RIEMANNIAN GEOMETRY OF THE REAL FLAG MANIFOLD OF TYPE A 2 AND GEODESIC PROPERTIES OF A WEYL-EQUIVARIANT CELLULAR DECOMPOSITION THEREOF

Arthur Garnier

RIEMANNIAN GEOMETRY OF THE REAL FLAG MANIFOLD OF TYPE A_2 AND GEODESIC PROPERTIES OF A WEYL-EQUIVARIANT CELLULAR DECOMPOSITION THEREOF

ARTHUR GARNIER

Abstract. Using three-dimensional spherical space forms, Chirivi, Spreafico and the author found a cellular structure on the flag manifold $SO(3)/S(O(1)^3)$, equivariant with respect to the action of the Weyl group $W = \mathfrak{S}_3$. In this note, we give some Riemannian geometry properties of this decomposition.

Contents

0. Introduction

Given a compact semisimple Lie group G and a maximal torus T of G, we can form the *flag manifold* G/T. Furthermore, the factor group $W := N_G(T)/T$ is a finite group, known as the *Weyl group* of G. An element $w \in W$ acts naturally on G/T by multiplication on the right by a representative of w in $N_G(T)$. This gives a free action of W on G/T. Moreover, the Bruhat decomposition of G/T (see [Bum13, §27]) provides G/T the structure of a cellular complex. Since W is finite, the quotient G/T admits a W-equivariant cellular decomposition (see [Mat73, Proposition 0.5]). Unfortunately, this is not explicit so a natural question is to find explicitly such a cellular structure.

In [CGS20], the authors obtain an explicit cellular decomposition for the real points $\mathcal{F}(\mathbb{R})$ of the flag manifold $\mathcal{F} := SU_3(\mathbb{C})/T$, equivariant with respect to the Weyl group $W = \mathfrak{S}_3$, using the so-called *octahedral spherical space form* $\mathbb{S}^3/\mathcal{O}$, where $\mathcal{O}$ is the binary octahedral group of order 48. More precisely, they first obtain an $\mathcal{O}$-equivariant cellular structure on $\mathbb{S}^3$ and then, taking the intermediate quotient of $\mathbb{S}^3$ by the quaternion group $\mathcal{Q}_8 \lhd \mathcal{O}$, noticing that $\mathcal{O}/\mathcal{Q}_8 \simeq \mathfrak{S}_3$ and that there is an $\mathfrak{S}_3$-equivariant diffeomorphism $\mathbb{S}^3/\mathcal{Q}_8 \simeq \mathcal{F}(\mathbb{R})$, they get the $\mathfrak{S}_3$-equivariant decomposition of $\mathcal{F}(\mathbb{R})$.

It is worth noticing that the cellular structure on $\mathbb{S}^3$ is constructed using geodesics in $\mathbb{S}^3$, with respect to the usual metric on $\mathbb{S}^3$ induced by the one on $\mathbb{R}^4$. Thus, a natural question is to look for a Riemannian metric on the real flag manifold $\mathcal{F}(\mathbb{R})$ giving "nice" geodesic properties to the cells in $\mathcal{F}(\mathbb{R})$; for example such that a 1-cell is the image of some (open) geodesic in $\mathcal{F}(\mathbb{R})$. This is the aim of this note. In fact, we shall see that there is a unique (up to scalar) bi-invariant Riemannian metric on $\mathcal{F}(\mathbb{R})$ and a suitable normalization of it will make the equivariant diffeomorphism $\mathcal{F}(\mathbb{R}) \simeq \mathbb{S}^3/\mathcal{Q}_8$ into an isometry. The resulting

Date: October 9, 2021.
2010 *Mathematics Subject Classification.* Primary 14M25, 53C22, 57M60 ; Secondary 57N60, 57R91.

metric will be called the *quaternionic metric* on $\mathcal{F}(\mathbb{R})$ and denoted by $\overline{g}^8$.

Let us briefly outline the content of this note. First of all, we give a reminder on elementary Riemannian geometry and in particular on invariant metrics on compact Lie groups and associated homogeneous spaces. Then we define the quaternionic metric on $\mathcal{F}(\mathbb{R})$ using the Killing form on the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ of skew-symmetric 3×3 matrices. Next, we prove that, endowed with this normalized metric, the equivariant diffeomorphism $\mathbb{S}^3/\mathcal{Q}_8 \xrightarrow{\sim} \mathcal{F}(\mathbb{R})$ constructed in [CGS20] is isometric. Finally, we describe the cells of $\mathcal{F}(\mathbb{R})$ constructed in [CGS20] in terms of unions of images of $\overline{g}^8$-geodesics. In particular, 1-cells are the images in $\mathcal{F}(\mathbb{R})$ of one-parameter subgroups of $SO_3(\mathbb{R})$.

1. Invariant Riemannian structures on Lie groups and homogeneous spaces

We start this note by giving some reminders on Riemannian manifolds and in particular on bi-invariant metrics on flag manifolds. For more details on Riemannian manifolds, the reader is invited to have a look at [Lee] or [GHL04].

Recall that a *Riemannian manifold* is a pair (M,g), where M is a smooth manifold and g is a symmetric positive-definite $(2,0)$-tensor on M. To simplify notations, we use freely the *Einstein convention* for repeated indices. For instance, we simply write x_ie^i to mean $\sum_i x_ie^i$. Denote by $\mathcal{X}(M) := \Gamma(TM)$ the set of vector fields on M (i.e. the sections of the tangent bundle TM of M) and recall that an *affine connection* on M is a bilinear map $\nabla : \mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M)$ which is $\mathcal{C}^\infty(M,\mathbb{R})$-linear on the left and which satisfies the Leibniz rule on the right, i.e. such that

$$\forall f\in\mathcal{C}^\infty(M,\mathbb{R}),\ \forall X,Y\in\mathcal{X}(M),\ \begin{cases}\nabla_{fX}Y = f\nabla_XY,\\ \nabla_X(fY) = \mathrm{d}f(X)Y + f\nabla_XY.\end{cases}$$

If (M,g) is a Riemannian manifold, then there exists a unique affine connection on M such that

$$\begin{cases} Z(g(X,Y)) = g(\nabla_ZX,Y) + g(X,\nabla_ZY),\\ \nabla_XY - \nabla_YX = [X,Y].\end{cases}$$

This connection is called the *Levi-Civita connection* on M. It may be implicitly defined by the Koszul formula

$$\begin{aligned}2g(\nabla_XY,Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X).\end{aligned}\tag{1}$$

It may be useful to express these objects in local charts. If $p\in M$, take $(x^1,\dots,x^n)$ a local system of coordinates around p and define the *metric coefficients*

$$g_{ij} := g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)$$

as well as (g^{ij}) the inverse matrix of (g_{ij}). If ∇ is the Levi-Civita connection on M, we define the *Christoffel symbols* Γ_{ij}^k by

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) =: \Gamma_{ij}^k\frac{\partial}{\partial x^k},$$

i.e.

