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1 **Effects of smooth divergence-free flows on tracer gradients and**
2 **spectra: Eulerian prognosis description**

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ABSTRACT

10 Ocean eddies play an important role in the transport of heat, salt, nutrients or pollutants.
11 During a finite-time advection, the gradients of these tracers can increase or decrease, de-
12 pending on a growth rate and the angle between flow gradients and initial tracer gradients.
13 The growth rate is directly related to finite-time Lyapunov exponents. Numerous studies on
14 mixing and/or tracer downscaling methods rely on satellite altimeter-derived ocean veloc-
15 ities. Filtering most oceanic small-scale eddies, those resulting smooth Eulerian velocities
16 are often stationary during the characteristic time of tracer gradient growth. While smooth,
17 these velocity fields are still locally misaligned, and thus uncorrelated, to many coarse-scale
18 tracers observations amenable to downscaling (e.g. SST, SSS). Using finite-time advectons,
19 the averaged squared norm of tracer gradients can then only increase, with local growth rate
20 independent of the initial coarse-scale tracer distribution. The key mixing processes are then
21 only governed by locally uniform shears and foldings around stationary convective cells. To
22 predict the tracer deformations and the evolution of their 2nd-order statistics, an efficient
23 proxy is proposed. Applied to a single velocity snapshot, this proxy extends the Okubo-
24 Weiss criterion. For the Lagrangian-advection-based downscaling methods, it successfully
25 predicts the evolution of tracer spectral energy density after a finite time, and the optimal
26 time to stop the downscaling operation. A practical estimation can then be proposed to
27 define an effective parameterization of the horizontal eddy diffusivity.

28 *Significance statement.* An analytical formalism is adopted to derive new exact and ap-
29 proximate relations that express the clustering of tracers transported by upper ocean flows.
30 This formalism bridges previous Eulerian and Lagrangian approaches. Accordingly, for slow
31 and smooth upper ocean flows, a rapid prognosis estimate can solely be performed using
32 single-time velocity field observations. Well suited to satellite-altimeter measurements, it
33 will help rapidly identify and monitor mixing regions occurring in the vicinity of ocean eddy
34 boundaries.

35 **1. Introduction**

36 Since the first images from space, the attention of both theoreticians and remote sensing
37 scientists has been triggered by the abundance of various ocean tracer patterns and signa-
38 tures in the mesoscale and sub-mesoscale (1-50 km) ranges (e.g. Gower et al. 1980; Lesieur
39 and Sardouy 1981). From precise satellite measurements of the ocean topography and its
40 related dynamics, coherent eddies have since been identified to stretch and fold tracers, lead-
41 ing to generate often very spectacular upper ocean intricate tracer distributions. Nowadays,
42 combined satellite altimeter measurements satisfactorily detail the large-scale ocean dynam-
43 ics (Klein et al. 2019). But the ocean’s mesoscale (10-100 km) and submesoscale (< 10
44 km) variability and energy are still challenging to map with conventional radar altimeters.
45 Indeed, the narrow illuminated swath of each instrument precludes precise mapping, regard-
46 less of the orbital configuration (Dufau et al. 2016). To date, global direct quantification of
47 horizontal dispersion and mixing at such scales is thus not available.

48 Nonetheless, a now-common strategy is to derive small-scale tracer structures and so-called
49 Lagrangian coherent structures, from the available smooth altimeter-derived velocities (e.g.
50 Price et al. 2006; Lehahn et al. 2007). Indeed, using a Lagrangian-dynamical framework, an

51 initial larger-scale tracer field can be advected on higher-resolution grids, generating much
52 smaller-scale patterns (Aref 1984; Pierrehumbert and Yang 1993). Typical moderate to
53 large-scale ocean cyclonic and anti-cyclonic eddies trap and advect fluid parcels over weeks
54 to months. As pictured, with time, these fluid parcels with different origins, temperature,
55 salinity and possibly different biogeochemical properties and/or contaminant loadings, come
56 closer, to sharpen fronts but also to possibly dilute their properties, and promote transfor-
57 mative chemical reactions. Stirring effect first characterizes the development of elongated
58 structures, well illustrated by Welander (1955) (see its Fig. 2), using a simple velocity field
59 to produce spectacular distortions. Initial patches, small compared to the length scale of
60 the deforming flow field, become subject to translation, rotation and shearing. With time,
61 deformation is significant. Increasingly long and thin filaments wrap around the eddy, and
62 possibly fold. Folds appear where the velocity gradient is perpendicular to the stream direc-
63 tion. Accordingly, at a given scale of observation, mixing can be associated with processes
64 that act to minimize filament thinning and dilute sharp differences (gradients). The result-
65 ing deformation of tracer isolines are thus associated to strengthening or weakening of small
66 scales structures, to be also captured in the high-wavenumber part of the tracer spectra.

67 In this paper, the motivation is first to present an analytical framework to derive exact and
68 approximate results for the evolution of tracer gradients after a finite-time advection. After
69 a long-time advection by a smooth and slowly-varying incompressible flow, the expected
70 growth of passive tracer gradients can also be theoretically obtained, and subsequently the
71 related evolution of tracer high-wavenumber spectra. Analytically, it is demonstrated that
72 local and global stretching and folding properties can be diagnosed without time-integration.
73 From a practical point of view, only a single snapshot of a velocity field is required. This
74 result provides a convenient diagnosis that fully applies to estimated smooth velocities from

75 altimeter-derived sea surface height (SSH) measurements. The Eulerian prognosis descrip-
76 tion then explains how an initial tracer field, sea surface temperature (SST) or salinity (SSS),
77 must be low-pass filtered in forward-backward Lagrangian-advection operations (Rogé et al.
78 2015). Accordingly, the time of advection and the low-pass filter bandwidth are directly
79 linked. Following this development, an exact relation can further be determined to provide
80 more rigorous constrains to the heuristic choices used in Dencausse et al. (2014) and Rogé
81 et al. (2015). This can also be compared to estimates inferred from the knowledge of the
82 Rossby deformation radius or the mean squared vorticity (Berti and Lapeyre 2014).

83 In section 2, we recall and propose exact theoretical results to study tracer mixing. Section
84 3 focuses in our case study : downscaling and mixing analyses with coarse-scale tracers and
85 surface current observations. Associated stretching and folding diagnostics are derived, and
86 folding and shearing time defined. Besides, the proposed analysis also conveniently provides
87 further understandings to identify regions with motions either dominated by rotation or
88 by stretching where two points become closer or diverge. Mixing can then occur when
89 folding is associated with stretching effects to strongly strengthen tracer gradients. In section
90 4, the evolution of the tracer high-wavenumber spectral tail is presented. The norm of
91 the averaged tracer gradient is shown to control the evolution of the tracer spectral tail.
92 Eulerian descriptors are then proposed to monitor the aforementioned Lagrangian advection
93 downscaling methodology. Based on these proposed developments, a practical estimation of
94 the horizontal diffusivity is derived to help constrain subgrid parameterizations of large-scale
95 flow simulations. In section 5, numerical experiments are used to illustrate these analytical
96 developments. Analysis are also performed using altimeter-derived smooth ocean velocities.
97 Conclusion follows, section 5.

98 **2. Exact mixing properties**

99 Hereafter, exact results are derived to describe the evolution of the average of the gradient
 100 squared norm of an advected tracer T :

$$\overline{\|\nabla T\|^2}, \quad (2.1)$$

101 where the averaging operator $\bar{\bullet}$ is defined for every function q as $\bar{q} = \frac{1}{S} \int_{\Omega} q$ with integration
 102 over the two-dimensional spatial domain Ω of finite area S .

103 *a. Stretching and the Cauchy-Green tensor*

104 Given a two-dimensional velocity field \mathbf{v} , the flow ϕ – also called Lagrangian displacement
 105 – is defined as:

$$\phi(\mathbf{x}_0) = \phi(\mathbf{x}_0, t) = \mathbf{x}_0 + \int_0^t dt' \mathbf{v}(\phi(\mathbf{x}_0, t'), t'). \quad (2.2)$$

106 For a divergence-free velocity, $\nabla \cdot \mathbf{v} = 0$, we have $\det(\nabla \phi^T) = 1$, where $\nabla \phi^T$ is the spa-
 107 tial gradient tensor of the flow. Subsequently, the right Cauchy-Green deformation tensor,
 108 $\nabla \phi^T (\nabla \phi^T)^T$, and its inverse, shall have two real and identical strictly positive eigenvalues.
 109 Only the stable direction, corresponding to the eigenvector associated with the eigenvalue
 110 smaller than 1, and the unstable direction, corresponding to the eigenvector associated with
 111 the eigenvalue larger than 1, are switched. Along the stable (resp. unstable) direction,
 112 the distance between two points decreases (resp. increases). More details of this classical
 113 analysis of the Cauchy-Green tensor is recalled in Appendix A1.

114 From there, we derive in Appendix A2 the new following compact expression of the trans-
 115 ported tracer gradient, ∇T as a function of the initial gradient field, ∇T_0 , and the flow,

116 ϕ :

$$\overline{\|\nabla T\|^2} - \overline{\|\nabla T_0\|^2} = \overline{\|\nabla T_0\|^2} \alpha^2 \left(1 + \frac{\beta}{\alpha} \cos\left(2\theta_{T_0}^\phi\right)\right), \quad (2.3)$$

117 where

$$\alpha^2 = \frac{1}{2}\|\nabla\phi^T\|^2 - 1 \geq 0, \quad \beta^2 = \alpha^2 + 2, \quad (2.4)$$

118 using the Frobenius matrix norm, and $\theta_{T_0}^\phi$ stands for the angle between the tracer gradient
 119 and the compressive (stable) direction of the direct flow. The Cauchy-Green tensor and the
 120 initial tracer gradient completely determine the averaged squared norm of advected tracer
 121 gradients. The advection acts to globally increase (decrease) the tracer gradient norm if
 122 the initial tracer gradient is locally close enough to the stable (unstable) direction of the
 123 direct flow. This corresponds to $\theta_{T_0}^\phi$ close to 0 modulo π or $\frac{\pi}{2}$ modulo π , respectively. This
 124 is modulated by the initial amplitude of the tracer gradients, a growth rate α^2 and a factor
 125 $\frac{\beta}{\alpha} = \sqrt{1 + \frac{2}{\alpha^2}} \geq 1$. Note, α and β do not explicitly depend on the tracer.

126 The largest Finite Time Lyapunov Exponents (FTLE) (Haller and Yuan 2000; Thiffeault
 127 and Boozer 2001; Haller 2005; Haller and Sapsis 2011) is $\Lambda = \frac{1}{2t} \log\left(1 + \alpha^2\left(1 + \frac{\beta}{\alpha}\right)\right)$. In
 128 particular, when both the largest FTLE, Λ , and the time, t , are large, the term α^2 is large
 129 and $\beta/\alpha = \sqrt{1 + \frac{2}{\alpha^2}}$ is small, leading to the approximation $\Lambda \approx \frac{1}{t} \log(\alpha)$. Therefore, the
 130 FTLE ridges – often considered as proxies of mixing barriers – coincide with the α ridges.

131 *b. Stretching expressed with mesochronic velocity*

132 Mezić et al. (2010) introduce a mesochronic velocity defined as the velocity, time-averaged
 133 along a trajectory:

$$\check{\mathbf{v}}(\mathbf{x}_0, t) = \begin{pmatrix} \check{u}(\mathbf{x}_0, t) \\ \check{v}(\mathbf{x}_0, t) \end{pmatrix} \triangleq \frac{1}{t} \int_0^t dt' \mathbf{V}(\mathbf{x}_0, t') = \frac{\phi(\mathbf{x}_0, t) - \mathbf{x}_0}{t}, \quad (2.5)$$

134 where \mathbf{V} is the Lagrangian velocity. The authors then separate mesoelliptic areas – areas over
 135 which the tracer gradients turn while keeping their norm unchanged – and mesohyperbolic
 136 areas – areas over which the gradients increase or decrease – depending on the sign of the
 137 following criterion:

$$Q_{\text{Mez}} = \det(\nabla \check{\mathbf{v}}^T) \left(\det(\nabla \check{\mathbf{v}}^T) - \frac{4}{t^2} \right). \quad (2.6)$$

138 Working with the mesochronic velocity $\check{\mathbf{v}}$ or with the flow ϕ is mathematically equivalent,
 139 expressed by the definition (2.5). The above criterion is thus similar in spirit to Cauchy-
 140 Green-tensor-based analyses. Nevertheless, that criterion is frame-dependent unlike Cauchy-
 141 Green-tensor-based metrics like FTLE or the growth rate α^2 (Karrasch 2015; Hadjighasem
 142 et al. 2017). Consequently, the classification of trajectories proposed by Mezić et al. (2010)
 143 will change under changes of a moving observer. We will still use mesochronic notations to
 144 provide deeper insights on mixing in general and on our growth rate α^2 in particular. In
 145 Appendix A3, we rewrite it as:

$$Q_{\text{Mez}} t^2 = \underbrace{(\partial_x \check{u} - \partial_y \check{v})^2 + (\partial_y \check{u} + \partial_x \check{v})^2}_{=2(\alpha/t)^2} - \check{\omega}^2, \quad (2.7)$$

146 where we introduce the mesochronic vorticity $\check{\omega} \triangleq \nabla^\perp \cdot \check{\mathbf{v}}$ and $\nabla^\perp \triangleq (-\partial_y, \partial_x)^T$ the orthog-
 147 onal gradient operator in 2D. Note that the mesochronic vorticity is not the time-averaged
 148 vorticity along a trajectory, and is frame-dependent. The above interpretation (2.7) then
 149 becomes reminiscent to the Okubo-Weiss criterion (Okubo 1970; Weiss 1991; Shivamoggi
 150 and van Heijst 2011). Indeed, it explicits the competition between the strain and the rota-
 151 tion of the mesochronic velocity, encoded by α and $\check{\omega}$, respectively. Moreover, according to
 152 (2.3), mesoelliptic regions must thus be associated with zero growth rate α^2 , and expression

153 (2.7) leads to a $Q_{\text{Mez}} = -(\dot{\omega}/t)^2 \leq 0$. From our proposed approach, the stretching criterion
 154 introduced by Mezić et al. (2010) is thus retrieved with deeper understanding.

155 *c. Folding*

156 Stretching occurs when two points, $\phi(\mathbf{x})$ and $\phi(\mathbf{x} + \delta\mathbf{x})$, become closer or diverge, strength-
 157 ening tracer gradients. This property is again naturally encoded in the Cauchy-Green tensor:

$$\|\phi(\mathbf{x} + \delta\mathbf{x}) - \phi(\mathbf{x})\|^2 \approx \|(\nabla\phi^T)^T \delta\mathbf{x}\|^2 = \delta\mathbf{x}^T \nabla\phi^T (\nabla\phi^T)^T \delta\mathbf{x}. \quad (2.8)$$

158 Mixing can occur when folding is associated with stretching. Folding is thus associated with
 159 a three-points kinematic property. Indeed, at least three points – e.g., $\phi(\mathbf{x})$, $\phi(\mathbf{x} + \delta\mathbf{x})$ and
 160 $\phi(\mathbf{x} - \delta\mathbf{x})$ – are needed to represent a folding. First, the three points are separated by
 161 stretching, creating a filament. Then, the filaments folds bringing the two opposite points
 162 ($\phi(\mathbf{x} + \delta\mathbf{x})$ and $\phi(\mathbf{x} - \delta\mathbf{x})$) closer again. This folding can trap an area having a distinct
 163 tracer value (squeezing) creating strong tracer gradients. The folding is encoded by the
 164 relative evolution of positions increments ($\phi(\mathbf{x} + \delta\mathbf{x}) - \phi(\mathbf{x})$) and ($\phi(\mathbf{x} - \delta\mathbf{x}) - \phi(\mathbf{x})$).

