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# Smash and Grab: the $0 \cdot 6$ Scoring Game on Graphs\*

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## Abstract

In this paper, we introduce and study a new scoring game on graphs called SMASH AND GRAB. In this game, two players, called Left and Right, take turns removing a vertex of the graph as well as all of its neighbours that become isolated by this removal. For each player and each of their turns, they score the number of vertices that were removed on their turn. The game ends when there are no more vertices remaining, and the player with the highest final score wins. We denote by  $Ls(G)$  the difference between Left and Right's final scores in  $G$  when Left starts and both players play optimally (they both aim to maximise their scores).

We mainly study this parameter for different graph classes. We notably prove that  $Ls(F) \geq 0$  for any forest  $F$  (*i.e.*, the first player cannot lose). We then use this result to compute the exact value of  $Ls(G)$  for particular forests such as unions of paths and subdivided stars. The result in paths then solves the case of a unique cycle. Finally, we prove that, for a generalisation of the game, computing the score is PSPACE-complete.

**Keywords:** Scoring game, Games on graphs, Combinatorial game theory, PSPACE-complete

## 1 Introduction

Scoring game theory was introduced by Milnor [12] and Hanner [8] in the 1950s. For the rest of the 20th century, their work was not followed up, but rather inspired both the construction of the so-called economic and combinatorial game theories. It is only in the last

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two decades that new results appeared on the topic, in particular, with the introduction of general frameworks of resolutions for particular families (also called *universes*) of scoring games.

Combinatorial game theory deals with finite 2-player games with perfect information and where players alternate turns. The main difference between scoring and non-scoring combinatorial game theory lies in the winning convention. In non-scoring combinatorial games, the winner only depends on who makes the last move. In scoring combinatorial games, points are awarded to the players as the game progresses, and the winner is the player having the highest number of points when there is no more move available. In Larsson et al. [10], the authors summarise the main differences between the two families of games and explain why non-scoring games are far better understood than scoring ones. They also give a short survey about the different universes of scoring games studied so far, as well as a general framework (including scoring notations) to deal with them.

In parallel to these general studies, several particular scoring games played on graphs have been considered in the literature (see [10] for a list, and, *e.g.*, [1, 3] for more recent papers). Yet, for almost all these games, the formalism described in [10], as well as the frameworks of the different universes of the literature, were not considered. To the best of our knowledge, the work presented in [6] is the first study of a scoring game within the recent scoring framework of Larsson et al. In the current paper, we propose to study a second game according to this general framework, indifferently called SMASH AND GRAB or the 0·6 scoring game. Before giving the rules, we first give the main motivations of our study:

- this game naturally extends another well-studied scoring game of the literature, namely the *graph-grabbing game* [17];
- this game can also be considered as a natural scoring version of the well-known octal game 0·6 [4], where isolated pins are points for the players;
- this game will be studied within the formalism of [10], that should now be considered as a reference for scoring game theory;
- within this formalism, this game can be embedded in the so-called Ettinger’s universe (see Section 2 for more details). To the best of our knowledge, there is no other concrete game studied in the literature that explicitly refers to it;
- in its generalised version, a proof of the PSPACE-completeness of the game is given, which is rare for combinatorial games (especially in a scoring context).

**Definition 1** (SMASH AND GRAB). The game SMASH AND GRAB is played on an undirected graph  $G$  without isolated vertices. Two players, called Left and Right, take turns removing a vertex  $x$  of  $G$ , as well as all the vertices that become isolated by this removal. The number of vertices removed by each move of a player is added to his/her score. The objective of each player is to maximise his/her score.

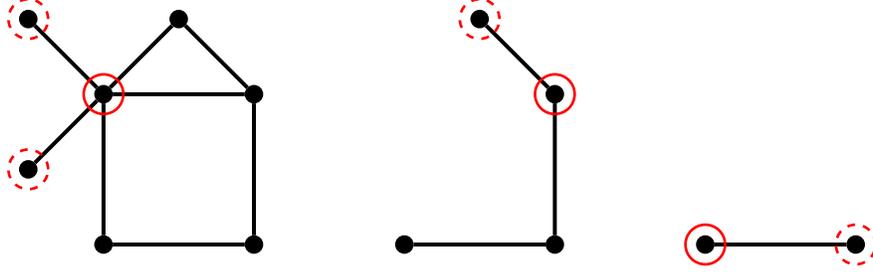


Figure 1: An example of SMASH AND GRAB, where the game ends after three moves. If Left starts, she scores  $3 + 2$  points (first and third figures), while her opponent scores 2 points (second figure). It can be proved that these moves are optimal for both players. Hence,  $Ls(G) = 5 - 2 = 3$ .

As usual for finite 2-player games with perfect information, games will be studied by assuming that both players are playing optimally. By following the definitions of Larsson et al. [10], we define the Left (Right, resp.) score of a scoring game  $G$ , denoted by  $Ls(G)$  ( $Rs(G)$ , resp.) as the difference between the number of points won by Left and Right, assuming that Left (Right, resp.) starts. Consequently, a positive (negative, resp.) score means that the game is winning for Left (Right, resp.). In addition, SMASH AND GRAB has the specificity that Left and Right always have the same available moves during the game. With respect to the combinatorial game terminology, such games are called *impartial*. We denote by  $\mathcal{U}$  the set of all impartial scoring games. Consequently, we have for any game  $G$  in  $\mathcal{U}$ :

$$Ls(G) = -Rs(G). \quad (1)$$

As a direct consequence, an impartial scoring game satisfying  $Ls(G) > 0$  (or, equivalently,  $Rs(G) < 0$ ) is a game for which the first player to move has a winning strategy.

Figure 1 gives an example of an optimal sequence of moves in SMASH AND GRAB. Note that, for the rest of the paper, since a starting instance of SMASH AND GRAB is a graph  $G$ , the notation  $G$  will be used both to define a graph and an instance of the game. Thus,  $Ls(G)$  will denote the Left score of the game played on the graph  $G$ .

Other instances of SMASH AND GRAB show that the first or second player may win, or the game may end in a draw. Indeed, it suffices to consider small sizes of paths and cycles:

- $Ls(P_4) = 0$ ;
- $Ls(P_5) = 1$ ;
- $Ls(C_4) = -2$ .

In what follows, we will denote by  $s(x, G)$  the number of points earned by a player after playing a vertex  $x$  in  $G$ , and by  $N^1[x]$  the set  $\{x \cup D_1\}$ , where  $D_1$  is the set of neighbours of  $x$  of degree 1. In other words,  $s(x, G)$  is equal to  $|N^1[x]|$ .

Note that a naïve strategy for SMASH AND GRAB that would consist in choosing, at each step, a vertex  $x$  that maximises  $s(x, G)$  may not be productive. As an example, the optimal

move on  $P_5$  consists in playing the middle vertex or an extremity (yielding only one point immediately, but two points afterwards). The other possible move (that gives two points immediately) is losing since the second player can end and win the game with his answering move. However, the score of SMASH AND GRAB can be computed recursively as follows:

$$Ls(G) = \max_{x \in V(G)} \{s(x, G) - Ls(G \setminus N^1[x])\},$$

with  $Ls(G) = 0$  when  $G$  is empty.

The main issue of the current work is to determine  $Ls(G)$  as the above formula does not yield a tractable algorithm. In some cases, we will exhibit an optimal strategy that corresponds to the score. Recall that, like many combinatorial games, there is no trivial correlation between these two problems.

As announced previously, this game is at the crossroads of several types of games. Firstly, it can be considered as a variation of well-known scoring games. In the *graph-grabbing game* [11], played on a connected graph with coins on its vertices, each player removes a non-cut vertex  $x$  of the graph and pockets the coins on  $x$ . By adding to each vertex of an instance of SMASH AND GRAB a number of coins equal to one plus its number of neighbors of degree 1 (and by updating this number when playing), one gets an extension of the *graph-grabbing game*, where any vertex  $x$  can be removed. For this game, there have been several results ensuring that the first player wins the game when  $G$  is an even tree (with a final proof in [14]), and it is conjectured that it remains true if the graph is bipartite and of even size. Compared to this game, a high interest of SMASH AND GRAB is about non-connected graphs, as disjoint unions of components are generally the major pitfall when solving combinatorial games. Another very close game is STRING AND COINS [4], that is a dual version of the well-known DOTS AND BOXES. In STRING AND COINS, the players take turns removing one edge of a graph, and each time a vertex becomes isolated, the player gets one point and plays again. SMASH AND GRAB is the same as STRING AND COINS, except that vertices are removed instead of edges, and the players do not play again when they get points (which is a more standard rule).

Secondly, SMASH AND GRAB can also be considered as a natural scoring version of the octal game  $0 \cdot 6$  [5]. In [2], octal games played on graphs were considered. For the game  $0 \cdot 6$ , a move consists in removing any vertex of the graph, except those that are isolated. The first player unable to move loses. The use of graphs is a nice way to analyse the algorithmic complexity of such games in a more generic structure than a simple path. The resolution of the game  $0 \cdot 6$  (also called OFFICERS) is still open on paths. However, transforming this game into a scoring version (by getting points for each isolated vertex), radically modifies the complexity, as we will see that the game will be more easily solved on paths. In addition, note that this transformation can be generalised for several octal games (not only for  $0 \cdot 6$ ), and thus, opens the door to a new family of scoring games.

The paper is organised as follows. In Section 2, some background on scoring combinatorial theory will be given in order to set the context in which the game is played. Section 3 is

about forests, for which it is proved that  $Ls(G)$  is non-negative. The exact value of  $Ls(G)$  will then be given for particular forests, *i.e.*, for unions of paths in Section 4 and subdivided stars in Section 5. In Section 6, the case of a unique cycle is solved, and the cases where the first player loses are fully identified. Finally, the algorithmic complexity of computing the score is examined in Section 7, with a proof of the PSPACE-completeness of an extended version of the game. The last section is about perspectives, in particular about a characterisation of the equivalence class of 0.

## 2 Background on scoring game theory

In the literature, different universes of scoring games have been defined in the last decades according to structural properties of the games. As pioneers of this theory, Milnor and Hanner restricted their study to nonzugzwang dicot games, also called Milnor's universe:

- a game  $G$  has no zugzwang if each player always prefers moving rather than missing his turn. In other words, it satisfies  $Ls(G) \geq Rs(G)$  and  $Ls(G') \geq Rs(G')$  for every subposition  $G'$  of  $G$  (*i.e.*, there exists a sequence of legal moves where  $G'$  can be reached from  $G$ ).
- a game is a dicot if, for every subposition of the game, a player can move if and only if his opponent also can.

We will denote by  $\mathcal{U}_{\geq 0}$  the set of all impartial scoring games belonging to the class of nonzugzwang dicot games. From Equation 1, an impartial scoring game  $G$  belongs to  $\mathcal{U}_{\geq 0}$  if and only if  $Ls(G) \geq 0$  and  $Ls(G') \geq 0$  for every subposition  $G'$  of  $G$ . Indeed, only the nonzugzwang property must be verified since every impartial game is a dicot.

SMASH AND GRAB does not belong to  $\mathcal{U}_{\geq 0}$  in general since it has zugzwang instances like  $C_4$ . Yet, there are particular instances of SMASH AND GRAB that belong to  $\mathcal{U}_{\geq 0}$  (*e.g.*,  $P_5$ ).

Later, in his thesis [7], Ettinger proposed general results for the universe of dicot scoring games (so-called Ettinger's universe), where zugzwang positions are allowed. Hence,  $\mathcal{U}$ , the universe of all impartial games, corresponds to the subset of Ettinger's universe with impartial games.

*Remark 2.* Note that, for a better consistency of the paper, all the results will be presented for the impartial universes  $\mathcal{U}$  and  $\mathcal{U}_{\geq 0}$ . In this paper, we will refer to Milnor's and Ettinger's universes for the extended universes  $\mathcal{U}_{\geq 0}$  and  $\mathcal{U}$  that are not restricted to impartial games. According to [12] and [7], if not mentioned explicitly, all the results of the current section remain true for Milnor's and Ettinger's universe.

Clearly, the game SMASH AND GRAB belongs to  $\mathcal{U}$ . In what follows, we will only present the material required for solving the so-called *sums of games*. Given two scoring games  $G_1$  and  $G_2$ , the sum  $G_1 + G_2$  is the game where a move consists in choosing to play either on  $G_1$

or on  $G_2$ . The game ends when no move is available, neither on  $G_1$  nor on  $G_2$ . Naturally, if  $G_1$  and  $G_2$  are dicots, then  $G_1 + G_2$  is also a dicot. Ettinger's universe is thus closed for the sum operator.

