Adaptive Hybrid Control for Robust Global Phase Synchronization of Kuramoto Oscillators

Alessandro Bosso, Ilario Antonio Azzollini, Simone Baldi, Luca Zaccarian

Adaptive Hybrid Control for Robust Global Phase Synchronization of Kuramoto Oscillators

Alessandro Bosso[1], *Member, IEEE*, Ilario A. Azzollini[1], Simone Baldi[2], *Senior Member, IEEE*, and Luca Zaccarian[3], *Fellow, IEEE*

***Abstract*— A distributed controller is designed for robust global phase synchronization of a network of uncertain second-order Kuramoto oscillators with a leader system, modeled as a nonlinear autonomous exosystem. The phase angles being elements of the unit circle, we propose an adaptive hybrid strategy based on a hysteresis mechanism to obtain global results despite the well-known topological obstructions. Only an upper bound on the unknown parameters of the oscillators is required to keep the adaptive estimates in a compact set. Since the reference signal is not available to each network node, we design a distributed observer of the leader exosystem. Leveraging the results of hybrid systems theory, including reduction theorems, Lyapunov techniques, and properties of ω-limit sets, we prove robust global asymptotic stability of the closed-loop dynamics, despite the presence of an adaptive control law.**

***Index Terms*— Kuramoto oscillators, distributed control, hybrid control, adaptive control, robustness.**

I. Introduction

SYNCHRONIZATION and coordination phenomena are ubiquitous in several application domains, including physics, engineering, biology, and social sciences. Particularly studied, in this context, are the dynamical behaviors arising from networks of interacting oscillators. To describe these behaviors, the Kuramoto model [1] is certainly the most popular model due to its ability to capture complex nonlinear phenomena with appealing mathematical simplicity. The study of power networks [2], [3] or of connectivity patterns in the human brain [4], [5] are just some examples where Kuramoto oscillators have been adopted.

In general, synchronization of Kuramoto oscillators may occur with or without a control input affecting the network. Concerning the uncontrolled scenario, significant efforts have been dedicated to studying the impact of couplings (either the network topology or the intensity of connections) on the synchronization properties of the trajectories [6]–[12]. In the controlled scenario, the emphasis is on finding an appropriate input to achieve synchronization [13]. In this context, the objective of leader-follower synchronization, also known as pacemaker-based synchronization [14], becomes particularly relevant. The typical challenge in achieving leader-follower synchronization is that the controller of each node should employ only locally available quantities and variables shared according to a communication topology.

This work considers a second-order version of the original first-order Kuramoto model, where each oscillator has its own inertia [3], [15] and is characterized by a phase angle and an angular frequency. We remark that further extensions have been recently proposed, including the third-order Kuramoto model [16], inspired by the transient behavior of power networks, or the generalization of the phase state space given by the Kuramoto model on Stiefel manifolds [17], capable of including in a unified framework both the classical model and more complex structures such as the Lohe model [18].

The focus of this work is to achieve *global* leader-follower phase synchronization in a network of second-order Kuramoto oscillators, without precise information of the model parameters. In the following, we review some representative results in the field, which clarify the motivations for our study.

A. Related works

It has been well recognized in the literature that the non-Euclidean nature of the state space of a Kuramoto model is the main obstruction for achieving global asymptotic convergence to the leader's phase reference. Several strategies have been proposed to deal with this obstruction. For example, a natural approach is to represent the phase of each oscillator as an element of the unit circle. It follows that the ensemble of the phase angles is an element of the N-torus [19].

One of the main advantages of employing the unit circle formalism is that phase synchronization can be reformulated as the attractivity of a compact set. Although this property is beneficial for control design, the N-torus is a non-Euclidean set, meaning that synchronization cannot be handled with the same tools used in linear consensus. In particular, the topological properties of a non-contractible space (i.e., not diffeomorphic to any Euclidean space) pose significant obstacles to *global stabilization through continuous feedback*. For instance, the continuous-time algorithms in [19] (and their corresponding discrete-time versions) lead to multiple equilibria in the state

[1]A. Bosso and I. A. Azzollini are with the Department of Electrical, Electronic, and Information Engineering (DEI), University of Bologna, Italy. Email: {alessandro.bosso, ilario.azzollini}@unibo.it

[2]S. Baldi is with the School of Mathematics, Southeast University, Nanjing, China and with the Delft Center for Systems and Control, Delft University of Technology, The Netherlands. Email: s.baldi@tudelft.nl

[3]L. Zaccarian is with LAAS-CNRS, Université de Toulouse, and the Department of Industrial Engineering, University of Trento, Italy. Email: zaccarian@laas.fr

space, where only one of them corresponds to the desired configuration. The same issue is shared by several applications involving rotations. In the context of rigid body dynamics, only *almost global* results can be achieved with continuous laws for control [20] and observation [21].

In recent years, it has been shown that robust global stabilization can be achieved on non-contractible spaces through dynamic *hybrid* (instead of continuous) feedback [22]. Meaningful results have been proposed, e.g., for unit quaternions [22] through hysteresis-based techniques and for the N-sphere [23] via synergistic potential functions. Some efforts have also been dedicated to the unit circle [24]. However, all of the above solutions have been developed in a single-agent scenario and in the absence of uncertain dynamics. One of the first attempts to present hybrid feedback in a multi-agent setting can be found in [25], for the special case of acyclic communication graphs.

Despite the progress in the field, some additional vital elements are needed for achieving global leader-follower synchronization of uncertain Kuramoto oscillators. Since no specific communication topology is imposed for the network, while the reference is not assumed to be globally available, it is necessary to ensure that each node reconstructs the leader signals. Additionally, the above-cited works have been developed under the assumption of complete knowledge of the parameters. The presence of model uncertainties complicates the asymptotic synchronization goal. Therefore, specific control solutions are needed to ensure robust asymptotic stability of the synchronization set.

B. Main contribution of this work

Motivated by the previous overview, we propose here a distributed scheme that solves the leader-follower problem by combining three components: (i) a distributed observer, used to reconstruct the reference in the nodes not directly connected to the leader; (ii) a hybrid stabilizer used to track the locally estimated reference and ensure, under parametric uncertainties, phase synchronization in a global sense; (iii) an adaptive mechanism to suitably handle the parametric uncertainties. Besides the technological interest of the synchronization problem at stake, for each one of the above components, we provide a solution of independent interest, whose novelty is highlighted next.

(i) About the distributed observer, we follow the idea that the unit circle, used to represent the phase angles, can be naturally embedded in $\mathbb{R}^2$. With this embedding, since the estimates are designed without being constrained on the unit circle, consensus techniques for Euclidean spaces can be employed to achieve global estimation of the leader signals. As compared with other solutions in the literature that follow the embedding approach [26], [27], here we allow for more general structures of the exosystem: in particular, we consider exosystems admitting a feedback interconnection between the phase and frequency subsystems, whereas the literature in this field only handles cascaded interconnections. Exploiting input-to-state stability (ISS) and small-gain arguments, we prove global asymptotic reconstruction of the reference for a fairly general class of exosystems, which is a contribution of independent interest.

(ii) About the hybrid stabilizers at each node, to ensure compatibility with the adaptive mechanism, we revisit and extend the hysteresis-based hybrid solution originally proposed in [22] to deal with the topological obstructions associated with the unit circle. In particular, we augment the hybrid feedback with a first-order filter, so that the stabilizing input does not change across jumps, a key property for interlacing the hybrid stabilizer with the continuous-time adaptation commented below. Due to the simplicity of the condition on the filter time constant under which we prove stability, this dynamic extension is of independent interest and can be exploited in future works.

(iii) Finally, about our hybrid adaptation mechanism, we show that, with an appropriate robust modification of the adaptive law, it is possible to ensure the existence of a *robustly globally asymptotically stable attractor* for the tracking error system, without requiring any persistency of excitation. This result, which may sound atypical as compared to standard results in the adaptive control literature, represents a notable by-product of this work. Specifically, the powerful characterization of ω-limit sets of well-posed hybrid systems given in the hybrid systems formalism [28], together with a simple dead-zone-based projection mechanism for keeping the parameter estimates in a compact set, enables proving the existence of such a compact globally asymptotically stable attractor.

To conclude, we emphasize that the closed-loop asymptotic stability of the overall control scheme is analyzed through reduction theorems for hybrid systems. In particular, we prove that global phase synchronization is well represented as robust global asymptotic stabilization of a suitable compact set. We remark that the robustness of asymptotic stability is guaranteed in this context by the regularity properties of the feedback law and compactness of the characterized attractor. With respect to this point, let us comment on the interesting distributed quaternion synchronization in [27], achieved by combining a sliding-mode distributed observer and a hybrid stabilizer. In that work, unfortunately, the presence of static discontinuities makes it impossible to ensure the robustness properties established in this work.

A preliminary version of this study has been published in [29]. In this paper we improve the work [29] in several directions. First of all, [29] considers the simplified case of known parameters of the Kuramoto oscillators. Therefore, several challenges related to including adaptation laws in a hybrid setting are addressed and solved here for the first time. Moreover, [29] only considers a simplified cascaded exosystem structure as in [26], [27], whereas in this paper we address nontrivial challenges emerging from feedback interconnections, involving suitable small-gain approaches not required in [29]. For example, the results of [29] cannot be applied to our simulation example of Section VII.

The paper is organized as follows. Sections II and III are dedicated, respectively, to formulating the model and presenting a formal statement of the control problem. Then, Section IV defines the distributed observer and provides a detailed stability analysis of the estimation error dynamics. On the other hand, Section V introduces the hybrid controller for reference tracking, along with a preliminary stability analysis

in conditions of global knowledge of the reference signal. The results of the previous sections are collected and exploited in Section VI, which presents the overall stability analysis based on reduction theorems. In Section VII, we validate the theoretical results through some numerical simulations that confirm the effectiveness of the proposed solution. Finally, Section VIII concludes the article.

Notation

$\mathbb{R}$ and $\mathbb{Z}$ are the sets of real and integer numbers, while $\mathbb{R}_{\geq 0} := [0, \infty)$. The transpose of real-valued vectors and matrices is denoted by $(\cdot)^\top$, while $\otimes$ is the Kronecker matrix product. For any positive integer n, I_n is the identity matrix of dimension n and $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of all ones. Given column vectors v and w, the notation (v, w) denotes the concatenated vector $[v^\top \ w^\top]^\top$. Finally, $\mathrm{diag}(a_1, \dots, a_n)$ denotes the block-diagonal matrix with diagonal elements a_i, $i \in \{1, \dots, n\}$.

1) Graph Theory: An undirected graph of order N is defined as $\mathcal{G} := \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} := \{1, \dots, N\}$ is a finite non-empty set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of non-ordered pairs of nodes, called edges. For each $i \in \mathcal{V}$, $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ is the set of neighbors of i. An undirected graph $\mathcal{G}$ is connected if, taken any arbitrary pair of nodes (i, j), $i, j \in \mathcal{V}$, there is a path from i to j. Given a leader node not included in $\mathcal{V}$, we denote with $\mathcal{T} \subseteq \mathcal{V}$ the set of target nodes, i.e., the set of nodes that receive information from the leader. For an undirected graph $\mathcal{G}$ with target nodes $\mathcal{T}$, the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, $i \neq j$, and $a_{ij} = 0$ otherwise; the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $l_{ii} = \sum_j a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$, while the target matrix $T = [\tau_{ij}] \in \mathbb{R}^{N \times N}$ is a diagonal matrix such that $\tau_{ii} = 1$ if $i \in \mathcal{T}$ and $\tau_{ii} = 0$ otherwise. Finally, the matrix $B := L + T$ is denoted leader-follower matrix. For an undirected and connected graph $\mathcal{G}$ with $\mathcal{T} \neq 0$ (equivalently, such that $T \neq 0$), B is positive definite [30].

2) Hybrid Dynamical Systems: A hybrid dynamical system can be compactly described as [28]:

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in C \\ x^+ \in G(x), & x \in D \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ is the jump set, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the jump map. A solution of (1) can either flow according to the differential inclusion $\dot{x} \in F(x)$ when $x \in C$, or jump according to the difference inclusion $x^+ \in G(x)$ when $x \in D$. We refer to [28], [31] for the main definitions and tools for the analysis of hybrid systems.

II. Model Description

A. Second-Order Kuramoto Network

In this article, we consider a generalization of the celebrated Kuramoto model [1], based on the *swing equations* described in [3]. More specifically, the *second-order Kuramoto network* is a system of N nonlinear oscillators, coupled through an undirected and connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$:

$$\begin{aligned} \dot{\theta}_i &= \omega_i, && i \in \mathcal{V} \\ m_i \dot{\omega}_i &= -d_i \omega_i + \omega_{\mathrm{n}i} + u_i - \sum_{j \in \mathcal{N}_i} k_{ij} \sin(\theta_i - \theta_j - \varphi_{ij}), \end{aligned} \tag{2}$$

where, for each $i \in \mathcal{V}$, $\theta_i \in \mathbb{R}$ and $\omega_i \in \mathbb{R}$ are the phase and the frequency, respectively, u_i is the control input, $m_i > 0$ is the oscillator's inertia, $d_i > 0$ is a damping constant, and $\omega_{\mathrm{n}i}$ is the oscillator's natural frequency. In addition, $k_{ij} = k_{ji} > 0$ and $\varphi_{ij} = \varphi_{ji} \in [0, 2\pi)$ are, respectively, the coupling weight and the phase shift between oscillators i and j. Suppose that the graph $\mathcal{G}$, associated with the physical couplings in (2), also defines the communication topology among the nodes.