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Now, a curve $\gamma :]a,b[\to M$ is a *geodesic* if and only if the covariant derivative of $\gamma' =: \dot\gamma$ vanishes, that is

$$\nabla_{\dot\gamma}\dot\gamma = 0,\quad \text{where}\quad \nabla_{\dot\gamma} := (\gamma^*\nabla)_{\frac{d}{dt}}.$$

Here, the connection $\gamma^*\nabla$ is defined as the only connection on $\gamma^*(TM)$ such that, for $x\in]a,b[$, $v\in\mathbb{R} = T_x(]a,b[)$ and for a vector field $X\in\mathcal{X}(M)$, one has

$$(\gamma^*\nabla)_v(\gamma^*X) = \gamma^*\left(\nabla_{d_x\gamma(v)}(X)\right).\tag{2}$$

Using the Christoffel symbols, this can be rephrased in the following differential system of $\dim M$ equations:

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0.$$

Using the Picard-Lindelöf theorem, given $m_0 \in M$, there exists an open neighborhood $U \subset M$ of m_0 and $\varepsilon > 0$ such that, for $m \in U$ and $v \in T_pM$ with $|v| < \varepsilon$, there is a unique geodesic $c_v :]-1, 1[\to M$ such that $c_v(0) = m$ and $c_v'(0) = v$. Like the maximal solutions of an ordinary differential equation, the maximal geodesics need not to be defined for all $t \in \mathbb{R}$. If so, the manifold M is said to be *geodesically complete*.

Next, we define the *length* of a curve $\gamma : [a, b] \to M$ by the integral

$$L(\gamma) := \int_a^b \sqrt{g(\dot\gamma(t), \dot\gamma(t))}\mathrm{d}t = \int_a^b \|\dot\gamma(t)\|_g \mathrm{d}t.$$

For $p, q \in M$ two points on a connected Riemannian manifold (M, g), denote by $\mathcal{C}(p, q)$ the set of piecewise smooth curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and define the quantity

$$d_g(p, q) := \inf_{\gamma\in\mathcal{C}(p,q)} L(\gamma).$$

This is well-defined since $\mathcal{C}(p, q) \neq \emptyset$ (see [Lee, Proposition 2.50]) and it is easy to see that the function $d_g : M \times M \to \mathbb{R}_+$ is a distance on M, making (M, d_g) into a metric space. Furthermore, we say that a geodesic γ between two points p and q of M is *minimal* if $L(\gamma) = d_g(p, q)$. Moreover, the topology induced by this distance is the original topology of M (see [GHL04, Definition-Proposition 2.91]). In fact, according to the Theorem 6.15 from [Lee], every geodesic is locally minimal and every minimal curve is a geodesic, when it is given the unit-speed parametrization ([Lee, Theorem 6.4]). However, it is typically not true that any two points of M can be joined by a minimal geodesic.

A *local isometry* between Riemannian manifolds (M, g) and (N, h) is a smooth map $f : M \to N$ such that

$$\forall p \in M,\ \forall u, v \in T_pM,\ h_{f(p)}(\mathrm{d}_p f(u), \mathrm{d}_p f(v)) = g_p(u, v).$$

Note that this condition implies that f is a local diffeomorphism, by the inverse function theorem. Moreover, a local isometry is called an *isometry* if it is a diffeomorphism. Note that an isometry preserves Riemannian distances between points.

Let $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ be a smooth submersion between Riemannian manifolds. For $x \in \widetilde{M}$, we define the following subspaces of $T_x\widetilde{M}$

$$V_x := \ker(\mathrm{d}_x\pi) = T_x(\pi^{-1}(\pi(x))) \quad \text{and} \quad H_x := V_x{}^{\perp},$$

the orthogonal being taken with respect to the inner product $\widetilde{g}_x$. These are respectively called the *vertical* and *horizontal* tangent spaces. We say that π is a *Riemannian submersion* if $\mathrm{d}_x\pi$ restricts to a linear isometry from H_x to $T_{\pi(x)}M$, i.e.

$$\forall u, v \in H_x,\ \widetilde{g}_x(u, v) = g_{\pi(x)}(\mathrm{d}_x\pi(u), \mathrm{d}_x\pi(v)).$$

We say that π is a *Riemannian covering map* if it is a covering map, that is also a Riemannian submersion.

Proposition 1.1. ([GHL04, Proposition 2.81]) *If $\pi : \widetilde{M} \to M$ is a Riemannian covering map, then the geodesics of M are the projections of the geodesics of $\widetilde{M}$ and conversely, every geodesic of M lifts to a geodesic of $\widetilde{M}$.*

Finally, recall the Hopf-Rinow theorem:

Theorem 1.2. (Hopf-Rinow, [Lee, Theorem 6.19])
Metric and geodesic completeness are equivalent in a connected Riemannian manifold. Moreover, if the manifold is complete, then any two points can be joined by a minimal geodesic.

We shall need the following fundamental result:

Theorem 1.3. *Let (M, g) be a connected Riemannian manifold and G be a Lie group acting freely, properly and isometrically on M. Then, there exists a unique Riemannian metric $\overline{g}$ on M/G such that the projection $\pi : M \twoheadrightarrow M/G$ is a Riemannian submersion.*

If moreover M and M/G are geodesically complete (which is the case for instance if M is compact and G is finite, by the Hopf-Rinow theorem), then the geodesic distance on M/G is given by

$$\forall x, y \in M,\ d_{\overline{g}}(\pi(x), \pi(y)) = \inf_{h \in G} d_g(x, hy).$$

Proof. The existence and uniqueness of $\overline{g}$ is a standard fact and can be found for instance in [Lee], Corollary 2.29 or in [Bes87, §9.12]. Only the statement about distance remains to be proved. Take $x, y \in M$ and $\overline{x} := \pi(x)$, $\overline{y} := \pi(y)$. Fix some $h \in G$. Since M is complete, there exists a minimizing geodesic arc $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = hy$. Then $\pi \circ \gamma$ is a geodesic linking $\overline{x}$ and $\overline{y}$ and since π is a Riemannian submersion, it is a local isometry so one has

$$L(\gamma) \overset{\text{def}}{=} \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\mathrm{d}t = L(\pi \circ \gamma).$$

Hence, $\pi \circ \gamma$ is a geodesic between $\overline{x}$ and $\overline{y}$ of the same length as γ, so by definition of the geodesic distance, one gets $d_{\overline{g}}(\overline{x}, \overline{y}) \leq d_g(x, hy)$. Because $h \in G$ is arbitrary, we get $d_{\overline{g}}(\overline{x}, \overline{y}) \leq \inf_h d_g(x, hy)$. We have to prove the converse inequality to conclude. Consider then a minimizing geodesic arc $\widetilde{\gamma} : [0,1] \to M/G$ between $\overline{x}$ and $\overline{y}$. Using again the Proposition 1.1, there exists a geodesic arc $\gamma : [0,1] \to M$ such that $\pi \circ \gamma = \widetilde{\gamma}$ and we have $L(\gamma) = L(\widetilde{\gamma}) = d_{\overline{g}}(\overline{x}, \overline{y})$. By construction, there exist $h_0, h_1 \in G$ such that $\gamma(0) = h_0 x$ and $\gamma(1) = h_1 y$ we have $d_g(x, h_0^{-1} h_1 y) = d_g(h_0 x, h_1 y) \leq L(\gamma) = d_{\overline{g}}(\overline{x}, \overline{y})$. □

