



**HAL**  
open science

# $\zeta$ -functions and the topology of superlevel sets of stochastic processes

Daniel Perez

► **To cite this version:**

Daniel Perez.  $\zeta$ -functions and the topology of superlevel sets of stochastic processes. 2021. hal-03372822

**HAL Id: hal-03372822**

**<https://hal.science/hal-03372822>**

Preprint submitted on 19 Oct 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# $\zeta$ -functions and the topology of superlevel sets of stochastic processes

Daniel Perez<sup>\*1,2,3</sup>

<sup>1</sup>Département de mathématiques et applications, École normale supérieure, CNRS, PSL University, 75005 Paris, France

<sup>2</sup>Laboratoire de mathématiques d'Orsay, Université Paris-Saclay, CNRS, 91405 Orsay, France

<sup>3</sup>DataShape, Centre Inria Saclay, 91120 Palaiseau, France

October 11, 2021

## Abstract

We describe the topology of superlevel sets of ( $\alpha$ -stable) Lévy processes  $X$  by introducing so-called stochastic  $\zeta$ -functions, which are defined in terms of the widely used  $\text{Pers}_p$ -functional in the theory of persistence modules. The latter share many of the properties commonly attributed to  $\zeta$ -functions in analytic number theory, among others, we show that for  $\alpha$ -stable processes, these (tail)  $\zeta$ -functions always admit a meromorphic extension to the entire complex plane with a single pole at  $\alpha$ , of known residue and that the analytic properties of these  $\zeta$ -functions are related to the asymptotic expansion of a dual variable, which counts the number of variations of  $X$  of size  $\geq \varepsilon$ . Finally, using these results, we devise a new statistical parameter test using the topology of these superlevel sets.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Our contribution	2
<b>2</b>	<b>Generalities</b>	<b>3</b>
2.1	The Mellin transform	3
2.1.1	Analytic continuation	5
2.2	Connected components of superlevel sets of stochastic processes	7
2.2.1	Integration on trees	8
2.2.2	Duality between $N^\varepsilon$ and $\ell_p^p$	8
2.2.3	Calculation of $N^\varepsilon$ in dimension one	9
2.3	$\zeta$ -functions associated to stochastic processes	10
<b>3</b>	<b><math>\zeta</math>-functions of Lévy processes</b>	<b>11</b>
3.1	Lévy processes	12
3.1.1	Exponential corrections	19
3.1.2	Distribution of the length of branches	20
3.1.3	Statistical parameter testing for $\alpha$ -stable processes and perspectives	21
<b>4</b>	<b>Examples of applications</b>	<b>24</b>
4.1	Brownian motion	24
4.1.1	$\zeta_B$ -function and asymptotic expansions for $N^\varepsilon$	24
4.1.2	Distribution of the length of the $k$ th longest branch	27
4.2	Reflected Brownian motion	29

---

\*Email: [daniel.perez@ens.fr](mailto:daniel.perez@ens.fr)

## 1 Introduction

The problem of the characterization of the topology of superlevel sets of random functions has been a long studied topic in the theory of random fields. While a complete description has been thus far unknown, partial descriptors of the topology of superlevel sets, such as their Euler characteristic, have been described for certain classes of random processes [2–4, 8, 17, 22]. Thus far, the study of the homology of superlevel sets of random functions in dimension one has focused on either smooth random (Gaussian) fields [2, 3], or irregular processes which are in some sense canonical, such as Brownian motion [4, 8, 22]. In this paper, following the universality reasoning detailed in [21, §3], we will adopt the second point of view while enlarging the category of processes considered to objects acting as universal limits of random processes in 1D.

One important tool introduced in this paper are the so-called  $\zeta$ -functions associated to a stochastic process. For a stochastic process  $X$ , these functions are constructed by taking the expectation of a functional which will be denoted  $\ell_p$  for the rest of this paper. The inspiration for this functional comes from the  $\text{Pers}_p$  functional, which is the one classically used in topological data analysis (TDA) [7, 9, 11, 16, 23]. The main, and perhaps most important, departure from the conventional TDA theory is that we will consider this quantity for complex  $p$ , for reasons which will become evident throughout this paper, but which are analogous to the ones behind the complexification of the Riemann  $\zeta$ -function in analytic number theory. The efficiency of the  $\ell_p$  functional in practice and the stability of the so-called Wasserstein- $p$  distance remain important open problems in TDA. While deterministic results are unknown, the results of this paper suggest a hint of possible probabilistic explanation for the latter. Indeed, we may describe  $\ell_p^p$  in terms of a dual variable,  $N^\varepsilon$ , which exhibits robust statistical behaviour for a wide variety of processes in dimension 1. Since both functionals contain the same information, the statistical robustness of  $N^\varepsilon$  may explain the effectiveness of  $\ell_p$  in practice, where we in effect sample diagrams from a given distribution.

This work is the final stage in a program started in [20] and later continued in [21], which aimed to characterize the barcodes of random functions as completely as possible (in dimension one). To do this, we adopted the tree formalism originally developed by Le Gall [13, 14], which brings benefits in the probabilistic setting. This formalism allowed us to partially study the case of Markov, self-similar and processes admitting the two latter as limits was studied [21]. In this paper, we further develop the theory to describe almost completely the case of ( $\alpha$ -stable) Lévy processes.

### 1.1 Our contribution

More precisely, our contribution can be split along the following lines:

1. We establish a duality relation with respect to the Mellin transform between the study of  $\ell_p^p$  and the number of leaves of a  $\varepsilon$ -trimmed tree  $\geq \varepsilon$ ,  $N^\varepsilon$  (*cf.* section 2.2.1). With the help of a correct notion of integration on trees developed in [22], it is possible to prove an interpolation theorem for  $\ell_p^p$  (proposition 2.15);
2. We introduce  $\zeta$ -functions for stochastic processes (*cf.* section 2.3). We show that in the context of  $\alpha$ -stable Lévy processes, the associated (tail)  $\zeta$ -functions always admit a meromorphic extension to the entire complex plane, with a unique pole at  $p = \alpha$  with known residue (theorem 3.18). By duality, this meromorphic extension implies the existence of an asymptotic series for  $N^\varepsilon$  as  $\varepsilon \rightarrow 0$ , which we explicitly calculate up to superpolynomial (*i.e.* smaller than any polynomial) corrections (theorem 3.8). An explicit form of the meromorphic continuation of  $\hat{\zeta}$  is shown to be related to the superpolynomial corrections to the asymptotic expansion of theorem 3.8 (*cf.* section 2.1.1). We also define a generating function for the length of the  $k$ th longest bar (*cf.* section 3.1.2);

3. We design a statistical test for the parameter  $\alpha$  of  $\alpha$ -stable Lévy processes by using the theory previously described (*cf.* section 3.1.3);
4. Finally, we apply the theory above to different stochastic processes, such as Brownian motion, reflected Brownian motion. We derive explicit formulæ for the respective  $\zeta$ -functions of these processes and infer the associated asymptotic expansions of  $N^\varepsilon$  (theorems 4.2, 4.7 propositions 4.3 and 4.8) and in the case of Brownian motion, the explicit distribution of the length of the  $k$ th longest bar (*cf.* section 4.1.2).

## 2 Generalities

### 2.1 The Mellin transform

**Definition 2.1.** Let  $f$  be a locally integrable function over the ray  $]0, \infty[$ . The **Mellin transform** of  $f$  is

$$\mathcal{M}[f(x)](s) := \int_0^\infty x^{s-1} f(x) dx. \quad (2.1)$$

Note that  $d \log(x) = \frac{dx}{x}$  is the Haar measure of  $(\mathbb{R}_+, \times)$ . The Mellin transform reflects the Pontryagin duality with respect to this locally compact abelian group. Its theory is analogous to that of the bilateral Laplace transform, as the map  $\log : (\mathbb{R}_+, \times) \rightarrow (\mathbb{R}, +)$  induces an isomorphism of abelian groups.

**Notation 2.2.** For convenience, we will also employ the shorthand notation  $\mathcal{M}[f](s) = f^*(s)$ .

**Definition 2.3.** The **fundamental strip** of  $f$ ,  $\langle \alpha, \beta \rangle$  is the maximal set

$$\langle \alpha, \beta \rangle := \{z \in \mathbb{C} \mid \alpha < \operatorname{Re}(z) < \beta\} \quad (2.2)$$

where  $f^*(s)$  is well defined.

The Mellin transform can be inverted by virtue of the following theorem, which follows from the Laplace inversion theorem.

**Theorem 2.4** (Mellin inversion, [12, 18]). *Let  $f$  have fundamental strip  $\langle \alpha, \beta \rangle$  and let  $c \in ]\alpha, \beta[$ . Then*

1. *If  $f$  is integrable and  $f^*(c + it)$  is integrable, then for almost every  $x$*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds \quad (2.3)$$

*If  $f$  is continuous, the equality holds everywhere.*

2. *If  $f$  is locally integrable and of bounded variation in a neighbourhood of  $x$ , then*

$$\frac{f(x^+) + f(x^-)}{2} = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f^*(s) x^{-s} ds \quad (2.4)$$

A sufficient condition for the Mellin transform to be well-defined on  $\langle \alpha, \beta \rangle$  is that the function is such that

$$f(x) = O(x^{-\alpha}) \text{ as } x \rightarrow 0 \quad \text{and} \quad f(x) = O(x^{-\beta}) \text{ as } x \rightarrow \infty. \quad (2.5)$$

In fact, Mellin transforms are a good tool to study asymptotic expansions as suggested by the following theorem.

**Theorem 2.5** (Fundamental correspondence, [15]). *Let  $f : ]0, \infty[ \rightarrow \mathbb{C}$  be a continuous function with non-empty fundamental strip  $\langle \alpha, \beta \rangle$ . Then,*

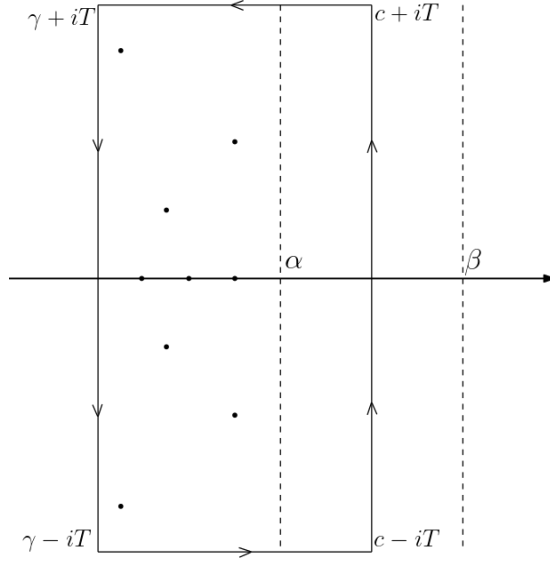


Figure 1: Contour for the evaluation of the Bromwich integral of the inverse Mellin transform.

- Assume that  $f^*(s)$  admits a meromorphic continuation to the strip  $\langle \gamma, \beta \rangle$  for  $\gamma < \alpha$ , that it has only a finite amount of poles there and that it is analytic on  $\text{Re}(s) = \gamma$ . Assume also that there exists  $\eta \in ]\alpha, \beta[$  such that along a denumerable set of horizontal segments with  $|\text{Im}(s)| = T_i$  where  $T_i \rightarrow \infty$ , we have

$$f^*(s) = O(|s|^{-r}) \text{ with } r > 1 \text{ as } |s| \rightarrow \infty \text{ and } s \in \langle \gamma, \eta \rangle. \quad (2.6)$$

Indexing the poles on  $\langle \gamma, \beta \rangle$  by their location  $\xi$  and by their order  $k$  and denoting  $c_{\xi,k}$  the  $k$ th coefficient in the Laurent expansion around  $\xi$  of  $f^*(s)$ , we have an asymptotic expansion of  $f$  around 0

$$f(x) \sim \sum_{(\xi,k)} c_{\xi,k} \frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} \log^k(x) + O(x^{-\gamma}) \text{ as } x \rightarrow 0. \quad (2.7)$$

- Conversely, if the function  $f$  has such an asymptotic expansion around 0, then  $f^*(s)$  has a meromorphic continuation to the strip  $\langle \gamma, \beta \rangle$ .

Furthermore, an analogous statement holds true for asymptotic expansions around  $\infty$  and meromorphic continuations beyond  $\beta$ .

*Sketch of proof.* It suffices to perform contour integration using the contour of figure 1. The estimates of the theorem allow us to discard the top and bottom integrals and to state that the integral of the path along  $\text{Re}(p) = \gamma$  is  $O(x^{-\gamma})$ . Conversely, consider

$$f(x) \sim \sum_{(\xi,k)} c_{\xi,k} x^\xi \log^k(x) + O(x^{-\gamma}) \text{ as } x \rightarrow 0 \quad (2.8)$$

for some  $\gamma < \alpha$ . It follows that

$$\begin{aligned} f^*(s) &= \sum_{(\xi,k)} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} + \int_1^\infty x^{s-1} f(x) dx \\ &+ \int_0^1 x^{s-1} \underbrace{\left[ f(x) - \sum_{(\xi,k)} c_{\xi,k} x^\xi \log^k(x) \right]}_{=O(x^{-\gamma})} dx, \end{aligned} \quad (2.9)$$

which is well-defined on the strip  $\langle \gamma, \beta \rangle$ . ■

$f(x)$	$f^*(s)$	$\langle \alpha, \beta \rangle$
$x^\nu f(x)$	$f^*(s + \nu)$	$\langle \alpha - \nu, \beta - \nu \rangle$
$f(x^\nu)$	$\frac{1}{\nu} f^*\left(\frac{s}{\nu}\right)$	$\langle \nu\alpha, \nu\beta \rangle$
$f(x^{-1})$	$f^*(-s)$	$\langle -\beta, -\alpha \rangle$
$f(\lambda x)$	$\lambda^{-s} f^*(s)$	$\langle \alpha, \beta \rangle$
$\frac{\partial}{\partial x} f(x)$	$-(s-1)f^*(s-1)$	
$\int_0^x f(t) dt$	$-\frac{1}{s} f^*(s+1)$	

Table 1: Functional properties of the Mellin transform

$f(x)$	$f^*(s)$	$\langle \alpha, \beta \rangle$
$e^{-x}$	$\Gamma(s)$	$\langle 0, \infty \rangle$
$e^{-x^2}$	$\frac{1}{2} \Gamma\left(\frac{s}{2}\right)$	$\langle 0, \infty \rangle$
$\operatorname{erfc}(x)$	$2^{-s} \frac{\Gamma(s)}{\Gamma(1+\frac{s}{2})}$	$\langle 0, \infty \rangle$
$\operatorname{csch}(x)$	$2^{1-s} (2^s - 1) \Gamma(s) \zeta(s)$	$\langle 1, \infty \rangle$
$\operatorname{csch}^2(x)$	$2^{2-s} \Gamma(s) \zeta(s-1)$	$\langle 2, \infty \rangle$
$\frac{1}{e^x - 1}$	$\Gamma(s) \zeta(s)$	$\langle 1, \infty \rangle$

Table 2: A short dictionary of Mellin transforms.

