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ON THE STRUCTURE OF SOME p-ADIC PERIOD DOMAINS

MIAOFEN CHEN, LAURENT FARGUES AND XU SHEN

Abstract. We study the geometry of the p-adic analogues of the complex analytic period spaces first introduced by Griffiths. More precisely, we prove the Fargues-Rapoport conjecture for p-adic period domains: for a reductive group G over a p-adic field and a minuscule cocharacter μ of G, the weakly admissible locus coincides with the admissible one if and only if the Kottwitz set $B(G,\mu)$ is fully Hodge-Newton decomposable.

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Introduction

Starting with the work of Griffiths ([14]), the study of the geometry of complex analytic period mappings and their images, the so called period domains, has been a recurrent theme in complex algebraic geometry ([48] and [13] for example for a recent review). In this article we study the p-adic analogue of this problem starting with the questions asked by Grothendieck ([17]) and the partial answers given by Drinfeld ([18]), Serre-Tate and Katz ([39]), Gross-Hopkins ([16]), Rapoport-Zink ([53]) and Hartl ([36]).

Let F be a finite degree extension of $\mathbb{Q}_p$, G a connected reductive group over F, and $\{\mu\}$ the geometric conjugacy class of a minuscule cocharacter μ of G. Attached to the datum $(G,\{\mu\})$, we have the flag variety $\mathcal{F}(G,\mu)$ defined over the reflex field $E = E(G,\{\mu\})$, the field of definition of $\{\mu\}$, a finite degree extension of F. We will consider the associated adic space $\mathcal{F}(G,\mu)$ over $\breve{E}$, the completion of the maximal

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unramified extension of E. After fixing an element $b \in G(\breve{F})$, Rapoport and Zink constructed in [53] (see also [50, 12]) an open subspace

$$\mathcal{F}(G,\mu,b)^{wa}$$

inside $\mathcal{F}(G,\mu)$, which is called a p-adic period domain, as a vast generalization of Drinfeld upper half spaces ([18]). The name "p-adic period domain" comes as follows. For any finite extension $K|\breve{E}$, the points in $\mathcal{F}(G,\mu,b)^{wa}(K)$ correspond to weakly admissible filtered isocrystals equipped with a G-structure, which are then admissible (as $K|\breve{E}$ is finite) by a fundamental result in p-adic Hodge theory ([8]). We thus get crystalline representations with additional structures attached to points in $\mathcal{F}(G,\mu,b)^{wa}(K)$. Here in order to get non empty $\mathcal{F}(G,\mu,b)^{wa}$, we have to assume that b satisfies a certain condition with respect to $\{\mu\}$, cf. prop. 2.2 for some background on this.

If $K|\breve{E}$ is an arbitrary complete extension, then it is not clear whether we still get Galois representations attached to points in $\mathcal{F}(G,\mu,b)^{wa}(K)$. This led Rapoport and Zink to conjecture that there exists an open subspace

$$\mathcal{F}(G,\mu,b)^{a}$$

inside $\mathcal{F}(G,\mu,b)^{wa}$, such that there exists a p-adic local system with additional structures over $\mathcal{F}(G,\mu,b)^{a}$ which interpolates the crystalline representations attached to all classical points, cf. [12] conj. 11.4.4 and [36] conj. 2.3. Contrary to $\mathcal{F}(G,\mu,b)^{wa}$, it is difficult to give a direct construction (and explicit description) for the desired $\mathcal{F}(G,\mu,b)^{a}$.

For certain triples (G,μ,b) (those so called of PEL type), Hartl ([36, 35]) and Faltings ([20]) constructed the space $\mathcal{F}(G,\mu,b)^{a}$ by using the Robba ring $\widetilde{B}^{\dagger}_{rig}$ and the crystalline period ring B_{cris} respectively. If $K|\breve{E}$ is finite, we have

$$\mathcal{F}(G,\mu,b)^{a}(K) = \mathcal{F}(G,\mu,b)^{wa}(K)$$

as explained above. But it turns out that in general the inclusion

$$\mathcal{F}(G,\mu,b)^{a} \subset \mathcal{F}(G,\mu,b)^{wa}$$

is strict.

By the recent progress in p-adic Hodge theory, thanks to the works of Fargues-Fontaine [29] and Fargues [25], we can now construct $\mathcal{F}(G,\mu,b)^{a}$ as in the Rapoport-Zink conjecture for any triple $(G,\{\mu\},b)$ (compatible with the constructions of Hartl and Faltings above), by using G-bundles on the Fargues-Fontaine curve, cf. def. 3.1. In [36] sec. 9 and [51] A.20, Hartl and Rapoport asked when we do have

$$\mathcal{F}(G,\mu,b)^{a} = \mathcal{F}(G,\mu,b)^{wa} \quad ?$$

For $G = \mathrm{GL}_n$, Hartl gave a complete solution to this question ([36] theo. 9.3).

Let $b \in G(\breve{F})$ be an element such that its associated σ-conjugacy class

$$[b] \in B(G,\mu)$$

is the unique *basic* element in the Kottwitz set $B(G,\mu)$ ([42] sec. 6). In this paper we prove the following theorem that was conjectured by Fargues and Rapoport ([32] conj. 0.1).

Theorem 0.1 (theo. 6.1)**.** *The equality $\mathcal{F}(G,\mu,b)^{wa} = \mathcal{F}(G,\mu,b)^{a}$ holds if and only if $B(G,\mu)$ is fully Hodge-Newton decomposable.*

Recall that, roughly, the set $B(G,\mu)$ is fully Hodge-Newton decomposable if for any non basic element of $B(G,\mu)$ its Newton polygon, seen as an element of a positive Weyl chamber, touches the Hodge polygon defined by μ outside its extremities.

We refer to section 4 and more precisely 4.3 for the meaning of this notion. In [32] theorem 2.5 there is a purely group theoretical classification of all the fully Hodge-Newton decomposable pairs $(G, \{\mu\})$, and in loc. cit. theo. 2.3 one can find further equivalent conditions for $(G, \{\mu\})$ being fully Hodge-Newton decomposable. In the following, Hodge-Newton will be usually abbreviated to HN for simplicity.

To prove the theorem, we make intensive use of the theory of G-bundles on the Fargues-Fontaine curve ([25]). More precisely, let $C|\breve{E}$ be a complete algebraically closed extension and $X_{C^\flat}$ be the Fargues-Fontaine curve attached to $C^\flat$ equipped with a closed point ∞ with residue field C. Let us recall that the main theorem of [25] says that

$$\begin{aligned} B(G) &\xrightarrow{\sim} H^1_{\text{ét}}(X, G) \\ [b'] &\longmapsto [\mathcal{E}_{b'}]. \end{aligned}$$

To each point $x \in \mathcal{F}(G, \mu)(C)$, we can attach a modified G-bundle at ∞ of $\mathcal{E}_b$

$$\mathcal{E}_{b,x}$$

on $X_{C^\flat}$. We assume that G is quasi-split to simplify. Then we can use the notion of a semi-stable G-bundle and the theory of Harder-Narasimhan reduction on $X_{C^\flat}$ to study the modification $\mathcal{E}_{b,x}$ and the geometry of $\mathcal{F}(G, \mu)$. The subspace $\mathcal{F}(G, \mu, b)^a$ is defined as the locus where $\mathcal{E}_{b,x}$ is a semi-stable G-bundle. The isomorphism class of $\mathcal{E}_{b,x}$ defines a stratification of $\mathcal{F}(G, \mu)$ indexed by another Kottwitz set $B(J_b, \mu^{-1})$ (cf. sec. 5). We prove that this other Kottwitz set is fully HN decomposable if and only if $B(G, \mu)$ is fully HN decomposable (coro. 4.16). To compare $\mathcal{F}(G, \mu, b)^a$ with the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$, we also need to describe $\mathcal{F}(G, \mu, b)^{wa}$ in terms of G-bundles on $X_{C^\flat}$. This is given in proposition 2.7: x is weakly admissible if and only if for any parabolic reduction $(\mathcal{E}_{b,x})_P$ coming from a reduction of b to the parabolic subgroup P, the usual numerical semi-stability condition is satisfied. With these ingredients at hand, we can prove the Fargues-Rapoport conjecture by comparing parabolic reductions of $\mathcal{E}_{b,x}$ with the parabolic reductions coming from a reduction of b.

The reader who wants to have a feeling of how this type of proof works in a particular case can read the case of $SO(2, n-2)$ treated in [27] (see the appendix of [59]). This case is instructive and served as a starting point for the general proof of the implication that fully HN decomposability implies $\mathcal{F}(G, \mu, b)^a = \mathcal{F}(G, \mu, b)^{wa}$. It gives in particular the computation of the p-adic period space of K3 surfaces with supersingular reduction. We also deal with the case of GL_n in remark 6.2.

We briefly describe the structure of this article. In section 1, we review the basic facts about G-bundles on the Fargues-Fontaine curve which we will use. In section 2, we review the definition of the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$. In the quasi-split case, we can give an equivalent definition for $\mathcal{F}(G, \mu, b)^{wa}$ by using the theory of G-bundles on the Fargues-Fontaine curve. In section 3, we give the definition of the admissible locus $\mathcal{F}(G, \mu, b)^a$ by using semi-stable G-bundles on the Fargues-Fontaine curve. In section 4, we describe a generalized Kottwitz set for general groups, and we also discuss the fully HN decomposability condition and related properties that will be used in the sequel. In section 5, we first explain the twin towers principle which is an important tool in the proof of the main theorem. Then we construct the Harder-Narasimhan stratification of the flag variety $\mathcal{F}(G, \mu)$ and describe each stratum. With all these preparations, we finally prove the Fargues-Rapoport conjecture in section 6. Finally, in section 7, we discuss the

asymptotic geometry of period spaces. We introduce in particular a new conjecture (7.2) saying that $\mathcal{F}(G,\mu,b)^a = \mathcal{F}(G,\mu,b)^{wa}$ if and only if there exists a quasicompact fundamental domain for the action of $J_b(F)$ on $\mathcal{F}(G,\mu,b)^a$. Moreover we give a cohomological application of some results obtained during the proof of the main theorem 6.1. In fact, when $B(G,\mu)$ is fully HN decomposable, the boundary of $\mathcal{F}(G,\mu,b)^a$ is parabolically induced (prop. 7.1). One should then deduce that in this case the "supercuspidal part" of the compactly supported cohomology and the cohomology without support coincide (conj. 7.3). Conjecturally this is only true when one restricts to supercuspidal L-packets (a cleanliness assumption about supercuspidal L-packets showing up in the geometrization conjecture formulated by the second author) but in this case this is true for *any* supercuspidal representation. This should have consequences on the realization of the local Langlands correspondence for supercuspidal representations in the cohomology of those period spaces.

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Notations

We use the following notations:

- F is a finite degree extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$.
- $\overline{F}$ is an algebraic closure of F and $\Gamma = \mathrm{Gal}(\overline{F}|F)$.
- $\breve{F} = \widehat{F^{un}}$ is the completion of the maximal unramified extension F^{un} of F with Frobenius σ.
- G is a connected reductive group over F.
- H is a quasi-split inner form of G equipped with an inner twisting $G_{\overline{F}} \xrightarrow{\sim} H_{\overline{F}}$.
- $\boldsymbol{A} \subset \boldsymbol{T} \subset \boldsymbol{B}$ are a maximal split torus, $\boldsymbol{T} = C_H(\boldsymbol{A})$ a minimal Levi and $\boldsymbol{B}$ a Borel subgroup in H. Note that $\boldsymbol{T}$ is a maximal torus of H over F (cf. [62] page 271). We reserve the notation T for a maximal torus in G.
- $(X^*(\boldsymbol{T}), \Phi, X_*(\boldsymbol{T}), \Phi^\vee)$ is the root datum given by the adjoint action of T with positive roots Φ^+ and simple roots Δ with respect to the choice of $\boldsymbol{B}$. Denote by W the absolute Weyl group of $\boldsymbol{T}$ in H.
- $(X^*(\boldsymbol{A}), \Phi_0, X_*(\boldsymbol{A}), \Phi_0^\vee)$ is the root datum given by the adjoint action of A with positive roots Φ_0^+ and simple (reduced) roots Δ_0.
- If M is a standard Levi subgroup in H with respect to the preceding choice of Borel subgroup, we note by a subscript M, for example Φ_M, the corresponding roots or coroots showing up in $\mathrm{Lie}\, M$. For example $M \mapsto \Delta_{0,M}$ induces a bijection between the standard Levi subgroups and subsets of Δ_0.
- If $\mathbb{D}$ is the slope pro-torus with characters $X^*(\mathbb{D}) = \mathbb{Q}$, we set
$$\mathcal{N}(G) = \left[\mathrm{Hom}(\mathbb{D}_{\overline{F}}, G_{\overline{F}}) \,/\, G(\overline{F})\text{-conjugacy}\right]^\Gamma,$$
the Newton chamber. Via the inner twisting between G and H, there is an identification
$$\mathcal{N}(G) = \mathcal{N}(H) = X_*(\boldsymbol{A})^+_{\mathbb{Q}}.$$
This is equipped with the usual order $\nu_1 \leq \nu_2$ if and only if $\nu_2 - \nu_1 \in \mathbb{Q}_{\geq 0}\Phi_0^+$.
- $\pi_1(H) = X_*(\boldsymbol{T})/\langle \Phi^\vee \rangle$ is the algebraic fundamental group of H, and $\pi_1(H)_\Gamma$ is its Galois coinvariant. Via the inner twisting between G and H, there are identifications
$$\pi_1(G) = \pi_1(H), \quad \pi_1(G)_\Gamma = \pi_1(H)_\Gamma.$$

- G_{ad} is the adjoint group associated to G, $G_{der} \subset G$ is the derived subgroup, and $G_{sc} \to G_{der}$ is the simply connected cover of G_{der}.

1. G-bundles on the Fargues-Fontaine curve

In this section, we review some basic facts about the Fargues-Fontaine curve and G-bundles on it, cf. [29, 25]. This theory will be the basic tool for our study of p-adic period domains. We change slightly the notations from [29] and [25] to be in accordance with [51].

1.1. **The Fargues-Fontaine curve.** The Fargues-Fontaine curve X over F is associated to the choice of a characteristic p perfectoid field $K|\mathbb{F}_q$. We note π_F for a uniformising element of F. It has several incarnations.

1.1.1. *The adic curve.* The adic curve X^{ad} admits the following adic uniformization

$$X^{ad} = Y/\varphi^{\mathbb{Z}},$$

where $Y = \mathrm{Spa}(W_{\mathcal{O}_F}(\mathcal{O}_K)) \setminus V(\pi_F[\varpi_K])$, with $\varpi_K \in K$ satisfying $0 < |\varpi_K| < 1$. The action of the Frobenius φ on the ramified Witt vectors is given by

$$\varphi\Big(\sum_{n\geq 0}[x_n]\pi_F^n\Big) = \sum_{n\geq 0}[x_n^q]\pi_F^n.$$

It induces a totally discontinuous action on Y without fixed point.

1.1.2. *The algebraic curve.* There is a natural line bundle $\mathcal{O}(1)$ on X^{ad}, corresponding to the φ-equivariant line bundle on Y whose underlying line bundle is trivial and for which the Frobenius is $\pi_F^{-1}\varphi$. Set $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, and

$$P = \bigoplus_{n\geq 0} H^0(X^{ad}, \mathcal{O}(n)).$$

We have

$$H^0(X^{ad}, \mathcal{O}(n)) = \mathcal{O}(Y)^{\varphi=\pi_F^n}.$$

Let

$$X = \mathrm{Proj}(P).$$

By [29], this is a one dimensional noetherian regular scheme over F. There exists a morphism of ringed spaces

$$X^{ad} \longrightarrow X,$$

and X^{ad} may be viewed as the analytification of X in some generalized sense. As for rigid spaces, there is a natural subset $|X^{ad}|^{cl} \subset |X^{ad}|$ of "classical Tate points" and the preceding induces a bijection

$$|X^{ad}|^{cl} \xrightarrow{\sim} |X|,$$

where $|X|$ denotes the closed points of X.

Let F be fixed. Suppose that instead of beginning with the characteristic p datum $K|\mathbb{F}_q$ we start with $K^\sharp|F$ a perfectoid field and set $K = K^{\sharp,\flat}$. Then, the curve X is equipped with a closed point ∞ with residue field $k(\infty) = K^\sharp$ and $\widehat{\mathcal{O}}_{X,\infty} = B_{dR}^+(K^\sharp)$, cf. [29]. Presently K will be supposed to be algebraically closed (we will test the semi-stability of some vector bundles by "specializing at a geometric point"). This is equivalent to $K^\sharp$ being algebraically closed.

