x$ 's,

$$K_{\text{tot};21}(x, y) = \begin{cases} K_{L;21}(x+w, y+w) & \text{if } y \leq w - M_R \\ m_\alpha(g_R(y-w) + w - x)g'_R(y-w) & \text{if } y \geq w - M_R \end{cases} \quad (5.13)$$

and

$$K_{\text{tot};11}(x, y) = \begin{cases} K_{L;11}(x+w, y+w) & \text{if } y \leq w - M_R \\ m_0(g_R(y-w) + 2w - g_L(x+w))g'_R(y-w) - m_0(y-x) & \text{if } y \geq w - M_R \end{cases}. \quad (5.14)$$

Thus, adding up the three pieces, one gets that for  $x \leq w - M_R$ ,

$$K_{\text{tot}}(x, y) = K_L(x+w, y+w)\mathbf{1}_{]-\infty;w[}(y) + G_L(x, y) \quad (5.15)$$



with  $G_L$  as defined in (5.7).

Analogously,  $g(x) = g_R(x - w) + w$  for  $-w + M_L \leq x$ . Hence, in this range of  $x$ 's,

$$K_{\text{tot};12}(x, y) = K_{R;12}(x - w, y - w) \quad \text{for any } y \in \mathbb{R}. \quad (5.16)$$

Also, in the same range of  $x$ 's, one has

$$K_{\text{tot};21}(x, y) = \begin{cases} m_\alpha(g_L(y + w) - w - x)g'_L(y + w) & \text{if } y \leq -w + M_L \\ K_{R;21}(x - w, y - w) & \text{if } y \geq -w + M_L \end{cases} \quad (5.17)$$

and

$$K_{\text{tot};11}(x, y) = \begin{cases} m_0(g_L(y + w) - 2w - g_R(x - w))g'_L(y + w) - m_0(y - x) & \text{if } y \leq -w + M_L \\ K_{R;11}(x - w, y - w) & \text{if } y \geq -w + M_L \end{cases}. \quad (5.18)$$

Thus, adding up the three pieces, one gets that for  $x \leq w - M_R$ ,

$$K_{\text{tot}}(x, y) = K_R(x - w, y - w)\mathbf{1}_{]-w; +\infty[}(y) + G_R(x, y) \quad (5.19)$$

with  $G_R$  as defined in (5.8).

Observe that the kernels  $K_{L/R}$  may be further expressed, following (2.38), as

$$K_L(x + w, y + w) = L_{L;w}^-(x, y) + L_{L;w}^{++}(x, y) + B_L(x + w, y + w) \quad (5.20)$$

$$K_R(x - w, y - w) = L_{R;w}^-(x, y) + L_{R;w}^{++}(x, y) + B_R(x - w, y - w). \quad (5.21)$$

There, one has

$$L_{L;w}^-(x, y) = L^{(e)}(x - y) \cdot \mathbf{1}_{I_w^- \times I_w^-}(x, y) \quad \text{and} \quad L_{L;w}^{++}(x, y) = L^{(0)}(x - y) \cdot \mathbf{1}_{]-w; +\infty[ \times ]-w; +\infty[}(x, y). \quad (5.22)$$

Likewise,

$$L_{R;w}^-(x, y) = L^{(0)}(x - y) \cdot \mathbf{1}_{]-\infty; w[ \times ]-\infty; w[}(x, y) \quad \text{and} \quad L_{R;w}^{++}(x, y) = L^{(e)}(x - y) \cdot \mathbf{1}_{I_w^+ \times I_w^+}(x, y). \quad (5.23)$$

The rest follows upon straightforward calculations starting from the decomposition, valid almost everywhere,

$$K_{\text{tot}}(x, y) = \mathbf{1}_{\mathbb{R}^+}(x) \cdot \left\{ K_R(x - w, y - w)\mathbf{1}_{]-w; +\infty[}(y) + G_R(x, y) \right\} \\ + \mathbf{1}_{\mathbb{R}^-}(x) \cdot \left\{ K_L(x + w, y + w)\mathbf{1}_{]-\infty; w[}(y) + G_L(x, y) \right\}. \quad (5.24)$$

■

From now on, it appears useful to introduce the following notation for the projections of  $f \in L^2(\mathbb{R})$  subordinate to the intervals  $I_w^\alpha$ ,  $\alpha \in \{+, -, 0\}$ :

$$f^- = f\mathbf{1}_{I_w^-}, \quad f^0 = f\mathbf{1}_{I_w^0}, \quad f^+ = f\mathbf{1}_{I_w^+}. \quad (5.25)$$

The decomposition of the operator  $K_{\text{tot}}$  achieved in Lemma 5.1 allows one to decompose naturally  $\text{id} - K_{\text{tot}}$  into a matrix bloc operator relative to the direct sum decomposition of the space  $\mathcal{E}(\mathbb{R})$  of (2.20) induced by the above projection operators:

$$\mathcal{E}(\mathbb{R}) = L_C^2(I_w^-) \oplus L^2(I_w^0) \oplus L^2(I_w^+), \quad (5.26)$$

where

$$L_C^2(I_w^-) = \left\{ f \in L_{\text{loc}}^2(I_w^-) : \exists C_f \text{ and } \alpha > 0 \quad f(x) = C_f + O(e^{\alpha x}) \right\}. \quad (5.27)$$

The main reason for doing so is that the operators  $\text{id} - L_w^{--}$ ,  $\text{id} - L_w^0$ ,  $\text{id} - L_w^{++}$  which encapsulate the continuous part of the spectrum of  $\text{id} - \mathbb{K}_{\text{tot}}$  and arise in the diagonal block subordinate to the splitting (5.26) may be explicitly inverted. This simplifies the analysis of the original equation  $(\text{id} - \mathbb{K}_{\text{tot}})[f] = h$  permitting to map it into a one whose non-trivial piece is governed by a compact operator whose large- $w$  behaviour may be controlled.

Furthermore, denoting by

$$B_{\text{tot}}^{\epsilon\sigma}(x, y) = B_{\text{tot}}(x, y) \mathbf{1}_{I_w^\epsilon \times I_w^\sigma}(x, y) \quad \text{with} \quad \epsilon, \sigma \in \{+, -, 0\}, \quad (5.28)$$

the integral kernels of the appropriate projections of the operator  $B_{\text{tot}}$ , the equation  $(\text{id} - \mathbb{K}_{\text{tot}})[f] = h$  may be recast into a block-matrix form subordinate to the direct sum decomposition (5.26)

$$\begin{pmatrix} \text{id} - L_w^{--} - B_{\text{tot}}^{--} & -B_{\text{tot}}^{0-} & -B_{\text{tot}}^{+-} \\ -B_{\text{tot}}^{0-} & \text{id} - L_w^0 - B_{\text{tot}}^{00} & -B_{\text{tot}}^{0+} \\ -B_{\text{tot}}^{+-} & -B_{\text{tot}}^{0+} & \text{id} - L_w^{++} - B_{\text{tot}}^{++} \end{pmatrix} \cdot \begin{pmatrix} f^- \\ f^0 \\ f^+ \end{pmatrix} = \begin{pmatrix} h^- \\ h^0 \\ h^+ \end{pmatrix}. \quad (5.29)$$

The above matrix operator equations may be rewritten as

$$(\text{id} - L_w^{--})[f^-] = H^- = h^- + \sum_{\sigma \in \{\pm, 0\}} B_{\text{tot}}^{-\sigma}[f^\sigma], \quad (5.30)$$

$$(\text{id} - L_w^0)[f^0] = H^0 = h^0 + \sum_{\sigma \in \{\pm, 0\}} B_{\text{tot}}^{0\sigma}[f^\sigma], \quad (5.31)$$

$$(\text{id} - L_w^{++})[f^+] = H^+ = h^+ + \sum_{\sigma \in \{\pm, 0\}} B_{\text{tot}}^{+\sigma}[f^\sigma]. \quad (5.32)$$

Since  $(\text{id} - \mathbb{K}_{\text{tot}})[f] = 0$  for any constant function, it is convenient, owing to the setting that was analysed in the previous sections, to extend the equation to the space

$$\mathcal{E}'(\mathbb{R}) = L_C^2(I_w^-) \oplus L_C^2(I_w^0) \oplus L_C^2(I_w^+), \quad (5.33)$$

where

$$L_C^2(I_w^+) = \left\{ f \in L_{\text{loc}}^2(I_w^+) : \exists C_f \text{ and } \alpha > 0 \quad f(x) = C_f + O(e^{-\alpha x}) \right\}. \quad (5.34)$$

Clearly, any solution obtained in  $\mathcal{E}'(\mathbb{R})$  gives rise to the solution in  $\mathcal{E}(\mathbb{R})$  by performing a global translation by a constant. The main point is that one may apply the results of the previous analysis in the case of  $\mathcal{E}'(\mathbb{R})$ , as the invertibility of  $(\text{id} - L_w^{\pm\pm})$  has been formulated on the spaces  $L_C^2(I_w^\pm)$ . Hence, considering (5.30), (5.31) and (5.32) as a system of equations on the space (5.33) and observing that the functions  $H^\pm, H^0$  do enjoy the properties stated in Propositions B.1 and B.2, as may be inferred from direct bounds and Proposition 2.3, one may apply the results of Sections B and C of the appendix in order to invert the operators appearing in the *lhs* of (5.30), (5.31), (5.32)

so as to get

$$\mathcal{F}[f^-](k) = -\alpha_{\downarrow}^{(e)}(k)e^{-ikw} \int_{\mathbb{R}+iv} \frac{ds}{2i\pi} \frac{\{\alpha_{\uparrow}^{(e)}(s)\}^{-1} \cdot \mathcal{F}[H^-](s)}{s-k} \cdot e^{isw} \quad \text{with} \quad k \in \mathbb{R} - iv, \quad (5.35)$$

$$\mathcal{F}[f^0](k) = \mathcal{F}[H^0](k) - \int_{\mathbb{R}+iv} d\mu R(k, \mu) \mathcal{F}[H^0](\mu) \quad \text{with} \quad k \in \mathbb{R} + iv, \quad (5.36)$$

$$\mathcal{F}[f^+](k) = \frac{e^{ikw}}{\alpha_{\uparrow}^{(e)}(k)} \int_{\mathbb{R}-iv} \frac{ds}{2i\pi} \frac{\alpha_{\downarrow}^{(e)}(s) \cdot \mathcal{F}[H^+](s)}{s-k} \cdot e^{-isw} \quad \text{with} \quad k \in \mathbb{R} + iv. \quad (5.37)$$

Here,  $\alpha_{\uparrow/\downarrow}^{(e)}$  are given by (2.54)-(2.55) upon the substitution  $\kappa^{\nu} \mapsto 0$  and  $R$  is the resolvent kernel of the operator  $\text{id} + \mathbf{V}$  on  $L^2(\mathbb{R} + iv)$  introduced in (C.1)-(C.2), *c.f.* Theorem C.1.

## 5.2 Preliminary estimates for $\mathbf{B}_{\text{tot}}$

In this subsection, we provide estimates for the Fourier transform of the  $\pm$  and  $0$  projections of the operator  $\mathbf{B}_{\text{tot}}$  which will then allow to study the large- $w$  behaviour of the solutions to the system (5.35), (5.36), (5.37).

**Proposition 5.2.** *Let*

$$\widehat{B}_{\text{tot}}^{\epsilon\sigma}(k, s) = \int_{\mathbb{R}} dx \int_{\mathbb{R}} \frac{dy}{2\pi} e^{ikx - isy} B_{\text{tot}}^{\epsilon\sigma}(x, y). \quad (5.38)$$

with  $B_{\text{tot}}^{\epsilon\sigma}(x, y)$  as introduced in (5.28). One has

$$\begin{pmatrix} \widehat{B}_{\text{tot}}^{--}(k, s) & \widehat{B}_{\text{tot}}^{-0}(k, s) & \widehat{B}_{\text{tot}}^{-+}(k, s) \\ \widehat{B}_{\text{tot}}^{0-}(k, s) & \widehat{B}_{\text{tot}}^{00}(k, s) & \widehat{B}_{\text{tot}}^{0+}(k, s) \\ \widehat{B}_{\text{tot}}^{+-}(k, s) & \widehat{B}_{\text{tot}}^{+0}(k, s) & \widehat{B}_{\text{tot}}^{++}(k, s) \end{pmatrix} = \begin{pmatrix} e^{i(s-k)w} \cdot \widehat{B}_L^-(k, s) & e^{i(s-k)w} \cdot \widehat{B}_L^+(k, s) + \mathfrak{B}^{-0}(k, s) \\ \mathfrak{B}^{0-}(k, s) + e^{i(s-k)w} \cdot \widehat{B}_L^-(k, s) & e^{i(s-k)w} \cdot \widehat{B}_L^+(k, s) + e^{i(k-s)w} \cdot \widehat{B}_R^-(k, s) + \mathfrak{B}^{00}(k, s) \\ \mathfrak{B}^{+-}(k, s) & e^{i(k-s)w} \cdot \widehat{B}_R^+(k, s) + \mathfrak{B}^{+0}(k, s) \\ & \mathfrak{B}^{-+}(k, s) \\ & e^{i(k-s)w} \cdot \widehat{B}_R^-(k, s) + \mathfrak{B}^{0+}(k, s) \\ & e^{i(k-s)w} \cdot \widehat{B}_R^+(k, s) \end{pmatrix}. \quad (5.39)$$

Above, the kernels  $\widehat{B}_{L/R}^{\sigma\epsilon}(k, s)$ , with  $\sigma, \epsilon \in \{\pm\}$ , are as introduced in (2.67), under the substitution  $g \mapsto g_{L/R}$  with  $g_{L/R}$  given by (3.1). The kernels  $\mathfrak{B}^{\sigma\epsilon}(k, s)$ , with  $\sigma, \epsilon \in \{\pm, 0\}$ , are all holomorphic in an open,  $w$ -independent, neighbourhood of  $\mathbb{R}^2$  and they satisfy the bounds

$$\mathfrak{B}^{\sigma\epsilon}(k, s) = \mathcal{O}\left(\frac{e^{-2w\eta}}{(1+|k|)(1+|s|)}\right) \quad (5.40)$$

uniformly in  $(k, s) \in \mathbb{C}^2$  such that  $|\Im(k)| < 2v$ ,  $|\Im(s)| < 2v$ , with  $v$  small enough, and for some  $\eta$  much larger than  $v$ .

*Proof*—

We only discuss the case of the coefficient  $\widehat{B}_{\text{tot}}^{0-}(k, s)$  which already contains all the features of the analysis. Indeed, one has

$$\begin{aligned} \widehat{B}_{\text{tot}}^{0-}(k, s) &= \int_{-w}^w dx \int_{-\infty}^{-w} \frac{dy}{2\pi} e^{ikx-is y} \left\{ \mathbf{1}_{\mathbb{R}^-}(x) \cdot B_L(x+w, y+w) + \mathbf{1}_{\mathbb{R}^+}(x) G_R(x, y) \right\} \\ &= e^{iw(s-k)} \widehat{B}_L^{+-}(k, s) + \mathfrak{B}^{0-}(k, s) \end{aligned} \quad (5.41)$$

with

$$\mathfrak{B}^{0-}(k, s) = e^{isw} \int_0^w dx \int_{-\infty}^0 \frac{dy}{2\pi} e^{ikx-is y} G_R(x, y-w) - e^{i(s-k)w} \int_w^{+\infty} dx \int_{-\infty}^0 \frac{dy}{2\pi} e^{ikx-is y} B_L(x, y). \quad (5.42)$$

Note that all integrals do converge either due to the exponential decay of  $B_L$ , *c.f.* (2.43), or to the estimates (5.9) for the decay of  $G_R(x, y)$ . We now discuss how to estimate the first term appearing in the definition of  $\mathfrak{B}^{0-}(k, s)$ .

For  $\max\{|\Im(k)|, |\Im(s)|\} < 2\nu$ , one sets  $\widetilde{G}_R(x, y-w) = e^{3\nu(x-y+w)} G_R(x, y-w)$ , so that, by carrying out integrations by parts,

$$\begin{aligned} e^{isw} \int_0^w dx \int_{-\infty}^0 \frac{dy}{2\pi} e^{ikx-is y} G_R(x, y-w) &= \frac{e^{i(s+3i\nu)w}}{2\pi(s+3i\nu)(k+3i\nu)} \left\{ e^{i(k+3i\nu)w} \widetilde{G}_R(w, -w) - \widetilde{G}_R(0, -w) \right. \\ &- \int_{-\infty}^0 dy e^{-i(s+3i\nu)y} \partial_y \left[ e^{i(k+3i\nu)w} \widetilde{G}_R(w, y-w) - \widetilde{G}_R(0, y-w) \right] - \int_0^w dx e^{i(k+3i\nu)x} \partial_x \widetilde{G}_R(x, -w) \\ &\left. + \int_0^w dx \int_{-\infty}^0 dy e^{i(k+3i\nu)x-i(s+3i\nu)y} \partial_x \partial_y \widetilde{G}_R(x, y-w) \right\}. \end{aligned} \quad (5.43)$$

Then, one estimates each term separately by using directly the bounds (5.9), where we remind that the remainder is differentiable. For instance, one has

$$\begin{aligned} &\left| \frac{e^{i(s+3i\nu)w}}{(s+3i\nu)(k+3i\nu)} \cdot \int_0^w dx \int_{-\infty}^0 \frac{dy}{2\pi} e^{i(k+3i\nu)x-i(s+3i\nu)y} \partial_x \partial_y \widetilde{G}_R(x, y-w) \right| \\ &\leq \frac{C e^{-\nu w}}{|s+3i\nu| |k+3i\nu|} \cdot \int_0^w dx \int_{-\infty}^0 dy e^{-\nu(x-y)} e^{-\left(\frac{\pi}{7}-3\nu\right)(x-y+w)} = O\left(\frac{e^{-2\eta w}}{(1+|k|)(1+|s|)}\right) \end{aligned} \quad (5.44)$$

for some  $\eta > 0$  and for  $\nu$  small enough. The remaining terms in (5.43) are estimated along the same lines. Finally, the estimation of the last term appearing in the *rhs* of (5.41) follows exactly the same philosophy.  $\blacksquare$

### 5.3 Finer direct sum decomposition of the Hilbert space

While effective for the operator inversion, the direct sum decomposition (5.33) is however not fine enough to effectively grasp the large- $w$  asymptotics of the integral operators appearing in (5.35)-(5.37). For this, as will

become apparent in the following, one should further partition the central interval  $I_w^0$  as

$$I_w^0 = I_w^L \cup I_w^R \quad \text{with} \quad I_w^L = ]-w; 0[ \quad \text{and} \quad I_w^R = [0; w[. \quad (5.45)$$

Accordingly, we introduce a notation for the projections of  $f \in L^2(\mathbb{R})$  subordinate to the new intervals  $I_w^{L/R}$ :

$$f^L = f \mathbf{1}_{I_w^L} \quad \text{and} \quad f^R = f \mathbf{1}_{I_w^R}. \quad (5.46)$$

Since the equations (5.35)-(5.37) are already in Fourier space, it is convenient to introduce the projection operators from  $\mathcal{F}[L^2(I_w^0)] \subset L^2(\mathbb{R} + i\nu)$  onto  $\mathcal{F}[L^2(I_w^{R/L})]$  which, for the moment, we continue to think of as a subspace of  $L^2(\mathbb{R} + i\nu)$

$$\mathcal{P}^R[f] = \mathcal{C}_+^{(+)}[f] \quad \text{and} \quad \mathcal{P}^L[f] = -\mathcal{C}_-^{(+)}[f] \quad (5.47)$$

in which  $\mathcal{C}^{(+)}$  is the Cauchy transform on  $L^2(\mathbb{R} + i\nu)$

$$\mathcal{C}^{(+)}[f](k) = \int_{\mathbb{R}-i\nu} \frac{ds}{2i\pi} \frac{f(s)}{s-k}. \quad (5.48)$$

Finally, it will appear convenient to introduce the shorthand notation

$$u^\alpha(k) = \mathcal{F}[f^\alpha](k) \quad \text{for} \quad \alpha \in \{\pm, 0, L, R\}. \quad (5.49)$$

for the Fourier transforms appearing in (5.35)-(5.37). Obviously,  $u^0 = u^L + u^R$ . Furthermore,  $u^{L/R}$  are entire and, in particular, analytic in a tubular neighbourhood of  $\mathbb{R}$ . This makes it possible to identify  $\mathcal{F}[L^2(I_w^L)]$  as a subspace of  $L^2(\mathbb{R} + i\nu)$  and  $\mathcal{F}[L^2(I_w^R)]$  as a subspace of  $L^2(\mathbb{R} - i\nu)$ , even though the splitting  $u^0 = u^L + u^R$  would suggest an identification of  $\mathcal{F}[L^2(I_w^R)]$  as being a subspace of  $L^2(\mathbb{R} + i\nu)$ . The former, however, appears to be more useful for the purposes of the analysis to come. On the practical side, this identification with a subspace of  $L^2(\mathbb{R} - i\nu)$  simply means a shift of the integration domain in the terms involving  $u^R$  from  $\mathbb{R} + i\nu$  to  $\mathbb{R} - i\nu$  what is possible owing to the analyticity of the integrand.

#### 5.4 Decomposition in the – sector

In this subsection, we recast the equation in the – sector, *viz.* (5.35), in a form convenient for the further analysis. In particular, we explicitly implement the changes issuing from the use of the decomposition  $u^0 = u^L + u^R$ .

It is readily seen that

$$\mathcal{F}[H^-](k) = \mathcal{F}[h^-](k) + \sum_{\alpha \in \{\pm, 0\}} \widehat{\mathbf{B}}_{\text{tot}}^{-\sigma} [u^\alpha](k) \quad (5.50)$$

with

- $H^-$  as defined in (5.30);
- $\widehat{\mathbf{B}}_{\text{tot}}^{-\sigma} : L^2(\mathbb{R} + i\nu) \rightarrow L^2(\mathbb{R} - i\nu)$ ,  $\sigma \in \{0, +\}$ , acting with the integral kernel  $\widehat{B}_{\text{tot}}^{-\sigma}(k, s)$  as defined in (5.38);
- $\widehat{\mathbf{B}}_{\text{tot}}^{-\sigma} : L^2(\mathbb{R} - i\nu) \rightarrow L^2(\mathbb{R} - i\nu)$  acting with the integral kernel  $\widehat{B}_{\text{tot}}^{-\sigma}(k, s)$ .