Remark 1.4. *If we take $M = \mathbb{S}^{2n+1}$ endowed with its natural round metric and $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{S}^{2n+1}$ as the antipode, then $M/G = \mathbb{P}^n(\mathbb{C})$ and there is a unique metric on $\mathbb{P}^n(\mathbb{C})$ making the projection $\mathbb{S}^{2n+1} \twoheadrightarrow \mathbb{P}^n(\mathbb{C})$ into a Riemannian submersion. This metric is called the* Fubini-Study metric*. Using the previous Theorem, we can easily see that the induced distance d_{FS} on $\mathbb{P}^n(\mathbb{C})$ is given by the following*

$$\forall p, q \in \mathbb{P}^n(\mathbb{C}),\ d_{FS}(p, q) = \arccos \frac{|\langle p, q \rangle|}{\|p\| \|q\|}.$$

We now review some basic facts about invariant Riemannian metrics on Lie groups and their flag manifolds. Let G be a Lie group and $\mathfrak{g} := T_1 G$ be its Lie algebra. For an element $p \in G$, denote by $L_p : G \to G$ and $R_p : G \to G$ left multiplication maps $q \mapsto pq$ and $q \mapsto qp$, respectively. Recall Of course, there may exists many Riemannian metrics on G, but a natural restriction is to look for *invariant* metrics. More precisely, a Riemannian metric g on G is said to be *left-invariant* if

$$\forall p \in G,\ \forall X, Y \in \mathfrak{g},\ g_p(\mathrm{d}_1 L_p(X), \mathrm{d}_1 L_p(Y)) = g_1(X, Y).$$

Denoting by $\mathcal{X}_\ell(G)$ the set of *left-invariant* vector fields on G (i.e. the set of all $X \in \mathcal{X}(G)$ such that $\mathrm{d}_q L_p(X_q) = X_{pq}$ for all $p, q \in G$) and using the bijection $\mathcal{X}_\ell(G) \to \mathfrak{g}$ defined by $X \mapsto X_1$, we see that g is left-invariant if and only if the following is true:

$$\forall p \in G,\ \forall X, Y \in \mathcal{X}_\ell(G),\ g_p(X_p, Y_p) = g_1(X_1, Y_1).$$

Analogously, g is *right-invariant* if the above condition is verified for right-invariant vector fields on G. Finally, the metric g is *bi-invariant* if it is both left and right-invariant.

Lemma 1.5. *Let G be a Lie group.*

The map $g \mapsto g_1$ is a bijective correspondence between the set of left-invariant (resp. right-invariant) metrics on G and the set of inner products on $\mathfrak{g}$.

Furthermore, the same map restricts to a bijective correspondence between the set bi-invariant metrics on G and the set of ad*-invariant inner products on $\mathfrak{g}$.*

In particular, if G is compact then the Killing form $\kappa(X, Y) := \mathrm{tr}\,(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ on $\mathfrak{g}$ is negative definite ([Bes87, Lemma 7.36]*) and thus there exists a bi-invariant metric on G.*

A first convenient fact about bi-invariant metrics is that the associated Levi-Civita connection is easily computed on invariant vector fields.

Lemma 1.6. *If g is a bi-invariant Riemannian metric on a Lie group G and if ∇ is the associated Levi-Civita connection, then*

$$\forall X, Y \in \mathcal{X}_\ell(G),\ \nabla_X Y = \frac{1}{2}[X, Y].$$

Proof. Let $Z \in \mathcal{X}_\ell(G)$ and note that, since g is bi-invariant, the function $g(X, Y)$ is constant on G and hence $Z(g(X, Y)) = 0$. Also, since g_1 is ad-invariant on $\mathfrak{g}$, we have $g(X, [Y, Z]) = g([X, Y], Z)$. Hence, the Koszul formula (1) reads

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \\ &= g([X, Y], Z) - g([X, Z], Y) + g(X, [Z, Y]) = g([X, Y], Z). \end{aligned}$$

Since this is true for arbitrary Z, the result follows. □

We now come to the following important result:

Theorem 1.7. [YWL19, Theorem 2.5]
If G is a compact Lie group endowed with a bi-invariant metric g and $H \leq G$ is a closed subgroup then the orbit space G/H, endowed with the induced Riemannian metric given by the Theorem 1.3, is a geodesic orbit space*, meaning that every geodesic on G/H is the orbit of a one-parameter subgroup of G.*

Proof. Using [GHL04, Proposition 2.81], we only have to prove that G is a geodesic orbit space. Denote by $\mathfrak{g}$ the Lie algebra of G. Let γ be the curve defined on $\mathbb{R}$ by $\gamma : t \mapsto pe^{tX}$. Then γ is an X-integral curve, i.e. $\gamma' = \widetilde{X} \circ \gamma$, where $\widetilde{X} \in \mathcal{X}_\ell(G)$ is the left-invariant vector field associated to $X \in \mathfrak{g}$. Then, one calculates

$$\forall t \in \mathbb{R},\ \nabla_{\dot\gamma}\dot\gamma(t) \overset{\text{def}}{=} (\gamma^*\nabla)_{\frac{d}{dt}}(\gamma^*\widetilde{X})(t) \overset{(2)}{=} \left(\gamma^*\left(\nabla_{\dot\gamma(t)}\widetilde{X}\right)\right)(t) = \left(\nabla_{\widetilde{X}}\widetilde{X}\right)(\gamma(t)) = 0,$$

since $\nabla_{\widetilde{X}}(\widetilde{X}) = \frac{1}{2}[\widetilde{X}, \widetilde{X}] = 0$ by the Lemma 1.6, we get that $\nabla_{\dot\gamma}\dot\gamma = 0$ and hence γ is a geodesic and by the Picard-Lindelöf theorem, this is the only geodesic on G such that $\gamma(0) = p$ and $\gamma'(0) = pX = d_1 L_p(X)$. Then, we have proved that any geodesic on G is of the form $t \mapsto pe^{tX}$ for some $p \in G$ and $X \in \mathfrak{g}$ and hence is a one-parameter subgroup of G. □

2. The quaternionic bi-invariant Riemannian metric on the flag manifold of $SL_3(\mathbb{R})$

We shall now equip the manifold $\mathcal{F}(\mathbb{R})$ with a bi-invariant Riemannian metric. An SU_3-invariant Riemannian metric on $\mathcal{F} = SU_3(\mathbb{C})/T$ is easily seen to be determined by its value on the tangent space $T_1\mathcal{F}$ (this is a general fact about homogeneous spaces which relies on Lemma 1.5). Now,