165 In the next section, we will show that folding is a key aspect of tracer gradient strength-
 166 ening, even with stationary Eulerian velocity. Indeed, the minimal requirement for folding
 167 to occur is the non-linearity in space of that velocity field.

168 To identify mixing zones, Mezić et al. (2010) further separate two types of mesohyperbolic-
 169 ity (i.e. stretching): the couples of points which have turned ($\delta\mathbf{x}^T (\phi(\mathbf{x} + \delta\mathbf{x}) - \phi(\mathbf{x})) < 0$)
 170 and the others ($\delta\mathbf{x}^T (\phi(\mathbf{x} + \delta\mathbf{x}) - \phi(\mathbf{x})) > 0$). An area, where both types of mesohyperbol-
 171 icity are present and adjacent, must have been folded, and hence corresponds to a mixing
 172 zone. To separate these two types of mesohyperbolicity, the authors study the – possibly
 173 complex – eigenvalues of the evolution matrix $(\nabla\phi^T)^T$ instead of its singular values. Yet, the

174 evolution matrix and thus the associated mixing criterion is frame-dependent unlike Cauchy-
 175 Green-tensor-based analyses (Karrasch 2015; Hadjighasem et al. 2017). Consequently, the
 176 classification of trajectories, obtained from this method, will change under changes of the
 177 observer.

178 To recall, many mixing diagnostics exist in the literature (Hadjighasem et al. 2017). Most
 179 are stretching proxies. However, few methods exist to diagnose folding and its relation to
 180 stretching. For instance, Ma et al. (2016) directly measure folding of material lines through
 181 an analysis of their curvature variations. In the following, we provide new relationships to
 182 further relate stretching and gradient of the curvature of streamlines in the case of a slowly
 183 varying Eulerian velocity field.

184 3. Approximations for coarse-scale observations

185 a. Decorrelation approximations

186 Over the space, flows encompass several eddies, e.g. flows are not laminar, and the angle
 187 $\theta_{T_0}^\phi$ appearing in (2.3) will take different values. If the flow gradients and the initial tracer
 188 gradients are not locally correlated, i.e. are oriented with various angles over the space, the
 189 variance of $\theta_{T_0}^\phi$ will likely be large. For large enough areas, the distribution of $2\theta_{T_0}^\phi [2\pi]$ over
 190 the space will then become close to an uniform law on $[0, 2\pi]$. In (2.3), the average over the
 191 space of the term $\cos\left(2\theta_{T_0}^\phi\right)$ will then become close to zero, and finally, the absence of salient
 192 alignments between the initial tracer gradient and the flow gradient (i.e. cross-correlations
 193 between the initial tracer and the flow) leads to:

$$\frac{\overline{\|\nabla T\|^2}}{\overline{\|\nabla T_0\|^2}} \approx 1 + \overline{\alpha^2}. \quad (3.1)$$

194 On average, the tracer gradients will thus always increase by stretching. Welander (1955)
195 already well illustrated the process. The tracer – a dye patch or an oil spill introduced at
196 time $t = 0$ – is completely passive. In this case, the tracer and the flow are locally completely
197 misaligned (i.e. uncorrelated in the above sense), and the initial structure of the tracer is
198 quickly stretched and folded to fill a broad range of scales.

199 In contrast, geophysical tracers are generally correlated, i.e. more or less aligned, with
200 some flow-dependent directions. Even passive tracers may be correlated to the flow due to
201 their long lasting patterns induced by past advection history. The effect of those correlations
202 is expressed by the angle $\theta_{T_0}^\phi$ in the right-hand-side integrand of equation (2.3) which can
203 be locally positive or negative. Accordingly, local correlations can restrict or reduce the
204 emergence of locally enhanced strong gradients. Berti and Lapeyre (2014), Dencausse et al.
205 (2014), and Rogé et al. (2015) applied Lagrangian advection method to passive and active
206 tracers – Sea Surface Temperature (SST) or Sea Surface Salinity (SSS) – to possibly recon-
207 struct finer scale ocean tracer patterns. However, submesoscales and also some mesoscales
208 of the initial tracer and of the flow used in these Lagrangian methods are often missing,
209 especially using interpolated ocean products. This is likely to strongly reduce local align-
210 ments. So, (3.1) generally holds, to predict a strengthening of mesoscale and submesoscale
211 tracer gradients, well confirmed in the results of Berti and Lapeyre (2014), Dencausse et al.
212 (2014) and Rogé et al. (2015).

213 *b. Time dependency*

214 In line with the geostrophy assumption used to estimate velocities from satellite sea surface
215 height measurements, the resulting large-scale Eulerian ocean flows are also slowly varying
216 (with characteristic time-scale of about 10 days). The Lagrangian downscaling methods of

217 Dencausse et al. (2014) and Rogé et al. (2015) aimed to consistently down-scale tracer fields,
 218 SST and SSS, respectively. These methods use quasi-stationary large-scale velocity fields
 219 when applying Lagrangian-advection schemes over one or two weeks. For ocean scales of
 220 order 100 km, a typical velocity correlation time is about 1 month. Hence, for such spatial
 221 scales, the flow field can well be assumed stationary. This assumption determines a specific
 222 form for the flow. In particular, the flow is not chaotic (Thiffeault 2004).

223 Furthermore, we will show that for such a flow the time dependency analysis of the mixing
 224 can be separated between two typical classes: open straight streamlines and closed curved
 225 streamlines. For both cases, the growth rate α^2 is proportional to t^2 .

226 1) LOCALLY UNIFORM SHEAR

227 Let us first focus on locally straight streamlines (*i.e.* streamlines with zero curvature). In
 228 such a case, the strengthening of tracer gradients results from a velocity shear, similarly to
 229 usual infinitesimal-time stretching. We denote by x the local axis of the straight streamline
 230 and by $u = \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{v} = \|\mathbf{v}\|$, the velocity component on this direction. The divergence-free
 231 assumption imposes

$$\partial_x u = \nabla \cdot \mathbf{v} = 0. \quad (3.2)$$

232 Since the Eulerian velocity is stationary, the Lagrangian velocity is stationary as well:

$$\frac{d\mathbf{V}}{dt}(\mathbf{x}_0, t) = \frac{d}{dt}(\mathbf{v}(\phi(\mathbf{x}_0, t))) = \left(u \partial_x u \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) (\phi(\mathbf{x}_0, t)) = 0, \quad (3.3)$$

233 and the flow simplifies to

$$\phi(\mathbf{x}_0, t) = \mathbf{x}_0 + \int_0^t dt' \mathbf{V}(\mathbf{x}_0, t') = \mathbf{x}_0 + \mathbf{v}(\mathbf{x}_0)t = \mathbf{x}_0 + \begin{pmatrix} u(\mathbf{x}_0)t \\ 0 \end{pmatrix}. \quad (3.4)$$

234 This so-called ballistic regime is superdiffusive (Vallis 2006; Falkovich et al. 2001). Taking
 235 the gradient of the above expression with the divergence-free condition (3.2), the stretching
 236 rate reads

$$\alpha^2 \triangleq \frac{1}{2} \|\nabla \phi^T\|^2 - 1 = \left(\frac{t}{\tau_s} \right)^2, \quad (3.5)$$

237 with the shearing time τ_s

$$1/\tau_s = \frac{1}{\sqrt{2}} \partial_y u. \quad (3.6)$$

238 For a computation independent of any specific local axis x , we rewrite

$$1/\tau_s = \frac{1}{\sqrt{2}} \frac{\mathbf{v}^\perp}{\|\mathbf{v}^\perp\|} \cdot \nabla \|\mathbf{v}\|, \quad (3.7)$$

239 with \mathbf{v}^\perp the $\frac{\pi}{2}$ rotation of \mathbf{v} .

240 2) STATIONARY CONVECTIVE CELLS

241 Close to rotating eddies, streamlines are often closed or at least curved, and the previ-
 242 ous development cannot be applied. Let us focus on closed streamlines. Since the flow is
 243 incompressible, fluid parcels cannot accumulate. Therefore, those streamlines define loops,
 244 called stationary convective cells (Falkovich et al. 2001), where fluid parcels rotate period-
 245 ically. Accordingly, the flow and thus the Lagrangian velocity are periodic and the flow is
 246 called subdiffusive (Vallis 2006; Falkovich et al. 2001). This geometry can nevertheless create
 247 strong stretching effect in finite-time. Indeed, two concentric closed streamlines can define
 248 Lagrangian loops associated with different rotation periods. Rotations after rotations, a
 249 fluid parcel on the fastest loop will deviate from its initial neighboring parcel on the slowest
 250 loop. This differential rotation thus creates stretching. Moreover, it also induces folding.
 251 A filament distributed perpendicular to streamlines will be deformed by the continuous dif-
 252 ferential rotation. After a finite time, the filament will wrap around the convective cell

253 creating spirals. Lehahn et al. (2007) illustrate a similar process with the action of stable
 254 and unstable manifolds on phytoplankton patches. In the same idea, Haller et al. (2016),
 255 and Haller (2016) propose two Lagrangian mixing diagnoses – a variant of the polar rotation
 256 angle (PRA) and the Lagrangian-averaged vorticity deviation (LAVD) – defining coherent
 257 sets with points having similar rotations.

258 To express the stretching induced by those convective cells, the flow is written as follows:

$$\boldsymbol{\phi}(\mathbf{x}_0) = \boldsymbol{\phi}(\mathbf{x}_0, t) \approx \mathbf{x}_0 + \mathbf{g}(\mathbf{x}_0, f(\mathbf{x}_0)t), \quad (3.8)$$

259 where \mathbf{g} is 1–periodic with respect to its second variable and $f(\mathbf{x}_0)$ is the local temporal
 260 frequency. For a point initially on \mathbf{x}_0 in a closed streamline \mathcal{C} , the trajectory $t \mapsto \boldsymbol{\phi}(\mathbf{x}_0, t)$
 261 runs from \mathbf{x}_0 to \mathbf{x}_0 through a path \mathcal{P} embedded in \mathcal{C} with a temporal period $1/f(\mathbf{x}_0)$ defined
 262 by:

$$\frac{1}{f(\mathbf{x}_0)} = \int_0^{1/f(\mathbf{x}_0)} dt = \int_{\mathcal{P}} \frac{dl}{\|\mathbf{v}\|} = \oint_{\mathcal{C}} \frac{\mathbf{v}}{\|\mathbf{v}\|^2} \cdot d\mathbf{l}. \quad (3.9)$$

263 The last integral only depends on the streamline \mathcal{C} and not on the precise initial condition
 264 \mathbf{x}_0 , and the local frequency inherits from the same invariance. Besides, the points \mathbf{x}_0 and
 265 $\boldsymbol{\phi}(\mathbf{x}_0, t)$ are on the same streamline, and thus:

$$f(\boldsymbol{\phi}(\mathbf{x}_0, t)) = f(\mathbf{x}_0). \quad (3.10)$$

266 This frequency can be approximated by a local angular velocity $\dot{\theta}$, estimated using the
 267 streamline curvature, denoted $1/R$, as:

$$f \approx \frac{\dot{\theta}}{2\pi} \approx \frac{\|\mathbf{v}\|}{2\pi R} = \frac{1}{2\pi} \left[(\mathbf{v} \cdot \nabla) \frac{\mathbf{v}}{\|\mathbf{v}\|} \right] \cdot \frac{\mathbf{v}^\perp}{\|\mathbf{v}^\perp\|}. \quad (3.11)$$

268 In practice, the exact formula (3.9) can be difficult to evaluate numerically, and we will
 269 instead use the above approximation. In the following derivation, we however keep the

270 exact definition (3.9). In particular, we still assume the frequency invariance along the
 271 streamline (3.10). The first coordinate of \mathbf{g} encodes the spatial dependency of the loop
 272 (vectorial) amplitudes. Note that the model (3.8) is very general, only assuming periodicity
 273 of Lagrangian trajectories. It helps to partially decouple flow variations associated with
 274 different streamlines (*i.e.* different local frequencies f) and flow variations associated with
 275 different temporal phase shift along the streamline (*i.e.* different times t). To some extent,
 276 this second type of variation can be understood as different initial conditions in the same
 277 streamline, due to the periodicity assumption. Similar decomposition ideas were proposed
 278 by Thiffeault (2004) for chaotic (non-periodic) flows.

279 In Appendix A4, it is shown that time dependency of the growth rate in the final grid
 280 (points \mathbf{x}) reads:

$$\alpha^2(\phi^{-1}(\mathbf{x}, t), t) = \left(\frac{t}{\tau_f(\mathbf{x})} \right)^2, \quad (3.12)$$

281 with the folding time τ_f

$$1/\tau_f = \frac{\|\nabla f\| \|\mathbf{v}\|}{\sqrt{2}f}. \quad (3.13)$$

282 As (3.1) only involves the spatial average of α^2 , we can further simplify the model by
 283 spatial integration. Indeed, integrating equation (3.12) over a specific domain Ω_f , we obtain
 284 with the variable change defined by the incompressible flow:

$$\int_{\Omega_f} d\mathbf{x}_0 \alpha^2(\mathbf{x}_0, t) = \int_{\phi(\Omega_f)} d\mathbf{x} \alpha^2(\phi^{-1}(\mathbf{x}, t), t) = t^2 \int_{\phi(\Omega_f)} \frac{d\mathbf{x}}{\tau_f^2(\mathbf{x})}. \quad (3.14)$$

285 The subspace Ω_f is a subset of Ω where the concept of wrapping convective cells is relevant.
 286 Since we consider closed streamlines, we assume that $\phi(\Omega_f) = \Omega_f$. This subspace will be
 287 properly defined in the following.

288 3) GLOBAL TIME DEPENDENCY

289 To combine the folding time, τ_f , and the shearing time, τ_s , a local stretching time τ is
 290 defined depending upon the local streamline curvature:

$$\tau(\mathbf{x}_0) \triangleq \begin{cases} \tau_f(\mathbf{x}_0) & \text{if } R(\mathbf{x}_0) \leq \frac{L}{2} \\ \tau_s(\mathbf{x}_0) & \text{if } R(\mathbf{x}_0) > \frac{L}{2} \end{cases}, \quad (3.15)$$

291 where $1/R(\mathbf{x}_0)$ is the streamline curvature on \mathbf{x}_0 , and L the average diameter of a vortex.

292 Following the previous models of shearing and folding, the stretching rate becomes:

$$\alpha = \frac{t}{\tau}. \quad (3.16)$$

293 Where gradients are created by uniform shears, streamlines are straight, the curvature $1/R$
 294 is small and $\tau = \tau_s$, whereas, at locations where gradients are strengthened by wrapping, the
 295 curvature is large and $\tau = \tau_f$.