Observe that, upon using  $P^R[u^R] = u^R$  one has

$$\begin{aligned}\Psi^{-R}[u^R](k) &= -\alpha_{\downarrow}^{(e)}(k)e^{-ikw} \int_{\mathbb{R}+iv} \frac{dt}{2i\pi} \frac{\{\alpha_{\uparrow}^{(e)}(t)\}^{-1} e^{itw} \widehat{B}_{\text{tot}}^{-0}[u^R](t)}{t-k} \\ &= \lim_{\epsilon \rightarrow 0^+} \left\{ -\alpha_{\downarrow}^{(e)}(k)e^{-ikw} \int_{\mathbb{R}+iv} \frac{dt}{2i\pi} \frac{\{\alpha_{\uparrow}^{(e)}(t)\}^{-1}}{t-k} \int_{\mathbb{R}+iv} ds \left[ e^{itw} \mathfrak{B}^{-0}(t,s) + \int_{\mathbb{R}+iv} \frac{dx}{2i\pi} \frac{\widehat{B}_L^{-+}(t,x)e^{ixw}}{s-x-i\epsilon} \right] u^R(s) \right\} \quad (5.51)\end{aligned}$$

Using that  $u_R$  is entire, and that  $\mathfrak{B}^{-0}(t,s)$  is analytic in a tubular neighbourhood of  $\mathbb{R}^2$ , one may deform the  $s$  integrals to  $\mathbb{R} - iv$ . Furthermore, the analytic structure of the integrand allows one to deform the  $x$ -integrations to  $\mathbb{R} + i\varrho$  for some fixed  $\varrho > 0$  that is  $\nu$ -independent. This entails that

$$\Psi^{-R}[u^R](k) = \int_{\mathbb{R}-iv} \Psi^{-R}(k,s) u^R(s) \quad , \quad k \in \mathbb{R} - iv \quad , \quad (5.52)$$

with

$$\Psi^{-R}(k,s) = -\alpha_{\downarrow}^{(e)}(k)e^{-ikw} \int_{\mathbb{R}+iv} \frac{dt}{2i\pi} \frac{\{\alpha_{\uparrow}^{(e)}(t)\}^{-1}}{t-k} \left[ e^{itw} \mathfrak{B}^{-0}(t,s) + \int_{\mathbb{R}+i\varrho} \frac{dx}{2i\pi} \frac{\widehat{B}_L^{-+}(t,x)e^{ixw}}{s-x} \right]. \quad (5.53)$$

The decay estimates for  $\widehat{B}_L^{\sigma\epsilon}$  (2.68) and  $\mathfrak{B}^{\sigma\epsilon}$  (5.40) along with Lemma D.1 readily entail that, for  $0 < \nu$  small enough,

$$\Psi^{-R}(k,s) = \mathcal{O}\left( e^{-\eta w} \frac{\ln(1+|s|) \cdot \ln(1+|k|)}{(1+|s|) \cdot (1+|k|)} \right) \quad \text{with} \quad (k,s) \in \{\mathbb{R} - iv\}^2 \quad , \quad (5.54)$$

where  $\eta > 0$  is fixed and  $\nu$  independent. To take into account the other quantities arising in (5.50), we introduce the integral kernels

$$\begin{pmatrix} \Psi^{-L}(k,s) \\ \Psi^{-+}(k,s) \end{pmatrix} = -\alpha_{\downarrow}^{(e)}(k)e^{-ikw} \int_{\mathbb{R}+iv} \frac{dt}{2i\pi} \frac{\{\alpha_{\uparrow}^{(e)}(t)\}^{-1} e^{itw}}{t-k} \begin{pmatrix} \mathfrak{B}^{-0}(t,s) \\ \mathfrak{B}^{-+}(t,s) \end{pmatrix} \quad (5.55)$$

which, owing to Lemma D.1 and (5.40), enjoy the bounds,  $\sigma \in \{L, +\}$ ,

$$\Psi^{-\sigma}(k,s) = \mathcal{O}\left( \frac{e^{-\eta w} \cdot \ln(1+|k|)}{(1+|s|) \cdot (1+|k|)} \right) \quad \text{with} \quad (k,s) \in \{\mathbb{R} - iv\} \times \{\mathbb{R} + iv\} \quad . \quad (5.56)$$

The functions  $\Psi^{-\sigma}(k,s)$ ,  $\sigma \in \{+, L, R\}$ , then allow one to introduce the integral operators  $\Psi^{-\sigma} : L^2(\mathbb{R} + i\varepsilon_{\sigma}\nu) \rightarrow L^2(\mathbb{R} + iv)$  where  $\varepsilon_{\sigma}$  is defined as

$$\varepsilon_{-} = \varepsilon_R = - \quad \text{and} \quad \varepsilon_{+} = \varepsilon_L = + \quad . \quad (5.57)$$

Also, one introduces

$$M_L^{-\sigma}(k,s) = -\alpha_{\downarrow}^{(e)}(k) \int_{\mathbb{R}+i\varrho} \frac{dt}{2i\pi} \frac{\{\alpha_{\uparrow}^{(e)}(t)\}^{-1} \widehat{B}_L^{-\sigma}(t,s)}{t-k} \quad , \quad \sigma = \pm \quad (5.58)$$

and

$$\mathfrak{d}_w^-[h^-](k) = -\alpha_{\downarrow}^{(e)}(k)e^{-ikw} \int_{\mathbb{R}+iv} \frac{ds}{2i\pi} \frac{\{\alpha_{\uparrow}^{(e)}(s)\}^{-1} \cdot e^{isw}}{s-k} \cdot \mathcal{F}[h^-](s). \quad (5.59)$$

Furthermore, we introduced the integral operator  $\mathbf{M}_L^{-\sigma} : L^2(\mathbb{R} + i\varepsilon_{\sigma}v) \rightarrow L^2(\mathbb{R} - iv)$ ,  $\sigma \in \{\pm\}$ , characterised by the integral kernel  $M_L^{-\sigma}(k, s)$ .

Finally, denote by  $e$  the operator of multiplication by the function  $e$ , viz.

$$e[f](\lambda) = e(\lambda)f(\lambda) \quad \text{with} \quad e(\lambda) = e^{i\lambda w}. \quad (5.60)$$

Then, by using the decomposition (5.50) one may recast the representation (5.35) in the operator form

$$u^-(k) = \left( e^{-1}\mathbf{M}_L^{-} e \right) [u^-](k) + \left( e^{-1}\mathbf{M}_L^{+} e \right) [u^L](k) + \sum_{\sigma \in \{L, R, +\}} \Psi^{-\sigma} [u^{\sigma}](k) + \mathfrak{d}_w^-[h^-](k). \quad (5.61)$$

Note that equation (5.61) may already be interpreted as holding for the first component  $u^-$  of the vector

$$\mathbf{u} = \begin{pmatrix} u^- \\ u^L \\ u^R \\ u^+ \end{pmatrix} \in L^2(\mathbb{R} - iv) \oplus L^2(\mathbb{R} + iv) \oplus L^2(\mathbb{R} - iv) \oplus L^2(\mathbb{R} + iv). \quad (5.62)$$

Finally, by introducing the operators

$$\left( \Omega^{-L} \quad \Omega^{-R} \quad \Omega^{-+} \right) = \left( e \Psi^{-L} e^{-1} \quad e \Psi^{-R} e \quad e \Psi^{-+} e \right) \quad (5.63)$$

one may recast (5.61) in the form of a line vector of operators times a column vector of functions, which will be best suited for the later handling:

$$\left( \text{id} - e^{-1}\mathbf{M}_L^{-} e; \quad -e^{-1}[\mathbf{M}_L^{+} + \Omega^{-L}] e; \quad -e^{-1}\Omega^{-R} e^{-1}; \quad -e^{-1}\Omega^{-+} e^{-1} \right) [\mathbf{u}] = \mathfrak{d}_w^-[h^-](k). \quad (5.64)$$

## 5.5 Decomposition in the + sector

In this subsection, we provide the appropriate operator rewriting of the equation in the + sector, viz. (5.37), after implementing the decomposition  $u^0 = u^L + u^R$ .

Analogously to the - sector, one has that

$$\mathcal{F}[H^+](k) = \mathcal{F}[h^+](k) + \sum_{\alpha \in \{\pm, 0\}} \widehat{\mathbf{B}}_{\text{tot}}^{+\sigma} [u^{\sigma}](k) \quad (5.65)$$

with

- $H^+$  as defined in (5.32);
- $\widehat{\mathbf{B}}_{\text{tot}}^{+\sigma} : L^2(\mathbb{R} + iv) \rightarrow L^2(\mathbb{R} + iv)$ ,  $\sigma \in \{0, +\}$ , acting with the integral kernel  $\widehat{B}_{\text{tot}}^{+\sigma}(k, s)$ , c.f. (5.38);
- $\widehat{\mathbf{B}}_{\text{tot}}^{+-} : L^2(\mathbb{R} - iv) \rightarrow L^2(\mathbb{R} + iv)$  acting with the integral kernel  $\widehat{B}_{\text{tot}}^{+-}(k, s)$ .

The obvious identity  $\mathcal{P}^L[u^L] = u^L$  leads to

$$\begin{aligned}\Psi^{+L}[u^L](k) &= \frac{e^{ikw}}{\alpha_{\uparrow}^{(e)}(k)} \int_{\mathbb{R}-iv} \frac{dt}{2i\pi} \frac{\alpha_{\downarrow}^{(e)}(t) e^{-itw}}{t-k} \widehat{B}_{\text{tot}}^{+0}[u^L](t) \\ &= \lim_{\epsilon \rightarrow 0^+} \left\{ \frac{e^{ikw}}{\alpha_{\uparrow}^{(e)}(k)} \int_{\mathbb{R}-iv} \frac{dt}{2i\pi} \frac{\alpha_{\downarrow}^{(e)}(t)}{t-k} \int_{\mathbb{R}+iv} ds \left[ e^{-itw} \mathfrak{B}^{+0}(t, s) - \int_{\mathbb{R}+iv} \frac{dx}{2i\pi} \frac{\widehat{B}_R^{+-}(t, x) e^{-ixw}}{s-x+i\epsilon} \right] u^L(s) \right\}. \quad (5.66)\end{aligned}$$

One then deforms the  $x$ -integrations to  $\mathbb{R} - i\rho$  for some fixed  $\rho > 0$  that is  $\nu$ -independent leading to

$$\Psi^{+L}[u^L](k) = \int_{\mathbb{R}+iv} \Psi^{+L}(k, s) u^L(s) \quad (5.67)$$

with

$$\Psi^{+L}(k, s) = \frac{e^{ikw}}{\alpha_{\uparrow}^{(e)}(k)} \int_{\mathbb{R}-iv} \frac{dt}{2i\pi} \frac{\alpha_{\downarrow}^{(e)}(t)}{t-k} \left[ e^{-itw} \mathfrak{B}^{+0}(t, s) - \int_{\mathbb{R}-i\rho} \frac{dx}{2i\pi} \frac{\widehat{B}_R^{+-}(t, x) e^{-ixw}}{s-x} \right]. \quad (5.68)$$

The decay estimates for  $\widehat{B}_L^{\sigma\epsilon}$  (2.68) and  $\mathfrak{B}^{\sigma\epsilon}$  (5.40) along with Lemma D.1 readily entail that, for  $0 < \nu$  small enough,

$$\Psi^{+L}(k, s) = \mathcal{O}\left( e^{-\eta w} \frac{\ln(1+|s|) \cdot \ln(1+|k|)}{(1+|s|) \cdot (1+|k|)} \right) \quad \text{with} \quad (k, s) \in \{\mathbb{R} + iv\}^2. \quad (5.69)$$

In order to take into account the other terms present in (5.65), one is lead to introduce the integral kernels

$$\begin{pmatrix} \Psi^{+-}(k, s) \\ \Psi^{+R}(k, s) \end{pmatrix} = \frac{e^{ikw}}{\alpha_{\uparrow}^{(e)}(k)} \int_{\mathbb{R}-iv} \frac{dt}{2i\pi} \frac{\alpha_{\downarrow}^{(e)}(t) e^{-itw}}{t-k} \begin{pmatrix} \mathfrak{B}^{+-}(t, s) \\ \mathfrak{B}^{+0}(t, s) \end{pmatrix} \quad (5.70)$$

which enjoy the bounds,

$$\Psi^{+\sigma}(k, s) = \mathcal{O}\left( \frac{e^{-\eta w} \cdot \ln(1+|k|)}{(1+|s|) \cdot (1+|k|)} \right) \quad \text{with} \quad (k, s) \in \{\mathbb{R} + iv\} \times \{\mathbb{R} - iv\} \quad \text{and} \quad \sigma \in \{-, R\}. \quad (5.71)$$

The functions  $\Psi^{+\sigma}(k, s)$ ,  $\sigma \in \{-, L, R\}$ , then allow one to introduce the integral operators  $\Psi^{+\sigma} : L^2(\mathbb{R} + i\varepsilon_{\sigma}\nu) \rightarrow L^2(\mathbb{R} + iv)$  in which  $\varepsilon_{\sigma}$  has been defined in (5.57).

Finally, one introduces

$$M_R^{+\sigma}(k, s) = \frac{1}{\alpha_{\uparrow}^{(e)}(k)} \int_{\mathbb{R}-iv} \frac{dt}{2i\pi} \frac{\alpha_{\downarrow}^{(e)}(t) \widehat{B}_R^{+\sigma}(t, s)}{t-k}, \quad \sigma = \pm, \quad (5.72)$$

and

$$\mathfrak{d}_w^+[h^+](k) = \frac{e^{ikw}}{\alpha_{\uparrow}^{(e)}(k)} \cdot \int_{\mathbb{R}-iv} \frac{ds}{2i\pi} \frac{\alpha_{\downarrow}^{(e)}(s) e^{-isw} \mathcal{F}[h^+](s)}{s-k}. \quad (5.73)$$



At this stage we introduced the integral operator  $\mathbf{M}_R^{+\sigma} : L^2(\mathbb{R} + i\varepsilon_\sigma\nu) \rightarrow L^2(\mathbb{R} + i\nu)$ ,  $\sigma \in \{\pm\}$ , characterised by the integral kernel  $M_R^{+\sigma}(k, s)$ .

Altogether, this recasts (5.35) in the following operator form

$$u^+(k) = \left( \mathbf{e} \mathbf{M}_R^{++} \mathbf{e}^{-1} \right) [u^+](k) + \left( \mathbf{e} \mathbf{M}_R^{+-} \mathbf{e}^{-1} \right) [u^R](k) + \sum_{\sigma \in \{L, R, -\}} \Psi^{+\sigma} [u^\sigma](k) + \mathfrak{d}_w^+ [h^+](k), \quad (5.74)$$

in which  $\Psi^{+\sigma}$  are integral operators acting with the integral kernels introduced earlier on and the operator  $\mathbf{e}$  has been introduced in (5.60). This equation now concerns the last component  $u^+$  of the vector appearing in (5.62).

Finally, by introducing the operators

$$\left( \begin{array}{ccc} \Omega^{+-} & \Omega^{+L} & \Omega^{+R} \end{array} \right) = \left( \begin{array}{ccc} \mathbf{e}^{-1} \Psi^{+-} \mathbf{e}^{-1} & \mathbf{e}^{-1} \Psi^{+L} \mathbf{e}^{-1} & \mathbf{e}^{-1} \Psi^{+R} \mathbf{e} \end{array} \right), \quad (5.75)$$

one may recast (5.74) in the following form that will be best suited for the later handling:

$$\left( \begin{array}{c} -\mathbf{e} \Omega^{+-} \mathbf{e} ; \quad -\mathbf{e} \Omega^{+L} \mathbf{e} ; \quad -\mathbf{e} [\mathbf{M}_R^{+-} + \Omega^{+R}] \mathbf{e}^{-1} ; \quad \text{id} - \mathbf{e} \mathbf{M}_R^{++} \mathbf{e}^{-1} \end{array} \right) [\mathbf{u}] = \mathfrak{d}_w^+ [h^+](k). \quad (5.76)$$

Above,  $\mathbf{u}$  is as given by (5.62).

## 5.6 Decomposition in the 0 sector

In this subsection, we provide an appropriate rewriting of the equation in the 0 sector, *viz.* (5.36), after incorporating the decomposition  $u^0 = u^L + u^R$ . In the case of the sector subordinate to the interval  $I_w^0$ , the Fourier transform of the function  $H^0$  defined in (5.31) takes the form

$$\mathcal{F}[H^0](k) = \mathcal{F}[h^0](k) + \sum_{\alpha \in \{\pm, 0\}} \widehat{\mathbf{B}}_{\text{tot}}^{0\sigma} [u^\sigma](k) \quad (5.77)$$

with

- $\widehat{\mathbf{B}}_{\text{tot}}^{0\sigma} : L^2(\mathbb{R} + i\nu) \rightarrow L^2(\mathbb{R} + i\nu)$ ,  $\sigma \in \{0, +\}$ , acting with the integral kernel  $\widehat{B}_{\text{tot}}^{0\sigma}(k, s)$ , *c.f.* (5.38);
- $\widehat{\mathbf{B}}_{\text{tot}}^{0-} : L^2(\mathbb{R} - i\nu) \rightarrow L^2(\mathbb{R} + i\nu)$  acting with the integral kernel  $\widehat{B}_{\text{tot}}^{0-}(k, s)$ .

It then follows from (5.39) that the latter may be further expressed as

$$\begin{aligned} \mathcal{F}[H^0](k) = \mathcal{F}[h^0](k) + \left( \mathbf{e}^{-1} \widehat{\mathbf{B}}_L^{+-} \mathbf{e} \right) [u^-] + \left( \mathbf{e}^{-1} \widehat{\mathbf{B}}_L^{++} \mathbf{e} + \mathbf{e} \widehat{\mathbf{B}}_R^{--} \mathbf{e}^{-1} \right) [u^0] \\ + \left( \mathbf{e} \widehat{\mathbf{B}}_R^{+-} \mathbf{e}^{-1} \right) [u^+] + \sum_{\alpha \in \{\pm, 0\}} \mathfrak{B}^{0\sigma} [u^\sigma](k). \end{aligned} \quad (5.78)$$

Thus equation (5.36) leads to the following expression for  $u^0(k)$

$$u^0(k) = \mathfrak{d}_w^0 [h^0](k) + \sum_{\alpha \in \{\pm, 0\}} \beta^{0\sigma} [u^\sigma](k) + \sum_{\alpha \in \{\pm, 0\}} \left( \widetilde{\mathfrak{B}}^{0\sigma} + \delta \beta^{0\sigma} \right) [u^\sigma](k). \quad (5.79)$$

There, we have introduced

$$\mathfrak{d}_w^0 [h^0](k) = \left( \text{id} - \mathbf{R} \right) \left[ \mathcal{F}[h^0] \right](k) \quad (5.80)$$

while the operators  $\widetilde{\mathfrak{B}}^{0\sigma}$ ,  $\sigma \in \{\pm, 0\}$ , act with the integral kernels

$$\widetilde{\mathfrak{B}}^{0\sigma}(k, s) = (\text{id} - \mathbf{R})[\mathfrak{B}^{0\sigma}(*, s)](k) = \mathfrak{B}^{0\sigma}(k, s) - \int_{\mathbb{R}+iv} d\mu R(k, \mu) \mathfrak{B}^{0\sigma}(\mu, s). \quad (5.81)$$

The expressions for the integral kernels  $\beta^{0\sigma}(k, s)$  and  $\delta\beta^{0\sigma}(k, s)$  involve the leading  $R_\infty(\lambda, \mu)$  and perturbative  $\delta R(\lambda, \mu)$  resolvent kernel, as introduced in (C.57) and (C.58). Indeed,

$$\beta^{0-}(k, s) = (\text{id} - \mathbf{R}_\infty)[e^{-1}(*)\widehat{B}_L^{+-}(*, s)e(s)](k), \quad \beta^{0+}(k, s) = (\text{id} - \mathbf{R}_\infty)[e(*)\widehat{B}_R^{+}(*, s)e^{-1}(s)](k), \quad (5.82)$$

as well as

$$\beta^{00}(k, s) = (\text{id} - \mathbf{R}_\infty)[e^{-1}(*)\widehat{B}_L^{++}(*, s)e(s) + e(*)\widehat{B}_R^{--}(*, s)e^{-1}(s)](k). \quad (5.83)$$

Above,  $*$  refers to the running variable on which the operator acts. Finally, one has

$$\delta\beta^{0-}(k, s) = -\delta\mathbf{R}[e^{-1}(*)\widehat{B}_L^{+-}(*, s)e(s)](k), \quad \delta\beta^{0+}(k, s) = -\delta\mathbf{R}[e(*)\widehat{B}_R^{+}(*, s)e^{-1}(s)](k), \quad (5.84)$$

as well as

$$\delta\beta^{00}(k, s) = -\delta\mathbf{R}[e^{-1}(*)\widehat{B}_L^{++}(*, s)e(s) + e(*)\widehat{B}_R^{--}(*, s)e^{-1}(s)](k). \quad (5.85)$$

The rewriting of the operators  $\delta\beta^{0\sigma}$  and  $\widetilde{\mathfrak{B}}^{0\sigma}$  in a form appropriate for the analysis to come is rather direct and we shall carry it out first. Then, we focus on the operators  $\beta^{0\sigma}$  whose large- $w$  asymptotics demand a deeper investigation.

### 5.6.1 Perturbing operators $\Psi_{\mathfrak{B}}^{\sigma\tau}$

It follows from the estimates (5.40) and (C.62), direct bounds and the possibility to deform slightly the  $\mu$ -integration contour in (5.81) that one has

$$\widetilde{\mathfrak{B}}^{0\sigma}(k, s) = \mathcal{O}\left(\frac{e^{-\eta w}}{(1+|k|)(1+|s|)}\right) \quad \text{for any} \quad |\Im(k)| \leq 2\nu, |\Im(s)| \leq 2\nu, \quad (5.86)$$

this provided that  $0 < \nu$  is sufficiently small. Furthermore, one also gets that

$$\begin{aligned} \widetilde{\mathfrak{B}}^{L\sigma}(k, s) &= \mathbf{P}^L[\widetilde{\mathfrak{B}}^{0\sigma}(*, s)](k) = -\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}+iv} \frac{dt}{2i\pi} \frac{\widetilde{\mathfrak{B}}^{0\sigma}(t, s)}{t - k + i\epsilon} \\ &= -\int_{\mathbb{R}+2iv} \frac{dt}{2i\pi} \frac{\widetilde{\mathfrak{B}}^{0\sigma}(t, s)}{t - k} = \mathcal{O}\left(\frac{e^{-\eta w} \ln(1+|k|)}{(1+|k|)(1+|s|)}\right), \end{aligned} \quad (5.87)$$

by virtue of Lemma D.1. Likewise,

$$\begin{aligned} \widetilde{\mathfrak{B}}^{R\sigma}(k, s) &= \mathbf{P}^R[\widetilde{\mathfrak{B}}^{0\sigma}(*, s)](k) = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}+iv} \frac{dt}{2i\pi} \frac{\widetilde{\mathfrak{B}}^{0\sigma}(t, s)}{t - k - i\epsilon} \\ &= \int_{\mathbb{R}-2iv} \frac{dt}{2i\pi} \frac{\widetilde{\mathfrak{B}}^{0\sigma}(t, s)}{t - k} = \mathcal{O}\left(\frac{e^{-\eta w} \ln(1+|k|)}{(1+|k|)(1+|s|)}\right). \end{aligned} \quad (5.88)$$

Since the integral kernel  $\mathfrak{B}^{00}(k, s)$  is holomorphic in a tubular neighbourhood of  $\mathbb{R}^2$ , the fact that the resolvent kernel  $R$  is also analytic in such a neighbourhood and the bounds (C.62) entail that  $\widetilde{\mathfrak{B}}^{00}(k, s)$  is also analytic in such a tubular neighbourhood. Then, the fact that  $u^{L/R}$  are entire allows one to deform the integration contours in the action below so as to get

$$\widetilde{\mathfrak{B}}^{00}[u^0](k) = \int_{\mathbb{R}+iv} ds \widetilde{\mathfrak{B}}^{00}(k, s) u^L(s) + \int_{\mathbb{R}-iv} ds \widetilde{\mathfrak{B}}^{00}(k, s) u^R(s). \quad (5.89)$$

Therefore, upon defining the integral kernels

$$\left( \Psi_{\mathfrak{B}}^{\sigma-}(k, s) \quad \Psi_{\mathfrak{B}}^{\sigma L}(k, s) \quad \Psi_{\mathfrak{B}}^{\sigma R}(k, s) \quad \Psi_{\mathfrak{B}}^{\sigma+}(k, s) \right) = \left( \widetilde{\mathfrak{B}}^{\sigma-}(k, s) \quad \widetilde{\mathfrak{B}}^{\sigma 0}(k, s) \quad \widetilde{\mathfrak{B}}^{\sigma 0}(k, s) \quad \widetilde{\mathfrak{B}}^{\sigma+}(k, s) \right), \quad (5.90)$$

with  $\sigma \in \{L, R\}$ , one may introduce the associated operators  $\Psi_{\mathfrak{B}}^{\sigma\tau} : L^2(\mathbb{R} + i\varepsilon_{\sigma}v) \rightarrow L^2(\mathbb{R} + i\varepsilon_{\tau}v)$ , with  $\sigma \in \{L, R\}$  and  $\tau \in \{\pm, L, R\}$  and where  $\varepsilon_{\sigma}$  is as given in (5.57).