Define $\theta := \begin{bmatrix} \theta_1 & \dots & \theta_N \end{bmatrix}^\top \in \mathbb{R}^N$ and $\omega := \begin{bmatrix} \omega_1 & \dots & \omega_N \end{bmatrix}^\top \in \mathbb{R}^N$, then denote by $(\theta(\cdot), \omega(\cdot)) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{2N}$ a solution of system (2), for some input signals $u_i(\cdot)$, $i \in \mathcal{V}$, and with initial conditions $(\theta(0), \omega(0))$. We say that $(\theta(\cdot), \omega(\cdot))$ achieves *phase synchronization* if

$$\lim_{t \to +\infty} \theta_i(t) - \theta_j(t) \in \left\{ \tilde{\theta} : \tilde{\theta} = 2k\pi, k \in \mathbb{Z} \right\}, \quad \forall i, j \in \mathcal{V}. \tag{3}$$

Similarly, the solution $(\theta(\cdot), \omega(\cdot))$ is said to achieve *frequency synchronization* if

$$\lim_{t \to +\infty} \omega_i(t) - \omega_j(t) = 0, \qquad \forall i, j \in \mathcal{V}. \tag{4}$$

For the network (2), our objective is to design a distributed strategy that ensures robust global phase synchronization to a reference trajectory. Namely, our aim is to define feedback laws for the inputs u_i based only on local information and network communication such that, for any initialization of system (2), the corresponding solution $(\theta(\cdot), \omega(\cdot))$ robustly achieves phase synchronization and convergence to the reference. When we refer to robust synchronization, we mean that (3) is obtained through asymptotic stability of a compact set, with appropriate robustness to perturbations of the closed-loop dynamics. A precise definition of this concept is presented in Section III.

Because we do not assume exact knowledge of the local parameters m_i, d_i, $\omega_{\mathrm{n}i}$, k_{ij}, and φ_{ij}, we design adaptive controllers that ensure asymptotic convergence in the presence of parametric uncertainties. At the same time, it is well known that the sensitivity of adaptive techniques to non-parametric (unmodeled) perturbations of the dynamics calls for a robust design of the adaptive law and some known bounds of the parametric uncertainty (see, e.g., [32, Chs. 8 and 9]). Accordingly, we impose the following assumption.

Assumption 1. *There exists a scalar $\rho > 0$, known to each node $i \in \mathcal{V}$, such that:*

$$\begin{aligned} &m_i \leq \rho, \ d_i \leq \rho, \ |\omega_{\mathrm{n}i}| \leq \rho, && \forall i \in \mathcal{V}, \\ &k_{ij} \leq \rho, && \forall i \in \mathcal{V}, \forall j \in \mathcal{N}_i, \end{aligned} \tag{5}$$

where the bound ρ is taken to be the same for all parameters for simplicity of notation.

B. Quaternion-Inspired Representation

For control design, we propose to rewrite system (2) in a more convenient form. Motivated by the equivalence modulo 2π of the phases θ_i, also reflected in the phase synchronization condition (3), we choose to represent θ_i on the unit circle $\mathbb{S}^1 := \left\{ \begin{bmatrix} \alpha & \beta \end{bmatrix}^\top \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1 \right\}$. Recall that the compact set $\mathbb{S}^1$ has Lie group structure that is isomorphic to the group of planar rotations $\mathrm{SO}(2) := \{R \in \mathbb{R}^{2\times 2} : R^\top R = I_2, \det(R) = 1\}$. In view of such an isomorphism, we define the function $\mathcal{R} : \mathbb{S}^1 \to \mathrm{SO}(2)$, which maps any $\begin{bmatrix} \alpha & \beta \end{bmatrix}^\top \in \mathbb{S}^1$ into the corresponding rotation matrix:

$$\mathcal{R}\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right) := \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{6}$$

Function $\mathcal{R}(\cdot)$ is useful to define the group multiplication between any $\xi, \hat{\xi} \in \mathbb{S}^1$ as $\mathcal{R}(\xi)\hat{\xi} = \mathcal{R}(\hat{\xi})\xi$ (note that $\mathbb{S}^1$ is Abelian, i.e., the group operation is commutative), where the identity element is given by

$$\boldsymbol{e} := \begin{bmatrix} 1 & 0 \end{bmatrix}^\top. \tag{7}$$

From the above definitions, we introduce the following representation for θ_i:

$$\zeta_i := \begin{bmatrix} \eta_i & \epsilon_i \end{bmatrix}^\top := \begin{bmatrix} \cos(\theta_i/2) & \sin(\theta_i/2) \end{bmatrix}^\top \in \mathbb{S}^1, \tag{8}$$

corresponding to a unit quaternion for planar rotations (cf. [22] for the parameterization adopted for 3D rotations). We refer to [29] for the relation between (8) and the representation employed in [33] and [31, Ex. 34]. Using (6) and (8), the phase dynamics on $\mathrm{SO}(2)$ and $\mathbb{S}^1$ is obtained as

$$\frac{d}{dt}\mathcal{R}(\zeta_i) = \frac{1}{2}\omega_i J \mathcal{R}(\zeta_i), \quad \dot{\zeta}_i = \frac{1}{2}\omega_i J \zeta_i, \qquad i \in \mathcal{V}, \tag{9}$$

where $J := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{SO}(2)$. Let $\mathbb{T}^N := \prod_{i=1}^N \mathbb{S}^1$ denote the N-torus. The network dynamics (2) can be conveniently rewritten on $\mathbb{T}^N \times \mathbb{R}^N$ as follows:

$$\begin{aligned} \dot{\zeta}_i &= \frac{1}{2}\omega_i J \zeta_i \\ m_i \dot{\omega}_i &= -d_i \omega_i + \omega_{\mathrm{n}i} + u_i - \sum_{j\in\mathcal{N}_i} k_{ij} \phi(\zeta_i)^\top J \phi(\zeta_j) \cos(\varphi_{ij}) \\ &\quad + \sum_{j\in\mathcal{N}_i} k_{ij} \phi(\zeta_i)^\top \phi(\zeta_j) \sin(\varphi_{ij}), \qquad i \in \mathcal{V}, \end{aligned} \tag{10}$$

where $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is defined as

$$\phi(\zeta_i) := \mathcal{R}(\zeta_i)\zeta_i = \begin{bmatrix} \eta_i^2 - \epsilon_i^2 \\ 2\eta_i\epsilon_i \end{bmatrix}, \quad \zeta_i := \begin{bmatrix} \eta_i \\ \epsilon_i \end{bmatrix} \tag{11}$$

and corresponds to the double angle formula from $\zeta_i := [\cos(\theta_i/2) \ \sin(\theta_i/2)]^\top$ to $[\cos(\theta_i) \ \sin(\theta_i)]^\top$. Note that, with the proposed representation (8), the condition (3) corresponding to phase synchronization coincides with

$$\lim_{t\to+\infty} \mathcal{R}(\zeta_i(t))^\top \zeta_j(t) \in \{-\boldsymbol{e}, \boldsymbol{e}\}, \quad \forall i, j \in \mathcal{V}. \tag{12}$$

Remark 1. *In some applications, such as those involving rotary encoders, θ_i is provided by sensors that "wrap" the angles in $[0, 2\pi)$ (equivalently, in $[-\pi, \pi)$). In this scenario, if (8) is used to compute ζ_i from the available sensor measurement, call it θ_i^{s}, special care must be taken to ensure that a continuous trajectory of the vector $[\cos(\theta_i^{\mathrm{s}}) \ \sin(\theta_i^{\mathrm{s}})]^\top$ (uniquely corresponding to any $\theta_i^{\mathrm{s}} \in [0, 2\pi)$) is mapped into a continuous trajectory of ζ_i. More specifically, for any $\theta_i^{\mathrm{s}} \in [0, 2\pi)$, there are two possible values of ζ_i, expressed through the half-angle formula:*

$$\zeta_i \in \{-\zeta_i^*, \zeta_i^*\}, \qquad \zeta_i^* := \begin{bmatrix} \sqrt{\frac{1+\cos(\theta_i^{\mathrm{s}})}{2}} \\ \sqrt{\frac{1-\cos(\theta_i^{\mathrm{s}})}{2}} \end{bmatrix}. \tag{13}$$

The same issue arises for unit quaternions. In that context, a path-lifting mechanism has been proposed in [34] to ensure that a continuous selection of the two quaternions is obtained for a "measured" rotation matrix. For simplicity, we avoid embedding a similar mechanism as [34] by considering ζ_i available for measurement. Including the path-lifting mechanism does not affect the results of this paper.

III. Problem Statement

A. Leader Exosystem

Since our objective involves the synchronization of the network to a reference signal, we consider a formulation of the tracking problem based on a pacemaker (see, e.g., [14], [33]). Specifically, the graph $\mathcal{G}$ is augmented with an additional node, named *leader system*, which delivers to the network some reference signals. The references are generated through an autonomous exosystem of the form

$$\left.\begin{aligned} \dot{\zeta}^\star &= \frac{1}{2} c^\top w^\star J \zeta^\star \\ \dot{w}^\star &= s(\zeta^\star, w^\star) \end{aligned}\right\} \quad (\zeta^\star, w^\star) \in \mathcal{K}^\star \subset \mathbb{S}^1 \times \mathbb{R}^n, \tag{14}$$

where $\zeta^\star \in \mathbb{S}^1$ is the phase reference, $w^\star \in \mathbb{R}^n$ is a state such that the frequency reference is given by $c^\top w^\star \in \mathbb{R}$, while $c \in \mathbb{R}^n$ is a constant vector and $s(\cdot) : \mathbb{S}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. Furthermore, $\mathcal{K}^\star$ is a compact set of admissible initial conditions $(\zeta^\star(0), w^\star(0))$.

The feedback structure in (14) suggests that, different from the solutions using unit quaternions (such as [26], [27]), we do not restrict the structure of exosystem (14) to a cascade between the $w^\star$-subsystem and the $\zeta^\star$-subsystem. The following assumption describes the properties related to (14).

Assumption 2. *For system (14), it holds that:*

1) *the compact set $\mathcal{K}^\star$ is forward invariant;*
2) *the map $s(\cdot)$ is globally Lipschitz, with Lipschitz constant $\ell_s \geq 0$;*
3) *c and $s(\cdot)$ are known to each node $i \in \mathcal{V}$.*

The global Lipschitz condition in Assumption 2 is instrumental in achieving global asymptotic stability, cf. [35]. As we shall see in Section IV, this Lipschitz continuity property allows designing the controllers for each node $i \in \mathcal{V}$ without the explicit knowledge of the compact set $\mathcal{K}^\star$, even though the knowledge of ℓ_s is required for tuning the controller gains.

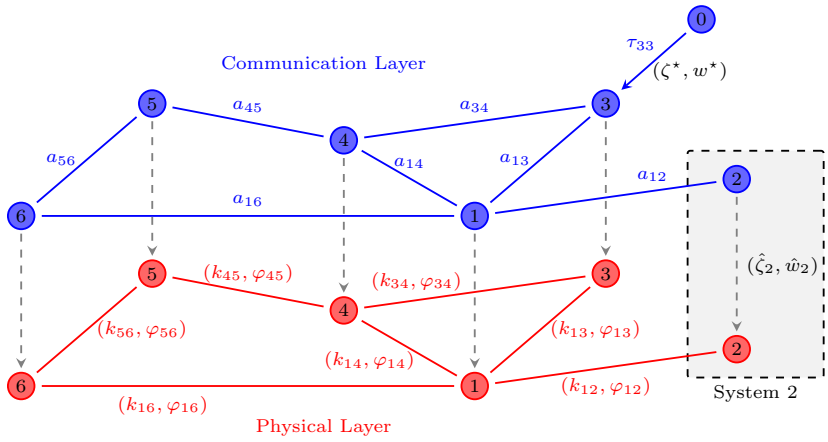

Fig. 1. Interaction and communication scheme. The same graph will be employed for the numerical example in Section VII.

As a final requirement for our design, we impose a standard assumption describing the communication topology among the leader (14) and the network (10).

Assumption 3. *System* (14) *interacts, by communicating the reference* $(\zeta^\star, w^\star)$*, with at least one node of graph* $\mathcal{G}$*, which defines both the physical couplings and the communication topology. More specifically, it holds that* $\mathcal{T} \neq \emptyset$ *(equivalently,* $T \neq 0$*).*

Remark 2. *Since* $\mathcal{G}$ *is undirected and connected, Assumption 3 implies that the leader-follower matrix* $B := L+T$ *is positive definite.*

Fig. 1 shows a scheme of the interaction and communication pattern underlying our distributed architecture.

The control problem of this work, stated in the following, aims at ensuring global asymptotic stability of a compact set corresponding to phase synchronization as in (12) and convergence to $\zeta^\star$, i.e.:

$$\lim_{t\to+\infty} \mathcal{R}(\zeta_i(t))^\top \zeta^\star \in \{-\boldsymbol{e}, \boldsymbol{e}\}, \quad \forall i \in \mathcal{V}. \tag{15}$$

In particular, we seek for a hybrid adaptive controller whose data satisfy the so-called hybrid basic conditions of [28, Assumption 6.5]. As a consequence, global asymptotic stability of a compact set is equivalent to the existence of a uniform class $\mathcal{KL}$ bound [28, Thm. 7.12]. Following the robustness results in [28, §7.3], this also implies robust $\mathcal{KL}$ asymptotic stability in the presence of fairly general perturbations of the dynamics.