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\delta \in \Phi^+} (\mathbb{C}e_\delta \oplus \mathbb{C}f_\delta)$$

is the root spaces decomposition of $\mathfrak{sl}_3$, with $(e_\delta, f_\delta, h_\delta)_{\delta\in\Phi^+}$ the Serre basis of $\mathfrak{sl}_3$, and $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$ is the set of positive roots, then one has the Cartan decomposition

$$\mathfrak{su}_3(\mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{\delta\in\Phi^+} \mathfrak{p}_\delta, \quad \text{with} \quad \mathfrak{p}_\delta := \mathbb{R}\underbrace{(e_\delta - f_\delta)}_{u_\delta} \oplus \mathbb{R}\underbrace{i(e_\delta + f_\delta)}_{v_\delta} \quad \text{and} \quad \mathfrak{t} = \bigoplus_{\delta\in\Phi^+} \mathbb{R}ih_\delta.$$

Now, one has $T_1\mathcal{F} \simeq \bigoplus_{\delta\in\Phi^+} \mathfrak{p}_\delta =: \mathfrak{p}$ and recalling that the Killing form $\kappa(X, Y) := 6\text{tr}\,(\text{ad}(X) \circ \text{ad}(Y)) = 6\text{tr}\,(XY)$ on $\mathfrak{su}_3(\mathbb{C})$ is a negative-definite symmetric bilinear form (since SU_3 is compact, see [Bes87, Lemma 7.36]), any left SU_3-invariant metric g on $\mathcal{F}$ may be written as

$$g = -\sum_{\delta\in\Phi^+} x_\delta \cdot \kappa(\cdot, \cdot)_{|\mathfrak{p}_\delta}, \quad \text{with} \quad x_\delta \in \mathbb{R}^+,\ \forall \delta \in \Phi^+$$

and this metric is bi-invariant if and only if $x_\delta = x_{\delta'}$ for all $\delta, \delta' \in \Phi^+$. Thus there is only one bi-invariant metric on $\mathcal{F}$, up to scalar. These standard considerations can be found in [Sak99] or [PS97]. Then, we take the *quaternionic bi-invariant metric*

$$g^8 := -\frac{1}{48}(\kappa_{|\mathfrak{p}_\alpha} + \kappa_{|\mathfrak{p}_\beta} + \kappa_{|\mathfrak{p}_{\alpha+\beta}}) = -\frac{1}{48}\kappa_{|\mathfrak{p}}$$

on $\mathcal{F}$, and restrict it to $\mathcal{F}(\mathbb{R})$. The reason of taking such a normalization will appear soon. Notice that this metric is *Einstein*, meaning that the Ricci tensor is a scalar multiple of the metric tensor, i.e. there exists a function λ such that $\mathrm{Ric}_{g^8} = \lambda g^8$ everywhere.

Proposition 2.1. *The metric g^8 on $SO(3)$ defined above induces a Riemannian metric $\overline{g}^8$ on $\mathcal{F}(\mathbb{R})$ making $(\mathcal{F}(\mathbb{R}), \overline{g}^8)$ into a geodesic orbit space. Moreover, for $p \in SO(3)$ and $X \in \mathfrak{so}(3) := \mathfrak{so}_3(\mathbb{R})$, the arc-length of the geodesic $\gamma : s \mapsto pe^{sX} \cdot S(O(1)^3)$ is given by*

$$\forall t \geq 0,\ \mathrm{L}(\gamma_{|[0,t]}) = \frac{t\|X\|_F}{2\sqrt{2}},$$

where $\|\cdot\|_F$ is the Frobenius norm, defined by $\|(a_{i,j})_{i,j}\|_F = \sqrt{\sum_{i,j} |a_{i,j}|^2}$.

Proof. The first statement is just a particular case of the Theorem 1.7. For the second statement we just calculate, for $t \in \mathbb{R}_+$,

$$\begin{aligned}\mathrm{L}(\gamma_{|[0,t]}) &\overset{\text{def}}{=} \int_0^t \sqrt{g^8_{\gamma(s)}(\gamma'(s), \gamma'(s))}\mathrm{d}s = \int_0^t \sqrt{g^8_{\gamma(s)}(pX\exp(sX), pX\exp(sX))}\mathrm{d}s \\ &= \int_0^t \sqrt{g^8_1(X,X)}\mathrm{d}s = t\sqrt{g^8_1(X,X)} = t\sqrt{\frac{\mathrm{tr}\,({}^tXX)}{8}} = \frac{t\|X\|_F}{2\sqrt{2}}.\end{aligned}$$

□

3. Isometry between $\mathcal{F}(\mathbb{R})$ and the quaternionic spherical space form $\mathbb{S}^3/\mathcal{Q}_8$

Recall the isomorphism

$$\begin{array}{rccc} \sigma & : & \mathcal{O}/\mathcal{Q}_8 & \xrightarrow{\sim} & \mathfrak{S}_3 \\ & & \frac{1+i}{\sqrt{2}} & \longmapsto & s_\beta \\ & & \frac{1+k}{\sqrt{2}} & \longmapsto & s_\alpha \end{array}$$

We now equip $\mathbb{S}^3/\mathcal{Q}_8$ with the quotient metric $q_{\mathcal{Q}_8}$ induced by the standard round metric on $\mathbb{S}^3$ and we shall prove that ϕ is in fact an isometry. For this, we need the following lemma:

Lemma 3.1. *The map*

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{\mathrm{B}} & SO(3) \\ q & \longmapsto & \mathrm{Mat}_{(i,j,k)}(L(q)R(q)) \end{array}$$

is smooth and we have

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\mathrm{d_1B}} & \mathfrak{so}(3) \\ (x,y,z) & \longmapsto & 2\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \end{array}$$

In particular, if $\mathbb{S}^3$ is equipped with the standard round metric induced from $\mathbb{R}^4$ and $SO(3)$ with the bi-invariant metric g^8 defined above, then we have an isometry

$$\overline{B} : \mathbb{S}^3/\{\pm 1\} \xrightarrow{\sim} SO(3).$$

Proof. Recall the space $V := \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ of pure quaternions. For $u, h \in V \simeq \mathbb{R}^3$, we simply compute

$$\begin{aligned}\mathrm{d_1B}(u)\cdot h &= \frac{d}{dt}\,\mathrm{B}(1+tu)(h)|_{t=0} = \frac{d}{dt}\,(1+tu)h\overline{(1+tu)}\Big|_{t=0} \\ &= \frac{d}{dt}\,(h + th\overline{u} + tuh + t^2uh\overline{u})\big|_{t=0} = h\overline{u} + uh = uh - hu = [u,h].\end{aligned}$$

Hence, by computing the matrix of $\mathrm{d}_1 B(u)$ with respect to the canonical basis (i, j, k) of V, one obtains the matrix from the first statement.