By virtue of the previous estimates, one has

$$\Psi_{\mathfrak{B}}^{\sigma\tau}(k, s) = \mathcal{O}\left(\frac{e^{-\eta w} \ln(1 + |k|)}{(1 + |k|)(1 + |s|)}\right) \quad \text{with} \quad (k, s) \in \{\mathbb{R} + i\varepsilon_{\sigma}v\} \times \{\mathbb{R} + i\varepsilon_{\tau}v\}. \quad (5.91)$$

### 5.6.2 Perturbing operators $\Psi_{\delta\beta}^{\sigma\tau}$

It follows from the estimates (2.68) and (C.61), direct bounds and the possibility to deform slightly the integration contour in the action of  $\delta R$  in (5.84)-(5.85) that one has the bounds

$$\delta\beta^{0\sigma}(k, s) = \mathcal{O}\left(\frac{e^{-\eta w}}{(1 + |k|)(1 + |s|)}\right) \quad \text{for any} \quad |\Im(k)| \leq 2v, \quad |\Im(s)| \leq 2v, \quad (5.92)$$

this provided that  $v$  is taken sufficiently small. Moreover, analogously to the previous reasonings

$$\delta\beta^{L\sigma}(k, s) = \mathbf{P}^L[\delta\beta^{0\sigma}(*, s)](k) = - \int_{\mathbb{R}+2iv} \frac{dt}{2i\pi} \frac{\delta\beta^{0\sigma}(t, s)}{t - k} = \mathcal{O}\left(\frac{e^{-\eta w} \ln(1 + |k|)}{(1 + |k|)(1 + |s|)}\right), \quad (5.93)$$

$$\delta\beta^{R\sigma}(k, s) = \mathbf{P}^R[\delta\beta^{0\sigma}(*, s)](k) = \int_{\mathbb{R}-2iv} \frac{dt}{2i\pi} \frac{\delta\beta^{0\sigma}(t, s)}{t - k} = \mathcal{O}\left(\frac{e^{-\eta w} \ln(1 + |k|)}{(1 + |k|)(1 + |s|)}\right). \quad (5.94)$$

Finally, one may present the action of  $\delta\beta^{00}$  on  $u^0$  as

$$\delta\beta^{00}[u^0](k) = \int_{\mathbb{R}+iv} ds \delta\beta^{00}(k, s) u^L(s) + \int_{\mathbb{R}-iv} ds \delta\beta^{00}(k, s) u^R(s). \quad (5.95)$$

Therefore, upon defining the integral kernels

$$\left( \Psi_{\delta\beta}^{\sigma-}(k, s) \quad \Psi_{\delta\beta}^{\sigma L}(k, s) \quad \Psi_{\delta\beta}^{\sigma R}(k, s) \quad \Psi_{\delta\beta}^{\sigma+}(k, s) \right) = \left( \delta\beta^{\sigma-}(k, s) \quad \delta\beta^{\sigma 0}(k, s) \quad \delta\beta^{\sigma 0}(k, s) \quad \delta\beta^{\sigma+}(k, s) \right), \quad (5.96)$$

with  $\sigma \in \{L, R\}$ , one gets the associated operators  $\Psi_{\delta\beta}^{\sigma\tau} : L^2(\mathbb{R} + i\varepsilon_{\sigma}v) \rightarrow L^2(\mathbb{R} + i\varepsilon_{\tau}v)$ , with  $\varepsilon_{\sigma}$  as defined in (5.57) and where  $\tau \in \{\pm, L, R\}$ .

By virtue of the previous estimates, one has

$$\Psi_{\delta\beta}^{\sigma\tau}(k, s) = \mathcal{O}\left(\frac{e^{-\eta w} \ln(1 + |k|)}{(1 + |k|)(1 + |s|)}\right) \quad \text{with} \quad (k, s) \in \{\mathbb{R} + i\varepsilon_{\sigma}v\} \times \{\mathbb{R} + i\varepsilon_{\tau}v\}. \quad (5.97)$$

### 5.6.3 Operator $\beta^{0+}$

In order to decompose the integral kernel  $\beta^{0+}(k, s)$  into its dominant and sub-dominant in  $w$  parts, by using the explicit expression for the leading resolvent (C.59), one first computes

$$\begin{aligned} \mathbb{P}^L[(\text{id} - \mathbb{R}_\infty)[e(*)\widehat{B}_R^{0+}(*, s)e^{-1}(s)](\cdot)](k) &= \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}+i\nu} \frac{d\lambda}{2i\pi} \frac{e(\lambda)\widehat{B}_R^{0+}(\lambda, s)e^{-1}(s)}{k - \lambda - i\epsilon} - \int_{\mathbb{R}+i\nu} \frac{d\lambda}{2i\pi} \frac{-\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda - i\epsilon} \\ &\times \int_{\mathbb{R}+i\nu} \frac{d\mu}{2i\pi} \left( -\{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1}, \alpha_\downarrow^{(0)}(\lambda)e(\lambda) \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu\mathfrak{b}'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu)e^2(\mu) \\ \{\alpha_\uparrow^{(0)}(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_R^{0+}(\mu, s)e^{-1}(s)}{\lambda - \mu} \end{aligned} \quad (5.98)$$

where  $\mathbb{D}$  is as defined in (C.33),  $\mathfrak{b}$  is as defined in (C.43), while  $\alpha_{\uparrow/\downarrow}^{(0)}$  are as described in Subsection C.2.1.

To proceed further, one splits the integral as follows

$$\mathbb{P}^L[(\text{id} - \mathbb{R}_\infty)[e(*)\widehat{B}_R^{0+}(*, s)e^{-1}(s)](\cdot)](k) = e^{-1}(k)\Phi^{L+}(k, s)e^{-1}(s) + \Psi_{\mathbb{R}_\infty}^{L+}(k, s) \quad (5.99)$$

in which, for  $\sigma \in \{+, R\}$  and  $\varepsilon_\sigma$  as in (5.57),

$$\begin{aligned} \Psi_{\mathbb{R}_\infty}^{L\sigma}(k, s) &= \int_{\mathbb{R}+2i\eta} \frac{d\lambda}{2i\pi} \frac{e(\lambda)\widehat{B}_R^{-\varepsilon_\sigma}(\lambda, s)e^{-1}(s)}{k - \lambda} \\ &+ \int_{\mathbb{R}+3i\frac{\eta}{2}} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \int_{\mathbb{R}+2i\eta} \frac{d\mu}{2i\pi} \left( 0, \alpha_\downarrow^{(0)}(\lambda)e(\lambda) \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu\mathfrak{b}'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu)e^2(\mu) \\ \{\alpha_\uparrow^{(0)}(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)e^{-1}(s)}{\lambda - \mu} \\ &+ \int_{\mathbb{R}+2i\nu} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \int_{\mathbb{R}+2i\eta} \frac{d\mu}{2i\pi} \left( -\{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1}, 0 \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu\mathfrak{b}'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu)e^2(\mu) \\ 0 \end{pmatrix} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)e^{-1}(s)}{\lambda - \mu}. \end{aligned} \quad (5.100)$$

Furthermore, upon using

$$(-a, b)\mathbb{D} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a + b, \quad (5.101)$$

one entails that

$$\Phi^{L\sigma}(k, s) = e(k) \int_{\mathbb{R}+2i\nu} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)e^{-1}(\lambda)}{(k - \lambda)\lambda\alpha_\uparrow^{(0)}(\lambda)} \times \int_{\mathbb{R}+i\nu} \frac{d\mu}{2i\pi} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)}{\mu\alpha_\uparrow^{(0)}(\mu)\mathfrak{b}'(0)} \quad (5.102)$$

$$= \left\{ e(k) \int_{\mathbb{R}+i\frac{\eta}{2}} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)e^{-1}(\lambda)}{(k - \lambda)\lambda\alpha_\uparrow^{(0)}(\lambda)} + \frac{\mathcal{F}[L^{(0)}](k)}{k\alpha_\uparrow^{(0)}(k)} \right\} \times \int_{\mathbb{R}+i\nu} \frac{d\mu}{2i\pi} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)}{\mu\alpha_\uparrow^{(0)}(\mu)\mathfrak{b}'(0)}. \quad (5.103)$$

It is direct to check that  $\Phi^{L\sigma}(k, s)$ ,  $\sigma \in \{+, R\}$  enjoys, for some  $c > 0$ , the bound

$$\Phi^{L\sigma}(k, s) = \mathcal{O}\left( \frac{e^{-\frac{wv}{2}}}{(1 + |k|)(1 + |s|)} + \frac{e^{-c|k|}}{w(1 + |s|)} \right) \quad \text{for} \quad (k, s) \in \{\mathbb{R} + i\nu\} \times \{\mathbb{R} + i\varepsilon_\sigma\nu\}. \quad (5.104)$$

Quite similarly, one has

$$\begin{aligned}
\mathbb{P}^R[(\text{id} - \mathbb{R}_\infty)[e(*)\widehat{B}_R^-(*, s)e^{-1}(s)](\cdot)](k) &= - \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}+iv} \frac{d\lambda}{2i\pi} \frac{e(\lambda)\widehat{B}_R^-(\lambda, s)e^{-1}(s)}{k - \lambda + i\epsilon} - \int_{\mathbb{R}+iv} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda + i\epsilon} \\
&\times \int_{\mathbb{R}+iv} \frac{d\mu}{2i\pi} \left( -\{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1}, \alpha_\downarrow^{(0)}(\lambda)e(\lambda) \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu b'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu)e^2(\mu) \\ \{\alpha_\uparrow^{(0)}(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_R^-(\mu, s)e^{-1}(s)}{\lambda - \mu} \\
&= U^{R+}(k, s) + \Psi_{R_\infty}^{R+}(k, s). \quad (5.105)
\end{aligned}$$

There, with  $\sigma \in \{+, R\}$  and  $\varepsilon_\sigma$  as in (5.57), we set

$$U^{R\sigma}(k, s) = e(k)\widehat{B}_R^{-\varepsilon_\sigma}(k, s)e^{-1}(s) - \int_{\mathbb{R}-2iv} \frac{d\lambda}{2i\pi} \int_{\mathbb{R}+2iv} \frac{d\mu}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)\alpha_\downarrow^{(0)}(\lambda)e(\lambda)}{(k - \lambda)\alpha_\uparrow^{(0)}(\mu)} \left( 1 + \frac{\lambda - \mu}{\lambda\mu b'(0)} \right) \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)e^{-1}(s)}{\lambda - \mu}. \quad (5.106)$$

Also,

$$\begin{aligned}
\Psi_{R_\infty}^{R\sigma}(k, s) &= - \int_{\mathbb{R}+2i\eta} \frac{d\lambda}{2i\pi} \frac{e(\lambda)\widehat{B}_R^{-\varepsilon_\sigma}(\lambda, s)e^{-1}(s)}{k - \lambda} \\
&- \int_{\mathbb{R}-2i\eta} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \cdot \int_{\mathbb{R}+2i\eta} \frac{d\mu}{2i\pi} \left( -\{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1}, 0 \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu b'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu)e^2(\mu) \\ \{\alpha_\uparrow^{(0)}(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)e^{-1}(s)}{\lambda - \mu} \\
&- \int_{\mathbb{R}-2iv} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \int_{\mathbb{R}+i\eta} \frac{d\mu}{2i\pi} \left( 0, \alpha_\downarrow^{(0)}(\lambda)e(\lambda) \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu b'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu)e^2(\mu) \\ 0 \end{pmatrix} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)e^{-1}(s)}{\lambda - \mu}. \quad (5.107)
\end{aligned}$$

Finally, for  $\tau \in \{L, R\}$  and  $\sigma \in \{+, R\}$ , direct bounds based on Lemma D.1 lead to

$$\Psi_{R_\infty}^{\tau\sigma}(k, s) = \mathcal{O}\left(\frac{e^{-\eta v} \ln(1 + |k|)}{(1 + |k|)(1 + |s|)}\right) \quad \text{with} \quad (k, s) \in \{\mathbb{R} + i\varepsilon_\tau v\} \times \{\mathbb{R} + i\varepsilon_\sigma v\} \quad (5.108)$$

where  $\varepsilon_\sigma$  are as defined in (5.57). For further convenience, we introduce the integral operators

$$\Psi_{R_\infty}^{\tau\sigma} : L^2(\mathbb{R} - i\varepsilon_\sigma v) \rightarrow L^2(\mathbb{R} + i\varepsilon_\tau v), \quad \tau \in \{L, R\} \quad \text{and} \quad \sigma \in \{+, R\}, \quad (5.109)$$

acting with the integral kernels  $\Psi_{R_\infty}^{\tau\sigma}(k, s)$ . We remind that  $\varepsilon_\sigma$  is as in (5.57).

One may push the chain of transformations further for  $U^{R\sigma}(k, s)$ . Indeed, for  $k \in \mathbb{R} - iv$  and  $\sigma \in \{+, R\}$ , contour deformations yield

$$e^{-1}(k)U^{R\sigma}(k, s)e(s) = \Phi^{R\sigma}(k, s) + \widehat{B}_R^{-\varepsilon_\sigma}(k, s) + \mathcal{F}[L^{(0)}](k)\alpha_\downarrow^{(0)}(k) \int_{\mathbb{R}+i\eta} \frac{d\mu}{2i\pi} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)}{(k - \mu)\alpha_\uparrow^{(0)}(\mu)}. \quad (5.110)$$

There, we have introduced

$$\begin{aligned}
\Phi^{R\sigma}(k, s) &= -e^{-1}(k) \int_{\mathbb{R}-i\frac{v}{2}} \frac{d\lambda}{2i\pi} \int_{\mathbb{R}+2iv} \frac{d\mu}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)\alpha_\downarrow^{(0)}(\lambda)e(\lambda)}{(k - \lambda)\alpha_\uparrow^{(0)}(\mu)} \left( 1 + \frac{\lambda - \mu}{\lambda\mu b'(0)} \right) \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)}{\lambda - \mu} \\
&+ \mathcal{F}[L^{(0)}](k) \frac{\alpha_\downarrow^{(0)}(k)}{k \cdot b'(0)} \int_{\mathbb{R}+i\eta} \frac{d\mu}{2i\pi} \frac{\widehat{B}_R^{-\varepsilon_\sigma}(\mu, s)}{\mu\alpha_\uparrow^{(0)}(\mu)}. \quad (5.111)
\end{aligned}$$

Then, by observing that  $\mathcal{F}[L^{(0)}](k)\alpha_{\downarrow}^{(0)}(k) = \alpha_{\downarrow}^{(0)}(k) - \alpha_{\uparrow}^{(0)}(k)$  and setting

$$M_R^{\pm}(k, s) = - \int_{\mathbb{R}+i\eta} \frac{d\mu}{2i\pi} \frac{\alpha_{\downarrow}^{(0)}(k) \cdot \widehat{B}_R^{\pm}(\mu, s)}{(\mu - k) \cdot \alpha_{\uparrow}^{(0)}(\mu)} \quad (5.112)$$

as well as

$$\begin{aligned} V_R^{\pm}(k, s) &= \widehat{B}_R^{\pm}(k, s) + \int_{\mathbb{R}+i\eta} \frac{d\mu}{2i\pi} \frac{\alpha_{\uparrow}^{(0)}(k) \cdot \widehat{B}_R^{\pm}(\mu, s)}{(\mu - k) \cdot \alpha_{\uparrow}^{(0)}(\mu)} \\ &= \alpha_{\uparrow}^{(0)}(k) \left\{ \int_{\mathbb{R}-i\eta} \frac{d\mu}{2i\pi} \frac{\widehat{B}_R^{\pm}(\mu, s)}{(\mu - k) \cdot \alpha_{\uparrow}^{(0)}(\mu)} + \frac{\widehat{B}_R^{\pm}(0, s)}{k \cdot \alpha_0^{(0)}} \right\}, \end{aligned} \quad (5.113)$$

with  $\alpha_0^{(0)}$  as defined in (C.14), one gets that, for  $\sigma \in \{+, R\}$  and  $\varepsilon_{\sigma}$  as in (5.57),

$$e^{-1}(k)U^{R\sigma}(k, s)e(s) = M_R^{-\varepsilon_{\sigma}}(k, s) + V_R^{-\varepsilon_{\sigma}}(k, s) + \Phi^{R\sigma}(k, s). \quad (5.114)$$

It is direct to check that  $\Phi^{R\sigma}(k, s)$ ,  $\sigma \in \{+, R\}$ , enjoy for some  $c > 0$  the bound

$$\Phi^{R\sigma}(k, s) = \mathcal{O}\left(\frac{e^{-\frac{vw}{2}}}{(1+|k|)(1+|s|)} + \frac{e^{-c|k|}}{w(1+|s|)}\right) \quad \text{for } (k, s) \in \{\mathbb{R} - iv\} \times \{\mathbb{R} + i\varepsilon_{\sigma}v\}. \quad (5.115)$$

Similarly as before, we introduce the integral operators

$$\Phi_{R_{\infty}}^{\tau\sigma} : L^2(\mathbb{R} - i\varepsilon_{\sigma}v) \rightarrow L^2(\mathbb{R} + i\varepsilon_{\tau}v), \quad \tau \in \{L, R\} \quad \text{and} \quad \sigma \in \{+, R\}, \quad (5.116)$$

for  $\varepsilon_{\sigma}$  as in (5.57), whose integral kernels are  $\Phi_{R_{\infty}}^{\tau\sigma}(k, s)$ . We also introduce the integral operators

$$M_R^{\pm} : L^2(\mathbb{R} \pm i_{\sigma}v) \rightarrow L^2(\mathbb{R} - iv) \quad \text{and} \quad V_R^{\pm} : L^2(\mathbb{R} \pm i_{\sigma}v) \rightarrow L^2(\mathbb{R} - iv) \quad (5.117)$$

having integral kernels  $M_R^{\pm}(k, s)$  and  $V_R^{\pm}(k, s)$ , respectively.

All in all, we have established that

$$\beta^{L+}(k, s) = e(k)\Phi^{L+}(k, s)e^{-1}(s) + \Psi_{R_{\infty}}^{L+}(k, s), \quad (5.118)$$

$$\beta^{R+}(k, s) = e(k)\left(M_R^{-+}(k, s) + V_R^{-+}(k, s) + \Phi^{R+}(k, s)\right)e^{-1}(s) + \Psi_{R_{\infty}}^{R+}(k, s). \quad (5.119)$$

#### 5.6.4 Operator $\beta^{0-}$

In order to decompose the integral kernel  $\beta^{0-}(k, s)$  into its dominant and sub-dominant in  $w$  parts, by using the explicit expression for the leading resolvent (C.59), one first computes

$$\begin{aligned} \mathbb{P}^R[(\text{id} - R_{\infty})[e^{-1}(\cdot)\widehat{B}_L^{+-}(\cdot, s)e(s)]](k) &= - \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}+iv} \frac{d\lambda}{2i\pi} \frac{e^{-1}(\lambda)\widehat{B}_L^{+-}(\lambda, s)e(s)}{k - \lambda + i\varepsilon} + \int_{\mathbb{R}+iv} \frac{d\lambda}{2i\pi} \frac{-\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda + i\varepsilon} \\ &\times \int_{\mathbb{R}+iv} \frac{d\mu}{2i\pi} \left( -\{\alpha_{\uparrow}^{(0)}(\lambda)e(\lambda)\}^{-1}, \alpha_{\downarrow}^{(0)}(\lambda)e(\lambda) \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu b'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_{\downarrow}^{(0)}(\mu) \\ \{\alpha_{\uparrow}^{(0)}(\mu)e^2(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_L^{+-}(\mu, s)e(s)}{\lambda - \mu} \\ &= e(k)\Phi^{R-}(k, s)e(s) + \Psi_{R_{\infty}}^{R-}(k, s). \end{aligned} \quad (5.120)$$

In this splitting, for  $\sigma \in \{-, L\}$ , we set

$$\begin{aligned}
\Psi_{R_\infty}^{R\sigma}(k, s) &= - \int_{\mathbb{R}-2i\eta} \frac{d\lambda}{2i\pi} \frac{e^{-1}(\lambda) \widehat{B}_L^{+\varepsilon\sigma}(\lambda, s) e(s)}{k - \lambda} \\
&- \int_{\mathbb{R}-3i\frac{\eta}{2}} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \int_{\mathbb{R}-2i\eta} \frac{d\mu}{2i\pi} \left( -\{\alpha_\uparrow^{(0)}(\lambda) e(\lambda)\}^{-1}, 0 \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda \mu b'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu) \\ \{\alpha_\uparrow^{(0)}(\mu) e^2(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_L^{+\varepsilon\sigma}(\mu, s) e(s)}{\lambda - \mu} \\
&- \int_{\mathbb{R}-2iv} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \int_{\mathbb{R}-2i\eta} \frac{d\mu}{2i\pi} \left( 0, \alpha_\downarrow^{(0)}(\lambda) e(\lambda) \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda \mu b'(0)} \mathbb{D} \right) \begin{pmatrix} 0 \\ \{\alpha_\uparrow^{(0)}(\mu) e^2(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_L^{+\varepsilon\sigma}(\mu, s) e(s)}{\lambda - \mu}.
\end{aligned} \tag{5.121}$$

Finally, upon using

$$(-a, b) \mathbb{D} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a + b, \tag{5.122}$$

$\Phi^{R\sigma}(k, s)$ ,  $\sigma \in \{-, L\}$ , may be recast as

$$\begin{aligned}
\Phi^{R\sigma}(k, s) &= -e^{-1}(k) \int_{\mathbb{R}-2iv} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda) \alpha_\downarrow^{(0)}(\lambda) e(\lambda)}{(k - \lambda) \lambda b'(0)} \times \int_{\mathbb{R}-iv} \frac{d\mu}{2i\pi \mu} \alpha_\downarrow^{(0)}(\mu) \widehat{B}_L^{+\varepsilon\sigma}(\mu, s) \\
&= - \left\{ e^{-1}(k) \int_{\mathbb{R}-i\frac{\eta}{2}} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda) \alpha_\downarrow^{(0)}(\lambda) e(\lambda)}{(k - \lambda) \lambda} - \frac{\mathcal{F}[L^{(0)}](k) \alpha_\downarrow^{(0)}(k)}{k} \right\} \int_{\mathbb{R}-iv} \frac{d\mu}{2i\pi} \frac{\alpha_\downarrow^{(0)}(\mu) \widehat{B}_L^{+\varepsilon\sigma}(\mu, s)}{\mu b'(0)}.
\end{aligned} \tag{5.123}$$

It is direct to check that  $\Phi^{R\sigma}(k, s)$ ,  $\sigma \in \{-, L\}$ , enjoy for some  $c > 0$  the bound

$$\Phi^{R\sigma}(k, s) = \mathcal{O} \left( \frac{e^{-\frac{v|k|}{2}}}{(1 + |k|)(1 + |s|)} + \frac{e^{-c|k|}}{w(1 + |s|)} \right) \quad \text{for } (k, s) \in \{\mathbb{R} - iv\} \times \{\mathbb{R} + i\varepsilon_\sigma v\}. \tag{5.124}$$

Furthermore, one also has

$$\begin{aligned}
\mathbb{P}^L \left[ (\text{id} - R_\infty) [e^{-1}(\cdot) \widehat{B}_L^{+\varepsilon\sigma}(\cdot, s) e(s)](\cdot) \right] (k) &= \int_{\mathbb{R}+iv} \frac{d\lambda}{2i\pi} \frac{e^{-1}(\lambda) \widehat{B}_L^{+\varepsilon\sigma}(\lambda, s) e(s)}{k - \lambda - i0^+} - \int_{\mathbb{R}+iv} \frac{d\lambda}{2i\pi} \frac{-\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda - i0^+} \\
&\times \int_{\mathbb{R}+iv} \frac{d\mu}{2i\pi} \left( -\{\alpha_\uparrow^{(0)}(\lambda) e(\lambda)\}^{-1}, \alpha_\downarrow^{(0)}(\lambda) e(\lambda) \right) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda \mu b'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu) \\ \{\alpha_\uparrow^{(0)}(\mu) e^2(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_L^{+\varepsilon\sigma}(\mu, s) e(s)}{\lambda - \mu} \\
&= U^{L^-}(k, s) + \Psi_{R_\infty}^{L^-}(k, s).
\end{aligned} \tag{5.125}$$

