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Dual kinematics

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Based on the concept of geometric duality, we argue that uniform synchrony provides non-causal time by symmetric extension. Galilean particle aberration and Doppler effect $c \rightarrow \infty$ coexists with conventional time and wave aberration in the Carrollian limit $c \rightarrow 0$. Linear canonical dispersion implies internal energy $E_0 = mc^2$, where antiparticles are space-time inversions and mass/energy conversion is prevented by infinite barrier. The Landé paradox is resolved and de Broglie waves are conceptually demystified as elsewhere time zones. Finally, we derive electric-magnetic duality from pseudoscalar charge symmetry.

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I. INTRODUCTION

Concepts of time usually involve some kind of change or motion [1–3] such as light pulses bouncing between parallel mirrors with isotropic two-way velocity according to the Michelson-Morley experiment. Given the frequency/time standard $f_{Cs} \equiv 9\,192\,631\,770$ Hz intrinsic to matter, household devices conveniently provide radar length in light seconds $2l = c\tau$ using a single clock and the assigned value $c \equiv 299\,792\,458$ m/s. One-way velocities, on the other hand, are linked to clock synchrony by a vicious circle [4–6] and the timing of each leg $l = c_+t_+ = c_-t_-$ remains *a priori* undefined constrained only by the harmonic mean $2/c = 1/c_+ + 1/c_-$. Uniform time is the simplest choice motivated by isotropic space $c_{\pm} = c$ as “stipulated” by Einstein in the introduction to special relativity [7]. Nevertheless, wrist-watch time may well disagree with local coordinate time as every traveller exposed to jet-lag will testify.

The subtleties of distributed time keeping were anticipated before 1905 as shown by the writings of Poincaré and Cohn [8]. Voigt had introduced non-uniform coordinate time in 1887 seeking a covariant wave equation [9] and group structure were provided by Larmor [10] and Lorentz [11] the following decade. While relativity was conceived prior to the idea of gravity as space-time curvature [12, 13], the four-dimensional continuum subsequently introduced by Minkowski [14] led to a shift of focus towards metric structure and coordinate based tensor calculus. The metric is, however, a second-order construct. The Lorentz transformation were derived by Furry invoking first-order generators [15] including “local time” in the so-called Carrollian limit $c \rightarrow 0$ identified by Lévy-Leblond [16]. In the following, we shall be concerned with pre-metric and coordinate-free kinematics.

Navigation in smooth manifolds relies on causal signals

propagating along geodesics. Assuming isotropic one-way velocity [17, 18], uniform synchrony is distributed among fixed clocks by converting radar length to light travel-time

$$s \leftrightarrow ct, \quad (1)$$

introducing a basic space-time symmetry [19] eventually leading to relativistic kinematics. Particle motion in pre-set time define instant velocities

$$ds(t) = vdt, \quad (2)$$

while light-time permutation

$$cdt = (v/c)ds \quad (3)$$

introduces the inverse function

$$dt(s) = V^{-1}ds \quad (4)$$

with characteristic velocity $V = c^2/v$. In this sense, linear elsewhere time set by “through-the-looking-glass” symmetry coexist with inertial motion through the geometric mean $c = \sqrt{vV}$ and the dimensionless ratio

$$\beta = v/c = c/V = \sqrt{v/V}. \quad (5)$$

We emphasize that no tachyon motion [20] suggested by relativistic conundrums such as the “scissor” pseudo-paradox is involved.

In tangent space, Galilean relativity in absolute time $t' = t$ must be accompanied by Carrollian resetting in absolute space $x' = x$ according to the symmetry

$$x' = x - v_G t, \quad \leftrightarrow \quad t' = t - x/v_C, \quad (6)$$

where parameters satisfy $v_G v_C = c^2$ and modified velocity shifts

$$v' = \frac{dx'}{dt'} = \frac{v - v_G}{1 - v/v_C}, \quad (7)$$

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allow the speed of light $v = c$ as a true invariant. The transformations differ from Lorentz kinematics only by rescaling

$$x' = \gamma(x - v_G t), \quad t' = \gamma(t - x/v_C) \quad (8)$$

to second order

$$\gamma = 1/\sqrt{1 - \beta^2} \sim 1 + \frac{1}{2}\beta^2, \quad (9)$$

where time dilation $dt = \gamma d\tau$ and length contraction is easily demonstrated using light clocks.