1.2. **G-bundles.** Let Bun_X and $\mathrm{Bun}_{X^{ad}}$ be the categories of vector bundles on X and X^{ad} respectively. The morphism $X^{ad} \to X$ induces a GAGA functor

$$\mathrm{Bun}_X \longrightarrow \mathrm{Bun}_{X^{ad}}.$$

Theorem 1.1 ([24, 40])**.** *The GAGA functor induces an equivalence of categories*

$$\mathrm{Bun}_X \xrightarrow{\sim} \mathrm{Bun}_{X^{ad}}.$$

We assume from now on that K is algebraically closed. For example, $K = C^\flat$ with $C|\overline{F}$ a complete algebraically closed field. Let

$$\varphi\text{–}\mathrm{mod}_{\breve{F}}$$

be the category of isocrystals relative to $\breve{F}|F$, that is to say couples (D, φ) where D is a finite dimensional $\breve{F}$-vector space together with a σ-linear automorphism φ, where σ is the Frobenius of $\breve{F}$ that reduces to Frob_q. For any $(D, \varphi) \in \varphi\text{–}\mathrm{mod}_{\breve{F}}$, we can construct a vector bundle $\mathcal{E}(D, \varphi)$ on X associated to the graded P-module

$$\bigoplus_{n \geq 0} \left(D \otimes_{\breve{F}} \mathcal{O}(Y)\right)^{\varphi \otimes \varphi = \pi_F^n}.$$

Via GAGA this corresponds to the vector bundle $Y \underset{\varphi^{\mathbb{Z}}}{\times} D$ on $X^{ad} = Y/\varphi^{\mathbb{Z}}$.

Theorem 1.2 ([29])**.** *The functor $\mathcal{E}(-) : \varphi\text{–}\mathrm{mod}_{\breve{F}} \longrightarrow \mathrm{Bun}_X$ is essentially surjective.*

We will need the following fact:

Theorem 1.3 ([29])**.** *The degree map of a line bundle induces an isomorphism*

$$\deg : \mathrm{Pic(X)} \xrightarrow{\sim} \mathbb{Z}.$$

Note that the fact the degree of a line bundle is well defined is not evident. This is a consequence of the fact that the curve is "complete": the degree of a principal divisor is zero. This is essential to develop a theory of Harder-Narasimhan reduction.

We have the following two equivalent definitions of a G-bundle on X:

(1) an exact tensor functor $\mathrm{Rep}\, G \to \mathrm{Bun}_X$, where $\mathrm{Rep}\, G$ is the category of rational algebraic representations of G,
(2) a G-torsor on X locally trivial for the étale topology.

Recall that an isocrystal with G-structure is an exact tensor functor

$$\mathrm{Rep}\, G \longrightarrow \varphi\text{–}\mathrm{mod}_{\breve{F}}.$$

Let $B(G)$ be the set of σ-conjugacy classes in $G(\breve{F})$ ([41, 42, 52]). Any $b \in G(\breve{F})$ defines an isocystal with G-structure

$$\begin{aligned} \mathcal{F}_b : \mathrm{Rep}\, G &\longrightarrow \varphi\text{–}\mathrm{mod}_{\breve{F}} \\ V &\longmapsto (V_{\breve{F}}, b\sigma). \end{aligned}$$

Its isomorphism class depends only on the σ-conjugacy class $[b] \in B(G)$ of b. Conversely, any isocrystal with G-structure arises in this way. Thus $B(G)$ is the set of isomorphism classes of isocrystals with G-structure ([52] rem. 3.4 (i)). For $b \in G(\breve{F})$, let $\mathcal{E}_b$ be the composition of the above functor $\mathcal{F}_b$ and

$$\mathcal{E}(-) : \varphi\text{–}\mathrm{mod}_{\breve{F}} \longrightarrow \mathrm{Bun}_X.$$

In this way, the set $B(G)$ also classifies G-bundles on X. In fact, we have

Theorem 1.4 ([25],[1]). *There is a bijection of pointed sets*

$$B(G) \xrightarrow{\sim} H^1_{\acute{e}t}(X, G)$$
$$[b] \longmapsto [\mathcal{E}_b].$$

1.3. **The Harder-Narasimhan reduction in the quasi-split case.** We assume that $G = H$ is quasi-split in this subsection. The theory of Harder-Narasimhan reduction ([3] for example) applies for G-bundles over the Fargues-Fontaine curve X ([25] sec. 5.1). If $G' \subset G$ is a closed subgroup, then a reduction of a G-bundle $\mathcal{E}$ to G' is a G'-bundle $\mathcal{E}_{G'}$ together with an isomorphism

$$\mathcal{E}_{G'} \overset{G'}{\times} G \xrightarrow{\sim} \mathcal{E}.$$

Recall the following definition of a semi-stable G-bundle, see for example [49] and [3].

Definition 1.5. *Let $\mathcal{E}$ be a G-bundle on X. It is called semi-stable if for any standard parabolic subgroup P of G, any reduction $\mathcal{E}_P$ of $\mathcal{E}$ to P, and any $\chi \in X^*(P/Z_G)^+$, we have*

$$\deg \chi_* \mathcal{E}_P \leq 0.$$

Let $M \subset P$ be the associated standard Levi subgroup of G, which we will identify with P/R_uP with R_uP the unipotent radical of P. Then as $X^*(P) = X^*(M)$, we have

$$\chi_* \mathcal{E}_P = \chi_*(\mathcal{E}_P \overset{P}{\times} M).$$

One proves, as usual, that $\mathcal{E}$ is semistable if and only if the associated adjoint bundle

$$\mathrm{Ad}(\mathcal{E}) := \mathcal{E} \overset{G,\mathrm{Ad}}{\times} \mathrm{Lie}\, G$$

is semi-stable. We will later use the following criterion.

Lemma 1.6. *The following are equivalent:*

(1) *The G-bundle $\mathcal{E}$ on X is semi-stable.*
(2) *For any standard parabolic subgroup P and any reduction $\mathcal{E}_P$ to P, one has*

$$\deg(\mathcal{E}_P \overset{P,\mathrm{Ad}}{\times} \mathrm{Lie}\, G/\mathrm{Lie}\, P) \geq 0.$$

(3) *The same holds for any maximal standard parabolic subgroup.*

Proof. This is a consequence of the fact that for any standard parabolic subgroup P, the adjoint representation of $\boldsymbol{T}$ on $\det(\mathrm{Lie}\, G/\mathrm{Lie}\, P)^{-1}$ lies in $X^*(\boldsymbol{T})^+$, and moreover as P runs through the set of maximal parabolic subgroups those are positive multiples of the fundamental weights. More precisely, if M is the standard Levi attached to P and $\rho_M = \frac{1}{2}\sum_{\alpha \in \Phi(\boldsymbol{T})^+_M} \alpha$, then for $\beta \in \Delta_M$, $\langle \beta^\vee, \rho_M \rangle = 1$ and for $\beta \in \Delta \backslash \Delta_M$, $\langle \beta^\vee, \rho_M \rangle \leq 0$. One concludes using that we have $\det(\mathrm{Lie}\, G/\mathrm{Lie}\, P)^{-1} = 2\rho - 2\rho_M$. □

One can rephrase this lemma in a more geometric way. In fact, if the reduction $\mathcal{E}_P$ corresponds to the section s of $P\backslash\mathcal{E} \to X$, then

$$\mathcal{E}_P \overset{P,\mathrm{Ad}}{\times} \mathrm{Lie}\, G/\mathrm{Lie}\, P = s^* T(P\backslash\mathcal{E})$$

is the pullback of the tangent bundle.

For a general G-bundle we have the following theorem.

Theorem 1.7. *Let $\mathcal{E}$ be a G-bundle on X. Then there exists a unique standard parabolic subgroup P of G and a unique reduction $\mathcal{E}_P$ to P, such that*

(1) *the associated M-bundle $\mathcal{E}_P \overset{P}{\times} M$ is semi-stable,*
(2) *for any $\chi \in X^*(P/Z_G) \setminus \{0\} \cap \mathbb{N}.\Delta$, we have $\deg \chi_* \mathcal{E}_P > 0$.*

Proof. See [25] 5.1, where one can apply the arguments of [3]. □

The reduction $\mathcal{E}_P$ in the above theorem is called the Harder-Narasimhan reduction of $\mathcal{E}$. Let $\mathcal{E}$ be a G-bundle on X with Harder-Narasimhan reduction $\mathcal{E}_P$. Then we get an element

$$\nu_{\mathcal{E}} \in X_*(\boldsymbol{A})_{\mathbb{Q}} = X_*(\boldsymbol{T})^{\Gamma}_{\mathbb{Q}}$$

by the Galois invariant morphism

$$\begin{aligned} X^*(P) &\longrightarrow \mathbb{Z} \\ \chi &\longmapsto \deg \chi_* \mathcal{E}_P \end{aligned}$$

and the inclusion

$$\mathrm{Hom}_{\mathbb{Z}}(X^*(P), \mathbb{Z}) \subset X_*(\boldsymbol{T})_{\mathbb{Q}}.$$

In fact, we have $\nu_{\mathcal{E}} \in X_*(\boldsymbol{A})^+_{\mathbb{Q}}$, and moreover M is the centralizer of $\nu_{\mathcal{E}}$. As in the classical theory of G-bundles on curves, the vector $\nu_{\mathcal{E}}$ is called the Harder-Narasimhan polygon of $\mathcal{E}$ as an element

$$\nu_{\mathcal{E}} \in \mathcal{N}(G) = X_*(\boldsymbol{A})^+_{\mathbb{Q}}.$$

Later we will need the following. Recall that if $\mathcal{E}$ is a vector bundle on X with Harder-Narasimhan filtration $(0) = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$ then:

(1) for any subbundle $\mathcal{F} \subset \mathcal{E}$, the point $(\mathrm{rk}\,\mathcal{F}, \deg \mathcal{F})$ lies on or under the Harder-Narasimhan polygon of $\mathcal{E}$,
(2) if $(\mathrm{rk}\,\mathcal{F}, \deg \mathcal{F})$ lies on this polygon, then there exists an index i such that $\mathcal{E}_i \subset \mathcal{F} \subset \mathcal{E}_{i+1}$.

Here is the generalization we will need.

Theorem 1.8 ([55] theo. 4.5.1)**.** *For $\mathcal{E}$ a G-bundle on X equipped with a reduction $\mathcal{E}_Q$ to the standard parabolic subgroup Q consider the vector*

$$\begin{aligned} v : X^*(Q) &\longrightarrow \mathbb{Z} \\ \chi &\longmapsto \deg \chi_* \mathcal{E}_Q \end{aligned}$$

seen as an element of $X_*(\boldsymbol{A})_{\mathbb{Q}}$. Then

(1) *One has $v \leq \nu_{\mathcal{E}}$.*
(2) *Moreover if this inequality is an equality, and $\mathcal{E}_P$ is the canonical reduction of $\mathcal{E}$, then $Q \subset P$ and $\mathcal{E}_P \simeq \mathcal{E}_Q \overset{Q}{\times} P$.*

As a corollary the vector $\nu_{\mathcal{E}}$ can be defined as being the supremum of all such vectors v associated to all possible reductions $\mathcal{E}_Q$ in the poset $X_*(\boldsymbol{A})_{\mathbb{Q}}$.

1.4. **The Harder-Narasimhan polygon in general and the first Chern class.** Suppose now that G is not necessarily quasi-split. Recall that the inner twisting $G_{\overline{F}} \xrightarrow{\sim} H_{\overline{F}}$ induces an identification

$$\mathcal{N}(G) \xrightarrow{\sim} \mathcal{N}(H) = X_*(\boldsymbol{A})^+_{\mathbb{Q}}.$$

There is a canonical $\mathbb{D}$-torsor $\mathscr{T}$ on $X \otimes_F F^{\mathrm{un}}$. More precisely, if $F_h|F$ is the degree h unramified extension, we have

$$X_F \otimes_F F_h = X_{F_h},$$

the curve attached to F_h. If $\pi_h : X_{F_h} \to X_F$ is the natural projection, one has a canonical identification

$$\pi_h^* \mathcal{O}_{X_F}(1) = \mathcal{O}_{X_{F_h}}(1)^{\otimes h}.$$

The compatible system of $\mathbb{G}_m$-torsors attached to $\big(\mathcal{O}(X_{F_h})(1)\big)_{h\geq 1}$ defines $\mathscr{T}$. As a consequence of the classification theorem of [25], any G-torsor on $X_F \otimes_F F^{un}$ has a reduction to a torus and we have

Theorem 1.9. *The pushforward of the $\mathbb{D}$-torsor $\mathscr{T}$ induces a bijection*

$$\mathrm{Hom}(\mathbb{D}_{F^{\mathrm{un}}}, G_{F^{\mathrm{un}}})/G(F^{\mathrm{un}}) \xrightarrow{\sim} H^1_{\acute{e}t}(X_F \otimes_F F^{un}, G).$$

Together with the pullback from X to $X_F \otimes_F F^{\mathrm{un}}$, this defines a map

$$\begin{aligned} H^1_{\text{ét}}(X, G) &\longrightarrow \mathcal{N}(G) \\ [\mathcal{E}] &\longmapsto \nu_{\mathcal{E}}. \end{aligned}$$

One can moreover define the G-equivariant first Chern class of $\mathcal{E}$ as a map ([25])

$$c_1^G : H^1_{\text{ét}}(X, G) \longrightarrow \pi_1(G)_\Gamma.$$

This generalizes the degree of a vector bundle for $G = \mathrm{GL}_n$. The quickest way to define it is through abelianized cohomology in the topos $X_{\text{ét}}$ ([6]) via the map

$$H^1_{\text{ét}}(X, G) \longrightarrow H^1_{\text{ét}}(X, [G_{sc} \to G]),$$

the homotopy equivalence $[T_{sc} \to T] \to [G_{sc} \to G]$ for a maximal torus T in G, and the canonical isomorphism

$$X_*(S)_\Gamma \xrightarrow{\sim} H^1_{\text{ét}}(X, S)$$

for a torus S ([25] theo. 2.8) (see the next subsection for more explanations in the case of $B(G)$ for this type of abelianization construction).

1.5. **Newton map and Kottwitz map.** We keep the same notations. The set $B(G)$ of σ-conjugacy classes in $G(\breve{F})$ can be described by two invariants. One invariant is the Newton map ([41] sec. 4). For each $b \in G(\breve{F})$ on can attach a slope morphism

$$\nu_b : \mathbb{D}_{\breve{F}} \longrightarrow G_{\breve{F}}.$$

Up to σ-conjugating b one can suppose that it is defined over F^{un} (the Tannakian category of isocrystals has a fiber functor over F^{un}). Moreover, since $\nu_b^\sigma = b^{-1}\nu_b b$, its conjugacy class is defined over F. We thus obtain a map

$$\begin{aligned} \nu : B(G) &\longrightarrow \mathcal{N}(G) \\ [b] &\longmapsto [\nu_b]. \end{aligned}$$

The other invariant is the Kottwitz map ([52] 1.15, [42] 4.9 and 7.5)

$$\begin{aligned} \kappa : B(G) &\longrightarrow \pi_1(G)_\Gamma \\ [b] &\longmapsto \kappa([b]). \end{aligned}$$

In the following we will simply write $\kappa([b])$ as $\kappa(b)$. The quickest way to define the Kottwitz map is via the abelianization of Kottwitz set à la Borovoï. More precisely, if we set

$$B_{ab}(G) := H^1(W_F, [G_{sc}(\overline{\breve{F}}) \to G(\overline{\breve{F}})])$$

(cohomology with coefficient in a crossed module where $G(\overline{\breve{F}})$ is in degree 0) there is an abelianization map

$$B(G) \longrightarrow B_{ab}(G)$$

induced by $[1 \to G(\overline{\breve{F}})] \to [G_{sc}(\overline{\breve{F}}) \to G(\overline{\breve{F}})]$. Moreover, if T is a maximal torus in G then $[T_{sc} \to T] \to [G_{sc} \to G]$ is an homotopy equivalence and this induces an isomorphism

$$H^1(W_F, [T_{sc}(\overline{\breve{F}}) \to T(\overline{\breve{F}})]) \xrightarrow{\sim} B_{ab}(G).$$

Now the left member is identified with the cokernel of

$$B(T_{sc}) \longrightarrow B(T)$$

which, according to Kottwitz, is the same as the cokernel of

$$X_*(T_{sc})_\Gamma \longrightarrow X_*(T)_\Gamma$$

that is to say $\pi_1(G)_\Gamma$. This is canonically defined independently of the choice of T since the Weyl group of T acts trivially on the cokernel of $X_*(T_{sc}) \to X_*(T)$.

The induced map

$$(\nu, \kappa) : B(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_\Gamma$$

is injective ([42] 4.13). Let $B(G)_{basic} \subset B(G)$ be the subset of basic elements. Then the restriction of κ to $B(G)_{basic}$ induces a bijection

$$\kappa : B(G)_{basic} \xrightarrow{\sim} \pi_1(G)_\Gamma$$

([41] prop. 5.6, [52] theo. 1.15). In view of Theorem 1.4, the two maps ν and κ have the following geometric interpretations in terms of G-bundles.

Theorem 1.10 ([25] prop. 6.6 and prop. 8.1)**.** *We have*

(1) $\nu_{\mathcal{E}_b} = w_0(-[\nu_b])$, *where* w_0 *is the element of longest length in the Weyl group acting on* $X_*(\boldsymbol{A})_{\mathbb{Q}}$, *and* $[\nu_b]$ *is the representative of the conjugacy class of* ν_b *that lies in* $X^*(\boldsymbol{A})^+_{\mathbb{Q}}$,

(2) $c_1^G(\mathcal{E}_b) = -\kappa(b)$.