There,  $U^{L\sigma}(k, s)$ , for  $\sigma \in \{-, L\}$  and  $\varepsilon_\sigma$  as in (5.57), is the integral kernel of the operator  $U^{L\sigma} : L^2(\mathbb{R} + \varepsilon_\sigma iv) \rightarrow L^2(\mathbb{R} + iv)$ :

$$U^{L\sigma}(k, s) = e^{-1}(k) \widehat{B}_L^{+\varepsilon\sigma}(k, s) e(s) - \int_{\mathbb{R}+2iv} \frac{d\lambda}{2i\pi} \int_{\mathbb{R}-i\eta} \frac{d\mu}{2i\pi} \left( 1 - \frac{\lambda - \mu}{\lambda \mu b'(0)} \right) \frac{\mathcal{F}[L^{(0)}](\lambda) \alpha_\downarrow^{(0)}(\mu) \widehat{B}_L^{+\varepsilon\sigma}(\mu, s) e(s)}{(k - \lambda)(\lambda - \mu) \alpha_\uparrow^{(0)}(\lambda) e(\lambda)}. \tag{5.126}$$

Moreover, we denote

$$\begin{aligned}
\Psi_{R_\infty}^{L\sigma}(k, s) &= \int_{\mathbb{R}-2i\eta} \frac{d\lambda}{2i\pi} \frac{e^{-1}(\lambda) \widehat{B}_L^{+\varepsilon\sigma}(\lambda, s) e(s)}{k - \lambda} \\
&+ \int_{\mathbb{R}+2i\eta} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \int_{\mathbb{R}-2i\eta} \frac{d\mu}{2i\pi} (0, \alpha_\downarrow^{(0)}(\lambda) e(\lambda)) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda \mu b'(0)} \mathbb{D} \right) \begin{pmatrix} \alpha_\downarrow^{(0)}(\mu) \\ \{\alpha_\uparrow^{(0)}(\mu) e^2(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_L^{+\varepsilon\sigma}(\mu, s) e(s)}{\lambda - \mu} \\
&+ \int_{\mathbb{R}+2i\eta} \frac{d\lambda}{2i\pi} \frac{\mathcal{F}[L^{(0)}](\lambda)}{k - \lambda} \int_{\mathbb{R}-2i\eta} \frac{d\mu}{2i\pi} (-\{\alpha_\uparrow^{(0)}(\lambda) e(\lambda)\}^{-1}, 0) \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda \mu b'(0)} \mathbb{D} \right) \begin{pmatrix} 0 \\ \{\alpha_\uparrow^{(0)}(\mu) e^2(\mu)\}^{-1} \end{pmatrix} \frac{\widehat{B}_L^{+\varepsilon\sigma}(\mu, s) e(s)}{\lambda - \mu}.
\end{aligned} \tag{5.127}$$

These representations entail, upon invoking Lemma D.1 that, for  $\tau \in \{L, R\}$  and  $\sigma \in \{-, L\}$  with  $\varepsilon_\sigma$  as defined in (5.57), one has

$$\Psi_{R_\infty}^{\tau\sigma}(k, s) = \mathcal{O}\left(\frac{e^{-\eta w} \ln(1 + |k|)}{(1 + |k|)(1 + |s|)}\right) \quad \text{with} \quad (k, s) \in \{\mathbb{R} + i\varepsilon_\tau v\} \times \{\mathbb{R} + i\varepsilon_\sigma v\}. \tag{5.128}$$

Transforming the integral kernel  $U^{L\sigma}(k, s)$  further, one gets for  $k \in \mathbb{R} + iv$ ,

$$e(k) U^{L\sigma}(k, s) e^{-1}(s) = \widehat{B}_L^{+\varepsilon\sigma}(k, s) - \frac{\mathcal{F}[L^{(0)}](k)}{\alpha_\uparrow^{(0)}(k)} \int_{\mathbb{R}-i\eta} \frac{d\mu}{2i\pi} \frac{\alpha_\downarrow^{(0)}(\mu) \widehat{B}_L^{+\varepsilon\sigma}(\mu, s)}{k - \mu} + \Phi^{L\sigma}(k, s), \tag{5.129}$$

where

$$\begin{aligned}
\Phi^{L\sigma}(k, s) &= -e(k) \int_{\mathbb{R}+i\frac{v}{2}} \frac{d\lambda}{2i\pi} \int_{\mathbb{R}-i\eta} \frac{d\mu}{2i\pi} \left(1 - \frac{\lambda - \mu}{\lambda \mu b'(0)}\right) \frac{\mathcal{F}[L^{(0)}](\lambda) \alpha_\downarrow^{(0)}(\mu) \widehat{B}_L^{+\varepsilon\sigma}(\mu, s)}{(k - \lambda)(\lambda - \mu) \alpha_\uparrow^{(0)}(\lambda) e(\lambda)} \\
&+ \frac{\mathcal{F}[L^{(0)}](k)}{k \alpha_\uparrow^{(0)}(k) b'(0)} \int_{\mathbb{R}-i\eta} \frac{d\mu}{2i\pi} \alpha_\downarrow^{(0)}(\mu) \widehat{B}_L^{+\varepsilon\sigma}(\mu, s).
\end{aligned} \tag{5.130}$$

It is direct to check that  $\Phi^{L\sigma}(k, s)$ ,  $\sigma \in \{-, L\}$ , enjoy for some  $c > 0$  the bound

$$\Phi^{L\sigma}(k, s) = \mathcal{O}\left(\frac{e^{-\frac{vw}{2}}}{(1 + |k|)(1 + |s|)} + \frac{e^{-c|k|}}{w(1 + |s|)}\right) \quad \text{for} \quad (k, s) \in \{\mathbb{R} + iv\} \times \{\mathbb{R} + i\varepsilon_\sigma v\}. \tag{5.131}$$

We now introduce the integral operators

$$\Psi_{R_\infty}^{\tau\sigma}, \Phi^{\tau\sigma} : L^2(\mathbb{R} + i\varepsilon_\sigma v) \rightarrow L^2(\mathbb{R} + i\varepsilon_\tau v) \quad \text{with} \quad \sigma \in \{-, L\}, \tau \in \{L, R\} \tag{5.132}$$

and  $\varepsilon_\sigma$  as in (5.57) with the integral kernels given by  $\Psi_{R_\infty}^{\tau\sigma}(k, s)$  and  $\Phi^{\tau\sigma}(k, s)$ , respectively.

Finally, since  $\mathcal{F}[L^{(0)}](k) \{\alpha_\uparrow^{(0)}(k)\}^{-1} = \{\alpha_\uparrow^{(0)}(k)\}^{-1} - \{\alpha_\downarrow^{(0)}(k)\}^{-1}$ , one infers that

$$e(k) U^{L\sigma}(k, s) e^{-1}(s) = M_L^{+\varepsilon\sigma}(k, s) + V_L^{+\varepsilon\sigma}(k, s) + \Phi^{L\sigma}(k, s), \tag{5.133}$$



where

$$M_L^{+\pm}(k, s) = \frac{1}{\alpha_{\uparrow}^{(0)}(k)} \int_{\mathbb{R}-i\eta} \frac{d\mu}{2i\pi} \frac{\alpha_{\downarrow}^{(0)}(\mu) \widehat{B}_L^{+\pm}(\mu, s)}{\mu - k} \quad (5.134)$$

while

$$V_L^{+\pm}(k, s) = \widehat{B}_L^{+\pm}(k, s) - \int_{\mathbb{R}-i\eta} \frac{d\mu}{2i\pi} \frac{\alpha_{\downarrow}^{(0)}(\mu) \widehat{B}_L^{+\pm}(\mu, s)}{\alpha_{\downarrow}^{(0)}(k) \cdot (\mu - k)} \quad (5.135)$$

$$= \left\{ \frac{\widetilde{\alpha}_0^{(0)} \widehat{B}_L^{+\pm}(0, s)}{k \alpha_{\downarrow}^{(0)}(k)} - \int_{\mathbb{R}+i\eta} \frac{d\mu}{2i\pi} \frac{\alpha_{\downarrow}^{(0)}(\mu) \widehat{B}_L^{+\pm}(\mu, s)}{\alpha_{\downarrow}^{(0)}(k) (\mu - k)} \right\}. \quad (5.136)$$

$\widetilde{\alpha}_0^{(0)}$  appearing above is defined in (C.14).

We introduce the integral operators

$$\mathbf{M}_L^{+\pm} : L^2(\mathbb{R} \pm i\nu) \rightarrow L^2(\mathbb{R} + i\nu) \quad \text{and} \quad \mathbf{V}_L^{+\pm} : L^2(\mathbb{R} \pm i\nu) \rightarrow L^2(\mathbb{R} + i\nu) \quad (5.137)$$

having integral kernels  $M_L^{+\pm}(k, s)$  and  $V_L^{+\pm}(k, s)$ , respectively.

All in all, we have established that

$$\beta^{L-}(k, s) = e^{-1}(k) \left( M_L^{+-}(k, s) + V_L^{+-}(k, s) + \Phi^{L-}(k, s) \right) e(s) + \Psi_{R_{\infty}}^{L-}(k, s), \quad (5.138)$$

$$\beta^{R-}(k, s) = e(k) \Phi^{R-}(k, s) e^{-1}(s) + \Psi_{R_{\infty}}^{R-}(k, s). \quad (5.139)$$

### 5.6.5 Operator $\beta^{00}$

One starts by decomposing  $\beta^{00}[u^0] = (\beta^{00\mathbf{P}^R})[u^R] + (\beta^{00\mathbf{P}^L})[u^L]$ . Furthermore, one has

$$(\beta^{00\mathbf{P}^R})[u^R](k) = - \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}+i\nu} \frac{ds d\mu}{2i\pi} \frac{\beta^{00}(k, s) u^R(\mu)}{s - \mu + i\epsilon} = \int_{\mathbb{R}-i\nu} ds \beta^{0R}(k, s) u^R(s) \quad (5.140)$$

$$(\beta^{00\mathbf{P}^L})[u^L](k) = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}+i\nu} \frac{ds d\mu}{2i\pi} \frac{\beta^{00}(k, s) u^L(\mu)}{s - \mu - i\epsilon} = \int_{\mathbb{R}+i\nu} ds \beta^{0L}(k, s) u^L(s), \quad (5.141)$$

in which, for  $k \in \mathbb{R} + i\nu$  and  $s \in \mathbb{R} + i\epsilon_{\sigma}\nu$ ,  $\sigma \in \{L, R\}$  and  $\epsilon_{\sigma}$  as in (5.57),

$$\beta^{0R}(k, s) = - \int_{\mathbb{R}+2i\nu} \frac{d\mu}{2i\pi} \frac{\beta^{00}(k, \mu)}{\mu - s} \quad \text{and} \quad \beta^{0L}(k, s) = \int_{\mathbb{R}-2i\nu} \frac{d\mu}{2i\pi} \frac{\beta^{00}(k, \mu)}{\mu - s}. \quad (5.142)$$

This representation allows one to identify the dominant contribution to  $\beta^{0\sigma}(k, s)$  in that a contour deformation entails that

$$\beta^{0L}(k, s) = (\text{id} - \mathbf{R}_{\infty}) \left[ e^{-1}(\ast) \widehat{B}_L^{++}(\ast, s) e(s) \right] (k) + \Psi_1^{0L}(k, s) \quad (5.143)$$

$$\beta^{0R}(k, s) = (\text{id} - \mathbf{R}_{\infty}) \left[ e(\ast) \widehat{B}_R^{--}(\ast, s) e^{-1}(s) \right] (k) + \Psi_1^{0R}(k, s) \quad (5.144)$$

in which

$$\Psi_1^{0L}(k, s) = \int_{\mathbb{R}+2i\eta} \frac{d\mu}{2i\pi} \frac{(\text{id} - \mathbf{R}_\infty) \left[ e^{-1}(\ast) \widehat{B}_L^{++}(\ast, \mu) e(\mu) \right](k)}{\mu - s} + \int_{\mathbb{R}-2i\eta} \frac{d\mu}{2i\pi} \frac{(\text{id} - \mathbf{R}_\infty) \left[ e(\ast) \widehat{B}_R^{--}(\ast, \mu) e^{-1}(\mu) \right](k)}{\mu - s} \quad (5.145)$$

while

$$\Psi_1^{0R}(k, s) = - \int_{\mathbb{R}+2i\eta} \frac{d\mu}{2i\pi} \frac{(\text{id} - \mathbf{R}_\infty) \left[ e^{-1}(\ast) \widehat{B}_L^{++}(\ast, \mu) e(\mu) \right](k)}{\mu - s} - \int_{\mathbb{R}-2i\eta} \frac{d\mu}{2i\pi} \frac{(\text{id} - \mathbf{R}_\infty) \left[ e(\ast) \widehat{B}_R^{--}(\ast, \mu) e^{-1}(\mu) \right](k)}{\mu - s}. \quad (5.146)$$

Due to (C.60), for  $\mu \in \mathbb{R} \pm 2i\eta$  and  $|\Im(k)| < 2\nu$  with  $k$  uniformly away from 0, one has

$$(\text{id} - \mathbf{R}_\infty) \left[ e^{\mp 1}(\ast) \widehat{B}_{L/R}^{\pm\pm}(\ast, \mu) \right](k) = \mathcal{O} \left( \frac{e^{5\nu w}}{(1 + |k|)(1 + |\mu|)} \right). \quad (5.147)$$

By virtue of Lemma D.1 one infers that, uniformly in  $(k, s)$  such that  $|\Im(k)| < 2\nu$ ,  $|\Im(s)| < 2\nu$  and  $k$  uniformly away from 0

$$\Psi_1^{0\sigma}(k, s) = \mathcal{O} \left( \frac{e^{-\eta w} \ln(1 + |s|)}{(1 + |k|)(1 + |s|)} \right), \quad \sigma \in \{L, R\} \quad (5.148)$$

and thus, uniformly in  $(k, s)$  such that  $|\Im(k)| \leq \nu$  and  $|\Im(s)| \leq \nu$ ,

$$\Psi_1^{\tau\sigma}(k, s) = \mathbf{P}^\tau \left[ \Psi_1^{0\sigma}(\ast, s) \right](k) = \mathcal{O} \left( \frac{e^{-\eta w} \ln(1 + |s|) \ln(1 + |k|)}{(1 + |k|)(1 + |s|)} \right), \quad \tau, \sigma \in \{L, R\}. \quad (5.149)$$

As earlier on, we introduce the integral operators characterised by the above integral kernels

$$\Psi_1^{\tau\sigma}, \Phi^{\tau\sigma} : L^2(\mathbb{R} + i\varepsilon_\sigma v) \rightarrow L^2(\mathbb{R} + i\varepsilon_\tau v) \quad \text{with} \quad \sigma, \tau \in \{L, R\} \quad (5.150)$$

and  $\varepsilon_\sigma$  as in (5.57).

The  $R$  and  $L$  projections of the first terms appearing in (5.143)-(5.144) can be computed exactly as in Subsections 5.6.3-5.6.4. All in all, one gets that

$$\begin{aligned} & \begin{pmatrix} \beta^{LL}(k, s) & \beta^{LR}(k, s) \\ \beta^{RL}(k, s) & \beta^{RR}(k, s) \end{pmatrix} = \begin{pmatrix} \mathbf{P}^L[\beta^{0L}(\ast, s)](k) & \mathbf{P}^L[\beta^{0R}(\ast, s)](k) \\ \mathbf{P}^R[\beta^{0L}(\ast, s)](k) & \mathbf{P}^R[\beta^{0R}(\ast, s)](k) \end{pmatrix} \\ & = \begin{pmatrix} e^{-1}(k) \left[ M_L^{++} + V_L^{++} + \Phi^{LL} \right](k, s) e(s) + \Psi_1^{LL}(k, s) + \Psi_{R_\infty}^{LL}(k, s) & \\ e(k) \Phi^{RL}(k, s) e(s) + \Psi_1^{RL}(k, s) + \Psi_{R_\infty}^{RL}(k, s) & \\ & e^{-1}(k) \Phi^{LR}(k, s) e^{-1}(s) + \Psi_1^{LR}(k, s) + \Psi_{R_\infty}^{LR}(k, s) \\ e(k) \left[ M_R^{--} + V_R^{--} + \Phi^{RR} \right](k, s) e^{-1}(s) + \Psi_1^{RR}(k, s) + \Psi_{R_\infty}^{RR}(k, s) & \end{pmatrix}. \quad (5.151) \end{aligned}$$

### 5.6.6 Matrix equation arising in the 0 sector

In order to write down the final form of the equation associated with the  $R$  and  $L$  projections of the 0-sector, by putting together the previous results, one first obtains the relation

$$\left( \begin{array}{cc} -e^{-1}[M_L^{+-} + V_L^{+-} + \Phi^{L-}]e - \Psi_{\mathfrak{B}}^{L-} - \Psi_{\delta\beta}^{L-} - \Psi_{R_\infty}^{L-} & ; \quad \text{id} - e^{-1}[M_L^{++} + V_L^{++} + \Phi^{LL}]e - \Psi_{\mathfrak{B}}^{LL} - \Psi_{\delta\beta}^{LL} - \Psi_{R_\infty}^{LL} - \Psi_1^{LL} \\ -e\Phi^{R-}e - \Psi_{\mathfrak{B}}^{R-} - \Psi_{\delta\beta}^{R-} - \Psi_{R_\infty}^{R-} & ; \quad -e\Phi^{RL}e - \Psi_{\mathfrak{B}}^{RL} - \Psi_{\delta\beta}^{RL} - \Psi_{R_\infty}^{RL} - \Psi_1^{RL} \\ -e^{-1}\Phi^{LR}e^{-1} - \Psi_{\mathfrak{B}}^{LR} - \Psi_{\delta\beta}^{LR} - \Psi_{R_\infty}^{LR} - \Psi_1^{LR} & ; \quad -e^{-1}\Phi^{L+}e^{-1} - \Psi_{\mathfrak{B}}^{L+} - \Psi_{\delta\beta}^{L+} - \Psi_{R_\infty}^{L+} \\ \text{id} - e[M_R^{-} + V_R^{-} + \Phi^{RR}]e^{-1} - \Psi_{\mathfrak{B}}^{RR} - \Psi_{\delta\beta}^{RR} - \Psi_{R_\infty}^{RR} - \Psi_1^{RR} & ; \quad -e[M_R^{+} + V_R^{+} + \Phi^{R+}]e^{-1} - \Psi_{\mathfrak{B}}^{R+} - \Psi_{\delta\beta}^{R+} - \Psi_{R_\infty}^{R+} \end{array} \right) [\mathbf{u}] \\ = \left( \begin{array}{c} \mathfrak{d}_w^L[h^0] \\ \mathfrak{d}_w^R[h^0] \end{array} \right) \quad \text{where} \quad \mathfrak{d}_w^\alpha[h^0] = P^\alpha[(\text{id} - R)[h^0]]. \quad (5.152)$$

We remind that  $\mathbf{u}$  appearing above was introduced in (5.62). Then, one defines the operators

$$\left( \begin{array}{cccc} \Omega^{L-} & \Omega^{LL} & \Omega^{LR} & \Omega^{L+} \\ \Omega^{R-} & \Omega^{RL} & \Omega^{RR} & \Omega^{R+} \end{array} \right) \\ = \left( \begin{array}{cc} \Phi^{L-} + e[\Psi_{\mathfrak{B}}^{L-} + \Psi_{\delta\beta}^{L-} + \Psi_{R_\infty}^{L-}]e^{-1} & ; \quad \Phi^{LL} + e[\Psi_{\mathfrak{B}}^{LL} + \Psi_{\delta\beta}^{LL} + \Psi_{R_\infty}^{LL} + \Psi_1^{LL}]e^{-1} \\ \Phi^{R-} + e^{-1}[\Psi_{\mathfrak{B}}^{R-} + \Psi_{\delta\beta}^{R-} + \Psi_{R_\infty}^{R-}]e^{-1} & ; \quad \Phi^{RL} + e^{-1}[\Psi_{\mathfrak{B}}^{RL} + \Psi_{\delta\beta}^{RL} + \Psi_{R_\infty}^{RL} + \Psi_1^{RL}]e^{-1} \\ \Phi^{LR} + e[\Psi_{\mathfrak{B}}^{LR} + \Psi_{\delta\beta}^{LR} + \Psi_{R_\infty}^{LR} + \Psi_1^{LR}]e & ; \quad \Phi^{L+} + e[\Psi_{\mathfrak{B}}^{L+} + \Psi_{\delta\beta}^{L+} + \Psi_{R_\infty}^{L+}]e \\ \Phi^{RR} + e^{-1}[\Psi_{\mathfrak{B}}^{RR} + \Psi_{\delta\beta}^{RR} + \Psi_{R_\infty}^{RR} + \Psi_1^{RR}]e & ; \quad \Phi^{R+} + e^{-1}[\Psi_{\mathfrak{B}}^{R+} + \Psi_{\delta\beta}^{R+} + \Psi_{R_\infty}^{R+}]e \end{array} \right). \quad (5.153)$$

This allows one to rewrite (5.152) in the form

$$\left( \begin{array}{cc} -e^{-1}[M_L^{+-} + V_L^{+-} + \Omega^{L-}]e & ; \quad \text{id} - e^{-1}[M_L^{++} + V_L^{++} + \Omega^{LL}]e \\ e\Omega^{R-}e & ; \quad e\Omega^{RL}e \\ e^{-1}\Omega^{LR}e^{-1} & ; \quad e^{-1}\Omega^{L+}e^{-1} \\ \text{id} - e[M_R^{-} + V_R^{-} + \Omega^{RR}]e^{-1} & ; \quad -e[M_R^{+} + V_R^{+} + \Omega^{R+}]e^{-1} \end{array} \right) [\mathbf{u}] = \left( \begin{array}{c} \mathfrak{d}_w^L[h^0] \\ \mathfrak{d}_w^R[h^0] \end{array} \right), \quad (5.154)$$

that is best suited for the later handling.