### 2.1.1 Analytic continuation

As stated by the fundamental correspondence (theorem 2.5), the existence of an asymptotic expansion around 0 of  $f(x)$  entails a meromorphic continuation of  $f^*(s)$  to a larger strip. If  $f(x)$  admits a converging Laurent series (with finite singular part) on some open disk around the origin, then this extension is in fact valid over all of  $\mathbb{C}$ , and the residues of the poles of  $f^*(s)$  will be related to the Laurent coefficients of  $f(x)$ . It turns out that in this context, one can even write an explicit integral representation for the extension of  $f^*(s)$ .

**Lemma 2.6** (Integral representation of  $f^*$ ). Let  $f$  be a meromorphic function admitting a Laurent series at 0, with singular part of degree  $n$ , holomorphic on a neighbourhood of  $\mathbb{R}_+^*$  and integrable over the Hankel contour (*cf.* figure 2). Suppose further that its fundamental strip  $\langle n, \beta \rangle$  is non-empty. Then, the function  $f^*$  admits a meromorphic continuation on  $\langle -\infty, \beta \rangle$  given by

$$f^*(s) = \frac{e^{-i\pi s}}{2i \sin(\pi s)} \oint_H z^{s-1} f(z) dz \quad (2.10)$$

$$= -\frac{\Gamma(s)\Gamma(1-s)}{2\pi i} \oint_H (-z)^{s-1} f(z) dz, \quad (2.11)$$

where  $H$  denotes the Hankel contour.

*Proof.* We start by splitting the Hankel contour into three pieces.

1. A segment from  $\infty + i\varepsilon$  to  $\nu + i\varepsilon$ ;
2. A circle  $C_\nu$  around the origin of radius  $\nu$ ;
3. A segment from  $\nu - i\varepsilon$  to  $\infty - i\varepsilon$ .

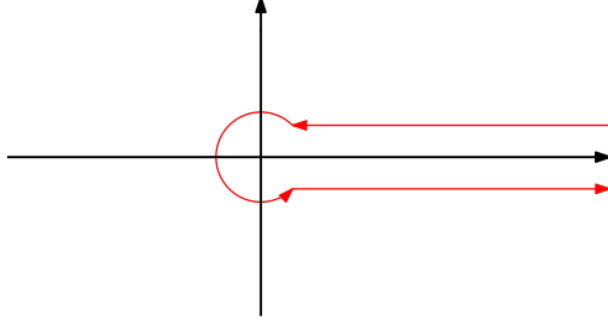


Figure 2: The Hankel contour  $H$ .

For  $s \in \langle n, \beta \rangle$ ,  $f$  is holomorphic everywhere on this contour, so that we may take  $\varepsilon = 0$  according to Cauchy's theorem. Notice also that

$$\int_{C_\nu} z^{s-1} B(z) dz = O(\nu^{\operatorname{Re}(s)-n}) \rightarrow 0 \quad \text{as } \nu \rightarrow 0. \quad (2.12)$$

It follows that for  $\operatorname{Re}(s) > n$

$$\begin{aligned} \oint_H z^{s-1} f(z) dz &= \lim_{\nu \rightarrow 0} \left\{ \int_\infty^\nu + \int_{C_\nu} + \int_{\nu e^{2\pi i}}^\infty \right\} z^{s-1} f(z) dz \\ &= (e^{2\pi i(s-1)} - 1) \int_0^\infty z^{s-1} f(z) dz \\ &= 2ie^{i\pi s} \sin(\pi s) f^*(s), \end{aligned} \quad (2.13)$$

as desired. Note that the integral over the complex contour  $H$  converges for all  $s \in \langle -\infty, \beta \rangle \setminus \{n\}$ . Finally, the second expression for  $f^*$  is obtained through Euler's reflection formula, namely

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}, \quad (2.14)$$

which after some simplification yields the desired expression.  $\blacksquare$

*Remark 2.7.* If  $f$  has fundamental strip  $\langle n, \infty \rangle$ , then the extension given by this procedure holds over  $\mathbb{C}$ .

Furthermore, if  $f$  possesses a meromorphic continuation to  $\mathbb{C} \setminus \mathbb{R}_+$  (i.e. we admit the possibility of a branch cut on the positive real axis), then we can find a more explicit formulation for the Hankel representation of  $f^*$ .

**Lemma 2.8** (Functional equation of  $f^*$ ). Suppose  $f$  possesses a meromorphic continuation to  $\mathbb{C} \setminus \mathbb{R}_+$  and denote  $\mathcal{P}$  the set of poles of  $f$  not including 0. Suppose further that  $f$  has the following decay condition : for all  $s \in \langle n, \beta \rangle$  and for some monotone increasing sequence of radii  $r_n \rightarrow \infty$  as  $n \rightarrow \infty$ ,

$$\int_{C_{r_n, \varepsilon}} |z^{s-1} f(z)| dz \xrightarrow{n \rightarrow \infty} 0. \quad (2.15)$$

where  $C_{r_n, \varepsilon}$  is the circle of radius  $r_n$  minus a small (symmetric) arc of length  $\varepsilon$  around the positive real axis (cf. figure 3). Then,

$$f^*(s) = \Gamma(s)\Gamma(1-s) \sum_{z_0 \in \mathcal{P}} \operatorname{Res}((-z)^{s-1} f(z); z_0). \quad (2.16)$$

*Proof.* The proof relies on the use of the residue theorem by completing the Hankel contour into a Pac-Man (cf. figure 3), whose circular contribution is going to zero, due to the assumption of the lemma. By the residue theorem, we then have

$$f^*(s) = \Gamma(s)\Gamma(1-s) \sum_{z_0 \in \mathcal{P}} \operatorname{Res}((-z)^{s-1} f(z); z_0), \quad (2.17)$$

as desired.  $\blacksquare$

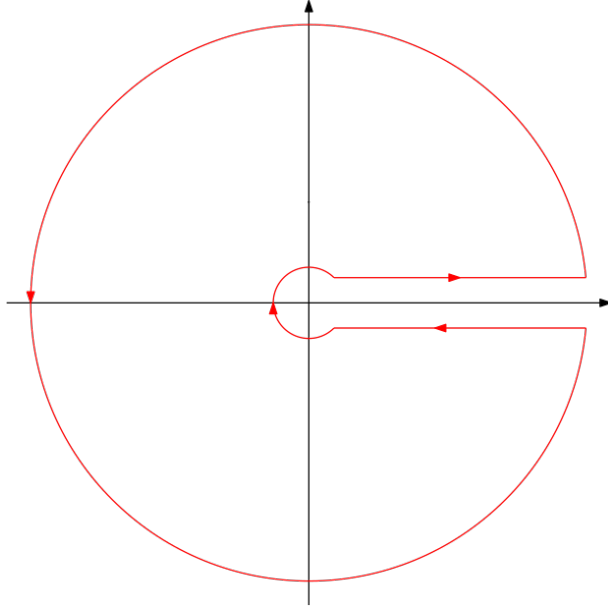


Figure 3: The Pac-Man contour.

## 2.2 Connected components of superlevel sets of stochastic processes

Let us briefly recall the construction of a tree from a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ . For a more complete description of this, the reader is welcome to consult [13, 20].

**Definition/Proposition 2.9** ([13]). Let  $x < y \in [0, 1]$ , the function

$$d_f(x, y) := f(x) + f(y) - 2 \min_{t \in [x, y]} f(t) \quad (2.18)$$

is a pseudo-distance on  $[0, 1]$  and the quotient metric space

$$T_f := [0, 1] / \{d_f = 0\} \quad (2.19)$$

with distance  $d_f$  is a rooted  $\mathbb{R}$ -tree, whose root coincides with the image in  $T_f$  of the point in  $[0, 1]$  at which  $f$  achieves its infimum.

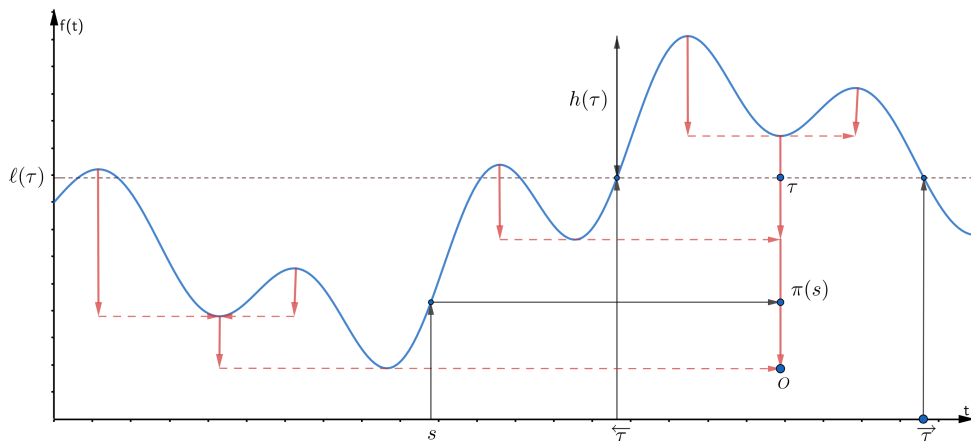


Figure 4: A function  $f$  and its associated tree  $T_f$  in red.

The tree  $T_f$  has the particularity that its branches correspond to connected components of the superlevel sets of  $f$ , as illustrated by figure 4. Let us now introduce the so-called  $\varepsilon$ -simplified or  $\varepsilon$ -trimmed tree of  $T_f^\varepsilon$ . This object is obtained by “giving a haircut” of length  $\varepsilon$  to  $T_f$ . More precisely, if we define a function  $h : T_f \rightarrow \mathbb{R}$  which to a point  $\tau \in T_f$  associates the distance from  $\tau$  to the highest leaf above  $\tau$  with respect to the filtration on  $T_f$  induced by  $f$ , then



**Definition 2.10.** Let  $\varepsilon \geq 0$ . An  $\varepsilon$ -**trimming** or  $\varepsilon$ -**simplification** of  $T_f$  is the metric subspace of  $T_f$  defined by

$$T_f^\varepsilon := \{\tau \in T_f \mid h(\tau) \geq \varepsilon\} \quad (2.20)$$

**Notation 2.11.** Let us denote  $N^\varepsilon$  the number of leaves of  $T_f^\varepsilon$ .

*Remark 2.12.* In the jargon of persistence homology, this  $N^\varepsilon$  corresponds to the number of bars of length  $\geq \varepsilon$  of the  $H_0$ -barcode of  $f$ . Equivalently, we can also understand  $N^\varepsilon$  as counting the number of variations of  $f$  of size at least  $\varepsilon$ .

### 2.2.1 Integration on trees

Let us recall the following simple remark made in [20]. On a tree  $T_f$ , we can define a notion of integration by defining the unique atomless Borel measure  $\lambda$  which is characterized by the property that every geodesic segment on  $T_f$  has measure equal to its length. Formally, we can express  $\lambda$  in two ways [22]

$$\lambda = \int_{\mathbb{R}} dx \sum_{\substack{\tau \in T_f \\ f(\tau)=x}} \delta_\tau \quad \text{and} \quad \lambda = \int_0^\infty d\varepsilon \sum_{\substack{\tau \in T_f \\ h(\tau)=\varepsilon}} \delta_\tau \quad (2.21)$$

By using the second way of writing  $\lambda$ , the identity

$$\lambda(T_f^\varepsilon) = \int_\varepsilon^\infty N^a da \quad (2.22)$$

is clear, as every sum in the second expression is finite for all  $\varepsilon > 0$  and has  $N^\varepsilon$  terms. Of course, we could very well have written it using the first sum, but this poses the difficulty that if  $T_f$  is infinite, so is the sum considered in this formal expression for at least some value of  $x$ .

### 2.2.2 Duality between $N^\varepsilon$ and $\ell_p^p$

Using the notion of integration on trees detailed above, it is possible to define the following functional.

**Definition 2.13.**

$$\ell_p(f) := \left[ p \int_{T_f} h(\tau)^{p-1} \lambda(d\tau) \right]^{1/p}. \quad (2.23)$$

*Remark 2.14.* With some work (cf. [20, 21]), it is possible to show that the  $\text{Pers}_p$  functional of [7, 9, 11, 16, 23] is equal to  $\ell_p$ . It is also possible to show that this  $\text{Pers}_p$  functional and  $\ell_p$  as defined above are the same.

The study of  $N^\varepsilon$  is in fact completely equivalent to the study of  $\ell_p^p(f)$ . Indeed,

$$\ell_p^p(f) = p \int_{T_f} h(\tau)^{p-1} \lambda(d\tau) = p \int_0^\infty \varepsilon^{p-1} N^\varepsilon d\varepsilon, \quad (2.24)$$

where  $h : T_f \rightarrow \mathbb{R}$  associating to  $\tau \in T_f$  the distance between  $\tau$  and the highest leaf (with respect to the filtration of  $f$ ) above  $\tau$  in  $T_f$ . We immediately recognize the above integral as being the Mellin transform of  $N^\varepsilon$ . Allowing for complex  $p$ , this integral relation can be inverted by virtue of the Mellin inversion theorem, provided that the fundamental strip of  $N^\varepsilon$  is not empty. For compact intervals and continuous functions  $f$ , this fundamental strip is never empty (provided  $\mathcal{L}(f) < \infty$ ) and in fact is exactly equal to  $\langle \mathcal{L}(f), \infty \rangle$ . Thus, for any real number  $c > \mathcal{L}(f)$ , we have

$$N^\varepsilon = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \ell_p^p(f) \varepsilon^{-p} \frac{dp}{p}. \quad (2.25)$$

Notice also that  $\ell_p^p$  is a norm in the sense that

$$\ell_p^p(f) = p \|h\|_{L^{p-1}(\lambda)}^{p-1}, \quad (2.26)$$

It follows from this observation that by the usual inequalities of  $L^p$ -spaces,

**Proposition 2.15.**  $\ell_p^p$  is almost log-convex, *i.e.* let  $p_0 < p_1$  and  $\theta \in [0, 1]$  and set  $p = (1 - \theta)p_0 + \theta p_1$ , then,

$$\ell_p^p \leq \frac{p}{p_0^{1-\theta} p_1^\theta} \ell_{p_0}^{p_0(1-\theta)} \ell_{p_1}^{p_1\theta}. \quad (2.27)$$

*Proof.* The statement follows directly from an application of Lyapunov's inequality.  $\blacksquare$

More generally, it is always true that one can express the  $L^p(\mu)$ -norm of a function  $f$  as the Mellin transform of the repartition function of  $|f|$ ,  $\mu(|f| > x)$ .