The kinematics defined by (6) does not represent a group at finite c . Symmetric domains $x \ll ct \leftrightarrow x \gg ct$ extend globally only in singular limits $\beta \rightarrow 0$ and $\gamma \rightarrow 1$, where Galilean relativity is contracted by instant signalling in absolute time

$$c \rightarrow \infty, \quad v_C \rightarrow \infty^2 \quad (10)$$

while time conventionality is introduced in the Carrollian limit

$$c \rightarrow 0, \quad v_C \rightarrow 0^2 \quad (11)$$

involving stationary frames with free parameter v_C .

Euclidean geometry is augmented by causal structure in Sec. II and dual vector spaces are treated in Sec. III. Based on geometric optics and linear wave structure, a dynamical principle of maximum aging is introduced in Sec. IV. Classical mechanics is augmented by wave aberration and Doppler effect in Sec. V. Kinetic and canonical momentum coexists identifying the celebrated term $E_0 = mc^2$, where mass/energy conversion based on classical antiparticles is prevented by infinite barrier. Matter waves are identified as elsewhere time zones and the Landé paradox is resolved in Sec. VI. Electric-magnetic duality is derived in Sec. VII, while Sec. VIII states our conclusions.

II. EUCLIDEAN AND PREMETRIC KINEMATICS

Locally, geodesics are Euclidean straight lines, where the (0,2) metric tensor converts infinitesimal displacements to distance $dl^2 = g_{ij} dx^i dx^j$ in arbitrary coordinates. A tangent basis is defined at every point

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \quad (12)$$

keeping all but one variable fixed $\mathbf{e}_i = \partial\mathcal{P}/\partial x^i$ [12]. The (2,0) inverse given by tensor contraction $g^{ij} g_{jk} = \delta^i_k$ introduces a Laplacian $g^{ij} \partial_i \partial_j$ and a measure of smoothness [21, 22]. The metric is diagonal in Cartesian coordinates and surface normals $\mathbf{e}^i = \epsilon^i_{jk} \mathbf{e}_j \times \mathbf{e}_k$ given by the antisymmetric Levi-Civita symbol coincide with tangents. More generally, directions of steepest ascent

$\mathbf{e}^j = \partial x^j / \partial x^i$ represent transverse surface elements defined by the (1,1) basis

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j \quad (13)$$

independent of metric. Bearing in mind that neither geometry nor physical law can depend on coordinate book-keeping [12], dual structure naturally introduces vector reciprocals $1/\mathbf{v} = \mathbf{v}/v^2$ allowing arbitrary rescaling. As an example, independent tangent displacements and distributed time

$$d\mathbf{x}(t) = \mathbf{v} dt, \quad dt(\mathbf{x}) = \mathbf{v}_s^{-1} \cdot d\mathbf{x}, \quad (14)$$

is related by the chain rule

$$1 = \mathbf{v}_s^{-1} \cdot \mathbf{v} = (\partial t / \partial \mathbf{x}) \cdot (d\mathbf{x} / dt), \quad (15)$$

where reciprocal phase velocities are referred to as slowness vectors by seismologists.

A. Causal structure

Although space-time navigation require four coordinates, a separate time dimension is not mandatory. Particles are mass points carrying proper de Broglie clocks [23] with associated light-spheres in Euclidean geometry rather than Minkowski light-cones. The traditional distinction between kinematics and dynamics can hardly be maintained [24], because synchrony itself requires a vacuum physics.

Local, coordinate-free foliations relative to a given reference clock provide causal structure of indefinite signature

$$\mp r^2 = \mathbf{x}^2 - c^2 t^2 \quad (16)$$

where $\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x} = \mathbf{xx}$ and metric structure need not be specified. Linear foliations

$$\mp r dr = \mathbf{x} \cdot d\mathbf{x} - c^2 t dt, \quad (17)$$

introduce time zones at constant parameter $dr = 0$,

$$t(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v} / c^2, \quad (18)$$

where dual rays $\mathbf{x} = \mathbf{v}_p t$ intersecting (phase) fronts implies $\mathbf{v} \cdot \mathbf{v}_p = c^2$. Provided velocities are parallel, dot products are replaced by algebra, Sec. III, introducing reciprocity

$$c/\mathbf{v}_p = \mathbf{v}/c. \quad (19)$$

In this case, space-time permutation along isotropic directions interchanges causal and elsewhere intervals

$$\mathbf{x} \leftrightarrow ct, \quad (20)$$

with no distinction between vector positions and light-time distance. Signature reversal invariance [25] appears as a byproduct.

The austerity principle “the boundary of a boundary is zero” coined by Wheeler [26] also known as Poincaré’s lemma implies $d(dr) = 0$. The resulting quadratic form

$$\mp ds^2 = d\mathbf{x}^2 - c^2 dt^2, \quad (21)$$

suggests a pseudo-Riemannian continuum, where Euclidean distance is defined instantly meaning $dt = 0$ and proper time intervals $ds^2/c^2 = d\tau^2$ are defined at coincidence $d\mathbf{x} = 0$.