### 5.7 Final form of the integral equation

By gathering the results of Subsections 5.6.3, 5.6.4, 5.6.5 and 5.6.6, in particular the equations (5.76), (5.64) and (5.154), one may recast the original system (5.35)-(5.37) into the following form

$$\mathbf{E}(\text{id} - \mathbf{0} - \mathbf{\Omega})\mathbf{E}^{-1}[\mathbf{u}] = \mathfrak{d}_w[h], \quad (5.155)$$

in which  $\mathbf{E} = \text{Diag}(e^{-1}, e^{-1}, e, e)$ ,  $\mathfrak{d}_w[h] = (\mathfrak{d}_w^-[h^-](k), \mathfrak{d}_w^L[h^0](k), \mathfrak{d}_w^R[h^0](k), \mathfrak{d}_w^+[h^+](k))^t$ , while

$$\mathbf{\Omega} = \left( \begin{array}{cccc} 0 & \Omega^{-L} & \Omega^{-R} & \Omega^{-+} \\ \Omega^{L-} & \Omega^{LL} & \Omega^{LR} & \Omega^{L+} \\ \Omega^{R-} & \Omega^{RL} & \Omega^{RR} & \Omega^{R+} \\ \Omega^{+-} & \Omega^{+L} & \Omega^{+R} & 0 \end{array} \right). \quad (5.156)$$

Finally, one has

$$\mathbf{0} = \left( \begin{array}{cc} 0_L & \mathbf{0} \\ \mathbf{0} & 0_R \end{array} \right) \quad (5.157)$$

in which  $O_{L/R}$  are integral operators on  $L^2(\mathbb{R} - iv) \oplus L^2(\mathbb{R} + iv)$  having the block-matrix form

$$O_L = \begin{pmatrix} M_L^{--} & M_L^{-+} \\ M_L^{+-} + V_L^{+-} & M_L^{++} + V_L^{++} \end{pmatrix} \quad \text{and} \quad O_R = \begin{pmatrix} M_R^{--} + V_R^{--} & M_R^{-+} + V_R^{-+} \\ M_R^{+-} & M_R^{++} \end{pmatrix}. \quad (5.158)$$

**Proposition 5.3.** *The operators  $O$  and  $\Omega$  appearing in (5.155) are compact Hilbert-Schmidt operators on*

$$L^2(\mathbb{R} - iv) \oplus L^2(\mathbb{R} + iv) \oplus L^2(\mathbb{R} - iv) \oplus L^2(\mathbb{R} + iv). \quad (5.159)$$

Moreover, the operator  $\text{id} - O - \Omega$  is invertible uniformly in  $w$  large enough. Its inverse is equal to  $\text{id} + \Delta$ , where the operator  $\Delta$  has the block-matrix form

$$\Delta = \begin{pmatrix} \Delta^{--} & \Delta^{-L} & \Delta^{-R} & \Delta^{-+} \\ \Delta^{L-} & \Delta^{LL} & \Delta^{LR} & \Delta^{L+} \\ \Delta^{R-} & \Delta^{RL} & \Delta^{RR} & \Delta^{R+} \\ \Delta^{+-} & \Delta^{+L} & \Delta^{+R} & \Delta^{++} \end{pmatrix}. \quad (5.160)$$

The integral kernels associated with this block decomposition enjoy the uniform in  $w$  bounds:

$$\Delta^{\sigma\tau}(k, s) = O\left(\frac{\ln(1 + |k|) \cdot \ln(1 + |s|)}{(1 + |k|) \cdot (1 + |s|)}\right) \quad \text{for} \quad (k, s) \in \{\mathbb{R} + i\varepsilon_\sigma v\} \times \{\mathbb{R} + i\varepsilon_\tau v\}, \quad (5.161)$$

with  $\{\sigma, \tau\} \in \{\pm, R, L\}$  and  $\varepsilon_\sigma$  is as introduced in (5.57).

*Proof* —

Using Proposition 2.3 and Lemma D.1 one may bound the integral kernels  $M_{L/R}^{\sigma\epsilon}$  and  $V_{L/R}^{\sigma\epsilon}$  of the operators  $M_{L/R}$  and  $V_{L/R}$  building up the operator  $O$  as

$$M_{L/R}^{\sigma\epsilon}(k, s) = O\left(\frac{\ln(1 + |k|)}{(1 + |k|) \cdot (1 + |s|)}\right) \quad \text{for} \quad (k, s) \in \{\mathbb{R} + i\sigma v\} \times \{\mathbb{R} + i\epsilon v\} \quad \text{with} \quad \sigma, \epsilon \in \{\pm\}, \quad (5.162)$$

and

$$V_L^{+\epsilon}(k, s) = O\left(\frac{\ln(1 + |k|)}{(1 + |k|) \cdot (1 + |s|)}\right) \quad \text{for} \quad (k, s) \in \{\mathbb{R} + iv\} \times \{\mathbb{R} + i\epsilon v\} \quad \text{with} \quad \epsilon \in \{\pm\}, \quad (5.163)$$

$$V_R^{-\epsilon}(k, s) = O\left(\frac{\ln(1 + |k|)}{(1 + |k|) \cdot (1 + |s|)}\right) \quad \text{for} \quad (k, s) \in \{\mathbb{R} - iv\} \times \{\mathbb{R} + i\epsilon v\} \quad \text{with} \quad \epsilon \in \{\pm\}. \quad (5.164)$$

Also, upon recalling the definition of various block operators building up  $\Omega$ , (5.63), (5.75) and (5.153), one may infer from the bounds (5.54), (5.56), (5.69), (5.71), (5.91), (5.97), (5.104), (5.108), (5.115), (5.124), (5.128), (5.131), (5.149) the relations

$$\Omega^{\sigma\tau}(k, s) = O\left(\frac{\ln(1 + |k|) \ln(1 + |s|)}{w \cdot (1 + |k|) \cdot (1 + |s|)}\right) \quad \text{for} \quad (k, s) \in \{\mathbb{R} + i\varepsilon_\sigma v\} \times \{\mathbb{R} + i\varepsilon_\tau v\} \quad \text{with} \quad \sigma, \tau \in \{\pm, R, L\}, \quad (5.165)$$

where  $\varepsilon_\sigma$  is as introduced in (5.57).

Put together, these pieces of information ensure that the operators  $O$  and  $\Omega$  appearing in (5.155) are compact, Hilbert-Schmidt operators having trace and leading to well-defined Fredholm determinants.

Since the Fredholm 2-determinants are continuous in the Hilbert-Schmidt norm [11] in the sense that for any  $C' > 0$  there exists  $C > 0$  such that

$$\left| \det_2[\text{id} - A] - \det_2[\text{id} - B] \right| \leq C \cdot \|A - B\|_{\text{HS}} \quad \text{for any} \quad \|A\|_{\text{HS}} + \|B\|_{\text{HS}} < C', \quad (5.166)$$

and since  $\det_2[\text{id} - 0] = \det[\text{id} - 0]e^{\text{tr}[0]}$ , it is enough to show that  $\det[\text{id} - 0]$  is away from 0 so as to have the uniform in  $w$  invertibility of  $\text{id} - 0 - \Omega$ . For this purpose, we compute the Fredholm determinant of  $\text{id} - 0$  in terms of other determinants which we know to be non-vanishing.

Clearly, one has

$$\det[\text{id} - 0] = \det[\text{id} - 0_L] \cdot \det[\text{id} - 0_R]. \quad (5.167)$$

In order to estimate those determinants, one first observes the block operator factorisation

$$\text{id} - 0_L = \begin{pmatrix} \text{id} - \mathbb{M}_L^- & -\mathbb{M}_L^+ \\ -\mathbb{M}_L^{+-} - \mathbb{V}_L^{+-} & \text{id} - \mathbb{M}_L^{++} - \mathbb{V}_L^{++} \end{pmatrix} \quad (5.168)$$

$$= \begin{pmatrix} \text{id} - \mathbb{M}_L^- & -\mathbb{M}_L^+ \\ -\mathbb{M}_L^{+-} & \text{id} - \mathbb{M}_L^{++} \end{pmatrix} \cdot \begin{pmatrix} \text{id} & 0 \\ -\mathbb{V}_L^{+-} & \text{id} - \mathbb{V}_L^{++} \end{pmatrix}. \quad (5.169)$$

The latter is a consequence of the identities

$$\mathbb{M}_L^{\sigma+} \cdot \mathbb{V}_L^{\tau+} = 0 \quad \text{for} \quad \sigma, \tau \in \{\pm\}. \quad (5.170)$$

established in Lemma 5.4 below.

Now, it follows from Proposition 2.4 that  $\det[\text{id} - \mathbb{M}_L] \neq 0$  where

$$\mathbb{M}_{L/R} = \begin{pmatrix} \mathbb{M}_{L/R}^- & \mathbb{M}_{L/R}^+ \\ \mathbb{M}_{L/R}^{+-} & \mathbb{M}_{L/R}^{++} \end{pmatrix}. \quad (5.171)$$

Moreover, owing to the identity  $\mathbb{V}_L^{++} \cdot \mathbb{V}_L^{++} = 0$  and the Plemelj-Smithies expansion for the determinant [11]:

$$\det[\text{id} + \mathbb{A}] = \sum_{n \geq 0} \frac{1}{n!} \det_n \begin{pmatrix} \text{tr}[\mathbb{A}] & n-1 & 0 & \dots & 0 \\ \text{tr}[\mathbb{A}^2] & \text{tr}[\mathbb{A}] & n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{tr}[\mathbb{A}^{n-1}] & \text{tr}[\mathbb{A}^{n-2}] & \text{tr}[\mathbb{A}^{n-3}] & \dots & 1 \\ \text{tr}[\mathbb{A}^n] & \text{tr}[\mathbb{A}^{n-1}] & \text{tr}[\mathbb{A}^{n-2}] & \dots & \text{tr}[\mathbb{A}] \end{pmatrix}, \quad (5.172)$$

one has

$$\det \begin{pmatrix} \text{id} & 0 \\ -\mathbb{V}_L^{+-} & \text{id} - \mathbb{V}_L^{++} \end{pmatrix} = 1 - \text{tr}[\mathbb{V}_L^{++}]. \quad (5.173)$$

The latter trace can be shown to vanish by deforming the integration contours to  $-i\infty$  in the expression below

$$\text{tr}[\mathbb{V}_L^{++}] = \int_{\mathbb{R}+iv} dk \left\{ \frac{\tilde{\alpha}_0^{(0)} \widehat{B}_L^{++}(0, k)}{k \alpha_\downarrow^{(0)}(k)} - \int_{\mathbb{R}+i\eta} d\mu \frac{\alpha_\downarrow^{(0)}(\mu) \widehat{B}_L^{++}(\mu, k)}{\alpha_\downarrow^{(0)}(k)(\mu - k)} \right\}, \quad (5.174)$$

since the integrand is analytic in  $\mathbb{H}_{2v}^- = \mathbb{H}^- + 2iv$  and goes to zero as  $o(k^{-3/2})$  at  $\infty$  in that domain.

Thus, all in all, one infers that  $\det[\text{id} - 0_L] = \det[\text{id} - \mathbb{M}_L] \neq 0$ .

Very analogous considerations lead to

$$\text{id} - 0_R = \begin{pmatrix} \text{id} - \mathbb{M}_R^- - \mathbb{V}_R^- & -\mathbb{M}_R^+ - \mathbb{V}_R^+ \\ -\mathbb{M}_R^{+-} & \text{id} - \mathbb{M}_R^{++} \end{pmatrix} \quad (5.175)$$

$$= \begin{pmatrix} \text{id} - \mathbb{M}_R^- & -\mathbb{M}_R^+ \\ -\mathbb{M}_R^{+-} & \text{id} - \mathbb{M}_R^{++} \end{pmatrix} \cdot \begin{pmatrix} \text{id} - \mathbb{V}_R^- & -\mathbb{V}_R^+ \\ 0 & \text{id} \end{pmatrix} \quad (5.176)$$

as follows from the identities

$$\mathbb{M}_R^{\sigma^-} \cdot \mathbb{V}_R^{-\tau} = 0 \quad \text{for} \quad \sigma, \tau \in \{\pm\}. \quad (5.177)$$

established in Lemma 5.4 below. Now, Proposition 2.4 implies that  $\det[\text{id} - \mathbb{M}_R] \neq 0$ . The identity  $\mathbb{V}_R^{-} \cdot \mathbb{V}_R^{-} = 0$  and the Plemelj-Smithies formula lead to

$$\det \begin{pmatrix} \text{id} - \mathbb{V}_R^{-} & -\mathbb{V}_R^{+} \\ 0 & \text{id} \end{pmatrix} = 1 - \text{tr}[\mathbb{V}_R^{-}]. \quad (5.178)$$

The latter trace can be shown to vanish by deforming the contour to  $+i\infty$  in the integral below

$$\text{tr}[\mathbb{V}_R^{-}] = \int_{\mathbb{R}-i\nu} dk \alpha_{\uparrow}^{(0)}(k) \left\{ \int_{\mathbb{R}-i\eta} d\mu \frac{\widehat{B}_R^{-}(\mu, k)}{2i\pi(\mu - k) \cdot \alpha_{\uparrow}^{(0)}(\mu)} + \frac{\widehat{B}_R^{-}(0, k)}{k \cdot \alpha_0^{(0)}} \right\}, \quad (5.179)$$

where the integrand is analytic in  $\mathbb{H}^+ - 2i\nu$  and goes to zero as  $o(k^{-3/2})$  at  $\infty$  in that domain.

Thus, all in all,  $\det[\text{id} - \mathbb{O}_R] = \det[\text{id} - \mathbb{M}_R] \neq 0$ .

It remains to establish the estimates (5.161) on the entries of the resolvent operator  $\Delta$ . Set  $\mathbb{Q} = \mathbb{O} + \Omega$  for short. The blocks of the resolvent kernel may be expressed as [11]

$$\Delta^{\tau\sigma}(\lambda, \mu) = \frac{\Delta_n^{\tau\sigma}(\lambda, \mu)}{\det[\text{id} - \mathbb{Q}]} \quad (5.180)$$

in which the numerator is expressed in terms of the Fredholm series

$$\Delta_n^{\tau\sigma}(\lambda, \mu) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{\substack{S_1, \dots, S_n \\ \in \{\pm, R, L\}}} \prod_{a=1}^n \int_{\mathbb{R} + i\varepsilon_{S_a} \nu} d\lambda_a \det_{n+1} \begin{bmatrix} Q^{\tau\sigma}(\lambda, \mu) & Q^{\tau S_b}(\lambda, \lambda_b) \\ Q^{S_a \sigma}(\lambda_a, \mu) & Q^{S_a S_b}(\lambda_a, \lambda_b) \end{bmatrix}. \quad (5.181)$$

Then, introduce the auxiliary kernel

$$\widetilde{Q}^{\tau\sigma}(\lambda, \mu) = Q^{\tau\sigma}(\lambda, \mu) \cdot \frac{(1 + |\lambda|) \cdot (1 + |\mu|)}{\ln(1 + |\lambda|) \ln(1 + |\mu|)} \quad (5.182)$$

which, by virtue of (5.162), (5.163), (5.164) and (5.165) is bounded on  $\{\mathbb{R} + i\varepsilon_{\tau} \nu\} \times \{\mathbb{R} + i\varepsilon_{\sigma} \nu\}$ , uniformly in  $w$ . This yields the representation

$$\begin{aligned} \Delta_n^{\tau\sigma}(\lambda, \mu) &= \frac{\ln(1 + |\lambda|) \ln(1 + |\mu|)}{(1 + |\lambda|) \cdot (1 + |\mu|)} \\ &\times \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{\substack{S_1, \dots, S_n \\ \in \{\pm, R, L\}}} \prod_{a=1}^n \int_{\mathbb{R} + i\varepsilon_{S_a} \nu} d\lambda_a \prod_{a=1}^n \left\{ \frac{\ln(1 + |\lambda_a|)}{1 + |\lambda_a|} \right\}^2 \cdot \det_{n+1} \begin{bmatrix} \widetilde{Q}^{\tau\sigma}(\lambda, \mu) & \widetilde{Q}^{\tau S_b}(\lambda, \lambda_b) \\ \widetilde{Q}^{S_a \sigma}(\lambda_a, \mu) & \widetilde{Q}^{S_a S_b}(\lambda_a, \lambda_b) \end{bmatrix}. \end{aligned} \quad (5.183)$$

A direct application of Hadamard's inequality for determinants allows one to infer that the above series converges on  $\{\mathbb{R} + i\varepsilon_{\tau} \nu\} \times \{\mathbb{R} + i\varepsilon_{\sigma} \nu\}$ , which also yields (5.161).  $\blacksquare$

**Lemma 5.4.** *Let  $M_R^{\sigma\tau}$ , resp.  $M_L^{\sigma\tau}$ , be the integral operators  $L^2(\mathbb{R} + i\tau\nu) \rightarrow L^2(\mathbb{R} + i\sigma\nu)$  defined by the integral kernels (5.72), (5.112), resp. (5.58), (5.134). Furthermore, let  $V_L^{+\tau}$ , resp.  $V_R^{-\tau}$ , be the integral operators  $L^2(\mathbb{R} + i\nu) \rightarrow L^2(\mathbb{R} + i\tau\nu)$ , resp.  $L^2(\mathbb{R} - i\nu) \rightarrow L^2(\mathbb{R} + i\tau\nu)$ , defined by the integral kernels (5.136), resp. (5.113). Then*

$$M_L^{\sigma+} \cdot V_L^{+\tau} = 0 \quad \text{for any} \quad \sigma, \tau \in \{\pm\} \quad (5.184)$$

$$M_R^{\sigma-} \cdot V_R^{-\tau} = 0 \quad \text{for any} \quad \sigma, \tau \in \{\pm\}. \quad (5.185)$$

*Proof*—

It follows from equation (5.136) that  $k \mapsto \mathbf{V}_L^{+\tau}(k, s)$  is analytic on  $\mathbb{H}_{2\nu}^- = \mathbb{H}^- + 2i\nu$  and that it falls-off at infinity in this domains like  $O(\ln(1 + |k|)/|k|)$ . Furthermore,  $s \mapsto \mathbf{M}_L^{\sigma+}(k, s)$  is also analytic on  $\mathbb{H}_{2\nu}^-$  and falls-off at infinity in this domains like  $O(1/|s|)$ . Hence, a direct contour deformation up to  $\mathbb{R} - i\infty$  entails that

$$\int_{\mathbb{R}+i\nu} ds M_L^{\sigma+}(k, s) \mathbf{V}_L^{+\tau}(s, t) = 0. \quad (5.186)$$

Similarly, it follows from equation (5.113) that  $k \mapsto \mathbf{V}_R^{-\tau}(k, s)$  is analytic on  $\mathbb{H}_{-2\nu}^+ = \mathbb{H}^+ - 2i\nu$  and that it falls off at infinity in this domains like  $O(\ln(1 + |k|)/|k|)$ . Furthermore,  $s \mapsto \mathbf{M}_R^{\sigma-}(k, s)$  is also analytic on  $\mathbb{H}_{-2\nu}^+$  and falls-off at infinity in this domains like  $O(1/|s|)$ . Hence, a direct contour deformation up to  $\mathbb{R} + i\infty$  entails that

$$\int_{\mathbb{R}-i\nu} ds M_R^{\sigma-}(k, s) \mathbf{V}_R^{-\tau}(s, t) = 0. \quad (5.187)$$

■

**Theorem 5.5.** *The solution  $u$  to the integral equation (5.155) satisfies to the bounds*

$$u^\sigma(k) = O\left(\frac{C_h e^{2\nu w} \ln(1 + |k|)}{1 + |k|}\right) \quad \text{with} \quad k \in \mathbb{R} + i\varepsilon_\sigma \nu, \quad (5.188)$$

provided that

$$\mathfrak{d}_w^\alpha[h](k) \leq \frac{C_h \ln(1 + |k|)}{1 + |k|}. \quad (5.189)$$

*Proof*—

This is an obvious consequence of Proposition 5.3. ■

## 6 Conclusion

In this paper we have carried out the  $w \rightarrow +\infty$  asymptotic analysis of the solution to a non-local Riemann–Hilbert problem characterising the conformal map from a welded cylinder onto the standard one in the case where the welding diffeomorphism is composed of two non-trivial bumps separated from one another by distance  $w$ . This problem was motivated by the study of the large-time behaviour of the generating function of full counting statistics of energy transfers in 1+1 dimensional non-equilibrium conformal field theories discussed in [10]. Our results allowed us to establish the large-time asymptotics of the generating function rigorously.

On technical ground, we have developed methods that allow to establish the existence and the uniqueness of solutions to non-local Riemann–Hilbert problems in the case where the "compact operator" arguments developed in the literature cannot be applied directly. Our analysis shows that it is still possible to study the asymptotic behaviour of solutions of such problems even if this is much more involved than in the local case.

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## A Conformal map of the welded cylinder

In this section we prove Proposition 1.1. Introduce the functions

$$\omega^{(L)}(z) = \frac{\widetilde{\gamma}_+ \cdot z}{1 + e^{-\frac{2\pi}{\tau}z}} \quad \text{with} \quad \omega^{(R)}(z) = \frac{\widetilde{\gamma}_- \cdot z}{1 + e^{\frac{2\pi}{\tau}z}}, \quad (\text{A.1})$$

with  $\widetilde{\gamma}_\pm$  as given in the statement of Proposition 1.1 and where  $\tau > 2\alpha$ . These admit the decomposition

$$\omega^{(L)}(z) = \widetilde{\gamma}_+ \cdot z + \omega_R^{(L)}(z) \quad \text{and} \quad \omega^{(R)}(z) = \widetilde{\gamma}_- \cdot z + \omega_L^{(R)}(z). \quad (\text{A.2})$$

Let us set  $\Omega(z | \kappa^+, \kappa^-) = \Upsilon(z) + \omega^{(L)}(z) + \omega^{(R)}(z)$ . Then,  $\Upsilon$  solves the non-local Riemann-Hilbert problem: find  $\Upsilon \in \mathcal{O}(\mathcal{S}_\alpha)$  such that

- $\Upsilon$  has smooth  $-$ , resp.  $+$ , boundary values on  $\mathbb{R}$ , resp.  $\mathbb{R} - i\alpha$ ;
- $\Upsilon_+(g(x) - i\alpha) = \Upsilon_-(x) + G_\Upsilon(x)$ , with  $x \in \mathbb{R}$ ;
- there exists a constants  $C_\Upsilon$  and  $\eta > 0$  such that

$$\Upsilon(z) = \begin{cases} -\omega_R^{(L)}(z) - \omega^{(R)}(z) + \mathcal{O}(e^{-\eta \Re(z)}) & \text{when } \Re(z) \rightarrow +\infty \\ -\omega^{(L)}(z) - \omega_L^{(R)}(z) + C_\Upsilon \delta_{\pm; -} + \mathcal{O}(e^{\eta \Re(z)}) & \text{when } \Re(z) \rightarrow -\infty \end{cases}, \quad (\text{A.3})$$

with an asymptotic expansion that is valid uniformly up to the boundary,

and where

$$G_\Upsilon(x) = \begin{cases} \omega^{(L)}(x) - \omega^{(L)}(g(x) - i\alpha) + \omega_L^{(R)}(x) - \omega_L^{(R)}(g(x) - i\alpha) & \text{for } x \leq -M \\ -i\alpha + \omega^{(L)}(x) - \omega^{(L)}(g(x) - i\alpha) + \omega^{(R)}(x) - \omega^{(R)}(g(x) - i\alpha) & \text{for } |x| \leq M \\ \omega_R^{(L)}(x) - \omega_R^{(L)}(g(x) - i\alpha) + \omega^{(R)}(x) - \omega^{(R)}(g(x) - i\alpha) & \text{for } x \geq M \end{cases} \quad (\text{A.4})$$

By virtue of Proposition 2.4, this non-local Riemann-Hilbert problem admits a unique solution. Hence, so does the one of  $\Omega$ .

Since  $\Omega \in \mathcal{O}(\mathcal{S}_\alpha)$ ,  $\Omega$  is open and thus  $\partial\Omega(\mathcal{S}_\alpha | \kappa^+, \kappa^-) = \Omega_+(\mathbb{R} - i\alpha | \kappa^+, \kappa^-) \cup \Omega_-(\mathbb{R} | \kappa^+, \kappa^-)$ . Thus, clearly,

$$\Omega(\mathcal{S}_\alpha | \kappa^+, \kappa^-) \ni \omega \mapsto \#\{\Omega^{-1}(\omega | \kappa^+, \kappa^-)\} = \int_{\{\mathbb{R}-i\alpha\} \cup \mathbb{R}} \frac{ds}{2i\pi} \frac{\Omega'(s | \kappa^+, \kappa^-)}{\Omega(s | \kappa^+, \kappa^-) - \omega} \quad (\text{A.5})$$

is continuous in  $\omega$  on  $\Omega(\mathcal{S}_\alpha | \kappa^+, \kappa^-)$ . Being integer valued, it is constant. The asymptotic behaviour of  $\Omega$  at infinity entails that  $\#\{\Omega^{-1}(\omega | \kappa^+, \kappa^-)\} = 1$  for  $\Re(\omega)$  large enough and such that  $\omega \in \Omega(\mathcal{S}_\alpha | \kappa^+, \kappa^-)$ . Hence,  $\Omega$  is injective and thus a biholomorphism on its image.  $\blacksquare$

## B Inversion of the operators $\text{id} - L^{uv}$ on $L^2(\mathbb{R}^v)$

We now discuss the invertibility of  $\text{id} - L^{uv}$  with the help of the Wiener-Hopf technique, see *e.g.* [8], as will be detailed in the two next subsections. The method builds on the solution of a multiplicative Riemann-Hilbert problem involving the Fourier transform of the kernel  $L^v(x)$ .