*Remark 2.16.* For the reader versed in persistent homology, notice that this definition of  $\ell_p^p$  coincides perfectly with a definition of  $\text{Pers}_p^p$  typically used in persistent homology [7, 9, 11, 16, 23], as long as we consider that the infinite bar has the length of the range (*i.e.* the sup – inf) of the function  $f$ . Of course, within this framework an equally valid definition for  $\ell_p^p$  would have been to exclude the infinite bar from being counted all-together, and to consider only the bars of finite length. This approach turns out to give the correct definition for the  $\ell_p^p$ -functional in the definition of *tail*  $\zeta$ -functions (*cf.* definition 3.17), which is necessary to study Lévy  $\alpha$ -stable processes for  $\alpha < 2$ .

### 2.2.3 Calculation of $N^\varepsilon$ in dimension one

In dimension one, it is possible to use the total order of  $\mathbb{R}$  and count  $N^\varepsilon$  by counting the number of times we go up by at least  $\varepsilon$  from a local minimum and down by at least  $\varepsilon$  from a local maximum. This idea can be formalized by the following sequence, originally introduced by Neveu *et al.* [17].

**Definition 2.17.** Setting  $S_0^\varepsilon = T_0^\varepsilon = 0$ , we define a sequence of times recursively

$$T_{i+1}^\varepsilon := \inf \left\{ t \geq S_i^\varepsilon \mid \sup_{[S_i^\varepsilon, t]} f - f(t) > \varepsilon \right\}$$

$$S_{i+1}^\varepsilon := \inf \left\{ t \geq T_{i+1}^\varepsilon \mid f(t) - \inf_{[T_{i+1}^\varepsilon, t]} f > \varepsilon \right\}$$

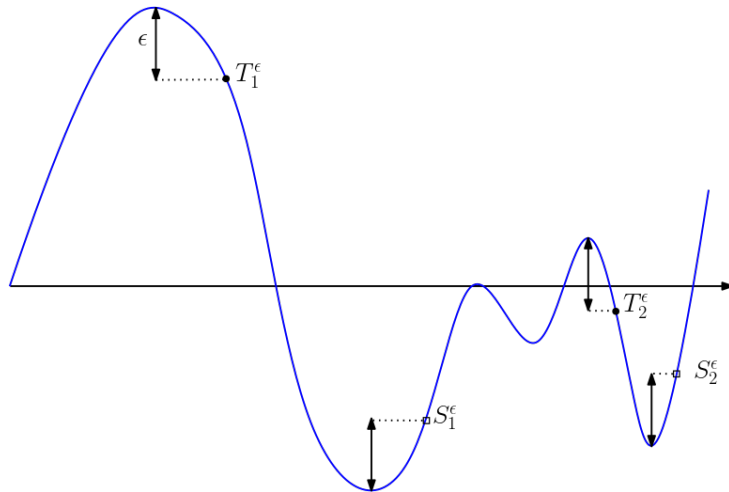


Figure 5: A function  $f$  in blue along with the times  $T_i^\varepsilon$  and  $S_i^\varepsilon$  indicated. Because of the boundary this function has exactly 3 bars of length  $\geq \varepsilon$  and not just 2.

Counting the number of bars of length  $\varepsilon$  is thus exactly to count the number of up and downs we make. More precisely

$$N^\varepsilon = \inf \{ i \mid T_i^\varepsilon \text{ or } S_i^\varepsilon = \inf \emptyset \} \quad (2.28)$$

by which we mean that it is the smallest  $i$  such that the set over which  $T_i^\varepsilon$  or  $S_i^\varepsilon$  are defined as infima is empty.

**Notation 2.18.** The range of the process  $X$  will play a considerable role in what will follow. Let us denote the range  $R$  of  $X$  by

$$R_t := \sup_{[0,t]} X - \inf_{[0,t]} X \quad (2.29)$$

Finally, we denote  $N_t^\varepsilon$  the number  $N^\varepsilon$  of the process  $X$  restricted to the interval  $[0, t]$ .

Intuitively, this calculation process hints at the fact that if  $\varepsilon$  is small, the number of bars  $N^\varepsilon$  should strongly depend on the regularity of the process, as ultimately  $N^\varepsilon$  counts the number of “oscillations” of size  $\varepsilon$ . In a very precise sense, regularity almost fully determines the asymptotics of  $N^\varepsilon$  in the  $\varepsilon \rightarrow 0$  regime. This intuition is corroborated by the following theorem.

**Theorem 2.19** (Picard, §3 [22] and [20]). *Given a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ ,*

$$\mathcal{V}(f) = \mathcal{L}(f) = \overline{\dim} T_f = \limsup_{\varepsilon \rightarrow 0} \frac{\log N^\varepsilon}{\log(1/\varepsilon)} \vee 1 \quad (2.30)$$

where  $\overline{\dim}$  denotes the upper-box dimension,  $a \vee b = \max\{a, b\}$ ,

$$\mathcal{V}(f) := \inf\{p \mid \|f\|_{p\text{-var}} < \infty\} \quad \text{and} \quad \mathcal{L}(f) := \inf\{p \mid \ell_p(f) < \infty\}. \quad (2.31)$$

For the rest of this paper it is exactly the functional  $\ell_p^p$  which shall occupy us.

*Remark 2.20.* Note that  $\ell_\infty$  is stable under  $L^\infty$  perturbations of  $f$ . However, it is unknown whether a similar stability result exists for  $p < \infty$ .

### 2.3 $\zeta$ -functions associated to stochastic processes

For any (deterministic) continuous function  $f$ ,  $\ell_p^p(f)$  is nothing other than a sum of lengths to the power  $p$ . An in depth explanation of this is provided in [20, §2.2], but let us briefly give some intuition for this. Starting from  $T_f$ , we can look at the longest branch (starting from the root) of  $T_f$  and remember the minimum and maximum value that  $f$  takes along this branch (we will call the difference between this maximum and minimum value the **length** of this branch). Next, we erase this longest branch and, on the remaining (rooted) forest, look for the next longest branch. This will be the second longest branch. An illustration of this procedure can be found in figure 6. By construction, if we denote  $b$  any of the branches defined by this procedure

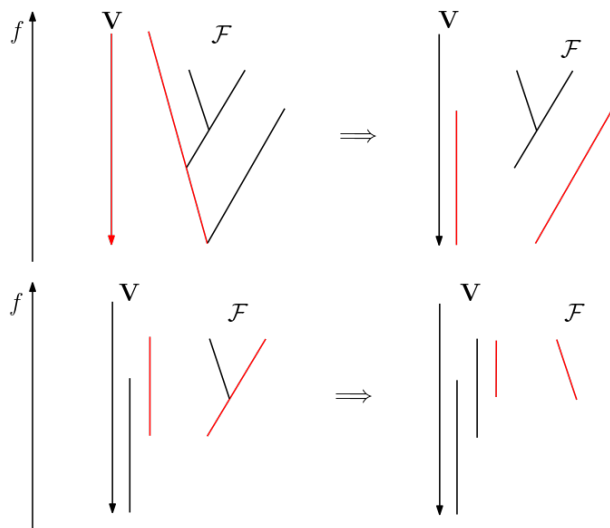


Figure 6: A depiction of the construction of [20].

and  $\ell(b)$  the length of the branch, notice that

$$p \int_b h(\tau)^{p-1} \lambda(d\tau) = \ell(b)^p. \quad (2.32)$$

But these branches partition the tree  $T_f$ , so that the integration present in the definition of  $\ell_p^p$  is nothing other than the sum of the  $\ell(b)^{p'}$ 's.

This is reminiscent of the structure of the  $\zeta$ -function, but *a priori* not enough to draw any parallels. However, it turns out that this nomenclature turns out to have a meaning for stochastic processes.

**Definition 2.21.** Let  $f$  be a stochastic process on some compact topological space  $X$ . Its  $\zeta$ -function  $\zeta_f$  is defined by:

$$\zeta_f(p) := \mathbb{E}[\ell_p^p(f)] = p \int_0^\infty \varepsilon^{p-1} \mathbb{E}[N^\varepsilon] d\varepsilon. \quad (2.33)$$

for  $p \in \langle \mathcal{L}(f), \infty \rangle$ .

A first important motivating result regarding  $\zeta$ -functions is the following.

**Proposition 2.22** (P., [19]). If  $X$  is a continuous semimartingale, then almost surely,  $\ell_p^p(X)$  admits a pole of order 1 at  $p = 2$  of residue  $[X]_t$ .

**Notation 2.23.** For the rest of this paper we will take the following conventions. First, we will sometimes omit the subscript  $t$  of  $N_t^\varepsilon$  whenever convenient. The Laplace transform  $\mathcal{L}$  is always taken with respect to the variable  $t$  and its conjugate variable will always be  $\lambda$ . Similarly, Mellin transforms will always be taken with respect to the variable  $\varepsilon$  and its conjugate variable will be  $p$ .

**Lemma 2.24.** Let  $f(x, t) : [0, \infty[^2 \rightarrow \mathbb{R}_+$  such that the functions  $f(x, -)$  and  $f(-, t)$  are monotone in their arguments. Then, denoting  $\mathcal{L}_t$  the Laplace transform with respect to  $t$ , we have

$$\mathcal{M}_x \mathcal{L}_t[f] = \mathcal{L}_t \mathcal{M}_x[f] \quad (2.34)$$

*Proof.* The monotonicity of  $f$  in its arguments ensures that  $f$  is a measurable, positive function. The statement holds by virtue of Tonelli's theorem.  $\blacksquare$

*Remark 2.25.* Notice this last lemma is applicable to  $N_t^\varepsilon$ ,  $\mathbb{E}[N_t^\varepsilon]$ ,  $\mathbb{P}(N_t^\varepsilon \geq k)$  and other such quantities.

### 3 $\zeta$ -functions of Lévy processes

In [21], we have already studied the functional  $N^\varepsilon$  for Markov processes. Let us briefly recall some useful known facts about  $N^\varepsilon$ .

**Proposition 3.1** (P, [21]). Using summation by parts, it is possible to write

$$\mathbb{E}[(N_t^\varepsilon)^s] = \sum_{k \geq 1} (k^s - (k-1)^s) \mathbb{P}(N_t^\varepsilon \geq k) \quad (3.35)$$

For processes on the interval which are not periodic (in the sense of [21]), if  $k \geq 2$

$$\mathbb{P}(N_t^\varepsilon \geq k) = \mathbb{P}(S_{k-1}^\varepsilon \leq t), \quad (3.36)$$

and  $\mathbb{P}(N_t \geq 1) = \mathbb{P}(R_t \geq \varepsilon)$ . Furthermore if  $X$  has the strong Markov property,

$$\mathbb{E}[N_t^\varepsilon] \sim \mathbb{P}(R_t \geq \varepsilon) \quad \text{as } \varepsilon \rightarrow \infty. \quad (3.37)$$

Finally, for  $k \geq 2$  the Laplace transform (with respect to time, as per our convention) of equation 3.36 is

$$\mathcal{L}(\mathbb{P}(N_t^\varepsilon \geq k))(\lambda) = \frac{\mathbb{E}\left[e^{-\lambda S_{k-1}^\varepsilon}\right]}{\lambda}. \quad (3.38)$$

*Proof.* The only thing to prove is the result of equation 3.38, as the previous statements are all proved in [21]. Since we are dealing with processes which are not periodic in the sense of [21], we have that

$$\mathbb{P}(N_t^\varepsilon \geq k) = \mathbb{P}(S_{k-1}^\varepsilon \leq t), \quad (3.39)$$

since as soon as the hitting time  $S_{k-1}^\varepsilon$  is attained we have at least  $k$  bars (due to the boundary of the interval). Using standard functional properties of the Laplace transform it is easy to see that

$$\mathcal{L}[\mathbb{P}(S_{k-1}^\varepsilon \leq t)](\lambda) = \frac{\mathcal{L}[\mathbb{P}(S_{k-1}^\varepsilon = t)](\lambda)}{\lambda}, \quad (3.40)$$

where  $\mathbb{P}(S_{k-1}^\varepsilon = t)$  denotes the probability density function of  $S_{k-1}^\varepsilon$ . However, the Laplace transform of this density function is nothing other than the moment generating function  $\mathbb{E}\left[e^{-\lambda S_{k-1}^\varepsilon}\right]$ , since  $S_{k-1}^\varepsilon$  is a positive random variable.  $\blacksquare$

*Remark 3.2.* If the process has the strong Markov property, we can write  $\mathbb{E}\left[e^{-\lambda S_{k-1}^\varepsilon}\right]$  as the product of the Laplace transform of the distribution of its increments

$$S_{k-1}^\varepsilon = \sum_{i=0}^{k-1} (S_i^\varepsilon - T_i^\varepsilon) + (T_i^\varepsilon - S_{i-1}^\varepsilon). \quad (3.41)$$

The expression of  $\mathbb{E}\left[e^{-\lambda S_{k-1}^\varepsilon}\right]$  is particularly simple as soon as these increments are independent and identically distributed.

*Remark 3.3.* Ordering the bars of the barcode of a function  $f$  by their length, and denoting the length of the  $k$ th longest branch by  $\ell_k$ , the following equivalence holds

$$N_t^\varepsilon \geq k \iff \ell_k \geq \varepsilon. \quad (3.42)$$

The probability distribution of both of the events above are thus the same. Consequently, there is a one-to-one correspondence between the elements of the sums

$$\mathbb{E}[\ell_p^p] = \sum_{k \geq 1} \mathbb{E}[\ell_k^p] = p \sum_{k \geq 1} \mathcal{M}[\mathbb{P}(N_t^\varepsilon \geq k)](p) \quad (3.43)$$

whenever these quantities are defined. In particular, the distribution of each bar is in principle readily available, since  $\mathbb{E}[\ell_k^{p-1}]$  is the Mellin transform of the distribution of  $\ell_k$ . We will later see that in particular cases, we can gain access to the explicit distribution of bars in this way (*cf.* section 3.1.2).

### 3.1 Lévy processes

For Lévy processes, the small scale asymptotics of  $N^\varepsilon$  can also be studied up to the following caveat : a wide range of Lévy processes have almost surely discontinuous paths (but nonetheless càdlàg), but our construction of trees (as done in [20]) is based on continuous functions. For this reason, it is necessary to define what tree we associate to a process  $X$  when  $X_t$  has almost surely discontinuous paths. Luckily, this caveat has been treated for càdlàg processes in [13, 22]. We will adopt the approach taken by Picard in [22], where the reader can find the details of the construction. Loosely speaking, Picard’s approach consists in “completing” the function at the discontinuity points by joining an imaginary line linking the points of discontinuity (*cf.* figure 7).

In any case, it has been shown that

**Proposition 3.4** (Picard, §3 [22]). Let  $X$  be a Lévy process and suppose that, almost surely  $X$  has no interval on which it is monotone. Define

$$\xi(\varepsilon) := \mathbb{E}[S^\varepsilon + T^\varepsilon] \quad (3.44)$$



To show the theorem, it is convenient to show first some technical lemmata, one of which is a slight refinement to a technical lemma proved in [22].