In SMASH AND GRAB, playing on a non-connected graph is equivalent to playing on the sum of its connected components. A major issue of combinatorial game theory is about the computation of the outcome (*i.e.*, who wins) of a sum of games, from the knowledge of each term of the sum. For that purpose, researchers introduced an equivalence relation, that also makes sense in the scoring context [7, 12]:

**Definition 3.** Let  $\mathcal{F}$  be a subset of  $\mathcal{U}$ , and let  $G$  and  $H$  be two games of  $\mathcal{U}$ . We say that  $G \equiv_{\mathcal{F}} H$  if, for any game  $X \in \mathcal{F}$ ,  $Ls(G + X) = Ls(H + X)$ .

When considering equivalence, one generally hopes to have the most general result, *i.e.*, for the case  $\mathcal{F} = \mathcal{U}$ . However, two games may not be equivalent over  $\mathcal{U}$  but may be equivalent in a smaller universe, as we will see later for some instances of SMASH AND GRAB. Such weaker results remain interesting, in practice, to simplify the computation of the score of particular games. In addition, note that the above definition can be extended to all scoring games (not necessarily impartial), by considering both Left and Right scores in the definition (see [7]).

In order to solve sums of games, combinatorial game theory (scoring or not) aims at finding a characterisation of the equivalence of two games  $G$  and  $H$  that would only require the examination of them (*i.e.*, without considering any game  $X$  as in the definition). If  $G$  and  $H$  belong to Milnor's universe, then the following nice result holds from [12]:

**Theorem 4** (Milnor [12]). *Let  $G$  and  $H$  be in  $\mathcal{U}_{\geq 0}$ . Then,  $G \equiv_{\mathcal{U}_{\geq 0}} H$  if and only if  $Ls(G + H) = 0$ .*

Roughly speaking, it suffices to compute the score of the sum  $G + H$  to decide whether  $G$  and  $H$  are equivalent in Milnor's universe. If 0 denotes the empty game with a score of zero, this result states that the equivalence class of 0 corresponds exactly to the games  $G$  satisfying  $Ls(G) = 0$ . It means that when playing a sum of games in  $\mathcal{U}_{\geq 0}$ , all of the components whose score is zero can be removed without changing the final score of the sum. This result also states that each game  $G$  of  $\mathcal{U}_{\geq 0}$  is its own inverse, *i.e.*,  $G + G \equiv_{\mathcal{U}_{\geq 0}} 0$ .

If the nonzuzwang property is removed, we fall into Ettinger's universe and the characterisation of the equivalence of two games is far trickier. Yet, some characterisations of the equivalence were given according to the structures of  $G$  and  $H$  [7]. Among them, the following nice property is given for the equivalence class of 0. First, we say that a game  $G$  is *Left-save* if  $Ls(G) \geq 0$  and, for any Right move  $G'$  from  $G$ , there exists a Left answer  $G''$  that is Left-save. Informally, it means that Left can force an even number of moves when playing second in  $G$ .

**Theorem 5** (Ettinger [7]). *Let  $G$  be in  $\mathcal{U}$ . Then,  $G \equiv_{\mathcal{U}} 0$  if and only if  $Ls(G) = 0$  and  $G$  is Left-save.*

As a preliminary result, we give a useful lemma about winning strategies for SMASH AND GRAB. It claims that if Left is playing optimally in an instance of  $\mathcal{U}_{\geq 0}$ , then after each of her turns, the current score must be at least  $Ls(G)$ . Its presentation requires the definition of the *actual* score of a game after  $k$  moves:

**Definition 6.** Let  $G$  be a position of SMASH AND GRAB, and  $x_1, x_2, \dots, x_{2k+1}$  a sequence of an odd number of moves from  $G$ , where  $x_{2i+1} \in V(G)$  are the vertices played by Left and  $x_{2i} \in V(G)$  are the vertices played by Right. Let  $G_0 = G$  and let  $G_i$  be the remaining graph after playing  $x_i$ . The actual score in  $G$  after the  $(2k+1)^{\text{th}}$  move is

$$s_{2k+1,G} = \sum_{i=0}^k s(x_{2i+1}, G_{2i}) - \sum_{i=1}^k s(x_{2i}, G_{2i-1})$$

Analogously, if  $x_1, x_2, \dots, x_{2k}$  is a sequence of an even number of moves from  $G$ , the actual score after the  $(2k)^{\text{th}}$  move is

$$s_{2k,G} = \sum_{i=1}^k s(x_{2i-1}, G_{2i-2}) - \sum_{i=1}^k s(x_{2i}, G_{2i-1})$$

**Lemma 7.** Let  $G$  be a position of SMASH AND GRAB that is in  $\mathcal{U}_{\geq 0}$ . Consider the first  $2k+1$  moves of a game in  $G$ :  $x_1, x_2, \dots, x_{2k}, x_{2k+1}$  according to Definition 6. If the vertices played by Left correspond to a strategy leading to a score  $Ls(G) \geq 0$ , then we have:

$$s_{2k+1,G} \geq Ls(G).$$

Analogously, if the vertices played by Right correspond to a strategy leading to a score  $Ls(G) \geq 0$ , then we have:

$$s_{2k,G} \leq Ls(G).$$

*Proof.* In the first case, we have that  $Ls(G) = s_{2k+1,G} + Rs(G_{2k+1})$ , since after an odd number of moves, it is Right's turn. Since the game is in  $\mathcal{U}_{\geq 0}$ , we have that  $Rs(G_{2k+1}) \leq 0$ , implying the desired result. Similarly, in the second case, we have  $Ls(G) = s_{2k,G} + Ls(G_{2k})$ , and since  $Ls(G_{2k}) \geq 0$ , the proof is completed. □

The next corollary is a simple application of the above lemma after the first move of Left.

**Corollary 8.** Let  $G$  be a position of SMASH AND GRAB that is in  $\mathcal{U}_{\geq 0}$ . In an optimal strategy for Left, her first move must score at least  $Ls(G)$  points.

In addition, since the total number of points won by both players equals  $|V(G)|$  (i.e., the total number of vertices), we have the following observation:

**Observation 9.** For any instance  $G$  of SMASH AND GRAB,  $Ls(G) \equiv |V(G)| \pmod{2}$ .

### 3 Forests are non-negative games

In this section, we first prove that the family of forests is included in  $\mathcal{U}_{\geq 0}$ . Recall that a forest is the disjoint union of trees. We then give sufficient conditions for the game to be a draw in forests  $F$ , *i.e.*,  $Ls(F) = 0$ . For a union of paths  $G$ , we use both of these results to characterise the score of  $G$  and the equivalence classes in  $G$  (modulo  $\mathcal{U}_{\geq 0}$ ) in Section 4.

**Theorem 10.** *For any forest  $F$ ,  $Ls(F) \geq 0$ .*

*Proof.* Towards a contradiction, assume that the result is not true and let  $F$  be the smallest forest for which  $Ls(F) < 0$ . In particular, with Right playing optimally, after any even number  $2i$  of moves in  $F$ , the actual score must satisfy  $s_{2i,F} < 0$ . Indeed, as in the proof of Lemma 7, we have  $Ls(F) = s_{2i,F} + Ls(F_{2i+1})$ , where  $F_{2i+1}$  is the forest remaining after the first  $2i$  moves. By the minimality of  $F$ , we have  $Ls(F_{2i+1}) \geq 0$ , and thus,  $s_{2i,F} < 0$ .

In particular, Right's first move must score at least one more than Left's first move. This means that any vertex  $u$  that maximises  $s(u, F)$ , that is, a vertex with a maximal number of leaves, must be connected by a vertex of degree 2 to another vertex with the same number of leaves. This way, if Left plays  $u$ , then there is always a move for Right that scores  $s(u, F) + 1$ .

Let  $u$  and  $v$  be two such vertices with a maximal number of leaves that are both adjacent to a vertex  $z$  of degree 2. Let  $F' = F - \{z\}$ , and note that we must have  $Ls(F') \geq 2$ . Indeed, imagine Left plays on  $z$  on her first turn removing only the vertex  $z$ . Then, since  $Ls(F) < 0$  and  $Ls(F) \geq 1 - Ls(F')$ , we necessarily have  $Ls(F') \geq 2$ .

We will now consider the two games  $F$  and  $F'$  played in parallel. Left starts in both games. We know that  $Ls(F) < 0$  and  $Ls(F') \geq 2$ . We will assume that Left plays optimally in  $F'$  in order to achieve  $Ls(F') \geq 2$ , and that Right plays optimally in  $F$  in order to guarantee  $Ls(F) < 0$ . We will denote by  $F_i$  and  $F'_i$  the remaining forests after move  $i$ . The method works as follows: Left chooses an optimal move  $x_1$  in  $F'_0 = F'$  and does the same move in  $F_0 = F$ . After that, Right chooses an optimal move  $x_2$  in the remaining forest  $F_1$  and mimics the move in  $F'_1$ . Then, Left again chooses an optimal move in the remaining forest  $F'_2$  and mimics the move in  $F_2$  and so on. This stops when  $z$  is removed.

We will prove the following facts:

1. We have  $s_{2i+1,F'} \geq 2$  and  $s_{2i,F} < 0$ .
2. If  $z$  is present, the actual scores in both games are equal: after  $k$  moves, we have  $s_{k,F} = s_{k,F'}$ .
3. When  $z$  is removed, we get a contradiction.

Item 1 is a consequence of Lemma 7 and the fact that Left plays optimally in  $F'$  and Right plays optimally in  $F$  (in the proof of Lemma 7, we just need that all the  $F_i$  for  $i \geq 1$  are in  $\mathcal{U}_{\geq 0}$ ).

To prove Item 2, we prove that before  $z$  is removed, the vertices removed in both games are the same, which in particular implies that the actual scores are equal. Consider the first move  $x_1$  of Left in  $F'$ . If playing  $x_1$  in  $F'$  removes  $u$  ( $v$ , resp.) and  $x_1 \neq u$  ( $x_1 \neq v$ , resp.),

then this component is  $K_2$ . Indeed, by the definition of  $u$ ,  $u$  has at least one leaf. Removing  $u$  when playing another vertex  $x_1$  means that  $u$  is a leaf of  $x_1$ , and thus,  $u$  has degree 1, and its component is  $K_2$ . Then, playing  $u$  is equivalent to playing  $x_1$ , and we can consider that  $u$  was played in this case. We can therefore assume that on the first move, if  $u$  ( $v$ , resp.) is removed, it is because the vertex played is  $u$  itself ( $v$ , resp.). Then, the vertices removed in  $F$  and  $F'$  are the same and  $s_{1,F} = s_{1,F'}$ .

After that, as long as  $z$  is still present, whenever Right plays on  $F_{2i+1}$ , after his move we should have  $s_{2i+2,F} < 0$  by Lemma 7 since  $Ls(F) < 0$  by assumption and Right plays optimally in  $F$ . Since  $s_{2i+1,F} = s_{2i+1,F'} \geq 2$ , Right must score at least three points, and thus, Right cannot remove a leaf of  $u$  or  $v$ . Similarly, Left must ensure the actual score  $s_{2i+2,F'}$  is at least 2 after her next move, and thus, she must also score at least three points. Again, no leaf of  $u$  or  $v$  can be Left's choice. Therefore, as long as  $z$  is still present, no leaf of  $u$  or  $v$  has been played. The only way these leaves can be removed is through moves on  $u$  or  $v$ . In particular,  $u$  and  $v$  can only be removed by playing them. As a consequence, if  $z$  has not yet been removed, the moves in  $F$  and  $F'$  are the same and the actual scores are equal, which proves Item 2.

At some point,  $z$  must be removed. It has to be done through a move on another vertex: Left never removes  $z$  because she is mimicking moves done in  $F'$ . Right never removes  $z$  since he must score at least three points (and  $u$  and  $v$  cannot be leaves). Therefore, removing  $z$  does not allow a difference of more than one point to Right, in relation to the parallel game. After that move, both forests are equivalent, and Left may obtain a final score of  $Ls(F) \geq 1$  by following her strategy in  $F'$  which guarantees  $Ls(F') \geq 2$ . A contradiction.  $\square$

In the next theorem, we give sufficient conditions for forests  $F$  to satisfy  $Ls(F) = 0$ .