296 To estimate the average eddy diameter L , a toy-approximation is used to locally define
 297 the velocity:

$$\mathbf{v} = U \begin{pmatrix} \cos\left(\frac{2\pi}{\lambda}x\right) \sin\left(\frac{2\pi}{\lambda}y\right) \\ \sin\left(\frac{2\pi}{\lambda}x\right) \cos\left(\frac{2\pi}{\lambda}y\right) \end{pmatrix}, \quad (3.17)$$

298 and the eddy diameter is identified to the size of the convective cell:

$$L = \frac{\lambda}{2} = \left(\frac{6\pi^2 \overline{\|\mathbf{v}\|^2}}{\overline{\|\nabla \mathbf{v}^T\|^2}} \right)^{1/2}. \quad (3.18)$$

299 We shall then use this diameter estimator in the general case. The condition (3.15) further
 300 defines a space partition $\Omega = \Omega_f \cup \Omega_s$ to integrate the growth rate α^2 :

$$\overline{\alpha^2} = \left(\frac{t}{\tau_G} \right)^2 \quad \text{with} \quad \frac{1}{\tau_G^2} \triangleq \overline{\left(\frac{1}{\tau^2} \right)} = \frac{1}{S} \left(\int_{\Omega_f} \frac{d\mathbf{x}}{\tau_f^2(\mathbf{x})} + \int_{\Omega_s} \frac{d\mathbf{x}_0}{\tau_s^2(\mathbf{x}_0)} \right), \quad (3.19)$$

301 where $\Omega_f \triangleq \{\mathbf{x} \in \Omega | R(\mathbf{x}) \leq \frac{L}{2}\}$ and $\Omega_s \triangleq \{\mathbf{x}_0 \in \Omega | R(\mathbf{x}_0) > \frac{L}{2}\}$. Again, $\phi(\Omega_f) = \Omega_f$ is
 302 assumed because the flow maps closed streamlines onto themselves. In the following, τ_G is
 303 referred as the global stretching time.

304 The model (3.19) together with the folding and shearing time definitions (3.13)-(3.7) thus
 305 specifies a global Eulerian estimate of finite-time stretching. Unlike usual Lagrangian diag-
 306 nosis, such as FTLE and Finite Size Lyapunov Exponents (FSLE) (d’Ovidio et al. 2009),
 307 the proposed global model does not require any integration of the flow. Finally, according
 308 to (3.1), the evolution law (3.19) determines the tracer gradient norm:

$$\frac{\|\nabla T\|^2}{\|\nabla T_0\|^2} \approx 1 + \left(\frac{t}{\tau_G}\right)^2. \quad (3.20)$$

309 4. Tracer spectral tail

310 It has long been realized (Batchelor 1959) that the general increase of gradients of T ,
 311 during the stirring action of an underlying flow field, is a consequence of local misalign-
 312 ments between tracer isolines and the velocity vector field, leading to a transfer of tracer
 313 variance from low wavenumber Fourier components to high ones. Mixing will thus be asso-
 314 ciated to strengthening processes acting on the tracer smallest scales. Mixing shall thus be
 315 characterized in the spectral domain, especially its high-wavenumber part. Overall Eulerian
 316 diagnostics of the tracer gradients norm evolution in the spatial domain must then be related
 317 to spectral diagnostics. In this section, this link is demonstrated.

318 After preliminary results related to tracer moments, we first derive a Gaussian approxima-
 319 tion for the evolution of the spectral tail, assuming spatial smoothness. This approximation
 320 is then applied to initial and advected tracers. Finally, we propose an alternative develop-
 321 ment for self-similar spectra.

322 Because of the incompressibility constraint ($\det(\nabla\phi^T) = 1$), as all scales are assumed to
323 be resolved, and since the molecular diffusion is ineffective on the length and time scales of
324 interest, there are no overlays of fluid parcels and no dilution of their properties. Each fluid
325 parcel conserves its tracer value while it is advected. Therefore, mean and variance, \bar{T} and
326 $\overline{(T - \bar{T})^2}$, are conserved. As a consequence, we will assume without loss of generality that
327 the tracer is centered.

328 *a. Locally smooth scalar approximation*

329 First, let us consider the covariance of a smooth scalar q , for small spatial distance $\|\delta\mathbf{x}\|$.
330 The scalar field q will represent here the initial tracer T_0 or the advected tracer T . We will
331 assume it is twice differentiable, which is a strong assumption on the scalar regularity. Yet,
332 for tracers measured at mesoscales and re-interpolated on a submesoscale spatial grid, this
333 assumption safely applies. Accordingly, the covariance of q – denoted γ_q – is four times
334 differentiable near the origin 0 and its Taylor expansion reads:

$$\gamma_q(\delta\mathbf{x}) \triangleq \frac{1}{S} \int_{\Omega} d\mathbf{x} q(\mathbf{x})q(\mathbf{x} + \delta\mathbf{x}), \quad (4.1)$$

$$= \overline{\|q\|^2} + \frac{1}{2} \delta\mathbf{x}^T \mathbf{H}_{\gamma_q}(0) \delta\mathbf{x} + \underset{\|\delta\mathbf{x}\| \rightarrow 0}{o}(\|\delta\mathbf{x}\|^3), \quad (4.2)$$

$$= \overline{\|q\|^2} \exp\left(-\frac{1}{2} \delta\mathbf{x}^T \left(\frac{-\mathbf{H}_{\gamma_q}(0)}{\overline{\|q\|^2}}\right) \delta\mathbf{x}\right) + \underset{\|\delta\mathbf{x}\| \rightarrow 0}{o}(\|\delta\mathbf{x}\|^3), \quad (4.3)$$

335 where \mathbf{H}_{γ_q} denotes the Hessian of the covariance γ_q . This Gaussian covariance approxi-
336 mation – valid near the origin 0 only – results from the absence of infinitely-small-scale
337 structures in the re-interpolated field q . This approximation is not instructive over longer
338 correlation distances (e.g., mesoscales), which often exhibit physical self-similar structures
339 and gradient singularities. Typically, in an isotropic case and for intermediate values of
340 spatial increments norms $\|\delta\mathbf{x}\|$, we would have $\gamma_q(\delta\mathbf{x}) \approx \overline{\|q\|^2} - C\|\delta\mathbf{x}\|^{\zeta-1}$ with $\zeta < 3$.

341 This would correspond to an omnidirectional spectrum proportional to $\kappa^{-\zeta}$ for intermediate
 342 wavenumbers κ and a diverging tracer gradient variance $\overline{\|\nabla q\|^2} = +\infty$. Here, we focus on
 343 the extremely-local behavior of the re-interpolated tracer, which can be well approximated
 344 by the above Gaussian covariance with $-\overline{\|q\|^2} \mathbf{H}_{\gamma_q}^{-1}(0)$ as squared correlation lengths. Note,
 345 the evolution of the Hessian $\mathbf{H}_{\gamma_q}^{-1}(0)$ is difficult to characterize whereas its trace is simple
 346 and reads $-\overline{\|\nabla q\|^2}$ (which is here finite and well-defined, see Appendix A5). To let this gra-
 347 dient norm explicitly appear instead of the covariance Hessian, we consider tracer statistics
 348 averaged over angles. In Fourier space, the omnidirectional spectrum is defined as follows:

$$\tilde{\Gamma}_q(\kappa) \triangleq \kappa \oint_{[0,2\pi]} d\theta_{\mathbf{k}} |\widehat{q}(\mathbf{k})|^2, \quad (4.4)$$

349 where the hat denotes spatial Fourier transform, $\mathbf{k} = \kappa (\cos(\theta_{\mathbf{k}}) \sin(\theta_{\mathbf{k}}))^T$ is the wavevector
 350 and $\kappa = \|\mathbf{k}\|$ the wavenumber. In the Appendix A5, we show that the local approximation
 351 (4.3) leads to:

$$\tilde{\Gamma}_q(\kappa) \underset{\kappa \rightarrow \infty}{\sim} C_q \exp\left(-\frac{1}{2} L_q^2 \kappa^2\right), \quad (4.5)$$

352 where

$$L_q^2 = \frac{\overline{\|q\|^2}}{\overline{\|\nabla q\|^2}} \text{ and } C_q = 2 \left(\frac{(2\pi)^3 (\overline{\|q\|^2})^3}{\overline{\|\nabla q\|^2}} \right)^{1/2}. \quad (4.6)$$

353 Similarly to the local covariance approximation (4.3), the above result is valid for $\kappa \gg 1/L_q$
 354 or at least $\kappa > 1/L_q$. The absence of infinitely-small-scale structures implies a spectral roll-
 355 off at the highest wave numbers. Equation (4.5) approximates this roll-off by a Gaussian
 356 decay and (4.6) precises the position of that roll-off.

357 *b. Tracer spectral tail evolution*

358 We now apply the tail approximation (4.5) to both the initial tracer, T_0 , and the advected
 359 tracer, T . Here again, T_0 and T are not the real full-scale ocean tracers, but correspond
 360 instead to coarse-scale observations under a fictitious smooth surface current advection.
 361 Because a fine interpolation grid is used for both T_0 and T , the spectrum Gaussian roll-off
 362 approximation (4.6) is valid for both fields and yields:

$$\tilde{\Gamma}_T(\kappa) \underset{\kappa \rightarrow \infty}{\sim} \tilde{\Gamma}_{T_0}(\kappa) \frac{C_T}{C_{T_0}} \exp\left(-\frac{1}{2} (L_T^2 - L_{T_0}^2) \kappa^2\right), \quad (4.7)$$

$$\underset{\kappa \rightarrow \infty}{\sim} \tilde{\Gamma}_{T_0}(\kappa) \left(\frac{\|\nabla T_0\|^2}{\|\nabla T\|^2}\right)^{1/2} \exp\left(\frac{1}{2} \overline{\|T_0\|^2} \left(\frac{1}{\|\nabla T_0\|^2} - \frac{1}{\|\nabla T\|^2}\right) \kappa^2\right), \quad (4.8)$$

363 where the simplification in the last asymptotic equivalence follows from the variance con-
 364 servation. As discussed in section 3.a, if the initial tracer T_0 and the flow display local
 365 misalignments, the tracer gradients strengthen: $\|\nabla T\|^2 > \|\nabla T_0\|^2$ (*i.e.* $L_T < L_{T_0}$) and by
 366 (4.8) the tracer spectral tail raises. Using the estimate (4.17), a final expression is derived:

$$\tilde{\Gamma}_T(\kappa) \underset{\kappa \rightarrow \infty}{\sim} \tilde{\Gamma}_{T_0}(\kappa) \left(1 + \left(\frac{t}{\tau_G}\right)^2\right)^{-1/2} \exp\left(\frac{1}{2} \frac{\overline{\|T_0\|^2}}{\|\nabla T_0\|^2} \frac{\kappa^2}{1 + \left(\frac{\tau_G}{t}\right)^2}\right). \quad (4.9)$$

367 It is thus possible to recover the initial spectrum tail by smoothing the advected tracer, as
 368 empirically noticed by Rogé et al. (2015) in their forward-backward Lagrangian advection
 369 method. Here, (4.9) provides a full parametrization of this low-pass Gaussian filter with
 370 squared length-scale:

$$\frac{\overline{\|T_0\|^2}}{\|\nabla T_0\|^2} \frac{1}{1 + \left(\frac{\tau_G}{t}\right)^2}. \quad (4.10)$$

371 The multiplicative constant of (4.9) decreases with the advection time t . Asymptotically,
 372 we have:

$$\left(\frac{\|\nabla T_0\|^2}{\|\nabla T\|^2}\right)^{1/2} \approx \left(1 + \left(\frac{t}{\tau_G}\right)^2\right)^{-1/2} = \begin{cases} 1 & \text{if } t \ll \tau_G \\ \frac{\tau_G}{t} & \text{if } t \gg \tau_G \end{cases}. \quad (4.11)$$

373 Note that this multiplicative constant does not provide information on large-scale tracer
374 structures because we here rely on spectrum roll-off approximations. This constant is pro-
375 portional to the advected tracer spectrum roll-off amplitude C_T . Since both this amplitude
376 and the squared length scale (4.10) decreases with time, the advected tracer spectrum roll-
377 off continuously moves towards higher wavenumbers and lower spectrum values during the
378 downscaling advection.

379 *c. A practical estimation of an effective horizontal eddy diffusivity*

380 Spectral fall-off of real oceanic tracers being relatively stable, the predicted changes under
381 multiple advection operations, shall thus be compensated. It can be hypothesized to result
382 from the combined antagonist effects of the well resolved, slow-varying and large-scale veloc-
383 ity properties, of the unresolved, fast-varying and likely small-scale velocities, and finally, of
384 smaller-scale tracer structures. In other words, the smoother velocity component will tend
385 to raise the high-wavenumber part of the spectrum (4.9), while the fast-varying velocity and
386 tracer components shall act to balance this rise. This last process can be simply accounted
387 for by considering the introduction of an effective spatially-uniform eddy diffusivity, ν . After
388 an advection of Δt , this effect leads to multiply the spectrum by $\exp(-\nu\Delta t\kappa^2)$. To exactly
389 balance the expected high-wavenumber spectral rise (4.9) during Δt , and thus to keep the
390 resulting advected tracer closer to its initial variance distribution over scales, the effective
391 horizontal eddy diffusivity can thus be defined according to:

$$\nu = \frac{1}{2\tau_G} \frac{\overline{\|T_0\|^2}}{\overline{\|\nabla T_0\|^2}} c(t) \quad \text{with} \quad c(t) = \frac{\frac{\Delta t}{\tau_G}}{\left(1 + \left(\frac{\Delta t}{\tau_G}\right)^2\right)} = \begin{cases} \frac{\Delta t}{\tau_G} & \text{if } \Delta t \ll \tau_G \\ \frac{\tau_G}{\Delta t} & \text{if } \Delta t \gg \tau_G \end{cases} . \quad (4.12)$$

392 For small time time step, Δt , the mixing is superdiffusive, whereas for large time step it
393 is subdiffusive. The superdiffusive regime is the usual ballistic regime observed for small

394 advection time Δt (Vallis 2006; Falkovich et al. 2001) whereas the subdiffusive regime is
 395 less straightforward to understand. A spectrum roll-off – by definition – is concave, possibly
 396 to flatten for large advection time. Still, it cannot be convex. Additional advection steps
 397 eventually get less and less efficient at modifying the evolving form of the spectrum roll-
 398 off. Mathematically, the squared length-scale of the low-pass Gaussian filter (4.10) cannot
 399 decrease less than its asymptotic value $\frac{\|T_0\|^2}{\|\nabla T_0\|^2}$. The eddy diffusivity ν – being given by this
 400 (bounded) squared length-scale by unit of time – gets infinitely small for large times.