**Lemma 3.9** (Picard, Proposition 3.14 [22]). The variables  $S^\varepsilon$  and  $T^\varepsilon$  admit finite moments of order  $k$  for all  $k$  and the moment generating function  $\mathbb{E}[e^{-\lambda U^\varepsilon}]$  is well defined on a neighborhood of zero and there exists  $1 \leq C_k \leq 2^k \text{Li}_{-k}(\frac{1}{2})$  such that

$$\mathbb{E}[U^\varepsilon]^k \leq \mathbb{E}\left[(U^\varepsilon)^k\right] \leq C_k \mathbb{E}[U^\varepsilon]^k. \quad (3.50)$$

*Remark 3.10.* The bound on the constant in this lemma is not optimal.

**Lemma 3.11.** Keeping the same notation (*cf.* notation 3.6) as before,

$$U^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{L^r} 0 \quad \text{and} \quad U^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} 0. \quad (3.51)$$

for every  $r \geq 1$ .

**Lemma 3.12.** For every  $k$ , we have that for some constant  $D_k$

$$1 - D_k e^{-\lambda} \mathbb{E}[U^\varepsilon]^k \lesssim \mathbb{E}\left[e^{-\lambda U^\varepsilon}\right] \leq 1 \quad \text{as } \varepsilon \rightarrow 0. \quad (3.52)$$

Furthermore, for real  $\gamma$  and  $\sigma$ , we have

$$1 - \mathbb{E}\left[e^{-\gamma U^\varepsilon}\right] \leq \left|1 - \mathbb{E}\left[e^{-(\gamma+i\sigma)U^\varepsilon}\right]\right|, \quad (3.53)$$

In particular,

$$\left|\frac{\mathbb{E}\left[e^{-(\gamma+i\sigma)U^\varepsilon}\right]}{1 - \mathbb{E}\left[e^{-(\gamma+i\sigma)U^\varepsilon}\right]}\right| \leq \frac{\mathbb{E}\left[e^{-\gamma U^\varepsilon}\right]}{1 - \mathbb{E}\left[e^{-\gamma U^\varepsilon}\right]}. \quad (3.54)$$

**Lemma 3.13.** For any  $\delta > 0$  weakly in  $L^2([\delta, \infty[)$  for any  $k \geq 0$  we have that

$$\frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{\lambda t} \lambda^k d\lambda \xrightarrow[T \rightarrow \infty]{\mathcal{D}} 0 \quad (3.55)$$

at a rate  $O(T^{-n})$  for any  $n \in \mathbb{N}$ . Furthermore, for any  $k \geq 0$  we have

$$\frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \frac{e^{\lambda t}}{\lambda^k} d\lambda \xrightarrow[T \rightarrow \infty]{\mathcal{D}} \frac{t^{k-1}}{(k-1)!} \quad (3.56)$$

weakly at a rate  $O(T^{-(n+k-1)})$  and the convergence is strong as soon as  $k \geq 2$ .

*Proof of lemma 3.9.* It is sufficient to show it for  $S^\varepsilon$  knowing that an analogous treatment is possible for  $T^\varepsilon$ . The points in  $[0, S^\varepsilon]$  are characterized by the fact that

$$X_t - \inf_{[0,t]} X > \varepsilon. \quad (3.57)$$

In particular, the supremum over all  $t$  ranging within  $[0, S^\varepsilon]$  of this quantity is also less than  $\varepsilon$ . From this, it follows that

$$\mathbb{P}(S^\varepsilon > a) = \mathbb{P}\left[\sup_{0 \leq t \leq a} \left(X_t - \inf_{[0,t]} X\right) < a\right]. \quad (3.58)$$

Consider now an interval  $[0, k\mu]$ , where  $\mu > 0$  and  $k$  is an integer and let us slice this interval into  $k$  segments of length  $\mu$ . It is clear that we have the following inequality

$$\sup_{0 \leq t \leq k\mu} \left(X_t - \inf_{[0,t]} X\right) \geq \sup_{1 \leq j \leq k} \left(\sup_{(j-1)\mu \leq t \leq j\mu} \left(X_t - \inf_{[(j-1)\mu, t]} X\right)\right), \quad (3.59)$$

since over each smaller interval we can have only smaller spread than over the entire interval. However, notice that now the right hand side is a supremum over i.i.d. random variables, so that

$$\begin{aligned} \mathbb{P}(S^\varepsilon > k\mu) &\leq \mathbb{P}\left[\sup_{1 \leq j \leq k} \left( \sup_{(j-1)\mu \leq t \leq j\mu} \left( X_t - \inf_{[(j-1)\mu, t]} X \right) \right) < \varepsilon\right] \\ &= \mathbb{P}\left[\sup_{0 \leq t \leq \mu} \left( X_t - \inf_{[0, t]} X \right) < \varepsilon\right]^k = \mathbb{P}(S^\varepsilon > \mu)^k. \end{aligned} \quad (3.60)$$

By the non-monotonicity of  $X$ ,  $\mathbb{P}(S^\varepsilon > \mu) < 1$ . In particular, if we let  $\mu = 1$ , and denote  $c = \mathbb{P}(S^\varepsilon > 1) < 1$ , then:

$$\lim_{k \rightarrow \infty} e^{\lambda k} \mathbb{P}(S^\varepsilon > k) \leq \lim_{k \rightarrow \infty} e^{\lambda k} \mathbb{P}(S^\varepsilon > 1)^k = \lim_{k \rightarrow \infty} (e^\lambda c)^k = 0 \quad (3.61)$$

as soon as  $\lambda < \log(1/c)$ . It follows that  $\mathbb{E}[e^{-\lambda S^\varepsilon}]$  is well-defined for  $\lambda$  in some neighbourhood of zero and in particular all moments of  $S^\varepsilon$  are well-defined and finite. Finally, combining the above remark with an application of Markov's inequality we get

$$\mathbb{P}(S^\varepsilon > 2k\mathbb{E}[S^\varepsilon]) \leq 2^{-k}. \quad (3.62)$$

It follows that  $\frac{S^\varepsilon}{2\mathbb{E}[S^\varepsilon]} \leq G$  almost surely, where  $G$  is a geometric random variable. The moments of  $G$  can easily be calculated, yielding the result.  $\blacksquare$

*Proof of lemma 3.11.* The statement in  $L^r$  follows from the following observation.

$$0 \leq \mathbb{E}[(U^\varepsilon)^r] \leq \sum_{k=1}^{\infty} k^r \mathbb{P}(U^\varepsilon \geq k) \leq \sum_{k=1}^{\infty} k^r \mathbb{P}(U^\varepsilon \geq 1)^k \quad (3.63)$$

by the arguments of lemma 3.9. This sum converges, since  $\mathbb{P}(U^\varepsilon \geq 1) < 1$ . As  $\varepsilon \rightarrow 0$ ,  $\mathbb{P}(U^\varepsilon > 1) \rightarrow 0$ , since  $X$  is almost surely nowhere monotone, so that the entire sum tends to 0. The almost sure statement follows from the fact that  $U^\varepsilon$  is monotone, since both  $T^\varepsilon$  and  $S^\varepsilon$  are monotone functions of  $\varepsilon$ . Indeed, notice that  $L^r$  convergence implies almost sure convergence along a subsequence  $\varepsilon_n$ , for  $\varepsilon_{n+1} < \varepsilon < \varepsilon_{n+1}$  by monotonicity of  $U^\varepsilon$  we have

$$U^{\varepsilon_{n+1}} < U^\varepsilon < U^{\varepsilon_n}, \quad (3.64)$$

so the convergence is almost sure.  $\blacksquare$

*Proof of lemma 3.12.* The first inequality of the lemma relies on the fact that

$$\begin{aligned} \mathbb{E}\left[e^{-\lambda T^\varepsilon}\right] &\leq \sum_{k \geq 0} e^{-\lambda k} \mathbb{P}(T^\varepsilon > k) \leq \sum_{k \geq 0} \left[ e^{-\lambda} \mathbb{P}(T^\varepsilon > 1) \right]^k \\ &= \frac{1}{1 - e^{-\lambda} \mathbb{P}(T^\varepsilon > 1)} \sim 1 - e^{-\lambda} \mathbb{P}(T^\varepsilon > 1) \quad \text{as } \varepsilon \rightarrow 0, \end{aligned} \quad (3.65)$$

since  $\mathbb{P}(T^\varepsilon > 1) \xrightarrow{\varepsilon \rightarrow 0} 0$ . Notice an analogous inequality holds for  $S^\varepsilon$ . By Markov's inequality, we know that

$$\mathbb{P}(T^\varepsilon > 1) \leq \mathbb{E}\left[(T^\varepsilon)^k\right] \leq C_k \mathbb{E}[T^\varepsilon]^k \quad (3.66)$$

from which the first inequality follows by lemma 3.11. The second and third inequalities follow from noticing that for any  $x$  and  $y$  we have

$$\left| |x| - |y| \right| \leq |x - y| \quad (3.67)$$

and applying Jensen's inequality.  $\blacksquare$



*Proof of lemma 3.13.* Consider a test function  $\varphi \in C_c^\infty([\delta, \infty[)$ , then integrating by parts

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{R}} \left[ \int_{\gamma-iT}^{\gamma+iT} e^{\lambda t} \lambda^k d\lambda \right] \varphi(t) dt &= \frac{(-1)^k}{2\pi i} \int_{\mathbb{R}} dt \varphi^{(k)}(t) \int_{\gamma-iT}^{\gamma+iT} e^{\lambda t} d\lambda \\ &= (-1)^k \int_{\mathbb{R}} \frac{e^{\gamma t} \varphi^{(k)}(t)}{\pi t} \sin(Tt) dt \end{aligned} \quad (3.68)$$

By performing the change of variables  $y = Tt$ , we see that the integral is weakly approaching 0, as  $\varphi$  is not supported at 0. Additionally, away from 0, the function

$$\frac{e^{\gamma t} \varphi^{(k)}(t)}{\pi t} \quad (3.69)$$

is a compactly supported  $C^\infty$ -function, integrating by parts  $n$  subsequent times equation 3.68 yields bounds of this integral by  $C_\varphi T^{-n}$ , where  $C_\varphi$  is a constant which depends on the test function and its support.

Let us now show that

$$\frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \frac{e^{\lambda t}}{\lambda^k} d\lambda \xrightarrow[T \rightarrow \infty]{\mathcal{D}} \frac{t^{k-1}}{(k-1)!}. \quad (3.70)$$

Once again integrating by parts, we have

$$\int_{\mathbb{R}} dt \varphi(t) \left[ \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \frac{e^{\lambda t}}{\lambda^k} d\lambda - \frac{t^{k-1}}{(k-1)!} \right] = (-1)^n \int_{\mathbb{R}} dt \varphi^{(n)}(t) \left[ \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \frac{e^{\lambda t}}{\lambda^{k+n}} d\lambda - \frac{t^{n+k-1}}{(n+k-1)!} \right].$$

Applying the residue theorem to evaluate the complex integral we have that for  $T > \gamma$

$$\frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \frac{e^{\lambda t}}{\lambda^{k+n}} d\lambda = \frac{t^{n+k-1}}{(n+k-1)!} + \frac{1}{2\pi i} \int_{C_T} \frac{e^{\lambda t}}{\lambda^{k+n}} d\lambda, \quad (3.71)$$

where  $C_T$  is the circle of center  $\lambda = \gamma$  and radius  $T$ . By the estimation lemma, the contribution of this integral is bounded by  $e^{\gamma t} T^{-(n+k-1)}$ . It follows that

$$\left| \int_{\mathbb{R}} dt \varphi(t) \left[ \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \frac{e^{\lambda t}}{\lambda^k} d\lambda - \frac{t^{k-1}}{(k-1)!} \right] \right| \leq T^{-(n+k-1)} \left\| e^{\gamma t} \varphi^{(n)}(t) \right\|_{L^1}, \quad (3.72)$$

thereby giving the speed of convergence desired. ■

*Proof of theorem 3.8.* Throughout this proof, we shall denote

$$F(\lambda, \varepsilon) := \frac{1}{\lambda} \frac{\mathbb{E}[e^{-\lambda U^\varepsilon}]}{1 - \mathbb{E}[e^{-\lambda U^\varepsilon}]}. \quad (3.73)$$

The assumption of non-monotonicity of the Lévy process ensures that, almost surely,  $S^\varepsilon$  and  $T^\varepsilon$  both tend to 0 as  $\varepsilon \rightarrow 0$ . Consider now the times  $T_i^\varepsilon$  and  $S_i^\varepsilon$  given in definition 2.17. Since  $X$  is Lévy,  $T_{i+1}^\varepsilon - S_i^\varepsilon$  and  $S_i^\varepsilon - T_i^\varepsilon$  are independent from one another, and are both equal in distribution to  $T^\varepsilon$  and  $S^\varepsilon$  respectively.

By lemma 3.9,  $S^\varepsilon$  and  $T^\varepsilon$  admit finite moments for all  $k$  and the function  $\mathbb{E}[e^{-\lambda U^\varepsilon}]$  is well defined, so, by equation 3.38

$$\begin{aligned} \mathcal{L}(\mathbb{E}[N_t^\varepsilon])(\lambda) &= \mathcal{L}(\mathbb{P}(R_t \geq \varepsilon))(\lambda) + \frac{1}{\lambda} \sum_{k \geq 1} \mathbb{E}[e^{-\lambda U^\varepsilon}]^k \\ &= \mathcal{L}(\mathbb{P}(R_t \geq \varepsilon))(\lambda) + \frac{1}{\lambda} \frac{1}{\mathbb{E}[e^{-\lambda U^\varepsilon}]^{-1} - 1} \end{aligned} \quad (3.74)$$

If we denote  $\rho_\varepsilon$  the radius of convergence of the Taylor series at zero associated to  $\mathbb{E}[e^{-\lambda U^\varepsilon}]$ , for  $|\lambda| < \rho_\varepsilon$ , we have

$$\mathbb{E}[e^{-\lambda U^\varepsilon}] = \sum_{k=0}^{\infty} \frac{(-\lambda)^k \mathbb{E}[(U^\varepsilon)^k]}{k!}. \quad (3.75)$$

This radius of convergence  $\rho_\varepsilon$  can be bounded below with the results of lemma 3.9 by

$$-\log(\mathbb{P}(S^\varepsilon > 1) \vee \mathbb{P}(T^\varepsilon > 1)) < \rho_\varepsilon, \quad (3.76)$$

which entails that  $\rho_\varepsilon \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ . We deduce from this series the Laurent series associated to  $\lambda^{-1}(\mathbb{E}[e^{-\lambda U^\varepsilon}]^{-1} - 1)^{-1}$ , namely

$$F(\lambda, \varepsilon) = \frac{1}{\lambda^2 \mathbb{E}[U^\varepsilon]} + \frac{1}{\lambda} \left[ \frac{\mathbb{E}[(U^\varepsilon)^2]}{2\mathbb{E}[U^\varepsilon]^2} - 1 \right] + \frac{3\mathbb{E}[(U^\varepsilon)^2]^2 - 2\mathbb{E}[U^\varepsilon] \mathbb{E}[(U^\varepsilon)^3]}{12\mathbb{E}[U^\varepsilon]^3} + O(\lambda). \quad (3.77)$$

where the remainder in  $\lambda$  is an analytic function of  $\lambda$  for  $|\lambda| < \rho_\varepsilon$ . Indeed, notice that by the inequalities of lemma 3.12, the function doesn't admit any poles on the half plane  $\operatorname{Re}(\lambda) > 0$ , so that the Taylor series above converges over the same disk as that of  $\mathbb{E}[e^{-\lambda U^\varepsilon}]$ .