## B.1 Inversion of the operators $\text{id} - \mathbf{L}^{++}$

For the purpose of the present section, we introduce the space

$$L_C^2(\mathbb{R}^+) = \left\{ f \in L_{\text{loc}}^2(\mathbb{R}^+) : \exists C_f \text{ and } \alpha > 0 \quad f(x) = C_f + O(e^{-\alpha x}) \right\}. \quad (\text{B.1})$$

**Proposition B.1.** *Let  $L^+$  be as defined through (2.42) and consider the integral equation*

$$f(x) - \int_0^{+\infty} L^+(x-y)f(y)dy = h(x), \quad x \in \mathbb{R}^+ \quad (\text{B.2})$$

on  $L_C^2(\mathbb{R}^+)$  with  $h$  such that there exist  $\eta > 0$  so that

$$h(x) = O(e^{-\eta x}), \quad \text{when } x \rightarrow +\infty. \quad (\text{B.3})$$

Then equation (B.2) is uniquely solvable on  $L_C^2(\mathbb{R}^+)$  and the Fourier transform of the solution takes the form

$$\mathcal{F}[f](k) = \frac{1}{\alpha_{\uparrow}^{(+)}(k)} \int_{\mathbb{R}-i\eta^-} \frac{ds}{2i\pi} \frac{\alpha_{\downarrow}^{(+)}(s) \cdot \mathcal{F}[h\mathbf{1}_{\mathbb{R}^+}](s)}{s-k} \quad \text{with } k \in \mathbb{R} + iv \quad (\text{B.4})$$

for any  $0 < \eta^- < \eta$  and with  $v > 0$ .

*Proof—*

Following the strategy of the Wiener-Hopf method, one starts by extending  $f$  and  $h$  to  $\mathbb{R}$  in such a way that equation (B.2) now holds on  $\mathbb{R}$ . We shall make the choice

$$h(x) = 0 \quad \text{and} \quad f(x) = \int_0^{+\infty} L^+(x-y)f(y)dy \quad \text{for } x < 0. \quad (\text{B.5})$$

Given the behaviour of  $f$  on  $\mathbb{R}^+$  and the explicit expression (2.42) for  $L^+(x)$ , it is easy to convince oneself that, upon reducing  $\eta$  if need be, these extensions satisfy

$$f(x) = O(e^{\eta x}) \quad \text{and} \quad h(x) = O(e^{\eta x}), \quad (\text{B.6})$$

when  $x \rightarrow -\infty$ . Actually, for the purpose of the analysis to come, it is convenient to introduce a specific notation for the restrictions of a function on  $\mathbb{R}$  to  $\mathbb{R}^{\pm}$ :  $f^{\pm} = f\mathbf{1}_{\mathbb{R}^{\pm}}$ . In particular, by construction, we have that  $h = h^+$ . The properties of the extended functions allow one to compute a well-defined Fourier transform provided that  $k \in \mathbb{R} + iv$ ,  $0 < v < \eta$ . Thus Fourier transforming (B.2) leads to

$$(1 - \mathcal{F}[L^+](k)) \cdot \mathcal{F}[f^+](k) + \mathcal{F}[f^-](k) = \mathcal{F}[h^+](k) \quad \text{with } k \in \mathbb{R} + iv. \quad (\text{B.7})$$

Then, by using the Wiener-Hopf factorisation of  $1 - \mathcal{F}[L^{++}]$  given in (2.53), one may recast the equation as

$$\alpha_{\uparrow}^{(+)}(k) \cdot \mathcal{F}[f^+](k) + \alpha_{\downarrow}^{(+)}(k) \cdot \mathcal{F}[f^-](k) = \alpha_{\downarrow}^{(+)}(k) \cdot \mathcal{F}[h^+](k). \quad (\text{B.8})$$

Given half-planes  $\mathcal{B}_{\uparrow/\downarrow}^{(+)}$  as introduced in (2.52), one may define  $U \in \mathcal{O}(\mathcal{B}_{\uparrow}^{(+)} \cup \mathcal{B}_{\downarrow}^{(+)} \setminus \{0\})$  and having a simple pole at 0 by the piecewise formula

$$U(z) = \begin{cases} \alpha_{\uparrow}^{(+)}(z) \cdot \mathcal{F}[f^+](z) - \mathcal{C}^{(+)}[\alpha_{\downarrow}^{(+)} \cdot \mathcal{F}[h^+]](z) & , \quad z \in \mathcal{B}_{\uparrow}^{(+)} \\ -\alpha_{\downarrow}^{(+)}(z) \cdot \mathcal{F}[f^-](z) - \mathcal{C}^{(+)}[\alpha_{\downarrow}^{(+)} \cdot \mathcal{F}[h^+]](z) & , \quad z \in \mathcal{B}_{\downarrow}^{(+)} \end{cases} \quad (\text{B.9})$$

where  $\mathcal{C}^{(+)}$  is the Cauchy transform on  $L^2(\mathbb{R} + iv)$ :

$$\mathcal{C}^{(+)}[u](z) = \int_{\mathbb{R}+iv} \frac{ds}{2i\pi} \frac{u(s)}{s-z} \quad \text{for} \quad z \in \mathbb{C} \setminus \{\mathbb{R} + iv\}. \quad (\text{B.10})$$

Then, by using the relation valid for any  $u \in L^p(\mathbb{R} + iv)$ ,  $+\infty > p > 1$ ,

$$\mathcal{C}_+^{(+)}[u](k) - \mathcal{C}_-^{(+)}[u](k) = u(k), \quad (\text{B.11})$$

one gets that  $U_+ = U_-$  on  $\mathbb{R} + iv$  and hence  $U$  extends into a meromorphic function on  $\mathbb{C}$  whose single pole is located at 0 and is simple. Moreover, it follows from (B.9) that

$$U(z) = -\frac{\tilde{\alpha}_0^{(+)}}{z} \mathcal{F}[f^-](0) + \mathcal{O}(1). \quad (\text{B.12})$$

It is easy to see that  $\mathcal{F}[f^{+/-}](k) \rightarrow 0$  when  $k \rightarrow \infty$  in  $\overline{\mathcal{B}_{\uparrow/\downarrow}^{(+)}}$  and this up to the boundary. Hence,  $U(k) \rightarrow 0$  as  $k \rightarrow \infty$ . Since the constant  $\mathcal{F}[f^-](0)$  is part of the unknowns in the problem, we conclude that there exists a constant  $\mathcal{K}^{(+)}$  such that

$$U(z) = -\frac{\tilde{\alpha}_0^{(+)} \mathcal{K}^{(+)}}{z}. \quad (\text{B.13})$$

This explicit expression for  $U$  entails that, for any  $k \in \mathcal{B}_{\uparrow}^{(+)}$ ,

$$\mathcal{F}[f^+](k) = \frac{1}{\alpha_{\uparrow}^{(+)}(k)} \cdot \left\{ \mathcal{C}^{(+)}[\alpha_{\downarrow}^{(+)} \cdot \mathcal{F}[h^+]](k) - \frac{\tilde{\alpha}_0^{(+)} \mathcal{K}^{(+)}}{k} \right\}. \quad (\text{B.14})$$

Note that, owing to (B.3), one may meromorphically continue  $\mathcal{F}[f^+](k)$  from  $\mathcal{B}_{\uparrow}^{(+)}$  up to  $\{z \in \mathbb{C} : \Im(z) > -\eta\}$  by the expression

$$\mathcal{F}[f^+](k) = \frac{1}{\alpha_{\uparrow}^{(+)}(k)} \cdot \left\{ \mathcal{C}^{(+)}[\alpha_{\downarrow}^{(+)} \cdot \mathcal{F}[h^+]](k) + \alpha_{\downarrow}^{(+)}(k) \cdot \mathcal{F}[h^+](k) - \frac{\tilde{\alpha}_0^{(+)} \mathcal{K}^{(+)}}{k} \right\}. \quad (\text{B.15})$$

Since  $\alpha_{\uparrow}^{(+)}(k)$  has a simple zero at  $k = 0$ , the expression above entails that  $\mathcal{F}[f^+]$  may have a double pole at  $k = 0$ , and that it is its sole pole in the domain  $\Im(k) > -\eta'$ , for some  $\eta' > 0$  and small enough.

Now assume that one is given a meromorphic function  $w$  in the tubular neighbourhood  $|\Im(z)| < 2\eta'$  of  $\mathbb{R}$  having one pole of order  $r + 1$  at  $k = 0$ :

$$w(k) = \sum_{p=0}^r \frac{w_p}{k^{p+1}} + \mathcal{O}(1) \quad k \rightarrow 0, \quad (\text{B.16})$$

and decaying at least as  $1/k$  at infinity. Then, it is easy to convince oneself that, for  $x \neq 0$ , one has

$$\int_{\mathbb{R}+i\eta'} \frac{dk}{2\pi} e^{-ikx} w(k) = -i \sum_{p=0}^r \frac{w_p}{p!} \cdot (-ix)^p + \int_{\mathbb{R}-i\eta'} \frac{dk}{2\pi} e^{-ikx} w(k). \quad (\text{B.17})$$

The integral appearing on the *rhs* of the above identity produces a  $O(e^{-\eta'x})$  behaviour when  $x \rightarrow +\infty$ .

$f^+$  can be reconstructed from (B.14) by taking the inverse Fourier transform on  $\mathbb{R} + iv$ . One infers from (B.17) that the only way to give rise to a solution  $f^+$  to (B.2) enjoying the asymptotic behaviour that is compatible with  $f \in L^2_C(\mathbb{R}^+)$ , *c.f.* (B.1), is that the meromorphic continuation of  $\mathcal{F}[f^+](k)$  has at most a simple pole at  $k = 0$ . This entails that

$$\widetilde{\mathcal{K}}^{(+)} = \mathcal{F}[h^+](0). \quad (\text{B.18})$$

Thus, if a solution to (B.2) exists in the class (B.1), then it is unique and necessarily takes the form

$$\mathcal{F}[f^+](k) = \frac{1}{\alpha_{\uparrow}^{(+)}(k)} \cdot \left\{ \mathcal{C}^{(+)}[\alpha_{\downarrow}^{(+)} \cdot \mathcal{F}[h^+]](k) - \frac{\widetilde{\alpha}_0^{(+)} \mathcal{F}[h^+](0)}{k} \right\}, \quad (\text{B.19})$$

with  $k \in \mathcal{B}_{\uparrow}^{(+)}$ . By deforming the contour in the Cauchy transform from  $\mathbb{R} + iv$  to  $\mathbb{R} - i\eta^-$  with  $0 < \eta^- < \eta$  one obtains the representation (B.4).

Reciprocally, it is easy to see that the function  $f$  defined as

$$f^{\pm}(x) = \int_{\mathbb{R}+iv} \frac{dk}{2\pi} e^{-ikx} \gamma^{\pm}(k) \quad \text{with} \quad \begin{cases} \gamma^+(k) = \frac{1}{\alpha_{\uparrow}^{(+)}(k)} \cdot \left\{ \mathcal{C}^{(+)}[\alpha_{\downarrow}^{(+)} \cdot \mathcal{F}[h^+]](k) - \frac{\widetilde{\alpha}_0^{(+)} \mathcal{F}[h^+](0)}{k} \right\} \\ \gamma^-(k) = \frac{1}{\alpha_{\downarrow}^{(+)}(k)} \cdot \left\{ \frac{\widetilde{\alpha}_0^{(+)} \mathcal{F}[h^+](0)}{k} - \mathcal{C}^{(+)}[\alpha_{\downarrow}^{(+)} \cdot \mathcal{F}[h^+]](k) \right\} \end{cases} \quad (\text{B.20})$$

solves the linear integral equation (B.2) on  $\mathbb{R}^+$ .

Indeed, since  $\gamma^+$ , resp.  $\gamma^-$ , admits a holomorphic continuation to  $\mathcal{B}_{\uparrow}^{(+)}$ , resp.  $\mathcal{B}_{\downarrow}^{(+)}$ , that decays as  $O(1/k)$  at infinity, one readily shows that, indeed, the function

$$x \mapsto \int_{\mathbb{R}+iv} \frac{dk}{2\pi} e^{-ikx} \gamma^{\pm}(k) \quad (\text{B.21})$$

are supported on  $\mathbb{R}^{\pm}$  and that they exhibit the required asymptotic behaviour. The previous reasonings taken backwards then ensure that

$$(1 - \mathcal{F}[L^+](k)) \cdot \gamma^+(k) + \gamma^-(k) = \mathcal{F}[h^+](k) \quad \text{for} \quad k \in \mathbb{R} + iv. \quad (\text{B.22})$$

Upon taking the inverse Fourier transform, the above relation leads to equation (B.2), hence proving the existence of solutions in  $L^2_C(\mathbb{R}^+)$ .  $\blacksquare$

## B.2 Inversion of the operators $\text{id} - L^-$

Analogously to the previous setting, we introduce the space

$$L_C^2(\mathbb{R}^-) = \left\{ f \in L^2(\mathbb{R}^-) : \exists C_f \text{ and } \alpha > 0 \quad f(x) = C_f + O(e^{\alpha x}) \right\}. \quad (\text{B.23})$$

**Proposition B.2.** *Let  $L^-$  be as defined through (2.42) and consider the integral equation*

$$f(x) - \int_{-\infty}^0 L^-(x-y)f(y)dy = h(x) \quad \text{for} \quad x \in \mathbb{R}^-, \quad (\text{B.24})$$

on  $L_C^2(\mathbb{R}^-)$  with  $h$  such that there exist  $\eta > 0$  so that

$$h(x) = O(e^{\eta x}) \quad (\text{B.25})$$

when  $x \rightarrow -\infty$ .

Then, equation (B.2) is uniquely solvable on  $L_C^2(\mathbb{R}^-)$  and the Fourier transform of the solution takes the form

$$\mathcal{F}[f](k) = -\alpha_{\downarrow}^{(-)}(k) \int_{\mathbb{R}+i\eta^-} \frac{ds}{2i\pi} \frac{\{\alpha_{\uparrow}^{(-)}(s)\}^{-1} \cdot \mathcal{F}[h\mathbf{1}_{\mathbb{R}^-}](s)}{s-k} \quad \text{with} \quad k \in \mathbb{R} - iv \quad (\text{B.26})$$

for any  $0 < \eta^- < \eta$  and with  $v > 0$ .

*Proof—*

One extends the functions  $f$  and  $h$  to  $\mathbb{R}$  as

$$f(x) = \int_{-\infty}^0 L^-(x-y)f(y)dy \quad \text{and} \quad h(x) = 0 \quad \text{for} \quad x > 0, \quad (\text{B.27})$$

so that, reducing  $\eta > 0$  if need be, these extensions possess the  $x \rightarrow +\infty$  asymptotic behaviour

$$f(x) = O(e^{-\eta x}) \quad \text{and} \quad h(x) = O(e^{-\eta x}). \quad (\text{B.28})$$

One may then take the Fourier transform of (B.24) extended to  $\mathbb{R}$ , provided that the Fourier variable  $k$  satisfies  $k \in \mathbb{R} - iv$ , with  $0 < v \ll 1$ . This leads to

$$\mathcal{F}[f^+](k) + (1 - \mathcal{F}[L^-](k))\mathcal{F}[f^-](k) = \mathcal{F}[h](k). \quad (\text{B.29})$$

Using the Wiener-Hopf factorisation of  $1 - \mathcal{F}[L^-]$  relatively to  $\mathbb{R} - iv$ , one may recast the last equation as

$$\{\alpha_{\uparrow}^{(-)}(k)\}^{-1} \cdot \mathcal{F}[f^+](k) + \{\alpha_{\downarrow}^{(-)}(k)\}^{-1} \cdot \mathcal{F}[f^-](k) = \{\alpha_{\uparrow}^{(-)}(k)\}^{-1} \cdot \mathcal{F}[h](k). \quad (\text{B.30})$$

Define  $U \in \mathcal{O}(\mathcal{B}_{\uparrow}^{(-)} \cup \mathcal{B}_{\downarrow}^{(-)} \setminus \{0\})$  by the piecewise formula

$$U(k) = \begin{cases} C^{(-)}[\{\alpha_{\uparrow}^{(-)}\}^{-1} \cdot \mathcal{F}[h]](k) - \{\alpha_{\uparrow}^{(-)}(k)\}^{-1} \cdot \mathcal{F}[f^+](k) & , \quad z \in \mathcal{B}_{\uparrow}^{(-)} \\ \{\alpha_{\downarrow}^{(-)}(k)\}^{-1} \cdot \mathcal{F}[f^-](k) + C^{(-)}[\{\alpha_{\uparrow}^{(-)}\}^{-1} \cdot \mathcal{F}[h]](k) & , \quad z \in \mathcal{B}_{\downarrow}^{(-)} \end{cases} \quad (\text{B.31})$$

where  $\mathcal{C}^{(-)}$  is the Cauchy transform on  $L^2(\mathbb{R} - i\nu)$ :

$$\mathcal{C}^{(-)}[u](z) = \int_{\mathbb{R} - i\nu} \frac{ds}{2i\pi} \frac{u(s)}{s - z} \quad \text{for } z \in \mathbb{C} \setminus \{\mathbb{R} - i\nu\}. \quad (\text{B.32})$$

Since  $\alpha_{\uparrow}^{(-)}(k)$  admits a simple zero at  $k = 0$ , one gets that  $U$  is meromorphic on  $\mathcal{B}_{\uparrow} \cup \mathcal{B}_{\downarrow}$ . Its sole pole is located at  $k = 0$  and is simple. Moreover  $U$  vanishes at  $\infty$  and satisfies  $U_+ = U_-$  on  $\mathbb{R} - i\nu$ . All of this allows one to infer that

$$U(k) = -\frac{\mathcal{F}[f^+](0)}{k\alpha_0^{(-)}}. \quad (\text{B.33})$$

However, since  $\mathcal{F}[f^+](0)$  is part of the unknowns in the problem, it is more convenient to set  $\mathcal{K}^{(-)} = \mathcal{F}[f^+](0)$ .

The expression (B.33) allows one to reconstruct the Fourier transform of  $f^-$  for  $k \in \mathcal{B}_{\downarrow}^{(-)}$  as:

$$\mathcal{F}[f^-](k) = -\alpha_{\downarrow}^{(-)}(k) \left\{ \frac{\mathcal{K}^{(-)}}{k \cdot \alpha_0^{(-)}} + \mathcal{C}^{(-)}[\{\alpha_{\uparrow}^{(-)}\}^{-1} \cdot \mathcal{F}[h^-]](k) \right\}. \quad (\text{B.34})$$

The meromorphic continuation of  $\mathcal{F}[f^-](k)$  to  $\mathcal{B}_{\uparrow}^{(-)}$  takes the form

$$\mathcal{F}[f^-](k) = -\alpha_{\downarrow}^{(-)}(k) \left\{ \frac{\mathcal{K}^{(-)}}{k \cdot \alpha_0^{(-)}} - \{\alpha_{\uparrow}^{(-)}(k)\}^{-1} \cdot \mathcal{F}[h^-](k) + \mathcal{C}^{(-)}[\{\alpha_{\uparrow}^{(-)}\}^{-1} \cdot \mathcal{F}[h^-]](k) \right\} \quad \text{for } k \in \mathcal{B}_{\uparrow}^{(-)}. \quad (\text{B.35})$$

The function  $\alpha_{\downarrow}^{(-)}(k)$  admits a simple pole at  $k = 0$ . For generic  $\mathcal{K}^{(-)}$ , the term under the bracket also admits a simple pole at  $k$ , so that the meromorphic continuation has a double pole at  $k = 0$ . As in the case of the Wiener-Hopf equation on  $\mathbb{R}^+$ , contour displacements in the inverse Fourier transform ensure that if  $f^-$  has at most constant asymptotics at  $-\infty$  then the meromorphic continuation of  $\mathcal{F}[f^-](k)$  must have at most a simple pole at  $k = 0$ . This unambiguously fixes the unknown constant as  $\mathcal{K}^{(-)} = \mathcal{F}[h^-](0)$ , leading to

$$\mathcal{F}[f^-](k) = -\alpha_{\downarrow}^{(-)}(k) \left\{ \frac{\mathcal{F}[h^-](0)}{k \cdot \alpha_0^{(-)}} + \mathcal{C}^{(-)}[\{\alpha_{\uparrow}^{(-)}\}^{-1} \cdot \mathcal{F}[h^-]](k) \right\} \quad (\text{B.36})$$

for any  $k \in \mathcal{B}_{\downarrow}^{(-)}$ . Upon deforming the contour in the Cauchy transform  $\mathcal{C}^{(-)}$  up to  $\mathbb{R} + i\eta^-$  with  $0 < \eta^- < \eta$ , one arrives to (B.26).

It is easy to see, proceeding similarly as before, that the above expression does give rise to a solution to (B.24). ■

## C Inversion of $\text{id} - \mathbf{L}_w^{(0)}$

### C.1 Characterisation in terms of a Riemann-Hilbert problem

The operator  $\text{id} - \mathbf{L}_w^{(0)}$  on  $L^2(\cdot - w; w[\cdot])$ , as defined through (5.3) and (5.5), is a truncated Wiener-Hopf operator and, as such, can be explicitly inverted in terms of the solution to an auxiliary Riemann-Hilbert problem. Consider the operator  $\mathbb{V}$  on  $L^2(\mathbb{R} + i\nu)$  with the kernel

$$V(k, s) = -\mathcal{F}[L^{(0)}](k) \cdot \frac{e^{i(k-s)w} - e^{-i(k-s)w}}{2i\pi(k-s)} \quad \text{where} \quad \mathcal{F}[L^{(0)}](k) = \frac{\cosh[k(\tau/2 - \alpha - i\alpha)]}{\cosh[k\tau/2]}. \quad (\text{C.1})$$

Then, it is easy to see that  $\mathcal{F}^{-1}(\text{id} + \mathbb{V})\mathcal{F} = \text{id} - L_w^{(0)}$  or, more precisely, if  $f$  solves  $(\text{id} - L_w^{(0)})[f] = h$  with  $h \in L^2(-w; w]$ , then

$$(\text{id} + \mathbb{V})[\mathcal{F}[f]](k) = \mathcal{F}[h](k) \quad (\text{C.2})$$

for an appropriate extension of  $f$  outside  $[-w; w]$ .

Observe that

$$V(\lambda, \mu) = \frac{(\mathbf{E}_L(\lambda), \mathbf{E}_R(\mu))}{\lambda - \mu} \quad (\text{C.3})$$

where, upon setting  $e(\lambda) = e^{iw\lambda}$ ,

$$\mathbf{E}_R(\mu) = \frac{1}{2i\pi} \begin{pmatrix} e(\mu) \\ e^{-1}(\mu) \end{pmatrix} \quad \text{and} \quad \mathbf{E}_L(\lambda) = -\mathcal{F}[L^{(0)}](\lambda) \begin{pmatrix} -e^{-1}(\lambda) \\ e(\lambda) \end{pmatrix}, \quad (\text{C.4})$$

so that  $(\mathbf{E}_L(\lambda), \mathbf{E}_R(\lambda)) = 0$ . This means that  $\mathbb{V}$  is an integrable integral operator. As such, it can be studied by means of an associated Riemann–Hilbert problem as first observed in [13].