A particular case of the above (1) says that $\mathcal{E}$ *is semi-stable if and only if* $[b_{\mathcal{E}}] \in B(G)$ *is basic.*

Example 1.11. *If* G_{der} *is simply connected one has*

$$\begin{tikzcd} \{G\text{-bundles}/X\}/\sim \arrow[r, "\text{push forward}"] \arrow[rr, bend right, "c_1^G"'] & \{G/G_{der}\text{-bundles}/X\}/\sim \arrow[r, "-\kappa", "\sim"'] & \pi_1(G)_\Gamma \end{tikzcd}$$

where, when G *is quasi-split, the first map is a bijection when restricted to semi-stable bundles.*

2. The weakly admissible locus

2.1. **Background.** Let $\{\mu\}$ be a geometric conjugacy class of a cocharacter $\mu : \mathbb{G}_m \to G_{\overline{F}}$. Unless clearly stated otherwise μ is supposed to be minuscule. The pair $(G, \{\mu\})$ will be fixed in the rest of this paper. We will sometimes see $\{\mu\}$ as an element of $X_*(\boldsymbol{T})^+$ that we again denote μ by abuse of notation. We get the following associated objects:

- the local reflex field $E = E(G, \{\mu\})$, a finite extension of F inside $\overline{F}$, which is the field of definition of $\{\mu\}$,
- the flag variety $\mathcal{F}(G, \mu)$ over $\breve{E}$, a twisted form of $G_{\overline{\breve{E}}}/P_\mu$, that classifies parabolic subgroups of type μ, where $\breve{E}$ is the completion of the maximal unramified extension of E (in fact $\mathcal{F}(G, \mu)$ is defined over E, but we will not need this fact),
- and the Kottwitz set

$$B(G, \mu) = \{[b] \in B(G) | [\nu_b] \leq \mu^\diamond, \kappa(b) = \mu^\sharp\},$$

which is a finite subset of $B(G)$; as usual we have put

$$\mu^\diamond = [\Gamma : \Gamma_\mu]^{-1} \sum_{\tau \in \Gamma/\Gamma_\mu} \mu^\tau \in X_*(\boldsymbol{A})^+_{\mathbb{Q}} = \big(X_*(\boldsymbol{T})^+_{\mathbb{Q}}\big)^\Gamma,$$

and $\mu^\sharp \in \pi_1(G)_\Gamma$ is the image of μ via $\pi_1(G) = X_*(\boldsymbol{T})/X_*(\boldsymbol{T}_{sc})$.

In the preceding definition of $B(G,\mu)$, the condition $[\nu_b] \leq \mu^\diamond$ implies that $\kappa(b) - \mu^\sharp \in \pi_1(G)_{\Gamma,tor} = H^1(F,G)$. The condition $\kappa(b) = \mu^\sharp$ requires that this cohomology class is trivial.

In the following we will also denote by $\mathcal{F}(G,\mu)$ the associated adic space over $\mathrm{Spa}(\breve{E})$. Fix an element

$$[b] \in B(G).$$

We get the reductive group J_b over F, the σ-centralizer of b, which only depends on $[b]$ up to isomorphism. For any F-algebra R, we have

$$J_b(R) = \{g \in G(\breve{F} \otimes_F R) |\, gb\sigma(g)^{-1} = b\}.$$

The group $J_b(F)$ acts naturally on the flag variety $\mathcal{F}(G,\mu)$ over $\breve{E}$ via $J_b(F) \subset G(\breve{F})$. For the triple $(G,\{\mu\},[b])$ as above, let us recall the definition of the weakly admissible locus ([53] chap. 1)

$$\mathcal{F}(G,\mu,b)^{wa} \subset \mathcal{F}(G,\mu).$$

First we recall the definition of a weakly admissible filtered isocrystal. Let $K|\breve{F}$ be a complete field extension. A filtered isocrystal $\mathcal{V} = (V,\varphi,\mathrm{Fil}^\bullet V_K)$ is called weakly admissible if for any subobject $\mathcal{V}' = (V',\varphi,\mathrm{Fil}^\bullet V'_K)$ of $\mathcal{V}$, with V' a φ-stable $\breve{F}$-subspace of V and $\mathrm{Fil}^\bullet V'_K = V'_K \cap \mathrm{Fil}^\bullet V_K$, we have

$$t_H(\mathcal{V}) = t_N(\mathcal{V}) \quad \text{and} \quad t_H(\mathcal{V}') \leq t_N(\mathcal{V}'),$$

where $t_N(\mathcal{V}) = v_{\pi_F}(\det\varphi)$ is the π_F-adic valuation of $\det\varphi$, and

$$t_H(\mathcal{V}) = \sum_{i\in\mathbb{Z}} i \cdot \dim_K gr^i_{\mathrm{Fil}^\bullet}(V_K).$$

Here the notation t_N, resp. t_H, means the terminal point of the Newton polygon, resp. the Hodge polygon. If we define the slope of $\mathcal{V}$ as

$$\mu(\mathcal{V}) = \frac{t_H(\mathcal{V}) - t_N(\mathcal{V})}{\dim V},$$

then $\mathcal{V}$ is weakly admissible if and only if it is semi-stable (for the slope function μ) and $\mu(\mathcal{V}) = 0$.

If $K|\breve{F}$ is a finite extension, then the category of weakly admissible filtered isocrystals is equivalent to the category of crystalline representations of $\mathrm{Gal}(\overline{K}|K)$ with coefficients in F whose Sen operator is F-linear ([8]); that is, weakly admissible is equivalent to admissible. We refer to chapter 10 of [29] for a proof of this result using the curve (this proof was a huge inspiration to study modifications of vector bundles on the curve).

Let us return to the general setting. Let $\mu' \in \{\mu\}$ be defined over $K|\breve{F}$. There is a functor

$$\begin{aligned} \mathcal{I}_{b,\mu'} : \mathrm{Rep}\, G &\longrightarrow \varphi\text{–}\mathrm{FilMod}_{K/\breve{F}} \\ (V,\rho) &\longmapsto (V_{\breve{F}}, \rho(b)\sigma, \mathrm{Fil}^\bullet_{\rho\circ\mu'} V_K). \end{aligned}$$

Let us recall that the category of weakly admissible filtered isocrystals is Tannakian.

Definition 2.1 ([53] def. 1.18)**.** *We call the pair (b,μ') weakly admissible if one of the following equivalent conditions is satisfied:*

(1) *for any $(V,\rho) \in \mathrm{Rep}\, G$, the filtered isocrystal $\mathcal{I}_{b,\mu'}(V,\rho)$ is weakly admissible;*
(2) *there is a faithful representation (V,ρ) of G such that $\mathcal{I}_{b,\mu'}(V,\rho)$ is weakly admissible.*

It is also equivalent to the condition that its image in G/G_{der} is weakly admissible (i.e. $[b] \in A(G,\mu)$, cf. prop. 2.2) and $\mathcal{I}_{b,\mu'}(\mathrm{Lie}\, G)$ is weakly admissible for the adjoint representation $\mathrm{Lie}\, G$ of G, cf. [12] def. 9.2.14 and coro. 9.2.26. We will give an equivalent geometric definition of weakly admissible in terms of G-bundles on the Fargues-Fontaine curve later.

Now for any point $x \in \mathcal{F}(G,\mu)(K)$ we have the associated filtration Fil_x of $\mathrm{Rep}\, G$ coming from a cocharacter $\mu_x \in \{\mu\}$ defined over K. Then x is called weakly admissible if the pair (b, μ_x) is weakly admissible in the above sense. By proposition 1.36 of [53], this defines a partially proper open subspace

$$\mathcal{F}(G,\mu,b)^{wa} \subset \mathcal{F}(G,\mu),$$

such that $\mathcal{F}(G,\mu,b)^{wa}(K)$ is the set of weakly admissible points in $\mathcal{F}(G,\mu)(K)$. It is of the form

$$\mathcal{F}(G,\mu) \setminus \bigcup_{i\in I} J_b(F).Z_i$$

where $(Z_i)_{i\in I}$ is a finite collection of Zariski closed Schubert varieties. The action of $J_b(F)$ on $\mathcal{F}(G,\mu)$ stabilizes the subspace $\mathcal{F}(G,\mu,b)^{wa}$.

Recall the following basic fact.

Proposition 2.2 ([54] prop. 3.1)**.** *The open subset $\mathcal{F}(G,\mu,b)^{wa}$ is non empty if and only if $[b] \in A(G,\mu) := \{[b] \in B(G) |\ [\nu_b] \leq \mu^\diamond\}$.*

As $B(G,\mu) \subset A(G,\mu)$ and we will be interested in the case $[b] \in B(G,\mu)$, our $\mathcal{F}(G,\mu,b)^{wa}$ will be non empty.

2.2. **Weak admissiblity and the curve.** The preceding definition of a weakly admissible point is Tannakian. We now give a geometric weak admissibility criterion in terms of the curve when G is quasi-split. Let $C|\overline{F}$ be algebraically closed complete and consider the curve attached to $C^\flat$ equipped with its point $\infty \in |X|$ with residue field C and

$$\mathrm{B}^+_{\mathrm{dR}} := \mathrm{B}^+_{\mathrm{dR}}(C) = \widehat{\mathcal{O}}_{X,\infty}.$$

Consider the C-points of the B_{dR}-affine Grassmannian attached to G ([57], [30], we only need its C-points, not the geometric diamond structure on it for what we do)

$$\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}}(C) := G(\mathrm{B}_{\mathrm{dR}})/G(\mathrm{B}^+_{\mathrm{dR}}).$$

Since we only consider the C-points of $\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}}$, the reader who believes in Zorn's lemma can fix an isomorphism $C[\![t]\!] \xrightarrow{\sim} \mathrm{B}^+_{\mathrm{dR}}$, that is to say a section of $\theta : \mathrm{B}^+_{\mathrm{dR}} \to C$. After such a choice this is reduced to the C-points of the "classical" affine Grassmannian. Hopefully we don't need this. In fact there is a canonical section over $\overline{F}$ and thus $\overline{F} \subset \mathrm{B}^+_{\mathrm{dR}}$ canonically, in particular we can define $\mu(t) \in G(\mathrm{B}^+_{\mathrm{dR}})$.

Choose $b \in G(\breve{F})$ that gives us the G-bundle $\mathcal{E}_b$. Its pullback via $\mathrm{Spec}(\mathrm{B}^+_{\mathrm{dR}}) \to X$ is canonically trivialized. For each $x \in \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}}(C)$ one can construct a modification

$$\mathcal{E}_{b,x}$$

of $\mathcal{E}_b$ à la Beauville-Laszlo ([9] 3.4.5, [24] 4.2, [26] 3.20). This is given by gluing $\mathcal{E}_{b|X\setminus\{\infty\}}$ and the trivial G-bundle on $\mathrm{Spec}(\mathrm{B}^+_{\mathrm{dR}})$ via the gluing datum given by x.

For $\mu \in X_*(\boldsymbol{T})^+$ the corresponding affine Schubert cell is

$$\mathrm{Gr}_{G,\mu}^{\mathrm{B}_{\mathrm{dR}}}(C) = G(\mathrm{B}^+_{\mathrm{dR}})\mu(t)^{-1}G(\mathrm{B}^+_{\mathrm{dR}})/G(\mathrm{B}^+_{\mathrm{dR}}) \subset \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}}(C).$$

For $x \in \mathrm{Gr}^{\mathrm{B_{dR}}}_{G,\mu}(C)$ one has (see [9] lemma 3.5.5)

$$\begin{aligned} c_1^G(\mathcal{E}_{b,x}) &= \mu^\sharp + c_1^G(\mathcal{E}_b) \\ &= \mu^\sharp - \kappa(b). \end{aligned}$$

Remark 2.3. *In terms of the stack* Bun_G *([26], [30]) the preceding formula gives the way the Hecke correspondence* $\mathrm{Hecke}_\mu \rightrightarrows \mathrm{Bun}_G$ *permute the factors in the decomposition* $\mathrm{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Bun}_G^\alpha$ *where* Bun_G^α *is the open/closed substack where* $c_1^G = \alpha$*. It says that*

$$\mathrm{Hecke}_\mu(\mathrm{Bun}_G^\alpha) = \mathrm{Bun}_G^{\alpha+\mu^\sharp}.$$

Recall that for any μ we have the *Bialynicki-Birula map* ([9] prop. 3.4.3 in general, we don't need its diamond version but just its evaluation on C-points)

$$\pi_{G,\mu} : \mathrm{Gr}^{\mathrm{B_{dR}}}_{G,\mu}(C) \longrightarrow \mathcal{F}(G,\mu)(C).$$

When μ is minuscule this is an isomorphism induced by applying θ via

$$G(\mathrm{B}^+_{\mathrm{dR}}) \cap \mu(t)^{-1} G(\mathrm{B}^+_{\mathrm{dR}}) \mu(t) = \{g \in G(\mathrm{B}^+_{\mathrm{dR}}) \mid \theta(g) \in P_\mu(C)\},$$

a parahoric subgroup in $G(\mathrm{B}^+_{\mathrm{dR}})$. Let us recall the following well-known lemma that is deduced from the properness of G/P together with the fact that X is a Dedekind scheme.

Lemma 2.4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two G-bundles on X with a modification $\mathcal{E}|_{X\setminus\{\infty\}} \xrightarrow{\sim} \mathcal{E}'|_{X\setminus\{\infty\}}$. Then for any parabolic subgroup P of G, we have a bijection*

$$\{\textit{Reductions of } \mathcal{E} \textit{ to } P\} \longrightarrow \{\textit{Reductions of } \mathcal{E}' \textit{ to } P\}.$$

We will need the following key definition.

Definition 2.5. *Let $b \in G(\breve{F})$ be an element. For a Levi subgroup M of G, a reduction of b to M is an element $b_M \in M(\breve{F})$ together with an element $g \in G(\breve{F})$, such that $b = g b_M \sigma(g)^{-1}$. Such a reduction (b_M, g) is considered to be equivalent to $(h b_M h^{-\sigma}, g h^{-1})$ for any $h \in M(\breve{F})$. We use the same notation and terminology for parabolic subgroups.*

The reductions of b to M and the reductions of $\mathcal{E}_b$ to M are almost the same. However, for the reductions to parabolic subgroups, the situation is different. Any reduction b_P of b to a parabolic subgroup P induces a reduction $\mathcal{E}_{b_P}$ of $\mathcal{E}_b$ to P. *But the converse is false and this is the main reason why in general the weakly admissibility and the admissibility conditions differ.* For example for GL_2, there is an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0,$$

although all sub-isocrystals of $(\breve{F}^2, \sigma \oplus \sigma)$ are isoclinic with slope 0. Let us notice that if M is a Levi subgroup of the parabolic subgroup P (semi-simplicity of the category of isocrystals)

$$B(M) \xrightarrow{\sim} B(P).$$

Thus, reductions of b to M or P are essentially the same.

Suppose now that μ is minuscule, b_M is a reduction of b to M with associated reduction b_P to P. For $x \in \mathcal{F}(G,\mu)(C)$ we deduce a reduction $(\mathcal{E}_{b,x})_P$ of $\mathcal{E}_{b,x}$ via lemma 2.4. This is a modification of $\mathcal{E}_{b_P}$. Now, we have the decomposition in Schubert cells of $\mathcal{F}(G,\mu)(C) = G(C)/P_\mu(C)$ according to the P-orbits

$$\mathcal{F}(G,\mu)(C) = \coprod_{w \in W_P \backslash W / W_{P_\mu}} \mathcal{F}(G,\mu)(C)^w$$

where, if $\dot{w} \in N_G(\boldsymbol{T})(C)$ has image w,

$$\begin{aligned}\mathcal{F}(G,\mu)(C)^w &= P(C)\dot{w}P_\mu(C)/P_\mu(C)\\ &= P(C)/P(C)\cap P_{\mu^w}(C).\end{aligned}$$

Projection to the Levi quotient induces an affine fibration

$$\mathrm{pr}_w : \mathcal{F}(G,\mu)(C)^w \longrightarrow \mathcal{F}(M,\mu^w)(C).$$

Lemma 2.6. *Suppose μ is minuscule. For $x \in \mathcal{F}(G,\mu)(C)^w$ there is an isomorphism*

$$(\mathcal{E}_{b,x})_P \overset{P}{\times} M \simeq \mathcal{E}_{b_M,\mathrm{pr}_w(x)}.$$

Proof. We use the Iwasawa decomposition

$$G(\mathrm{B}_{\mathrm{dR}}) = P(\mathrm{B}_{\mathrm{dR}})G(\mathrm{B}_{\mathrm{dR}}^+).$$

This induces an identification

$$P(\mathrm{B}_{\mathrm{dR}})/P(\mathrm{B}_{\mathrm{dR}}^+) \xrightarrow{\sim} G(\mathrm{B}_{\mathrm{dR}})/G(\mathrm{B}_{\mathrm{dR}}^+). \tag{1}$$

Now if $y \mapsto x$ via this bijection then

$$(\mathcal{E}_{b,x})_P = \mathcal{E}_{b_P,y}.$$

We use now that

$$G(\mathrm{B}_{\mathrm{dR}}^+) = \coprod_{w\in W_P\backslash W/W_{P_\mu}} P(\mathrm{B}_{\mathrm{dR}}^+)\dot{w}P_\mu(\mathrm{B}_{\mathrm{dR}}^+)$$

(this is deduced from the étaleness of the scheme $P_{F_\mu}\backslash G_{F_\mu}/P_\mu$ (coarse quotient) where F_μ is the field of definition of μ). Now, using the identification $\mathcal{F}(G,\mu)(C) = \mathrm{Gr}_{G,\mu}^{\mathrm{B}_{\mathrm{dR}}}(C)$, write

$$x = g\mu(t)^{-1}G(\mathrm{B}_{\mathrm{dR}}^+) \in \mathrm{Gr}_{G,\mu}^{\mathrm{B}_{\mathrm{dR}}}(C)$$

with $g \in G(\mathrm{B}_{\mathrm{dR}}^+)$. Write $g = a\dot{w}c$ with $a \in P(\mathrm{B}_{\mathrm{dR}}^+)$ and $c \in P_\mu(\mathrm{B}_{\mathrm{dR}}^+)$. We have

$$g\mu(t)^{-1} = \underbrace{a\mu^w(t)^{-1}}_{\in P(\mathrm{B}_{\mathrm{dR}})}\underbrace{\dot{w}\mu(t)c\mu(t)^{-1}}_{\in G(\mathrm{B}_{\mathrm{dR}}^+)}.$$

Thus via the bijection (1), $y \mapsto x$ with $y = a\mu^w(t)^{-1}$. Thus, $(\mathcal{E}_{b,x})_P = \mathcal{E}_{b_P,y}$ and the result is obtained by projection y from $P(\mathrm{B}_{\mathrm{dR}})$ to $M(\mathrm{B}_{\mathrm{dR}})$. □

We can now state the main result of this section.