Observe now that for some small  $\gamma > 0$ , the inverse Laplace transform of  $F(\lambda, \varepsilon)$  can be written as

$$\mathcal{L}^{-1}[F](t, \varepsilon) = \frac{1}{2\pi i} \left\{ \int_{\gamma - i\rho_\varepsilon}^{\gamma + i\rho_\varepsilon} + \int_{\gamma + i\rho_\varepsilon}^{\gamma + i\infty} + \int_{\gamma - i\infty}^{\gamma - i\rho_\varepsilon} \right\} e^{\lambda t} F(\lambda, \varepsilon) d\lambda. \quad (3.78)$$

Weakly, the integrals going off to infinity are of order  $o(\rho_\varepsilon^{-n})$  for any  $n \in \mathbb{N}$ . Indeed, notice that for any test function  $\varphi \in C_c^\infty([0, \infty[)$ , integrating by parts

$$\int_{\mathbb{R}} dt \varphi(t) \left[ \frac{1}{2\pi i} \int_{\gamma + i\rho_\varepsilon}^{\gamma + i\infty} e^{\lambda t} F(\lambda, \varepsilon) d\lambda \right] = \frac{(-1)^n}{2\pi i} \int_{\mathbb{R}} dt \varphi^{(n)}(t) \int_{\gamma + i\rho_\varepsilon}^{\gamma + i\infty} d\lambda \frac{e^{\lambda t}}{\lambda^n} F(\lambda, \varepsilon) d\lambda \quad (3.79)$$

But using lemma 3.12, we have

$$\left| \int_{\gamma + i\rho_\varepsilon}^{\gamma + i\infty} d\lambda \frac{e^{\lambda t}}{\lambda^n} F(\lambda, \varepsilon) d\lambda \right| \leq e^{\gamma t} \int_{\gamma + i\rho_\varepsilon}^{\gamma + i\infty} \left| \frac{F(\lambda, \varepsilon)}{\lambda^n} \right| d\lambda = O(\rho_\varepsilon^{-n-2}), \quad (3.80)$$

which entails that the integrals going to infinity in equation 3.78 converge weakly to 0 at a rate  $o(\rho_\varepsilon^{-n})$  for any  $n \in \mathbb{N}$ . Thus, asymptotically as  $\varepsilon \rightarrow 0$ , for  $t > 0$ ,

$$\mathbb{E}[N_t^\varepsilon] = \frac{t}{\mathbb{E}[U^\varepsilon]} + \left( \frac{\mathbb{E}[(U^\varepsilon)^2]}{2\mathbb{E}[U^\varepsilon]^2} - 1 \right) + \mathbb{P}(R_t \geq \varepsilon) + o(\rho_\varepsilon^{-n}), \quad (3.81)$$

for any  $n \in \mathbb{N}$ . If the process is  $\alpha$ -stable, then  $(X_{c^\alpha t})_{t \geq 0} = (cX_t)_{t \geq 0}$  in distribution for all  $c$ , so that  $U^\varepsilon = \varepsilon^\alpha U$  in distribution and

$$\mathbb{E}[N_t^\varepsilon] = \frac{t}{\mathbb{E}[U] \varepsilon^\alpha} + \frac{\mathbb{E}[U^2]}{2\mathbb{E}[U]^2} + o(\varepsilon^{\alpha n}) \quad \text{as } \varepsilon \rightarrow 0, \quad (3.82)$$

for all  $n \in \mathbb{N}$ . Note that as  $\varepsilon \rightarrow 0$ ,  $1 - \mathbb{P}(R_t \geq \varepsilon) = o(\varepsilon^n)$  for any  $n$ , since

$$\mathbb{P}(R_t \leq \varepsilon) \leq \mathbb{P}(T^\varepsilon > t) \leq \frac{\varepsilon^{\alpha k} \mathbb{E}[(T^1)^k]}{t^k} \quad (3.83)$$

for any  $k$  by Markov's inequality. ■

*Remark 3.14.* A similar theorem can be proven in  $L^s(\Omega)$  for  $\alpha$ -stable processes. For instance, if  $X$  is  $\alpha$ -stable and  $s = 2$  one obtains that for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \operatorname{Var}(N_t^\varepsilon) &\sim \left[ \frac{\operatorname{Var}(U) - 2\mathbb{E}[U]^2}{\mathbb{E}[U]^3} \right] \frac{t}{\varepsilon^\alpha} \\ &+ \frac{5\operatorname{Var}(U)^2}{4\mathbb{E}[U]^4} + \frac{\operatorname{Var}(U)}{\mathbb{E}[U]^2} - \frac{2\mathbb{E}[U^3]}{3\mathbb{E}[U]^3} + \frac{7}{4} + o(\varepsilon^{\alpha n}) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (3.84)$$

Interestingly, there is a constant term appearing in this expansion, which can be understood as induced by the boundary. This interpretation comes from Picard's analysis of the problem [22], in which the first term of this asymptotic series was also derived (cf. proposition 3.4).

If  $X$  has almost surely discontinuous paths,  $X_t$  exhibits macroscopic jumps. These will turn out to bring significative contributions, so much so that

**Corollary 3.15.** *If  $\alpha \neq 2$ , the  $\zeta$ -function of any  $\alpha$ -stable Lévy process is ill-defined for any  $p \in \mathbb{C}$ .*

*Proof.* The  $\zeta$ -function of a stochastic process  $X$  can be written as

$$\zeta_X(p) = \mathbb{E}[R_t^p] + \mathbb{E} \left[ \sum_{k \geq 2} \ell_k^p(X_t) \right], \quad (3.85)$$

if  $X$  is  $\alpha$ -stable, it follows that the first term can be written as

$$\mathbb{E}[R_t^p] = t^{\frac{p}{\alpha}} \mathbb{E}[R_1^p] \geq t^{\frac{p}{\alpha}} \mathbb{E}[|X_1|^p], \quad (3.86)$$

where we have momentarily taken  $p \in \mathbb{R}$ . Since  $X_1$  has a Lévy  $\alpha$ -stable distribution, taking  $p$  now complex, it follows that  $\mathbb{E}[R_t^p]$  is infinite as soon as  $\text{Re}(p) \geq \alpha$ , since  $X_1$  does not admit any moments of order (of real part) larger than  $\alpha$ . Applying theorem 3.8, we know that the second term in the above decomposition of  $\zeta_X$  is only defined for  $\text{Re}(p) > \alpha$ , so the fundamental strip of  $\mathcal{M}\mathbb{E}[N_t^\varepsilon]$  is empty. ■

In fact, it is possible to show that  $\mathbb{P}(X_1 > \varepsilon) \sim \mathbb{P}(R_1 > \varepsilon)$  as  $\varepsilon \rightarrow \infty$ . It turns out that the distribution of  $R_1$  is dominated by the probability of having one large jump, which indeed confirms our previous statement on the effect of the discontinuity of Lévy processes on the distribution of  $R$ . This is the so-called single big jump principle.

**Proposition 3.16** (Single big jump principle, Bertoin, [5]). Let  $\Pi$  denote the Lévy-Khinchine measure of  $X$ . Then

$$\mathbb{P}(R_1 \geq \varepsilon) \sim \Pi([- \infty, -\varepsilon] \cup [\varepsilon, \infty]) \text{ as } \varepsilon \rightarrow \infty. \quad (3.87)$$

In particular if  $X$  is an  $\alpha$ -stable process, there exists a constant  $k$  such that

$$\mathbb{P}(R_1 \geq \varepsilon) \sim \frac{k}{\varepsilon^\alpha} \text{ as } \varepsilon \rightarrow \infty. \quad (3.88)$$

Loosely speaking, it is intuitive to think that a corrective asymptotic power series for  $\mathbb{P}(R_t \geq \varepsilon)$  of the form

$$\mathbb{P}(R_1 \geq \varepsilon) \sim \sum_{k \geq 1} a_k \varepsilon^{-k\alpha} \text{ as } \varepsilon \rightarrow \infty \quad (3.89)$$

should exist for the following reason. By the single big jump principle, the probability that the range exceeds  $\varepsilon$  for large  $\varepsilon$  is dominated by the probability of a single big jump. However, it is also possible to have  $n$  large jumps of size  $J_k \varepsilon$  where  $\sum_k^n J_k \geq 1$ . The probability of each of these jumps happening is of order  $O(\varepsilon^{-\alpha})$  and by independence, the probability that  $k$  jumps of size  $O(\varepsilon)$  happen is  $O(\varepsilon^{-\alpha k})$ . In general, we cannot expect these events to be disjoint from one another, so the coefficients  $a_k$  of this sum may be negative. Finally, by the scaling invariance it is sufficient to show that this is so for  $R_1$ . Corrective terms to the above asymptotic relation should thus in principle exist, but the explicit calculation of these terms is out of the scope of this paper.

By contrast, we will now show that  $\mathbb{E}[N_t^\varepsilon] - \mathbb{P}(R_t \geq \varepsilon)$  is well-behaved. This motivates the following definition

**Definition 3.17.** The **tail  $\zeta$ -function** of the stochastic process  $X$  on  $[0, t]$  is defined as

$$\hat{\zeta}_X(p) := \mathbb{E}[\ell_p^p(X) - R_t^p]. \quad (3.90)$$

**Theorem 3.18.** *The tail  $\zeta$ -function associated to an  $\alpha$ -stable Lévy process is given by*

$$\hat{\zeta}_X(p) = \frac{t_\alpha^{\frac{p}{\alpha}}}{\Gamma(\frac{p}{\alpha})} B^*\left(\frac{p}{\alpha}\right), \quad (3.91)$$

which extends to a meromorphic function of  $p$  to the entire complex plane (since  $B^*$  is itself meromorphic), with a unique simple pole at  $p = \alpha$  of residue  $\mathbb{E}[U]^{-1} \alpha t$ .

*Proof of theorem 3.18.* To show that this quantity is well-defined, let us start by noticing that

$$\mathcal{L}(\mathbb{E}[N_t^\varepsilon] - \mathbb{P}(R_t \geq \varepsilon))(\lambda) = \frac{\mathbb{E}[e^{-\lambda\varepsilon^\alpha U}]}{\lambda(1 - \mathbb{E}[e^{-\lambda\varepsilon^\alpha U}])} \quad (3.92)$$

which for  $\text{Re}(\lambda) > 0$  goes to zero (uniformly in  $\lambda$ ) exponentially fast as  $\varepsilon \rightarrow \infty$ , showing that  $\mathbb{E}[N_t^\varepsilon] - \mathbb{P}(R_t \geq \varepsilon)$  does as well for  $\varepsilon \rightarrow \infty$  by an application of Markov's inequality. We can also compute the contribution of the second term of equation 3.85. First, notice that

$$\mathcal{L}\left(\mathbb{E}\left[\sum_{k \geq 2} \ell_k^p(X_t)\right]\right)(\lambda) = \frac{p}{\lambda} \mathcal{M}\left[\frac{\mathbb{E}[e^{-\lambda\varepsilon^\alpha U}]}{1 - \mathbb{E}[e^{-\lambda\varepsilon^\alpha U}]}](p) \quad (3.93)$$

Now, first, notice that using the scaling property of the Mellin transform  $\mathcal{M}_z[f(\lambda z)](p) = \lambda^{-p} f^*(p)$  and inverting the Laplace transform we have

$$\mathbb{E}[\ell_p^p(X) - R_t^p] = \frac{pt_\alpha^{\frac{p}{\alpha}}}{\Gamma(1 + \frac{p}{\alpha})} \mathcal{M}\left[\frac{\mathbb{E}[e^{-\varepsilon^\alpha U}]}{1 - \mathbb{E}[e^{-\varepsilon^\alpha U}]}](p). \quad (3.94)$$

Finally, setting

$$B(z) := \frac{\mathbb{E}[e^{-zU}]}{1 - \mathbb{E}[e^{-zU}]} \quad \text{and} \quad B^*(p) := \mathcal{M}_z[B(z)](p), \quad (3.95)$$

the polynomial scaling property of the Mellin transform,  $\mathcal{M}_z[f(z^\alpha)](p) = \frac{1}{\alpha} f^*\left(\frac{p}{\alpha}\right)$  yields the final result.  $\blacksquare$

*Remark 3.19.* Theorem 3.18 can be used to give an alternative proof for the series expansion of theorem 3.8.

*Alternate proof of theorem 3.8.* By lemma 3.9 and the analyticity of the expression of  $B$  with respect to  $\mathbb{E}[e^{-zU}]$  note that  $B$  admits a Laurent series on some non-trivial annulus around zero with a single simple pole at  $z = 0$ . By the fundamental correspondence (theorem 2.5), the existence of this Laurent expansion guarantees that  $B^*\left(\frac{p}{\alpha}\right)$  admits a meromorphic continuation to the whole complex plane with only simple poles at every  $p = -n\alpha$  for every  $n \in \mathbb{N}$  and at  $p = \alpha$ . The poles at the negative integer multiples of  $\alpha$  are compensated exactly by those of the  $\Gamma$ -function in the denominator of the expression of  $\hat{\zeta}$ , leaving only a pole at  $\alpha$ . Now, recalling that

$$\hat{\zeta}(p) = p \mathcal{M}[\mathbb{E}[N_t^\varepsilon] - \mathbb{P}(R_t \geq \varepsilon)](p), \quad (3.96)$$

$\mathcal{M}[\mathbb{E}[N_t^\varepsilon] - \mathbb{P}(R_t \geq \varepsilon)]$  has a supplementary pole at  $p = 0$ . Admitting that  $\hat{\zeta}(p)/p$  has the decay condition to apply the fundamental correspondence by inverting the Mellin transform we get the asymptotic relation desired.  $\blacksquare$

### 3.1.1 Exponential corrections

The fundamental correspondence is limited in that it allows us only to describe  $\mathbb{E}[N^\varepsilon]$  asymptotically up to terms smaller than any polynomial. However, in accordance to the discussion of section 2.1.1, a finer study of the analytic properties of  $\hat{\zeta}$  can yield the superpolynomial

corrections to our estimate, assuming that  $B(z)$  admits a meromorphic extension to the whole complex plane. Using lemmata 2.6 and 2.8, we have

$$\hat{\zeta}_X(p) = t^{\frac{p}{\alpha}} \Gamma\left(1 - \frac{p}{\alpha}\right) \sum_{z_0 \in \mathcal{P}} \text{Res}\left((-z)^{\frac{p}{\alpha}-1} B(z); z_0\right) \quad (3.97)$$

$$\mathcal{M}(\mathbb{E}[N_t^\varepsilon] - \mathbb{P}(R_t \geq \varepsilon))(p) = -\frac{t^{\frac{p}{\alpha}} \Gamma\left(-\frac{p}{\alpha}\right)}{\alpha} \sum_{z_0 \in \mathcal{P}} \text{Res}\left((-z)^{\frac{p}{\alpha}-1} B(z); z_0\right). \quad (3.98)$$

Recognizing that

$$\mathcal{M}_z \left[ \frac{e^{z_0/z}}{z_0} \right] (p) = -\Gamma(-p) \left( \frac{-1}{z_0} \right)^{1-p}, \quad (3.99)$$

we may formally invert the Mellin transform if all the  $z_0$ 's are simple poles to obtain the exponentially small corrections

$$\mathbb{E}[N_t^\varepsilon] - \mathbb{P}(R_t \geq \varepsilon) - \frac{t}{\mathbb{E}[U] \varepsilon^\alpha} - \left[ \frac{\mathbb{E}[U^2]}{2\mathbb{E}[U]^2} - 1 \right] \sim \sum_{z_0 \in \mathcal{P}} \frac{e^{tz_0/\varepsilon^\alpha}}{\alpha z_0} \text{Res}(B(z); z_0) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.100)$$

Generally, the poles are not simple so the corrective terms to this series stem from residues of higher order poles (the corrections remain nonetheless superpolynomially small as  $\varepsilon \rightarrow 0$ ).