401 *d. Self-similar approximation for intermediate wavenumbers*

402 While the Gaussian approximation is useful to link advection and filtering operations,
 403 outside the roll-off and the planetary scales, spectra of geophysical tracer fields are more
 404 likely self-similar. Moreover, one may wish to target specific spectral slopes using the La-
 405 grangian advection method. The following alternative form for the scalar spectrum is hence
 406 now considered:

$$\tilde{\Gamma}_q(\kappa) = \begin{cases} A \left(1 + \frac{\kappa}{\kappa_m}\right)^{-\zeta} & \text{if } \kappa \leq \kappa_\infty \\ 0 & \text{otherwise} \end{cases}, \quad (4.13)$$

407 where κ_∞ is set by the numerical resolution. For intermediate wave numbers $\kappa_m \ll \kappa \ll$
 408 κ_∞ , the spectrum exhibits an inertial range $\tilde{\Gamma}_q(\kappa) \approx A \left(\frac{\kappa}{\kappa_m}\right)^{-\zeta}$. Unlike the previous local
 409 smoothness approximation, this inertial range is not an artifact of the field processing. It is
 410 a physical phenomenon induced by the advection by real oceanic currents, which also exhibit

411 energy scale invariance. Accordingly,

$$L_q^2 = \frac{\overline{\|q\|^2}}{\overline{\|\nabla q\|^2}}, \quad (4.14)$$

$$= \frac{\int_0^{\kappa_\infty} \left(1 + \frac{\kappa}{\kappa_m}\right)^{-\zeta}}{\int_0^{\kappa_\infty} \kappa^2 \left(1 + \frac{\kappa}{\kappa_m}\right)^{-\zeta}}, \quad (4.15)$$

$$= \frac{(\zeta - 2)(\zeta - 3)}{\kappa_m^2 \left(2 - \left(1 + \frac{\kappa_\infty}{\kappa_m}\right)^{1-\zeta} \left((\zeta - 1)(\zeta - 2) \left(\frac{\kappa_\infty}{\kappa_m}\right)^2 + 2(\zeta - 1)\frac{\kappa_\infty}{\kappa_m} + 2 \right) \right)}. \quad (4.16)$$

412 As long as the width of the inertial range, $\kappa_\infty - \kappa_m$, is large enough, the above function is
 413 strictly positive and continuous w.r.t. the spectral slope, ζ , for all $\zeta > 1$.

414 Setting q to the advected tracer ($q = T$), the wavenumber κ_m can encompass planetary
 415 length scales which does not vary much during the advection process. The resolution, κ_∞ ,
 416 is constant as well. So, a targeted spectral slope, ζ , conveniently provides a length scale L_T
 417 to be reached over a given advection time. Using (4.17), it can be estimated

$$t = \tau_G \sqrt{\frac{1}{L_T^2(\zeta)} \frac{\overline{\|T_0\|^2}}{\overline{\|\nabla T_0\|^2}} - 1}. \quad (4.17)$$

418 5. Numerical results

419 a. Illustrative toy-model

420 These analytical developments can be first illustrated using a simplified toy-model. We
 421 define an ellipsoidal eddy from the following vorticity field:

$$\omega(\mathbf{x}) = A_\omega \exp\left(-\frac{1}{2} \left(\frac{\|\mathbf{x}\|_e - r_0}{r_\omega}\right)^2 \mathbf{1}_{\{\|\mathbf{x}\|_e > r_0\}}\right), \quad (5.1)$$

422 with

$$\|\mathbf{x}\|_e^2 = (e(x + y))^2 + (x - y)^2, \quad (5.2)$$

423 an eccentricity $e = 1.7$, $r_0 = 23.0$ km, $r_\omega = 76.8$ km and $A_\omega = 6.43 \times 10^{-6} \text{s}^{-1}$. The vorticity
 424 is constant at the ellipse center ($\|\mathbf{x}\|_e \leq r_0$) and smoothly decreases to zero outside.

425 A large tracer filament is advected by a stationary velocity field using a backward La-
 426 grangian advection (Figure 1). For technical details, we refer to Berti and Lapeyre (2014)
 427 and Dencausse et al. (2014). The tracer progressively wraps, eventually creating infinitely
 428 long filaments. Following (3.19), an estimate of the global stretching time is 13.36 days. It
 429 roughly corresponds to half a rotation.

430 Figure 2 represents the spatial distribution of the time-normalized stretching rate $(\alpha/t)^2$,
 431 the factor β/α and the mesochronic vorticity $\check{\omega}$ at several times. As found, the spatial
 432 distribution of α/t becomes nearly constant after one week only. This number is significant
 433 on the folding area, i.e. the border of the vortex. The ratio $\beta/\alpha = \sqrt{1 + \frac{2}{(\alpha/t)^2} \frac{1}{t^2}}$ – which
 434 quantifies the significance of the orientation of tracer gradient – decreases with time in the
 435 mixing area. It stabilizes to its minimum value, say 1, at $t \approx \tau_G$. The mesochronic vorticity
 436 is first concentrated in the center of the vortex. Then, after each global stretching time, a
 437 new ring of mesochronic vorticity adds to the mixing area.

438 Figure 3 displays the spatial distribution of the squared inverse of the folding time, τ_f , of
 439 the shearing time, τ_s , and of the stretching time, τ , for this toy model. Folding and stretching
 440 time are represented both in the initial grid (\mathbf{x}_0) and in the advected grid ($\mathbf{x} = \phi(\mathbf{x}_0, t)$).
 441 This remapping on the initial grid is needed as the folding time is locally defined in the
 442 advected grid (see (3.12) and (3.14)). For this remapping, we integrated the forward flow
 443 $\mathbf{x}_0 \mapsto \phi(\mathbf{x}_0, t)$. The remapping provides a better visualization of the stretching spatial
 444 distribution, but is not necessary for the global stretching time computation (3.19). For this
 445 toy-model, the folding effects are dominant, and the inverse folding time well captures the
 446 spatial structure of α/t . Yet, the inverse folding time diverges outside of the vortex where

447 the streamline curvatures tend to zero. Indeed, according to (3.11) zero curvature implies
448 zero local frequency f , and thus infinite folding time (see (3.13)). For such a weak curvature,
449 the relevant model is the uniform shear. Following the space partition (3.15), the stretching
450 time is chosen as a shearing time in these areas. The global time evolution models for the
451 averaged stretching rate (3.19) and for the tracer gradients (4.17) are also successfully tested
452 in Figure 4.

453 The spectral roll-off proxy (4.9) is illustrated for the toy flow in Figure 5. The local
454 Gaussian approximation successfully captures the spectral tail shift towards small scales.
455 The associated spatial fields have been presented in Figure 1.

456 We also exemplify the adaptive filtering of the advected tracer (with squared correlation
457 length (4.10)) in Figure 6. The tracer is advected during a time t , and then smoothed by a
458 Gaussian filter with the width (4.10). Hence, small-scale tracer structures are created by the
459 advection and are then filtered out. However, the transport of large-scale tracer structures
460 due to advection remains after filtering. The combined effect of advection and filtering moves
461 the large-scale structures, but keeps the global amount of small-scale structures stationary.

462 *b. Ocean applications*

463 A similar analysis is performed using satellite data. Following geostrophic assumption,
464 velocities are estimated from altimeter-derived sea surface height (SSH) fields. We em-
465 ploy pre-computed gridded geostrophic velocities from AVISO. The altimeter products were
466 produced by SSALTO/Developing Use of Altimetry for Climate Studies (DUACS) and dis-
467 tributed by AVISO, with support from CNES (<http://www.aviso.altimetry.fr/duacs/>). The
468 velocity field, on which the Eulerian diagnostics will be estimated, corresponds to January
469 1st, 2011, in the Antarctic Circumpolar Current (ACC) region, south of Australia. Never-

470 theless, for the reference Lagrangian advection, we will rely on (slowly) time-varying daily
471 AVISO dataset. A small spatial window of $10^\circ \times 10^\circ$ is first considered. For sea surface
472 temperature, the Ifremer/ODYSSEA SST Level 4 product is considered. Produced daily
473 using optimal interpolation (OI) on a global 0.1 degree grid, it provides a daily cloud-free
474 field of foundation sea surface temperature at approximately 10 km resolution (0.1 degree)
475 over the full globe. It is generated by merging microwave and infrared satellite observations.

476 Figure 7 delineates the Kinetic energy (KE) and the vorticity fields. The KE shows the
477 ACC eastward jet between latitudes -50° and -48° for January 1st, 2011. Two (warm)
478 anticyclones and a (cold) depression are visible both in the vorticity and initial SST fields
479 at $(129^\circ, -51^\circ)$, $(131^\circ, -54^\circ)$ and $(130.5^\circ, -49.5^\circ)$, respectively. The January 1st, 2011, SST
480 field is then advected. Similarly to Dencausse et al. (2014), time-interpolation is performed
481 to obtain the velocity fields between two daily data. The dipole closed to the jet creates
482 a mushroom-like structure in the advected tracer. Each vortex wraps the tracer, creating
483 spirals. The small southern anticyclone $(131^\circ, -54^\circ)$ seems weaker than the other anticyclone
484 $(129^\circ, -51^\circ)$. Yet, it faster wraps the tracer, as velocities are certainly larger than over the
485 dipole area.

486 Figure 8 shows the time-normalized growth rate, $(\alpha/t)^2$, the mesochronic vorticity, $\tilde{\omega}$, and
487 the weighting of the tracer/flow correlation, β/α . The squared inverse of folding, shearing
488 and stretching times are also presented. A slight low-pass spatial filtering (2-km filter width)
489 is applied to the stretching time to help distinguish the filamentary structures. The spatial
490 distributions of time-normalized stretching rate and inverse squared stretching time are
491 found very similar. The amplitude of the stretching time is slightly underestimated (ratio
492 of about 2). In the aforementioned vortex boundaries, intense mixing occurs, whereas the

493 inverse shearing time is weak. Comparable to the toy model results, folding effect due to
 494 differential rotations near the vortex boundaries is the leading mixing processes.

495 A larger ocean extend is now considered to encompass a broader variety of structures
 496 and dynamical processes. The spatial location and the date remain the same. Figure 9
 497 displays the KE and the vorticity. The jet and many eddies are visible. The SST is again
 498 advected (Figure 10). The advection creates small-scale structures, becoming non-physical
 499 spirals when the advection time is too long. After 48 days, the advected domain is strongly
 500 deformed, especially by the eastward jet. Figure 11 compares the time-normalized stretching
 501 ratio, $(\alpha/t)^2$, and the estimated inverse squared stretching time in this larger spatial window.
 502 As found, most stretching structures are well predicted by the proposed model.

503 Finally, Figure 12 presents the time evolution of the averaged stretching rate (3.19) and of
 504 the averaged tracer gradients norm (4.17). The reference plots clearly exhibit the structures
 505 prescribed by the derived models :

$$\overline{\alpha^2} = \left(\frac{t}{\tau_G}\right)^2 \text{ and } \frac{\overline{\|\nabla T\|^2}}{\overline{\|\nabla T_0\|^2}} = \begin{cases} 1 & \text{if } t \ll \tau_G \\ \left(\frac{t}{\tau_G}\right)^2 & \text{if } t \gg \tau_G \end{cases} . \quad (5.3)$$

506 A global stretching time is estimated to 1.67 days. The plots reveal a good match, though
 507 the stretching time seems slightly underestimated (by a factor of about ~ 1.7). The small
 508 shift between predicted and reference averaged tracer gradients norm may also be explained
 509 by a residual correlation between the tracer and the underlying flow as explained previously
 510 with equation (2.3).

511 Figure 13 illustrates the self-similarity-based estimate (4.16), applied to the ACC SST
 512 field. As prescribed, the spectrum tail slope of the advected SST reaches the value -3 after
 513 5 days of advection, -2.5 after 10 days and -2 after 48 days. -3 , -2.5 or -2 spectrum
 514 slope observed in "high-resolution" tracer data can be physically relevant. For instance, a

515 surface quasi-geostrophic dynamics (Held et al. 1995) would lead to a $-5/3$ slope for the
516 SST spectrum (if the salinity contribution to buoyancy is neglected). However, here -3 ,
517 -2.5 and -2 are arbitrarily chosen values. Theoretically, any spectrum slope (< -1 and $>$
518 to the initial slope) could be reached by this downscaling process. Figure 10 displays the
519 spatial SST fields before and after advection. These data correspond to the summer season,
520 January in the Southern hemisphere.

521 A seasonal variation can then be studied. Figure 14 displays, for each day of the year
522 2011, the global stretching time, τ_G , the spectrum slope of the measured SST and the
523 prescribed advection time to reach a -2.5 spectrum slope. As found, the stretching is faster
524 during the winter. The measured SST spectral slopes are relatively stationary (close to -4).
525 Accordingly, the prescribed advection time is smaller in wintertime.

526 Berti and Lapeyre (2014) proposed other Eulerian estimates to prescribe the advection
527 time: the inverse of the vorticity Root Mean Square (RMS), $\left(\overline{(\nabla^\perp \cdot \mathbf{v})^2}\right)^{-1/2}$, and of the
528 velocity gradient RMS, $\left(\overline{\|\nabla \mathbf{v}\|^2}\right)^{-1/2}$. The latter is directly linked to the shearing time
529 (3.7). Yet, these estimates can encode shearing but not folding. Indeed, folding involves
530 2^{nd} order derivatives of the velocity, such as to describe the curvature variation of adjacent
531 streamlines (3.13). Moreover, these criteria do not depend on the initial nor on the resulting
532 spectral slope. So, these criteria cannot fully control the necessary advection time, and
533 may not be sufficient to control the resulting tracer spectral slopes. For instance, Figure 14
534 demonstrates that these criteria strongly underestimate the advection time needed to reach
535 a -2.5 spectrum slope. Accordingly, these criteria mostly apply to very short advection
536 time, with a resulting advected tracer already close to the true SST.

537 **Conclusion**

538 In this paper, we analyzed how fluid parcels are stretched and folded by a smoothed
539 velocity field, creating strong tracer gradients and raising the high-wavenumber part of the
540 tracer spectral distribution. That is the case for SSH-inferred surface currents and the
541 associated advection of coarse-scale tracer observations. Lagrangian methods, such as the
542 determination of finite-time Lyapunov exponent (FTLE), are now very popular means to
543 infer upper ocean transport properties of heat, salt, nutrients or pollutants from such velocity
544 fields. Still, Lagrangian methods can be computationally expensive to perform integration
545 of particle along trajectories, and may not be suitable to quickly assess short-term material
546 transport. Using Eulerian quantities from a single snapshot of velocities, our proposed
547 development is more practical and fully exploit the connection between the Cauchy-Green
548 deformation tensor and the evolution of the averaged squared norm of the advected tracer
549 gradients.

550 Recently, following a variational theory to objectively define Eulerian coherent structures,
551 Serra and Haller (2016) and Nolan et al. (2020) exploit the fact that for infinitesimally small
552 integration times, the eigenvectors of the right Cauchy-Green strain tensor are equal to those
553 of the Eulerian rate-of-strain tensor. This property can thus already provide an Eulerian
554 diagnostic, i.e. the instantaneous Lyapunov exponent structure, to help identify major flow
555 features dominating short-time particle deformation patterns.

556 The present developments are not limited to small integration times. Two characteristics
557 of the flow influence the norm of the advected tracer gradients: a local growth rate, asso-
558 ciated with the eigenvalues of the Cauchy-Green tensor, and the orientation of the stable
559 direction, eigenvector of the Cauchy-Green tensor. Integrated over space, the influence of lo-

560 cal orientation disappears if the initial tracer is not correlated to the flow. This decorrelation
561 can be due to a strong spatial smoothing induced by initial tracer observation process. It
562 is then demonstrated that when the initial correlations between smooth divergent-free flow
563 fields and tracers are weak the overall gradients can only strengthen. The local growth rate
564 of the tracer gradients is independent of the initial tracer distribution, and is directly related
565 to FTLEs and along-trajectory time-integrated velocities, i.e. mesochronic velocities.