Now, since $\{\pm 1\}$ acts freely and isometrically on $\mathbb{S}^3$, the map $\mathbb{S}^3 \twoheadrightarrow \mathbb{S}^3/\{\pm 1\}$ is a Riemannian covering, hence a local isometry (in particular, a local diffeomorphism). Therefore, if we prove that B is a local isometry, then $\overline{\mathrm{B}}$ will be a bijective local isometry, hence an isometry by the inverse function theorem, as required. But since B is a homomorphism of Lie groups, it suffices to show that $\mathrm{d}_1\mathrm{B}$ is a linear isometry. This is where the normalization by $\frac{1}{48}$ comes into the game. Since we have endowed $\mathbb{S}^3$ with the round metric, we can compute for $u := (x, y, z) \in \mathbb{R}^3 = T_1\mathbb{S}^3$,

$$g_1^8(\mathrm{d}_1\mathrm{B}(u), \mathrm{d}_1\mathrm{B}(u)) \overset{\text{df}}{=} -\frac{6\mathrm{tr}\,(\mathrm{d}_1\mathrm{B}(u)^2)}{48} = -\frac{4}{8}\mathrm{tr}\left(\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}^2\right)$$

$$= -\frac{1}{2}\mathrm{tr}\begin{pmatrix} -y^2-z^2 & xy & xz \\ xy & -x^2-z^2 & yz \\ xz & yz & -x^2-y^2 \end{pmatrix} = x^2+y^2+z^2 = g_1^{\mathbb{S}^3}(u,u).$$

□

Proposition 3.2. *If we endow respectively $\mathbb{S}^3/\mathcal{Q}_8$ and $\mathcal{F}(\mathbb{R})$ with the metrics $g_{\mathcal{Q}_8}$ and $\overline{g}^8$, then the map ϕ of Lemma* **??** *is an isometry.*

Proof. The quotient map

$$\mathbb{S}^3/\{\pm 1\} \twoheadrightarrow (\mathbb{S}^3/\{\pm 1\})/K_4 \simeq \mathbb{S}^3/\mathcal{Q}_8$$

is a Riemannian covering, hence a local isometry. On the other hand, the map

$$SO(3) \twoheadrightarrow SO(3)/S(O(1)^3) = \mathcal{F}(\mathbb{R})$$

is a Riemannian covering too. Now, since $\overline{\mathrm{B}} : \mathbb{S}^3/\{\pm 1\} \to SO(3)$ is an isometry by the previous Lemma and since the following diagram commutes

$$\begin{array}{ccc} \mathbb{S}^3/\{\pm 1\} & \xrightarrow[\sim]{\overline{\mathrm{B}}} & SO(3) \\ \downarrow & & \downarrow \\ \mathbb{S}^3/\mathcal{Q}_8 & \xrightarrow[\phi]{\sim} & \mathcal{F}(\mathbb{R}) \end{array}$$

one concludes that ϕ is a bijective local isometry, hence a global isometry. □

In particular, combining Proposition 3.2 and Theorem 1.3 yields the following corollary:

Corollary 3.3. *For $q := a + bi + cj + dk \in \mathbb{S}^3$, one has*

$$d_{g^8}(1, \mathrm{B}(q)) = \min_{\varepsilon=\pm 1} d_{\mathbb{S}^3}(1, \varepsilon q) = \arccos|a|$$

and

$$d_{\overline{g^8}}(1, \phi(\overline{q})) = \min_{g\in\mathcal{Q}_8} d_{\mathbb{S}^3}(1, gq) = \min_{x=\pm a,\pm b,\pm c,\pm d}(\arccos(x)) = \min_{x=a,b,c,d}(\arccos|x|).$$

4. Geodesics in $\mathcal{F}(\mathbb{R})$ as projections of geodesics in $\mathbb{S}^3$

Now that we know what geodesics look like and that we can compute the distance between two flags, we can start describing the cells. But before that, we have to adapt the curved join construction to $\mathcal{F}(\mathbb{R})$. This is not as easy as in the case of $\mathbb{S}^3$, since there can exist many minimizing geodesics between two points in $\mathcal{F}(\mathbb{R})$ (as for two antipodal points in $\mathbb{S}^3$). Since $SO(3)$ acts transitively by isometries on $\mathcal{F}(\mathbb{R})$, it suffices to look at geodesics starting at 1 and translate them. It turns out that, if a matrix in $SO(3)$, seen as a rotation, has angle different from π, then there will be a unique minimizing geodesic linking it to 1. For this, we shall use the matrix logarithm.

Recall that, given $X \in \mathfrak{so}(3)$ and $\theta \in [0, 2\pi]$, we have the *Rodrigues formula* (see[CL10, §2])

$$e^{\theta X} = I_3 + \sin(\theta)X + (1 - \cos(\theta))X^2,$$

hence we obtain $\sin(\theta)X = \frac{e^{\theta X} - {}^t(e^{\theta X})}{2}$ and if $\theta \neq 0, \pi$, then

$$X = \frac{1}{2\sin(\theta)}(e^{\theta X} - e^{-\theta X}).$$

Thus, if $R \in SO(3)$ is a rotation with $\mathrm{tr}\,(R) \neq -1, 3$, then there is a unique $X \in \mathfrak{so}(3)$ such that $e^X = R$ and X is given by

$$X = \frac{\theta}{2\sin(\theta)}(R - {}^tR),\ \theta = \arccos\left(\frac{\mathrm{tr}\,(R) - 1}{2}\right).$$

We shall denote $X := \log(R)$. This is uniquely defined as soon as $\theta \neq 0, \pi$. If $\theta = 0$, we can just take $\log(R) = 0$. With this notion, we see that the curve $\gamma_R : t \mapsto e^{t\log(R)}$ is a geodesic from 1 to R in $SO(3)$ and hence its projection $\overline{\gamma_R} : t \mapsto e^{t\log(R)}S(O(1)^3)$ is a geodesic from 1 to $R \cdot S(O(1)^3)$ in $\mathcal{F}(\mathbb{R})$.

Now, we have to prove that the images of the geodesics we used in $\mathbb{S}^3$ to construct our $\mathcal{O}$-cellular decomposition go to geodesics in $\mathcal{F}(\mathbb{R})$. Denote by $\pi_{\mathcal{Q}_8} : \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^3/\mathcal{Q}_8$ the natural projection and recall the isometry $\phi : \mathbb{S}^3/\mathcal{Q}_8 \to \mathcal{F}(\mathbb{R})$. We have the following result:

Proposition 4.1. *Let*

$$q := (\cos\omega, \sin\omega\cos\varphi, \sin\omega\sin\varphi\cos\theta, \sin\omega\sin\varphi\sin\theta) \in \mathbb{S}^3$$

be a point expressed in spherical coordinates, with $0 \leq \omega, \varphi \leq \pi$ *and* $0 \leq \theta \leq 2\pi$. *Suppose* $0 < \omega < \frac{\pi}{2}$ *and denote by* $\widetilde{\gamma_q}$ *the unique minimizing geodesic such that* $\widetilde{\gamma_q}(0) = 1$ *and* $\widetilde{\gamma_q}(1) = q$. *Then one has*

$$\forall 0 \leq t \leq 1,\ (\phi \circ \pi_{\mathcal{Q}_8})\widetilde{\gamma_q}(t) = \exp(tX_q) \cdot S(O(1)^3) =: \gamma_q(t),$$

where

$$X_q := 2\omega \begin{pmatrix} 0 & -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & 0 & -\cos(\varphi) \\ -\sin(\varphi)\cos(\theta) & \cos(\varphi) & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

In particular, one has

$$\mathrm{L}(\gamma_q) = \mathrm{L}(\widetilde{\gamma_q}) = \omega.$$

Moreover, $B \circ \widetilde{\gamma_q}$ *is the only geodesic (up to reparametrization) in* $SO(3)$ *from* 1 *to* $\mathrm{B}(q)$.