### 3.1.2 Distribution of the length of branches

The distribution of the length of the  $k$ th branch (in the sense of figure 6) can be calculated. Recall that

$$\mathbb{E}[\ell_k^p(X)] = p \mathcal{M}[\mathbb{P}(N_t^\varepsilon \geq k)](p), \quad (3.101)$$

for  $k \geq 2$ , we have

$$\mathcal{L}[\mathbb{E}[\ell_k^p(X)]](\lambda) = \frac{p}{\lambda} \mathcal{M} \left[ \mathbb{E} \left[ e^{-\lambda U^\varepsilon} \right]^{k-1} \right] (p) \quad (3.102)$$

If we suppose once again that  $X$  is a Lévy  $\alpha$ -stable process, this can be simplified to yield

$$\mathbb{E}[\ell_k^p(X)] = \frac{t^{\frac{p}{\alpha}}}{\Gamma\left(\frac{p}{\alpha}\right)} \mathcal{M} \left[ \mathbb{E} \left[ e^{-\varepsilon U} \right]^{k-1} \right] \left( \frac{p}{\alpha} \right), \quad (3.103)$$

Taking the Mellin transform of a power is in general difficult. Furthermore, note that inversion is also in general complicated due to the presence of the  $\Gamma$ -function in the denominator of the above expression. To remediate the first problem, we can form the generating function yielding the distribution for the  $k$ th bar,

$$G_\alpha(z; p) := \sum_{k \geq 2} \mathbb{E}[\ell_k^p(X)] z^k = \frac{t^{\frac{p}{\alpha}}}{\Gamma\left(\frac{p}{\alpha}\right)} \mathcal{M} \left[ \frac{z}{(z \mathbb{E}[e^{-\varepsilon U}] - 1) - 1} \right] \left( \frac{p}{\alpha} \right), \quad (3.104)$$

which allows us to express

**Proposition 3.20.** For  $k \geq 2$ , the distribution of the length of the  $k$ th longest branch is characterized by its Mellin transform which is given by

$$\mathbb{E}[\ell_k^p(X)] = \frac{1}{k!} \left. \frac{\partial^k}{\partial z^k} \right|_{z=0} G_\alpha(z; p). \quad (3.105)$$

Whenever convenient, the expression above can also be evaluated by considering a circular contour of small enough radius  $r$  around the origin  $C_r$  and evaluating

$$\mathbb{E}[\ell_k^p(X)] = \frac{1}{2\pi i} \oint_{C_r} \frac{G_\alpha(z; p)}{z^{k+1}} dz. \quad (3.106)$$

### 3.1.3 Statistical parameter testing for $\alpha$ -stable processes and perspectives

What we will aim to do in this section is to illustrate by example why barcodes can be a robust statistical tools for parameter testing. Parameter testing is a widely studied subject, notably for self-similar processes, where the problem has been treated in dimension 1 (a non-comprehensive list of references is [10] and the references therein). A variety of different methods, such as multi-scale wavelet analysis, have been used to produce these results (although other methods such as the ones of [10] have also been used), so our approach does not offer anything new in this respect. The interest of our method lies in possible applications to higher dimensional random fields, for which wavelet analysis is not an effective tool. A complete theoretical framework for this would require the study of the trees of higher dimensional random fields, which are out of the scope of this paper : instead, this section acts as a proof of concept for the use of topological estimators and their utility, by studying what happens in dimension 1.

In what follows, we will consider  $X$  to be an  $\alpha$ -stable Lévy process, of which we will aim to estimate the parameter  $\alpha$ . From proposition 3.4 we know that almost surely

$$N_t^\varepsilon \sim Ct\varepsilon^{-\alpha} \quad \text{as } \varepsilon \rightarrow 0. \quad (3.107)$$

In particular, given some sample we may compute the sampled value of  $N_t^\varepsilon$ , which we will denote  $\hat{N}_t^\varepsilon$  explicitly. A close inspection of the behaviour of the sample mean  $\overline{N}_t^\varepsilon$  should thus yield an estimation for the parameter  $\alpha$  of the process  $X$ .

*Remark 3.21.* In fact, the same reasoning allows us to estimate the Hurst parameter  $H$  of a fractional Brownian motion (fBM), which also exhibits self-similarity. In this case, the analogue of the asymptotic result of proposition 3.4 is [22, §3]

$$\text{a.s. } N^\varepsilon \sim Ct\varepsilon^{-\frac{1}{H}} \quad \text{as } \varepsilon \rightarrow 0. \quad (3.108)$$

More precisely, given a sample, our test consists in performing the following steps.

1. Sample  $M$  paths of the stochastic process  $X$  (for example at regular intervals of size  $\frac{1}{N}$  for some  $N$ ) ;
2. Compute the barcode of the sampled paths. To do this, first construct a filtered simplicial complex (which is in this case nothing other than a chain with  $\sim N$  links) by taking each point to be a vertex of a complex and joining adjacent sampling points with an edge. The filtration on this complex is the value of the process at the edge (for an edge connecting vertex  $a$  to vertex  $b$ , the value of the filtration is  $X_a \wedge X_b$ ). Finally, the persistent homology of this complex can be computed with the `gudhi` package [1], which incidentally also offers a convenient implementation of filtered simplicial complexes due to Boissonnat and Maria [6].
3. For *some range of small enough*  $\varepsilon$ , and for some positive constant  $c > 1$  compute the quantity

$$\hat{\alpha}_M := \log_c \left[ \frac{\overline{N}_t^{\varepsilon/c} - \overline{N}_t^{2\varepsilon/c}}{\overline{N}_t^\varepsilon - \overline{N}_t^{2\varepsilon}} \right]. \quad (3.109)$$

Here, the notion of *some range of small enough*  $\varepsilon$  and the constant  $c$  both depend on  $N$ , with the limiting condition that as  $N \rightarrow \infty$ , the lower bound on the range of valid  $\varepsilon$  goes to zero.

Our claim is that the computed quantity  $\hat{\alpha}$  is a valid estimation of the parameter  $\alpha$  (for fBM, the quantity obtained in this way is an estimate of  $\frac{1}{H}$ ).

**Lemma 3.22** (Convergence of the sample means). The quotient

$$\frac{\overline{N}_t^{\varepsilon/c} - \overline{N}_t^{2\varepsilon/c}}{\overline{N}_t^\varepsilon - \overline{N}_t^{2\varepsilon}} \xrightarrow[M \rightarrow \infty]{\mathbb{P}} \frac{\mathbb{E}[N_t^{\varepsilon/c} - N_t^{2\varepsilon/c}]}{\mathbb{E}[N_t^\varepsilon - N_t^{2\varepsilon}]} \quad (3.110)$$

at a rate  $C_s M^{-s}$ , for every  $s \geq 1$  where  $C_s$  is a constant depending on  $s$  and the  $s$ th moment of  $N^{\varepsilon/c}$ . In particular,

$$\hat{\alpha}_M \xrightarrow[M \rightarrow \infty]{\mathbb{P}} \alpha + \xi(\varepsilon) \quad (3.111)$$

at the same rate, where  $\xi(\varepsilon)$  is a superpolynomially small function of  $\varepsilon$ .

*Remark 3.23.* The at first seemingly convoluted expression for the estimator  $\hat{\alpha}_M$  can be explained due to the results of theorem 3.8. Indeed, the subtraction present in the numerator and denominator is performed so that the constant terms of equation 3.49 vanish. Ignoring the superpolynomial contributions to this expression which remain small, we then have that the argument inside the log of the estimator is roughly

$$c^{\hat{\alpha}_M} \approx \frac{\frac{t}{\mathbb{E}[U](\varepsilon/c)^\alpha} - \frac{t}{\mathbb{E}[U](2\varepsilon/c)^\alpha}}{\frac{t}{\mathbb{E}[U]\varepsilon^\alpha} - \frac{t}{\mathbb{E}[U](2\varepsilon)^\alpha}} \approx c^\alpha. \quad (3.112)$$

With this in mind, let us now formally prove the statement of lemma 3.22.

*Proof.* That the numerator and the denominator tend to the respective expected values holds by a simple application of the weak law of large numbers, since  $N_t^\varepsilon$  is a random variable in  $L^s$  for  $s \geq 1$ . The rate of convergence of this limit can be obtained via a simple application of Markov's inequality, by noting first that the summands in the denominator tend to their limits faster than those of the numerator, as the latter's  $s$ th moments are always larger than the former's. From theorem 3.8, we see that the limit can be expressed as

$$\frac{\mathbb{E}[N_t^{\varepsilon/c} - N_t^{2\varepsilon/c}]}{\mathbb{E}[N_t^\varepsilon - N_t^{2\varepsilon}]} = c^\alpha \frac{1 + g(c\varepsilon)}{1 + g(\varepsilon)}, \quad (3.113)$$

where  $g$  is a function tending to 0 superpolynomially fast as  $\varepsilon \rightarrow 0$ , determined by the superpolynomial corrections to the results of theorem 3.8. The statement of the lemma ensues. ■

**Lemma 3.24** (Probable  $L^\infty$ -distance of sampling). Denote  $\hat{X}$  the trajectory samples of the  $\alpha$ -stable process  $X$  at every interval of length  $\frac{1}{N}$ . More precisely, somewhat abusing the notation we can write,

$$\hat{X}_t = \sum_{n=0}^{N-1} 1_{[\frac{n}{N}, \frac{n+1}{N}[}(t) X_{\frac{n}{N}}. \quad (3.114)$$

There exists a constant  $k$  such that

$$\mathbb{P}\left(\sup_{t \in \mathbb{R}} |X_t - \hat{X}_t| \leq \varepsilon\right) \gtrsim 1 - \left(\frac{k}{N\varepsilon^\alpha}\right)^N \quad \text{as } \varepsilon N^{1/\alpha} \rightarrow \infty. \quad (3.115)$$

*Remark 3.25.* The asymptotic dependence above fixes admissible values of  $\varepsilon$  as a function of  $N$  as holding whenever the asymptotic dependence above is valid (it must be valid between  $\varepsilon/c$  and  $2\varepsilon$ ). Furthermore, the parameter  $c$  we chose above is also further constrained by the requirement that the asymptotic relation of theorem 3.8 holds between  $\varepsilon/c$  and  $2\varepsilon$ . More precisely, we fix  $c$  and  $\varepsilon$  such that the superpolynomial contributions in the expansion of theorem 3.8 are negligible with respect to the term in  $\varepsilon^{-\alpha}$  and by imposing that  $\varepsilon N^{1/\alpha}$  is large enough so that the asymptotic relation of lemma 3.24 simultaneously holds within the range  $[\frac{\varepsilon}{c}, 2\varepsilon]$ . In practice, we may fix  $c$  and  $\varepsilon$  by looking at a log-log chart of  $N^\varepsilon$ , the regime of validity of  $\varepsilon$  and the value of  $c$  become clear, as shown in figure 8.

*Remark 3.26.* With respect to the barcode, linear interpolation between values of  $X$  or the consideration of the process  $\hat{X}$  is equivalent.

*Remark 3.27.* It is not a priori obvious that the event above is measurable. However, continuity in probability and the a.s. existence of a càdlàg modification of the process allows us to interpret this event to be a supremum over every  $t \in \mathbb{Q}$ , rendering the event measurable.

*Proof.* It suffices to show the result over the interval  $[0, 1]$ . Noticing that the sampling coincides with the value of the path at every  $t = \frac{1}{N}$ , it suffices to evaluate the probability that over  $N$  intervals of length  $\frac{1}{N}$  the real sampled path  $X_t(\omega)$  (notice the absence of a hat) strays away from the sampled path  $\hat{X}_t(\omega)$ . Focusing on a single interval  $[0, \frac{1}{N}]$ , the single big jump principle 3.16 states that there exists a constant  $k$  such that this probability of straying away in this interval is

$$\mathbb{P}(R_{\frac{1}{N}} \geq \varepsilon) \sim \frac{k\varepsilon^{-\alpha}}{N} \quad \text{as } \varepsilon N^{1/\alpha} \rightarrow \infty. \quad (3.116)$$

By independence, it follows that over  $N$  such intervals

$$\mathbb{P}\left(\sup_{t \in \mathbb{R}} |X_t - \hat{X}_t| \leq \varepsilon\right) \gtrsim 1 - \left(\frac{k}{N\varepsilon^\alpha}\right)^N \sim 1 \quad \text{as } \varepsilon N^{1/\alpha} \rightarrow \infty, \quad (3.117)$$

as desired. ■

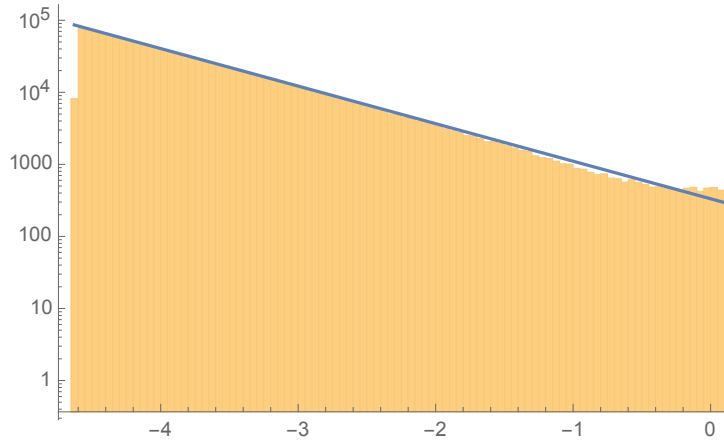


Figure 8: In orange, a histogram of the number of bars of length  $\geq \varepsilon$ ,  $N^\varepsilon$  found as a function of  $\log \varepsilon$  from a simulation of a Lévy 1.2-stable process as a random walk. In blue, the function  $C_{1.2}\varepsilon^{-1.2}$ .