**Theorem 11.** *For any forest  $F$ , if the number of vertices of  $F$  is a multiple of 4 and  $F$  admits a perfect matching, then  $Ls(F) = 0$ .*

*Proof.* We will prove the result by induction on  $|V(F)|$ . When  $|V(F)| = 4$ , since  $F$  admits a perfect matching,  $F$  is either  $P_4$  or the disjoint union of two  $K_2$ 's. In both cases, it is trivial to see that  $Ls(F) = 0$ . Let  $k > 1$  be an integer and assume now that the result is true for all forests of order  $4k'$  with  $k > k' \geq 0$  and  $k'$  an integer. Let  $F$  be a forest such that  $|V(F)| = 4k$ . First, note that Left cannot score more than two points on her first turn. Indeed, if she can score at least 3, there would be a vertex adjacent to at least two leaves, and then  $F$  would not admit a perfect matching, a contradiction. Hence,  $Ls(F) \leq 2$  by Lemma 7. Secondly,  $Ls(F) \geq 0$  by Theorem 10 since  $F$  is a forest, and thus, by Observation 9, we have that  $Ls(F) \in \{0, 2\}$  since  $|V(F)|$  is even. Thus, we just need to show that  $Ls(F) \neq 2$ .

Assume by contradiction that  $Ls(F) = 2$ . Let  $M$  be a perfect matching of  $F$ . By Lemma 7, Left must score two points on her first turn. To do so, Left must remove a vertex  $v$  adjacent to a leaf  $u$ . The edge  $uv$  must be in  $M$  since  $u$  is a leaf. Let  $F_1$  be the graph remaining after Left's first turn. Since  $uv \in M$ ,  $F_1$  admits a perfect matching  $M_1 = M \setminus \{uv\}$ . Since  $F_1$  is a forest, it must contain a leaf  $u_1$  adjacent to a vertex  $v_1$ . Right removes  $v_1$  and scores at least 2 since  $u_1$  is also removed in the process. Again,  $u_1v_1$  must be in  $M_1$  since  $u_1$  is a leaf. Let  $F_2$  be the graph remaining after Right's first turn. It is clear that  $F_2$  is a forest with a

perfect matching  $M_2 = M_1 \setminus \{u_1v_1\}$ . Since  $|V(F)| - |V(F_2)| = 4$ , we have that  $Ls(F_2) = 0$  by the inductive hypothesis. Thus,  $Ls(F) \leq Ls(F_2) = 0$  since the relative score after the first turns of Left and Right is at most 0.  $\square$

Note that these conditions are not necessary since  $Ls(P_3 + P_3) = 0$ . Thus, there exist forests  $F$  such that  $Ls(F) = 0$  and  $F$  does not admit a perfect matching, and forests  $F$  such that  $Ls(F) = 0$  and  $|V(F)| \neq 4k$  for some integer  $k \geq 0$ .

## 4 Complete characterisation for unions of paths

When  $G$  is a union of paths, we are able to completely characterise the score of  $G$  and the equivalence classes in  $G$  (modulo  $\mathcal{U}_{\geq 0}$ ). In particular, the score of  $G$  can be computed in linear time. We denote by  $P_i$  the path on  $i$  vertices. By convention,  $P_0$  is the empty graph.

**Theorem 12.** *Let  $\mathcal{P}$  be the class of unions of paths. For any  $i \geq 0$ , with  $i \neq 3$ ,  $P_i \equiv_{\mathcal{U}_{\geq 0}} P_{i+4}$ , and  $P_1 + P_2 \equiv_{\mathcal{U}_{\geq 0}} P_7$ . Consequently, there are only eight equivalence classes in  $\mathcal{P}$  modulo  $\mathcal{U}_{\geq 0}$  which can be represented by the following graphs (grouped by their scores):*

- Score 0:  $P_0$ ;
- Score 1:  $P_1, P_2 + P_3, P_1 + P_2$ ;
- Score 2:  $P_2, P_1 + P_3, P_1 + P_2 + P_3$ ;
- Score 3:  $P_3$ .

*Proof.* First, recall that, by Theorem 10, we have that  $G \in \mathcal{U}_{\geq 0}$  for any graph  $G \in \mathcal{P}$ . For all  $n \geq 0$ , let  $v_1, \dots, v_n$  be the vertices of  $P_n$ . By induction on  $i$ , we first prove that, for any  $i \geq 0$ , with  $i \neq 3$ ,  $P_i \equiv_{\mathcal{U}_{\geq 0}} P_{i+4}$ . Recall that, by Theorem 4, we just need to prove that  $Ls(P_i + P_{i+4}) = 0$ . For the base cases, we have that  $Ls(P_0) = Ls(P_4) = 0$ ,  $Ls(P_1) = 1$ ,  $Ls(P_2) = 2$ ,  $Ls(P_3) = 3$ , and  $Ls(P_7) = 1$ . Now, as the inductive hypothesis, suppose that  $P_j \equiv_{\mathcal{U}_{\geq 0}} P_{j+4}$  for all  $0 \leq j < i$ , with  $j \neq 3$ . In what follows, for any two graphs  $G$  and  $H$ , where  $|V(G)| \geq |V(H)|$ , whenever  $G \equiv_{\mathcal{U}_{\geq 0}} H$ , we replace  $G$  by  $H$ . There are 3 cases:

1.  $i = 0 \pmod 4$  or  $i = 2 \pmod 4$ :  $Ls(P_i + P_{i+4}) = 0$  by Theorem 11.
2.  $i = 1 \pmod 4$ : by the inductive hypothesis, we have that  $P_i \equiv_{\mathcal{U}_{\geq 0}} P_1$ . We just have to prove that  $Ls(P_{i+4} + P_1) = 0$ . By Theorem 10, we have  $Ls(P_{i+4} + P_1) \geq 0$ . To prove that  $Ls(P_{i+4} + P_1) \leq 0$ , we give a strategy for Right. If Left first plays  $v_1$  (or  $v_n$  by symmetry) in  $P_{i+4}$ , Right can take the isolated vertex of  $P_1$ . Then  $P_{i+3}$  is remaining. By Theorem 11,  $Ls(P_{i+3}) = 0$  and we are done. Similarly, if Left first plays on  $P_1$ , then Right plays  $v_1$  in  $P_{i+4}$ , and as above,  $Ls(P_{i+4} + P_1) \leq 0$ . If Left first plays  $v_2$  (or  $v_{n-1}$  by symmetry) in  $P_{i+4}$ , then she scores 2 and this leaves behind  $P_{i+2} + P_1$ . Since  $i + 2 = 3 \pmod 4$ , by the inductive hypothesis, either  $P_{i+2} \equiv_{\mathcal{U}_{\geq 0}} P_7$  or  $P_{i+2} = P_3$ . In the latter case, Right plays on the center vertex of  $P_3$ , scoring 3, and

thus,  $Ls(P_{i+4} + P_1) \leq 0$ . In the former case, Right plays on  $v_2$  in  $P_7$ , scoring 2, which leaves behind  $P_5 + P_1$ , but  $P_5 \equiv_{\mathcal{U}_{\geq 0}} P_1$  by the inductive hypothesis and the two  $P_1$ 's cancel each other out. Hence,  $Ls(P_{i+4} + P_1) \leq 0$ . Thus, by symmetry, we can assume that Left first plays on  $v_x$  in  $P_{i+4}$  for some  $2 < x \leq \lceil (i+4)/2 \rceil$ . Hence, Left scores 1 on her first turn. Since  $i = 1 \pmod 4$ , this leaves behind  $P_y + P_z$  ( $y, z \geq 0$ ), where  $y = 2 \pmod 4$  and  $z = 2 \pmod 4$  or  $y = 0 \pmod 4$  and  $z = 0 \pmod 4$  or  $y = 3 \pmod 4$  and  $z = 1 \pmod 4$ . In the first (second, resp.) case,  $P_y$  and  $P_z$  cancel out since they are both equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to  $P_2$  ( $P_0$ , resp.) by the inductive hypothesis, and thus, all that remains is  $P_1$ , and hence,  $Ls(P_{i+4} + P_1) \leq 0$ . In the last case,  $P_z \equiv_{\mathcal{U}_{\geq 0}} P_1$  by the inductive hypothesis, and thus, the two  $P_1$ 's cancel out, leaving behind  $P_y$ , and  $Ls(P_y) \geq 1$  by the inductive hypothesis, and so,  $Ls(P_{i+4} + P_1) \leq 0$ .

3.  $i = 3 \pmod 4$ : by the inductive hypothesis, we have that  $P_i \equiv_{\mathcal{U}_{\geq 0}} P_7$  since  $i \neq 3$ . Now, we just have to prove that  $Ls(P_{i+4} + P_7) = 0$ , and actually, using Theorem 10, we just need to prove that  $Ls(P_{i+4} + P_7) \leq 0$ .

For that, we give a strategy for Right. If Left first plays on  $v_2$  (or  $v_{n-1}$  by symmetry) of  $P_{i+4}$ , then Right can play similarly on the resulting  $P_{i+2}$  to leave behind  $P_i + P_7$ , and  $Ls(P_i + P_7) = 0$  by the inductive hypothesis. Thus, by symmetry, if Left first plays on  $P_{i+4}$ , then we can assume that Left first plays on  $v_x$  of the  $P_{i+4}$  for some  $1 \leq x \leq \lceil (i+4)/2 \rceil$  and  $x \neq 2$ . Hence, Left scores 1 on her first turn. Since  $i = 3 \pmod 4$ , this leaves behind  $P_y + P_z$  ( $y, z \geq 0$ ), where  $y = 2 \pmod 4$  and  $z = 0 \pmod 4$  or  $y = 1 \pmod 4$  and  $z = 1 \pmod 4$  or  $y = 3 \pmod 4$  and  $z = 3 \pmod 4$ . In the first case, by the inductive hypothesis, we have that  $P_z \equiv_{\mathcal{U}_{\geq 0}} P_0$  and  $P_y \equiv_{\mathcal{U}_{\geq 0}} P_2$ , and thus, all that remains is  $P_2 + P_7$ , and hence,  $Ls(P_{i+4} + P_7) \leq 0$  since Right will play on the  $P_2$  and  $Ls(P_7) = 1$ . In the second case,  $P_y$  and  $P_z$  cancel out since they are both equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to  $P_1$  by the inductive hypothesis, and thus, all that remains is  $P_7$ , hence,  $Ls(P_{i+4} + P_1) \leq 0$  since  $Ls(P_7) = 1$ . In the last case, by the inductive hypothesis, either one of  $P_y$  and  $P_z$  is equivalent to  $P_7$  and the other is equivalent to  $P_3$  or  $P_y$  and  $P_z$  cancel out since they are both equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to  $P_7$  or  $P_3$ . Since the two  $P_7$ 's cancel out in the former case, all that remains in either case is either a  $P_3$  or a  $P_7$ , and thus,  $Ls(P_{i+4} + P_7) \leq 0$  since  $Ls(P_3) = 3$  and  $Ls(P_7) = 1$ .

The other case is when Left first plays on the  $P_7$ . Then, there is  $P_5 + P_{i+4}$  remaining if Left scored 2, and  $P_5 + P_{i+4} \equiv_{\mathcal{U}_{\geq 0}} P_1 + P_{i+4}$  since  $P_5 \equiv_{\mathcal{U}_{\geq 0}} P_1$  by the inductive hypothesis. Right then plays on the  $v_2$  of the  $P_{i+4}$ , leaving behind  $P_1 + P_{i+2}$ , and since  $i + 2 = 1 \pmod 4$ , by the inductive hypothesis,  $P_{i+2} \equiv_{\mathcal{U}_{\geq 0}} P_1$ , and thus, all that remains is  $P_1 + P_1 \equiv_{\mathcal{U}_{\geq 0}} P_0$ , and hence,  $Ls(P_{i+4} + P_7) \leq 0$ . Otherwise, Left scored 1, in which case, there is  $P_{i+4} + P_6$  or  $P_{i+4} + P_2 + P_4$  or  $P_{i+4} + P_3 + P_3$  remaining. By the inductive hypothesis, the first two cases are equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to  $P_{i+4} + P_2$  (since  $P_6 \equiv_{\mathcal{U}_{\geq 0}} P_2$  and  $P_4 \equiv_{\mathcal{U}_{\geq 0}} P_0$ ), and the last case is equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to  $P_{i+4}$ . For the first two cases, Right plays on the  $v_1$  of the  $P_{i+4}$ , leaving behind  $P_{i+3} + P_2$ , which is equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to  $P_2 + P_2 \equiv_{\mathcal{U}_{\geq 0}} P_0$  since  $P_{i+3} \equiv_{\mathcal{U}_{\geq 0}} P_2$  by the inductive hypothesis as  $i + 3 = 2 \pmod 4$ , hence,  $Ls(P_{i+4} + P_7) \leq 0$ . In the last case, Right plays

on the  $v_2$  of the  $P_{i+4}$ , scoring 2, and leaving behind  $P_{i+2}$ , which is equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to  $P_1$  since  $P_{i+2} \equiv_{\mathcal{U}_{\geq 0}} P_1$  by the inductive hypothesis as  $i + 2 = 1 \pmod 4$ , and hence,  $Ls(P_{i+4} + P_7) \leq 0$ .