Problem 1. *Under Assumptions 1, 2, and 3, consider the following synchronization set to the reference* $\zeta^\star$*:*

$$\begin{aligned}\mathcal{A}_{\mathrm{s}} := \big\{&(\zeta^\star, w^\star), (\zeta_i, \omega_i), i \in \mathcal{V} : \\ &\mathcal{R}(\zeta_i)^\top \zeta^\star \in \{-\boldsymbol{e}, \boldsymbol{e}\}, \omega_i = c^\top w^\star\big\},\end{aligned} \tag{16}$$

where $\boldsymbol{e}$ *is defined in* (7)*, and note that* $\mathcal{A}_{\mathrm{s}}$ *is compact because* (14) *evolves in the compact set* $\mathcal{K}^\star$*. Then, design a distributed adaptive strategy, only based on the local measurements* (ζ_i, ω_i) *and the information exchange according to graph* $\mathcal{G}$*, such that the second-order Kuramoto network* (10) *achieves robust global phase synchronization to the reference* $\zeta^\star$*. Namely, the closed-loop dynamics is such that there exists a robustly globally* $\mathcal{KL}$ *asymptotically stable compact set (in the sense of [28, Def. 7.18]), whose projection in the plant-exosystem direction coincides with the compact set* $\mathcal{A}_{\mathrm{s}}$ *in* (16)*.*

B. Control Architecture

Through the parametrization (8), (10), (11), Problem 1 addresses the synchronization goal in (3), (4), in a convenient scenario wherein the set to be stabilized is compact. Topological obstructions associated to the non-Euclidean nature of the phase dynamics make Problem 1 challenging. In fact, the N-torus is non-contractible, i.e., it is not diffeomorphic to any Euclidean space, and convergence to (16) requires convergence to a disconnected set of points. Two main issues arise in this context.

- If the control laws u_i are designed to stabilize only one of the points of (16), the trajectories in the coordinates θ_i display the so-called *unwinding phenomenon* [34], which causes unnecessary motion in cases where the system is initialized close to synchronization.
- If a static discontinuous feedback is employed, it is not possible to ensure robust $\mathcal{KL}$ asymptotic stability because the closed-loop system does not satisfy the hybrid basic assumptions. This fact, in practice, translates into chattering and high disturbance sensitivity [22].

In view of these considerations, we employ a hybrid dynamic feedback to robustly globally asymptotically stabilize a compact set comprising $\mathcal{A}_{\mathrm{s}}$ in (16). As discussed in the introduction, the proposed control strategy is built upon the interconnection of a distributed observer for exosystem (14), a hybrid stabilizer for globally tracking the observer estimates, and an adaptive law to handle parametric uncertainties under Assumption 1. More specifically, our design is based on the following steps.

- A distributed observer is designed so that certain local estimates $(\hat{\zeta}_i, \hat{w}_i)$ of $(\zeta^\star, w^\star) \in \mathcal{K}^\star$ are defined as elements of $\mathbb{R}^{2+n}$. The ensuing estimation error dynamics is described by two feedback-interconnected subsystems, associated with the phase and the frequency estimation errors, respectively. These subsystems are proven to be

ISS and then combined through small-gain arguments. The design and the stability analysis of the observer are discussed in Section IV.

- For each agent i, we implement an adaptive hybrid mechanism to ensure $\mathcal{R}(\zeta_i)^\top \hat{\zeta}_i \to \{-\boldsymbol{e}, \boldsymbol{e}\}$, $\omega_i \to c^\top \hat{w}_i$, for all $i \in \mathcal{V}$. The design is first performed assuming global knowledge of the leader signals (Section V). Supposing that ω_i can be assigned as a virtual input $\omega_{\mathrm{v}i}$, a hysteresis-based controller is used to show global phase synchronization while ensuring that $\omega_{\mathrm{v}i}$ is constant across jumps. Then, a backstepping-based adaptive controller is designed to guarantee $\omega_i \to \omega_{\mathrm{v}i}$. Finally, exploiting the cascade structure between the estimation error subsystem and the tracking subsystem, the effectiveness of the overall control solution is proven through reduction theorems (Section VI).

IV. Distributed Observer

In order to solve Problem 1, we propose the following distributed observer:

$$\begin{aligned} \dot{\hat{\zeta}}_i &= \frac{1}{2} c^\top \hat{w}_i J \hat{\zeta}_i - k_\zeta e_{\zeta_i} \\ \dot{\hat{w}}_i &= s(\hat{\zeta}_i, \hat{w}_i) - k_w e_{w_i} \end{aligned} \qquad i \in \mathcal{V}, \tag{17}$$

where $\hat{\zeta}_i \in \mathbb{R}^2$, $\hat{w}_i \in \mathbb{R}^n$ are the local estimates of $(\zeta^\star, w^\star)$ (14) at node i, k_ζ and $k_w \in \mathbb{R}$ are gains to be designed, while

$$\begin{aligned} e_{\zeta_i} &\coloneqq \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\zeta}_i - \hat{\zeta}_j) + \tau_{ii}(\hat{\zeta}_i - \zeta^\star) \\ e_{w_i} &\coloneqq \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{w}_i - \hat{w}_j) + \tau_{ii}(\hat{w}_i - w^\star) \end{aligned} \qquad i \in \mathcal{V}, \tag{18}$$

are the *local estimation errors*, in which a_{ij} and τ_{ii} are the entries of the adjacency matrix A and the target matrix T, respectively.

Observer (17) is distributed as it is only driven by locally available quantities (18). To represent the variables for the overall network in a compact form, it is convenient to use the Kronecker product. In particular, define the overall states $\hat{\zeta} \coloneqq [\hat{\zeta}_1^\top \ \dots \ \hat{\zeta}_N^\top]^\top \in \mathbb{R}^{2N}$ and $\hat{w} \coloneqq [\hat{w}_1^\top \ \dots \ \hat{w}_N^\top]^\top \in \mathbb{R}^{Nn}$, so that the overall estimation errors are $\tilde{\zeta} \coloneqq \hat{\zeta} - \mathbf{1}_N \otimes \zeta^\star$ and $\tilde{w} \coloneqq \hat{w} - \mathbf{1}_N \otimes w^\star$. Furthermore, define $e_\zeta \coloneqq [e_{\zeta_1}^\top \ \dots \ e_{\zeta_N}^\top]^\top \in \mathbb{R}^{2N}$ and $e_w \coloneqq [e_{w_1}^\top \ \dots \ e_{w_N}^\top]^\top \in \mathbb{R}^{Nn}$, which from (18) can be written as [36]:

$$e_\zeta = (B \otimes I_2)\tilde{\zeta}, \qquad e_w = (B \otimes I_n)\tilde{w}, \tag{19}$$

where $B \coloneqq L + T$ is the leader-follower matrix, which satisfies $B = B^\top > 0$ as discussed in Remark 2. In the following, denote with $\underline{\sigma}(B) > 0$ the smallest singular value of B.

A. Phase Subnetwork

We start by analyzing the $(\zeta^\star, \hat{\zeta})$-subsystem (referred to as phase subnetwork) and the phase estimation error $\tilde{\zeta}$. From (14), (17), the phase subnetwork obeys dynamics

$$\begin{aligned} \dot{\zeta}^\star &= \frac{1}{2} c^\top w^\star J \zeta^\star \\ \dot{\hat{\zeta}}_i &= \frac{1}{2} c^\top w^\star J \hat{\zeta}_i + \frac{1}{2} c^\top \tilde{w}_i J \hat{\zeta}_i - k_\zeta e_{\zeta_i}, \quad i \in \mathcal{V}. \end{aligned} \tag{20}$$

For notational convenience, define

$$\tilde{\Omega} \coloneqq \mathrm{diag}(c^\top \tilde{w}_1, \dots, c^\top \tilde{w}_N) = \mathrm{diag}((I_N \otimes c^\top)\tilde{w}), \tag{21}$$

which allows writing the dynamics of $\hat{\zeta}$ in compact form as follows

$$\dot{\hat{\zeta}} = \frac{1}{2}\left(c^\top w^\star (I_N \otimes J) + (\tilde{\Omega} \otimes J)\right)\hat{\zeta} - k_\zeta e_\zeta. \tag{22}$$

As a consequence, the dynamics of the phase estimation error $\tilde{\zeta} \coloneqq \hat{\zeta} - \mathbf{1}_N \otimes \zeta^\star$ can be computed from (19), (20), and (22) as:

$$\begin{aligned} \dot{\tilde{\zeta}} = &\left(\frac{1}{2} c^\top w^\star (I_N \otimes J) - k_\zeta (B \otimes I_2) + \frac{1}{2}(\tilde{\Omega} \otimes J)\right)\tilde{\zeta} \\ &+ \frac{1}{2}(\tilde{\Omega} \otimes J)(\mathbf{1}_N \otimes \zeta^\star), \end{aligned} \tag{23}$$

with inputs given by $\zeta^\star$, $w^\star$, and $\tilde{w}$ (through $\tilde{\Omega}$ in (21)). The next proposition provides an ISS characterization for system (23).

Proposition 1. *For any scalar gain $k_\zeta > 0$, system (23) is finite-gain exponentially input-to-state stable with respect to the input $\tilde{w}$, uniformly in the inputs $(\zeta^\star, w^\star)$. Namely, for any solution $(\zeta^\star(\cdot), w^\star(\cdot))$ of the exosystem (14) and any $\tilde{w}(\cdot) \in \mathcal{L}_\infty$, the solutions of (23) satisfy, for all $t \geq 0$:*

$$|\tilde{\zeta}(t)| \leq \max\left\{ \mathrm{e}^{-\frac{1}{2}\underline{\sigma}(B)k_\zeta t}|\tilde{\zeta}(0)|, \frac{|c|\|\tilde{w}(\cdot)\|_\infty}{\underline{\sigma}(B)k_\zeta} \right\}. \tag{24}$$

Proof: For any solution $(\zeta^\star(\cdot), w^\star(\cdot))$ of the exosystem (14), system (23) can be regarded as a time-varying system with input $\tilde{w}$. It is convenient to rewrite the last term of (23) as

$$\begin{aligned} (\tilde{\Omega} \otimes J)(\mathbf{1}_N \otimes \zeta^\star) &= \tilde{\Omega}\mathbf{1}_N \otimes J\zeta^\star \\ &= \mathrm{diag}((I_N \otimes c^\top)\tilde{w})\mathbf{1}_N \otimes J\zeta^\star \\ &= \mathrm{diag}(\mathbf{1}_N)(I_N \otimes c^\top)\tilde{w} \otimes J\zeta^\star \\ &= \underbrace{(I_N \otimes J\zeta^\star c^\top)}_{\coloneqq Z^\star}\tilde{w} = Z^\star \tilde{w}, \end{aligned} \tag{25}$$

where we used the identity $((I_N \otimes c^\top)w) \otimes v = (I_N \otimes vc^\top)w$, which holds for any vectors $c^\top$, w, v, of compatible dimensions. Consider the Lyapunov function candidate

$$V_\zeta \coloneqq \frac{1}{2}|\tilde{\zeta}|^2 \tag{26}$$

whose derivative along the system trajectories results in

$$\begin{aligned} \dot{V}_\zeta = &-k_\zeta \tilde{\zeta}^\top (B \otimes I_2)\tilde{\zeta} + \frac{1}{2}\tilde{\zeta}^\top Z^\star \tilde{w} \\ &+ \frac{1}{2}\tilde{\zeta}^\top \left(c^\top w^\star (I_N \otimes J) + (\tilde{\Omega} \otimes J)\right)\tilde{\zeta}, \\ = &-k_\zeta \tilde{\zeta}^\top (B \otimes I_2)\tilde{\zeta} + \frac{1}{2}\tilde{\zeta}^\top Z^\star \tilde{w}, \end{aligned} \tag{27}$$

where we employed the fact that $I_N \otimes J$ and $\tilde{\Omega} \otimes J$ are skew symmetric. Since $k_\zeta > 0$, we obtain

$$\dot{V}_\zeta \leq -\underline{\sigma}(B)k_\zeta |\tilde{\zeta}|^2 + \frac{1}{2}|Z^\star||\tilde{\zeta}||\tilde{w}|. \tag{28}$$

The following computations yield $|Z^\star| = |c|$:

$$\begin{aligned} |Z^\star| &= |I_N||J\zeta^\star c^\top| = |J\zeta^\star c^\top| \\ &= |J\zeta^\star c^\top|_F = \sqrt{\mathrm{Tr}(c\zeta^{\star\top} J^\top J \zeta^\star c^\top)} \\ &= \sqrt{\mathrm{Tr}(cc^\top)} = |c|, \end{aligned} \tag{29}$$

where $|J\zeta^\star c^\top| = |J\zeta^\star c^\top|_F$ since the rank of $J\zeta^\star c^\top$ is 1 by construction. Applying (28) and (29) yields

$$|\tilde\zeta| \geq \frac{|c|}{\underline\sigma(B)k_\zeta}|\tilde w| \implies \dot V_\zeta \leq -\frac{\underline\sigma(B)k_\zeta}{2}|\tilde\zeta|^2, \tag{30}$$

which leads to (24) via standard ISS calculations [37, Thm. 10.4.1]. □

B. Frequency Subnetwork

Starting again from (14), (17), the frequency subnetwork obeys dynamics

$$\begin{aligned} \dot w^\star &= s(\zeta^\star, w^\star) \\ \dot{\hat w}_i &= s(\hat\zeta_i, \hat w_i) - k_w e_{w_i}, \quad i \in \mathcal{V}. \end{aligned} \tag{31}$$

We can then write the dynamics of $\hat w$ as

$$\dot{\hat w} = S(\hat\zeta, \hat w) - k_w e_w, \tag{32}$$

where

$$S(\hat\zeta, \hat w) := \begin{bmatrix} s(\hat\zeta_1, \hat w_1) \\ \vdots \\ s(\hat\zeta_N, \hat w_N) \end{bmatrix}. \tag{33}$$

Therefore, using (19), the dynamics of the frequency estimation error $\tilde w := \hat w - (\mathbf{1}_N \otimes w^\star)$ is given by

$$\dot{\tilde w} = S(\hat\zeta, \hat w) - \mathbf{1}_N \otimes s(\zeta^\star, w^\star) - k_w(B \otimes I_n)\tilde w, \tag{34}$$

which, in view of $\hat\zeta = \tilde\zeta + \mathbf{1}_N \otimes \zeta^\star$, $\hat w = \tilde w + \mathbf{1}_N \otimes w^\star$, is a non-autonomous system with inputs given by $\zeta^\star$, $w^\star$, and $\tilde\zeta$. In the following, we present a result that follows the same structure as Proposition 1, now applied to the frequency subnetwork.