Proof. The round metric on $\mathbb{S}^3$ is given in spherical coordinates (around 1) by the matrix (g_{ij}) where $g_{ij} = 0$ for $i \neq j$ and

$$g_{\omega\omega} = 1, \quad g_{\varphi\varphi} = \sin^2\omega, \quad g_{\theta\theta} = \sin^2\omega\sin^2\varphi.$$

hence, the Christoffel symbols Γ_{ij}^k are easily computed and the geodesic equations $\ddot{\gamma^k} + \Gamma_{ij}^k\dot{\gamma^i}\dot{\gamma^j} = 0$ for a curve $t \mapsto (\omega(t), \varphi(t), \theta(t))$ are given by the system

$$\begin{cases} \ddot{\omega} - \sin(\omega)\cos(\omega)(\dot{\varphi}^2 + \sin^2(\varphi)\dot{\theta}^2) = 0, \\ \ddot{\varphi} + \cot(\omega)\dot{\varphi}\dot{\omega} - \sin(\varphi)\cos(\varphi)\dot{\theta}^2 = 0, \\ \ddot{\theta} + \dot{\theta}(\cot(\varphi)\dot{\varphi} + \cot(\omega)\dot{\omega}) = 0. \end{cases}$$

Hence, the curve

$$\widetilde{\gamma_q} : t \mapsto (\cos(t\omega), \sin(t\omega)\cos\varphi, \sin(t\omega)\sin\varphi\cos\theta, \sin(t\omega)\sin\varphi\sin\theta)$$

is a geodesic, with $\widetilde{\gamma_q}(0) = (1, 0, 0, 0)$ and $\widetilde{\gamma_q}(1) = q$. Moreover, it is minimizing since

$$\mathrm{L}(\widetilde{\gamma_q}) = \int_0^1 \sqrt{g^{\mathbb{S}^3}_{\widetilde{\gamma_q}(t)}(\dot{\widetilde{\gamma_q}}(t), \dot{\widetilde{\gamma_q}}(t))}dt = \int_0^1 \sqrt{\dot{\omega}(t)^2 + \sin^2\omega(t)\underbrace{\left(\dot{\varphi}(t)^2 + \sin^2\varphi(t)\dot{\theta}(t)^2\right)}_{=0}}dt$$

$$= \int_0^1 \dot{\omega}(t)dt = \omega = d_{\mathbb{S}^3}(1, q).$$

Now, since $0 < \omega < \frac{\pi}{2}$, we have $2t\omega < \pi$ and hence, we can compute

$$\operatorname{tr}(\mathrm{B}(\widetilde{\gamma_q}(t))) = 2\cos^2(t\omega) - 1 = \cos(2t\omega) \neq -1.$$

Thus, the logarithm $\log(\mathrm{B}(\widetilde{\gamma_q}(t)))$ is well-defined and the Rodrigues formula yields

$$\log(\mathrm{B}(\widetilde{\gamma_q}(t))) = 2t\omega \begin{pmatrix} 0 & -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & 0 & -\cos(\varphi) \\ -\sin(\varphi)\cos(\theta) & \cos(\varphi) & 0 \end{pmatrix} \stackrel{\text{def}}{=} tX_q,$$

so that $\mathrm{B}(\widetilde{\gamma_q}(t)) = e^{tX_q}$. Finally, since $\phi \circ \pi_{\mathcal{Q}_8} = \pi \circ \mathrm{B}$ where $\pi : SO(3) \twoheadrightarrow \mathcal{F}(\mathbb{R})$, we have the result. The statement about uniqueness follows immediately from the fact that $\log(\mathrm{B}(q))$ is uniquely defined and that $SO(3)$ is a geodesic-orbit space. □

Recall that in the Section **??**, we have denoted $\widehat{\mathscr{D}_{\mathcal{O}}} := \bigcup_i \widehat{\Delta_i}$ and $\widehat{\mathscr{F}_{\mathcal{O}}} := \mathrm{pr}(\widehat{\mathscr{D}_{\mathcal{O}}})$.

Corollary 4.2. *For every $q \in \widehat{\mathscr{F}_{\mathcal{O}}}$, the logarithm $\log(\mathrm{B}(q)) \in \mathfrak{so}(3)$ is well-defined and the curve $t \mapsto \exp(t \log \mathrm{B}(q))$ is the only minimal geodesic in $SO(3)$ from 1 to $\mathrm{B}(q)$. Furthermore, its projection γ_q is a geodesic in $\mathcal{F}(\mathbb{R})$.*

Proof. In view of Proposition 4.1, we only have to prove that $\Re(q) > 0$, because in this case we will have $\omega_q = \arccos(\Re(q)) < \frac{\pi}{2}$. Hence, we have to prove that for $1 \leq i \leq 6$ and for $x = (x_1, x_2, x_3, x_4) \in \widehat{\Delta_i}$, we have $x_1 > 0$; given that the $\widehat{\Delta_i}$'s are defined as convex hulls, it suffices to show that their vertices have positive first coordinates. But since these vertices are among

$$\left\{ \frac{1}{2}(1, \pm 1, \pm 1, \pm 1), \frac{1}{\sqrt{2}}(1, -1, 0, 0), \frac{1}{\sqrt{2}}(1, 0, -1, 0), \frac{1}{\sqrt{2}}(1, 0, 0, -1) \right\},$$

the result is now clear. □

5. The cells of the $\mathfrak{S}_3$-equivariant cellular structure of $\mathcal{F}(\mathbb{R})$ as unions of open geodesics

We shall now describe the cells in $\mathcal{F}(\mathbb{R})$ from Theorem 3.4.6 of [CGS20] as unions of images of geodesics in $\mathcal{F}(\mathbb{R})$, with respect to the quaternionic metric $\overline{g}^8$. First, we briefly recall the curved join construction. Given two points x and $y \neq -x$ in $\mathbb{S}^3$, we write $x * y$ to denote the image $\gamma_{x,y}([0, 1])$ of the unique minimal geodesic $\gamma_{x,y} : [0, 1] \to \mathbb{S}^3$ joining them. The resulting curve is called the *curved join* of x and y. Also, $x \overset{\circ}{*} y$ denotes the image $\gamma_{x,y}(]0, 1[)$ that is, the image of the geodesic $\gamma_{x,y}$ with endpoints removed. We can extend the curved join to subsets of $\mathbb{S}^3$: if $U, V \subset \mathbb{S}^3$ are such that $U \cap (-V) = \emptyset$, then we can define

$$U * V := \bigcup_{\substack{u \in U \\ v \in V}} u * v.$$

This is easily seen to be associative on subsets. We may also define $U \overset{\circ}{*} V := \bigcup_{u,v} u \overset{\circ}{*} v$.