Proposition 2.7. *Assume that G is quasi-split and $[b] \in A(G,\mu)$ with μ minuscule. Then $x \in \mathcal{F}(G,\mu)(C)$ is weakly admissible if and only if for any standard parabolic P with associated standard Levi M, any reduction b_M of b to M, and any $\chi \in X^*(P/Z_G)^+$, we have*

$$\deg \chi_*(\mathcal{E}_{b,x})_P \leq 0,$$

where $(\mathcal{E}_{b,x})_P$ is the reduction to P of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b_M} \overset{M}{\times} P$ of $\mathcal{E}_b$ and lemma 2.4.

Proof. We use lemma 2.6. The character χ factorizes through the Levi quotient M and thus

$$\chi_*(\mathcal{E}_{b,x})_P = \chi_*\mathcal{E}_{b_M,\mathrm{pr}_w(x)}$$

if x is in the Schubert cell associated to w. Now,

$$c_1^M(\mathcal{E}_{b_M,\mathrm{pr}_w(x)}) = (\mu^w)^\sharp - \kappa_M(b_M) \in \pi_1(M)_\Gamma.$$

In particular, for any $\chi \in X^*(M)$,

$$\deg \chi_*\mathcal{E}_{b_M,\mathrm{pr}_w(x)} = \deg(\chi(b_M), \chi\circ\mu^w)$$

where the degree of the right member has to be taken in the sense of filtered φ-modules of rank 1. The result is then a consequence of the corollary 9.2.30 of [12] (together with lemma 1.6). □

3. The admissible locus

As before, we fix the triple $(G, \{\mu\}, [b])$. We will assume that $\{\mu\}$ is *minuscule* until clearly stated otherwise, and also we assume $[b] \in B(G, \mu)$ (the triple $(G, \{\mu\}, [b])$ is called a local Shimura datum, cf. [54] sec. 5).

Let $K|\breve{E}$ be a finite extension and $x \in \mathcal{F}(G, \mu, b)^{wa}(K)$. We get the following diagram of functors

$$\begin{tikzcd} \mathrm{Rep}\, G \arrow[r, "\mathcal{I}_{b,x}"] \arrow[dr, "\omega_G"'] & \varphi\text{-}\mathrm{FilMod}^{wa}_{K/\breve{F}} \arrow[r, "V_{cris}"] & \mathrm{Rep}_{cris}\mathrm{Gal}(\overline{K}/K) \arrow[dl, "\omega_{cris}"] \\ & \mathrm{Vect}_F & \end{tikzcd}$$

where $\varphi\text{-}\mathrm{FilMod}^{wa}_{K/\breve{F}}$ is the category of weakly admissible filtered isocrystals, $\mathrm{Rep}_{cris}(\mathrm{Gal}(\overline{K}/K))$ is the category of crystalline representations of $\mathrm{Gal}(\overline{K}/K)$ with coefficients in F whose Sen operator is F-linear, and ω_G and ω_{cris} are the natural fiber functors. This diagram is commutative since the class of the torsor

$$\underline{\mathrm{Isom}}^{\otimes}(\omega_G, \omega_{cris} \circ V_{cris} \circ \mathcal{I}_{b,x})$$

is given by $\kappa(b) - \mu^{\sharp}$ and is thus trivial ([54] prop. 2.7, see also 1.20 of [53] in case G_{der} is simply connected, and [63] in the general case). Thus, the choice of an isomorphism between ω_G and $\omega_{cris} \circ V_{cris} \circ \mathcal{I}_{b,x}$ induces the homomorphism

$$\rho_x : \mathrm{Gal}(\overline{K}/K) \longrightarrow G(F).$$

Rapoport and Zink conjectured the existence of an open subspace

$$\mathcal{F}(G, \mu, b)^a \subset \mathcal{F}(G, \mu, b)^{wa}$$

together with an étale G-local system $\mathscr{E}$ on $\mathcal{F}(G, \mu, b)^a$ such that for all finite extension $K|\breve{E}$ we have

$$\mathcal{F}(G, \mu, b)^a(K) = \mathcal{F}(G, \mu, b)^{wa}(K),$$

and $\mathscr{E}$ interpolates the preceding Galois representation $\rho_x : \mathrm{Gal}(\overline{K}/K) \to G(F)$, cf. [12] conj. 11.4.4 and [36] conj. 2.3. The associated spaces of lattices will give the desired tower of local Shimura varieties, ([50] hope 4.2, [54] conj. 5.1, [59] sec. 3.1).

Now we will use the theory of G-bundles on the Fargues-Fontaine curve to define the admissible locus

$$\mathcal{F}(G, \mu, b)^a \subset \mathcal{F}(G, \mu, b)^{wa}.$$

Let $C|\breve{E}$ be any complete algebraically closed extension containing $\overline{F}$. We consider the curve $X = X_{C^\flat}$ over F with the canonical point $\infty \in X$.

Definition 3.1 (See also [51] def. A.6)**.**

(1) *We set*

$$\mathcal{F}(G, \mu, b)^a(C) = \{x \in \mathcal{F}(G, \mu)(C) | \nu_{\mathcal{E}_{b,x}} \text{ is trivial}\}.$$

In other words, if G becomes quasi-split over $F'|F$ we ask that $\mathcal{E}_{b,x} \otimes_F F'$ is semi-stable on the curve $X_{F'}$ and that, for all $\chi : G \to \mathbb{G}_m$, $\deg \chi_* \mathcal{E}_{b,x} = 0$.

(2) *We define $\mathcal{F}(G, \mu, b)^a$ as the subset of $\mathcal{F}(G, \mu)$ stable under generalization whose C-points are given by the preceding for any C as before.*

Since

$$c_1^G(\mathcal{E}_{b,x}) = \mu^\sharp - \kappa(b)$$

the isomorphism class of $\mathcal{E}_{b,x}$ does not depend on $x \in \mathcal{F}(G,\mu,b)^a(C)$. In particular, if $[b] \in B(G,\mu)$ one has

$$\mathcal{F}(G,\mu,b)^a(C) = \{x \in \mathcal{F}(G,\mu)(C) \mid \mathcal{E}_{b,x} \text{ is trivial}\}.$$

Let us recall some basic properties of $\mathcal{F}(G,\mu,b)^a$.

Proposition 3.2. *The admissible locus satisfies the following properties:*

(1) *$\mathcal{F}(G,\mu,b)^a$ is a partially proper open subset of $\mathcal{F}(G,\mu)$.*
(2) *We have the inclusion*

$$\mathcal{F}(G,\mu,b)^a \subset \mathcal{F}(G,\mu,b)^{wa},$$

such that for any finite extension $K|\breve{E}$

$$\mathcal{F}(G,\mu,b)^a(K) = \mathcal{F}(G,\mu,b)^{wa}(K).$$

In particular, $\mathcal{F}(G,\mu,b)^a \neq \emptyset$ if and only if $[b] \in A(G,\mu)$, cf. prop. 2.2.
(3) *When $(G,\{\mu\},[b])$ is a Hodge type local Shimura datum (see [59] 3.2), then the subspace $\mathcal{F}(G,\mu,b)^a$ coincides with those introduced by Hartl [36] (via the Robba ring $\widetilde{B}^\dagger_{rig}(C)$) and Faltings [20] (via the crystalline period ring $B_{cris}(C)$).*

Proof. (1) follows from the work of Kedlaya-Liu [40]. For (2) and (3) see [51] Remarks A.5. □

For the more advanced reader here is how to prove point (1) in the preceding. One can consider Scholze's B_{dR} affine Grassmannian $\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}}$ over $\mathrm{Spa}(\breve{E})^\diamond$. This is the étale sheaf associated to the presheaf $(R,R^+) \mapsto G(\mathrm{B}_{\mathrm{dR}}(R))/G(\mathrm{B}^+_{\mathrm{dR}}(R))$ on affinoid perfectoid $\breve{E}$-algebras. There is a Bialynicki-Birula morphism that is an isomorphism thanks to our minuscule hypothesis ([9]). Its inverse gives an embedding

$$\mathcal{F}(G,\mu)^\diamond \hookrightarrow \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}}.$$

Now for any $\breve{F}$-perfectoid space S together with an element $x \in \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}}(S)$ one can define

$$\mathcal{E}_{b,x},$$

a G-bundle on the relative adic Fargues-Fontaine curve $X_{S^\flat}$. This is defined using the "degree one Cartier divisor"

$$S \hookrightarrow X_{S^\flat}$$

and a gluing "à la Beauville-Laszlo" by modifying $\mathcal{E}_b$ on X_S. In fact, when $S = \mathrm{Spa}(R,R^+)$, "the formal completion along this Cartier divisor" is $\mathrm{Spf}(\mathrm{B}^+_{\mathrm{dR}}(R))$. According to Kedlaya and Liu ([40] theo. 7.4.5),

$$S^{ss} := \{s \in S \mid \mathcal{E}_{b,x|X_{k(s)^\flat}} \text{ is semi-stable}\}$$

is open in S. Now if x is given by a morphism $S \to \mathcal{F}(G,\mu)$ that is quasicompact quasiseparated surjective, then $|S| \to |\mathcal{F}(G,\mu)|$ is a quotient map such that S^{ss} is the pullback of $\mathcal{F}(G,\mu,b)^a$. Choosing $S \to \mathcal{F}(G,\mu)$ surjective, we conclude.

Let us now state the following result since this is not written anywhere explicitly. It says that one can construct the local Shimura varieties as pro-étale covers of $\mathcal{F}(G,\mu,b)^a$ as conjectured by Rapoport and Zink. We suppose μ minuscule as before.

Theorem 3.3 (Scholze [57] sec. 23.3)**.** *Suppose $[b] \in B(G,\mu)$. Then one can construct a pro-étale $\underline{G(F)}$-local system on $\mathcal{F}(G,\mu,b)^a$ such that for any compact open subgroup $K \subset G(F)$, the moduli of its K-trivializations $\mathcal{M}_K(G,\mu,b) \to \mathcal{F}(G,\mu,b)^a$ is represented by a rigid analytic space.*

Proof. Let $S \to \mathcal{F}(G,\mu,b)^a$ be as before. Kedlaya and Liu prove in [40] that rank n vector bundles on $X_{S^\flat}$ that are geometrically fiberwise on $S^\flat$ semi-stable of slope zero are the same as rank n pro-étale $\underline{F}$-local systems on $S^\flat$. Using this one can prove that the preceding G-bundle $\mathcal{E}_{b,x}$ on $X_{S^\flat}$ gives rise to a pro-étale $\underline{G(F)}$-torsor on $S^\flat$

$$\underline{\mathrm{Isom}}(\mathcal{E}_1, \mathcal{E}_{b,x})$$

where $\underline{G(F)} = \underline{\mathrm{Aut}}(\mathcal{E}_1)$ (we refer to [30]). This defines a pro-étale $\underline{G(F)}$-torsor

$$\mathcal{M}_\infty \to \mathcal{F}(G,\mu,b)^{a,\diamond}.$$

Now one uses [56] (pro-étale descent of separated étale morphisms) that says that $\mathcal{M}_\infty/K$ is representated by a rigid analytic space separated and étale over $\mathcal{F}(G,\mu,b)^a$. □

4. Hodge-Newton decomposability

In [38] (Theorem 1.1 & Lemma 2.5) He and Nie give a description of Kottwitz set $B(G,\mu)$ as a subset of the positive Weyl chamber via the Newton map. This is applied in [32] to give a criterion for HN decomposability in terms of the Hodge polygon $\mu^\diamond$. He and Nie's description relies on their description of $B(G)$ via an affine root system derived from Bruhat-Tits theory applied to $G_{\breve{F}}$. In this section we give a description of $B(G,\mu)$ that only relies on the usual relative and absolute spherical root systems associated to G. He and Nie's description of $B(G)$ is well adapted to problems concerning integral models of Rapoport-Zink spaces and their mod p special fiber. The problem we are interested in concerns only the generic fiber and there is no reason to use Bruhat-Tits theory and integral structure on Dieudonné modules for this. We will take an approach developed by Chai in [10] to make a link with He and Nie's work. The two really original results of this section are proposition 4.14 and corollary 4.16 that will be a key point in the proof of our main theorem.

4.1. A description of a generalized Kottwitz set in the quasi-split case.

In this subsection $G = H$ is a quasi-split group over F. Since $C_G(\boldsymbol{A}) = \boldsymbol{T}$ the restriction of any root of $\boldsymbol{T}$ to $\boldsymbol{A}$ is a root and this induces a bijection ([62] 15.5.3)

$$\Phi/\Gamma \xrightarrow{\sim} \Phi_0.$$

One verifies moreover that

$$\Delta/\Gamma \xrightarrow{\sim} \Delta_0.$$

Let us fix $\mu \in X_*(\boldsymbol{T})^+$ *not necessarily minuscule.* As before, we note $\mu^\diamond \in X_*(\boldsymbol{A})^+_{\mathbb{Q}}$ the Galois average of μ and $[\nu_b] \in X_*(\boldsymbol{A})^+_{\mathbb{Q}} = \mathcal{N}(G)$ for $[b] \in B(G)$.

For $\alpha \in \Delta_0$ and $\beta \in \Phi$ such that $\beta_{|\boldsymbol{A}} = \alpha$, one has $\beta \in \Delta$. We note

$$\omega_\beta \in \langle \Phi \rangle_{\mathbb{Q}}$$

the corresponding fundamental weight, that is to say for $\gamma \in \Delta$,

$$\langle \gamma^\vee, \omega_\beta \rangle = \delta_{\gamma,\beta}.$$

Now, for $\alpha \in \Delta_0$, we set

$$\tilde{\omega}_\alpha = \sum_{\substack{\beta \in \Phi \\ \beta_{|A} = \alpha}} \omega_\beta \in X^*(\boldsymbol{T})^\Gamma_{\mathbb{Q}} = X^*(\boldsymbol{A})_{\mathbb{Q}}.$$

This satisfies the following property: for $\gamma \in \Delta$ we have

$$\langle (\gamma^\vee)^\diamond, \tilde{\omega}_\alpha \rangle = \begin{cases} 0, & \text{if } \gamma_{|A} \neq \alpha \\ 1, & \text{otherwise} \end{cases}$$

where $(\gamma^\vee)^\diamond \in \langle \Phi_0^\vee \rangle_{\mathbb{Q}}$ is the Galois average of $\gamma^\vee$.

We will need the following generalized Kottwitz set later.