Proposition 2. *For any scalar gain $k_w > \ell_s/\underline\sigma(B)$, system (34) is finite-gain exponentially input-to-state stable with respect to the input $\tilde\zeta$, uniformly in the inputs $(\zeta^\star, w^\star)$. Namely, for any solution $(\zeta^\star(\cdot), w^\star(\cdot))$ of the exosystem (14) and any $\tilde\zeta(\cdot) \in \mathcal{L}_\infty$, the solutions of system (34) satisfy, for all $t \geq 0$:*

$$|\tilde w(t)| \leq \max\left\{ \mathrm{e}^{-\frac{1}{2}(\underline\sigma(B)k_w - \ell_s)t}|\tilde w(0)|, \frac{2\ell_s \|\tilde\zeta(\cdot)\|_\infty}{\underline\sigma(B)k_w - \ell_s} \right\}. \tag{35}$$

Proof: For any solution $(\zeta^\star(\cdot), w^\star(\cdot))$ of the exosystem (14), system (34) can be regarded as a time-varying system with input $\tilde\zeta$. Consider the Lyapunov function candidate

$$V_w := \frac{1}{2}|\tilde w|^2, \tag{36}$$

whose derivative along (34) is

$$\begin{aligned} \dot V_w &= -k_w \tilde w^\top (B \otimes I_n)\tilde w + \tilde w^\top \left(S(\hat\zeta, \hat w) - \mathbf{1}_N \otimes s(\zeta^\star, w^\star) \right) \\ &= -k_w \tilde w^\top (B \otimes I_n)\tilde w + \sum_{i=1}^N \tilde w_i^\top \left(s(\hat\zeta_i, \hat w_i) - s(\zeta^\star, w^\star) \right). \end{aligned} \tag{37}$$

By Assumption 2, it holds that

$$|s(\hat\zeta_i, \hat w_i) - s(\zeta^\star, w^\star)| \leq \ell_s(|\tilde\zeta_i| + |\tilde w_i|), \tag{38}$$

therefore we conclude that

$$\begin{aligned} \dot V_w &\leq -\underline\sigma(B)k_w|\tilde w|^2 + \ell_s \sum_{i=1}^N \left(|\tilde w_i|^2 + |\tilde w_i||\tilde\zeta_i| \right) \\ &\leq -\left(\underline\sigma(B)k_w - \ell_s \right)|\tilde w|^2 + \ell_s|\tilde w||\tilde\zeta|. \end{aligned} \tag{39}$$

Finally, from (39) we obtain the following ISS characterization:

$$|\tilde w| \geq \frac{2\ell_s}{\underline\sigma(B)k_w - \ell_s}|\tilde\zeta| \implies \dot V_w \leq -\frac{\underline\sigma(B)k_w - \ell_s}{2}|\tilde w|^2, \tag{40}$$

which proves the finite-gain exponential ISS bound (35) through [37, Thm. 10.4.1]. □

C. Overall Observer Analysis

We conclude the section with a stability result for the feedback interconnection between the phase estimation error dynamics (23) and the frequency estimation error dynamics (34).

Theorem 1. *For any choice of the scalar gains k_ζ and k_w such that*

$$\begin{aligned} &k_\zeta > 0, \qquad k_w > \ell_s/\underline\sigma(B), \\ &k_\zeta \underline\sigma(B)(k_w\underline\sigma(B) - \ell_s) - 2\ell_s|c| > 0, \end{aligned} \tag{41}$$

the zero-equilibrium $(\tilde\zeta, \tilde w) = 0$ of the overall estimation error system (23), (34) is globally exponentially stable.

Proof: From Proposition 1, which holds for $k_\zeta > 0$, and Proposition 2, valid for $k_w\underline\sigma(B) > \ell_s$, we obtain that both (23) and (34) are finite-gain exponentially ISS. Therefore, global exponential stability is ensured from (24), (35), and [37, Thm. 10.6.1], through the following small-gain condition:

$$\frac{2\ell_s|c|}{k_\zeta\underline\sigma(B)(k_w\underline\sigma(B) - \ell_s)} < 1, \tag{42}$$

which is ensured by (41). □

Remark 3. *In the special case where exosystem (14) is a cascade, i.e., $s = s(w^\star)$, conditions (41) collapse to $k_\zeta > 0$, $k_w > \ell_s/\underline\sigma(B)$. Additionally, if $s(w^\star) = S_w w^\star$, where S_w is a Poisson stable matrix as in [26], [27], [29], (37) becomes $\dot V_w = -k_w \tilde w^\top (B \otimes I_n)\tilde w$, therefore conditions (41) become $k_\zeta > 0$, $k_w > 0$.*

V. Synchronization with Global Knowledge of the Leader Signals

In this section, we design a tracking controller for the simplified setup where the observer estimation errors are zero. This approach will be motivated in Section VI by the reduction arguments of the stability analysis.

Firstly, we compute the local tracking error dynamics. Define the phase and frequency tracking errors as

$$\begin{aligned} &\bar\zeta_i := \begin{bmatrix} \bar\eta_i & \bar\epsilon_i \end{bmatrix}^\top := \mathcal{R}(\zeta_i)^\top \hat\zeta_i \in \mathbb{R}^2 \\ &\bar\omega_i := c^\top \hat w_i - \omega_i \in \mathbb{R} \end{aligned} \qquad i \in \mathcal{V}. \tag{43}$$

In these coordinates, the control objective in Problem 1 corresponds to imposing $(\bar{\epsilon}_i, \bar{\omega}_i) \to 0$, for all $i \in \mathcal{V}$. From (10), (17), and $\mathcal{R}(\zeta_i)^\top J = J\mathcal{R}(\zeta_i)^\top$, we can compute the phase error dynamics as

$$\begin{aligned}\dot{\bar{\zeta}}_i &= \frac{d}{dt}\left(\mathcal{R}(\zeta_i)^\top\right)\hat{\zeta}_i + \mathcal{R}(\zeta_i)^\top\dot{\hat{\zeta}}_i \\ &= -\mathcal{R}(\zeta_i)^\top\frac{d\mathcal{R}(\zeta_i)}{dt}\mathcal{R}(\zeta_i)^\top\hat{\zeta}_i + \mathcal{R}(\zeta_i)^\top\dot{\hat{\zeta}}_i \quad i \in \mathcal{V} \\ &= \frac{1}{2}\bar{\omega}_i J\bar{\zeta}_i - k_\zeta\mathcal{R}(\zeta_i)^\top e_{\zeta_i}.\end{aligned} \tag{44}$$

Similarly, the dynamics of the frequency error $\bar{\omega}_i$ is computed from (10), (14), and (17) as

$$m_i\dot{\bar{\omega}}_i = \psi_i - u_i - m_i k_w c^\top e_{w_i}, \qquad i \in \mathcal{V}, \tag{45}$$

where we defined

$$\begin{aligned}\psi_i \coloneqq\ & m_i c^\top s(\hat{\zeta}_i, \hat{w}_i) + d_i\omega_i - \omega_{\mathrm{n}i} \\ & + \sum_{j\in\mathcal{N}_i} k_{ij}\phi(\zeta_i)^\top J\phi(\zeta_j)\cos(\varphi_{ij}) \\ & - \sum_{j\in\mathcal{N}_i} k_{ij}\phi(\zeta_i)^\top\phi(\zeta_j)\sin(\varphi_{ij}).\end{aligned} \tag{46}$$

Observe that, with $\hat{\zeta}_i = \zeta^\star$ and $\hat{w}_i = w^\star$ (i.e., $\tilde{\zeta} = 0$, $\tilde{w} = 0$, equivalently, $e_\zeta = 0$, $e_w = 0$), the quantities in (43) become $\bar{\zeta}_i = \mathcal{R}(\zeta_i)^\top\zeta^\star \in \mathbb{S}^1$ and $\bar{\omega}_i = c^\top w^\star - \omega_i \in \mathbb{R}$. In view of this reduction argument, we begin the design by assuming that the exosystem signals $(\zeta^\star, w^\star)$ are globally known for feedback. This scenario corresponds to the requirement $\mathcal{T} = \mathcal{V}$, which will be removed in Section VI.

A. Phase Synchronization

Assume initially that ω_i can be arbitrarily assigned by the feedback controller as a virtual input $\omega_{\mathrm{v}i}$. With $e_\zeta = 0$, the dynamics (44) thus reduces to

$$\dot{\bar{\zeta}}_i = \frac{1}{2}(c^\top\hat{w}_i - \omega_{\mathrm{v}i})J\bar{\zeta}_i, \qquad i \in \mathcal{V}, \tag{47}$$

where $\omega_{\mathrm{v}i}$ is the virtual input that should ensure $\bar{\epsilon}_i \to 0$. We refer to this objective as *phase synchronization* with the reference $\zeta^\star$.

Define $\mathcal{Q} \coloneqq \{-1, 1\}$ and choose any gain $k > 0$ and a hysteresis margin $\delta \in (0, 1)$. For each $i \in \mathcal{V}$, a hysteresis-based hybrid dynamic controller that achieves global phase synchronization is given by

$$\begin{aligned}&\begin{cases}\dot{q}_i = 0, & (\bar{\zeta}_i, q_i) \in C_\kappa \\ q_i^+ = -q_i, & (\bar{\zeta}_i, q_i) \in D_\kappa\end{cases} \qquad i \in \mathcal{V} \\ &\omega_{\mathrm{v}i} = c^\top\hat{w}_i + kq_i\bar{\epsilon}_i,\end{aligned} \tag{48}$$

where $q_i \in \mathcal{Q}$ is the controller state and

$$\begin{aligned}C_\kappa &\coloneqq \{(\bar{\zeta}_i, q_i) \in \mathbb{S}^1 \times \mathcal{Q} : \bar{\eta}_i q_i \geq -\delta\} \\ D_\kappa &\coloneqq \{(\bar{\zeta}_i, q_i) \in \mathbb{S}^1 \times \mathcal{Q} : \bar{\eta}_i q_i \leq -\delta\}.\end{aligned} \tag{49}$$

The closed-loop error dynamics then corresponds to

$$\begin{cases}\dot{\bar{\zeta}}_i = -\frac{1}{2}kq_i\bar{\epsilon}_i J\bar{\zeta}_i \\ \dot{q}_i = 0,\end{cases} \begin{bmatrix}\bar{\zeta}_i \\ q_i\end{bmatrix} \in C_\kappa, \begin{cases}\bar{\zeta}_i^+ = \bar{\zeta}_i \\ q_i^+ = -q_i,\end{cases} \begin{bmatrix}\bar{\zeta}_i \\ q_i\end{bmatrix} \in D_\kappa, \tag{50}$$

which provides an autonomous hybrid dynamics having state $(\bar{\zeta}_i, q_i) = ((\bar{\eta}_i, \bar{\epsilon}_i), q_i) \in \mathbb{S}^1 \times \mathcal{Q}$ and such that $q_i\bar{\eta}_i = -1$ is not included in the flow set C_κ (because $\delta < 1$). The next lemma is a straightforward generalization of [22].

Lemma 1. *The attractor $\mathcal{A}_\kappa \coloneqq \{(\zeta, q) \in \mathbb{S}^1 \times \mathcal{Q} : \zeta = q\boldsymbol{e}\}$ is uniformly globally asymptotically stable for* (50).