This concludes the proof that, for any  $i \geq 0$ , with  $i \neq 3$ ,  $P_i \equiv_{\mathcal{U}_{\geq 0}} P_{i+4}$ .

Now, we prove that  $P_1 + P_2 \equiv_{\mathcal{U}_{\geq 0}} P_7$ . Recall that, by Theorem 4, we need to prove that  $Ls(P_1 + P_2 + P_7) = 0$  and using Theorem 10, we just need to prove that  $Ls(P_1 + P_2 + P_7) \leq 0$ . We give a strategy for Right. If Left plays in  $P_1$  first, she scores 1, and then, Right plays in  $P_2$  and scores 2. Then,  $Ls(P_1 + P_2 + P_7) \leq 0$  since  $Ls(P_7) = 1$ . If Left plays in  $P_2$  first, she scores 2. Then, Right plays on  $v_2$  of  $P_7$ , scoring also 2. Hence, there is  $P_1 + P_5 \equiv_{\mathcal{U}_{\geq 0}} P_1 + P_1 \equiv_{\mathcal{U}_{\geq 0}} P_0$  remaining, and thus,  $Ls(P_1 + P_2 + P_7) \leq 0$ . Lastly, if Left plays in  $P_7$  first, then Right plays on  $P_2$ , scoring 2. Then, there is  $P_1 + P_5$  remaining if Left scored 2, and, as in the previous case,  $Ls(P_1 + P_2 + P_7) \leq 0$ . Otherwise, Left scored 1, in which case, there is  $P_1 + P_6 \equiv_{\mathcal{U}_{\geq 0}} P_1 + P_2$  or  $P_1 + P_2 + P_4 \equiv_{\mathcal{U}_{\geq 0}} P_1 + P_2$  or  $P_1 + P_3 + P_3 \equiv_{\mathcal{U}_{\geq 0}} P_1$  remaining. Hence, in each of the cases,  $Ls(P_1 + P_2 + P_7) \leq 0$ .

It is then easy to see that any for any  $P \in \mathcal{P}$ ,  $P$  is equivalent (modulo  $\mathcal{U}_{\geq 0}$ ) to one of the eight equivalence classes in the statement of the theorem. To finish, one can check that none of these eight equivalence classes are equivalent (modulo  $\mathcal{U}_{\geq 0}$ ). Indeed, summing any two elements of different classes with the same score leads to a graph with positive score.  $\square$

In particular, we have the following scores for paths:

**Corollary 13.** *Let  $n \geq 1$ . Then,*

$$Ls(P_n) = \begin{cases} 0 & \text{if } n = 0 \pmod 4 \\ 1 & \text{if } n = 1 \pmod 2 \text{ and } n \neq 3 \\ 2 & \text{if } n = 2 \pmod 4 \\ 3 & \text{if } n = 3 \end{cases}$$

## 5 Relative scores of subdivided stars

In this section, we give closed formulas to compute the score of subdivided stars. A subdivided star is a tree with at most one vertex of degree greater than 2. This vertex is called the root. A subdivided star is characterised by the lengths of the paths that are starting from the root (without counting the root as a vertex of the path). We first deal with subdivided stars with at least two paths of length 1.

**Proposition 14.** *Let  $G$  be a subdivided star with root  $r$ ,  $k \geq 2$  paths of length 1, and  $t$  paths of length  $\ell_1, \dots, \ell_t$  with  $\ell_i \geq 2$  for all  $1 \leq i \leq t$ . Then, an optimal first move for Left is to remove  $r$ , leading to a score of  $Ls(G) = k + 1 - Ls(\sum_{i=1}^t P_{\ell_i})$ .*

Note that the score of  $\sum_{i=1}^t P_{\ell_i}$  can be directly computed using Theorem 12: reduce the paths modulo 4 (except for  $P_7$  that must be reduced to  $P_1 + P_2$ ) and remove the pairs of paths of the same size.

*Proof.* First, note that playing the central vertex  $r$  ensures a non-negative relative score for Left since she scores at least 3, and, in a union of paths, the relative score is at most 3.

Assume that playing  $r$  is not an optimal first move for Left. In particular, this means that  $Ls(G) > 0$ . Consider an optimal first move  $x$  for Left.

If this move scores 1, then by Corollary 8,  $Ls(G) = 1$ . Thus, by Observation 9,  $G$  has an odd number of vertices. But, since  $r$  is not an optimal move, it means that the score given by playing  $r$ , which must be non-negative, is necessarily 0. This would imply, again by Observation 9, an even number of vertices in  $G$ , a contradiction.

Therefore, playing  $x$  scores 2 (there is no vertex other than  $r$  that gives a score larger than 2) and does not remove  $r$  nor one of its leaves. Hence,  $x$  is a vertex of a path  $P_j$  for some  $j$ , adjacent to an extremity of  $P_j$ . Consider that Right answers by playing  $r$ , and note that he scores at least  $k + 1$  this way. Let  $H$  be the graph obtained after these two moves.

We have that  $Ls(G) \leq 2 - (k + 1) + Ls(H)$  since  $x$  is an optimal first move in  $G$ . By our hypothesis, since  $r$  was not an optimal first move, we have that  $Ls(G) > k + 1 - Ls(\sum_{i=1}^t P_{\ell_i})$ . From these two inequalities, we obtain that  $Ls(H) + Ls(\sum_{i=1}^t P_{\ell_i}) > 2k \geq 4$ . Note that  $H$  is a union of paths, hence,  $Ls(H) \leq 3$  by Theorem 12. Furthermore,  $H$  is exactly  $\sum_{i=1, \ell_i \neq j}^t P_{\ell_i} + P_{j-2}$ . Thus, by Observation 9, the relative score of  $H$  and the relative score of  $\sum_{i=1}^t P_{\ell_i}$  must be of the same parity, and so, we must have that  $Ls(H) + Ls(\sum_{i=1}^t P_{\ell_i}) = 6$ , meaning that  $Ls(H) = Ls(\sum_{i=1}^t P_{\ell_i}) = 3$ . But, using Theorem 12, since there is a unique equivalence class of unions of paths of score 3, it means that  $H$  and  $\sum_{i=1}^t P_{\ell_i}$  are equivalent. This is clearly a contradiction, since it would imply that  $P_j$  and  $P_{j-2}$  are equivalent (which is also not true by Theorem 12).  $\square$

We now consider subdivided stars with at most one path of length 1.

**Proposition 15.** *Let  $G$  be a subdivided star with root  $r$  and  $t$  paths attached to  $r$  of length  $1 \leq \ell_1 \leq \dots \leq \ell_t$  with  $\ell_i \geq 2$  for  $i > 1$ . Let  $n = \sum_{i=1}^t \ell_i + 1$  be the total number of vertices in  $G$  and let  $t_{\text{odd}}$  be the number of paths of odd length. Then:*

$$Ls(G) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ (n - 3(t_{\text{odd}} - 1)) \bmod 4 & \text{if } n \text{ is even.} \end{cases}$$

Note that this result includes the case where  $G$  is a path.

*Proof.* Since Left scores at most 2 on her first turn, we have that  $Ls(G) \leq 2$  by Lemma 7. Since the relative score must be non-negative (Theorem 10) and of the same parity as  $n$  (Observation 9), it is necessarily equal to 1 if  $n$  is odd and to 0 or 2 if  $n$  is even. This proves the odd part of the result.

Consider now that  $n$  is even. In particular,  $t_{\text{odd}}$  is odd. Let  $p(G) = n - 3(t_{\text{odd}} - 1)$ . This number corresponds to the number of vertices that remain after removing a  $P_3$  in all the paths of odd length except one. Since there is at most one path of odd length of size 1, a  $P_3$  is counted for  $t_{\text{odd}} - 1$  paths. In particular,  $p(G)$  is even since we remove an even number of vertices, and  $p(G) \geq 2$  since at least one path of odd length and the root remain.

Note that if we consider that the  $P_3$ 's we removed all start by an adjacent vertex of the root, then there exists a perfect matching  $M(G)$  in the remaining vertices. The proposition says that if the number of edges in this matching is odd, then the score is 2, and otherwise the score is 0.

We prove by induction on  $p(G)$  that  $Ls(G) = p(G) \bmod 4$ . The base case is  $p(G) = 2$ . In this case, there are only paths of odd length. In particular, there is one path of length 1, and all the other (an even number) are of length 3. Left can score 2 by playing on the root (and thus, also scoring the leaf adjacent to it). Now what remains is a union of an even number of  $P_3$ 's, and so, the relative score in the graph that remains is 0. Thus,  $Ls(G) = 2 = p(G) \bmod 4$ .

Let  $k > 2$  be an even integer and assume now that the result is true for any subdivided star  $H$  of even order with at most one attached path of length 1 and with  $p(H) < k$ . Let  $G$  be a subdivided star of even order  $n$  with at most one attached path of length 1 and such that  $p(G) = k$ .

Assume first that  $p(G) = 2 \bmod 4$ . Since  $p(G) > 2$ , there is a vertex  $x \neq r$  adjacent to a leaf that is not in a path of length 3. Indeed, if all the paths were of length 3, then we would have  $p(G) = 4 \not\equiv 2 \bmod 4$ . Since  $p(G) > 2$ , we are also not in the case where all the paths are of length 3 except for one, which is of length 1. Thus, the vertex  $x$  always exists. By playing on  $x$ , Left scores 2 and the remaining graph  $H$  is still a subdivided star of even size with at most one attached path of length 1 (note that  $H$  could be a path). We have  $p(H) = p(G) - 2$ , and, by induction,  $Ls(H) = 0$ . Thus, playing on  $x$  ensures a score of at least 2, which is optimal by the first remark of the proof.

Assume now that  $p(G) = 0 \bmod 4$ . Assume for the sake of contradiction that  $Ls(G) \neq 0$ . Then, it must be that  $Ls(G) = 2$ , and Left must score 2 on her first move. There are several possibilities for Left:

- a. She plays on the root and scores a leaf adjacent to the root.
- b. She plays on a neighbour of a leaf not in a  $P_3$ .
- c. She plays on a neighbour of a leaf in a  $P_3$ .

In case (a), the remaining graph  $H$  is a union of paths with an even number of paths of odd length and it must be that  $Ls(H) = 0$ . Since  $Ls(H) = 0$ , then by Theorem 12, this means that the paths of  $H$  that have length not equal to 0 modulo 4 can be partitioned into pairs of paths of equal length modulo 4, or into triples  $(P_i, P_j, P_7)$ , where  $i = 1 \bmod 4$  and  $j = 2 \bmod 4$  (since  $P_7 \equiv P_1 + P_2$ ). Besides, by definition,  $p(G)$  is invariant modulo 4 by removing paths of length equal to 0 or 3 modulo 4, by removing pairs of paths of the same length modulo 4, and by removing a triple of the form  $(P_i, P_j, P_7)$  as above. Hence,  $p(G)$  has the same value (modulo 4) as a graph with a root and a leaf, *i.e.*,  $p(G) = 2 \bmod 4$ , which is a contradiction.

In case (b), the remaining graph  $H$  is still a subdivided star of even size with at most one attached path of length 1. By induction, since  $p(H) = p(G) - 2$ , we have  $Ls(H) = 2$ , and thus, Right can ensure a final score of 0, a contradiction.