Definition 4.1. *For $\epsilon \in \pi_1(G)_\Gamma$ and $\delta \in X_*(\boldsymbol{A})^+_{\mathbb{Q}}$ we set*

$$B(G, \epsilon, \delta) = \{[b] \in B(G) \mid \kappa(b) = \epsilon \text{ and } [\nu_b] \leq \delta\}.$$

Of course if this set is non empty then $\epsilon \equiv \delta$ in $\pi_1(G)_\Gamma \otimes \mathbb{Q}$. One has

$$B(G, \mu) = B(G, \mu^\sharp, \mu^\diamond).$$

Proposition 4.2. *If $\epsilon = \mu^\sharp$ then, as a subset of $\mathcal{N}(G)$, $B(G, \epsilon, \delta)$ is the set of $v \in X_*(\boldsymbol{A})^+_{\mathbb{Q}}$ satisfying*

- *$\delta - v \in \langle \Phi_0^\vee \rangle_{\mathbb{Q}}$,*
- *$\forall \alpha \in \Delta_0$ s.t. $\langle v, \alpha \rangle \neq 0, \langle \delta - v, \tilde{\omega}_\alpha \rangle \geq 0$ and $\langle \mu^\diamond - v, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$.*

Proof. One has

$$B(G) = \bigcup_M \mathrm{Im}\big(B(M)_{basic} \to B(G)\big)$$

where M goes through the standard Levi subgroups (G included). Moreover

$$\kappa_M : B(M)_{basic} \xrightarrow{\sim} \pi_1(M)_\Gamma.$$

Via this isomorphism the slope morphism associated to an element of $B(M)_{basic}$ is given by

$$\pi_1(M)_\Gamma \otimes \mathbb{Q} = X_*(Z_M)^\Gamma_{\mathbb{Q}}.$$

For $v \in B(G, \epsilon, \delta) \subset X_*(\boldsymbol{A})^+_{\mathbb{Q}}$, by definition $\delta - v \in \mathbb{Q}_{\geq 0}(\Phi_0^\vee)^+$. Now for $\alpha \in \Delta_0$ and $z \in \mathbb{Q}_{\geq 0}(\Phi^\vee)^+$

$$\langle z, \tilde{\omega}_\alpha \rangle \geq 0$$

since

$$\mathbb{Q}_{\geq 0}(\Phi_0^\vee)^+ = \mathbb{Q}_{\geq 0}\{(\gamma^\vee)^\diamond \mid \gamma \in \Phi^+\}.$$

One deduces that $\langle \delta - v, \tilde{\omega}_\alpha \rangle \geq 0$. Now consider an element of $B(M)_{basic}$ given by the class of some $\mu' \in X_*(\boldsymbol{T})$ in $\big[X_*(\boldsymbol{T})/\langle \Phi_M^\vee \rangle\big]_\Gamma$. One has a decomposition

$$X_*(\boldsymbol{T})_{\mathbb{Q}} = \langle \Phi_M^\vee \rangle_{\mathbb{Q}} \oplus \langle \Phi_M \rangle^\perp_{\mathbb{Q}}$$

that gives rise to a projection morphism

$$\mathrm{pr}_M : X_*(\boldsymbol{T})_{\mathbb{Q}} \longrightarrow \langle \Phi_M \rangle^\perp_{\mathbb{Q}}.$$

Then the slope morphism in $X_*(\boldsymbol{A})_{\mathbb{Q}}$ associated to our element of $B(M)_{basic}$ is given by

$$v = \mathrm{pr}_M(\mu')^\diamond$$

the Galois average of $\mathrm{pr}_M(\mu')$. We can suppose that M is the centralizer of v. Then v defines a parabolic subgroup P_v with Levi subgroup M. One can find $w \in W^\Gamma$ such that P_v^w is a standard parabolic subgroup and M^w a standard Levi. Up to replacing M by M^w and μ' by μ'^w we can thus suppose that $v \in X_*(\boldsymbol{A})^+_{\mathbb{Q}}$. Suppose

now that the image of our element in $B(M)_{basic}$ in $B(G)$ lies in $B(G,\epsilon,\delta)$. Then μ and μ' have the same image in $\pi_1(G)_\Gamma$,

$$\mu-\mu'\in\langle\Phi^\vee\rangle+I_\Gamma.X_*(\boldsymbol{T})$$

where $I_\Gamma\subset\mathbb{Z}[\Gamma]$ is the augmentation ideal. Let $\alpha\in\Delta_0$ be such that $\langle v,\alpha\rangle\neq 0$ and thus $\alpha\in\Delta_0\setminus\Delta_{0,M}$. If

$$\mu-\mu'=\sum_{\gamma\in\Delta}\lambda_\gamma\gamma^\vee+z,\ \lambda_\gamma\in\mathbb{Z},\quad z\in I_\Gamma.X_*(\boldsymbol{T})$$

we have

$$\begin{aligned}\langle\mu^\diamond-v,\tilde\omega_\alpha\rangle &= \langle\mu^\diamond-\mathrm{pr}_M(\mu')^\diamond,\tilde\omega_\alpha\rangle\\ &= \langle\mu-\mathrm{pr}_M(\mu'),\tilde\omega_\alpha\rangle\\ &= \langle\mu-\mu',\tilde\omega_\alpha\rangle\quad\text{since }\alpha\notin\Delta_{0,M}\\ &= \sum_{\substack{\gamma\in\Delta\\ \gamma_{|A}=\alpha}}\lambda_\gamma\in\mathbb{Z}.\end{aligned}$$

This proves that $B(G,\epsilon,\delta)$ is contained in the announced subset of $X_*(\boldsymbol{A})^+_{\mathbb{Q}}$. Reciprocally, let $v\in X_*(A)^+_{\mathbb{Q}}$ satisfying the conditions of the statement. Let M be the centralizer of v. Define

$$\mu'=\mu-\sum_{\alpha\in\Delta_0\setminus\Delta_{0,M}}\langle\mu^\diamond-v,\tilde\omega_\alpha\rangle\gamma_\alpha^\vee\in X_*(\boldsymbol{T}).$$

where γ_α is any element in Φ such that $\gamma_{\alpha|A}=\alpha$. One checks that the image of μ' in $\pi_1(M)_\Gamma$ defines an element of $B(M)_{basic}$ whose image in $B(G)$ lies in $B(G,\epsilon,\delta)$ and whose associated slope is v. □

One deduces the following description for the usual Kottwitz set.

Corollary 4.3. *As a subset of $X_*(\boldsymbol{A})^+_{\mathbb{Q}}$, $B(G,\mu)$ equals*

$$\big\{v\in X_*(\boldsymbol{A})^+_{\mathbb{Q}}\ \big|\ \mu^\diamond-v\in\langle\Phi_0^\vee\rangle_{\mathbb{Q}}\text{ and }\forall\alpha\in\Delta_0\ s.t.\ \langle v,\alpha\rangle\neq 0,\ \langle\mu^\diamond-v,\tilde\omega_\alpha\rangle\in\mathbb{N}\big\}.$$

Later we will need the following.

Corollary 4.4. *Suppose $[b]\in B(G,\mu)$ is the basic element. Let*

$$B(G,0,\nu_b\mu^{-1}):=B(G,0,\nu_b(w_0\mu^{-1})^\diamond).$$

Then we have

$$\begin{aligned}B(G,0,\nu_b\mu^{-1})=\big\{v\in X_*(\boldsymbol{A})^+_{\mathbb{Q}}\ \big|\ v\in\langle\Phi_0^\vee\rangle_{\mathbb{Q}}\text{ and }\forall\alpha\in\Delta_0\ s.t.\ \langle v,\alpha\rangle\neq 0,\\ \langle\nu_b-w_0\mu^\diamond-v,\tilde\omega_\alpha\rangle\geq 0\ \textit{and}\ \langle v,\tilde\omega_\alpha\rangle\in\mathbb{Z}\big\}.\end{aligned}$$

Remark 4.5. *The preceding set $B(G,0,\nu_b\mu^{-1})$ is the one denoted $B(G,\nu_b\mu^{-1})$ in* [51]. *This notation may be confusing since if $\nu_b=1$, for example if G is adjoint, then $B(G,0,\nu_b\mu^{-1})$ is not equal to $B(G,\mu^{-1})$ is general. This is why we introduced this more precise notation.*

4.2. **The non-quasi-split case.** Suppose now G is not necessarily quasi-split. We have

$$H^1(F,H_{ad})=\pi_1(H_{ad})_\Gamma=\big[\langle\Phi\rangle^\vee/\langle\Phi^\vee\rangle\big]_\Gamma$$

We see $\langle\Phi\rangle^\vee$ as a lattice in $\langle\Phi^\vee\rangle_{\mathbb{Q}}$. The isomorphism class of the inner form G is then given by the class of some element

$$\xi\in\langle\Phi\rangle^\vee\subset\langle\Phi^\vee\rangle_{\mathbb{Q}}.$$

Projection to the adjoint group induces a bijection (see [42] 4.11)

$$B(G,\epsilon,\delta)\xrightarrow{\sim}B(G_{ad},\epsilon_{ad},\delta_{ad}).$$

Moreover

$$H^1(F, H_{ad}) = B(H_{ad})_{basic}$$

and the isomorphism class of G as an inner form of H is given by some $[b_G] \in B(H_{ad})_{basic}$ for which $G_{ad} = J_{b_G}$. There is then a bijection

$$B(G_{ad}) \xrightarrow{\sim} B(H_{ad})$$

that sends $[1]$ to $[b_G]$. We can thus see $B(G, \epsilon, \delta)$ as a subset of $B(H_{ad})$. We have moreover an identification $\pi_1(G_{ad}) = \pi_1(H_{ad})$. Via this bijection the following diagram commutes

$$\begin{array}{ccc} B(G_{ad}) & \xrightarrow{\sim} & B(H_{ad}) \\ \downarrow{\scriptstyle \kappa_{G_{ad}}} & & \downarrow{\scriptstyle \kappa_{H_{ad}}} \\ \pi_1(H_{ad})_\Gamma & \xrightarrow{\bullet+\xi} & \pi_1(H_{ad})_\Gamma \end{array}$$

We thus have

$$B(G, \epsilon, \delta) = B(H_{ad}, \epsilon_{ad} + \xi, \delta_{ad}).$$

From proposition 4.2 we deduce the following.

Proposition 4.6. *If $\epsilon = \mu^\sharp$, as a subset of $X_*(\boldsymbol{A})^+_{\mathbb{Q}}$, the set $B(G, \epsilon, \delta)$ is given by the vectors v such that*

(1) $\delta - v \in \langle \Phi_0^\vee \rangle_{\mathbb{Q}}$,
(2) *for all $\alpha \in \Delta_0$ such that $\langle v, \alpha \rangle \neq 0$ one has $\langle \delta - v, \tilde{\omega}_\alpha \rangle \geq 0$,*
(3) *for all $\alpha \in \Delta_0$ such that $\langle v, \alpha \rangle \neq 0$, $\langle \mu^\diamond + \xi^\diamond - v, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$.*

This specializes to the two following statements.

Corollary 4.7. *As a subset of $X_*(\boldsymbol{A})^+_{\mathbb{Q}}$, the set $B(G, \mu)$ is given by the vectors v such that*

(1) $\mu^\diamond - v \in \langle \Phi_0^\vee \rangle_{\mathbb{Q}}$,
(2) *for all $\alpha \in \Delta_0$ such that $\langle v, \alpha \rangle \neq 0$ one has $\langle \mu^\diamond - v, \tilde{\omega}_\alpha \rangle \geq 0$,*
(3) *for all $\alpha \in \Delta_0$ such that $\langle v, \alpha \rangle \neq 0$, $\langle \mu^\diamond + \xi^\diamond - v, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$.*

Corollary 4.8. *Suppose $[b] \in B(G, \mu)$ is the basic element. Then*

$$B(G, 0, \nu_b \mu^{-1}) = \big\{ v \in X_*(\boldsymbol{A})^+_{\mathbb{Q}} \;\big|\; v \in \langle \Phi_0^\vee \rangle_{\mathbb{Q}} \text{ and } \forall \alpha \in \Delta_0 \text{ s.t. } \langle v, \alpha \rangle \neq 0, \\ \langle \nu_b - w_0 \mu^\diamond - v, \tilde{\omega}_\alpha \rangle \geq 0 \text{ and } \langle v - \xi^\diamond, \tilde{\omega}_\alpha \rangle \in \mathbb{Z} \big\}.$$

Remark 4.9. *When G is unramified the root system we use is identical to the one used by He and Nie, and the statement of the preceding proposition is identical to theirs. This is not the case anymore in general, even if G is quasi-split. One can compare He's result with ours using theorem 6.1 of [33] to obtain that the $(\tilde{\omega}_\alpha)_{\alpha \in \Delta_0}$ are exactly the $\omega_{\mathcal{O}}$ of [38] when $\mathcal{O}$ goes through the set of σ_0-orbits of simple roots in the Bruhat-Tits échelonnage root system attached to $G_{\breve{F}}$. An analysis of the construction of this root systems shows that we can take $\xi = \sigma(0)$ with the notations of [38].*

4.3. **HN decomposability.** Let us recall the following definition. Here G is not necessarily quasi-split.

Definition 4.10. *The set $B(G, \mu)$ is fully HN decomposable if for any non-basic $[b] \in B(G, \mu)$ there exists a standard strict Levi subgroup M of the quasi-split inner form H such that:*

(1) *the centralizer of $[\nu_b]$ is contained in M,*
(2) $\mu^\diamond - [\nu_b] \in \langle \Phi_{0,M}^\vee \rangle_{\mathbb{Q}}$.

In the quasi-split case we have the following equivalent definition. As before we suppose $\mu \in X_*(\boldsymbol{T})^+$ which defines a cocharacter with values in M for any standard Levi subgroup M (the conjugacy class $\{\mu\}$ does not define a unique conjugacy class in such an M, it is important to fix this).

Lemma 4.11. *For G quasi-split the following are equivalent:*

(1) *$B(G,\mu)$ is fully HN decomposable,*
(2) *for any non basic $[b] \in B(G,\mu)$ there exists a strict standard Levi subgroup M containing M_b the centralizer of $[\nu_b]$ such that $[b_M] \in B(M,\mu)$,*
(3) *for any non basic $[b] \in B(G,\mu)$ there exists a strict standard Levi subgroup M containing M_b the centralizer of $[\nu_b]$ such that $\kappa_M(b_M) = \mu^\sharp \in \pi_1(M)_\Gamma$.*

where b_M is the reduction of b to M deduced from its canonical reduction to M_b and the inclusion $M_b \subset M$.

Proof. This is easily deduced from the fact that $\pi_1(M)_{\Gamma,tor} \hookrightarrow \pi_1(G)_{\Gamma,tor}$ since this is identified with the injective map $H^1(F,M) \to H^1(F,G)$ ([58] ex. 1 p. 136). □

Before going further, let us remark that for $\alpha \in \Delta_0$, $\tilde{\omega}_\alpha \in \mathbb{Q}_{\geq 0}\Phi_0^+$ since for $\beta \in \Delta$, $\omega_\beta \in \mathbb{Q}_{\geq 0}\Phi^+$. In particular,

$$\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle \geq 0.$$

We remark too that if $\xi \in \langle \Phi \rangle^\vee$ is as in the preceding section, then one can form $\langle \xi^\diamond, \tilde{\omega}_\alpha \rangle \in \mathbb{Q}$. The reduction modulo $\mathbb{Z}$ of this quantity depends only on the class of ξ in $H^1(F, H_{ad}) = \pi_1(H_{ad})_\Gamma$ and this defines a character

$$\langle (-)^\diamond, \tilde{\omega}_\alpha \rangle : H^1(F, H_{ad}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

We note $\{.\} : \mathbb{Q}/\mathbb{Z} \to [0,1[$ the fractional part lift. The proof of the following proposition is then strictly identical to the one of the equivalence between (1) and (2) in theorem 2.3 of [32] (cf. the proof of 4.14 for this type of proof with our notations).

Proposition 4.12 (Minute criterion[1])**.** *The set $B(G,\mu)$ is fully HN decomposable if and only if for all $\alpha \in \Delta_0$, $\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle + \{\langle \xi^\diamond, \tilde{\omega}_\alpha \rangle\} \leq 1$.*

In particular if G is quasi-split this is reduced to the condition

$$\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle \leq 1.$$

Of course, in this case, since $\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle = \langle \mu, \tilde{\omega}_\alpha \rangle$, this can be rephrased in the following way:

$$\forall\, \mathcal{O},\ \text{a } \Gamma\text{-orbit in } \Delta,\quad \sum_{\beta \in \mathcal{O}} \langle \mu, \omega_\beta \rangle \leq 1.$$

We will need the following later. Let $[b] \in B(G,\mu)$ be the basic element. Let us first remark the following.

Lemma 4.13. *The set $B(J_b, \mu^{-1})$ is fully HN decomposable if and only if for any $[b'] \in B(G, 0, \nu_b\mu^{-1})$ non basic there exists a standard strict Levi subgroup M of the quasisplit inner form H such that:*

(1) *the centralizer of $[\nu_{b'}]$ is contained in M*
(2) $[\nu_b] + w_0(-\mu) - [\nu_{b'}] \in \langle \Phi_{0,M}^\vee \rangle_{\mathbb{Q}}$

Proof. If $[b'] \in B(G, 0, \nu_b\mu^{-1})$ corresponds to $[b''] \in B(J_b, \mu^{-1})$ via the identification $B(G, 0, \nu_b\mu^{-1}) = B(J_b, \mu^{-1})$, then $[\nu_{b'}] = [\nu_{b''}] + [\nu_b]$. We can then apply definition 4.10. □

[1] As we learned from Eva Viehmann, here minute is the adjective, to be pronounced maɪ'nju:t.

Proposition 4.14. *The set $B(J_b,\mu^{-1})$ is fully HN decomposable if and only if for all $\alpha\in\Delta_0$ one has $\langle\mu^\diamond,\tilde\omega_\alpha\rangle+\{\langle\xi^\diamond,\tilde\omega_\alpha\rangle\}\leq 1$.*

Proof. We use corollary 4.8. We can suppose G is adjoint and thus $\nu_b=1$.