Proof: Choose the Lyapunov function

$$V_\kappa(\bar{\zeta}_i, q_i) \coloneqq 2(1 - q_i\bar{\eta}_i), \tag{51}$$

which is positive definite and radially unbounded with respect to $\mathcal{A}_\kappa$. Denoting $\dot{V}_\kappa = \left\langle \nabla V_\kappa, [\dot{\bar{\zeta}}_i\ \dot{q}_i]^\top \right\rangle$ and $\Delta V_\kappa = V_\kappa(\bar{\zeta}_i^+, q_i^+) - V_\kappa(\bar{\zeta}_i, q_i)$, straightforward calculations yield

$$\begin{aligned}\dot{V}_\kappa &= -k\bar{\epsilon}_i^2 < 0, \qquad \forall(\bar{\zeta}_i, q_i) \in C_\kappa\backslash\mathcal{A}_\kappa \\ \Delta V_\kappa &= 4q_i\bar{\eta}_i \leq -4\delta < 0, \quad \forall(\bar{\zeta}_i, q_i) \in D_\kappa,\end{aligned} \tag{52}$$

implying UGAS from standard hybrid Lyapunov theory. □

For a convenient design of the backstepping-based adaptive controller defined in the next subsection, we propose now a dynamically extended version of (48) to ensure that $\omega_{\mathrm{v}i}$ remains constant across jumps. Specifically, we augment the controller with a first-order filter of the feedback $kq_i\bar{\epsilon}_i$:

$$\begin{cases}\dot{q}_i = 0 \\ \dot{\lambda}_i = -h(\lambda_i - kq_i\bar{\epsilon}_i)\end{cases} \quad (\bar{\zeta}_i, q_i, \lambda_i) \in C_\lambda \qquad i \in \mathcal{V}, \\ \begin{cases}q^+ = -q_i \\ \lambda_i^+ = \lambda_i\end{cases} \quad (\bar{\zeta}_i, q_i, \lambda_i) \in D_\lambda \tag{53}$$

where h is a positive gain and the sets C_λ, D_λ are defined as the next generalization of (49):

$$\begin{aligned}C_\lambda &\coloneqq \left\{(\bar{\zeta}_i, q_i, \lambda_i) \in \mathbb{S}^1 \times \mathcal{Q} \times \mathbb{R} : \left(\bar{\eta}_i + \frac{\lambda_i\bar{\epsilon}_i}{k}\right)q_i \geq -\delta\right\} \\ D_\lambda &\coloneqq \left\{(\bar{\zeta}_i, q_i, \lambda_i) \in \mathbb{S}^1 \times \mathcal{Q} \times \mathbb{R} : \left(\bar{\eta}_i + \frac{\lambda_i\bar{\epsilon}_i}{k}\right)q_i \leq -\delta\right\}.\end{aligned} \tag{54}$$

We can then replace by λ_i the term $kq_i\bar{\epsilon}_i$ in the selection of $\omega_{\mathrm{v}i}$ of (48), namely we choose:

$$\omega_{\mathrm{v}i} = c^\top\hat{w}_i + \lambda_i, \qquad i \in \mathcal{V}, \tag{55}$$

which remains constant across jumps. The closed-loop error dynamics for each node i, obtained from the interconnection of (47), (53), and (55), is described by:

$$\begin{cases}\dot{\bar{\zeta}}_i = -\frac{1}{2}\lambda_i J\bar{\zeta}_i \\ \dot{q}_i = 0 \\ \dot{\lambda}_i = -h(\lambda_i - kq_i\bar{\epsilon}_i)\end{cases} \begin{bmatrix}\bar{\zeta}_i \\ q_i \\ \lambda_i\end{bmatrix} \in C_\lambda, \begin{cases}\bar{\zeta}_i^+ = \bar{\zeta}_i \\ q_i^+ = -q_i \\ \lambda_i^+ = \lambda_i\end{cases} \begin{bmatrix}\bar{\zeta}_i \\ q_i \\ \lambda_i\end{bmatrix} \in D_\lambda. \tag{56}$$

The next result generalizes the argument of Lemma 1.

Proposition 3. *For any $h > k$, the attractor $\mathcal{A}_\lambda \coloneqq \{(\zeta, q, \lambda) \in \mathbb{S}^1 \times \mathcal{Q} \times \mathbb{R} : \zeta = q\boldsymbol{e}, \lambda = 0\}$ is uniformly globally asymptotically stable for the hybrid system* (56).

Proof: Define $\tilde{\lambda}_i := \lambda_i - kq_i\bar{\epsilon}_i$, then consider the Lyapunov function

$$V_\lambda(\bar{\zeta}_i, q_i, \lambda_i) := 2k^2(1 - q_i\bar{\eta}_i) + \tilde{\lambda}_i^2. \tag{57}$$

Note that V_λ is positive definite with respect to $\mathcal{A}_\lambda$ and radially unbounded relative to $\mathbb{S}^1 \times \mathcal{Q} \times \mathbb{R}$. Denote $\dot{V}_\lambda = \left\langle \nabla V_\lambda, [\dot{\bar{\zeta}}_i \ \dot{q}_i \ \dot{\lambda}_i]^\top \right\rangle$. For all $(\bar{\zeta}_i, q_i, \lambda_i) \in C_\lambda$, it holds that

$$\begin{aligned}\dot{V}_\lambda &= -k^2 q_i\lambda_i\bar{\epsilon}_i + \tilde{\lambda}_i(-2h\tilde{\lambda}_i + kq_i\lambda_i\bar{\eta}_i) \\ &= -k^3\bar{\epsilon}_i^2 + k^2\tilde{\lambda}_i\bar{\epsilon}_i(\bar{\eta}_i - q_i) - (2h - kq_i\bar{\eta}_i)\tilde{\lambda}_i^2 \\ &\leq -\begin{bmatrix}|\bar{\epsilon}_i| \\ |\tilde{\lambda}_i|\end{bmatrix}^\top \begin{bmatrix}k^3 & -k^2 \\ -k^2 & 2h - k\end{bmatrix} \begin{bmatrix}|\bar{\epsilon}_i| \\ |\tilde{\lambda}_i|\end{bmatrix}.\end{aligned} \tag{58}$$

From $\delta < 1$, for any point in C_λ we have that $\tilde{\lambda}_i = 0$ and $\bar{\epsilon}_i = 0$ implies that $(\bar{\zeta}_i, q_i, \lambda_i) \in \mathcal{A}$ (in particular, the point with $\lambda_i = 0$ and $q_i = -\bar{\eta}_i$ does not belong to C_λ), then $\dot{V}_\lambda < 0$, for all $(\bar{\zeta}_i, q_i, \lambda_i) \in C_\lambda \backslash \mathcal{A}_\lambda$, if $k^3(2h - k) - k^4 = 2k^3(h-k) > 0$, i.e., $h > k$. On the other hand, denote $\Delta V_\lambda = V_\lambda(\bar{\zeta}_i^+, q_i^+, \lambda_i^+) - V_\kappa(\bar{\zeta}_i, q_i, \lambda_i)$, then for all $(\bar{\zeta}_i, q_i, \lambda_i) \in D_\lambda$ we have:

$$\begin{aligned}\Delta V_\lambda &= 4k^2 q_i\bar{\eta}_i + (\tilde{\lambda}_i + 2kq_i\bar{\epsilon}_i)^2 - \tilde{\lambda}_i^2 \\ &= 4k^2 q_i\left(\bar{\eta}_i + \frac{\lambda_i\bar{\epsilon}_i}{k}\right) \leq -4k^2\delta < 0,\end{aligned} \tag{59}$$

thus concluding UGAS for the attractor $\mathcal{A}_\lambda$. □

B. Global Adaptive Synchronization

Taking advantage of the hybrid system defined in (53), we propose to achieve global synchronization to the reference $\zeta^\star$ using an adaptive backstepping controller where, in place of the feedback $\omega_{\mathrm{v}i} = c^\top\hat{w}_i + \lambda_i$ in (55), we ensure $\omega_i \to \omega_{\mathrm{v}i}$ by design of the control input u_i.

In place of the frequency tracking error $\bar{\omega}_i$ in (43), consider the error variable

$$z_i := c^\top\hat{w}_i + \lambda_i - \omega_i = \bar{\omega}_i + \lambda_i \in \mathbb{R}, \qquad i \in \mathcal{V}. \tag{60}$$

We can rewrite the error dynamics (44) and (45) using variables z_i as follows:

$$\begin{aligned}\dot{\bar{\zeta}}_i &= \frac{1}{2}(z_i - \lambda_i)J\bar{\zeta}_i - k_\zeta\mathcal{R}(\zeta_i)^\top e_{\zeta i} \\ m_i\dot{z}_i &= \psi_i - u_i - m_i\left(k_w c^\top e_{w_i} + h(\lambda_i - kq_i\bar{\epsilon}_i)\right)\end{aligned} \quad i \in \mathcal{V}. \tag{61}$$

Using (46), the second equation can also be rewritten as follows

$$m_i\dot{z}_i = \Psi_i^\top p_i - u_i - m_i k_w c^\top e_{w_i}, \qquad i \in \mathcal{V}, \tag{62}$$

with regressor Ψ_i and parameter vector $p_i \in \mathbb{R}^{3+2|\mathcal{N}_i|}$ given by:

$$\Psi_i := \begin{bmatrix} c^\top s(\hat{\zeta}_i, \hat{w}_i) - h(\lambda_i - kq_i\bar{\epsilon}_i) \\ \omega_i \\ 1 \\ \phi(\zeta_i)^\top J\phi(\zeta_{j_1}) \\ \vdots \\ \phi(\zeta_i)^\top J\phi(\zeta_{j_{|\mathcal{N}_i|}}) \\ \phi(\zeta_i)^\top\phi(\zeta_{j_1}) \\ \vdots \\ \phi(\zeta_i)^\top\phi(\zeta_{j_{|\mathcal{N}_i|}}) \end{bmatrix}, \ p_i := \begin{bmatrix} m_i \\ d_i \\ -\omega_{\mathrm{n}i} \\ k_{ij_1}\cos(\varphi_{ij_1}) \\ \vdots \\ k_{ij_{N_i}}\cos(\varphi_{ij_{|\mathcal{N}_i|}}) \\ -k_{ij_1}\sin(\varphi_{ij_1}) \\ \vdots \\ -k_{ij_{N_i}}\sin(\varphi_{ij_{|\mathcal{N}_i|}}) \end{bmatrix} \tag{63}$$

where we denoted $\mathcal{N}_i = \{j_1, \ldots, j_{|\mathcal{N}_i|}\}$. By Assumption 1, it follows that

$$|p_i|_\infty = \max\{|p_{i1}|, \ldots, |p_{i(3+2|\mathcal{N}_i|)}|\} \leq \rho. \tag{64}$$

The control of system (61) is based on the augmentation of control law (53) with the following adaptive state-input selections:

$$\begin{aligned}u_i &= \Psi_i^\top\hat{p}_i + k_z z_i \\ \dot{\hat{p}}_i &= \gamma\Psi_i z_i - \gamma\nu\,\mathrm{dz}(\hat{p}_i) \qquad i \in \mathcal{V}, \\ \hat{p}_i^+ &= \hat{p}_i,\end{aligned} \tag{65}$$

where k_z, γ, and ν are positive gains, while $\mathrm{dz} : \mathbb{R}^{3+2|\mathcal{N}_i|} \to \mathbb{R}^{3+2|\mathcal{N}_i|}$ is a decentralized dead-zone function defined as [38, §3.4]:

$$\mathrm{dz}(\xi) := \begin{bmatrix} \xi_1 - \rho\mathrm{sat}\left(\frac{\xi_1}{\rho}\right) \\ \vdots \\ \xi_{3+2|\mathcal{N}_i|} - \rho\mathrm{sat}\left(\frac{\xi_{3+2|\mathcal{N}_i|}}{\rho}\right) \end{bmatrix}, \tag{66}$$

where $\mathrm{sat}(y) := \max\{-1, \min\{1, y\}\}$. Exploiting (64), it can be verified that, for all p_i and all $\xi \in \mathbb{R}^{3+2|\mathcal{N}_i|}$:

$$(\xi - p_i)^\top\,\mathrm{dz}(\xi) \geq 0. \tag{67}$$

Moreover, there exist positive scalars r and μ such that, for all p_i and all $\xi \in \mathbb{R}^{3+2|\mathcal{N}_i|}$:

$$|\xi| \geq r \implies (\xi - p_i)^\top\,\mathrm{dz}(\xi) \geq \mu|\xi|^2. \tag{68}$$

In view of our reduction arguments ($e_{\zeta_i} = 0$ and $e_{w_i} = 0$ in (61)), the closed-loop dynamics obtained from the interconnection of (61), (62), (53), and (65), having state $x_i := (\bar{\zeta}_i, q_i, \lambda_i, z_i, \hat{p}_i)$, is expressed, for each $i \in \mathcal{V}$, as follows:

$$\begin{cases} \dot{\bar{\zeta}}_i = \frac{1}{2}(z_i - \lambda_i)J\bar{\zeta}_i \\ \dot{q}_i = 0 \\ \dot{\lambda}_i = -h(\lambda_i - kq_i\bar{\epsilon}_i) \\ m_i\dot{z}_i = -k_z z_i - \Psi_i^\top(\hat{p}_i - p_i) \\ \dot{\hat{p}}_i = \gamma\Psi_i z_i - \gamma\nu\,\mathrm{dz}(\hat{p}_i) \end{cases} \quad x_i \in C_i, \quad \begin{cases} \bar{\zeta}_i^+ = \bar{\zeta}_i \\ q_i^+ = -q_i \\ \lambda_i^+ = \lambda_i \\ z_i^+ = z_i \\ \hat{p}_i^+ = \hat{p}_i \end{cases} \quad x_i \in D_i, \tag{69}$$

where $C_i := C_\lambda \times \mathbb{R}^{4+2|\mathcal{N}_i|}$ and $D_i := C_z \times \mathbb{R}^{4+2|\mathcal{N}_i|}$. In the following, we focus on the stability properties of the closed-loop system obtained through the interconnection of

the exosystem (14) and the local error dynamics (69). For this interconnection, we are going to show that there exists a globally asymptotically stable compact attractor wherein all oscillators are synchronized with the reference $\zeta^\star$. More specifically, this compact attractor, named $\mathcal{A}_0$, is a set wherein $\bar{\zeta}_i = q_i \mathbf{e}$, $z_i = 0$, for all $i \in \mathcal{V}$, and satisfying $\mathcal{A}_0 \subset \mathcal{K}_0$, with $\mathcal{K}_0$ compact, defined as:

$$\begin{aligned}\mathcal{K}_0 \coloneqq \{(\zeta^\star, w^\star, x_1, \dots, x_N) \in \mathcal{K}^\star \times \prod_{i\in\mathcal{V}} (\mathbb{S}^1 \times \mathcal{Q} \times \mathbb{R}^{5+2|\mathcal{N}_i|}): \\ \bar{\zeta}_i = q_i \mathbf{e}, \lambda_i = 0, z_i = 0, |\hat{p}_i| \le r, \forall i \in \mathcal{V}\},\end{aligned} \tag{70}$$

where $r > 0$ as per (68). In the sequel, we call $\mathcal{K}_0$ *synchronization set*.