We take some notation. If $q \in \mathbb{S}^3$ with $\operatorname{tr}(q) > 0$, recall the unique geodesic $\widetilde{\gamma_q}$ from 1 to q on $\mathbb{S}^3$ and its image $\gamma_q := \phi \circ \pi_{\mathcal{Q}_8} \circ \widetilde{\gamma_q}$ on $\mathcal{F}(\mathbb{R})$ defined by $\gamma_q(t) = \exp(t \log(\mathrm{B}(q)))S(O(1)^3)$. We shall denote by $\Gamma_q := \gamma_q(]0, 1[)$ the image of the *open* geodesic $(\gamma_q)_{]0,1[}$. Next, for $u \neq v \in \{i, j, k\}$, let

$$e_v^u := \bigcup_{q \in \tau_u \overset{\circ}{*} \omega_v} \widetilde{\gamma_{q^{-1}}}(]0, 1[) \quad \text{and} \quad e^{uv} := \bigcup_{q \in \tau_u \overset{\circ}{*} \tau_v} \widetilde{\gamma_{q^{-1}}}(]0, 1[),$$

as well as

$$\underline{e}_v^u := \phi(\pi_{\mathcal{Q}_8}(e_v^u)) = \bigcup_{q \in \tau_u \overset{\circ}{*} \omega_v} \Gamma_{q^{-1}} \quad \text{and} \quad \underline{e}^{uv} := \phi(\pi_{\mathcal{Q}_8}(e^{uv})) = \bigcup_{q \in \tau_u \overset{\circ}{*} \tau_v} \Gamma_{q^{-1}}.$$

Note that we may of course define also, for $u \in \{i,j,k\}$,

$$e_0^u := \bigcup_{q \in \tau_u \overset{\circ}{*} \omega_0} \widetilde{\gamma_{q^{-1}}}(]0,1[) \quad \text{and} \quad \underline{e}_0^u := \phi(\pi_{\mathcal{Q}_8}(e_0^u)) = \bigcup_{q \in \tau_u \overset{\circ}{*} \omega_0} \Gamma_{q^{-1}}.$$

With this notation we can determine the images $\widetilde{\Delta_i} := \phi(\pi_{\mathcal{Q}_8}(\widehat{\Delta_i}))$ as

$$\begin{cases} \widetilde{\Delta_1} = \bigcup_{q \in e_j^k} \Gamma_{q\tau_j}, \\ \widetilde{\Delta_2} = \bigcup_{q \in e_k^i} \Gamma_{q\tau_k}, \\ \widetilde{\Delta_3} = \bigcup_{q \in e_i^j} \Gamma_{q\tau_i}, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{\Delta_4} = \bigcup_{q \in e_j^i} \Gamma_{q\tau_j}, \\ \widetilde{\Delta_5} = \bigcup_{q \in e_k^j} \Gamma_{q\tau_k}, \\ \widetilde{\Delta_6} = \bigcup_{q \in e_i^k} \Gamma_{q\tau_i}. \end{cases}$$

Remark 5.1. *We have used quaternions to define these subsets, however, it should be remarked that one can write them using only the exponential. For instance, one has*

$$\underline{e}_j^i = \bigcup_{q \in \tau_i \overset{\circ}{*} \omega_j} \Gamma_{q^{-1}}$$

$$= \left\{ \exp\left(\frac{2s \arccos\left(\frac{\cos\frac{t\pi}{4} + \sin\frac{t\pi}{4}}{2} \right)}{\sqrt{3 - \sin\frac{t\pi}{2}}} \begin{pmatrix} 0 & \sqrt{1-\sin\frac{t\pi}{2}} & -\cos\frac{t\pi}{4} - \sin\frac{t\pi}{4} \\ -\sqrt{1-\sin\frac{t\pi}{2}} & 0 & \sin\frac{t\pi}{4} - \cos\frac{t\pi}{4} \\ \cos\frac{t\pi}{4} + \sin\frac{t\pi}{4} & \cos\frac{t\pi}{4} - \sin\frac{t\pi}{4} & 0 \end{pmatrix} \right) S(O(1)^3),\ 0 < s,t < 1 \right\}$$

To see this, first notice that

$$\underline{e}_j^i = \bigcup_{q \in \tau_j \overset{\circ}{*} \omega_i} \Gamma_{q^{-1}} = \bigcup_{q \in \tau_j^{-1} \overset{\circ}{*} \omega_i^{-1}} \Gamma_q = \bigcup_{q \in (1 \overset{\circ}{*} \tau_i^{-1}) \cdot \omega_i^{-1}} \Gamma_q = \bigcup_{q \in \operatorname{im} \widetilde{\gamma_{\tau_i^{-1}}}^{\circ}} \Gamma_{q\omega_i^{-1}}.$$

But, one has that $q \in \operatorname{im}(\widetilde{\gamma_{\tau_i^{-1}}}^{\circ})$ *if there exists* $0 < t < 1$ *such that* $q = \cos\frac{t\pi}{4} - i\sin\frac{t\pi}{4}$. *To simplify notations, denote* $c_t := \cos\frac{t\pi}{4}$ *and* $s_t := \sin\frac{t\pi}{4}$. *Then, one has*

$$q\omega_i^{-1} = \frac{(c_t + s_t) + i(c_t - s_t) - j(c_t + s_t) - k(c_t - s_t)}{2}$$
$$= \cos(\omega_t) + i\sin(\omega_t)\cos(\varphi_t) + j\sin(\omega_t)\sin(\varphi_t)\cos(\theta_t) + k\sin(\omega_t)\sin(\varphi_t)\sin(\theta_t),$$

where

$$\omega_t = \arccos\left(\frac{c_t + s_t}{2}\right), \quad \varphi_t = \arccos\left(\frac{c_t - s_t}{\sqrt{3 - \sin\frac{t\pi}{2}}}\right), \quad \theta_t = \arccos\left(\frac{c_t + s_t}{\sqrt{2}}\right) - \pi.$$

Now, we have that $z \in \Gamma_{q\omega_i^{-1}} = \operatorname{im} \gamma_{q\omega_i^{-1}}^{\circ}$ *if there exists* $0 < s < 1$ *such that* $z = e^{sX_{q\omega_i^{-1}}} \cdot S(O(1)^3)$ *and since we have*

$$X_{q\omega_i^{-1}} = 2\omega_t \begin{pmatrix} 0 & -\sin(\varphi_t)\sin(\theta_t) & \sin(\varphi_t)\cos(\theta_t) \\ \sin(\varphi_t)\sin(\theta_t) & 0 & -\cos(\varphi_t) \\ -\sin(\varphi_t)\cos(\theta_t) & \cos(\varphi_t) & 0 \end{pmatrix}$$
$$= \frac{2\arccos\left(\frac{c_t + s_t}{2}\right)}{\sqrt{3 - \sin\frac{t\pi}{2}}} \begin{pmatrix} 0 & \sqrt{1-\sin\frac{t\pi}{2}} & -c_t - s_t \\ -\sqrt{1-\sin\frac{t\pi}{2}} & 0 & s_t - c_t \\ c_t + s_t & c_t - s_t & 0 \end{pmatrix}$$

and we find indeed the announced description.