566 In the case of SSH-inferred surface currents, Eulerian velocity fields are almost station-
567 ary during one or two weeks. As such, a simple and efficient prognosis can be derived.
568 It extends the widely used Okubo-Weiss method – also known as the Q-criterion – which
569 instantaneously compares relative vorticity to strain properties. Computationally cheap
570 and perfectly suited to Eulerian snapshots of satellite-derived large-scale ocean flows, our
571 proposed criterion identifies regions where mixing can occur and quantifies it. The Eule-
572 rian descriptors can well separate mesoelliptic regions, rotating areas over which the tracer
573 gradient norm is conserved, from mesohyperbolic regions where motion is dominated by
574 stretching in one direction and contraction in the other. Over these latter areas, strain and
575 mesochronic vorticity compete and tracer gradients tend to increase. Moreover, our Eulerian
576 proxies quantifies the tracer gradients growth.

577 Tracer gradients also control and specify the high-wavenumber tracer spectra. As pre-
578 sented and discussed, our prognostic analysis helps consistently determining the advection
579 time and low-pass filter to apply when using Lagrangian downscaling advection methods.
580 Based on these developments, a practical estimation of the horizontal diffusivity is also de-
581 rived to help constraining subgrid parameterizations of large-scale flow simulations. The
582 smooth velocity component acting to raise the high-wavenumber part of the spectrum (4.9),
583 the horizontal effective diffusivity will balance this rise to best take into account unresolved

584 small-scale components. Further investigations could thus extend the proposed criterion to
585 stochastic flows, possibly exhibiting preferred sense of rotation, leading to quasi-stationary
586 drift terms superposed to rapidly time-uncorrelated terms. The stochastic Eulerian frame-
587 work proposed in Bauer et al. (2020); Mémin (2014); Resseguier et al. (2017a,b, 2020) shall
588 be suited for such a purpose. Indeed, the slow-fast decomposition of the velocity can lead
589 to introduce a smooth drift component, adding to the mesochronic velocity, and a random
590 highly oscillating velocity component, acting to consistently define the diffusivity. The in-
591 troduction of so-called polarized small-scale fluctuations (Middleton and Loder 1989) might
592 then be tested to provide meaningful information on the additional folding/shearing effects
593 associated to local statistical drifts attached to spatially-distributed small-scale fluctuations
594 (Bauer et al. 2020).

595 To further note, the Eulerian prognosis descriptors are certainly well suited to present-day
596 large-scale altimeter-derived velocity estimates. Satellite-derived maps are weekly available,
597 and time sequences of Eulerian estimates of flow mixing can be performed. Accordingly,
598 seasonal to inter-annual variations of derived time evolution of folding and shearing proper-
599 ties of upper ocean flows in different basins can also be derived. The simple prognostic shall
600 then serve to possibly assess readjustment of the surface mesoscale ocean circulation over
601 the last 3 decades (Martínez-Moreno et al. 2021).

602 Finally, the presented analytical and numerical results have strongly highlighted the im-
603 portance of folding induced by upper ocean eddies in tracers' dynamics. The actual satellite
604 altimeter constellation will soon include a future wide-swath Surface Water & Ocean To-
605 pography (SWOT) altimeter (Morrow et al. 2019), to more precisely characterize ocean
606 sea surface height variability. Specifically, ocean spatial structures will thus be better re-
607 solved. New Eulerian prognosis descriptors will thus be derived from these snapshot flow

608 fields, and the impact of spatial resolution better assessed when compared to standard actual
 609 altimeter-derived products.

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616 *Data availability statement.* These data were produced by ESA / GlobCurrent project and
 617 obtained from Ifremer/CERSAT. They are free and open.

618 APPENDIX

619 **A1. Recap on Cauchy-Green tensor analysis**

620 The Cauchy-Green tensor diagonalization writes as follows:

$$\nabla\phi^T (\nabla\phi^T)^T = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad \text{with} \quad D_{ii} = 1 + \alpha^2 \left(1 - (-1)^i \frac{\beta}{\alpha} \right), \quad (\text{A1})$$

621 where \mathbf{P} is an orthogonal matrix, $\alpha^2 = \frac{1}{2} \|\nabla\phi^T\|^2 - 1 \geq 0$ and $\beta^2 = \alpha^2 + 2$, using the Frobe-
 622 nius matrix norm. The eigenvalues D_{ii} define the Finite Time Lyapunov Exponents (FTLE)
 623 (Haller and Yuan 2000; Thiffeault and Boozer 2001; Haller 2005; Haller and Sapsis 2011).

624 Note that Pierrehumbert and Yang (1993) introduced this notion but provided an erroneous
 625 way of calculating it (referring to the frame-dependent flow map gradient eigenvalues instead
 626 of its frame-independent singular values). The largest and the smallest FTLEs are:

$$\Lambda = \frac{1}{2t} \log(D_{11}) \quad \text{and} \quad -\Lambda = \frac{1}{2t} \log(D_{22}). \quad (\text{A2})$$

627 The Cauchy-Green tensor encodes insightful and frame-independent information of the flow,
 628 further controlling the tracer gradient norm evolution (Haller and Yuan 2000).

629 A2. Exact formula for the tracer gradient norm

630 Note that at a given time, the transported tracer gradient, ∇T , can be written from the
 631 initial gradient field, ∇T_0 , as:

$$\nabla T(\mathbf{x}) = \nabla(T_0(\phi^{-1}(\mathbf{x}))) = [\nabla\phi^T]^{-1}(\phi^{-1}(\mathbf{x}))\nabla T_0(\phi^{-1}(\mathbf{x})). \quad (\text{A1})$$

632 Using the divergence-free assumption in the variable change and the matrix diagonalization
 633 (A1) with the identities $D_{11}^{-1} = D_{22}$ and $D_{22}^{-1} = D_{11}$, an exact expression of the averaged
 634 squared norm of tracer gradients reads:

$$\overline{\|\nabla T\|^2} - \overline{\|\nabla T_0\|^2} = \frac{1}{S} \int_{\Omega} d\mathbf{x} \|\nabla T(\mathbf{x})\|^2 - \frac{1}{S} \int_{\Omega} d\mathbf{x}_0 \|\nabla T_0(\mathbf{x}_0)\|^2, \quad (\text{A2})$$

$$= \frac{1}{S} \int_{\Omega} d\mathbf{x}_0 \|[\nabla\phi^T]^{-1}(\mathbf{x}_0)\nabla T_0(\mathbf{x}_0)\|^2 - \frac{1}{S} \int_{\Omega} d\mathbf{x}_0 \|\nabla T_0(\mathbf{x}_0)\|^2, \quad (\text{A3})$$

$$= \overline{(\nabla T_0)^T \left([\nabla\phi^T (\nabla\phi^T)^T]^{-1} - \mathbb{I}_d \right) \nabla T_0}, \quad (\text{A4})$$

$$= \alpha^2 \left(\underbrace{\left(1 - \frac{\beta}{\alpha}\right)}_{<0} (\mathbf{P}^T \nabla T_0)_1^2 + \underbrace{\left(1 + \frac{\beta}{\alpha}\right)}_{>0} (\mathbf{P}^T \nabla T_0)_2^2 \right), \quad (\text{A5})$$

635 where $(\mathbf{P}^T \nabla T_0)_i$ is the i -th component of the vector $\mathbf{P}^T \nabla T_0$. Thus, the Cauchy-Green
 636 tensor and the initial tracer gradient completely determine the averaged squared norm of
 637 advected tracer gradients. To simplify the above expression, we define the angle between
 638 the tracer gradient and the compressive (stable) direction of the direct flow as:

$$\cos(\theta_{T_0}^{\phi}) = \frac{(\mathbf{P}^T \nabla T_0)_2}{\|\nabla T_0\|}. \quad (\text{A6})$$

639 Finally, we infer the following compact expression:

$$\overline{\|\nabla T\|^2} - \overline{\|\nabla T_0\|^2} = \overline{\|\nabla T_0\|^2} \alpha^2 \left(1 + \frac{\beta}{\alpha} \cos\left(2\theta_{T_0}^\phi\right)\right). \quad (\text{A7})$$

640 **A3. Mixing criterion of Mezić et al. (2010)**

641 As derived in Mezić et al. (2010), the incompressibility of the flow yields:

$$1 = \det(\nabla \phi^T) = \det(\mathbb{I}_d + t \nabla \check{\mathbf{v}}^T) = 1 + t \operatorname{tr}(\nabla \check{\mathbf{v}}^T) + t^2 \det(\nabla \check{\mathbf{v}}^T). \quad (\text{A1})$$

642 For an incompressible flow, the mesochronic velocity follows:

$$t \det(\nabla \check{\mathbf{v}}^T) = -\operatorname{tr}(\nabla \check{\mathbf{v}}^T) = -\nabla \cdot \check{\mathbf{v}} \neq 0. \quad (\text{A2})$$

643 The definition of the local growth rate, α^2 then reads:

$$\alpha^2 \triangleq \frac{1}{2} \|\nabla \phi^T\|^2 - 1, \quad (\text{A3})$$

$$= \frac{1}{2} \|\mathbb{I}_d + t \nabla \check{\mathbf{v}}^T\|^2 - 1, \quad (\text{A4})$$

$$= -t (t \det(\nabla \check{\mathbf{v}}^T)) + \frac{t^2}{2} \|\nabla \check{\mathbf{v}}^T\|^2, \quad (\text{A5})$$

$$= \frac{t^2}{2} ((\partial_x \check{u} - \partial_y \check{v})^2 + (\partial_y \check{u} + \partial_x \check{v})^2). \quad (\text{A6})$$

644 Then, the incompressibility constraint (A2) helps rewrite (A6) as a function of the determi-

645 nant $\det(\nabla \check{\mathbf{v}}^T)$:

$$\alpha^2 = \frac{t^2}{2} ((\nabla \cdot \check{\mathbf{v}})^2 - 4 \det(\nabla \check{\mathbf{v}}^T) + \check{\omega}^2), \quad (\text{A7})$$

$$= \frac{t^2}{2} \left(t^2 \det(\nabla \check{\mathbf{v}}^T) \left(\det(\nabla \check{\mathbf{v}}^T) - \frac{4}{t^2} \right) + \check{\omega}^2 \right). \quad (\text{A8})$$

646 **A4. Folding time**

647 Denotes:

$$(\partial_1 \mathbf{g}^T)(\mathbf{z}_1, z_2) = \nabla_{\mathbf{z}_1}(\mathbf{g}^T(\mathbf{z}_1, z_2)) \text{ and } (\partial_2 \mathbf{g})(\mathbf{z}_1, z_2) = \partial_{z_2}(\mathbf{g}(\mathbf{z}_1, z_2)). \quad (\text{A1})$$

648 with both terms 1–periodic with respect to its second variable. Using frequency invariance
 649 (3.10), we can replace $f(\mathbf{x}_0)$ by $f(\phi(\mathbf{x}_0, t))$ in the model (3.8):

$$\phi(\mathbf{x}_0, t) = \mathbf{x}_0 + \mathbf{g}(\mathbf{x}_0, f(\phi(\mathbf{x}_0, t))t). \quad (\text{A2})$$

650 Then, replacing back $f(\phi(\mathbf{x}_0, t))$ by $f(\mathbf{x}_0)$ after evaluating the gradient, the stretching of
 651 the flow reads:

$$\begin{aligned} \nabla \phi^T(\mathbf{x}_0, t) &= \mathbb{I}_d + (\partial_1 \mathbf{g}^T)(\mathbf{x}_0, f(\phi(\mathbf{x}_0, t))t) \\ &\quad + t \nabla \phi^T(\mathbf{x}_0, t) \nabla f(\phi(\mathbf{x}_0, t)) (\partial_2 \mathbf{g}^T)(\mathbf{x}_0, f(\phi(\mathbf{x}_0, t))t), \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} &= \mathbb{I}_d + (\partial_1 \mathbf{g}^T)(\mathbf{x}_0, f(\mathbf{x}_0)t) \\ &\quad + t \nabla \phi^T(\mathbf{x}_0, t) \nabla f(\phi(\mathbf{x}_0, t)) (\partial_2 \mathbf{g}^T)(\mathbf{x}_0, f(\mathbf{x}_0)t). \end{aligned} \quad (\text{A4})$$

652 In the last equality, the second right-hand term is time-periodic and thus bounded. If we
 653 neglect its time variation (e.g., for large advection time t), it writes

$$(\partial_1 \mathbf{g}^T)(\mathbf{x}_0, f(\mathbf{x}_0)t) \approx (\partial_1 \mathbf{g}^T)(\mathbf{x}_0, 0) = \nabla \phi^T(\mathbf{x}_0, 0) - \mathbb{I}_d = 0. \quad (\text{A5})$$

654 Introducing the original periodic model (3.8) into its definition, the Lagrangian velocity \mathbf{V}
 655 reads:

$$\mathbf{V}(\mathbf{x}_0, t) = \frac{d\phi(\mathbf{x}_0, t)}{dt} = f(\mathbf{x}_0)(\partial_2 \mathbf{g})(\mathbf{x}_0, f(\mathbf{x}_0)t). \quad (\text{A6})$$

656 Finally, the flow gradient expression (A4) can be rewritten using equations (A5) and (A6):

$$\nabla \phi^T(\mathbf{x}_0, t) = \mathbb{I}_d + t \nabla \phi^T(\mathbf{x}_0, t) \frac{\nabla f(\phi(\mathbf{x}_0, t))}{f(\mathbf{x}_0)} \mathbf{V}^T(\mathbf{x}_0, t), \quad (\text{A7})$$

$$= \mathbb{I}_d + t \nabla \phi^T(\mathbf{x}_0, t) \left(\frac{1}{f} \nabla f \mathbf{v}^T \right) (\phi(\mathbf{x}_0, t)), \quad (\text{A8})$$

657 where the frequency invariance (3.10) was used in the last equality. Factorizing terms in
 658 $\nabla \phi^T$,

$$\mathbb{I}_d = \nabla \phi^T(\mathbf{x}_0, t) \left(\mathbb{I}_d - t \left(\frac{1}{f} \nabla f \mathbf{v}^T \right) (\phi(\mathbf{x}_0, t)) \right). \quad (\text{A9})$$

659 we then inverse the matrix equation and remap with the inverse flow, ϕ^{-1} :

$$(\nabla\phi^T)^{-1}(\phi^{-1}(\mathbf{x}, t), t) = \mathbb{I}_d - t \left(\frac{1}{f} \nabla f \mathbf{v}^T \right) (\mathbf{x}). \quad (\text{A10})$$

660 Since the frequency is a function of the streamline (3.10), we have

$$0 = \frac{d}{dt}(f(\mathbf{x}_0)) = \frac{d}{dt}(f(\phi(\mathbf{x}_0, t))) = (\mathbf{v} \cdot \nabla f)(\phi(\mathbf{x}_0, t), t). \quad (\text{A11})$$

661 Therefore, the frequency gradient is orthogonal to the velocity, and

$$\|\nabla f\| \approx \left| \frac{\mathbf{v}^\perp}{\|\mathbf{v}^\perp\|} \cdot \nabla f \right|. \quad (\text{A12})$$

662 Eigenvalues of the flow gradients $\nabla\phi^T$ are the inverse of one another, and the matrix and
 663 its inverse have the same Frobenius norm. So, the time dependency of the growth rate in
 664 the final grid (points \mathbf{x}) follows from its definition and from (A10):

$$\alpha^2(\phi^{-1}(\mathbf{x}, t), t) \triangleq \frac{1}{2} \|\nabla\phi^T(\phi^{-1}(\mathbf{x}, t), t)\|^2 - 1, \quad (\text{A13})$$

$$= \frac{1}{2} \|(\nabla\phi^T)^{-1}(\phi^{-1}(\mathbf{x}, t), t)\|^2 - 1, \quad (\text{A14})$$

$$= - \underbrace{\left(\frac{t}{f} \nabla f \cdot \mathbf{v} \right)}_{=0 \text{ by (A11)}} (\mathbf{x}) + \left(\frac{t^2}{2f^2} \|\nabla f\|^2 \|\mathbf{v}\|^2 \right) (\mathbf{x}). \quad (\text{A15})$$