Remark 4. *The set $\mathcal{K}_0$ is compact. Indeed, $(\zeta^\star, w^\star) \in \mathcal{K}^\star$ is in a compact by assumption, while the only components of $x_i \coloneqq (\bar{\zeta}_i, q_i, \lambda_i, z_i, \hat{p}_i)$ that are possibly unbounded are λ_i, z_i, and $\hat{p}_i$. Therefore, from the conditions in* (70)*, compactness follows immediately.*

Remark 5. *Since no persistency of excitation is necessarily satisfied by the regressor Ψ_i in* (63)*, it might be surprising that a globally asymptotically stable attractor can be found with the considered adaptive controller. This result is possible because we make use of the analysis tools in [28, Chs. 6 and 7] instead of the standard tools for adaptive control (see, e.g., [39, §8.3]). In particular, we leverage the result [28, Cor. 7.7], which states that, under some regularity properties including well posedness, the ω-limit set from a compact set of initial conditions is locally asymptotically stable.*

Theorem 2. *For any selection of the tuning parameters $k > 0$, $\delta \in (0, 1)$, $h > k$, $k_z > 0$, $\gamma > 0$, and $\nu > 0$, there exists a compact set $\mathcal{A}_0$, contained in the synchronization set $\mathcal{K}_0$ of* (70)*, that is robustly globally $\mathcal{KL}$ asymptotically stable in the sense of [28, Def. 7.18] for the interconnection among* (14) *and* (69)*.*

Robustness of our stability result follows from compactness of $\mathcal{A}_0$ and well posedness of the considered hybrid dynamics.

Proof: First, we prove that the closed-loop solutions are bounded and forward complete. The state $(\zeta^\star, w^\star)$ of the exosystem (14) evolves in the bounded forward invariant set $\mathcal{K}^\star$, thus it is bounded. Moreover, q_i is bounded by construction. In the scenario with known leader signals considered in this section, it holds that $\bar{\zeta}_i \coloneqq \mathcal{R}(\zeta_i)^\top \hat{\zeta}_i = \mathcal{R}(\zeta_i)^\top \zeta_i \in \mathbb{S}^1$, thus $\bar{\zeta}_i$ is bounded. Since $|k q_i \bar{\epsilon}_i| \le k$ we obtain, for $|\lambda_i| \ge k$:

$$\begin{aligned}\frac{d}{dt}|\lambda_i| &= -h \frac{\lambda_i}{|\lambda_i|}(\lambda_i - k q_i \bar{\epsilon}_i) \qquad \forall i \in \mathcal{V} \\ &\le -h(|\lambda_i| - k) \le 0,\end{aligned} \tag{71}$$

therefore λ_i is bounded. Boundedness of $(z_i, \hat{p}_i)$ is established by using the following Lyapunov function

$$V_i(z_i, \hat{p}_i) \coloneqq \frac{1}{2} m_i z_i^2 + \frac{1}{2\gamma} |\hat{p}_i - p_i|^2, \qquad \forall i \in \mathcal{V}. \tag{72}$$

Along the closed-loop solutions, we obtain from (69),

$$\begin{aligned}\dot{V}_i &= -k_z z_i^2 - z_i \Psi_i^\top (\hat{p}_i - p_i) + (\hat{p}_i - p_i)^\top \left[\Psi_i z_i - \nu \, \mathrm{dz}(\hat{p}_i)\right] \\ &= -k_z z_i^2 - \nu (\hat{p}_i - p_i)^\top \mathrm{dz}(\hat{p}_i), \qquad \forall i \in \mathcal{V}.\end{aligned} \tag{73}$$

Therefore, using (68) and the properties of the dead-zone function, we obtain, for all $i \in \mathcal{V}$,

$$\begin{aligned}\dot{V}_i &\le -k_z z_i^2 \le 0, && \text{if } |\hat{p}_i| \le r, \\ \dot{V}_i &\le -k_z z_i^2 - \nu\mu |\hat{p}_i|^2 < 0, && \text{if } |\hat{p}_i| \ge r.\end{aligned} \tag{74}$$

Property (74) shows forward invariance of the sublevel sets of V_i, $i \in \mathcal{V}$, thus $(z_i, \hat{p}_i)$ is contained in a compact set, for all $i \in \mathcal{V}$. Exploiting boundedness of the components of x_i, $i \in \mathcal{V}$, established above, we also conclude by [28, Prop. 6.10] that solutions are forward complete, thus they are precompact.

From (72)–(74) and using the fact that z_i and $\hat{p}_i$ do not change across jumps, we can apply [28, Cor. 8.4] to obtain that all solutions approach the largest weakly invariant subset of the set

$$\begin{aligned}\mathcal{U} \coloneqq \{(\zeta^\star, w^\star, x_1, \dots, x_N) \in \mathcal{K}^\star \times \prod_{i\in\mathcal{V}} (\mathbb{S}^1 \times \mathcal{Q} \times \mathbb{R}^{5+2|\mathcal{N}_i|}): \\ z_i = 0, |\hat{p}_i| \le r, \forall i \in \mathcal{V}\}.\end{aligned} \tag{75}$$

The closed-loop dynamics restricted to the set $\mathcal{U}$ in (75) is given, for all $i \in \mathcal{V}$, by $z_i = 0$, $\dot{\hat{p}}_i = 0$, and (56). In view of Proposition 3, the solutions globally approach $\mathcal{K}_0 \subset \mathcal{U}$ as per (70) and (75).

For a set of initial conditions of the form $\mathcal{K}_\varepsilon \coloneqq \mathcal{K}_0 + \varepsilon \mathbb{B}$, where $\varepsilon > 0$ is arbitrary, it holds that $\mathcal{A}_0 \coloneqq \Omega(\mathcal{K}_\varepsilon) \subset \mathcal{K}_0 \subset \mathrm{Int}(\mathcal{K}_\varepsilon)$, where $\Omega(\mathcal{K}_\varepsilon)$ denotes the ω-limit set of $\mathcal{K}_\varepsilon$. By [28, Cor. 7.7], $\mathcal{A}_0$ is asymptotically stable (therefore Lyapunov stable) with basin of attraction containing $\mathcal{K}_\varepsilon$. From the previous arguments, $\mathcal{A}_0$ is globally attractive, which, together with its Lyapunov stability, gives GAS. Since the hybrid dynamics satisfies the hybrid basic conditions of [28, Assumption 6.5] and $\mathcal{A}_0$ is compact, then GAS of $\mathcal{A}_0$ implies robust global $\mathcal{KL}$ asymptotic stability from [28, Thm. 7.21]. □

As customary in adaptive control, convergence of the estimated parameters $\hat{p}_i$ to the true parameters p_i cannot be guaranteed. This in turn makes it difficult, if at all possible, to give an explicit representation of the attractor $\mathcal{A}_0$. Even without its explicit representation, the mere existence of $\mathcal{A}_0$ is sufficient to complete the design through reduction theorems in the next section.

VI. Main Result

We finally present the complete hybrid observer-based controller for each node i, obtained by combining the distributed observer (17) and the local hysteresis-based controller (53), and the local adaptive controller (65). Note that, in this context, we no longer assume $\tilde{\zeta} = 0$, $\tilde{w} = 0$ (equivalently, $e_\zeta = 0$, $e_w = 0$), thus the dynamics of the tracking errors $(\bar{\zeta}_i, z_i)$ in (61) is not simplified as in the scenario with known leader signals. The robustness property established in Theorem 2 is naturally inherited here due to well posedness of the hybrid dynamics.

Define the overall state at node i as

$$\chi_i \coloneqq (\hat{\zeta}_i, \hat{w}_i, \underbrace{\bar{\zeta}_i, q_i, \lambda_i, z_i, \hat{p}_i}_{x_i}) \in \mathbb{R}^{n+4} \times \mathcal{Q} \times \mathbb{R}^{5+2|\mathcal{N}_i|}, \tag{76}$$

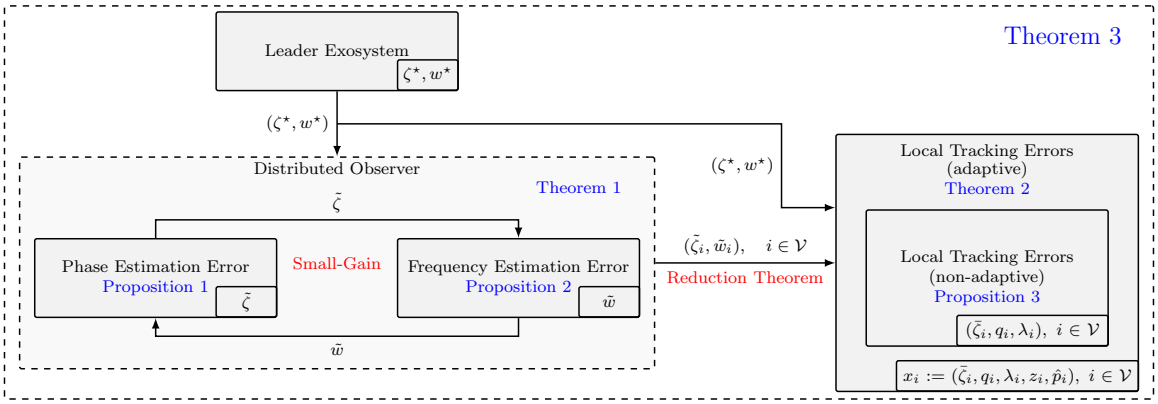

Fig. 2. Sketch of the closed-loop error subsystems, with their interconnections and the related stability results.

then the local controllers that solve Problem 1 are given as follows, for each $i \in \mathcal{V}$:

$$\begin{cases} \dot{\hat{\zeta}}_i = \frac{1}{2} c^\top \hat{w}_i J \hat{\zeta}_i - k_\zeta e_{\zeta_i} \\ \dot{\hat{w}}_i = s(\hat{\zeta}_i, \hat{w}_i) - k_w e_{w_i} \\ \dot{q}_i = 0 \\ \dot{\lambda}_i = -h(\lambda_i - k q_i \bar{\epsilon}_i) \\ \dot{\hat{p}}_i = \gamma \Psi_i z_i - \gamma \nu \, \mathrm{dz}(\hat{p}_i) \end{cases} \chi_i \in C_{\chi i}, \quad \begin{cases} \hat{\zeta}_i^+ = \hat{\zeta}_i \\ \hat{w}_i^+ = \hat{w}_i \\ q_i^+ = -q_i \\ \lambda_i^+ = \lambda_i \\ \hat{p}_i^+ = \hat{p}_i \end{cases} \chi_i \in D_{\chi i},$$

$$\text{with: } C_{\chi i} := \left\{ \chi_i \in \mathbb{R}^{n+4} \times \mathcal{Q} \times \mathbb{R}^{5+2|\mathcal{N}_i|} : \left(\bar{\eta}_i + \frac{\lambda_i \bar{\epsilon}_i}{k} \right) q_i \geq -\delta \right\},$$

$$D_{\chi i} := \left\{ \chi_i \in \mathbb{R}^{n+4} \times \mathcal{Q} \times \mathbb{R}^{5+2|\mathcal{N}_i|} : \left(\bar{\eta}_i + \frac{\lambda_i \bar{\epsilon}_i}{k} \right) q_i \leq -\delta \right\},$$

$$\text{and:} \quad u_i = \Psi_i^\top \hat{p}_i + k_z z_i, \tag{77}$$

where e_{ζ_i}, e_{w_i} are given in (19), $\bar{\eta}_i$, $\bar{\epsilon}_i$ are defined in (43), z_i is defined in (60), Ψ_i is given in (63), dz is given in (66), while the tuning parameters are the stabilizer gains k, h, k_z, γ, ν, the observer gains k_ζ, k_w, and the hysteresis margin δ.