Now, let us recall the following theorem.

**Theorem 3.28** (P, Theorem 3.1 [21]). *Let  $\delta_N := \|X - \hat{X}\|_{L^\infty}$ , then there exists a  $\delta_N$ -matching between the barcodes of  $\hat{X}$  and  $X$ . In particular, for any  $\varepsilon \geq 2\delta_N$*

$$N_X^{\varepsilon+\delta_N} \leq N_{\hat{X}}^\varepsilon \leq N_X^{\varepsilon-\delta_N} \quad (3.118)$$

Moreover, if  $\mathbb{E}[N_X^\varepsilon]$  is continuous with respect to  $\varepsilon$ , then

$$N_{\hat{X}}^\varepsilon \xrightarrow[N \rightarrow \infty]{L^1} N_X^\varepsilon \quad \text{and} \quad N_{\hat{X}}^\varepsilon \xrightarrow[n \rightarrow \infty]{\mathbb{P}} N_X^\varepsilon,$$

which at fixed  $N$  quantitatively translates to

$$\mathbb{E}\left[|N_{\hat{X}}^\varepsilon - N_X^\varepsilon|\right] \leq 2\omega_\varepsilon(\delta_N) \quad \text{and} \quad \mathbb{P}\left(|N_{\hat{X}}^\varepsilon - N_X^\varepsilon| \geq k\right) \leq \frac{2\omega_\varepsilon(\delta_N)}{k} \quad (3.119)$$

where  $\omega_\varepsilon$  is the modulus of continuity of  $\mathbb{E}[N_X^\varepsilon]$  on the interval  $[\varepsilon - \delta_N, \varepsilon + \delta_N]$ . Finally, the following inequalities also hold

$$N_{\hat{X}}^{\delta_N} \geq N_X^{2\delta_N} \quad \text{and} \quad N_X^{\delta_N} \geq N_{\hat{X}}^{2\delta_N}.$$



This statement can be specialized given our two lemmas above. The theorem provides bounds on  $N_X^\varepsilon$  provided that we know the value of  $\delta_N$ , since if  $\varepsilon$  is small enough,  $N_X^\varepsilon$  has some almost sure asymptotic behaviour. On the other hand, by virtue of lemma 3.24 we have a *probable* estimate of  $\delta_N$ , *i.e.* with probability  $q$ , we may give a bound of  $\delta_N$ , rendering the statement quantitative. The second part of the statement of theorem 3.28 provides bounds on the  $L^1$  distance between  $N_X^\varepsilon$  and  $N_X$ , provided that we know that  $\mathbb{E}[N_X^\varepsilon]$  is continuous. This happens to be the case for Brownian motion, as shown in [21]. Showing it in full generality for Lévy processes requires a closer study of the range of Lévy processes and the continuity of the inverse Mellin transform of  $\hat{\zeta}_X(p)/p$ . However, for the purposes of the construction of our statistical test, lemma 3.22 suffices, as it provides us with a quantitative guarantee that the parameter  $\alpha$  is well estimated by our estimator  $\hat{\alpha}_M$ .

## 4 Examples of applications

### 4.1 Brownian motion

For the rest of this section,  $B$  will denote a standard Brownian motion started at 0.

#### 4.1.1 $\zeta_B$ -function and asymptotic expansions for $N^\varepsilon$

Let us start by remarking that, in distribution

$$\sup_{[0,t]} B - B_t = |B_t|. \quad (4.120)$$

It follows that the stopping times  $T^\varepsilon$  and  $S^\varepsilon$  of theorem 3.8 are identically distributed and are distributed as the hitting times of  $\varepsilon$  by a reflected Brownian motion. An application of Doob's stopping theorem (*cf.* section ??) shows that

$$\mathbb{E}\left[e^{-\lambda U^\varepsilon}\right] = \operatorname{sech}^2(\varepsilon\sqrt{2\lambda}). \quad (4.121)$$

The term  $\mathbb{P}(N_t^\varepsilon \geq 1) = \mathbb{P}(R_t \geq \varepsilon)$  can also be computed by considering the fundamental solution of the corresponding heat equation with Dirichlet boundary conditions. We obtain [21]

$$\mathbb{P}(R_t \geq \varepsilon) = 4 \sum_{k=1}^{\infty} (-1)^{k-1} k \operatorname{erfc} \left[ \frac{k\varepsilon}{\sqrt{2t}} \right]. \quad (4.122)$$

Respectively, since Brownian motion is a 2-stable Lévy process, using equation 3.91 (here,  $B(z) = \operatorname{csch}^2(\sqrt{2z})$ ) we can write

$$\hat{\zeta}_B(p) = 2^{3-\frac{3p}{2}} t^{\frac{p}{2}} \frac{\Gamma(p)}{\Gamma(\frac{p}{2})} \zeta(p-1). \quad (4.123)$$

*Remark 4.1.* This can be obtained by using the functional properties of the Mellin transform (scaling, power of the argument) shown in table 1 and by the results of table 2.

Putting everything together, we have

**Theorem 4.2.** *The  $\zeta$ -function of Brownian motion on the interval  $[0, t]$  admits an meromorphic extension to the whole complex plane. Furthermore, it is exactly equal to*

$$\zeta_B(p) = \frac{4(2^p - 3)}{\sqrt{\pi}} \left(\frac{t}{2}\right)^{\frac{p}{2}} \Gamma\left(\frac{p+1}{2}\right) \zeta(p-1) \quad (4.124)$$

for all  $p$  and has a unique simple pole at  $p = 2$  of residue  $t$ .

*Proof.* Taking the Mellin transform of  $\mathbb{P}(R_t \geq \varepsilon)$

$$\mathcal{M}(\mathbb{P}(R_t \geq \varepsilon))(p) = 2^{2-\frac{3p}{2}} (2^p - 4) t^{\frac{p}{2}} \frac{\Gamma(p)}{\Gamma(\frac{p}{2} + 1)} \zeta(p-1). \quad (4.125)$$

Multiplying the above expression by  $p$  and adding both terms and using the fact that  $\Gamma(z+1) = z\Gamma(z)$  and the Legendre duplication formula, we obtain the result.  $\blacksquare$

The meromorphic extension of  $\zeta$  allows us to directly compute correction terms for the asymptotic series given in [21].  $\mathcal{M}(\mathbb{E}[N_t^\varepsilon])(p)$  has only two poles, one at  $p = 0$  and one at  $p = 2$ . Furthermore, along a vertical strip,  $\mathcal{M}(\mathbb{E}[N_t^\varepsilon])(p)$  decays rapidly enough to use the fundamental correspondence (theorem 2.5). Using contour integration and the Mellin inversion theorem, we can conclude that

$$\mathbb{E}[N_t^\varepsilon] = \frac{t}{2\varepsilon^2} + \frac{2}{3} + O(\varepsilon^n) \quad \text{as } \varepsilon \rightarrow 0, \quad (4.126)$$

for any  $n \in \mathbb{N}$ , as prescribed by theorem 3.8. The expectations in the expression of the theorem can be read on the expansion

$$\operatorname{sech}^2(\sqrt{2\varepsilon}) = 1 - 2\varepsilon + \frac{1}{2!} \frac{16}{3} \varepsilon^2 + O(\varepsilon^3). \quad (4.127)$$

As previously shown, the analyticity of  $\zeta_B$  beyond  $\operatorname{Re}(p) = 2$  guarantees that there are no polynomial corrections in  $\varepsilon$  to  $\mathbb{E}[N_t^\varepsilon]$  as  $\varepsilon \rightarrow 0$ . The analyticity of  $\zeta_B$  on the half plane  $\operatorname{Re}(p) > 2$  suggests that  $\mathbb{E}[N_t^\varepsilon]$  is rapidly decreasing as  $\varepsilon \rightarrow \infty$ . This is corroborated by the more general approximation of proposition 3.1 for Markov processes found in [21], namely  $\mathbb{E}[N_t^\varepsilon] \sim \mathbb{P}(R_t \geq \varepsilon)$  as  $\varepsilon \rightarrow \infty$ .

Applying the observations made in section 2.1.1, the superpolynomial corrections to the asymptotic series can be found by looking carefully at the meromorphic extension of  $\zeta_B$ .

**Proposition 4.3.** For Brownian motion  $\mathbb{E}[N_t^\varepsilon]$  admits the following series representations which converge well for large and small  $\varepsilon$  respectively

$$\mathbb{E}[N_t^\varepsilon] = 4 \sum_{k \geq 1} (2k - 1) \operatorname{erfc}\left(\frac{(2k - 1)\varepsilon}{\sqrt{2t}}\right) - k \operatorname{erfc}\left(\frac{2k\varepsilon}{\sqrt{2t}}\right) \quad (4.128)$$

$$= \frac{t}{2\varepsilon^2} + \frac{2}{3} + 2 \sum_{k \geq 1} (2(-1)^k - 1) \frac{e^{-\pi^2 k^2 t / 2\varepsilon^2}}{\varepsilon^2} \left[1 + \frac{\varepsilon^2}{\pi^2 k^2 t}\right]. \quad (4.129)$$

*Proof.* This asymptotic formula for  $\mathbb{E}[N_t^\varepsilon]$  can be obtained by using the arguments of section 2.1.1. Indeed,  $B(z) = \operatorname{csch}^2(\sqrt{2z})$  admits a meromorphic continuation to the entire complex plane and has poles of order two at  $z = -\frac{\pi^2 n^2}{2}$  for every  $n \in \mathbb{Z} \setminus \{0\}$ . It follows that

$$\operatorname{Res}\left(\left(-z\right)^{\frac{p}{2}-1} B(z), -\frac{\pi^2 n^2}{2}\right) = 2^{1-\frac{p}{2}} (2\pi)^{p-2} (p-1). \quad (4.130)$$

Taking the inverse Mellin transform of equation 3.98, we obtain the desired result. The second expression converging fast for large  $\varepsilon$  is obtained by using the functional equation of the  $\zeta$ -function and taking the inverse Mellin transform of the expression obtained. ■

*Alternative proof of proposition 4.3.* We have

$$\mathcal{L}(\mathbb{E}[N_t^\varepsilon])(\lambda) = \frac{4}{\lambda} \sum_{k \geq 1} (2k - 1) e^{-(2k-1)\varepsilon\sqrt{2\lambda}} - k e^{-2k\varepsilon\sqrt{2\lambda}} \quad (4.131)$$

$$= \left[ \frac{2 \cosh(\varepsilon\sqrt{2\lambda}) - 1}{\lambda} \right] \operatorname{csch}^2(\varepsilon\sqrt{2\lambda}). \quad (4.132)$$

By inverting the Laplace transform in equation 4.131 (this can be done by first decomposing the hyperbolic expressions into a series of exponential terms, of which the inverse Laplace transform can be found by virtue of a table or using some computational software such as `Mathematica`. The normal convergence of the resulting series guarantees that the inverse transform of the expression is exactly the series of the inverse transforms of the resulting exponentials), we obtain

$$\mathbb{E}[N_t^\varepsilon] = 4 \sum_{k \geq 1} (2k - 1) \operatorname{erfc}\left(\frac{(2k - 1)\varepsilon}{\sqrt{2t}}\right) - k \operatorname{erfc}\left(\frac{2k\varepsilon}{\sqrt{2t}}\right), \quad (4.133)$$

which converges quickly for large  $\varepsilon$ . For  $\varepsilon \rightarrow 0$ , we can get a quickly converging expression by recalling the Mittag-Leffler expansion of the hyperbolic cosecant,

$$\begin{aligned} \frac{\operatorname{csch}^2(\varepsilon\sqrt{2\lambda})}{\lambda} &= \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \frac{1}{(\varepsilon\sqrt{2\lambda} - i\pi k)^2} \\ &= \frac{1}{2\varepsilon^2\lambda^2} + \frac{1}{\lambda} \sum_{k \geq 1} \frac{4\varepsilon^2\lambda - 2\pi^2 k^2}{(2\varepsilon^2\lambda + \pi^2 k^2)^2}. \end{aligned} \quad (4.134)$$

We can take the inverse Laplace transform termwise by using the residue theorem, due to the absolute and uniform convergence of the expression. After some algebra, this operation results in

$$\mathbb{E}[N_t^\varepsilon] = \frac{t}{2\varepsilon^2} + \frac{2}{3} + 2 \sum_{k \geq 1} (2(-1)^k - 1) \frac{e^{-\pi^2 k^2 t / 2\varepsilon^2} t}{\varepsilon^2} \left[ 1 + \frac{\varepsilon^2}{\pi^2 k^2 t} \right], \quad (4.135)$$

which confirms that  $\mathbb{E}[N_t^\varepsilon]$  is extremely well approximated by  $\frac{t}{2\varepsilon^2} + \frac{2}{3}$  when  $\varepsilon$  is small.  $\blacksquare$

The alternative proof of proposition 4.3 and the formulas above give the functional equation of  $\zeta_B$  and by extension, also provide a proof for the functional equation of the Riemann  $\zeta$ -function. Indeed, taking the Mellin transform of the representation of  $\mathbb{E}[N_t^\varepsilon]$  converging rapidly close to zero one obtains

$$\zeta_B(p) = -2^{-\frac{p}{2}} \pi^{p-2} t^{\frac{p}{2}} (2^p - 3) p(p-1) \Gamma\left(-\frac{p}{2}\right) \zeta(2-p). \quad (4.136)$$

Equating this expression with the expression of the  $\zeta$ -function found in theorem 4.2, after some algebra we obtain the functional equation of the Riemann  $\zeta$ -function, namely

$$\zeta(p) = 2^p \pi^{p-1} \sin\left(\frac{\pi p}{2}\right) \Gamma(1-p) \zeta(1-p). \quad (4.137)$$

*Remark 4.4.* This is not an isolated case. The functional equation of the Riemann  $\zeta$ -function can be obtained by taking the image under the Mellin transform of other functions expressible as geometric series and their corresponding Mittag-Leffler expansion. The Poisson summation formula provides an alternative way of writing these expansions, so that both under  $\mathcal{M}$  yield the extension for  $\zeta$ . For example the image under  $\mathcal{M}$  of

$$\pi \coth \pi x = \pi + \pi \sum_{k \geq 1} e^{-2\pi k x} = \frac{1}{x} + \sum_{k \geq 1} \frac{2x}{x^2 + k^2}. \quad (4.138)$$

also yields the functional equation of  $\zeta$ .

**Proposition 4.5.** Defining

$$\eta_B(p) := (3 \cdot 2^p - 8)(p-2)(2\pi t)^{-\frac{p}{2}} \zeta_B(p), \quad (4.139)$$

The functional equation of  $\zeta_B$  is given by

$$\eta_B(p) = \eta_B(3-p). \quad (4.140)$$

In particular, as expected from the symmetry of  $\zeta$ , the axis of symmetry of  $\eta_B$  is  $\operatorname{Re}(p) = \frac{3}{2}$ .