First, let us notice that the condition of the statement is equivalent to

$$\forall\alpha\in\Delta_0,\quad \langle -w_0\mu^\diamond,\tilde\omega_\alpha\rangle+\{-\langle\xi^\diamond,\tilde\omega_\alpha\rangle\}\leq 1.$$

In fact, if $*$ is the involution of Dynkin diagram induced by $-w_0$ then this last condition is equivalent to $\langle\mu^\diamond,\tilde\omega_{\alpha^*}\rangle+\{\langle w_0\xi^\diamond,\tilde\omega_{\alpha^*}\rangle\}\leq 1$. But $\{\langle w_0\xi^\diamond,\tilde\omega_{\alpha^*}\rangle\}=\{\langle\xi^\diamond,\tilde\omega_{\alpha^*}\rangle\}$ since $w_0\xi-\xi\in\langle\Phi^\vee\rangle$.

Suppose thus that $\forall\alpha\in\Delta_0,\langle -w_0\mu^\diamond,\tilde\omega_\alpha\rangle+\{-\langle\xi^\diamond,\tilde\omega_\alpha\rangle\}\leq 1$ and let $v\in B(G,0,\mu^{-1})$ be non basic. If $\langle v,\alpha\rangle\neq 0$ then

$$\begin{aligned}&\langle -w_0\mu^\diamond,\tilde\omega_\alpha\rangle+\{-\langle\xi^\diamond,\tilde\omega_\alpha\rangle\}\\ =&\underbrace{\langle -w_0\mu^\diamond-v,\tilde\omega_\alpha\rangle}_{\geq 0}+\underbrace{\underbrace{\langle\xi^\diamond,\tilde\omega_\alpha\rangle+\{-\langle\xi^\diamond,\tilde\omega_\alpha\rangle\}}_{\in\mathbb{Z}}+\underbrace{\langle v-\xi^\diamond,\tilde\omega_\alpha\rangle}_{\in\mathbb{Z}}}_{\geq 0\text{ thus }\in\mathbb{N}}\leq 1.\end{aligned}$$

Thus, if $\langle -w_0\mu^\diamond-v,\tilde\omega_\alpha\rangle\neq 0$ then $\langle v,\tilde\omega_\alpha\rangle+\{-\langle\xi^\diamond,\tilde\omega_\alpha\rangle\}=0$ and thus $\langle v,\tilde\omega_\alpha\rangle=0$ which is impossible since v is dominant and $\langle v,\alpha\rangle\neq 0$. We thus have

$$-w_0\mu^\diamond-v\in\langle\Phi^\vee_{0,M}\rangle_{\mathbb{Q}}$$

where $\Delta_{0,M}=\Delta_0\setminus\{\alpha\}$ for any α such that $\langle v,\alpha\rangle\neq 0$.

Reciprocally, suppose $\langle -w_0\mu^\diamond,\tilde\omega_{\alpha_0}\rangle+\{-\langle\xi^\diamond,\tilde\omega_{\alpha_0}\rangle\}>1$ for some $\alpha_0\in\Delta_0$. Let v be such that $\langle v,\alpha\rangle=0$ if $\alpha\neq\alpha_0$, $\alpha\in\Delta_0$, and

$$\langle v,\tilde\omega_{\alpha_0}\rangle=1-\{-\langle\xi^\diamond,\tilde\omega_{\alpha_0}\rangle\}.$$

Since $\tilde\omega_{\alpha_0}\in\mathbb{Q}_{\geq 0}\Delta_0$, v is dominant. Moreover,

$$\langle -w_0\mu^\diamond-v,\tilde\omega_{\alpha_0}\rangle>0$$

and

$$\langle v-\xi^\diamond,\tilde\omega_{\alpha_0}\rangle=1-\langle\xi^\diamond,\tilde\omega_\alpha\rangle-\{-\langle\xi^\diamond,\tilde\omega_{\alpha_0}\rangle\}\in\mathbb{Z}.$$

Thus, $v\in B(G,0,\mu^{-1})$ and is not HN decomposable since its centralizer is the maximal Levi subgroup M with $\Delta_{0,M}=\Delta_0\setminus\{\alpha_0\}$ and $\langle -w_0\mu^\diamond-v,\tilde\omega_{\alpha_0}\rangle\neq 0$. □

Remark 4.15. *An analysis of the proof of the preceding proposition shows that if $B(G,0,\nu_b\mu^{-1})$ is fully HN decomposable, then for all non basic $[b']$ in this set $\nu_b-w_0\mu^\diamond-[\nu_{b'}]\in\langle\Phi^\vee_{0,M}\rangle_{\mathbb{Q}}$, where M is the centralizer of $[\nu_{b'}]$. The same holds for $B(G,\mu)$.*

Here is the corollary we will use. This is a key point in the proof of our main theorem. We don't know a direct proof of this in the sense that there is à priori no direct relation between $B(G,\mu)$ and $B(J_b,\mu^{-1})$.

Corollary 4.16. *The following are equivalent:*

(1) *the set $B(G,\mu)$ is fully HN decomposable,*
(2) *the set $B(J_b,\mu^{-1})$ is fully HN decomposable.*

5. Harder-Narasimhan stratification of the flag variety

5.1. **The twin towers principle ([21], [20], [22]).** Let $[b]\in B(G)$ be a basic element. What we call the "twin towers principle" is the identification

$$\mathrm{Bun}_{J_b}\cong\mathrm{Bun}_G,$$

that is to say there is an equivalence of groupoids between G-bundles and J_b-bundles on the curve. Here and in the following we give sometimes statements that are true

at the level of perfectoid v-stacks of bundles like in [26] and [30] or objects like the diamond $\mathcal{F}(G,\mu)^\diamond$. Nevertheless the reader not familiar with those notions should not be frightened; at the end, for the proof of our main theorem, we only need the evaluation on C-points of those objects and he can work in this context.

In fact, $J_b \times X$ is the twisted pure inner form of $G \times X$ obtained by twisting by the G-torsor $\mathcal{E}_b$,

$$J_b \times X = \underline{\mathrm{Aut}}(\mathcal{E}_b)$$

as a group over the curve. If $\mathcal{E}$ is a G-bundle on X one associates to it the J_b-bundle

$$\underline{\mathrm{Isom}}(\mathcal{E}_b, \mathcal{E}).$$

This is what Serre calls "torsion au moyen d'un cocyle" in sec. 5.3 of [58]. At the level of points of the preceding perfectoid v-stacks this gives the well known bijection

$$B(J_b) \xrightarrow{\sim} B(G)$$

that sends $[1]$ to $[b]$.

Example 5.1. *For $\lambda \in \mathbb{Q}$ the functor $\mathcal{E} \mapsto \underline{\mathrm{Hom}}(\mathcal{O}(\lambda), \mathcal{E})$ induces an equivalence between semi-stable vector bundles of slope λ and vector bundles equipped with an action of $D_\lambda^{\mathrm{op}} = \mathrm{End}(\mathcal{O}(\lambda))^{\mathrm{op}}$, where D_λ is the division algebra with invariant λ.*

The identification $\mathrm{Bun}_{J_b} = \mathrm{Bun}_G$ respects modifications of a given type μ that is to say it identifies the corresponding Hecke stacks of modifications. Suppose $[b] \in B(G,\mu)$ is the basic element. Let

$$[b''] \in B(J_b, \mu^{-1})$$

be the basic element, $[b''] \mapsto [1]$ via $B(J_b) \xrightarrow{\sim} B(G)$. One thus has

$$J_{b''} = G.$$

The preceding considerations give an isomorphism of moduli spaces over $\mathrm{Spa}(\breve{E})^\diamond$:

$$\begin{array}{c}
\text{modifications of type } \mu \text{ between } \mathcal{E}_b^G \text{ and } \mathcal{E}_1^G \\
\Big\downarrow \sim \\
\text{modifications of type } \mu \text{ between } \mathcal{E}_1^{J_b} \text{ and } \mathcal{E}_{b''}^{J_b} \\
\Big\| \\
\text{modifications of type } \mu^{-1} \text{ between } \mathcal{E}_{b''}^{J_b} \text{ and } \mathcal{E}_1^{J_b}.
\end{array}$$

In the end this induces a $J_b(F) \times G(F)$-isomorphism of local Shimura varieties with infinite level

$$\mathcal{M}(G,\mu,b)_\infty \xrightarrow{\sim} \mathcal{M}(J_b, \mu^{-1}, b'')_\infty$$

as pro-étale sheaves on $\mathrm{Spa}(\breve{E})$ (that are representable by diamonds). This fits into a twin towers diagram using the de Rham and Hodge-Tate period morphisms that allow us to collapse each tower on its base

$$\begin{tikzcd}
\mathcal{M}(G,\mu,b)_\infty \arrow[rr, "\sim"] \arrow[d, two heads, "\pi_{dR}"] \arrow[dr] & & \mathcal{M}(J_b,\mu^{-1},b'')_\infty \arrow[dl] \arrow[d, two heads, "\pi_{dR}"] \\
\mathcal{F}(G,\mu,b)^a & \pi_{HT} & \mathcal{F}(J_b,\mu^{-1},b'')^a
\end{tikzcd}$$

$G(F)$ $\qquad$ $J_b(F)$

where:

- $\mathcal{M}(G,\mu,b)_\infty$ classifies modifications of type μ between $\mathcal{E}_b^G$ and $\mathcal{E}_1^G$.

- For such a modification its image by π_{dR} is x if $\mathcal{E}_1^G = \mathcal{E}_{b,x}^G$. Its image by π_{HT} is y if $\mathcal{E}_b^G = \mathcal{E}_{1,y}^G$.
- $\mathcal{M}(J_b, \mu^{-1}, b'')_\infty$ classifies modifications of type μ^{-1} between $\mathcal{E}_{b''}^{J_b}$ and $\mathcal{E}_1^{J_b}$.
- For such a modification, its image by π_{dR} is x if $\mathcal{E}_1^{J_b} = \mathcal{E}_{b'',x}^{J_b}$. Its image by π_{HT} is y if $\mathcal{E}_{b''}^{J_b} = \mathcal{E}_{1,y}^{J_b}$.

We will extend this type of diagram outside the admissible locus in section 5.3.

5.2. **Computation of the modifications of $\mathcal{E}_b$.** Let $[b] \in B(G)$ be any basic element.

Proposition 5.2 ([51] A.10)**.** *As a subset of $B(G)$ there is an equality*

$$\{\mathcal{E}_{b,x} | x \in \mathcal{F}(G,\mu)(C)\}/\sim = B(G, \kappa(b) - \mu^\sharp, \nu_b\mu^{-1}).$$

Proof. Let $f : B(G) \xrightarrow{\sim} B(J_b)$. According to the twin towers principle

$$\{\mathcal{E}_{b,x} \mid x \in \mathcal{F}(G,\mu)(C)\}/\sim = f^{-1}\big(\{\mathcal{E}_{1,x} \mid x \in \mathcal{F}(J_b,\mu)(C)\}/\sim\big).$$

Now,

$$\{\mathcal{E}_{1,x} \mid x \in \mathcal{F}(J_b,\mu)(C)\}/\sim = \{[b'] \in B(J_b) \mid \exists y \in \mathcal{F}(J_b,\mu^{-1})(C),\ \mathcal{E}_{b',y} \simeq \mathcal{E}_1\}.$$

The condition on the right hand side means $y \in \mathcal{F}(J_b,\mu^{-1},b')^a$ and $\kappa(b') = (\mu^{-1})^\sharp$. Now,

$$\mathcal{F}(J_b,\mu^{-1},b')^a \neq \emptyset \Leftrightarrow [b'] \in A(J_b,\mu^{-1})$$

(see 3.2). Thus,

$$\{\mathcal{E}_{1,x} \mid x \in \mathcal{F}(J_b,\mu)(C)\}/\sim = B(J_b,\mu^{-1}).$$

Moreover, via the identifications $\mathcal{N}(G) = \mathcal{N}(J_b)$ and $\pi_1(G) = \pi_1(J_b)$, we have

$$\nu \circ f^{-1} = \nu_b + \nu$$
$$\kappa \circ f^{-1} = \kappa(b) + \kappa.$$

The result follows immediately. □

5.3. **The Harder-Narasimhan stratification.** Suppose now that $[b] \in B(G,\mu)$ is the basic element. According to proposition 5.2 there is a stratification

$$\mathcal{F}(G,\mu) = \coprod_{[b'] \in B(G,0,\nu_b\mu^{-1})} \mathcal{F}(G,\mu,b)^{[b']},$$

where $\mathcal{F}(G,\mu,b)^{[b']}$ is a locally closed generalizing subset of the adic space $\mathcal{F}(G,\mu)$ that defines a locally spatial sub diamond of $\mathcal{F}(G,\mu)^\diamond$. Here the fact that each stratum is locally closed can be deduced from Kedlaya-Liu's semi-continuity theorem of the Harder-Narasimhan polygon ([40]). The open stratum is

$$\mathcal{F}(G,\mu,b)^{[1]} = \mathcal{F}(G,\mu,b)^a,$$

the admissible locus.

One can describe each stratum in the following way. Fix $[b'] \in B(G,0,\nu_b\mu^{-1})$ and let $[b''] \in B(J_b,\mu^{-1})$ be the corresponding element. We note

$$\widetilde{J}_{b'} = \underline{\mathrm{Aut}}(\mathcal{E}_{b'})$$

the pro-étale sheaf of automorphisms of $\mathcal{E}_{b'}$ on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$; that is, $S \mapsto \mathrm{Aut}(\mathcal{E}_{b'|X_S})$. One has

$$\widetilde{J}_{b'} = \widetilde{J}_{b'}^0 \rtimes \underline{J_{b'}(F)},$$

where $\widetilde{J}^0_{b'}$ is a connected unipotent diamond that is a successive extension of effective Banach-Colmez spaces (see [26] and [30]). This is identified with the same object for b'':

$$\widetilde{J}_{b''} = \widetilde{J}_{b'}.$$

Now let

$$\mathcal{T} \longrightarrow \mathcal{F}(G,\mu,b)^{[b'],\diamond}$$

be the pro-étale sheaf of isomorphisms between $\mathcal{E}_{b'}$ and $\mathcal{E}_{b,x}$, $x \in \mathcal{F}(G,\mu,b)^{[b'],\diamond}$. Using again a result of Kedlaya-Liu ([40]) one can check this is a $\widetilde{J}_{b'}$-torsor. More precisely, this is deduced from the fact that the Harder-Narasimhan stratum in the perfectoid stack Bun_G attached to b' is the v-stack classifying $\widetilde{J}_{b'}$-torsors ([30]). Now, using the twin towers principle, $\mathcal{T}$ is identified with

$$\mathcal{M}(J_b,\mu^{-1},b'')_\infty$$

and the morphism to $\mathcal{F}(G,\mu)$ with the Hodge-Tate period morphism π_{HT}.

Proposition 5.3. *If $[b''] \in B(J_b,\mu^{-1})$ corresponds to $[b'] \in B(G,0,\nu_b\mu^{-1})$ then*

$$\mathcal{F}(G,\mu,b)^{[b']} = \mathrm{Im}\big(\mathcal{M}(J_b,\mu^{-1},b'')_\infty \xrightarrow{\pi_{HT}} \mathcal{F}(J_b,\mu) = \mathcal{F}(G,\mu)\big).$$

The morphism π_{HT} is a $\widetilde{J}_{b''}$-torsor and thus

$$\mathcal{F}(G,\mu,b)^{[b'],\diamond} \simeq \mathcal{M}(J_b,\mu^{-1},b'')_\infty \,/\, \widetilde{J}_{b''}.$$

One has

$$\dim \mathcal{F}(G,\mu,b)^{[b']} = \dim \mathcal{F}(G,\mu) - \langle [\nu_{b'}], 2\rho\rangle.$$

Here the dimension of a locally spatial diamond is the maximal length of a chain of specializations in its underlying locally spectral space ([56] sec. 21).

Proof. The fact that π_{HT} is a $\widetilde{J}_{b''}$-torsor is a consequence of the structure of the HN stratum corresponding to b'' in Bun_{J_b} recalled before. For the assertion about the dimension we use lemma 3.2.5 in [2]. Using it we deduce that

$$\dim \mathcal{F}(G,\mu,b)^{[b']} = \dim \mathcal{M}(J_b,\mu^{-1},b'')_\infty - \dim \widetilde{J}_{b''}.$$

Since $\pi_{dR} : \mathcal{M}(J_b,\mu^{-1},b'')_\infty \to \mathcal{F}(J_b,\mu^{-1})$ is pro-étale one has

$$\dim \mathcal{M}(J_b,\mu^{-1},b'')_\infty = \dim \mathcal{F}(G,\mu).$$

The formula

$$\dim \widetilde{J}_{b''} = \langle [\nu_{b'}], 2\rho\rangle$$

is obtained by writing $\widetilde{J}^0_{b''}$, the neutral connected component, as a successive extension of Banach-Colmez spaces, a use of lemme 3.2.5 in [2], and the fact that $\widetilde{J}_{b''} = \widetilde{J}^0_{b''} \rtimes \underline{J_{b''}(\mathbb{Q}_p)}$. □

We can thus again collapse the tower on two different bases

$$\begin{tikzcd}
 & \mathcal{M}(J_b,\mu^{-1},b'')_\infty \arrow[dl, "\pi_{dR}"'] \arrow[dr, "\pi_{HT}"] \arrow[dl, bend right, dotted, "J_b(F)"'] \arrow[dr, bend left, dotted, "\widetilde{J}_{b''}=\widetilde{J}_{b'}"] & \\
\mathcal{F}(J_b,\mu^{-1},b'')^a & & \mathcal{F}(G,\mu,b)^{[b'],\diamond}
\end{tikzcd}$$

6. Proof of the main theorem

As before, we consider a triple $(G, \{\mu\}, [b])$ with $\{\mu\}$ minuscule. In sections 2 and 3 we have introduced two open subspaces $\mathcal{F}(G,\mu,b)^a \subset \mathcal{F}(G,\mu,b)^{wa}$ of $\mathcal{F}(G,\mu)$. In general, the inclusion $\mathcal{F}(G,\mu,b)^a \subset \mathcal{F}(G,\mu,b)^{wa}$ is strict, see [35] Example 3.6, [36] Example 6.7. In [36] section 9 and [51] A.20, Hartl and Rapoport asked when do we have

$$\mathcal{F}(G,\mu,b)^a = \mathcal{F}(G,\mu,b)^{wa} \ ?$$

For $G = \mathrm{GL}_n$, Hartl gave a complete solution of this question in Theorem 9.3 of [36]. We give a complete solution to this problem for any G when $[b]$ is basic.