665 A5. Spectra of tracers for smooth flow

666 With Dirichlet boundary conditions for the tracer, integration by parts leads to:

$$-\mathbf{H}_{\gamma_q}(0) = -(\nabla_{\delta\mathbf{x}} \nabla_{\delta\mathbf{x}}^T \gamma_q(\delta\mathbf{x}))|_{\delta\mathbf{x}=0}, \quad (\text{A1})$$

$$= -\frac{1}{S} \int_{\Omega} d\mathbf{x} q(\mathbf{x}) (\nabla_{\delta\mathbf{x}} \nabla_{\delta\mathbf{x}}^T q(\mathbf{x} + \delta\mathbf{x}))|_{\delta\mathbf{x}=0}, \quad (\text{A2})$$

$$= -\frac{1}{S} \int_{\Omega} d\mathbf{x} q(\mathbf{x}) \mathbf{H}_q(\mathbf{x}), \quad (\text{A3})$$

$$= \frac{1}{S} \int_{\Omega} d\mathbf{x} \nabla q(\mathbf{x}) (\nabla q(\mathbf{x}))^T, \quad (\text{A4})$$

$$= \overline{\nabla q (\nabla q)^T} > 0 \text{ (in the Lowner sense)}. \quad (\text{A5})$$

667 In particular, the Hessian trace simplified to:

$$\text{tr}(\mathbf{H}_{\gamma_q}(0)) = -\overline{\text{tr}(\nabla q(\nabla q)^T)} = -\overline{\|\nabla q\|^2}. \quad (\text{A6})$$

668 The Fourier transform of the covariance expression (4.3) provides the approximation of
669 the spectrum tail:

$$\Gamma_q(\mathbf{k}) \triangleq |\widehat{q}(\mathbf{k})|^2 = \widehat{\gamma}_q(\mathbf{k}) \underset{\|\mathbf{k}\| \rightarrow \infty}{\sim} \frac{2\pi \left(\overline{\|\nabla q\|^2}\right)^2}{\det(\mathbf{H}_{\gamma_q}(0))^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{k}^T \left(-\overline{\|\nabla q\|^2} \mathbf{H}_{\gamma_q}^{-1}(0)\right) \mathbf{k}\right), \quad (\text{A7})$$

670 where the hat denotes spatial Fourier transform. Nevertheless, $\nabla q(\nabla q)^T$ and thus \mathbf{H}_{γ_q} is
671 not convenient to manipulate. Accordingly, hereafter, we will focus on the omnidirectional
672 spectrum of a scalar q . This will enable us to replace $\nabla q(\nabla q)^T$ by $\|\nabla q(\mathbf{x})\|^2$ in the expression
673 of the spectrum (A7).

$$\widetilde{\Gamma}_q(\kappa) = \kappa \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \Gamma_f(\mathbf{k}), \quad (\text{A8})$$

$$= \kappa \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\delta\mathbf{x} \gamma_q(\delta\mathbf{x}) e^{-i\mathbf{k}\cdot\delta\mathbf{x}}, \quad (\text{A9})$$

$$= \kappa \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\delta\mathbf{x} \left(\gamma_q(0) - \frac{1}{2}\delta\mathbf{x}^T \frac{1}{S} \int_{\Omega} d\mathbf{x} \nabla q(\mathbf{x})(\nabla q(\mathbf{x}))^T \delta\mathbf{x} + \underset{\|\delta\mathbf{x}\| \rightarrow 0}{o}(\|\delta\mathbf{x}\|^3) \right) \times e^{-i\mathbf{k}\cdot\delta\mathbf{x}}, \quad (\text{A10})$$

$$= \frac{\kappa}{S} \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\delta\mathbf{x} \int_{\Omega} d\mathbf{x} \left(q^2(\mathbf{x}) - \frac{1}{2}\delta\mathbf{x}^T \nabla q(\mathbf{x})(\nabla q(\mathbf{x}))^T \delta\mathbf{x} + \underset{\|\delta\mathbf{x}\| \rightarrow 0}{o}(\|\delta\mathbf{x}\|^3) \right) \times e^{-i\mathbf{k}\cdot\delta\mathbf{x}}, \quad (\text{A11})$$

$$= \frac{\kappa}{S} \int_{\Omega} d\mathbf{x} \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\delta\mathbf{x} \left(q^2(\mathbf{x}) - \frac{1}{2} \left(\frac{\nabla q(\mathbf{x})}{\|\nabla q(\mathbf{x})\|_2} \cdot \delta\mathbf{x} \right)^2 \|\nabla q(\mathbf{x})\|_2^2 + \underset{\|\delta\mathbf{x}\| \rightarrow 0}{o}(\|\delta\mathbf{x}\|^3) \right) e^{-i\mathbf{k}\cdot\delta\mathbf{x}}. \quad (\text{A12})$$

674 Locally in \mathbf{x} , we can define a variable change for $\delta\mathbf{x} = (\delta x_1 \ \delta x_2)^T$. We apply the rotation
675 matrix $\mathbf{U}(\mathbf{x}) = \frac{1}{\|\nabla q(\mathbf{x})\|_2} [\nabla q(\mathbf{x}) \ \nabla^\perp q(\mathbf{x})]$ to $\delta\mathbf{x}$ to align $\delta\mathbf{x}$ with the tracer gradient and

676 denote $\theta(\mathbf{x})$ the angle of the associated rotation:

$$\begin{aligned} \tilde{\Gamma}_q(\kappa) &= \frac{\kappa}{S} \int_{\Omega} d\mathbf{x} \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\boldsymbol{\delta x} \left(q^2(\mathbf{x}) - \frac{1}{2}(\delta x_1)^2 \|\nabla q(\mathbf{x})\|_2^2 + \underset{\|\boldsymbol{\delta x}\| \rightarrow 0}{o}(\|\boldsymbol{\delta x}\|^3) \right) \\ &\quad \times e^{-i(\mathbf{U}^T \mathbf{k}) \cdot \boldsymbol{\delta x}}, \quad (\text{A13}) \end{aligned}$$

$$\begin{aligned} &= \frac{\kappa}{S} \int_{\Omega} d\mathbf{x} \oint_{[\theta(\mathbf{x}), \theta(\mathbf{x})+2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\boldsymbol{\delta x} \left(q^2(\mathbf{x}) - \frac{1}{2}(\delta x_1)^2 \|\nabla q(\mathbf{x})\|_2^2 + \underset{\|\boldsymbol{\delta x}\| \rightarrow 0}{o}(\|\boldsymbol{\delta x}\|^3) \right) \\ &\quad \times e^{-i\mathbf{k} \cdot \boldsymbol{\delta x}}, \quad (\text{A14}) \end{aligned}$$

$$\begin{aligned} &= \frac{\kappa}{S} \int_{\Omega} d\mathbf{x} \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\boldsymbol{\delta x} \left(q^2(\mathbf{x}) - \frac{1}{2}(\delta x_1)^2 \|\nabla q(\mathbf{x})\|_2^2 + \underset{\|\boldsymbol{\delta x}\| \rightarrow 0}{o}(\|\boldsymbol{\delta x}\|^3) \right) \\ &\quad \times e^{-i\mathbf{k} \cdot \boldsymbol{\delta x}}, \quad (\text{A15}) \end{aligned}$$

$$= \kappa \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\boldsymbol{\delta x} \left(\overline{\|q\|^2} - \frac{1}{2}(\delta x_1)^2 \overline{\|\nabla q\|^2} + \underset{\|\boldsymbol{\delta x}\| \rightarrow 0}{o}(\|\boldsymbol{\delta x}\|^3) \right) e^{-i\mathbf{k} \cdot \boldsymbol{\delta x}}. \quad (\text{A16})$$

677 The third equality above is due to the averaging over the spatial frequency angle $\theta_{\mathbf{k}}$. Indeed,
 678 $\mathbf{U}^T \mathbf{k}$ is just a rotation of \mathbf{k} . And, integrating over $[0, 2\pi]$ or over $[\theta(\mathbf{x}), 2\pi + \theta(\mathbf{x})]$ is the
 679 same thing, since it leads to the same closed line: a circle of radius κ .

$$\begin{aligned} \tilde{\Gamma}_q(\kappa) &= \kappa \oint_{[0,2\pi]} d\theta_{\mathbf{k}} \int_{\Omega} d\boldsymbol{\delta x}_1 d\boldsymbol{\delta x}_2 \left(\overline{\|q\|^2} \exp\left(-\frac{1}{2} \frac{\overline{\|\nabla q\|^2}}{\overline{\|q\|^2}} (\delta x_1)^2\right) + \underset{\|\boldsymbol{\delta x}\| \rightarrow 0}{o}(\|\boldsymbol{\delta x}\|^3) \right) \\ &\quad \times e^{-ik_1 \delta x_1} e^{-ik_2 \delta x_2}, \quad (\text{A17}) \end{aligned}$$

$$\underset{\|\mathbf{k}\| \rightarrow \infty}{\sim} \oint_{[0,2\pi]} \kappa d\theta_{\mathbf{k}} \frac{C_f}{2(2\pi)} \exp\left(-\frac{1}{2} L_f^2 k_1^2\right) (2\pi) \delta(k_2), \quad (\text{A18})$$

680 where $\mathbf{k} = (k_1 \ k_2)^T = (\kappa \cos(\theta_{\mathbf{k}}) \ \kappa \sin(\theta_{\mathbf{k}}))^T$, $L_f^2 = \frac{\overline{\|q\|^2}}{\overline{\|\nabla q\|^2}}$ and $\frac{C_f}{2(2\pi)} = \left(\frac{2\pi(\overline{\|q\|^2})^3}{\overline{\|\nabla q\|^2}}\right)^{1/2}$. Note
 681 that the asymptotic equivalence is an approximation. Then, switching from cylindrical to
 682 Cartesian coordinates in each half ring $\{\mathbf{k} \in \mathbb{R}^2 | k_1 \leq 0, \|\mathbf{k}\| = \kappa\}$ and $\{\mathbf{k} \in \mathbb{R}^2 | k_1 \geq 0, \|\mathbf{k}\| =$
 683 $\kappa\}$ yields:

$$\tilde{\Gamma}_q(\kappa) \underset{\|\mathbf{k}\| \rightarrow \infty}{\sim} 2 \int_{-\kappa}^{\kappa} \frac{dk_2}{\sqrt{1 - \left(\frac{k_2}{\kappa}\right)^2}} \frac{C_f}{2} \exp\left(-\frac{1}{2} L_f^2 (\kappa^2 - k_2^2)\right) \delta(k_2), \quad (\text{A19})$$

$$\underset{\|\mathbf{k}\| \rightarrow \infty}{\sim} C_f \exp\left(-\frac{1}{2} L_f^2 \kappa^2\right). \quad (\text{A20})$$