Assume that  $\text{id} + \mathbb{V}$  is invertible. Then, define the functions  $\mathbf{F}_{R/L}(\lambda)$  as the solutions to the linear integral equations

$$[\mathbf{F}_R](\text{id} + \mathbb{V})(\lambda) = \mathbf{E}_R(\lambda) \quad \text{and} \quad (\text{id} + \mathbb{V})[\mathbf{F}_L](\lambda) = \mathbf{E}_L(\lambda). \quad (\text{C.5})$$

The first formula is to be understood as an action of the operator to the left and the second one as its action to the right.

We refer the reader to Subsection 1.4 where the notations used below are introduced.

**Theorem C.1.** *There exists  $w_0$  large enough such that the operator  $\text{id} + \mathbb{V}$  acting on  $L^2(\mathbb{R} + iv)$  with the integral kernel (C.3) is invertible for any  $w \geq w_0$  with inverse given by  $\text{id} - R$ . The integral kernel of the resolvent operator  $R$  is expressed as*

$$R(\lambda, \mu) = \frac{(\mathbf{F}_L(\lambda), \mathbf{F}_R(\mu))}{\lambda - \mu}. \quad (\text{C.6})$$

The vectors  $\mathbf{F}_{R/L}(\lambda)$  are given by

$$\mathbf{F}_R(\lambda) = \chi_+(\lambda) \cdot \mathbf{E}_R(\lambda) \quad \text{and} \quad \mathbf{F}_L^\dagger(\lambda) = \mathbf{E}_L^\dagger(\lambda) \cdot \chi_+^{-1}(\lambda). \quad (\text{C.7})$$

Above,  $\tau$  is the vector transposition while  $\chi$  corresponds to the unique solution to the matrix Riemann–Hilbert problem for <sup>3</sup> $\chi$ : find  $\chi \in \mathcal{M}_2(\mathcal{O}(\mathbb{C} \setminus \{\mathbb{R} + iv\}))$  such that

- $\chi(\lambda) = I_2 + \mathcal{O}\left(\frac{1}{\lambda}\right)$  when  $\lambda \rightarrow \infty$ ;
- $\chi$  admits continuous  $\pm$  boundary values on  $\mathbb{R}$  such that  $\chi_\pm - I_2 \in \mathcal{M}_2(L^2(\mathbb{R} + iv))$ . These boundary values are related by

$$\chi_+(\lambda) G_\chi(\lambda) = \chi_-(\lambda), \quad (\text{C.8})$$

---

<sup>3</sup>In Appendix C, symbols  $\chi, \Xi, \Upsilon, \Pi, \mathcal{P}$  and  $\mathcal{G}$  stand for matrix-valued functions defined below.

where the jump matrix takes the form

$$G_\chi(\lambda) = I_2 + 2i\pi \mathbf{E}_R(\lambda) \cdot \mathbf{E}_L^t(\lambda) = \begin{pmatrix} 1 + \mathcal{F}[L^{(0)}](\lambda) & -\mathcal{F}[L^{(0)}](\lambda) e^{2(\lambda)} \\ \mathcal{F}[L^{(0)}](\lambda) e^{-2(\lambda)} & 1 - \mathcal{F}[L^{(0)}](\lambda) \end{pmatrix}. \quad (\text{C.9})$$

The unique solution  $\chi$  takes the explicit form given in Fig. 9. It admits the integral representations

$$\chi(\lambda) = I_2 - \int_{\mathbb{R}} \frac{\mathbf{F}_R(\mu) \cdot \mathbf{E}_L^t(\mu)}{\mu - \lambda} d\mu \quad \text{and} \quad \chi^{-1}(\lambda) = I_2 + \int_{\mathbb{R}} \frac{\mathbf{E}_R(\mu) \cdot \mathbf{F}_L^t(\mu)}{\mu - \lambda} d\mu. \quad (\text{C.10})$$

Most results stated in Theorem C.1 are classic and go back to the work [13]. The representation given in Fig. 9 is established throughout Subsection (C.2) to come by a rather standard application of the non-linear steepest descent method [7]. It is a standard fact, which follows from  $\det G_\chi = 1$ , that the Riemann-Hilbert problem for  $\chi$  admits a unique solution, see *e.g.* [6]. Thus, we will not discuss this question further.

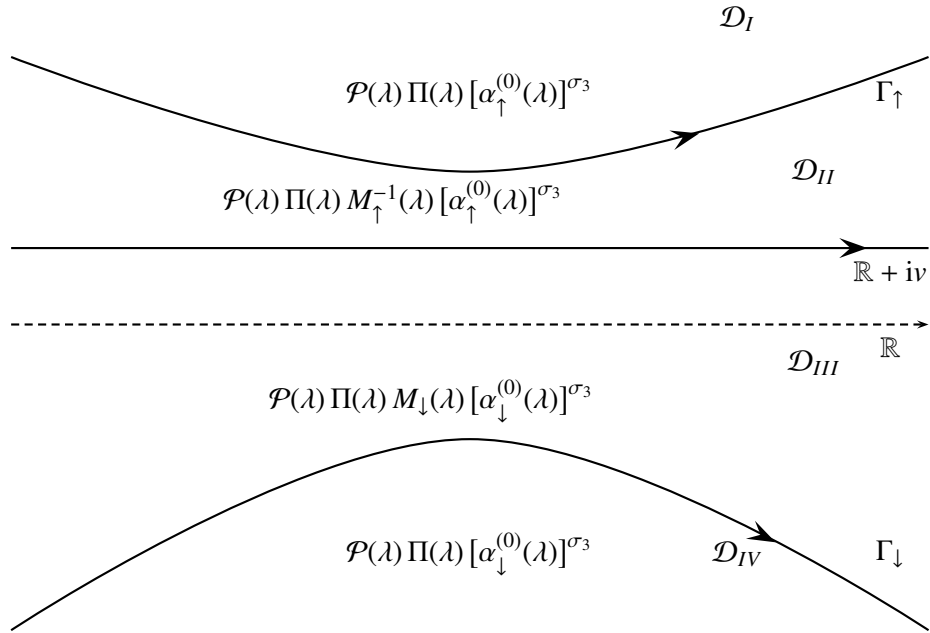


Figure 9: Piecewise definition of the matrix  $\chi$ . The curves  $\Gamma_{\uparrow/\downarrow}$  separate all poles, other than at 0, of  $\lambda \mapsto \mathcal{F}[L^{(0)}](\lambda) \cdot \{1 - \mathcal{F}[L^{(0)}](\lambda)\}^{-1}$  from  $\mathbb{R}$  and are such that  $\text{dist}(\Gamma_{\uparrow/\downarrow}, \mathbb{R}) > \varrho$  for some  $\varrho > 0$ . The piecewise holomorphic matrix  $\Pi$  appearing in the above figure is as defined through (C.26).

## C.2 Asymptotic resolution of the Riemann-Hilbert problem

### C.2.1 Riemann-Hilbert problem for $\Xi$

First, we consider the solution to an auxiliary scalar Riemann-Hilbert problem. Let

$$\mathcal{B}_{\uparrow}^{(0)} = \{z \in \mathbb{C} : \Im z > \nu\} \quad \text{and} \quad \mathcal{B}_{\downarrow}^{(0)} = \{z \in \mathbb{C} : \Im z < \nu\}. \quad (\text{C.11})$$

One introduces the function

$$\alpha^{(0)} \in \mathcal{O}(\mathbb{C}^* \setminus \{\mathbb{R} + iv\}) \quad \text{with} \quad \alpha^{(0)}(\lambda) = \begin{cases} \alpha_{\uparrow}^{(0)}(\lambda) & \lambda \in \mathcal{B}_{\uparrow}^{(0)} \\ \alpha_{\downarrow}^{(0)}(\lambda) & \lambda \in \mathcal{B}_{\downarrow}^{(0)} \end{cases} \quad (\text{C.12})$$

in which<sup>4</sup>  $\alpha_{\uparrow}^{(0)} \in \mathcal{O}(\overline{\mathcal{B}_{\uparrow}^{(0)}})$ ,  $\alpha_{\downarrow}^{(0)} \in \mathcal{O}(\overline{\mathcal{B}_{\downarrow}^{(0)}} \setminus \{0\})$ ,  $\alpha_{\uparrow/\downarrow}^{(0)}(\lambda) \rightarrow 1$  when  $\lambda \rightarrow \infty$  in  $\overline{\mathcal{B}_{\uparrow/\downarrow}^{(0)}}$  and such that

$$\frac{\alpha_{\uparrow}^{(0)}(\lambda)}{\alpha_{\downarrow}^{(0)}(\lambda)} = 1 - \mathcal{F}[L^{(0)}](\lambda). \quad (\text{C.13})$$

$\alpha_{\uparrow/\downarrow}^{(0)}$  admit meromorphic continuations to  $\mathcal{B}_{\downarrow/\uparrow}^{(0)}$  such that

$$\alpha_{\uparrow}^{(0)}(k) \underset{k \rightarrow 0}{\sim} k \alpha_0^{(0)} \quad \text{and} \quad \alpha_{\downarrow}^{(0)}(k) \underset{k \rightarrow 0}{\sim} \frac{\tilde{\alpha}_0^{(0)}}{k}. \quad (\text{C.14})$$

Note that  $k = 0$  is the only zero and pole of  $\alpha_{\uparrow/\downarrow}^{(0)}$  in a fixed  $v$ -independent tubular neighbourhood of  $\mathbb{R}$ .

The functions  $\alpha_{\uparrow/\downarrow}^{(0)}$  can be read out from equations (2.54)-(2.55) upon the substitution  $\varkappa^v \leftrightarrow \varkappa$ .

Assume that one is given a solution  $\chi$  to the Riemann-Hilbert problem for  $\chi$ , and define

$$\Xi(\lambda) = \chi(\lambda) \cdot (\alpha^{(0)}(\lambda))^{-\sigma_3}. \quad (\text{C.15})$$

It is clear that the Riemann-Hilbert problem for  $\chi$  is in one-to-one correspondence with the Riemann-Hilbert problem for  $\Xi$ . The latter consists in finding  $\Xi \in \mathcal{M}_2(\mathcal{O}(\mathbb{C}^* \setminus \{\mathbb{R} + iv\}))$  such that

- $\Xi$  admits a simple pole at 0;
- $\Xi(\lambda) = I_2 + \mathcal{O}\left(\frac{1}{\lambda}\right)$  when  $\lambda \rightarrow \infty$ ;
- $\Xi(\lambda) \cdot (\alpha_{\downarrow}^{(0)}(\lambda))^{\sigma_3}$  is regular at  $\lambda = 0$ ;
- $\Xi$  admits continuous  $\pm$  boundary values on  $\mathbb{R} + iv$  such that  $\Xi_{\pm} - I_2 \in \mathcal{M}_2(L^2(\mathbb{R} + iv))$ . These boundary values are related as

$$\Xi_+(\lambda) G_{\Xi}(\lambda) = \Xi_-(\lambda), \quad \text{where} \quad G_{\Xi}(\lambda) = \begin{pmatrix} 1 + P(\lambda)Q(\lambda) & P(\lambda)e^2(\lambda) \\ Q(\lambda)e^{-2}(\lambda) & 1 \end{pmatrix}. \quad (\text{C.16})$$

Note that the jump matrix factorises as  $G_{\Xi}(\lambda) = M_{\uparrow}(\lambda) \cdot M_{\downarrow}(\lambda)$  in which

$$M_{\uparrow}(\lambda) = \begin{pmatrix} 1 & P(\lambda)e^2(\lambda) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{\downarrow}(\lambda) = \begin{pmatrix} 1 & 0 \\ Q(\lambda)e^{-2}(\lambda) & 1 \end{pmatrix}. \quad (\text{C.17})$$

The expression for these matrices involve the functions

$$P(\lambda) = -\alpha_{\uparrow}^{(0)}(\lambda) \cdot \alpha_{\downarrow}^{(0)}(\lambda) \cdot \mathcal{F}[L^{(0)}](\lambda) \quad \text{and} \quad Q(\lambda) = \frac{\mathcal{F}[L^{(0)}](\lambda)}{\alpha_{\uparrow}^{(0)}(\lambda) \cdot \alpha_{\downarrow}^{(0)}(\lambda)}. \quad (\text{C.18})$$

<sup>4</sup>Any function holomorphic on a closed set is, in fact understood to be holomorphic on an open neighbourhood thereof



In particular,  $Q$  is analytic on a tubular neighbourhood of  $\mathbb{R}$  and satisfies

$$Q(0) = \frac{1}{\alpha_0^{(0)} \cdot \widetilde{\alpha}_0^{(0)}} = -1, \quad (\text{C.19})$$

see (2.58).

The matrices  $M_{\uparrow/\downarrow}$  are such that their off-diagonal entries are exponentially small in  $w$  for  $\lambda$  belonging to  $\mathbb{H}^\pm$  and uniformly away from  $\mathbb{R}$ .

### C.2.2 Riemann-Hilbert problem for $\Upsilon$

Next, one defines  $\Upsilon$  as in Fig. 10. The contours  $\Gamma_{\uparrow/\downarrow}$  are chosen such that it holds  $\Gamma_\uparrow = -\Gamma_\downarrow$ . It is clear that the Riemann-Hilbert problems for  $\Xi$  is in one-to-one correspondence with the one for  $\Upsilon$ .

Find  $\Upsilon \in \mathcal{M}_2(\mathcal{O}(\mathbb{C}^* \setminus \{\Gamma_\uparrow \cup \Gamma_\downarrow\}))$  such that

- $\Upsilon$  admits a simple pole at 0;
- $\Upsilon(\lambda) = I_2 + \mathcal{O}\left(\frac{1}{\lambda}\right)$  when  $\lambda \rightarrow \infty$ ;
- $\Upsilon(\lambda) \cdot M_\downarrow(\lambda) \cdot \left(\alpha_\downarrow^{(0)}(\lambda)\right)^{\sigma_3}$  is regular at  $\lambda = 0$ ;
- $\Upsilon$  admits continuous  $\pm$  boundary values on  $\Gamma_\uparrow \cup \Gamma_\downarrow$  such that  $\Upsilon_\pm - I_2 \in \mathcal{M}_2(L^2(\Gamma_\uparrow \cup \Gamma_\downarrow))$ . These boundary values are related by

$$\Upsilon_+(\lambda) G_\Upsilon(\lambda) = \Upsilon_-(\lambda) \quad \text{with} \quad G_\Upsilon(\lambda) = M_\downarrow(\lambda) \cdot \mathbf{1}_{\Gamma_\downarrow}(\lambda) + M_\uparrow(\lambda) \cdot \mathbf{1}_{\Gamma_\uparrow}(\lambda). \quad (\text{C.20})$$

### C.2.3 Auxiliary Riemann-Hilbert problem for $\Pi$

To continue further, one first introduces  $\Pi$  as the unique solution to the below Riemann-Hilbert problem for  $\Pi$ . Find  $\Pi \in \mathcal{M}_2(\mathcal{O}(\mathbb{C} \setminus \{\Gamma_\uparrow \cup \Gamma_\downarrow\}))$  such that:

- $\Pi(\lambda) = I_2 + \mathcal{O}\left(\frac{1}{\lambda}\right)$  when  $\lambda \rightarrow \infty$ ;
- $\Pi$  admits continuous  $\pm$  boundary values on  $\Gamma_\uparrow \cup \Gamma_\downarrow$  such that  $\Pi_\pm - I_2 \in \mathcal{M}_2(L^2(\Gamma_\uparrow \cup \Gamma_\downarrow))$ . These boundary values are related by

$$\Pi_+(\lambda) G_\Upsilon(\lambda) = \Pi_-(\lambda). \quad (\text{C.21})$$

Again, there exists at most a one solution to the Riemann-Hilbert problem for  $\Pi$ . Existence may be established by the singular integral equation method introduced in [1].

Indeed, introduce the singular integral operator on the space  $\mathcal{M}_2(L^2(\Gamma_\uparrow \cup \Gamma_\downarrow))$  of  $2 \times 2$  matrix-valued  $L^2(\Gamma_\uparrow \cup \Gamma_\downarrow)$  functions by

$$\mathcal{C}_{\Gamma_\uparrow \cup \Gamma_\downarrow}^{(+)}[\Psi](\lambda) = \lim_{z \rightarrow \lambda} \int_{\Gamma_\uparrow \cup \Gamma_\downarrow} \frac{\Psi(t) \cdot (G_\Upsilon - I_2)(t)}{t - z} \cdot \frac{dt}{2i\pi}. \quad (\text{C.22})$$

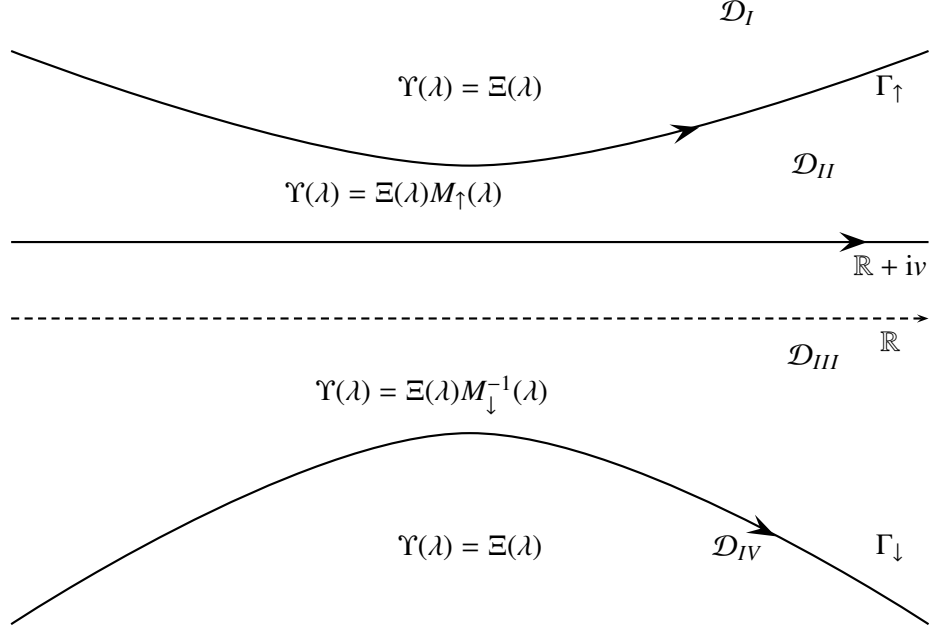


Figure 10: Piecewise definition of the matrix  $\Upsilon$  in terms of the matrix  $\Xi$ . The curves  $\Gamma_{\uparrow/\downarrow}$  separate all poles, other than the one at 0, of  $\lambda \mapsto \mathcal{F}[L^{(0)}](\lambda) \cdot \{1 - \mathcal{F}[L^{(0)}](\lambda)\}^{-1}$  from  $\mathbb{R}$  and are such that  $\text{dist}(\Gamma_{\uparrow/\downarrow}, \mathbb{R}) > \varrho$  for some  $\varrho > 0$ .

Since  $G_{\Upsilon} - I_2 \in \mathcal{M}_2((L^\infty \cap L^2)(\Gamma_{\uparrow} \cup \Gamma_{\downarrow}))$  and  $\Gamma_{\uparrow} \cup \Gamma_{\downarrow}$  is a Lipschitz curve, it follows from [4] that  $\mathcal{C}_{\Gamma_{\uparrow} \cup \Gamma_{\downarrow}}^{(+)}$  is continuous on  $\mathcal{M}_2(L^2(\Gamma_{\uparrow} \cup \Gamma_{\downarrow}))$  and fulfils:

$$\| \mathcal{C}_{\Gamma_{\uparrow} \cup \Gamma_{\downarrow}}^{(+)} \|_{\mathcal{M}_2(L^2(\Gamma_{\uparrow} \cup \Gamma_{\downarrow}))} \leq C e^{-\varrho w}. \quad (\text{C.23})$$

Hence, since

$$G_{\Upsilon} - I_2 \in \mathcal{M}_2(L^2(\Gamma_{\uparrow} \cup \Gamma_{\downarrow})) \quad \text{and} \quad \mathcal{C}_{\Gamma_{\uparrow} \cup \Gamma_{\downarrow}}^{(+)}[I_2] \in \mathcal{M}_2(L^2(\Gamma_{\uparrow} \cup \Gamma_{\downarrow})), \quad (\text{C.24})$$

provided that  $w$  is large enough, it follows that the singular integral equation

$$(I_2 + \mathcal{C}_{\Gamma_{\uparrow} \cup \Gamma_{\downarrow}}^{(+)})[\Pi_+] = I_2 \quad (\text{C.25})$$

admits a unique solution  $\Pi_+$  such that  $\Pi_+ - I_2 \in \mathcal{M}_2(L^2(\Gamma_{\uparrow} \cup \Gamma_{\downarrow}))$ . It is then a standard fact [1] in the theory of Riemann-Hilbert problems that the matrix

$$\Pi(\lambda) = I_2 - \int_{\Gamma_{\uparrow} \cup \Gamma_{\downarrow}} \frac{\Pi_+(t)(G_{\Upsilon} - I_2)(t)}{t - \lambda} \cdot \frac{dt}{2i\pi} \quad (\text{C.26})$$

is the unique solution to the Riemann-Hilbert problem for  $\Pi$ . It is a direct consequence of the Neumann expansion of the solution to the singular integral equation (C.25) for  $\Pi_+$  and of the local holomorphicity of the jump matrices

that, for some  $\varrho > 0$ ,

$$\Pi(\lambda) = I_2 + O\left(\frac{e^{-\varrho w}}{1 + |\lambda|}\right) \quad (\text{C.27})$$

uniformly on  $\mathbb{C}$  and with a differentiable remainder.

The piecewise holomorphic matrix  $\Pi$  thus constructed enjoys a few properties that will be useful below. Indeed, one readily infers from the identity  $M_{\uparrow}(-\lambda) = \sigma^x \cdot M_{\downarrow}^{-1}(\lambda) \cdot \sigma^x$ , adjoined to the contour symmetry  $\Gamma_{\uparrow} = \{-z : z \in \Gamma_{\downarrow}\}$  and the uniqueness of the Riemann–Hilbert problem for  $\Pi$  that the relation  $\Pi(\lambda) = \sigma^x \cdot \Pi(-\lambda) \cdot \sigma^x$  holds. In particular,

$$\Pi(0) = \sigma^x \cdot \Pi(0) \cdot \sigma^x \quad \text{and} \quad \Pi'(0) = -\sigma^x \cdot \Pi'(0) \cdot \sigma^x. \quad (\text{C.28})$$

These properties lead to the the  $\lambda \rightarrow 0$  expansion

$$\Pi(\lambda) = \begin{pmatrix} \Pi_{11}(0) & \Pi_{21}(0) \\ \Pi_{21}(0) & \Pi_{11}(0) \end{pmatrix} + \lambda \begin{pmatrix} \Pi'_{11}(0) & -\Pi'_{21}(0) \\ \Pi'_{21}(0) & -\Pi'_{11}(0) \end{pmatrix} + O(\lambda^2). \quad (\text{C.29})$$

In other words, by setting

$$c_1 = \Pi_{11}(0)\Pi'_{11}(0) - \Pi_{21}(0)\Pi'_{21}(0) \quad \text{and} \quad c_2 = \Pi_{11}(0)\Pi'_{21}(0) - \Pi_{21}(0)\Pi'_{11}(0), \quad (\text{C.30})$$

since  $\det \Pi(\lambda) = 1$ , one infers that

$$\Pi^{-1}(0)\Pi(\lambda) = I_2 + \lambda \begin{pmatrix} c_1 & -c_2 \\ c_2 & -c_1 \end{pmatrix} + O(\lambda^2). \quad (\text{C.31})$$

#### C.2.4 Solution of the Riemann–Hilbert problem for $\Upsilon$

With  $\Pi$  defined, the solution to the Riemann-Hilbert problem for  $\Upsilon$  can be constructed as  $\Upsilon(\lambda) = \mathcal{P}(\lambda) \cdot \Pi(\lambda)$ , where  $\mathcal{P}(\lambda)$  is a meromorphic matrix on  $\mathbb{C}$  whose only pole is located at  $\lambda = 0$ . Below, we establish that this meromorphic matrix takes the form

$$\mathcal{P}(\lambda) = \Pi(0) \cdot \left( I_2 + \frac{\theta}{\lambda} \mathbb{D} \right) \cdot \Pi^{-1}(0) \quad (\text{C.32})$$

where

$$\mathbb{D} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \theta = \frac{1}{(e^{-2}Q)'(0) + 2(c_1 + c_2)}, \quad (\text{C.33})$$

with  $c_1, c_2$  as introduced in (C.30). The matrix  $\mathcal{P}$  is constructed so that

$$\lambda \mapsto \mathcal{P}(\lambda)\Pi(\lambda)M_{\downarrow}(\lambda)\left(\alpha^{(0)}(\lambda)\right)^{\sigma_3} \quad (\text{C.34})$$

is regular at  $\lambda = 0$ .