Here is the main theorem of this article.

Theorem 6.1. *If μ minuscule and $[b] \in B(G,\mu)$ is basic, then the following are equivalent:*

(1) *$B(G,\mu)$ is fully HN decomposable,*
(2) *$\mathcal{F}(G,\mu,b)^a = \mathcal{F}(G,\mu,b)^{wa}$.*

Proof. We first treat the case when G is quasi-split.
(1)⇒(2) [quasi-split case]. Let $x \in \mathcal{F}(G,\mu)(C) \setminus \mathcal{F}(G,\mu,b)^a(C)$. We want to prove that $x \notin \mathcal{F}(G,\mu,b)^{wa}(C)$. Let $[b'] \in B(G, 0, \nu_b\mu^{-1})$ be such that

$$\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$$

(proposition 5.2). Let $M_{b'}$ be the centralizer of $|\nu_{b'}|$, a standard Levi subgroup. Up to replacing b' by a σ-conjugate element we can suppose $b' \in M_{b'}(\breve{E})$. Since x is not admissible, b' is non basic. According to corollary 4.16 and lemma 4.13, there exists a strict standard Levi subgroup M of G containing $M_{b'}$ such that

$$\kappa_M(b'_M) \otimes 1 = [\nu_b] + w_0(-\mu) \in \pi_1(M)_\Gamma \otimes \mathbb{Q} \tag{2}$$

where b'_M is the image in M of b'. Here we assume $\mu \in X_*(\boldsymbol{T})^+$ as usual and $w_0.(-\mu)$ is seen as a cocharacter of M. Let P be the standard parabolic subgroup associated to M. We note b'_P for the image of b'_M in P. By lemma 2.4, the reduction $\mathcal{E}_{b'_P}$ of $\mathcal{E}_{b'}$ induces a reduction

$$\mathcal{E}_{b,P}$$

of $\mathcal{E}_b$ to P. Let $[\tilde{b}] \in B(M)$ be such that

$$\mathcal{E}_{\tilde{b}} \simeq \mathcal{E}_{b,P} \overset{P}{\times} M.$$

Let us note $\tilde{b}_G$ for the image of $\tilde{b}$ in G. We are going to prove that $[\tilde{b}_G] = [b]$ in $B(G)$.

According to lemma 2.6 there exists $\mu_1 \in W.\mu$ and $y \in \mathcal{F}(M, \mu_1^{-1})$ such that

$$\mathcal{E}_{\tilde{b}} \simeq \mathcal{E}_{b'_M, y}.$$

We can suppose μ_1 is in the *negative* Weyl chamber associated to M. We have

$$\kappa_M(\tilde{b}) = \kappa_M(b'_M) + \mu_1^\sharp \in \pi_1(M)_\Gamma.$$

Using equation (2) this implies

$$\kappa_M(\tilde{b}) \otimes 1 = \nu_b + [w_0.(-\mu)]^\sharp \otimes 1 + \mu_1^\sharp \otimes 1 \in \pi_1(M)_\Gamma \otimes \mathbb{Q}. \tag{3}$$

One can identify

$$\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} = (\pi_1(M)_\Gamma \otimes \mathbb{Q})^\vee.$$

Via this equality (3) gives that for any $\chi : P/Z_G \to \mathbb{G}_m$,

$$\deg \chi_* \mathcal{E}_{b,P} = \underbrace{\langle \nu_b, \chi \rangle}_{0 \text{ since } \chi_{|Z_G}=1} + \langle w_0.(-\mu), \chi \rangle + \langle \mu_1, \chi \rangle.$$

Since $\mathcal{E}_b$ is semi-stable one obtains

$$\forall \chi \in X^*(P/Z_G)^{+,\Gamma}, \quad \langle w_0.(-\mu), \chi\rangle + \langle \mu_1, \chi\rangle \leq 0.$$

Lemma 6.3 then shows that

$$\mu_1 = -w_0.(-\mu).$$

Inserting this in equation (3) we obtain

$$\kappa_M(\tilde{b}) \otimes 1 = \nu_b \in \pi_1(M)_\Gamma \otimes \mathbb{Q}.$$

We can now conclude that $[\tilde{b}^G] = [b]$ using lemma 6.4 that follows.

Now, since $M_{b'} \subset M$, if we choose $\chi \in X^*(P/Z_G)^{+,\Gamma} \cap \mathbb{N}.\Delta_0$, then

$$\deg \chi_* \mathcal{E}_{b'_M} > 0.$$

Hence by proposition 2.7 x is not weakly admissible.

(2)⇒(1) [quasi-split case]. We show that if $B(G,\mu)$ is not fully HN decomposable, then $\mathcal{F}(G,\mu,b)^a \subsetneq \mathcal{F}(G,\mu,b)^{wa}$. We may assume that G is simple and adjoint. In fact, $\mathcal{F}(G,\mu,b)^{wa} = \mathcal{F}(G_{ad},\mu_{ad},b_{ad})^{wa}$ and $\mathcal{F}(G,\mu,b)^a = \mathcal{F}(G_{ad},\mu_{ad},b_{ad})^a$.

According to corollary 4.16, $B(J_b,\mu^{-1})$ is not fully HN decomposable. We now use the construction at the end of the proof of proposition 4.14. Let $\alpha \in \Delta_0$ be such that

$$\langle -w_0\mu^\diamond, \tilde{\omega}_\alpha\rangle > 1.$$

Let M and P be the associated standard maximal Levi and parabolic subgroups, $\Delta_{0,M} = \Delta_0 \setminus \{-w_0\alpha\}$. Let $b'_M \in B(M)_{basic}$ be such that

$$\kappa_M(b'_M) = (-\beta^\vee)^\sharp$$

with $\beta \in \Delta$ and $\beta_{|\boldsymbol{A}} = -w_0\alpha$. Let b' be the image of b'_M in G. Then $[\nu_{b'_M}] \in \mathcal{N}(M)$ is G-antidominant and $[\nu_{b'}] = w_0[\nu_{b'_M}]$. Moreover, the image of $[b'] \in B(G,0,\nu_b\mu^{-1})$ in $B(J_b,\mu^{-1})$ is HN-indecomposable.

Claim: We may assume that $[b'] \in B(G,0,\nu_b\mu^{-1})\backslash\{[1]\}$ is minimal (i.e., there doesn't exist $[\hat{b}'] \in B(G,0,\nu_b\mu^{-1})\backslash\{[1]\}$ satisfying $[\nu_{\hat{b}'}] \leq [\nu_{b'}]$ and $[\hat{b}'] \neq [b']$).

Suppose $[b'] \in B(G,0,\nu_b\mu^{-1})\backslash\{[1]\}$ is not minimal. Then there exists $[\hat{b}'] \in B(G,0,\nu_b\mu^{-1})\backslash\{[1]\}$ which is minimal and $[\nu_{\hat{b}'}] \leq [\nu_{b'}]$. Let $\hat{M} := \mathrm{Cent}(w_0\nu_{\hat{b}'})$. Then it's a maximal Levi subgroup by the minimality of $\hat{b}'$. Suppose $\Delta_{0,\hat{M}} = \Delta_0\backslash\{-w_0\hat{\alpha}\}$ for some $\hat{\alpha} \in \Delta_0$. Still by the minimality of $[\hat{b}']$, we can show that $[\hat{b}']$ is the image of some $[\hat{b}'_{\hat{M}}] \in B(\hat{M})_{basic}$ in $B(G)$, where $\hat{b}'_{\hat{M}}$ satisfies $\kappa_{\hat{M}}(\hat{b}'_{\hat{M}}) = (-\hat{\beta}^\vee)^\sharp$ with $\hat{\beta} \in \Delta$ and $\hat{\beta}_{|\boldsymbol{A}} = -w_0\hat{\alpha}$. Moreover,

$$\begin{aligned}&\langle -w_0\mu^\diamond, \tilde{\omega}_{\hat{\alpha}}\rangle \\ =&\underbrace{\langle -w_0\mu^\diamond - [\nu_{b'}], \tilde{\omega}_{\hat{\alpha}}\rangle}_{>0} + \underbrace{\langle [\nu_{b'}] - [\nu_{\hat{b}'}], \tilde{\omega}_{\hat{\alpha}}\rangle}_{\geq 0} + \underbrace{\langle [\nu_{\hat{b}'}], \tilde{\omega}_{\hat{\alpha}}\rangle}_{=1} > 1\end{aligned}$$

where the first part is positive since $[b']$ is HN-indecomposable in $B(G,0,\nu_b\mu^{-1})$ and the fact that G is simple, thus the relative root system of G is irreducible. Therefore, we may replace α, β, b', M, b'_M by $\hat{\alpha}$, $\hat{\beta}$, $\hat{b}'$, $\hat{M}$, $\hat{b}'_{\hat{M}}$ respectively and the Claim follows.

Let us note

$$Z = \{x \in \mathcal{F}(G,\mu)(C) \mid \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}.$$

We want to show $Z \cap \mathcal{F}(G,\mu,b)^{wa} \neq \emptyset$.

Let $x \in Z$ be such that $\mathcal{E}_{b,x}$ is not weakly admissible. Then there exists a standard maximal parabolic subgroup Q, a reduction $b_{M_Q}(x)$ (that depends on x) of b to M_Q and $\chi \in X^*(Q/Z_G)^+$ such that

$$\deg \chi_*(\mathcal{E}_{b,x})_Q > 0. \tag{4}$$

According to theorem 1.8 the vector

$$\begin{aligned} v : X^*(Q/Z_G) &\longrightarrow \mathbb{Z} \\ \chi &\longmapsto \deg \chi_*(\mathcal{E}_{b,x})_Q \end{aligned}$$

seen as an element of $X_*(A)_{\mathbb{Q}}$ satisfies $v \leq \nu_{\mathcal{E}_{b,x}}$. The existence of χ implies that $v \in \mathcal{N}(G)$ and it's non-trivial. Therefore

$$v \leq \nu_{\mathcal{E}_{b,x}} = -w_0[\nu_{b'}]$$

is the usual dominant order. It follows that $-w_0 v \leq [\nu_{b'}]$. By the description of $B(G, 0, \nu_b\mu^{-1})$ in terms of Newton map in corollary 4.4, we see that $-w_0 v \in B(G, 0, \nu_b\mu^{-1})$. By the minimality of $[b'] \in B(G, 0, \nu_b\mu^{-1}) \setminus \{[1]\}$ in the Claim, we deduce that $Q = P$ and $(\mathcal{E}_{b,x})_Q$ is the Harder-Narasimhan canonical reduction of $\mathcal{E}_{b,x}$.

We now note $b_M(x) := b_{M_Q}(x)$. Let $\mu_1 \in W.\mu$ be such that

$$\mathcal{E}_{b'_M} \simeq (\mathcal{E}_{b,x})_P \overset{P}{\times} M \simeq \mathcal{E}_{b_M(x),y}$$

with $y \in \mathcal{F}(M, \mu_1)$. One then has

$$\kappa_M(b'_M) = \kappa_M(b_M(x)) - \mu_1^\sharp \in \pi_1(M)_\Gamma.$$

Pushing forward this equality in $\pi_1(M)_\Gamma \otimes \mathbb{Q} = X_*(Z_M)^\Gamma_{\mathbb{Q}}$ one obtains

$$[\nu_{b'_M}] = [\nu_b] - \mu_1^\sharp \otimes 1 \in X_*(Z_M)^\Gamma_{\mathbb{Q}}.$$

Here we use that since b, and thus $b_M(x)$, is basic then $[\nu_{b_M(x)}] = [\nu_b]$. We now use the diagram at the end of section 5.3. Let $[b''] \in B(J_b)$ corresponding to $[b'] \in B(G)$. Let us look at

$$\begin{tikzcd} & \mathcal{M}(J_b, \mu^{-1}, b'')_\infty(C) \arrow[dl, two heads, "\pi_{dR}"'] \arrow[dr, two heads, "\pi_{HT}"] & \\ \mathcal{F}(J_b, \mu^{-1}, b'')^a(C) & & \mathcal{F}(G, \mu, b)^{[b']}(C) = Z \end{tikzcd}$$

The twin towers principle extends to P and M-torsors. More precisely, the reductions $b_M(x)$ and $b_P(x)$ define a Levi and parabolic subgroup of J_b i.e. M and P transfer to the inner form J_b. We still denote them M and P (we can fix them so that they do not depend on the choice of x, in fact a Levi subgroup, resp. a parabolic subgroup, of G transfers to at most one conjugacy class of Levi subgroup, resp. parabolic subgroup, of J_b). Moreover b'' admits a reduction $b''_M(x)$ to M. Then

$$\pi_{dR}\big(\pi_{HT}^{-1}(x)\big)$$

lies in the locus of points $z \in \mathcal{F}(J_b, \mu^{-1}, b'')^a(C)$ where

$$(\mathcal{E}_{b'',z})_P \overset{P}{\times} M \simeq \mathcal{E}_{b''_M(x),s} \simeq \mathcal{E}_1$$

for some $s \in \mathcal{F}(M, \mu_1^{-1})$. The open Schubert cell in $\mathcal{F}(J_b, \mu^{-1})$ with respect to the action of P is $Pw_0P_{w_0\mu^{-1}}/P_{w_0\mu^{-1}}$. The point z is thus in this Schubert cell if and only if

$$-\mu_1 \otimes 1 \equiv -\mu \otimes 1 \in \pi_1(M) \otimes \mathbb{Q}.$$

Projected to $\pi_1(M)_\Gamma \otimes \mathbb{Q}$ this is equivalent to

$$\mu \otimes 1 - \beta^\vee \otimes 1 \equiv \nu_b.$$

But this is impossible since

$$\langle \mu - \beta^\vee, \tilde{\omega}_{-w_0\alpha} \rangle = \langle -w_0\mu, \tilde{\omega}_\alpha \rangle - \langle \beta^\vee, \tilde{\omega}_{-w_0\alpha} \rangle > 0$$

and is thus non zero.

From this analysis we deduce that if $Z \subset \mathcal{F}(G,\mu) \setminus \mathcal{F}(G,\mu,b)^{wa}$ then

$$\mathrm{Im}(\pi_{dR}) = \mathcal{F}(J_b, \mu^{-1}, b'')^a$$

is contained in a profinite (indexed by $J_b(F)/P(F)$, the set of reductions of b'' to P) union of non-open Schubert cells in $\mathcal{F}(J_b, \mu^{-1})$. This is in contradiction with the openness of the admissible locus. Thus $Z \cap \mathcal{F}(G,\mu,b)^{wa} \neq \emptyset$ and therefore $\mathcal{F}(G,\mu,b)^a \subsetneq \mathcal{F}(G,\mu,b)^{wa}$.