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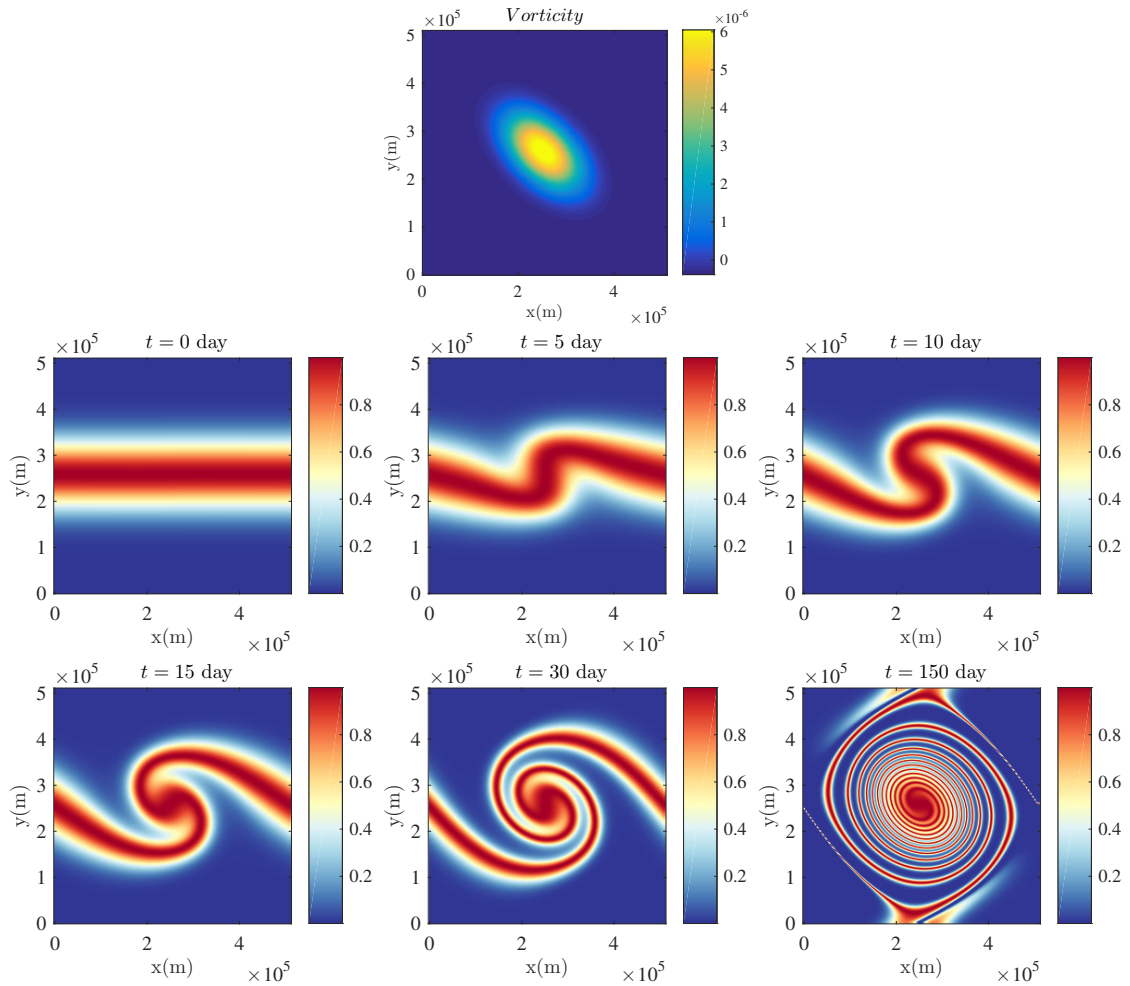
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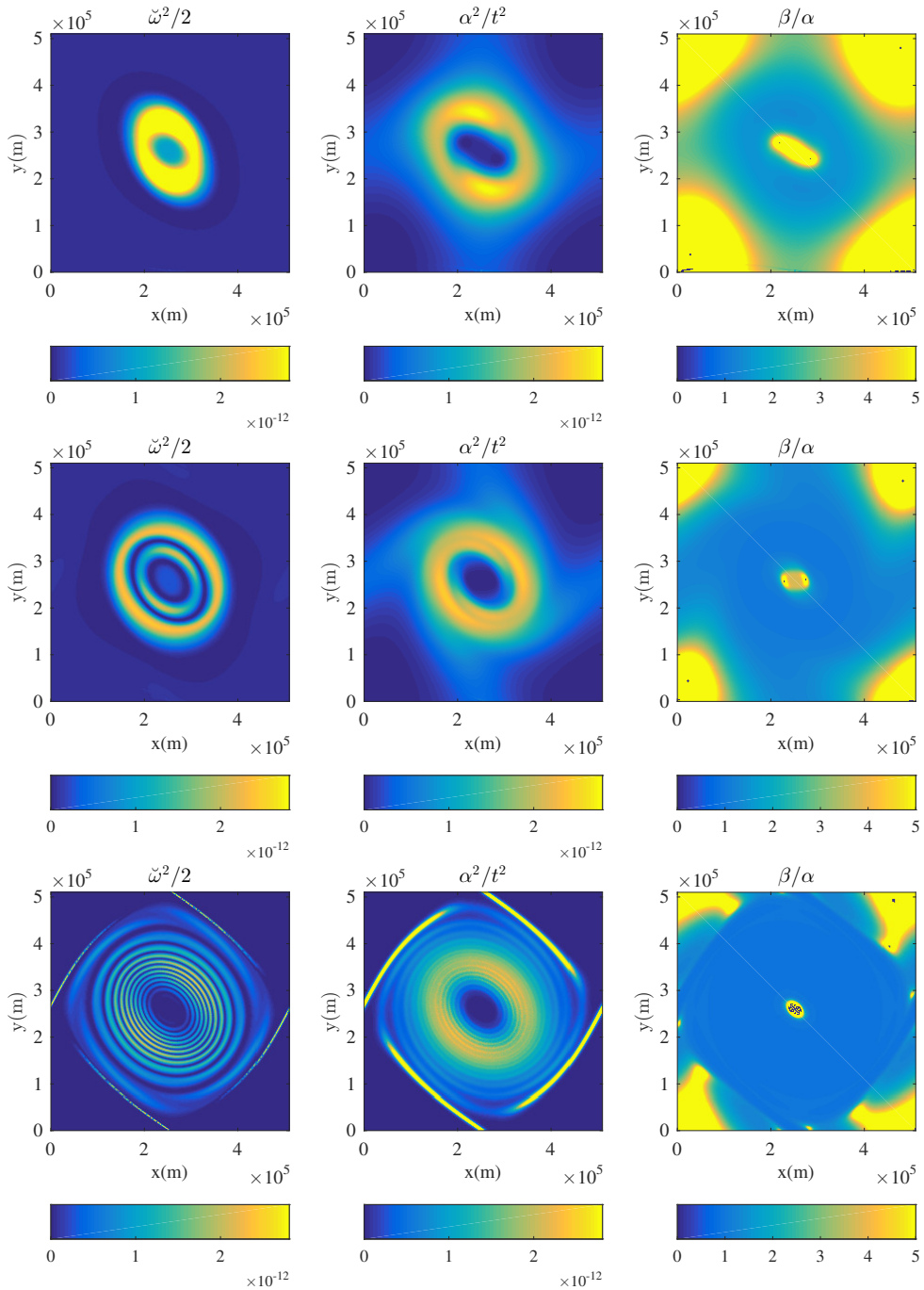
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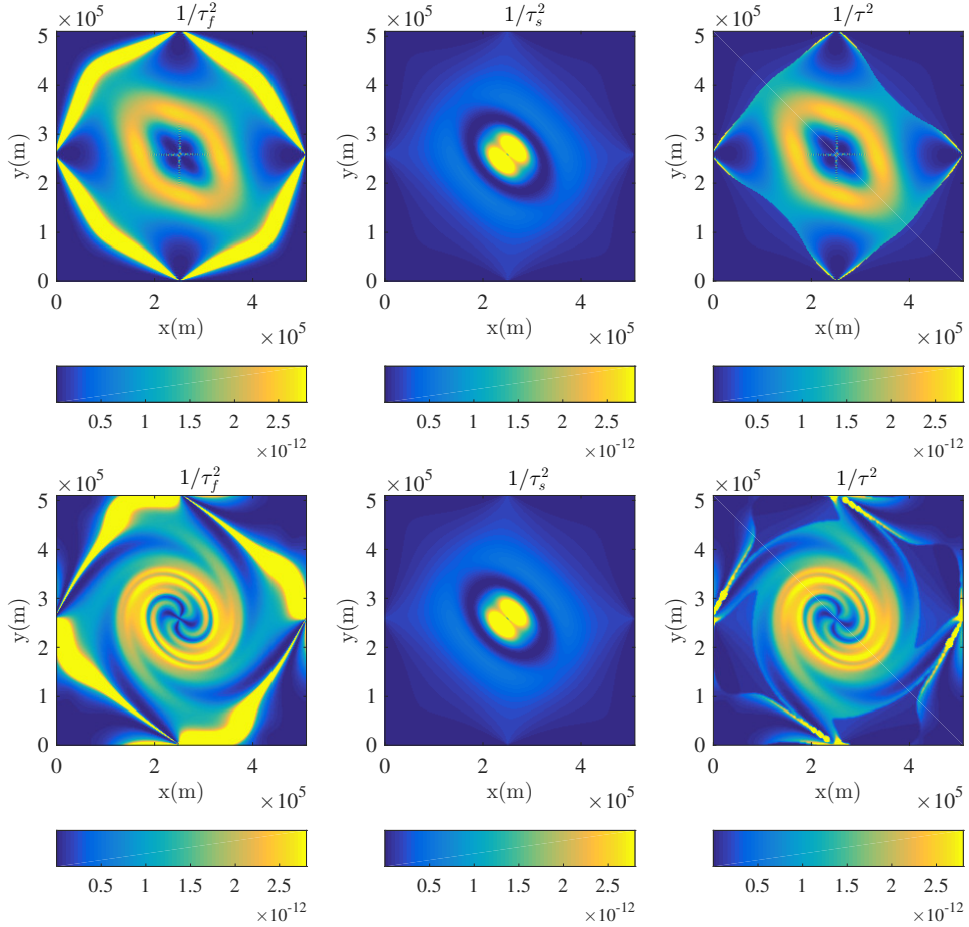
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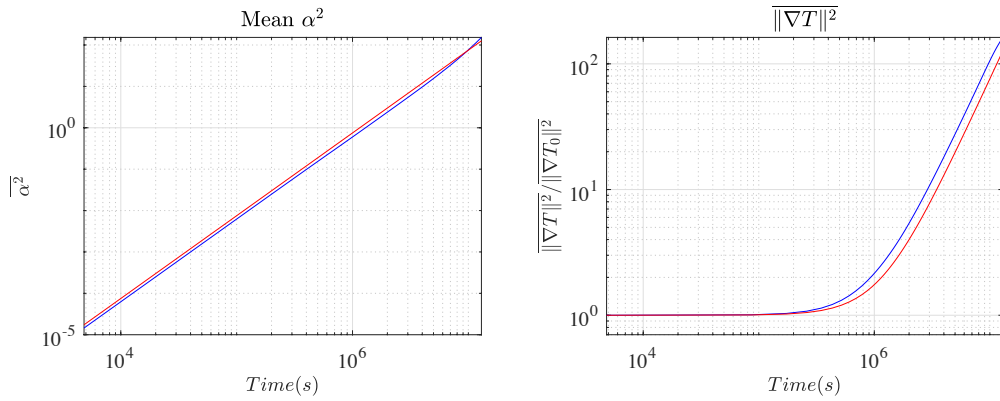
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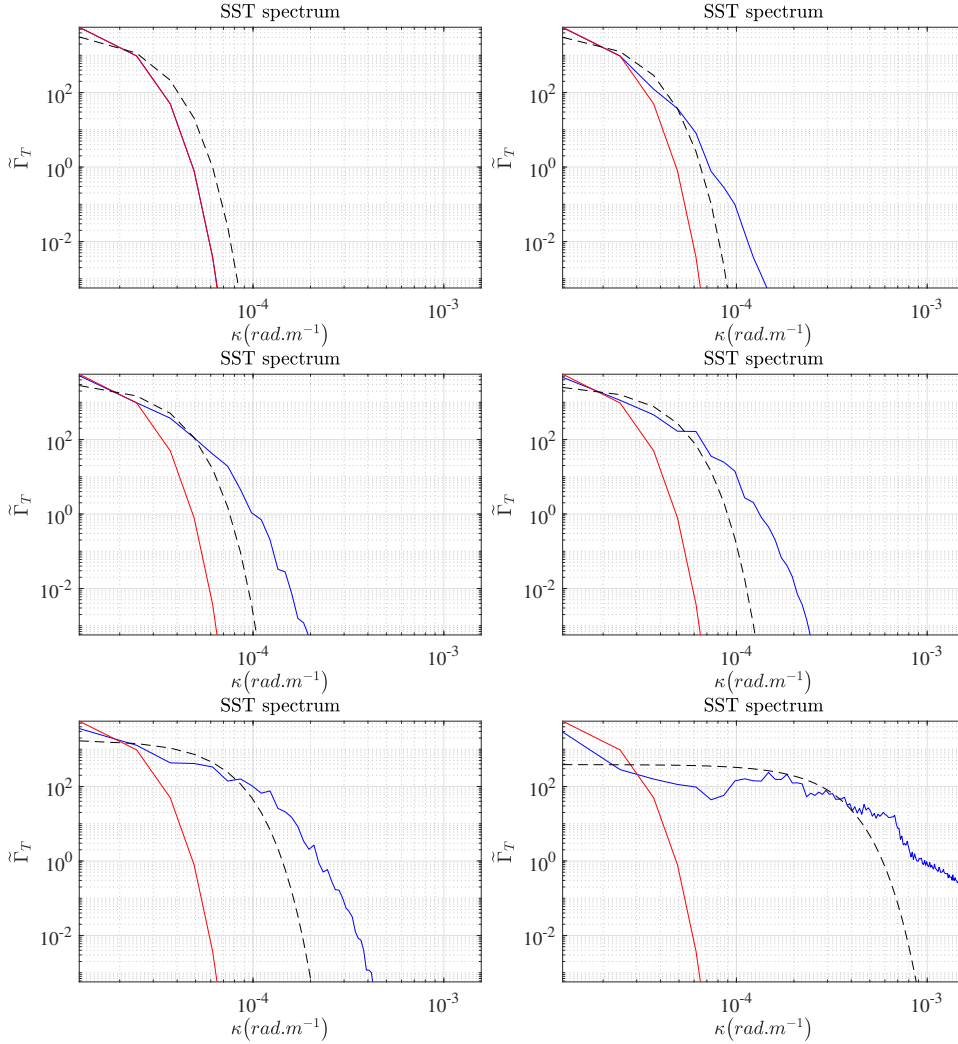
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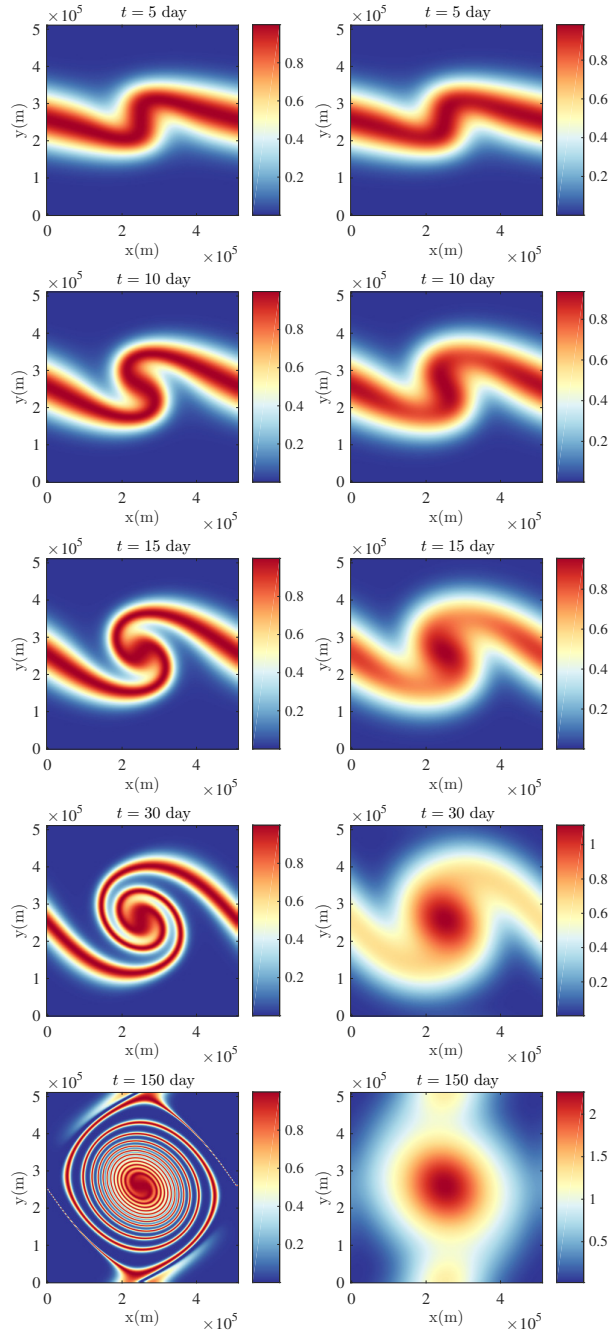
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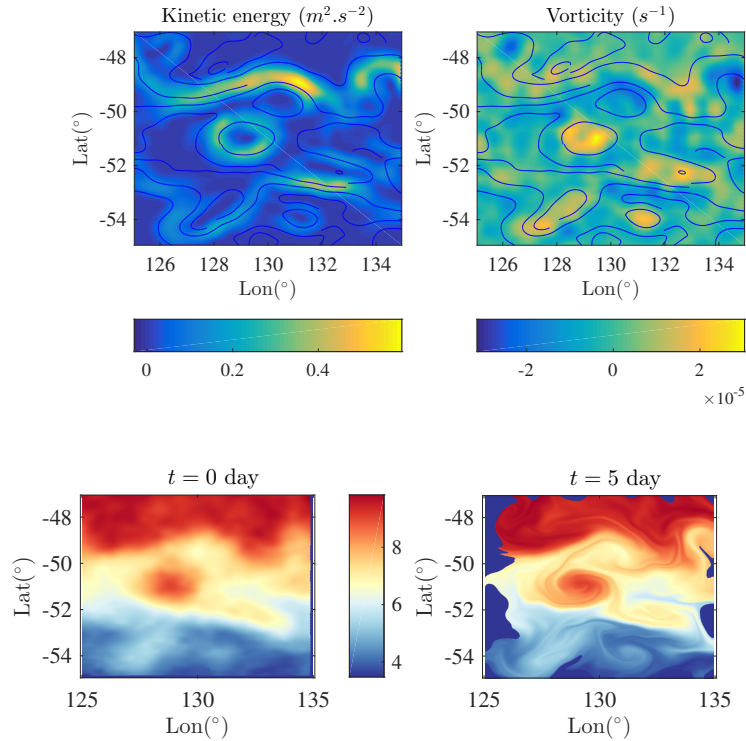