Finally, in case (c), there is one more attached path of length 1 in the remaining graph  $H$ . If  $H$  has only one attached path of length 1, then the reasoning is the same as in case (b). Otherwise, by Proposition 14, Right should answer by playing on the root, thus scoring 3, and leaving behind a union of paths  $H'$ . Since we assume that  $Ls(G) = 2$ , we must have  $Ls(H) = 0$  and  $Ls(H') = 3$ . By Theorem 12, there must be an odd number of  $P_3$ 's in  $H'$ , and every other path that has a length not equal to 0 modulo 4 must be either paired with another path of equal length modulo 4, or in a triple  $(P_i, P_j, P_7)$  with  $i = 1 \pmod 4$  and  $j = 2 \pmod 4$ . This means that  $p(G)$  has the same value modulo 4 as a graph that has a root, a leaf, and an adjoined  $P_3$ , *i.e.*, is equal to 2 mod 4, which is a contradiction.  $\square$

Note that it would be interesting (but seems difficult) to obtain a complete characterisation with equivalence classes, as in Theorem 12 for paths, which would permit us to compute the relative score for the disjoint unions of (subdivided) stars.

## 6 Unions of cycles

Since the first move in  $C_n$  always leads to  $P_{n-1}$ , Corollary 13 immediately gives the score of a unique cycle:

**Corollary 16.** *Let  $n \geq 3$ . Then,*

$$Ls(C_n) = \begin{cases} -2 & \text{if } n = 4 \\ 0 & \text{if } n = 0 \pmod 2 \text{ and } n > 4 \\ 1 & \text{if } n = 1 \pmod 4 \\ -1 & \text{if } n = 3 \pmod 4 \end{cases}$$

We can actually go further and decide which cycles are equivalent to 0:

**Theorem 17.** *Let  $n > 2$ . Then,  $C_{2n} \equiv_{\mathcal{U}} 0$ .*

*Proof.* By Theorem 5, it suffices to prove that  $Ls(C_{2n}) = 0$  and  $C_{2n}$  is Left-save. By Corollary 16,  $Ls(C_{2n}) = 0$ . Regardless of Right's first move in  $C_{2n}$ , Right scores 1 and what remains is  $P_{2n-1}$ . Left then plays the center vertex of the  $P_{2n-1}$ , *i.e.*, the vertex of minimal eccentricity. Then, what remains is  $P_{n-1} + P_{n-1}$ . It is clear that  $Ls(P_{n-1} + P_{n-1}) = 0$  and that  $P_{n-1} + P_{n-1}$  is Left-save (recall from Section 2 that,  $\forall G \in \mathcal{U}_{\geq 0}$ ,  $G + G \equiv_{\mathcal{U}_{\geq 0}} 0$ , and that  $P_{n-1} \in \mathcal{U}_{\geq 0}$  by Theorem 10).  $\square$

Studying the class of unions of paths and cycles (or just unions of cycles, since after one move on a cycle  $C_n$ , what remains is a path  $P_{n-1}$ ) is difficult, in the sense that it would solve the octal game  $0 \cdot 6$ :

**Theorem 18.** *For all  $n \geq 1$ ,  $Ls(P_n + nC_3) \geq 0$  if and only if Left wins in  $P_n$  in the game  $0 \cdot 6$ .*

*Proof.* First note that  $Ls(C_3) = -1$ , and that after a player plays in a  $C_3$ , what remains is a  $P_2$ , and so, the next player removes what remains of the  $C_3$  on his next turn. Thus, in  $P_n + nC_3$ , neither player wants to play first on any of the  $C_3$ 's since the other player will gain a score of 1 more than them once the  $C_3$  is removed and it will again be their turn. Hence, both players will play on  $P_n$  until it is no longer possible. If Left wins in  $P_n$  in the game  $0 \cdot 6$ , then Right is forced to be the first player to play on a  $C_3$ , and thus, Left will force him to play first on all  $n$  of the  $C_3$ 's, and hence, Left will gain a score of  $n$  over Right just with the  $C_3$ 's. Hence,  $Ls(P_n + nC_3) \geq 0$  in this case. If Right wins in the game  $0 \cdot 6$ , then Left will be forced to play first on all of the  $C_3$ 's, and hence,  $Ls(P_n + nC_3) < 0$  in this case.  $\square$

## 7 Complexity

In this section, we define a generalisation of SMASH AND GRAB, called GENERALISED SMASH AND GRAB, and show that determining the outcome of this new game is PSPACE-complete. This shows evidence that SMASH AND GRAB is most likely PSPACE-complete as well. GENERALISED SMASH AND GRAB is as follows. It is a two-player scoring game where, at each turn, a player must remove a vertex  $v$  from the graph, and, for all  $u \in N(v)$ , if  $deg(u) \leq d$  (for some integer  $d \geq 1$  taken as an input for the game) prior to  $v$  being removed, then  $u$  is also removed. The player scores the number of vertices removed from the graph on his turn. The player with the highest score at the end of the game (when no vertices remain) wins. Analogously to SMASH AND GRAB, for a graph  $G$ , we let  $Ls(G, d)$  be the relative score in GENERALISED SMASH AND GRAB, where the graph  $G$  and the integer  $d \geq 1$  are the inputs. Note that GENERALISED SMASH AND GRAB is equivalent to SMASH AND GRAB when  $d = 1$ .

We begin by defining the problem that we will modify and then reduce from. It was shown to be PSPACE-complete in [13], and is as follows:

**Definition 19 (The 6-uniform Maker-Breaker Game).** Two-player game in which the input consists of a set of variables  $X = \{x_1, \dots, x_n\}$  and a conjunctive normal form (CNF) formula  $\mathcal{F}$  consisting of clauses  $C_1, \dots, C_m$ , each containing exactly 6 variables from  $X$ , all of which appear in their positive form. At each turn, first, the player called Left must set a variable (that is not yet set) to true, and then, the player called Right must set a variable (that is not yet set) to false. Once all of the variables have been assigned a truth value, Left wins if the truth assignment has rendered  $\mathcal{F}$  true, and otherwise, Right wins.

From an instance  $\mathcal{F}$  of the 6-uniform Maker-Breaker Game, we will replace every clause  $C$  by all  $n - 6$  of the clauses with exactly 7 variables that are a superset of  $C$ . Hence, each clause now contains exactly 7 variables, and we call this new instance (of the now 7-uniform Maker-Breaker game)  $\phi$ . Left wins in  $\mathcal{F}$  if and only if she wins in  $\phi$ , and thus, the 7-uniform Maker-Breaker game is also PSPACE-complete. Indeed, the first direction is trivial, and if Left wins in  $\phi$ , then she must satisfy each clause  $C$  in  $\mathcal{F}$  since this is the only way to satisfy all of the supersets of each clause  $C$  as Right will set half the variables of  $X$  to false (and hence, for each  $C$  in  $\mathcal{F}$ , there exists a superset in  $\phi$  that will only be satisfiable in  $\phi$  by the variables of its subset in  $\mathcal{F}$ ). Note that 6 is the smallest integer  $k$  for which it is known that

the  $k$ -uniform Maker-Breaker Game is PSPACE-complete. In general, the main difficulty in the construction in the proof of the next theorem, is to be able to control the parity of the number of turns while ensuring some of the vertices are played on before others (with at most one exception). In order to do so, our construction requires that each clause contains an odd number of variables which is the reason why we reduce from the 7-uniform Maker-Breaker game and the reason why  $d \geq 15$ , since in the construction we will require that each of the clause vertices have degree  $d + 1$ .

As the proof that GENERALISED SMASH AND GRAB is PSPACE-complete is, unsurprisingly, quite involved, we first give some more intuition on the proof to come. In particular, as stated above, we need to ensure that some vertices are played on before others in optimal strategies for both players. Essentially, this divides the game into two phases: the initial phase that is then followed by the final phase. By the properties of the construction, we ensure that these vertices are played on in the initial phase, and we formally prove that the optimal strategies for the players is to (almost always) play in such a desired order in Claims 21–23. While these claims are sufficient to prove the first direction of the proof, this is not the case for the more difficult second direction of the proof (this is why we say “almost always” before). In the latter case, we need an additional claim (Claim 24) that roughly states that any optimal strategy for Left does not require her to deviate more than once from playing on the vertices in the order enforced by Claims 21–23. A case analysis based on whether Left deviated, and if so, how she deviated, then allows to prove the second direction of the proof, concluding the reduction. Further more detailed intuitions can be found just after the construction within the proof.

**Theorem 20.** *Given a graph  $G$  and a fixed integer  $d \geq 15$  with  $d \neq 16$ , determining if  $Ls(G, d) \geq 0$  is PSPACE-complete.*

*Proof.* Since the number of turns and the number of possible moves at each turn are both bounded above by  $|V(G)|$ , the game is in PSPACE. To show the problem is PSPACE-hard, we reduce from an instance  $\phi$  of the 7-uniform Maker-Breaker Game. We construct, in polynomial time, an instance  $G$  of GENERALISED SMASH AND GRAB such that Left wins in  $\phi$  if and only if  $Ls(G, d) \geq 0$ .

Let  $x_1, \dots, x_n$  be the variables and  $C_1, \dots, C_m$  be the clauses in the instance  $\phi$  of the 7-uniform Maker-Breaker Game. We assume that  $n$  is even since we can add a dummy variable if needed (e.g., a variable that is not contained in any of the clauses). The construction of  $G$  is as follows (see Figure 2 for an illustration): for each variable  $x_i$  ( $1 \leq i \leq n$ ), there are the vertices  $x_i$  and  $\tilde{x}_i$ , and an independent set of  $2m^8 n^8 d^8$  vertices. For each variable  $x_i$ , the vertices  $x_i$  and  $\tilde{x}_i$  are both adjacent to all of the vertices of the independent set of  $2m^8 n^8 d^8$  vertices associated to that variable  $x_i$ . For each clause  $C_j$  ( $1 \leq j \leq m$ ), there are  $2md$  vertices  $C_j^1, \dots, C_j^{2md}$ . For each  $1 \leq j \leq m$  and  $1 \leq t \leq 2md$ , a  $K_{\lceil (d-15)/2 \rceil}$  and a  $K_{\lfloor (d-15)/2 \rfloor}$  are added, and all of their vertices are made adjacent to the vertex  $C_j^t$  (if  $d = 15$ , then these are both  $K_0$ , and so, nothing is added, and recall that  $d \neq 16$ , so they both exist unless  $d = 15$ ). There are two vertices  $C^*$  and  $C'$  that are both adjacent to each of the vertices  $C_j^t$ . There are  $2m^2 + 2m$   $K_{d+2}$ 's (clique with  $d + 2$  vertices). Let  $H$  be a  $K_d$  with  $d - 3$  of its vertices adjacent to the two vertices of an additional  $K_2$ , one of which is called the