When looking for  $\mathcal{P}$ , it is convenient to parameterise

$$\mathcal{P}(\lambda) = \Pi(0)\mathcal{G}(\lambda)\Pi^{-1}(0) \quad \text{with} \quad \mathcal{G}(\lambda) = I_2 + \frac{1}{\lambda} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad (\text{C.35})$$

Then, one has  $\mathcal{P}(\lambda)\Pi(\lambda)M_{\downarrow}(\lambda)\left(\alpha_{\downarrow}^{(0)}(\lambda)\right)^{\sigma_3} = \Pi(0)H(\lambda)$ , with

$$H_{11}(\lambda) = \alpha_{\downarrow}^{(0)}(\lambda)\left[\mathcal{G}_{11}(\lambda) + Q(\lambda)e^{-2(\lambda)}\mathcal{G}_{12}(\lambda)\right] + c_1\lambda\alpha_{\downarrow}^{(0)}(\lambda)\left[\mathcal{G}_{11}(\lambda) - Q(\lambda)e^{-2(\lambda)}\mathcal{G}_{12}(\lambda)\right] \\ + c_2\lambda\alpha_{\downarrow}^{(0)}(\lambda)\left[\mathcal{G}_{12}(\lambda) - Q(\lambda)e^{-2(\lambda)}\mathcal{G}_{11}(\lambda)\right] + \mathcal{O}(1), \quad (\text{C.36})$$

as well as  $H_{21} = [H_{11}]_{\mathcal{G}_{1a} \leftrightarrow \mathcal{G}_{2a}}$  and  $H_{a2}(\lambda) = \mathcal{O}(1)$ , when  $\lambda \rightarrow 0$ .

In principle,  $H_{11}$  admits a second order pole at  $\lambda = 0$ . By imposing that  $H_{11}$  is regular at  $\lambda = 0$ , one obtains the system of equations on the coefficients  $g_{1a}$ :

$$g_{11} = -Q(0)g_{12} \quad \text{and} \quad g_{12} \cdot \left[ (e^{-2}Q)'(0) - 2c_1Q(0) + c_2[1 + Q^2(0)] \right] = -1. \quad (\text{C.37})$$

These equations are solvable owing to  $|c_1| + |c_2| = \mathcal{O}(e^{-\epsilon w})$ , what is in itself a consequence of (C.27).

Likewise, by requiring that  $H_{21}$  is regular at  $\lambda = 0$ , one obtains the system of equations on the coefficients  $g_{2a}$ :

$$g_{21} = -Q(0)g_{22} \quad \text{and} \quad g_{22} \cdot \left[ (e^{-2}Q)'(0) - 2c_1Q(0) + c_2[1 + Q^2(0)] \right] = -Q(0). \quad (\text{C.38})$$

All in all, this yields that

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \frac{1}{(e^{-2}Q)'(0) - 2c_1Q(0) + c_2[1 + Q^2(0)]} \cdot \begin{pmatrix} Q(0) & -1 \\ Q^2(0) & -Q(0) \end{pmatrix}. \quad (\text{C.39})$$

The form of  $\mathcal{P}(\lambda)$  then follows upon recalling that  $Q(0) = -1$ .

By tracing backwards the various transformations, one gets that the unique solution  $\chi$  to the Riemann-Hilbert problem for  $\chi$  takes the piecewise form as depicted in Fig. 9.

### C.3 Resolvent kernel of $\text{id} + \mathbf{V}$

It follows from the results of Theorem C.1 that the solution to (C.2) takes the form

$$\mathcal{F}[f](k) = \mathcal{F}[h](k) - \int_{\mathbb{R}+iv} d\mu R(k, \mu) \mathcal{F}[h](\mu). \quad (\text{C.40})$$

Since  $\chi_+(\lambda)\mathbf{E}_R(\lambda) = \chi_-(\lambda)\mathbf{E}_R(\lambda)$ , and since the vectors  $\mathbf{E}_{L/R}$  are analytic in a tubular neighbourhood of  $\mathbb{R}$ , it follows that  $R(\lambda, \mu)$  is also analytic in some open neighbourhood of  $\mathbb{R}^2$ .

#### C.3.1 Support restrictions

One may explicitly check that the integral term only involves the values of  $h$  inside of  $[-w; w]$ . Indeed, one has

$$\int_{\mathbb{R}+iv} d\mu e^{i\mu x} R(k, \mu) = \int_{\mathbb{R}+iv} d\mu e^{i\mu x} \frac{\left(\mathbf{E}_L(k), \chi_+^{-1}(k) \cdot \chi_+(\mu) \mathbf{E}_R(\mu)\right)}{k - \mu} \quad (\text{C.41})$$

If  $x > w$ , then  $e^{i\mu x} \mathbf{E}_R(\mu)$  is bounded on  $\mathbb{H}_v^+ = \mathbb{H}^+ + iv$ , and so, since the integrand vanishes at  $\infty$  in  $\mathbb{H}_v^+$ , one obtains zero by deforming the integration contour to  $+i\infty$ . One arrives to the same conclusion when  $x < -w$  upon using  $\chi_+(\mu)\mathbf{E}_R(\mu) = \chi_-(\mu)\mathbf{E}_R(\mu)$ . Hence, for *any* function  $h$  on  $\mathbb{R}$  with exponential decay at  $\pm\infty$ , one gets, for  $0 < v$  small enough, that

$$\int_{\mathbb{R}+iv} d\mu R(k, \mu) \mathcal{F}[h](\mu) = \int_{\mathbb{R}+iv} d\mu R(k, \mu) \mathcal{F}[h\mathbf{1}_{[-w; w]}](\mu). \quad (\text{C.42})$$

### C.3.2 Leading asymptotic form of the resolvent

The resolvent may be approximated, in the large- $w$  limit, by inserting the leading behaviour of the matrix  $\chi$  into the expression for the vectors  $F_{R/L}$  (C.7), and then inserting the latter into the formula for the resolvent kernel (C.6).

For further convenience, given  $D$  as in (C.33), set

$$\mathcal{P}_\infty(\lambda) = I_2 + \frac{1}{\lambda b'(0)} D \quad \text{with} \quad b(\lambda) = \frac{e^{-2}(\lambda)}{\alpha_\downarrow^{(0)}(\lambda)\alpha_\uparrow^{(0)}(\lambda)}, \quad (\text{C.43})$$

so that, by using that  $(e^{-2}Q)'(0) = b'(0)$ , one may decompose

$$\mathcal{P}(\lambda) = \mathcal{P}_\infty(\lambda) + \delta\mathcal{P}(\lambda) \quad \text{with} \quad \delta\mathcal{P}(\lambda) = O\left(\frac{e^{-\varrho w}}{|\lambda|}\right). \quad (\text{C.44})$$

It is as well convenient to introduce an analogous parameterisation gathering the exponentially small corrections to  $\Pi(\lambda) = I_2 + \delta\Pi(\lambda)$ , where, by virtue of (C.27), one has

$$\delta\Pi(\lambda) = O\left(\frac{e^{-\varrho w}}{1+|\lambda|}\right) \quad (\text{C.45})$$

uniformly on  $\mathbb{C}$ .

From there, one obtains that, uniformly throughout the region  $\mathcal{D}_{II}$ , as defined in Fig. 9, one has

$$\chi(\lambda) = \chi_\infty^{(II)}(\lambda) + \delta\chi^{(II)}(\lambda) \quad \text{with} \quad \chi_\infty^{(II)}(\lambda) = \mathcal{P}_\infty(\lambda)M_\uparrow^{-1}(\lambda)[\alpha_\uparrow^{(0)}(\lambda)]^{\sigma_3}, \quad (\text{C.46})$$

and

$$\delta\chi^{(II)}(\lambda) = \delta\mathcal{P}(\lambda)\Pi(\lambda)M_\uparrow^{-1}(\lambda)[\alpha_\uparrow^{(0)}(\lambda)]^{\sigma_3} + \mathcal{P}_\infty(\lambda)\delta\Pi(\lambda)M_\uparrow^{-1}(\lambda)[\alpha_\uparrow^{(0)}(\lambda)]^{\sigma_3}. \quad (\text{C.47})$$

By direct inspection, one obtains that uniformly in  $\lambda \in \overline{\mathcal{D}_{II}}$ ,

$$\delta\chi^{(II)}(\lambda) = O\left(\frac{e^{-\varrho w}}{1+|\lambda|}\right). \quad (\text{C.48})$$

Likewise, uniformly throughout the region  $\mathcal{D}_{III}$ , one has the decomposition

$$\chi(\lambda) = \chi_\infty^{(III)}(\lambda) + \delta\chi^{(III)}(\lambda) \quad \text{with} \quad \chi_\infty^{(III)}(\lambda) = \mathcal{P}_\infty(\lambda)M_\downarrow(\lambda)[\alpha_\downarrow^{(0)}(\lambda)]^{\sigma_3}, \quad (\text{C.49})$$

and

$$\delta\chi^{(III)}(\lambda) = \delta\mathcal{P}(\lambda)\Pi(\lambda)M_\downarrow(\lambda)[\alpha_\downarrow^{(0)}(\lambda)]^{\sigma_3} + \mathcal{P}_\infty(\lambda)\delta\Pi(\lambda)M_\downarrow(\lambda)[\alpha_\downarrow^{(0)}(\lambda)]^{\sigma_3}. \quad (\text{C.50})$$

Again, a direct analysis shows that for  $\lambda \in \mathcal{D}_{III}$  and uniformly away from 0, one has

$$\delta\chi^{(III)}(\lambda) = O\left(\frac{e^{-\varrho w+2vw}}{1+|\lambda|}\right). \quad (\text{C.51})$$

Note that the additional term  $e^{2vw}$  present in the estimates on the remainder is due to the presence of  $e^{-2}$  in the off-diagonal entry of  $M_\downarrow$  and the fact that  $\mathcal{D}_{III} \cap \mathbb{H}^+ = \{\lambda \in \mathbb{C} : 0 < \Im(\lambda) < v\}$ .

These formulae allow to compute the leading behaviour of the vector  $\mathbf{F}_R(\lambda)$  inside each of the domains. One infers that

$$\mathbf{F}_R(\lambda) = \mathbf{F}_{R;\infty}(\lambda) + \delta\mathbf{F}_R^{(A)}(\lambda) \quad \text{with} \quad \delta\mathbf{F}_R^{(A)}(\lambda) = \delta\chi^{(A)}(\lambda)\mathbf{E}_R(\lambda) \quad \text{for} \quad \lambda \in \mathcal{D}_A, A \in \{II, III\}. \quad (\text{C.52})$$

We stress that the expression for  $\mathbf{F}_{R;\infty}(\lambda)$  does not depend on whether  $\lambda \in \mathcal{D}_{II}$  or  $\lambda \in \mathcal{D}_{III}$ . A direct calculation shows that

$$\begin{aligned} \mathbf{F}_{R;\infty}(\lambda) &= \frac{1}{2i\pi} \mathcal{P}_\infty(\lambda) \cdot \begin{pmatrix} \alpha_\downarrow^{(0)}(\lambda)e(\lambda) \\ \{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1} \end{pmatrix} \\ &= \frac{1}{2i\pi} \begin{pmatrix} \alpha_\downarrow^{(0)}(\lambda)e(\lambda) - \frac{1}{\lambda b'(0)} [\alpha_\downarrow^{(0)}(\lambda)e(\lambda) + \{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1}] \\ \{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1} + \frac{1}{\lambda b'(0)} [\alpha_\downarrow^{(0)}(\lambda)e(\lambda) + \{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1}] \end{pmatrix} \\ &= \frac{1}{2i\pi} \begin{pmatrix} \alpha_\downarrow^{(0)}(\lambda)e(\lambda) \cdot \left[1 - \frac{1}{\lambda b'(0)}(1 + b(\lambda))\right] \\ \{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1} \cdot \left[1 + \frac{1}{\lambda b'(0)}(1 + \{b(\lambda)\}^{-1})\right] \end{pmatrix} = \frac{1}{2i\pi} \begin{pmatrix} f_{+;\infty}(\lambda) \\ f_{-;\infty}(\lambda) \end{pmatrix}. \quad (\text{C.53}) \end{aligned}$$

Above,  $b$  is as introduced in (C.43). It is easy to see that the above expression for  $f_{\pm;\infty}(\lambda)$  is analytic in a tubular neighbourhood of  $\mathbb{R}$ . In particular, there is no pole at  $\lambda = 0$  as follows from  $b(0) = -1$ .

Similarly, using the relation  $\det \chi(\lambda) = 1$ , one infers that

$$\mathbf{F}_L(\lambda) = \mathbf{F}_{L;\infty}(\lambda) + \delta\mathbf{F}_L^{(A)}(\lambda) \quad \text{with} \quad \delta\mathbf{F}_L^{(A)}(\lambda) = \text{CoMat}(\delta\chi^{(A)}(\lambda))\mathbf{E}_L(\lambda) \quad \text{for} \quad \lambda \in \mathcal{D}_A, A \in \{II, III\} \quad (\text{C.54})$$

where

$$\mathbf{F}_{L;\infty}(\lambda) = -\mathcal{F}[L^{(0)}](\lambda) \begin{pmatrix} -f_{-;\infty}(\lambda) \\ f_{+;\infty}(\lambda) \end{pmatrix} \quad (\text{C.55})$$

and  $\text{CoMat}(M)$  stands for the Comatrix of  $M$ .

From the above one infers that the resolvent admits the following expansion

$$R(\lambda, \mu) = R_\infty(\lambda, \mu) + \delta R(\lambda, \mu) \quad \text{uniformly in} \quad \mathcal{D}_{II} \cup \mathcal{D}_{III}, \quad (\text{C.56})$$

where

$$R_\infty(\lambda, \mu) = \frac{-\mathcal{F}[L^{(0)}](\lambda)}{2i\pi(\lambda - \mu)} \cdot \begin{pmatrix} -f_{-;\infty}(\lambda) & f_{+;\infty}(\lambda) \\ f_{+;\infty}(\mu) & -f_{-;\infty}(\mu) \end{pmatrix} \quad (\text{C.57})$$

while, for  $(\lambda, \mu) \in \mathcal{D}_A \times \mathcal{D}_B$  with  $A, B \in \{II, III\}$ ,

$$\begin{aligned} \delta R(\lambda, \mu) &= \frac{1}{\lambda - \mu} \left\{ \left( \mathbf{E}_L(\lambda), {}^t \text{CoMat}(\delta\chi^{(A)}(\lambda)) \mathbf{F}_{R;\infty}(\mu) \right) + \left( \mathbf{F}_{L;\infty}(\lambda), \delta\chi^{(B)}(\mu) \mathbf{E}_R(\mu) \right) \right. \\ &\quad \left. + \left( \mathbf{E}_L(\lambda), {}^t \text{CoMat}(\delta\chi^{(A)}(\lambda)) \cdot \delta\chi^{(B)}(\mu) \mathbf{E}_R(\mu) \right) \right\}. \quad (\text{C.58}) \end{aligned}$$

The leading resolvent may be explicitly cast as

$$R_\infty(\lambda, \mu) = \frac{-\mathcal{F}[L^{(0)}](\lambda)}{2i\pi(\lambda - \mu)} \begin{pmatrix} -\{\alpha_\uparrow^{(0)}(\lambda)e(\lambda)\}^{-1} & \alpha_\downarrow^{(0)}(\lambda)e(\lambda) \\ \alpha_\downarrow^{(0)}(\mu)e(\mu) & \{\alpha_\uparrow^{(0)}(\mu)e(\mu)\}^{-1} \end{pmatrix} \cdot \left( I_2 + \frac{\lambda - \mu}{\lambda\mu b'(0)} \mathbf{D} \right) \quad (\text{C.59})$$

as follows from  $D^2 = 0$ .

Obviously,  $R_\infty(\lambda, \mu)$  is analytic in a tubular neighbourhood of  $\mathbb{R}^2$  and satisfies, for some  $\alpha > 0$ , the bounds

$$|R_\infty(\lambda, \mu)| = \mathcal{O}\left(\frac{e^{-\alpha|\lambda|}}{|\lambda - \mu|} e^{w(|\Im(\lambda)| + |\Im(\mu)|)}\right), \quad (\text{C.60})$$

which is valid throughout  $\{\mathcal{D}_{II} \cup \mathcal{D}_{III} \cup \{\mathbb{R} + iv\}\}^2$ , provided that  $\lambda, \mu$  are both uniformly away from 0.

Since  $\delta R = R - R_\infty$ , one infers that  $\delta R$  is analytic in a tubular neighbourhood of  $\mathbb{R}^2$ . One can bound  $\delta R$ , globally on  $\{\mathcal{D}_{II} \cup \mathcal{D}_{III} \cup \{\mathbb{R} + iv\}\}^2$  by using its patch-wise valid decomposition. This yields that

$$|\delta R(\lambda, \mu)| = \mathcal{O}\left(\frac{e^{-\alpha|\lambda| - \varrho w + 4vw}}{|\lambda - \mu|} e^{w(|\Im(\lambda)| + |\Im(\mu)|)}\right). \quad (\text{C.61})$$

Upon putting these two bounds together, one concludes that for  $\lambda, \mu$  throughout  $\{\mathcal{D}_{II} \cup \mathcal{D}_{III} \cup \{\mathbb{R} + iv\}\}$  but both uniformly away from 0,

$$|R(\lambda, \mu)| \leq \frac{C e^{-\alpha|\lambda|}}{|\lambda - \mu|} e^{w(|\Im(\lambda)| + |\Im(\mu)|)}, \quad (\text{C.62})$$

for some  $\alpha > 0$ .

## D Auxiliary Lemma

**Lemma D.1.** *Given  $\sigma, v > 0$  and  $r \in \mathbb{N}$  there exists  $C > 0$  such that one has the upper bound*

$$\int_{\mathbb{R} \pm i(\sigma+v)} dt \cdot \frac{[\ln(1 + |t|)]^r}{(1 + |t|) \cdot |k - t|} \leq C \cdot \frac{[\ln(1 + |k|)]^{r+1}}{1 + |k|}, \quad (\text{D.1})$$

for any  $k \in \mathbb{C}$  satisfying  $|\Im k| \leq v$ .

*Proof*—

First of all, by changing  $\Re(t) \leftrightarrow -\Re(t)$  under the integral, one may always assume that  $\Re(k) > 0$ . Furthermore, for  $|\Re(k)| < M$  for some fixed  $M$ , the integral is well-defined and the bound (D.1) is obvious. Hence, from now on, one may assume  $\Re(k)$  to be large enough.

Given  $t = u \pm i(\sigma + v)$ , one has

$$[\ln(1 + |t|)]^r \leq [\ln(1 + |u| + \sigma + v)]^r \leq \sum_{\ell=0}^r C_\ell^r [\ln(1 + |u|)]^\ell \cdot [\ln(1 + \sigma + v)]^{r-\ell}, \quad (\text{D.2})$$

with  $C_\ell^r$  being the binomial coefficients.

Given the same parameterisation for  $t$ , since

$$|k - t| \geq \frac{1}{3} \{\sigma + |x - u|\} \quad \text{where} \quad k = x + i\Im(k) \quad \text{as well as} \quad 1 + |t| \geq 1 + |u|, \quad (\text{D.3})$$

one gets the upper bound

$$\int_{\mathbb{R} \pm i(\sigma+v)} dt \cdot \frac{[\ln(1 + |t|)]^r}{(1 + |t|) \cdot |k - t|} \leq \sum_{\ell=0}^r 3C_\ell^r [\ln(1 + \sigma + v)]^{r-\ell} \mathcal{I}_\ell \quad \text{with} \quad \mathcal{I}_\ell = \int_{\mathbb{R}} \frac{du \cdot [\ln(1 + |u|)]^\ell}{(1 + |u|) \cdot (\sigma + |x - u|)}. \quad (\text{D.4})$$

Then, one may decompose  $\mathcal{I}_\ell$  as

$$\mathcal{I}_\ell = \underbrace{\int_{-\infty}^0 \frac{du \cdot [\ln(1-u)]^\ell}{(1-u) \cdot (\sigma+x-u)}}_{=\mathcal{I}_\ell^{(1)}} + \underbrace{\int_0^x \frac{du \cdot [\ln(1+u)]^\ell}{(1+u) \cdot (\sigma+x-u)}}_{=\mathcal{I}_\ell^{(2)}} + \underbrace{\int_x^{+\infty} \frac{du \cdot [\ln(1+u)]^\ell}{(1+u) \cdot (\sigma+u-x)}}_{=\mathcal{I}_\ell^{(3)}}. \quad (\text{D.5})$$

$\mathcal{I}_\ell^{(2)}$  may be estimated by direct bounds

$$\mathcal{I}_\ell^{(2)} \leq \frac{[\ln(1+x)]^\ell}{1+x+\sigma} \int_0^x du \cdot \left( \frac{1}{(1+u)} + \frac{1}{(\sigma+x-u)} \right) = \frac{[\ln(1+x)]^\ell}{1+x+\sigma} \cdot (\ln(1+x) - \ln \sigma + \ln(\sigma+x)). \quad (\text{D.6})$$

To estimate  $\mathcal{I}_\ell^{(3)}$  one first decomposes it as

$$\mathcal{I}_\ell^{(3)} = \int_0^{+\infty} \frac{du \cdot [\ln(1+u+x)]^\ell}{(1+u+x) \cdot (\sigma+u)} = \int_0^x du \frac{[\ln(1+u+x)]^\ell}{1+x-\sigma} \left( \frac{1}{\sigma+u} - \frac{1}{1+u+x} \right) + \int_0^{+\infty} \frac{du \cdot [\ln(1+u+2x)]^\ell}{(1+u+2x) \cdot (\sigma+u+x)}. \quad (\text{D.7})$$

The logarithmic term in the first integral may be bounded by  $[\ln(1+2x)]^\ell$  while, in the second integral one uses the bound valid for  $x$  large enough

$$\frac{1+2x+u}{\sigma+x+u} \leq \frac{1+2x}{\sigma+x}, \quad (\text{D.8})$$

so as to integrate only a function of the variable  $1+2x+u$ . Then, the identity

$$\int_0^{+\infty} du \frac{[\ln(A+u)]^\ell}{(A+u)^2} = \frac{1}{A} \sum_{p=0}^{\ell} \frac{\ell!}{p!} [\ln A]^p \quad (\text{D.9})$$

leads to

$$\mathcal{I}_\ell^{(3)} \leq \frac{[\ln(1+u+x)]^\ell}{1+x-\sigma} \left\{ \ln(\sigma+x) - \ln(1+2x) - \ln \sigma + \ln(1+x) \right\} + \sum_{p=0}^{\ell} \frac{\ell!}{p!} \frac{[\ln(1+2x)]^p}{\sigma+x}. \quad (\text{D.10})$$

By using analogous techniques, one may bound  $\mathcal{I}_\ell^{(1)}$  concluding that

$$\mathcal{I}_\ell^{(1)} = \mathcal{O}\left(\frac{[\ln(1+x)]^{\ell+1}}{1+x}\right). \quad (\text{D.11})$$

All in all, this entails the claim. ■

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