Let us now explain how to treat the case of a general G which is not necessarily quasi-split. One still suppose G is adjoint. Since H is adjoint, $H^1(F,H) = B(H)_{basic}$ and G is an extended pure inner form of H, $G = J_{b^*}$ with $[b^*] \in B(H)$ basic. Via $B(G) \xrightarrow{\sim} B(H)$ let $[b] \mapsto [b^H]$. Then

$$\mathcal{F}(G,\mu,b)^{wa} = \mathcal{F}(H,\mu,b^H)^{wa},$$
$$\mathcal{F}(G,\mu,b)^a = \mathcal{F}(H,\mu,b^H)^a.$$

Here $[b^H] \in B(H, \mu^\sharp + \kappa(b^*), \mu^\diamond)$ is the basic element. Via $B(G) \xrightarrow{\sim} B(H)$ the set $B(G, 0, w_0\mu^{-1,\diamond})$ is sent to $B(H, \kappa(b^*), w_0\mu^{-1,\diamond})$. Then one checks that all the preceding arguments are valid in this context, working with $B(H, \mu^\sharp + \kappa(b^*), \mu^\diamond)$ and $B(H, \kappa(b^*), w_0\mu^{-1,\diamond})$ instead of $B(H,\mu)$ and $B(H, 0, w_0\mu^{-1,\diamond})$. □

Remark 6.2. *The case $G = \mathrm{GL}_n$ was dealt by Hartl in [36]. In this case, the proof of the main theorem can be illustrated and simplified as follows. Suppose $\mu = (1^r, 0^{n-r})$ with $0 < r < n$ and $[b] \in B(G,\mu)$ is the unique basic element. Let $d = \gcd(r,n)$. Then $\mathcal{E}_b$ can be viewed as the vector bundle $\mathcal{O}(-\frac{r}{n})^d$ on the Fargues-Fontaine curve X.*

Suppose (G,μ) is fully HN decomposable. For any $x \in \mathcal{F}(G,\mu)(C) \setminus \mathcal{F}(G,\mu,b)^a(C)$, Let

$$\mathcal{E}_{b,x} = \mathcal{E}_{b'}$$

with $[b'] \in B(G, 0, \nu_b\mu^{-1}) \setminus \{[1]\}$. The Newton polygon of $\nu_{b'}$ and the Newton polygon of $\nu_b\mu^{-1}$ has a touch point with integral coordinates by the fully HN decomposable condition. This touch point corresponds to a direct summand (subbundle) $\mathcal{F}$ of $\mathcal{E}_{b,x}$ with positive degree which comes from a subisocrystal of the isocrystal given by $[b]$. Therefore x is not weakly admissible.

If (G,μ) is not fully HN decomposable, then we can always find $[b'] \in B(G, 0, \nu_b\mu^{-1})$ with

$$\mathcal{E}_{b'} = \mathcal{O}(\frac{1}{u}) \oplus \mathcal{O}(-\frac{1}{n-u})$$

for some integer $u \in [1, n-1] \setminus \frac{n}{d}\mathbb{N}$. Choose any element $x \in \mathcal{F}(G,\mu)(C)$ such that $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$. For any sub-vector bundle $\mathcal{F}$ of $\mathcal{E}_{b,x}$ which comes from a subisocrystal of the isocrystal given by $[b]$, we have $\mathrm{rank}\,\mathcal{F} \in \frac{n}{d}\mathbb{N}$. In particular,

$$\mathcal{F} \not\simeq \mathcal{O}(\frac{1}{u}).$$

It follows that $\deg \mathcal{F} \leq 0$. Therefore x is weakly admissible but not admissible.

In the following lemmas G is quasi-split.

Lemma 6.3. *Consider $\mu \in X_*(\boldsymbol{T})^+$, M a standard Levi subgroup of G with associated standard parabolic subgroup P and $\mu_1 \in W.\mu$ that is in the positive Weyl chamber associated to M. If for all $\chi \in X^*(P/Z_G)^{+,\Gamma}$ one has $\langle \mu - \mu_1, \chi \rangle \leq 0$ then $\mu_1 = \mu$.*

Proof. The condition $\langle \mu - \mu_1, \chi \rangle \leq 0$ for all $\chi \in X^*(P/Z_G)^{+,\Gamma}$ implies that $\mu - \mu_1$ is a linear combination of positive simple coroots in M with positive coefficients. The result follows since μ_1 and μ are both M-dominant. □

Lemma 6.4. *Let $\mathcal{E}$ be a semi-stable G-bundle on X equipped with a reduction $\mathcal{E}_P$ to the standard parabolic subgroup P with standard Levi subgroup M_P such that $c_1^{M_P}(\mathcal{E}_P \overset{P}{\times} M_P) \equiv \nu_{\mathcal{E}}$ in $\pi_1(M_P)_\Gamma \otimes \mathbb{Q}$, then $\mathcal{E}_P \overset{P}{\times} M_P$ is in fact a reduction of $\mathcal{E}$ to M_P.*

Proof. Let us begin with some generalities about reductions to parabolic subgroups. Let us forget momentarily the hypothesis of the statement. Let $\mathcal{E}$ be a G-torsor on X. Let Q be a standard parabolic subgroup of G such that $Q \subset P$ with associated standard Levi subgroup M_Q. Then $Q \cap M_P$ is a standard parabolic subgroup of M_P whose standard Levi subgroup is M_Q and all of them are of this type. The morphism

$$Q\backslash\mathcal{E} \longrightarrow P\backslash\mathcal{E}$$

is a locally trivial fibration with fiber $Q\backslash P = M_P \cap Q\backslash M_P$. If $\mathcal{E}_P$ is a reduction of $\mathcal{E}$ to P corresponding to the section s of $P\backslash\mathcal{E} \to X$ then the pullback by s of the preceding fibration is

$$M_P \cap Q\backslash(\mathcal{E}_P \overset{P}{\times} M_P) \longrightarrow X.$$

As a consequence there is a bijection between

- reductions $\mathcal{E}_Q$ of $\mathcal{E}$ to Q
- reductions $\mathcal{E}_P$ of $\mathcal{E}$ to P together with a reduction $(\mathcal{E}_P \overset{P}{\times} M_P)_{M_P \cap Q}$ of $\mathcal{E}_P \overset{P}{\times} M_P$ to $M_P \cap Q$.

Let us come back to our statement. We first prove that $\mathcal{E}_P \overset{P}{\times} M_P$ is semi-stable. Thus, let $Q \subset P$ be as before and $(\mathcal{E}_P \overset{P}{\times} M_P)_{M_P \cap Q}$ be a reduction corresponding to the reduction $\mathcal{E}_Q$. One has

$$X^*(Q) \xrightarrow{\sim} X^*(M_P \cap Q) \xrightarrow{\sim} X^*(M_Q).$$

For $\chi \in X^*(Q)$ one has

$$\chi_* \mathcal{E}_Q = \chi_{|M_P \cap Q *}(\mathcal{E}_P \overset{P}{\times} M_P)_{M_P \cap Q}.$$

Now, suppose $\chi_{|M_P \cap Q} \in X^*(M_P \cap Q/Z_{M_P})^+$. Then one can write $\chi = \chi_1 + \chi_2$ with

$$\chi_1 \in X^*(Q/Z_G)^+ \quad \text{and} \quad \chi_2 \in X^*(P/Z_G).$$

Then,

$$\deg \chi_{|M_P \cap Q *}(\mathcal{E}_P \overset{P}{\times} M_P)_{M_P \cap Q} = \underbrace{\deg \chi_{1*}\mathcal{E}_Q}_{\leq 0 \text{ by s.s. of } \mathcal{E}} + \underbrace{\deg \chi_{2*}\mathcal{E}_P}_{\langle \nu_{\mathcal{E}}, \chi \rangle = 0} \leq 0.$$

Thus, $\mathcal{E}_P \overset{P}{\times} M_P$ is semi-stable with slope $\nu_{\mathcal{E}}$. Now, as

$$c_1^G\big((\mathcal{E}_P \overset{P}{\times} M_P) \overset{M_P}{\times} G\big) = c_1^G(\mathcal{E}),$$

one concludes that $(\mathcal{E}_P \overset{P}{\times} M_P) \overset{M_P}{\times} G \simeq \mathcal{E}$ using the injectivity of

$$(c_1^G, \nu) : H^1_{\text{ét}}(X, G) \longrightarrow \pi_1(G)_\Gamma \times \mathcal{N}(G).$$

□

Remark 6.5. *For* GL_n *the preceding lemma says that if* $\mathcal{E}$ *is a semi-stable vector bundle equipped with a finite filtration* $(\mathrm{Fil}^i\mathcal{E})_{i\in\mathbb{Z}}$ *whose graded pieces satisfy for all* i, $\mu(\mathrm{Gr}^i\mathcal{E}) = \mu(\mathcal{E})$, *then* $\mathcal{E} \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}^i\mathcal{E}$. *In fact, the category of slope* $\mu(\mathcal{E})$ *semi-stable vector bundles is abelian. From this one deduces by induction on* i *that the* $\mathrm{Fil}^i\mathcal{E}$ *and the* $\mathrm{Gr}^i\mathcal{E}$, $i \in \mathbb{Z}$, *are semi-stable of slope* $\mu(\mathcal{E})$. *One concludes using that if* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are semi-stable with* $\mu(\mathcal{F}_1) = \mu(\mathcal{F}_2)$ *then* $\mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}_2) = 0$.

7. Asymptotic geometry of the admissible locus

In this section we sketch some ideas about fully HN decomposable period spaces. We suppose μ minuscule, $[b] \in B(G,\mu)$ is the basic element and $B(G,\mu)$ is fully HN decomposable. Let us set

$$\partial\mathcal{F}(G,\mu,b)^a = \mathcal{F}(G,\mu,b) \setminus \mathcal{F}(G,\mu,b)^a.$$

One has the stratification

$$\partial\mathcal{F}(G,\mu,b)^a = \coprod_{[b']\in B(G,0,\nu_b\mu^{-1})\setminus\{[1]\}} \mathcal{F}(G,\mu,b)^{[b']}.$$

Suppose G is quasi-split. Then there is a bijection

$$\{\text{parabolic subgroups of } G \text{ which admit a reduction of } b\}/\sim$$
$$\xrightarrow{\sim}\{\text{parabolic subgroups of } J_b\}.$$

A parabolic subgroup of G transfer to the inner form J_b if and only if b has a reduction to this parabolic subgroup. Now for each $x \in \partial\mathcal{F}(G,\mu,b)^a(C)$ the canonical reduction $(\mathcal{E}_{b,x})_P$ defines a reduction $(\mathcal{E}_b)_P$. According to the first part of the proof of theorem 6.1 this corresponds to a reduction of b_P, $(\mathcal{E}_b)_P = \mathcal{E}_{b_P}$.

The same type of analysis can be carried out if G is non-quasi-split, writing $G_{ad} = J_{b^*}$ with $[b^*] \in B(H_{ad})_{basic}$ as at the end of the proof of theorem 6.1. At the end one obtains the following result.

Proposition 7.1. *Let* $\mathcal{P}$ *be a set of representatives of the conjugacy classes of proper parabolic subgroups of* J_b. *There is a* $J_b(F)$*-invariant stratification by locally closed generalizing subsets*

$$\partial\mathcal{F}(G,\mu,b)^a = \bigcup_{P\in\mathcal{P}} \partial_P\mathcal{F}(G,\mu,b)^a$$

together with a $J_b(F)$*-equivariant continuous map* $\partial_P\mathcal{F}(G,\mu,b)^a \to J_b(F)/P(F)$.

Thus, $\mathcal{F}(G,\mu,b)^a$ *shares a lot of similarities with hermitian symmetric spaces*: its boundary is parabolically induced. Another way to see this is via the *generalized Boyer's trick* also known as Hodge-Newton decomposition ([5] for the original trick, [37] for its Shimura varieties variant, [46], [47], [60] for generalizations in the PEL case, [32] for the special fiber in general, [34] and [31] for "modern" versions in the context of local Shtuka moduli spaces). In fact for $[b'] \in B(G,0,\nu_b\mu^{-1}) \setminus \{[1]\}$ we have the $\tilde{J}_{b''}$-torsor (see sec. 5.3)

$$\mathcal{M}(J_b,\mu^{-1},b'')_\infty \xrightarrow{\pi_{HT}} \mathcal{F}(G,\mu,b)^{[b']}.$$

Now $[b''] \in B(J_b,\mu^{-1})$ is HN decomposable (cf. corollary 4.16) and this local Shtuka moduli space is parabolically induced.

Let us now state the following conjecture about the existence of analogues of *Siegel domains* in hermitian symmetric spaces.

Conjecture 7.2. *For $[b] \in B(G,\mu)$ basic with μ minuscule the following are equivalent:*

(1) $\mathcal{F}(G,\mu,b)^a = \mathcal{F}(G,\mu,b)^{wa}$.
(2) *There exists a quasi-compact open subset $U \subset \mathcal{F}(G,\mu,b)^a$ such that*

$$J_b(F).U = \mathcal{F}(G,\mu,b)^a.$$

Condition (2) is equivalent to the fact that there exists a quasi-compact open subset $V \subset \mathcal{M}(G,\mu,b)_\infty$ such that $J_b(F) \times G(F).V = \mathcal{M}(G,\mu,b)_\infty$. Reduction theory and the construction of Siegel domains can be done via stability conditions in the Arakelov setting ([15]). In our setting this can be done in some particular cases, in the PEL case, using integral semi-stability conditions for finite flat group schemes ([23], see in particular corollary 11 where the link is made with the Laffaille/Gross-Hopkins fundamental domain ([43], [16]). For example, the preceding conjecture is solved in [61] for $U(1,n-1)$ using this technique and methods developed in [28].

Finally let us point out that the fact that the boundary of those spaces is parabolically induced should have some cohomological consequences.

Conjecture 7.3. *If π is a smooth representation with $\overline{\mathbb{Q}}_\ell$ coefficients of $G(F)$, $\mathcal{G}_\pi = \mathcal{M}(G,\mu,b)_\infty \times_{G(F)} \pi$ the corresponding pro-étale $J_b(F)$-equivariant local system on $\mathcal{F}(G,\mu,b)^a$ the natural map*

$$R\Gamma_c\big(\mathcal{F}(G,\mu,b)^a_{\mathbb{C}_p}, \mathcal{G}_\pi\big)_{cusp} \xrightarrow{\sim} R\Gamma\big(\mathcal{F}(G,\mu,b)^a_{\mathbb{C}_p}, \mathcal{G}_\pi)_{cusp}$$

is an isomorphism in the derived category of smooth representations of $J_b(F)$ with $\overline{\mathbb{Q}}_\ell$-coefficients.

Let us give more details on the meaning of the preceding conjecture in the context of [30]. The representation π defines a local system $\mathcal{F}_\pi$ on the smooth perfectoid v-stack

$$\big[\operatorname{Spa}(\overline{\mathbb{F}}_q)/\underline{G(F)}\big] = \operatorname{Bun}_G^{0,ss},$$

the semi-stable locus of the component $\{c_1^G = 0\}$ in the perfectoid v-stack Bun_G. Let

$$j : \operatorname{Bun}_G^{ss,0} \hookrightarrow \operatorname{Bun}_G,$$

an open immersion. Then according to [30], $j_!\mathcal{F}_\pi$ is a reflexive sheaf. Consider the Hecke correspondence

$$\begin{tikzcd} & \operatorname{Hecke}_\mu \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \operatorname{Bun}_G & & \operatorname{Bun}_G \times \operatorname{Div}^1 \end{tikzcd}$$

where $\operatorname{Div}^1 = \operatorname{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}$. Then, again according to [30], the Hecke transform

$$R\overrightarrow{h}_* \overleftarrow{h}^* j_!\mathcal{F}_\pi$$

is again reflexive. Let

$$x_b : \big[\operatorname{Spa}(\overline{\mathbb{F}}_q)/\underline{J_b(F)}\big] = \operatorname{Bun}_G^{c_1^G = -\kappa(b), ss} \hookrightarrow \operatorname{Bun}_G.$$

Then

$$x_b^*\big(R\overrightarrow{h}_* \overleftarrow{h}^* j_!\mathcal{F}_\pi\big) = R\Gamma_c\big(\mathcal{F}(G,\mu,b)^a_{\mathbb{C}_p}, \mathcal{G}_\pi\big)$$

as an admissible representation of $J_b(F)$. Moreover $\mathbb{D}(j_!\mathcal{F}_\pi) = Rj_*\mathcal{F}_\pi$ (Verdier dual) and

$$x_b^*\big(R\overrightarrow{h}_* \overleftarrow{h}^* Rj_*\mathcal{F}_\pi\big) = R\Gamma\big(\mathcal{F}(G,\mu,b)^a_{\mathbb{C}_p}, \mathcal{G}_\pi\big)$$

as an admissible representation. Now, if

$$\varphi_\pi : W_E \longrightarrow {}^L G$$

is the L-parameter of π, by definition φ_π *is cuspidal* if it is discrete ($S_{\varphi_\pi}/Z(\widehat{G})^\Gamma$ is finite) and the image of the inertia by φ_π in $\widehat{G}$ is finite (i.e. the monodromy operator is trivial). When G, resp. J_b, is quasi-split, conjecturally, φ_π is cuspidal if and only if *all* elements of the L-packet of π, resp. the packet or representations of $J_b(F)$ associated to π via generalized Jacquet-Langlands, are supercuspidal. Under this hypothesis $\mathcal{F}_\pi$ is *clean*, in the sense that

$$j_! \mathcal{F}_\pi = R j_* \mathcal{F}_\pi.$$

In this case conjecture 7.3 is thus immediate. Nevertheless it may happen that one element of the L-packet is supercuspidal and not the other one in which case $\mathcal{F}_\pi$ may not be clean anymore. *Conjecture 7.3 then says that, even if this cleanliness hypothesis is not satisfied, the cohomological consequence for the associated basic RZ spaces is satisfied, if $B(G,\mu)$ is fully HN decomposable.*

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School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, No. 500, Dong Chuan Road, Shanghai 200241, China
E-mail address: mfchen@math.ecnu.edu.cn

CNRS, Institut de Mathématiques de Jussieu, 4 place Jussieu, Paris 75252, France
E-mail address: laurent.fargues@imj-prg.fr

Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55, Zhongguancun East Road, Beijing 100190, China
E-mail address: shen@math.ac.cn