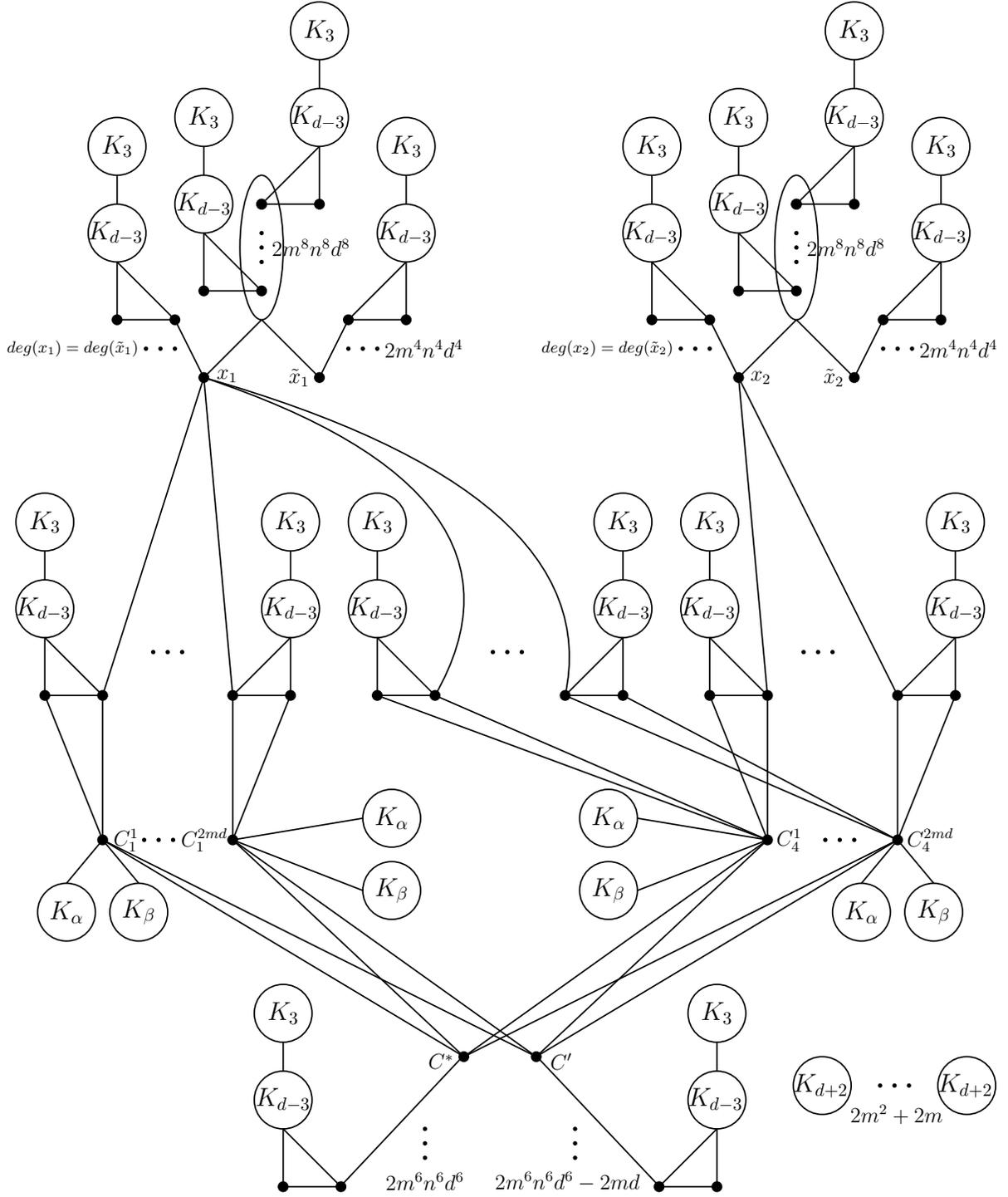


Figure 2: The construction of  $G$  from an instance  $\phi$  of the 7-uniform Maker-Breaker Game. For legibility, only two variables  $x_1$  and  $x_2$ , and two clauses  $C_1$  and  $C_4$  are shown. Furthermore,  $\alpha = \lceil (d-15)/2 \rceil$  and  $\beta = \lfloor (d-15)/2 \rfloor$ . Both  $C_1$  and  $C_4$  contain  $x_1$  in  $\phi$ , and  $C_4$  also contains  $x_2$  in  $\phi$ . Edges to blobs indicate adjacency to all of the vertices in the blob.

“attachment” vertex. Let the 3 other vertices of this  $K_d$  that are non-adjacent to the  $K_2$  be the “triangle”. For each variable  $x_i$  and each clause  $C_j$ , if the variable  $x_i$  is in the clause  $C_j$  in  $\phi$ , then, for each  $1 \leq t \leq 2md$ , there is an  $H$  whose attachment vertex is adjacent to  $x_i$  and  $C_j^t$ , and the other vertex of the  $K_2$  of the  $H$  is also adjacent to  $C_j^t$ . Then  $H$ 's are added and made adjacent to each of the  $x_i$  and  $\tilde{x}_i$  via their attachment vertices, so that all the  $x_i$  and  $\tilde{x}_i$  have degree  $2m^4n^4d^4 + 2m^8n^8d^8$ . An additional  $2m^6n^6d^6$   $H$ 's are added and each of their attachment vertices are made adjacent to  $C^*$ . Another  $2m^6n^6d^6 - 2md$   $H$ 's are added and each of their attachment vertices are made adjacent to  $C'$ . For every  $1 \leq i \leq n$  and for each vertex  $v$  that is adjacent to both  $x_i$  and  $\tilde{x}_i$ ,  $v$  is made to be the attachment vertex of a new  $H$ . This completes the construction.

Note that, for all  $1 \leq j \leq m$  and all  $1 \leq t \leq 2md$ , the vertex  $C_j^t$  has degree  $d + 1$  since each variable that appears in the clause  $C_j$  in  $\phi$  contributes 2 to the degree of each of the vertices  $C_j^1, \dots, C_j^{2md}$ , each clause in  $\phi$  contains exactly 7 variables,  $C^*$  and  $C'$  are adjacent to each clause vertex, and each clause vertex is adjacent to  $d - 15$  additional vertices through its adjacencies to its associated  $K_{\lceil (d-15)/2 \rceil}$  and  $K_{\lfloor (d-15)/2 \rfloor}$ . For each  $H$ , each of the vertices in the  $K_{d-3}$  also have degree  $d + 1$ . For all  $1 \leq i \leq n$ , the vertices  $x_i$  and  $\tilde{x}_i$  both have degree  $2m^4n^4d^4 + 2m^8n^8d^8$ . The vertex  $C^*$  has degree  $2m^6n^6d^6 + 2m^2d$ , and the vertex  $C'$  has degree  $2m^6n^6d^6 + 2m^2d - 2md$ . Lastly, each of the vertices in the  $K_{d+2}$ 's have degree  $d + 1$ . Every other vertex in  $G$  has degree at most  $d$ , and hence, is removed when any of its neighbours are played on.

The main idea behind the construction of  $G$ , is to initially incentivise the two players to play on the variable vertices in  $G$  as they would in the instance  $\phi$  of the 7-uniform Maker-Breaker Game. Then, to reward Left if she followed a winning strategy in  $\phi$  in GENERALISED SMASH AND GRAB in  $G$ , the vertices  $C^*$  and  $C'$  are adjacent to all of the clause vertices, which all have degree at most  $d$  if Left followed such a winning strategy, and so, if she plays on  $C^*$  or  $C'$ , she will score all of the clause vertices. The idea from there is to make the players remove the remaining variable vertices in order to make what remains all simple disconnected components, and from there, it is easy to decide the outcome of the game in  $G$ . To force the players to play on the vertices in a certain order, large amounts of vertices of degree at most  $d$  are made adjacent to the vertices in such a way that, the earlier we want the players to play on certain vertices, the greater the order of the number of vertices of degree at most  $d$  there are adjacent to those vertices.

The main difficulty of the proof is the second direction, *i.e.*, proving that if Right wins in  $\phi$ , then  $Ls(G, d) < 0$ . Essentially, in  $G$ , there are many  $K_{d+2}$ 's so that if Left does not score all of the clause vertices when she plays on  $C^*$  or  $C'$  (which is the case if she does not have a winning strategy in  $\phi$ ), then Right will win in  $G$  by making Left play first on all of the  $K_{d+2}$ 's. Indeed, no player wants to play first on a  $K_{d+2}$  since they will only score one vertex, while the other player can then play on the same  $K_{d+2}$  (now a  $K_{d+1}$ ) and score all  $d + 1$  of its vertices. Note that Left could also be forced to play first on all of the  $K_{d+2}$ 's in the proof of the first direction, but there are not enough of them for Right to win in  $G$ . The key difficulty of the proof of the second direction is that Left does not always have to play on the vertices in the order we want her to, since she is the first player. This is the reason

we add many copies of the graphs  $H$ . They are there to ensure that Right can always force Left to play first on all of the  $K_{d+2}$ 's, even if she deviates from playing on the vertices in the desired order. In particular, once the attachment vertex of an  $H$  is removed, its structure ensures that a player can either remove it entirely in one turn (by playing on a vertex of its  $K_{d-3}$ ), or play on one of its vertices and leave behind a graph that is removed in one turn, regardless of the vertex that is played on (by playing on one of the vertices of its triangle, which leaves behind a singleton). Later, we will see that once all of the variable vertices have been played on (the last of the vertices that we force the order on), there are no longer any  $H$ 's with attachment vertices. At this point, if it is necessary due to Left deviating earlier, Right can choose how to play on one of these  $H$ 's, in order to ensure Left plays first on the  $K_{d+2}$ 's.

In what follows, we will refer to two phases for the game in  $G$ : the initial phase consists of all the turns that take place while there still exists a variable vertex, while the final phase consists of the remainder of the turns after the initial phase.

Now that we have given some intuition regarding the construction of  $G$  and the strategies to follow, the next claims will be useful for proving that there is some structure in terms of the order the vertices should be played on in optimal strategies as discussed above.

**Claim 21.** *If a player plays on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist on each of their first  $n/2 + 1$  turns, then that player has a winning strategy in  $G$ .*

*Proof of the claim.* Assume, w.l.o.g., that on each of his first  $n/2 + 1$  turns, Right has played on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist. Then, on each of his first  $n/2 + 1$  turns, Right scored  $\Omega(m^8 n^8 d^8)$ . Note that playing on any other vertex in  $G$  only scores  $o(m^8 n^8 d^8)$ . Hence, after Right's  $(n/2 + 1)^{\text{th}}$  turn, Right has a score of  $\Omega(m^8 n^8 d^8)$  more than Left since Right played on at least one more such vertex  $x_i$  or  $\tilde{x}_i$  (for an  $i$  for which both still exist) than Left (recall that  $n$  is even). From then on, Right has a strategy that ensures that, at the end of the game, he will have a score of  $\Omega(m^8 n^8 d^8)$  more than Left. Indeed, while possible, for his next turns, Right continues playing on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist, and then, once this is no longer possible, Right plays on the  $H$ 's whose respective attachment vertices make up the independent sets of order  $2m^8 n^8 d^8$ . In particular, when playing on those  $H$ 's, Right first plays on the  $H$ 's for which only their attachment vertex has been removed, and he plays on a vertex of the  $K_{d-3}$  when doing so, which ensures scoring the entire remaining  $H$  on that turn. Since Right can ensure scoring at most  $n + 1$  less than Left of these  $2m^8 n^9 d^8$   $H$ 's (which each have order  $O(d)$ ), and the rest of the vertices in the graph (excluding the independent sets of order  $2m^8 n^8 d^8$  and the  $H$ 's associated with them) amount to  $o(m^8 n^8 d^8)$  vertices, then at the end of the game, Right's score will be  $\Omega(m^8 n^8 d^8)$  more than Left's score.  $\diamond$

**Claim 22.** *If a player plays on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist on each of their first  $n/2$  turns, and that same player plays on  $C^*$  and  $C'$  on their next two turns, then that player has a winning strategy in  $G$ .*

*Proof of the claim.* Assume, w.l.o.g., that on each of his first  $n/2$  turns, Right has played on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist, and that Right played on  $C^*$  and

$C'$  on his next two turns. Then, on each of his first  $n/2$  turns, Right scored  $\Omega(m^8 n^8 d^8)$ , and, on his next two turns, he scored  $\Omega(m^6 n^6 d^6)$ . Since  $n$  is even and playing on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both still exist scores the most possible in  $G$ , then Left's score is at most the same as Right's score after Right's  $(n/2)^{\text{th}}$  turn. Note that playing on any other vertex that is not  $C^*$  nor  $C'$ , nor a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both still exist, only scores  $o(m^6 n^6 d^6)$ . Hence, after Right's  $(n/2 + 2)^{\text{th}}$  turn, Right has a score of  $\Omega(m^6 n^6 d^6)$  more than Left, and by Claim 21, we can assume that after Left's next turn, there no longer exists an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist. From then on, Right has a strategy that ensures that, at the end of the game, he will have a score of  $\Omega(m^6 n^6 d^6)$  more than Left. Indeed, while possible, for his next turns, Right plays on the  $H$ 's whose respective attachment vertices make up the independent sets of order  $2m^8 n^8 d^8$  (as in the proof of Claim 21) and the  $H$ 's associated to the independent sets of order  $2m^6 n^6 d^6$  and  $2m^6 n^6 d^6 - 2md$  (those adjacent to  $C^*$  and those adjacent to  $C'$ , respectively). Since Left can have played in at most 2 of these  $H$ 's before Right plays in one, and Right can pair the  $H$ 's associated to the independent sets of order  $2m^6 n^6 d^6$  and  $2m^6 n^6 d^6 - 2md$ , so that he plays symmetrically to Left in the  $H$  paired with the one Left plays in, then, at the end of the game, Right's score will be  $\Omega(m^6 n^6 d^6)$  more than Left's score. Indeed, this follows since each of the  $H$ 's has order  $O(d)$  and the rest of the vertices in the graph amount to  $o(m^6 n^6 d^6)$  vertices.  $\diamond$

**Claim 23.** *If a player plays on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist on each of their first  $n/2$  turns,  $C^*$  or  $C'$  on their  $(n/2 + 1)^{\text{th}}$  turn, and a vertex  $x_i$  or  $\tilde{x}_i$  on each of their next  $n/2 + 1$  turns, then that player has a winning strategy in  $G$ .*