The closed-loop system is given by the interconnection of the second-order Kuramoto network (10), the exosytem (14), and the local controllers (77). For such system, we exploit reduction theorems to show that there exists a compact attractor $\mathcal{A}$ that is robustly globally $\mathcal{KL}$ asymptotically stable. As for Theorem 2, we show that $\mathcal{A}$ is a subset of a compact set $\mathcal{K}$ that we may call again *synchronization set*, with a slight abuse of notation, because its elements enjoy phase synchronization to the reference $\zeta^\star$:

$$\mathcal{K} := \{(\zeta^\star, w^\star, \chi_1, \dots, \chi_N) \in \mathcal{K}^\star \times \prod_{i \in \mathcal{V}} (\mathbb{R}^{n+4} \times \mathcal{Q} \times \mathbb{R}^{5+2|\mathcal{N}_i|}) : \hat{\zeta}_i = \zeta^\star, \hat{w}_i = w^\star, \bar{\zeta}_i = q_i \boldsymbol{e}, \lambda_i = z_i = 0, |\hat{p}_i| \leq r, \forall i \in \mathcal{V}\}. \tag{78}$$

Note that the projection of $\mathcal{K}$ in the direction of $(\zeta^\star, w^\star, x_1, \dots, x_N)$ corresponds to $\mathcal{K}_0$ in (70).

The main result of this work is given by the following statement, which provides formal guarantees for the effectiveness of the controllers (77).

Theorem 3. *For any selection of the tuning parameters $k > 0$, $\delta \in (0,1)$, $h > k$, $k_z > 0$, $\gamma > 0$, $\nu > 0$ and k_ζ, k_w satisfying* (41)*, there exists a compact set $\mathcal{A}$, contained in $\mathcal{K}$ of* (78)*, that is robustly globally $\mathcal{KL}$ asymptotically stable for the interconnection among* (10)*,* (14)*, and* (77)*.*

Proof: We begin by highlighting the cascade-structure of the closed-loop error dynamics. As shown in Section IV, the distributed observer dynamics is collected in the estimation error subsystems (23) and (34). We can establish a cascade interconnection between the system (23), (34), (14), with output $(\zeta^\star, w^\star, \tilde{\zeta}, \tilde{w})$, and the tracking error dynamics (69). We highlight that whenever $(\tilde{\zeta}, \tilde{w}) = 0$ the closed-loop system is described by the dynamics with known leader signals (14), (69). The overall interconnection of these subsystems is shown in Fig. 2.

Asymptotic stability of the attractor $\mathcal{A}$ is proven through reduction theorems. By Theorem 1, we showed that the closed (but not compact) attractor

$$\hat{\mathcal{A}} := \{(\zeta^\star, w^\star, \chi_1, \dots, \chi_N) \in \mathcal{K}^\star \times \prod_{i \in \mathcal{V}} (\mathbb{R}^{n+4} \times \mathcal{Q} \times \mathbb{R}^{5+2|\mathcal{N}_i|}) : \hat{\zeta}_i = \zeta^\star, \hat{w}_i = w^\star, \forall i \in \mathcal{V}\},$$

corresponding to the scenario with known leader signals, is UGAS. On the set $\hat{\mathcal{A}}$, we recover the dynamics (69), thus by Theorem 2 there exists an attractor $\mathcal{A} \subset \mathcal{K}$ that is UGAS relative to $\hat{\mathcal{A}}$. By [40, Cor 4.8], $\mathcal{A}$ is uniformly asymptotically stable for the overall closed-loop system, with basin of attraction given by all the initial conditions generating bounded solutions.

We conclude the proof by showing that all solutions of the closed-loop system are bounded, which then implies UGAS and then robust global $\mathcal{KL}$ asymptotic stability from [28, Thm. 7.21]. First note that the state $(\zeta^\star, w^\star)$ of the exosystem (14) evolves in the bounded forward invariant set $\mathcal{K}^\star$, thus it is bounded. Similarly, q_i is bounded by construction. Due to Theorem 1, $(\tilde{\zeta}, \tilde{w})$ converge to zero, therefore $(\hat{\zeta}_i, \hat{w}_i)$ are bounded for all $i \in \mathcal{V}$. It remains to show that $\bar{\zeta}_i$, λ_i, z_i, and $\hat{p}_i$ are bounded, for all $i \in \mathcal{V}$. Concerning $\bar{\zeta}_i$, recall that $\bar{\zeta}_i :=$

$\mathcal{R}(\zeta_i)^\top \hat{\zeta}_i$, where $\zeta_i \in \mathbb{S}^1$, therefore $\bar{\zeta}_i$ is bounded because $|\bar{\zeta}_i| \leq |\hat{\zeta}_i|$. Since $\bar{\zeta}_i$ is bounded, so is $kq_i\bar{\epsilon}_i$. Indicate with $\bar{k}_i$ the upper bound of $kq_i\bar{\epsilon}_i$, for a given set of initial conditions, and boundedness of λ_i is proven by parallel derivations to (71). To analyze $(z_i, \hat{p}_i)$, as in the proof of Theorem 2, consider the Lyapunov function

$$V_i(z_i, \hat{p}_i) := \frac{1}{2} m_i z_i^2 + \frac{1}{2\gamma} |\hat{p}_i - p_i|^2, \qquad i \in \mathcal{V}. \tag{79}$$

From (67) and (68), respectively for each $i \in \mathcal{V}$, similar steps to those in (73), (74) yield:

$$\begin{aligned} \dot{V}_i &= -k_z z_i^2 - \nu(\hat{p}_i - p_i)^\top \mathrm{dz}(\hat{p}_i) - m_i k_w z_i c^\top e_{w_i} \\ &\leq -\frac{k_z}{2} z_i^2 + \frac{1}{2k_z} |m_i k_w c^\top e_{w_i}|^2, \\ |\hat{p}_i| \geq r \implies \dot{V}_i &\leq -\frac{k_z}{2} z_i^2 - \nu\mu |\hat{p}_i|^2 + \frac{1}{2k_z} |m_i k_w c^\top e_{w_i}|^2. \end{aligned} \tag{80}$$

These two bounds provide, respectively,

$$\begin{aligned} |z_i| > \frac{m_i k_w |c|}{k_z} |e_{w_i}| \implies \dot{V}_i(z_i, \hat{p}_i) < 0, \\ |\hat{p}_i| > \max\left\{ r, \frac{m_i k_w |c|}{\sqrt{2k_z \nu\mu}} |e_{w_i}| \right\} \implies \dot{V}_i(z_i, \hat{p}_i) < 0. \end{aligned} \tag{81}$$

The two implications above prove that neither z_i nor $\hat{p}_i$ can grow unbounded because $e_w = (B \otimes I_n)\tilde{w}$ is bounded. Therefore, we conclude global boundedness of solutions. □

VII. Numerical Example

For the numerical analysis, we consider a Kuramoto model composed of six oscillators, whose parameters and initial conditions are reported in Tab. I. In particular, the graph of the network is depicted in Fig. 1, where the coupling parameters have been assigned as $k_{12} = 0.5$, $k_{13} = 3$, $k_{14} = 1$, $k_{16} = 1.5$, $k_{34} = 2$, $k_{45} = 2.5$, $k_{56} = 2$, $\varphi_{12} = \pi/2$, $\varphi_{13} = \pi/3$, $\varphi_{14} = \pi/4$, $\varphi_{16} = \pi/3$, $\varphi_{34} = \pi/5$, $\varphi_{45} = \pi/4$, $\varphi_{56} = \pi/2$. We suppose to have a rough knowledge of the parameter bounds by letting $\rho = 25$ in (5). It follows that Assumptions 1 and 3 hold. The leader exosystem (14) has been chosen as

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \zeta_1^\star \\ \zeta_2^\star \end{bmatrix} &= \frac{1}{2}(w_1^\star + w_3^\star) J \begin{bmatrix} \zeta_1^\star \\ \zeta_2^\star \end{bmatrix} \\ \frac{d}{dt} \begin{bmatrix} w_1^\star \\ w_2^\star \\ w_3^\star \end{bmatrix} &= \begin{bmatrix} 0 \\ w_3^\star \\ -w_2^\star + \left(1 - \frac{1}{2}|w_3^\star|\right) \tanh(w_3^\star) + \frac{3}{2}\zeta_2^\star \end{bmatrix}, \end{aligned} \tag{82}$$

with initial conditions $\zeta^\star(0) = [1\ 0]^\top$ and $w^\star(0) = [2\ 0\ 0]^\top$. For completeness, we briefly prove that Assumption 2 is satisfied. The existence of $\mathcal{K}^\star$ is guaranteed by proving boundedness of solutions of (82). Note that $(\zeta_1^\star, \zeta_2^\star, w_1^\star)$ are bounded by construction. On the other hand, boundedness of $(w_2^\star, w_3^\star)$ is proven by direct application of [41, Thm. 2]. We remark that, from the chosen initial conditions, the solution converges to a periodic orbit as depicted in Figs. 3, 4. It can be easily shown that $s(\hat{\zeta}^\star, w^\star)$ is globally Lipschitz, since the derivative of the nonlinear term is bounded for all $w_3^\star$. From the numerical evaluation of the differential of s over the values of $(\zeta^\star, w^\star)$, we established a Lipschitz constant $\ell_s = 2.129$.

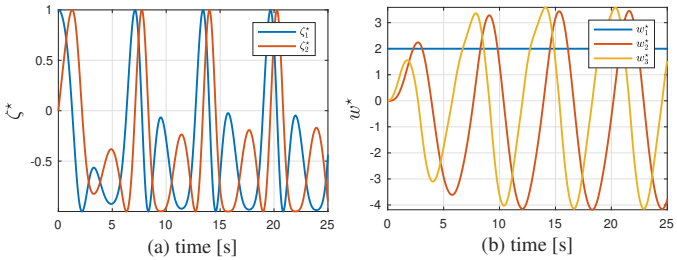

Fig. 3. Response of exosystem (82) initialized in $\boldsymbol{\zeta^\star(0) = [1\ 0]^\top}$, $\boldsymbol{w^\star(0) = [2\ 0\ 0]^\top}$.

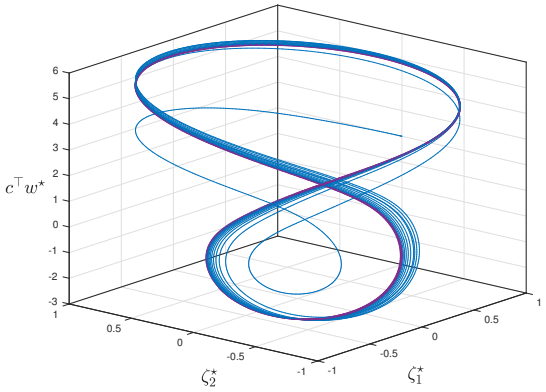

Fig. 4. Response of exosystem (82) (in blue) and corresponding asymptotic behavior (in violet).

The Kuramoto model has been implemented according to (2), with the angles θ_i wrapped between -2π and 2π in order to ensure boundedness of the simulation variables. Then, for the computation of the feedback laws, the variables ζ_i have been computed according to (8). The tuning parameters have been selected as $k_\zeta = 50$, $k_w = 50$, $\delta = 0.5$, $k = 1$, $k_z = 5$, $h = 2$, $\gamma = 1$, $\nu = 1$. Note that (41) is verified since $\underline{\sigma}(B) = 0.1136$. The initial conditions for controller (77) have been randomly chosen, where in particular the logic variables q_i have been initialized in the set $\mathcal{Q} := \{-1, 1\}$.

In Figs. 5, 6 we report the results of a simulation run. Fig. 5 shows the behavior of the distributed observer, which rapidly converges to the exosystem signals. On the other hand, Fig. 6 depicts the tracking performance. In Fig. 6-(e), we also report the evolution of $\hat{p}_1$, showing that the parameters of the adaptive controllers converge to constant values. Finally, we employ wrapped angles to depict the phase tracking performance in Figs. 6-(f), 6-(g). In particular, we define

$$\begin{aligned} \vartheta^\star &:= 2\mathrm{atan2}\left(\zeta_2^\star, \zeta_1^\star\right), \\ \vartheta_i &:= \mathrm{mod}\left(\theta_i + \pi, 2\pi\right) - \pi, \qquad i \in \mathcal{V}, \end{aligned} \tag{83}$$

where $\vartheta^\star$ is the angular reference corresponding to $\zeta^\star$, while ϑ_i is θ_i wrapped in the interval $[-\pi, \pi)$.