Finally, we can calculate the moments of  $N_t^\varepsilon$ . After some calculations, we obtain

$$\mathcal{L}(\mathbb{E}[(N^\varepsilon)^s])(\lambda) = \frac{1}{\lambda} \left[ \sinh^2(\varepsilon\sqrt{2\lambda}) \operatorname{Li}_{-s}(\operatorname{sech}^2(\varepsilon\sqrt{2\lambda})) - \tanh^2(\varepsilon\sqrt{\lambda/2}) \right] \quad (4.141)$$

Finally, we can somewhat generalize the result we just obtained to all semimartingales in accordance to the following proposition, which is proved in [19].

**Proposition 4.6** (P., [19]). If  $X$  is a continuous semimartingale, then in expectation,  $\ell_p^p(X)$  admits a pole of order 1 at  $p = 2$  of residue  $[X]_t$ . In particular, so does the  $\zeta$ -function of the continuous semimartingale.

### 4.1.2 Distribution of the length of the $k$ th longest branch

Following the discussion of section 3.1.2, we know that for  $k \geq 2$ , we can calculate the moment generating function  $G_\alpha(z; p)$ , noticing that Brownian motion is a 2-stable process ( $\alpha = 2$ ). We have,

$$\frac{z}{(z\mathbb{E}[e^{-\varepsilon U}])^{-1} - 1} = \frac{2z^2}{\cosh(2\sqrt{2\varepsilon}) - 2z + 1}. \quad (4.142)$$

However, taking the Mellin transform of this expression is not easily feasible. To do so, we will write the above expression as a geometric series of decaying exponentials. Denoting  $y := e^{-2\sqrt{2\varepsilon}}$ , we can write the expression above as

$$\frac{4z^2y}{y^2 - 2(2z - 1)y + 1} = \frac{4z^2y}{(y - y_+)(y - y_-)}, \quad (4.143)$$

where  $y_\pm$  are the roots of the polynomial in the denominator of the expression, namely

$$y_\pm = 2z - 1 \pm 2i\sqrt{z(1 - z)}. \quad (4.144)$$

Partial fraction decomposition entails

$$\frac{4z^2y}{y^2 - 2(2z - 1)y + 1} = \frac{A(z)y}{y - y_+} - \frac{A(z)y}{y - y_-}, \quad (4.145)$$

where

$$A(z) = \frac{4z^2}{y_+ - y_-} = \frac{z^2}{\sqrt{z(z - 1)}}. \quad (4.146)$$

Finally, we may express each of the terms above as a geometric series. Summing both terms,

$$-A(z) \sum_{k \geq 1} \left( \frac{y_-^k - y_+^k}{y_+^k y_-^k} \right) y^k = 4z^2 \sum_{k \geq 1} \frac{y_+^k - y_-^k}{y_+ - y_-} y^k \quad (4.147)$$

Recalling that  $y = e^{-2\sqrt{2\varepsilon}}$  and taking the Mellin transform with respect to  $\varepsilon$  we have

$$\mathcal{M} \left[ \frac{z}{(z\mathbb{E}[e^{-\varepsilon U}])^{-1} - 1} \right] (p) = 2^{3-3p} \Gamma(2p) z^2 \frac{\text{Li}_{2p}(y_+(z)) - \text{Li}_{2p}(y_-(z))}{y_+(z) - y_-(z)}. \quad (4.148)$$

Finally, the generating function can be written as

$$G_2(z; p) = 8 \frac{\Gamma(p)}{\Gamma(\frac{p}{2})} \left( \frac{t}{8} \right)^{\frac{p}{2}} z^2 \frac{\text{Li}_p(y_+(z)) - \text{Li}_p(y_-(z))}{y_+(z) - y_-(z)}. \quad (4.149)$$

When  $z$  is in the vicinity of 0,  $y_+$  and  $y_-$  are both complex, it is thus *a priori* not obvious that the quantity defined above should remain real for real  $p$ . However, this must be so, since

$$\frac{y_+^k - y_-^k}{y_+ - y_-} = a_k(z), \quad (4.150)$$

where  $a_k(z)$  is the solution to the following difference equation

$$a_k(z) = 2(2z - 1) a_{k-1} - a_{k-2}, \quad (4.151)$$

with seed  $a_0 = 0$  and  $a_1 = 1$ . In fact, it is possible to express  $a_k(z)$  as defined in equation 4.151 in terms of the Chebyshev polynomials of the second kind  $U_k$

$$a_k(z) = U_{k-1}(2z - 1) \quad \text{and} \quad a_k(0) = (-1)^{k-1} k, \quad (4.152)$$

and incidentally  $\frac{a_k}{a_{k-1}}$  corresponds to the  $k$ th convergent of the continuous fraction

$$2(2z-1) - \frac{1}{2(2z-1) - \frac{1}{2(2z-1) - \frac{1}{\ddots}}} \quad (4.153)$$

Using these relations, it is possible to rewrite  $G_2$  as

$$G_2(z; p) = 8 \frac{\Gamma(p)}{\Gamma(\frac{p}{2})} \left(\frac{t}{8}\right)^{\frac{p}{2}} z^2 \sum_{k \geq 1} \frac{a_k(z)}{k^p}. \quad (4.154)$$

Since ultimately what interests us are the derivatives of this function at 0, we can rewrite  $G_2(z; p)$  formally as

$$G_2(z; p) = 8 \frac{\Gamma(p)}{\Gamma(\frac{p}{2})} \left(\frac{t}{8}\right)^{\frac{p}{2}} z^2 \sum_{n=0}^{\infty} \sum_{k \geq 1} \frac{a_k^{(n)}(0)}{n!} \frac{z^n}{k^p}. \quad (4.155)$$

The problem thus boils down to effectively computing the coefficients  $a_k^{(n)}(0)$ . To do so, we can consider augmenting the recurrence problem to phase space  $(a_k, a_k^{(1)}, a_k^{(2)}, \dots, a_k^{(n)})$  noticing the following relation

$$a_k^{(n)}(z) = 2(2z-1)a_{k-1}^{(n)}(z) + 4na_{k-1}^{(n-1)}(z) - a_{k-2}^{(n)}(z) \quad (4.156)$$

Setting  $z = 0$  and  $a_k^{(n)}(0) = a_k^{(n)}$  in this relation yields

$$a_k^{(n)} = -2a_{k-1}^{(n)} + 4na_{k-1}^{(n-1)} - a_{k-2}^{(n)}. \quad (4.157)$$

For example, it is easy to verify that

$$a_k^{(1)} = \frac{2}{3}(-1)^k k(k^2 - 1) \quad \text{and} \quad a_k^{(2)} = \frac{2^2}{15}(-1)^{k-1} k(k^2 - 1)(k^2 - 2^2). \quad (4.158)$$

In general, the following formula holds

$$a_k^{(n)} = \frac{2^n (-1)^{k+n+1}}{k} \prod_{i=0}^n \frac{k^2 - i^2}{2i + 1}. \quad (4.159)$$

It follows from these observations that the Mellin transforms of the distributions of the longest branches may be expressed as linear combinations of shifted and twisted  $\zeta$ -functions, since the  $a_k^{(n)}$ 's are polynomials of degree  $2n + 1$  in  $k$ . Indeed, computing the first few terms, we have

$$\mathbb{E}[\ell_2^p(B)] = \frac{2^{3-\frac{5p}{2}} t^{\frac{p}{2}} \Gamma(p)}{\Gamma(\frac{p}{2})} (2^p - 2^2) \zeta(p-1) \quad (4.160)$$

$$\mathbb{E}[\ell_3^p(B)] = \frac{2^{4-\frac{5p}{2}} t^{\frac{p}{2}} \Gamma(p)}{3\Gamma(\frac{p}{2})} [(2^p - 2^2) \zeta(p-1) - (2^p - 2^4) \zeta(p-3)] \quad (4.161)$$

$$\mathbb{E}[\ell_4^p(B)] = \frac{2^{4-\frac{5p}{2}} t^{\frac{p}{2}} \Gamma(p)}{15\Gamma(\frac{p}{2})} [4(2^p - 2^2) \zeta(p-1) - 5(2^p - 2^4) \zeta(p-3) + (2^p - 2^6) \zeta(p-5)], \quad (4.162)$$

and so on. These Mellin transforms can be inverted to yield explicit expressions of  $\mathbb{P}(\ell_k \geq \varepsilon)$ ,

$$\mathbb{P}(\ell_2(B) \geq \varepsilon) = 4 \sum_{k \geq 1} k \left[ \operatorname{erfc} \left( k\varepsilon \sqrt{\frac{2}{t}} \right) - 4 \operatorname{erfc} \left( 2k\varepsilon \sqrt{\frac{2}{t}} \right) \right] \quad (4.163)$$

$$\mathbb{P}(\ell_3(B) \geq \varepsilon) = \frac{8}{3} \sum_{k \geq 1} k \left[ 4(4k^2 - 1) \operatorname{erfc} \left( 2k\varepsilon \sqrt{\frac{2}{t}} \right) - (k^2 - 1) \operatorname{erfc} \left( k\varepsilon \sqrt{\frac{2}{t}} \right) \right] \quad (4.164)$$

$$\mathbb{P}(\ell_4(B) \geq \varepsilon) = \frac{8}{15} \sum_{k \geq 1} k \left[ (k^4 - 5k^2 + 4) \operatorname{erfc} \left( k\varepsilon \sqrt{\frac{2}{t}} \right) - 16(4k^4 - 5k^2 + 1) \operatorname{erfc} \left( 2k\varepsilon \sqrt{\frac{2}{t}} \right) \right]. \quad (4.165)$$

These calculations can be performed for any  $\ell_k$  without any additional difficulty.

## 4.2 Reflected Brownian motion

**Theorem 4.7.** *The  $\zeta$ -function of the process  $|B|$  is*

$$\zeta_{|B|}(p) = \frac{2^{1-\frac{p}{2}}(2^p - 2)t^{\frac{p}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \zeta(p-1) \quad (4.166)$$

which has a unique pole at  $p = 2$  of residue  $t$ .

*Proof.* The theorem immediately follows from applying [19, Prop. 2.14]. ■

We immediately deduce by inverting the Mellin transform that

**Proposition 4.8.** The function  $\mathbb{E}[N_t^\varepsilon]$  admits the following representations for reflected Brownian motion

$$\mathbb{E}[N_t^\varepsilon] = \sum_{k \geq 1} 2k \left[ \operatorname{erfc}\left(\frac{k\varepsilon}{\sqrt{2t}}\right) - 2 \operatorname{erfc}\left(\frac{2k\varepsilon}{\sqrt{2t}}\right) \right] \quad (4.167)$$

$$= \frac{1}{2\varepsilon^2} + \frac{1}{6} + \sum_{k \geq 1} \frac{4\pi^2 k^2 e^{-\frac{2\pi^2 k^2}{\varepsilon^2}} + \varepsilon^2 e^{-\frac{2\pi^2 k^2}{\varepsilon^2}} - 2e^{-\frac{\pi^2 k^2}{2\varepsilon^2}} (\pi^2 k^2 + \varepsilon^2)}{\pi^2 k^2 \varepsilon^2}. \quad (4.168)$$

## 5 Acknowledgements

The author would like to thank Pierre Pansu and Claude Viterbo for helping with the redaction of the manuscript as well as their guidance. Many thanks are also owed to Pascal Massart, Robert Adler, Frédéric Chazal and Shmuel Weinberger for the fruitful discussions regarding this work.

## References

- [1] Gudhi library. <https://gudhi.inria.fr/>.
- [2] R. J. Adler and J. E. Taylor. *Random Fields and Geometry*. Springer New York, 2007.
- [3] J.-M. Azaïs and M. Wschebor. *Level Sets and Extrema of Random Processes and Fields*. John Wiley & Sons, Inc., Jul 2008.
- [4] Y. Baryshnikov. Time series, persistent homology and chirality. *arXiv:1909.09846*, 2019.
- [5] J. Bertoin. *Lévy Processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1996.
- [6] J.-D. Boissonnat and C. Maria. The simplex tree: An efficient data structure for general simplicial complexes. *Algorithmica*, 70(3):406–427, may 2014.
- [7] M. Carriere, S. Oudot, and M. Ovsjanikov. Sliced Wasserstein Kernel for Persistence Diagrams. In *ICML 2017 - Thirty-fourth International Conference on Machine Learning*, pages 1–10, Sydney, Australia, Aug. 2017.
- [8] F. Chazal and V. Divol. The density of expected persistence diagrams and its kernel based estimation. In B. Speckmann and C. D. Tóth, editors, *34th International Symposium on Computational Geometry (SoCG 2018)*, volume 99 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 26:1–26:15, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

- [9] D. Cohen-Steiner, H. Edelsbrunner, J. Harer, and Y. Mileyko. Lipschitz functions have  $L^p$ -stable persistence. *Foundations of Computational Mathematics*, 10(2):127–139, Jan 2010.
- [10] T. T. N. Dang and J. Istaş. Estimation of the Hurst and the stability indices of a  $h$ -self-similar stable process. 06 2015.
- [11] V. Divol and T. Lacombe. Understanding the topology and the geometry of the persistence diagram space via optimal partial transport. *CoRR*, abs/1901.03048, 2019.
- [12] G. Doetsch. *Handbuch der Laplace-Transformation*. Birkhäuser Basel, 1950.
- [13] T. Duquesne and J.-F. Le Gall. *Random trees, Lévy processes and spatial branching processes*. Number 281 in Astérisque. Société mathématique de France, 2002.
- [14] T. Duquesne and J.-F. Le Gall. Probabilistic and fractal aspects of Lévy trees. *Probability Theory and Related Fields*, 131(4):553–603, Nov 2004.
- [15] P. Flajolet, X. Gourdon, and P. Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoretical Computer Science*, 144(1):3–58, 1995.
- [16] Y. Mileyko, S. Mukherjee, and J. Harer. Probability measures on the space of persistence diagrams. *Inverse Problems*, 27(12):124007, Nov 2011.
- [17] J. Neveu and J. Pitman. Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. *Séminaire de probabilités de Strasbourg*, 23:239–247, 1989.
- [18] A. C. Offord. Introduction to the theory of Fourier integrals. *Nature*, 141(3561):183, jan 1938.
- [19] D. Perez. Local  $\zeta$ -functions in stochastic persistent homology.
- [20] D. Perez. On  $C^0$ -persistent homology and trees. <https://arxiv.org/abs/2012.02634>, Dec. 2020.
- [21] D. Perez. On the persistent homology of almost surely  $C^0$  stochastic processes. <https://arxiv.org/abs/2012.09459>, Dec. 2020.
- [22] J. Picard. A tree approach to  $p$ -variation and to integration. *The Annals of Probability*, 36(6):2235–2279, Nov 2008.
- [23] K. Turner, Y. Mileyko, S. Mukherjee, and J. Harer. Fréchet means for distributions of persistence diagrams. *Discrete & Computational Geometry*, 52(1):44–70, Jul 2014.