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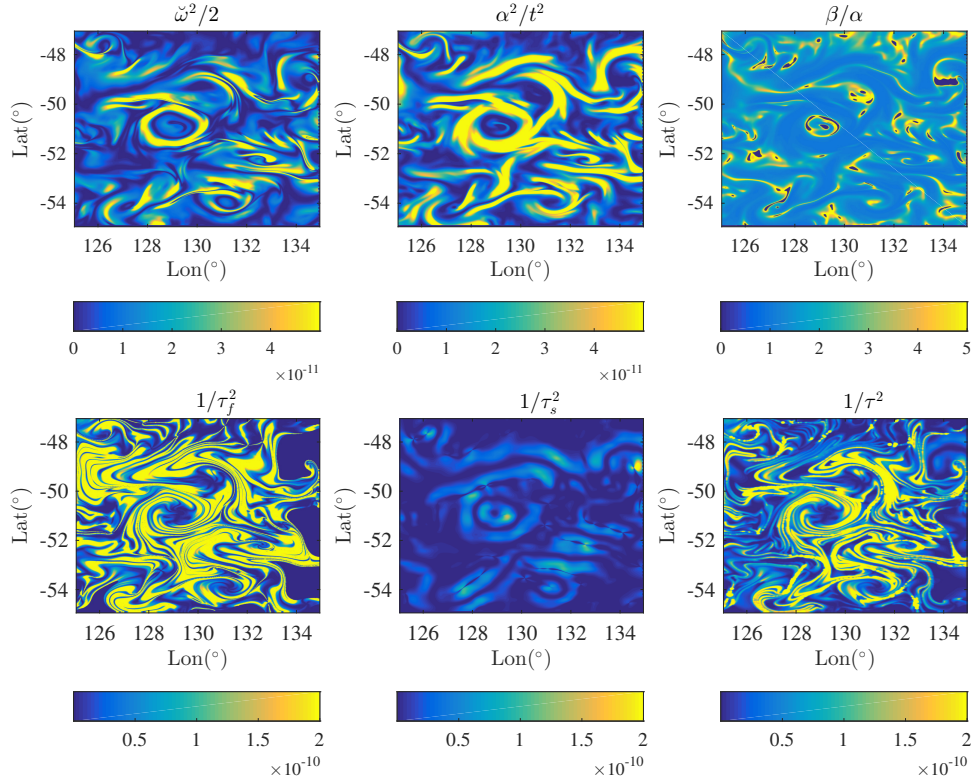
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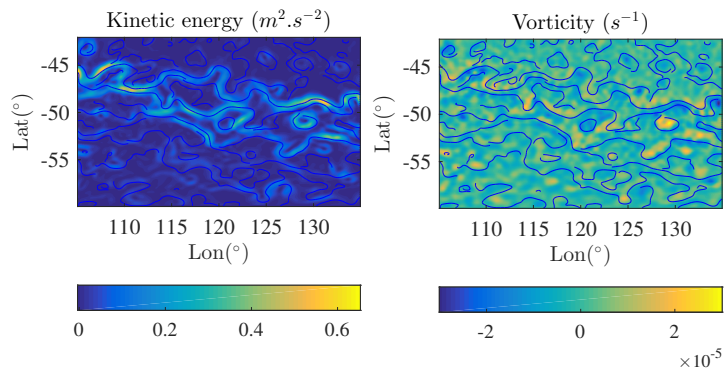
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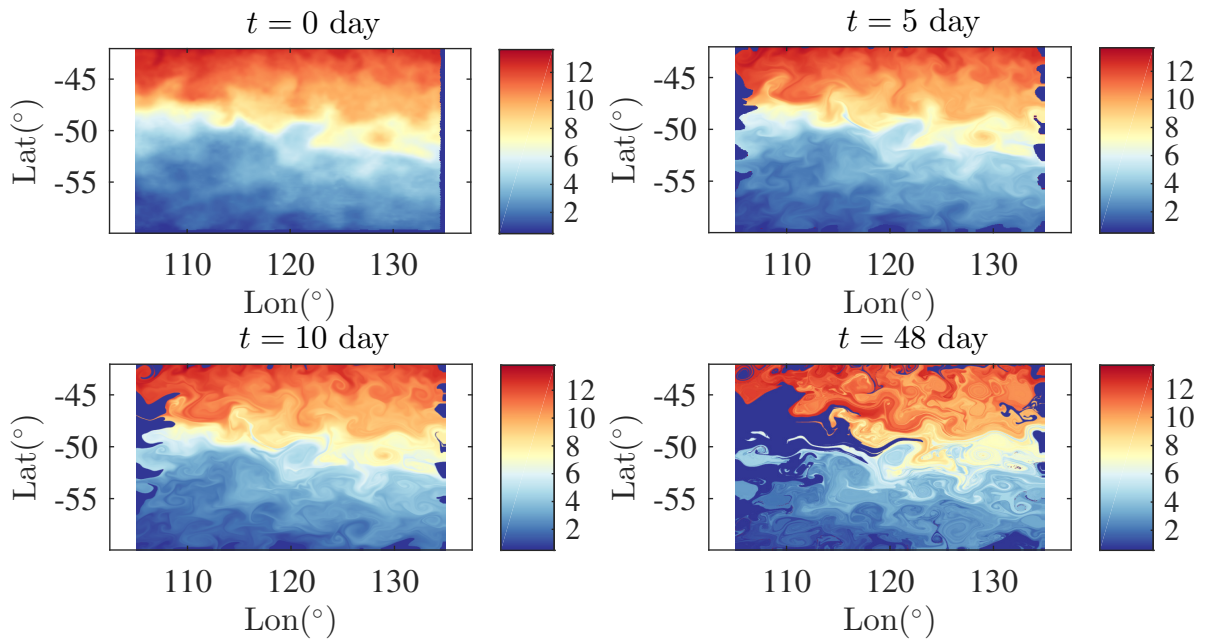
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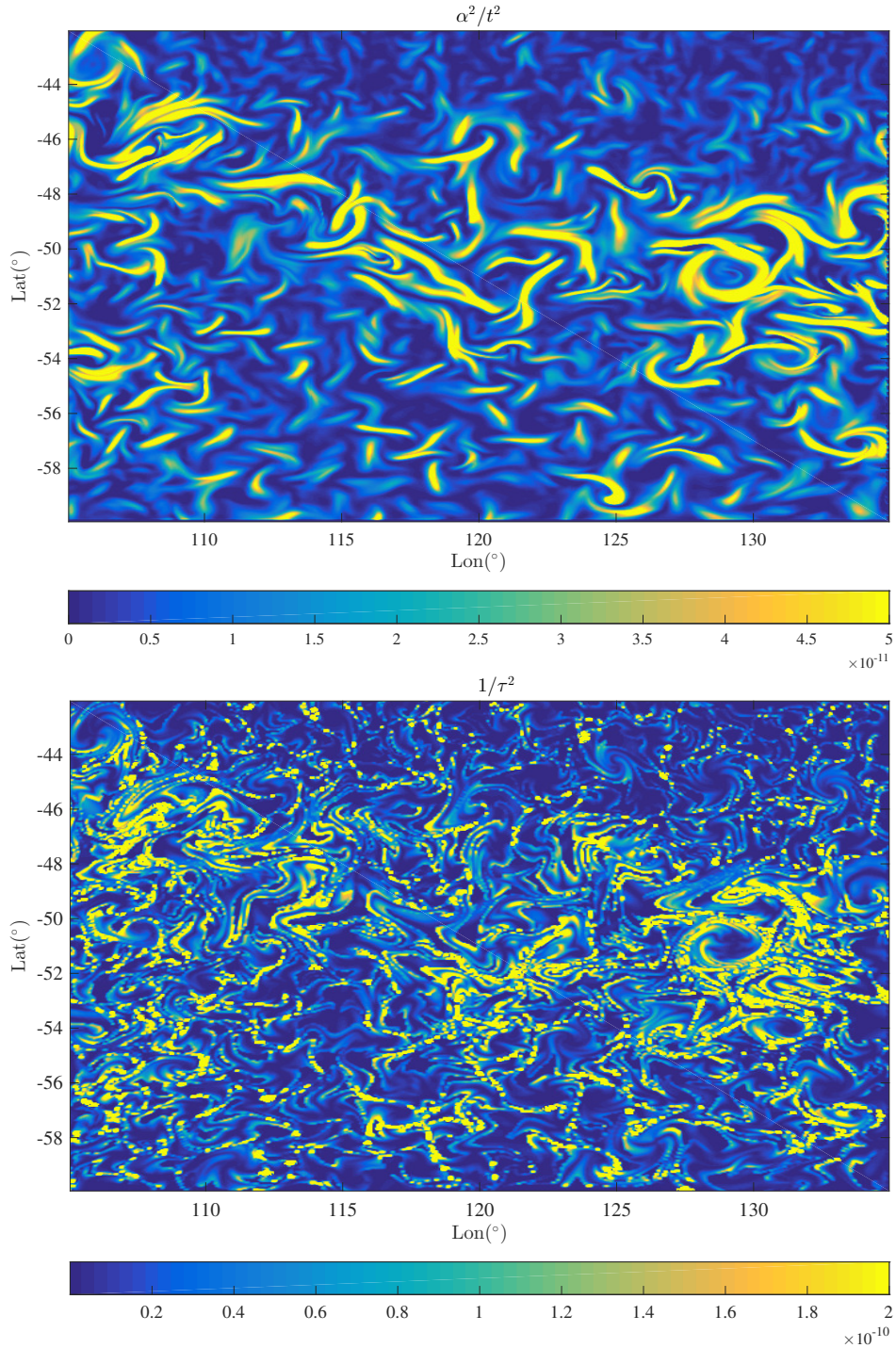
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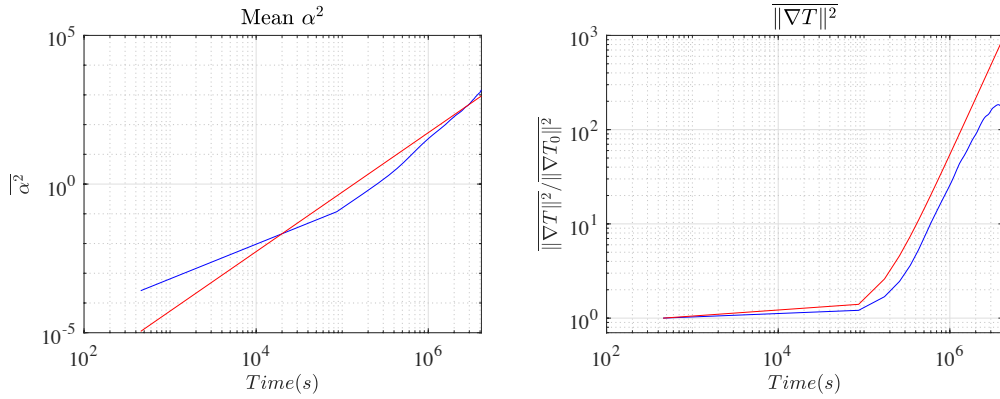
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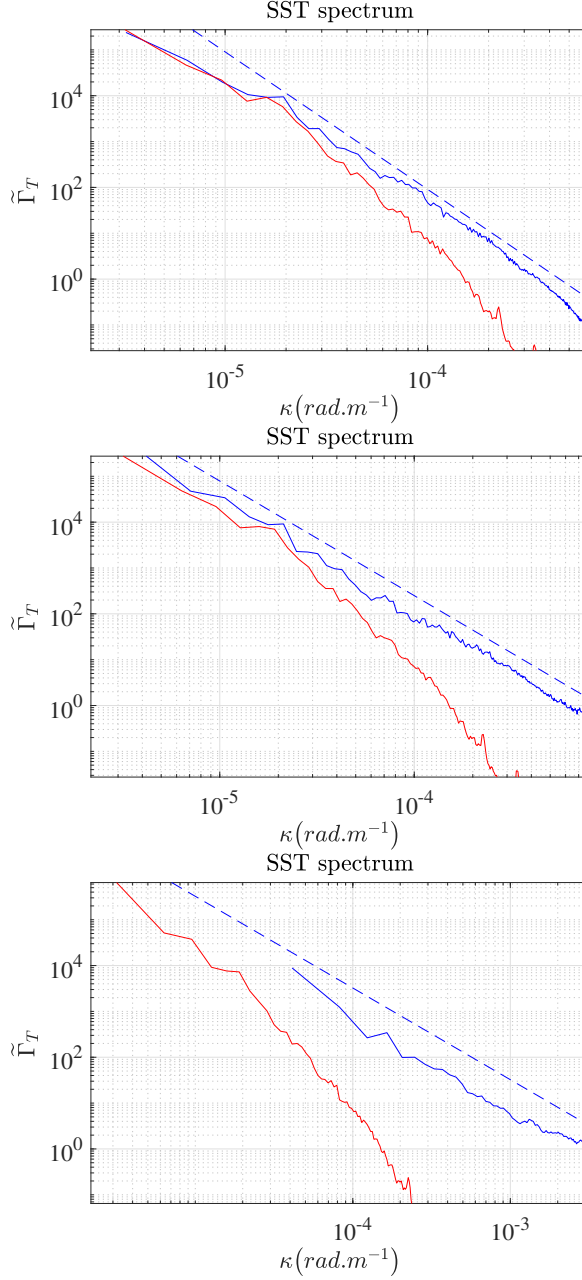
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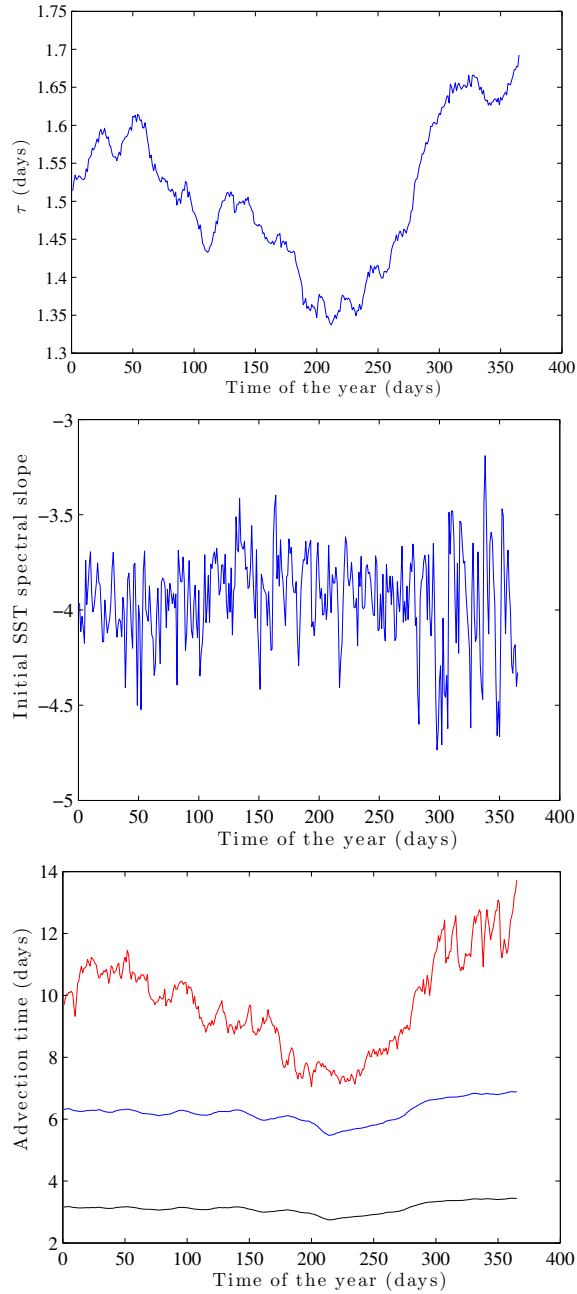
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