VIII. Conclusions

We introduced an adaptive hybrid control strategy for the robust global phase synchronization of second-order Kuramoto oscillators. The objective of phase synchronization was cast into a leader-follower tracking problem, where the leader system is modeled as an autonomous nonlinear exosystem. Under fairly mild assumptions on the network topology and

TABLE I

PARAMETERS AND INITIAL CONDITIONS OF THE OSCILLATORS

	m_i	d_i	$\omega_{\mathrm{n}i}$	$\theta_i(0)$	$\omega_i(0)$
oscillator #1	1.1	0.1	5	$-\pi$	2
oscillator #2	1.3	0.15	10	π	0.5
oscillator #3	1.2	0.2	15	$\pi/2$	1
oscillator #4	1.6	0.21	20	$-\pi/2$	0.3
oscillator #5	1.4	0.18	8	$\pi/3$	1.5
oscillator #6	1.5	0.13	18	$-\pi/3$	0.8

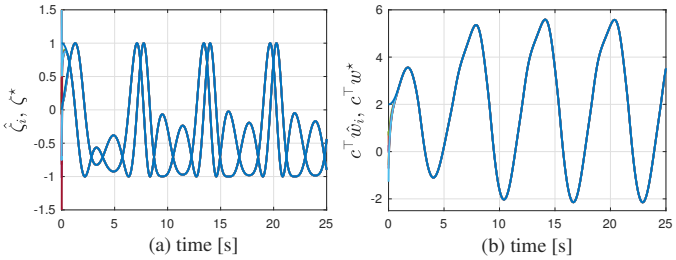

Fig. 5. Closed-loop simulation results. (a): distributed observer phase estimation (reference in blue); (b): distributed observer frequency estimation (reference in blue).

the exosystem dynamics, we proved that our design, which comprises a distributed observer and an adaptive hybrid stabilizer, ensures robust global stability of a compact synchronization set. In particular, robust adaptive stabilization was ensured without requiring persistency of excitation conditions. Future efforts will be dedicated to relaxing the information requirements (e.g., by removing the frequency measurements) and the connectivity properties of the network. Furthermore, it will be worth generalizing the approach to a broader class of nonlinear oscillators.

REFERENCES

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*. Springer, 1984.

[2] Y. Guo, D. Zhang, Z. Li, Q. Wang, and D. Yu, "Overviews on the applications of the Kuramoto model in modern power system analysis," *International Journal of Electrical Power & Energy Systems*, vol. 129, p. 106804, 2021.

[3] F. Dorfler and F. Bullo, "Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators," *SIAM Journal on Control and Optimization*, vol. 50, no. 3, pp. 1616–1642, 2012.

[4] T. Menara, G. Baggio, D. S. Bassett, and F. Pasqualetti, "A framework to control functional connectivity in the human brain," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 4697–4704.

[5] Y. Qin, Y. Kawano, O. Portoles, and M. Cao, "Partial phase cohesiveness in networks of networks of Kuramoto oscillators," *IEEE Transactions on Automatic Control*, 2021.

[6] S.-Y. Ha, T. Ha, and J.-H. Kim, "On the complete synchronization of the Kuramoto phase model," *Physica D: Nonlinear Phenomena*, vol. 239, no. 17, pp. 1692 – 1700, 2010.

[7] Y. Zhang and R. Xiao, "Synchronization of Kuramoto oscillators in small-world networks," *Physica A: Statistical Mechanics and its Applications*, vol. 416, pp. 33 – 40, 2014.

[8] G. S. Schmidt, A. Papachristodoulou, U. Münz, and F. Allgöwer, "Frequency synchronization and phase agreement in Kuramoto oscillator networks with delays," *Automatica*, vol. 48, no. 12, pp. 3008 – 3017, 2012.

[9] J. Zhang and J. Zhu, "Exponential synchronization of the high-dimensional Kuramoto model with identical oscillators under digraphs," *Automatica*, vol. 102, pp. 122 – 128, 2019.

[10] N. Chopra and M. W. Spong, "On exponential synchronization of Kuramoto oscillators," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 353–357, Feb 2009.

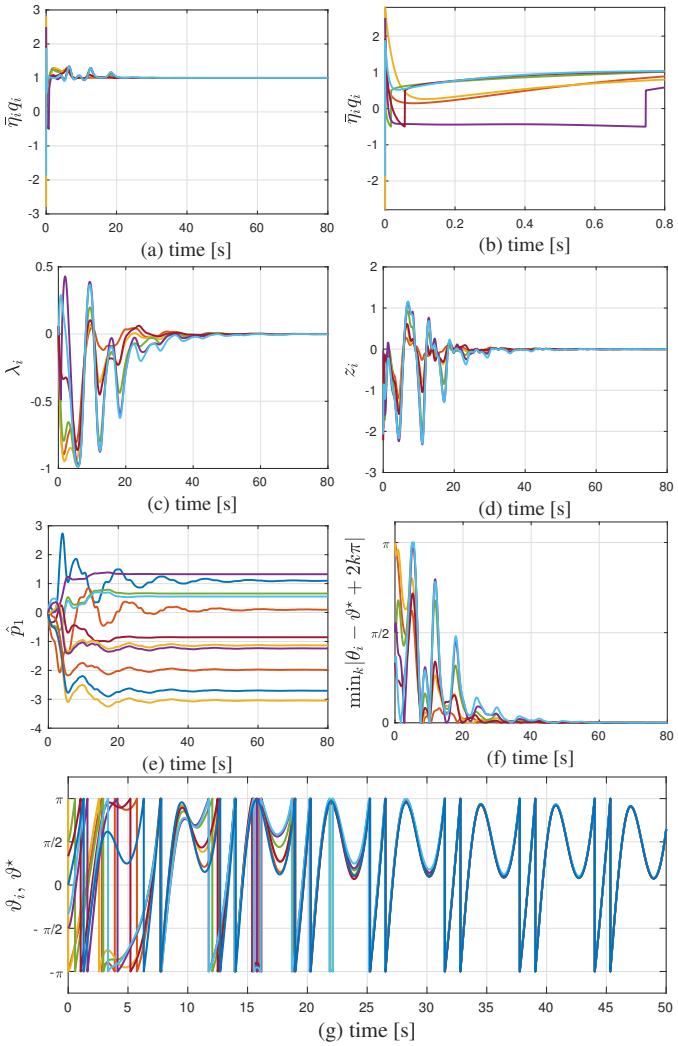

Fig. 6. Closed-loop simulation results. (a): phase tracking errors; (b): phase tracking errors, zoomed in $[\mathbf{0}, \mathbf{0.8}]$s to highlight the jumps during the initial transient; (c): filtered inputs $\boldsymbol{\lambda_i}$; (d): frequency tracking errors $\boldsymbol{z_i}$; (e): evolution of $\hat{\boldsymbol{p}}_\mathbf{1}$; (f): arc distance between $\boldsymbol{\theta_i}$ and $\boldsymbol{\vartheta^\star}$; (g): phase angles, wrapped in the interval $[-\boldsymbol{\pi}, \boldsymbol{\pi})$ (reference in blue).

[11] F. Dörfler and F. Bullo, "On the critical coupling for Kuramoto oscillators," *SIAM Journal on Applied Dynamical Systems*, vol. 10, no. 3, pp. 1070–1099, 2011.

[12] L. Zhu and D. J. Hill, "Synchronization of Kuramoto oscillators: A regional stability framework," *IEEE Transactions on Automatic Control*, vol. 65, no. 12, pp. 5070–5082, 2020.

[13] C. A. Moreira and M. A. de Aguiar, "Global synchronization of partially forced Kuramoto oscillators on networks," *Physica A: Statistical Mechanics and its Applications*, vol. 514, pp. 487 – 496, 2019.

[14] Y. Wang and F. J. Doyle, "Exponential synchronization rate of Kuramoto oscillators in the presence of a pacemaker," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 989–994, April 2013.

[15] Y.-P. Choi, S.-Y. Ha, and S.-B. Yun, "Complete synchronization of Kuramoto oscillators with finite inertia," *Physica D: Nonlinear Phenomena*, vol. 240, no. 1, pp. 32 – 44, 2011.

[16] L. Wu and H. Chen, "Synchronization conditions for a third-order Kuramoto network," in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 5834–5839.

[17] S.-Y. Ha, M. Kang, and D. Kim, "Emergent behaviors of high-dimensional Kuramoto models on Stiefel manifolds," *arXiv preprint arXiv:2101.04300*, 2021.

[18] J. Markdahl, D. Proverbio, and J. Goncalves, "Robust synchronization of heterogeneous robot swarms on the sphere," in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 5798–5803.

[19] L. Scardovi, A. Sarlette, and R. Sepulchre, "Synchronization and balanc-

ing on the N-torus," *Systems and Control Letters*, vol. 56, pp. 335–341, 2007.
[20] S. De Marco, L. Marconi, T. Hamel, and R. Mahony, "Output regulation on the special Euclidean group SE(3)," in *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 4734–4739.
[21] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, 2008.
[22] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, "Quaternion-based hybrid control for robust global attitude tracking," *IEEE Transactions on Automatic control*, vol. 56, no. 11, pp. 2555–2566, 2011.
[23] P. Casau, C. G. Mayhew, R. G. Sanfelice, and C. Silvestre, "Robust global exponential stabilization on the n-dimensional sphere with applications to trajectory tracking for quadrotors," *Automatica*, vol. 110, p. 108534, 2019.
[24] C. G. Mayhew and A. R. Teel, "Hybrid control of planar rotations," in *Proceedings of the 2010 American Control Conference*. IEEE, 2010, pp. 154–159.
[25] C. G. Mayhew, R. G. Sanfelice, J. Sheng, M. Arcak, and A. R. Teel, "Quaternion-based hybrid feedback for robust global attitude synchronization," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2122–2127, 2011.
[26] H. Cai and J. Huang, "Leader-following attitude consensus of multiple rigid body systems by attitude feedback control," *Automatica*, vol. 69, pp. 87–92, 2016.
[27] H. Gui and A. H. de Ruiter, "Global finite-time attitude consensus of leader-following spacecraft systems based on distributed observers," *Automatica*, vol. 91, pp. 225–232, 2018.
[28] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling Stability, and Robustness*. Princeton University Press, Princeton, NJ, 2012.
[29] A. Bosso, I. A. Azzollini, S. Baldi, and L. Zaccarian, "A hybrid distributed strategy for robust global phase synchronization of second-order Kuramoto oscillators," in *2021 IEEE 60th Conference on Decision and Control (CDC)*. IEEE, 2021.
[30] H. Zhang and F. L. Lewis, "Adaptive cooperative tracking control of higher-order nonlinear systems with unknown dynamics," *Automatica*, vol. 48, no. 7, pp. 1432–1439, 2012.
[31] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, no. 2, pp. 28–93, 2009.
[32] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Dover, 2012.
[33] A. Bosso, I. A. Azzollini, and S. Baldi, "Global frequency synchronization over networks of uncertain second-order Kuramoto oscillators via distributed adaptive tracking," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 1031–1036.
[34] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, "On path-lifting mechanisms and unwinding in quaternion-based attitude control," *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1179–1191, 2012.
[35] A. Isidori, L. Marconi, and G. Casadei, "Robust output synchronization of a network of heterogeneous nonlinear agents via nonlinear regulation theory," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2680–2691, 2014.
[36] I. A. Azzollini, W. Yu, S. Yuan, and S. Baldi, "Adaptive leader-follower synchronization over heterogeneous and uncertain networks of linear systems without distributed observer," *IEEE Transactions on Automatic Control*, 2020.
[37] A. Isidori, *Nonlinear control systems II*. Springer, 1999.
[38] A. Isidori, L. Marconi, and A. Serrani, *Robust Autonomous Guidance: an Internal Model Approach*. Springer Science & Business Media, 2012.
[39] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.
[40] M. Maggiore, M. Sassano, and L. Zaccarian, "Reduction theorems for hybrid dynamical systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2254–2265, 2018.
[41] M. Arcak and A. Teel, "Input-to-state stability for a class of Lurie systems," *Automatica*, vol. 38, no. 11, pp. 1945–1949, 2002.

Alessandro Bosso (Member, IEEE) received the master's degree in automation engineering and the Ph.D. degree in automatic control from the University of Bologna, Italy, in 2016 and 2020, respectively.

Currently, he is a Postdoctoral Researcher at the Department of Electrical, Electronic and Information Engineering (DEI), University of Bologna. His research interests include nonlinear control with input and state constraints, nonlinear and distributed adaptive control, hybrid systems, and sensorless control of electric machines.

Ilario Antonio Azzollini received the B.Sc. degree in automation engineering from Alma Mater Studiorum University of Bologna, Italy, in 2016, and the M.Sc. degree (cum laude) in systems and control from the Delft University of Technology, The Netherlands, in 2018. He is currently pursuing the Ph.D. degree in automatic control at the University of Bologna. His research interests include nonlinear optimization, nonlinear and adaptive control, and control of mobile robots.

Simone Baldi (Senior Member, IEEE) received the Ph.D. degree in automatic control systems engineering from the University of Florence, Italy, in 2011. He is currently professor at the School of Mathematics, Southeast University, with a guest position at the Delft Center for Systems and Control, Delft University of Technology, where he was assistant professor. He is a subject editor of International Journal of Adaptive Control and Signal Processing and an associate editor of IEEE Control Systems Letters. His research interests include adaptive and switched systems with applications in networked control and multi-agent systems.

Luca Zaccarian (Fellow, IEEE) received the Ph.D. from the University of Roma Tor Vergata (Italy) in 2000, where he then became assistant and then associate professor. Since 2011 he is Directeur de Recherche at LAAS-CNRS, Toulouse (France) and since 2013 he holds a part-time professor position at the University of Trento, Italy. Luca Zaccarian's main research interests include analysis and design of nonlinear and hybrid control systems, modeling and control of mechatronic systems. He was a recipient of the 2001 O. Hugo Schuck Best Paper Award given by the American Automatic Control Council.