*Proof of the claim.* Assume, w.l.o.g., that Right has played on a vertex  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist on each of his first  $n/2$  turns, that he played on  $C^*$  or  $C'$  on his next turn, and that he played on a vertex  $x_i$  or  $\tilde{x}_i$  for his subsequent  $n/2 + 1$  turns. Then, on each of his first  $n/2$  turns, Right scored  $\Omega(m^8 n^8 d^8)$ , on his next turn, he scored  $\Omega(m^6 n^6 d^6)$ , and on each of his subsequent  $n/2 + 1$  turns, he scored  $\Omega(m^4 n^4 d^4)$ . As in the proof of Claim 22, Left's score is at most the same as Right's score after Right's  $(n/2)^{\text{th}}$  turn. Furthermore, after Right's  $(n/2 + 1)^{\text{th}}$  turn, Left's score is at most  $2m^2 d + 2md$  more than Right's score (the case where Left played on  $C^*$  before Right played on  $C'$ , and all the clauses were satisfied by the variable vertices played on thus far). Note that playing on any other vertex that is not  $C^*$  nor  $C'$ , nor a vertex  $x_i$  or  $\tilde{x}_i$ , only scores  $o(m^4 n^4 d^4)$ . Hence, after Right's  $(n + 2)^{\text{th}}$  turn, Right has a score of  $\Omega(m^4 n^4 d^4)$  more than Left, and by Claims 21 and 22, we can assume that after Left's next turn,  $C^*$  and  $C'$  no longer exist, and that, for any  $i$ , at most one of  $x_i$  and  $\tilde{x}_i$  exist. From then on, Right has a strategy that ensures that, at the end of the game, he will have a score of  $\Omega(m^4 n^4 d^4)$  more than Left. Indeed, while possible, for his next turns, Right plays on a vertex  $x_i$  or  $\tilde{x}_i$ , and once this is no longer possible, Right plays on the  $H$ 's whose respective attachment vertices make up the independent sets of order  $2m^8 n^8 d^8$  (as in the proof of Claim 21), the  $H$ 's associated to the independent sets of order  $2m^6 n^6 d^6$  and  $2m^6 n^6 d^6 - 2md$  (those adjacent to  $C^*$  and those adjacent to  $C'$ , respectively), and the  $H$ 's associated to the independent sets of order  $2m^4 n^4 d^4$  whose attachment vertices were not adjacent to any clause vertex. Since Left can

have played in at most  $n + 1$  of these  $H$ 's before Right plays in one, and Right can pair the  $H$ 's associated to the independent sets of order  $2m^6n^6d^6$  and  $2m^6n^6d^6 - 2md$  (as in the proof of Claim 22), as well as the latter ones associated to the independent sets of order  $2m^4n^4d^4$ , so that he plays symmetrically to Left in the  $H$  paired with the one Left plays in, then, at the end of the game, Right's score will be  $\Omega(m^4n^4d^4)$  more than Left's score. Indeed, this follows since each of the  $H$ 's has order  $O(d)$  and the rest of the vertices in the graph amount to  $o(m^4n^4d^4)$  vertices.  $\diamond$

First, we prove the forward direction, that is, if Left wins in  $\phi$ , then  $Ls(G, d) \geq 0$ . Assume Left has a winning strategy in  $\phi$ . We give a strategy for Left ensuring that  $Ls(G, d) \geq 0$ . Left first only plays on vertices  $x_i$  for which neither of  $x_i$  and  $\tilde{x}_i$  have been played on yet. By Claim 21, we can assume Right always responds by playing on an  $x_\ell$  or  $\tilde{x}_\ell$  for an  $\ell$  for which neither of these vertices have been played on as of yet, as otherwise,  $Ls(G, d) \geq 0$ . Left first follows her winning strategy in  $\phi$  in the instance  $G$  of GENERALISED SMASH AND GRAB by removing the vertex  $x_i$  when she sets the variable  $x_i$  to true in  $\phi$ . Therefore, after  $n$  turns, for all  $1 \leq i \leq n$ , exactly one of  $x_i$  and  $\tilde{x}_i$  has been played on, and both players have the same score since all the  $x_i$  and  $\tilde{x}_i$  have the same degree. Also, since Left followed a winning strategy in  $\phi$ , each clause is satisfied, and thus, for all  $1 \leq j \leq m$  and all  $1 \leq t \leq 2md$ ,  $C_j^t$  has degree at most  $d$  (initially it has degree  $d + 1$ ). Indeed, for each  $C_j^t$ , there exists at least one  $\ell$  ( $1 \leq \ell \leq n$ ) such that the attachment vertex of the  $H$  that is adjacent to  $x_\ell$  and  $C_j^t$  was removed, since the vertex of at least one variable  $x_\ell$  from the clause  $C_j$  was played on and removed by Left (or Right if he did not play optimally) by her strategy (and this attachment vertex got removed in the same move since it had degree  $d$ ). Since  $n$  is even, it is Left's turn. Left plays on  $C^*$  and scores  $2m^6n^6d^6 + 2m^2d + 1$  since, for all  $1 \leq j \leq m$  and all  $1 \leq t \leq 2md$ ,  $C_j^t$  has degree at most  $d$  and each of the other neighbours of  $C^*$  has degree  $d - 1$ . By Claim 22, we can assume Right plays on  $C'$ , as otherwise,  $Ls(G, d) \geq 0$ . Left now has a score of  $2m^2d + 2md$  more than Right.

Left now plays on the remaining  $x_i$  and  $\tilde{x}_i$ . By Claim 23, we can assume Right always responds by playing on an  $x_\ell$  or  $\tilde{x}_\ell$  as well, as otherwise,  $Ls(G, d) \geq 0$ . Let  $H_1$  be the graph obtained from the graph  $H$  by removing its attachment vertex. Then, we can assume it is Left's turn, Left has a score of  $2m^2d + 2md$  more than Right, and there are an even number of  $H_1$ 's (since, initially, there were an even number of  $H$ 's in  $G$ ), an even number of  $K_{\lfloor (d-15)/2 \rfloor}$ 's, an even number of  $K_{\lfloor (d-15)/2 \rfloor}$ 's, and  $2m^2 + 2m$   $K_{d+2}$ 's, and all of these components are disconnected. Indeed, for each  $H$ , its attachment vertex was removed when a player played on either  $C^*$ , or  $C'$ , or a vertex of the form  $x_i$  or  $\tilde{x}_i$ . When excluding the  $2m^2 + 2m$   $K_{d+2}$ 's, since any of the other components can be removed in one turn and they are all disconnected, Left can ensure scoring at least the same number of vertices as Right in the final phase (excluding the  $K_{d+2}$ 's). However, note that  $Ls(K_{d+2}, d) = -d$  since the first player to play in a  $K_{d+2}$  only removes the vertex they play on, while the second player to play in a  $K_{d+2}$  removes the remainder of it. Thus, when including the  $2m^2 + 2m$   $K_{d+2}$ 's, it is possible that Left will be forced to play first on each of the  $K_{d+2}$ 's, but even in this case, Right will gain a score of only  $2m^2d + 2md$  back on Left, and thus,  $Ls(G, d) \geq 0$ .

Now, we prove the other direction, that is, if Right wins in  $\phi$ , then  $Ls(G, d) < 0$ . Assume

Right has a winning strategy in  $\phi$ . We give a strategy for Right ensuring that  $Ls(G, d) < 0$ . Right follows the following pairing strategy in the initial phase:

- While there exists an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  exist, Right plays on  $\tilde{x}_i$  according to his winning strategy in  $\phi$ . In the case Left just previously played on  $x_i$  for some  $i$ , Right assumes that Left set  $x_i$  to true in  $\phi$ . In any other case or if Right already played his desired move in  $\phi$  on a previous turn, then Right plays on any arbitrary  $\tilde{x}_i$  for an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  still exist.
- Otherwise, if there is no  $i$  for which both  $x_i$  and  $\tilde{x}_i$  exist, then while  $C^*$  or  $C'$  still exists, Right plays on  $C^*$  if possible, and if not, then he plays on  $C'$ .
- Otherwise, if there is no  $i$  for which both  $x_i$  and  $\tilde{x}_i$  exist, and  $C^*$  and  $C'$  do not exist, then while there exists any variable vertices  $x_i$  or  $\tilde{x}_i$ , Right plays on one of them.

The following claim proves that any optimal strategy for Left does not require her to deviate more than once from playing on the same “type” of vertices as Right in the initial phase. By “type”, we mean that variable vertices ( $x_i$  and  $\tilde{x}_i$ ) for which both  $x_i$  and  $\tilde{x}_i$  exist are of the same “type”,  $C^*$  and  $C'$  are of the same “type”, and variable vertices where only one of  $x_i$  and  $\tilde{x}_i$  exist are of the same “type”. Thus, to be more precise, in the initial phase, we say that Left deviated from playing on the same “type” of vertices as Right if Left does not play on the same “type” of vertex Right just played on and there still exists a vertex of that “type”. First, note the cases in which Left can deviate multiple times.

- While there exists an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  exist, Left must either play on  $C^*$ ,  $C'$  or a variable vertex  $x_i$  or  $\tilde{x}_i$  for which only one of them exists. That way, in the former case, when Right plays on  $C^*$  or  $C'$  (whichever is remaining), Left can again deviate. In the latter case, when Right plays on a variable vertex  $x_i$  or  $\tilde{x}_i$  for which only one of them exists, Left can again deviate.
- Otherwise, if there is no  $i$  for which both  $x_i$  and  $\tilde{x}_i$  exist, then while  $C^*$  and  $C'$  still exist, Left must play on a variable vertex  $x_i$  or  $\tilde{x}_i$  for which only one of them exists. That way when Right plays on a variable vertex  $x_i$  or  $\tilde{x}_i$  for which only one of them exists, Left can again deviate.

In any other case, by Right’s strategy in the initial phase and Claims 21 – 23, we can assume that Left can only deviate from playing on the same “type” of vertices as Right once in the initial phase, since otherwise  $Ls(G, d) < 0$ .

**Claim 24.** *Assuming Right follows the strategy above during the initial phase, for any strategy for Left where she deviates multiple times from playing on the same “type” of vertices as Right in the initial phase, there exists a strategy for Left where she deviates only once that results in at least the same final relative score for Left.*

*Proof of the claim.* Assume that Left deviates while there exists an  $i$  for which both  $x_i$  and  $\tilde{x}_i$  exist. If Left plays on  $C^*$  or  $C'$ , then she clearly scores at most the same on that turn as if she plays on that vertex in a strategy that does not deviate. Indeed, by Right's strategy in the initial phase and Claim 21, Left must then only play on an  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both exist until no such  $i$  exist, at which point it is Right's turn, and he plays on whichever of  $C^*$  and  $C'$  that is remaining. Moreover, she does not change the parity of the number of turns since all of the clause vertices are removed once  $C^*$  and  $C'$  are played on.

If Left plays on a variable vertex  $x_i$  or  $\tilde{x}_i$  for which only one of them exists, then she scores the same on that turn as if she plays on one of them in a strategy that does not deviate. Moreover, she does not change the parity of the number of turns since all of the vertices that will be played on until these variable vertices are played on, are at distance at least 3 from these variable vertices and all neighbours of these variable vertices have degree at most  $d$  to begin with. Indeed, by Right's strategy in the initial phase and Claims 21 and 22, Left must then only play on an  $x_i$  or  $\tilde{x}_i$  for an  $i$  for which both exist, and then once none of these exist anymore, she must play on  $C^*$  or  $C'$  (in this case it will be  $C'$ ).

The case where Left deviates while  $C^*$  and  $C'$  still exist is analogous to the case described in the paragraph above.  $\diamond$

Therefore, by Right's strategy in the initial phase and Claims 21 – 24, we can assume Left can only ever deviate from playing on the same "type" of vertices as Right once in the initial phase. Moreover, by Claim 24, we can assume Left never deviates by playing on a variable vertex,  $C^*$  or  $C'$ . There are two cases for the final phase (recall that the final phase begins after the initial phase, that is, when there are no longer any variable vertices left).