We are now in a position to state the main result:

Theorem 5.2. *With the above notation, the real flag manifold $\mathcal{F}(\mathbb{R}) = SO(3)/S(O(1)^3)$ admits an $\mathfrak{S}_3$-equivariant cellular decomposition with orbit representatives cells given by*

$$\mathfrak{e}^0 := \{1 \cdot S(O(1)^3)\},$$
$$\mathfrak{e}^1_1 := \Gamma_{\tau_i^{-1}}, \quad \mathfrak{e}^1_2 := \Gamma_{\tau_j^{-1}}, \quad \mathfrak{e}^1_3 := \Gamma_{\tau_k^{-1}},$$

and

$$\mathfrak{e}^2_1 := \bigcup_{q \in (\tau_i \overset{\circ}{*} \omega_k) \cup (\omega_k \overset{\circ}{*} \tau_j)} \Gamma_{q^{-1}}, \quad \mathfrak{e}^2_2 := \bigcup_{q \in (\tau_j \overset{\circ}{*} \omega_i) \cup (\omega_i \overset{\circ}{*} \tau_k)} \Gamma_{q^{-1}}, \quad \mathfrak{e}^2_3 := \bigcup_{q \in (\tau_k \overset{\circ}{*} \omega_j) \cup (\omega_j \overset{\circ}{*} \tau_i)} \Gamma_{q^{-1}},$$

as well as

$$\mathfrak{e}^3 := \widetilde{\Delta_4} \cup \underline{e}^{ij} \cup \widetilde{\Delta_1} \cup \underline{e}^j_0 \cup \widetilde{\Delta_2} \cup \underline{e}^{jk} \cup \widetilde{\Delta_5} \cup \underline{e}^k_0 \cup \widetilde{\Delta_3} \cup \underline{e}^{ki} \cup \widetilde{\Delta_6} \cup \underline{e}^i_0.$$

Moreover, the closures of the 1-cells $\mathfrak{e}^1_j$ are minimal geodesics from $\mathfrak{e}^0$ to $\mathfrak{e}^0 \cdot s_\beta$, $\mathfrak{e}^0 \cdot w_0$ and $\mathfrak{e}^0 \cdot s_\alpha$, respectively.

Proof. We just have to check that, if $\widehat{e}^i_j$ is a cell of the analogue of the cellular decomposition provided by the Theorem **??** for the action of $\mathcal{O}$ on $\mathbb{S}^3$ by multiplication on the *right*, then one has $\phi \circ \pi_{\mathcal{Q}_8}(\widehat{e}^i_j) = \mathfrak{e}^i_j$, in other words,

$$\mathfrak{e}^i_j = \phi\left(\pi_{\mathcal{Q}_8}\left((e^i_j)^{-1}\right)\right),$$

but we have defined the cells $\mathfrak{e}^i_j$ in this way.

Next, take for instance the closure $\overline{\mathfrak{e}^1_1} = \gamma_{\tau_i^{-1}}([0,1])$, the other two being treated in the same way. By the Corollary 4.2, $\gamma_{\tau_i^{-1}}$ is a geodesic in $\mathcal{F}(\mathbb{R})$ and by the Corollary 3.3, we have $d_{\overline{g}^8}(1, \phi(\overline{\tau_i^{-1}})) = \min\left(\arccos\left(\pm\frac{\sqrt{2}}{2}\right), \arccos(0)\right) = \frac{\pi}{4}$. Thus, we have to show that $\mathrm{L}(\gamma_{\tau_i^{-1}}) = \frac{\pi}{4}$. But since $\gamma_{\tau_i^{-1}}(1) = \pi_{\mathcal{Q}_8}(\tau_i^{-1}) = s_\beta$ and

$$\log(\dot{s_\beta}) = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

by the Proposition 2.1, we get $\mathrm{L}(\gamma_{\tau_i^{-1}}) = \frac{1}{2\sqrt{2}}\|\log(\dot{s_\beta})\|_F = \frac{\pi}{4} = d_{\overline{g}^8}(1, s_\beta)$, as required. □

Remark 5.3. *We can also describe more explicitly the 1-cells as*

$$\mathfrak{e}^1_1 = \left\{\exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{t\pi}{2} \\ 0 & \frac{t\pi}{2} & 0 \end{pmatrix} S(O(1)^3),\ 0 < t < 1\right\} = \left\{e^{\frac{t\pi}{2}u_\beta} \cdot S(O(1)^3),\ 0 < t < 1\right\},$$

$$\mathfrak{e}^1_2 = \left\{\exp\begin{pmatrix} 0 & 0 & -\frac{t\pi}{2} \\ 0 & 0 & 0 \\ \frac{t\pi}{2} & 0 & 0 \end{pmatrix} S(O(1)^3),\ 0 < t < 1\right\} = \left\{e^{\frac{t\pi}{2}u_{\alpha+\beta}} \cdot S(O(1)^3),\ 0 < t < 1\right\},$$

$$\mathfrak{e}^1_3 = \left\{\exp\begin{pmatrix} 0 & -\frac{t\pi}{2} & 0 \\ \frac{t\pi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S(O(1)^3),\ 0 < t < 1\right\} = \left\{e^{\frac{t\pi}{2}u_\alpha} \cdot S(O(1)^3),\ 0 < t < 1\right\}.$$

Notice that the closure $\overline{\mathfrak{e}^3}$ is a fundamental domain for $\mathfrak{S}_3$ acting on $\mathcal{F}(\mathbb{R})$.

References

[Bes87] A. L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1987.

[Bum13] D. Bump. *Lie groups*, volume 225 of *Graduate texts in Mathematics*. Springer, second edition edition, 2013.

[CGS20] R. Chirivi, A. Garnier, and M. Spreafico. Cellularization for exceptional spherical space forms and application to flag manifolds, 2020.

[CL10] J. R. Cardoso and F. Silva Leite. Exponentials of skew-symmetric matrices and logarithm of orthogonal matrices. *J. of comput. and applied math.*, 233(11):2867–2875, 2010.

[GHL04] D. Hulin, S. Gallot and J. Lafontaine. *Riemannian geometry*. Springer-Verlag, third edition edition, 2004.

[Lee] J. M. Lee. *Introduction to Riemannian manifolds*, volume 176 of *Graduate texts in Mathematics*. Springer-Verlag, 2nd ed. edition, 2018.

[Mat73] T. Matumoto. Equivariant cohomology theories on g-cw complexes. *Osaka J. Math.*, 10:51–68, 1973.
[PS97] J-S. Park and Y. Sakane. Invariant einstein metrics on certain homogeneous spaces. *Tokyo J. Math.*, 20(1), 1997.
[Sak99] Y. Sakane. Homogeneous einstein metrics on flag manifolds. *Lobachevskii J. Math.*, 4:71–87, 1999.
[YWL19] K. Ye, K. S.-W. Wong, and L.-H. Lim. Optimization on flag manifolds, 2019.

Université de Picardie,
Département de Mathématiques et LAMFA (UMR 7352 du CNRS),
33 rue St Leu,
F-80039 Amiens Cedex 1,
France
Email address: arthur.garnier@u-picardie.fr