**Case 1:** Left never deviated from playing on the same "type" of vertices as Right in the initial phase. Since Right followed a winning strategy in  $\phi$ , there was at least one clause  $C_\ell$  for which all of the vertices  $C_\ell^t$  had degree at least  $d + 1$  the first time one of  $C^*$  and  $C'$  was played on. Therefore, Left has a score of at most  $2m^2d - 2md$  more than Right, and, by parity, it is Left's turn. Indeed, if Left played on  $C^*$ , then Left did not score any of the  $2md$  copies of at least one of the clauses while Right did, and otherwise, if Right played on  $C^*$ , then Left played on  $C'$  and scored  $2md$  less attachment vertices of  $H$ 's since  $C'$  has  $2md$  less  $H$ 's whose attachment vertices are adjacent to it than  $C^*$ . Note that once both  $C^*$  and  $C'$  have been played on, there are no longer any vertices of the form  $C_j^t$  left (they all have degree  $d + 1$  initially and both  $C^*$  and  $C'$  are adjacent to all of them). Thus, there are an even number of  $H_1$ 's, an even number of  $K_{\lceil(d-15)/2\rceil}$ 's, an even number of  $K_{\lfloor(d-15)/2\rfloor}$ 's, and  $2m^2 + 2m$   $K_{d+2}$ 's, and all of these components are disconnected. Right employs a simple pairing strategy for the  $H_1$ 's, where he just plays on the same vertex Left played on during the previous turn, in the  $H_1$  paired with the one Left just played on. Similarly, whenever Left plays on a  $K_{\lceil(d-15)/2\rceil}$  ( $K_{\lfloor(d-15)/2\rfloor}$ , resp.), it is removed, and then, Right also plays on a  $K_{\lceil(d-15)/2\rceil}$  ( $K_{\lfloor(d-15)/2\rfloor}$ , resp.), which removes it. Thus, by parity, Right can ensure the same score as Left in the final phase when excluding the  $K_{d+2}$ 's, but Left will always be forced to play first on each of the  $K_{d+2}$ 's, since Right responds by playing on the same  $K_{d+2}$  (now a  $K_{d+1}$ ) each time, and thus, Right will gain a score of  $2m^2d + 2md$  back on Left in this way.

Hence,  $Ls(G, d) < 0$  since Left had a score of at most  $2m^2d - 2md$  more than Right at the beginning of the final phase.

**Case 2:** Left deviated from playing on the same “type” of vertices as Right once in the initial phase. There are 4 subcases.

**Case 2.1:** Left played on a clause vertex, w.l.o.g.,  $C_q^1$  for some  $1 \leq q \leq m$ . Hence, at the beginning of the final phase, Left has a score of at most  $2m^2d - 2md + O(d)$  more than Right, and, by parity, it is Right’s turn. Since  $\phi$  is an instance of the 7-uniform Maker-Breaker Game, 7  $K_d$ ’s are what remains of the  $H$ ’s whose respective attachment vertices were adjacent to  $C_q^1$ . Thus, there are 7  $K_d$ ’s, an odd number of  $H_1$ ’s, an odd number of  $K_{\lceil (d-15)/2 \rceil}$ ’s, an odd number of  $K_{\lfloor (d-15)/2 \rfloor}$ ’s, and  $2m^2 + 2m$   $K_{d+2}$ ’s, and all of these components are disconnected. Right plays on one of the vertices of the triangle of an  $H_1$ , which leaves behind a singleton. Now, there are 7  $K_d$ ’s, one singleton, an even number of  $H_1$ ’s, an odd number of  $K_{\lceil (d-15)/2 \rceil}$ ’s, an odd number of  $K_{\lfloor (d-15)/2 \rfloor}$ ’s, and  $2m^2 + 2m$   $K_{d+2}$ ’s, and all of these components are disconnected. For a single  $K_d$  and the singleton, when Left plays on one of them, Right plays on the other (which removes them both). For the other 6  $K_d$ ’s, whenever Left plays on one of them, it is removed, and then, Right plays on another one, which removes it. For a single  $K_{\lceil (d-15)/2 \rceil}$  and a single  $K_{\lfloor (d-15)/2 \rfloor}$ , when Left plays on one of them, Right plays on the other (which removes them both). The rest of what remains is analogous to what remains in **Case 1**, and so, Right follows his strategy as in **Case 1** for the rest. Thus,  $Ls(G, d) < 0$  since even though Left may gain a score of  $O(d)$  over Right in the final phase when not considering the  $K_{d+2}$ ’s, Right will gain a score of  $2m^2d + 2md$  back on Left because of the  $K_{d+2}$ ’s.

**Case 2.2:** Left played on a vertex of an  $H$ . Hence, at the beginning of the final phase, Left has a score of at most  $2m^2d - 2md + O(d)$  more than Right, and, by parity, it is Right’s turn. If the  $H$  that Left played in is completely gone, then Right plays on the  $K_{d-3}$  of another  $H_1$ , thereby removing it entirely. Right then follows his strategy as in **Case 1** since what remains is analogous to what remains in **Case 1**. Clearly,  $Ls(G, d) < 0$  since Right will gain a score of  $2m^2d + 2md$  back on Left because of the  $K_{d+2}$ ’s.

If the  $H$  that Left played in is not entirely gone, call it  $H'$ , then Right plays on the triangle of another  $H_1$ , call it  $H''$ . Now, what remains is analogous to what remains in **Case 1** except for  $H'$  and  $H''$ . Then, Right follows his strategy as in **Case 1**, except that when Left plays on  $H'$  or  $H''$  ( $H''$  is a singleton), Right plays on the other, and note that, no matter where a player plays in  $H'$ , all of  $H'$  is removed. Since  $H'$  and  $H''$  each have order  $O(d)$ , and Right will gain a score of  $2m^2d + 2md$  back on Left because of the  $K_{d+2}$ ’s, we have that  $Ls(G, d) < 0$ .

**Case 2.3:** Left played on a vertex of a  $K_{\lceil (d-15)/2 \rceil}$  or a  $K_{\lfloor (d-15)/2 \rfloor}$ . Hence, at the beginning of the final phase, Left has a score of at most  $2m^2d - 2md + O(d)$  more than Right, and, by parity, it is Right’s turn. If Left played on a  $K_{\lceil (d-15)/2 \rceil}$  ( $K_{\lfloor (d-15)/2 \rfloor}$ , resp.), then Right plays

on a  $K_{\lceil (d-15)/2 \rceil}$  ( $K_{\lfloor (d-15)/2 \rfloor}$ , resp.) and both are gone. Right then follows his strategy as in **Case 1** since what remains is analogous to what remains in **Case 1**. Clearly,  $Ls(G, d) < 0$  since Right will gain a score of  $2m^2d + 2md$  back on Left because of the  $K_{d+2}$ 's.

**Case 2.4:** Left played on a vertex of a  $K_{d+2}$ . Hence, at the beginning of the final phase, Left has a score of at most  $2m^2d - 2md + 1$  more than Right and, by parity, it is Right's turn. Right plays on another vertex of the same  $K_{d+2}$  that Left previously played on, and then Right follows his strategy as in **Case 1** since what remains is analogous to what remains in **Case 1** except that there is one less  $K_{d+2}$ . Clearly,  $Ls(G, d) < 0$  since Right will gain a score of  $2m^2d + 2md - d$  back on Left because of the remaining  $2m^2 + 2m - 1$   $K_{d+2}$ 's.

In all of the cases,  $Ls(G, d) < 0$ , and this completes the proof.  $\square$

## 8 Further work

### 8.1 Equivalence classes of 0

Playing SMASH AND GRAB on a graph  $G$  where each connected component belongs to  $\mathcal{U}_{\geq 0}$  allows for more simplifications according to Theorem 4. For example, we can immediately conclude that  $Ls(P_5 + P_4) = 1$  since  $P_4 \equiv_{\mathcal{U}_{\geq 0}} 0$ , or that  $Ls(P_5 + P_5 + P_4) = 0$  since  $P_5 + P_5 \equiv_{\mathcal{U}_{\geq 0}} 0$ .

For the instances of SMASH AND GRAB that do not belong to  $\mathcal{U}_{\geq 0}$ , similar simplifications turn out to be harder. For example, the game  $C_4 + C_4$  is not equivalent to 0 and has a score of  $-4$ . For this reason, characterising the equivalence class of zero is the first natural study that must be considered, as it allows to remove all such games in a sum without changing the score.

#### Games satisfying $G \equiv_{\mathcal{U}} 0$

From Theorem 4, we know that all the instances of SMASH AND GRAB that are in  $\mathcal{U}_{\geq 0}$  are equivalent to 0 if and only if their score is 0. But this equivalence is proved to be true only in the universe  $\mathcal{U}_{\geq 0}$ . Can it be extended to the whole universe  $\mathcal{U}$ ? For example, we have the following positive result:

$$P_4 \equiv_{\mathcal{U}} 0.$$

It means that all the connected components isomorphic to  $P_4$  can be removed from any instance of SMASH AND GRAB. The proof of this result comes from the fact that  $P_4$  is Left-save and  $Ls(P_4) = 0$  (from Theorem 5). However, it does not hold for all the instances of  $\mathcal{U}_{\geq 0}$  satisfying  $Ls(G) = 0$ . For example, consider the graph  $P_8$  that satisfies  $Ls(P_8) = 0$  and belongs to  $\mathcal{U}_{\geq 0}$ . This game is not equivalent to 0 since it is not Left-save. Indeed, if the first player in  $P_8$  plays in a such a way that  $P_5 + P_2$  remains after their turn, then they can force an odd number of moves by playing in such a way that  $P_2 + P_2$  remains after their second move. As a consequence, we have the somehow suprising result  $Ls(P_8 + C_4) = 2$  since Left

can force Right to start in the  $C_4$ .

Therefore, it would be interesting to characterise the instances of SMASH AND GRAB that are in  $\mathcal{U}_{\geq 0}$  and satisfy Ettinger’s property, and, as a first step, to do it for unions of paths. Indeed, in Section 4, we have characterised the equivalence classes of unions of paths modulo  $\mathcal{U}_{\geq 0}$ , but the problem remains open modulo  $\mathcal{U}$ , even for deciding what are the single paths equivalent to 0. Note however that this question has been solved in the case of cycles in Section 6.

### Games for which $G + G \equiv_{\mathcal{U}} 0$

A second interesting question concerns the inverse of a game  $G$ . From Theorem 4, we know that every instance  $G$  of SMASH AND GRAB belonging to  $\mathcal{U}_{\geq 0}$  satisfies  $G + G \equiv_{\mathcal{U}_{\geq 0}} 0$ . Moreover, from Theorem 5, this equivalence is extended to  $\mathcal{U}$ :

$$\forall G \in \mathcal{U}_{\geq 0}, G + G \equiv_{\mathcal{U}} 0.$$

This result holds since  $Ls(G + G) = 0$  and  $G + G$  is Left-save (it suffices that Left plays symmetrically to Right on the other component). If  $G$  belongs to  $\mathcal{U}$ , this result is no longer valid as shown by the example of  $C_4 + C_4$ . Solving this in general would imply first to characterise the graphs  $G$  for which  $Ls(G + G) = 0$  when  $G \in \mathcal{U} \setminus \mathcal{U}_{\geq 0}$ . Figure 3 below shows that  $Ls(G) = 0$  is not a sufficient condition for that. Indeed, for the graph  $G$  from this figure, we have that  $Ls(G) = 0$  and  $Ls(G + G) = -2$ . Indeed, on  $G + G$ , whatever the first moves of Left (playing first) are, Right can answer until reaching either the position  $C_3 + C_3$  (of value  $-2$ ) or  $C_3 + G$  (of value  $-1$ , but in that case, he has already won one additional point before).

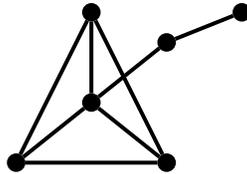


Figure 3: An example of a graph satisfying  $Ls(G) = 0$  and  $Ls(G + G) = -2$ .

## 8.2 Other perspectives

As we have shown for forests, it would be nice to find other graph classes in which Left never loses, assuming both players play optimally. This seems like a very difficult question in general, since already for cycles this is not true. Thus, an interesting direction to take would be to find necessary and sufficient conditions for this. It would also be intriguing to know if there exists a polynomial-time algorithm to determine  $LS(F)$  for any forest  $F$ . Another direction to take would be to study GENERALISED SMASH AND GRAB for at least

the same graph classes we considered in this paper. Lastly, we wonder in which complexity class SMASH AND GRAB lies in. Due to our result that GENERALISED SMASH AND GRAB is PSPACE-complete, it seems likely that SMASH AND GRAB would also be PSPACE-complete (note that it is in PSPACE). A first step towards answering this question would be to prove that SMASH AND GRAB is NP-hard.

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