B. Vector kinematics

A simple argument demonstrates that Galilean shifts to a new inertial frame

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}_G t, \quad t' = t, \quad (22)$$

must be accompanied by distributed elsewhere time. Completing the square using the algebra defined by the first Eq. (31) below [27],

$$\mathbf{x}'^2 = \mathbf{x}^2 + \mathbf{v}_G^2 t^2 - 2\mathbf{x} \cdot \mathbf{v}_G t, \quad (23)$$

implies that isotropic light spheres $c^2 t^2 = \mathbf{x}^2$ are transformed as

$$t' = t \sqrt{1 + \mathbf{v}_G^2/c^2 - 2\mathbf{v}_G \cdot \mathbf{x}/c^2 t}, \quad (24)$$

assuming positive time with off-set $t_0 = 0$ at the common origin $\mathbf{x} = 0$. The expansion

$$t'(\mathbf{x}') \sim t(1 + \mathbf{v}_G^2/2c^2) - \mathbf{v}_G \cdot \mathbf{x}/c^2. \quad (25)$$

shows that time remains first-order uniform along the Galilean trajectory $x = v_G t \ll ct$, where both terms contribute to second-order time dilation $t' \sim t/\gamma$. Symmetric events $ct \ll x$, on the other hand, allow inertial shifts to be neglected to second order introducing Carrollian resetting

$$\mathbf{x}' = \mathbf{x}, \quad t' = t - \mathbf{v}_G \cdot \mathbf{x}/c^2, \quad (26)$$

and the combination

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}_G t, \quad t' = t - \mathbf{x} \cdot \mathbf{v}_G^{-1}, \quad (27)$$

holds to first order, where the algebraic product

$$\mathbf{v}_G \mathbf{v}_G = c^2, \quad (28)$$

defines reciprocal parameters. Euclidean tangent space need not be assumed, while global symmetries are defined in singular limits according to Eqs. (10) and (11).

Dilated clocks passing a stationary ruler or contracted lengths passing a fixed clock introduces comoving velocities $\gamma \mathbf{v} = d\mathbf{x}/d\tau$. The transformation

$$\mathbf{v}' = \frac{\mathbf{v} - \mathbf{v}_G}{1 - \mathbf{v} \cdot \mathbf{v}_G^{-1}}, \quad (29)$$

is consistent with Lorentz kinematics at $\gamma_G \sim 1$,

$$\gamma' \mathbf{v}' = \gamma(\mathbf{v} - \mathbf{v}_G), \quad \gamma' = \gamma(1 - \mathbf{v} \cdot \mathbf{v}_G^{-1}), \quad (30)$$

as can be seen by eliminating Lorentz factors. First-order rescaling of γ -factors for $v \gg c$ follows from the expansion (9) when exposed to a Galilean shift. Most importantly, Carrollian invariance $\gamma' \mathbf{v}' \sim \gamma \mathbf{v}$ allows Doppler-like energy-momentum, Sec. V.

III. DUAL VECTOR SPACES

The concept of geometric duality is most easily developed in generalized vector spaces. Symmetric contraction is augmented by anti-symmetric Grassmann extension as envisioned by Clifford [28–30]

$$2\mathbf{a} \cdot \mathbf{b} = \mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a}, \quad 2\mathbf{a} \wedge \mathbf{b} = \mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}, \quad (31)$$

introducing 2-blade parallelograms as geometric primitives. The resulting algebra

$$\mathbf{a}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \quad (32)$$

compounds scalar projections and simple bivectors into objects of even or axial symmetry under sign reversal $\mathbf{a}^P = -\mathbf{a}$. In contrast, polar vectors and 3-blade volumes must be odd under parity. As an example, Euclidean trigonometry in the plane

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta, \quad \mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}| \iota \sin \theta, \quad (33)$$

is spanned by non-parallel vectors providing unit squares $\iota = \mathbf{e}_1 \mathbf{e}_2$ in Cartesian coordinates, where non-simple bivectors are denoted in boldface italic. It is easily seen that rectangular 2-blades must be pseudovectors with negative square

$$(\mathbf{a} \wedge \mathbf{b})^2 = \mathbf{a}\mathbf{b}\mathbf{a}\mathbf{b} = -\mathbf{a}^2 \mathbf{b}^2, \quad (34)$$

implying $\iota^2 = -1$. The complex-like product

$$\mathbf{a}\mathbf{b}/|\mathbf{a}||\mathbf{b}| = \cos \theta + \mathbf{i} \sin \theta = e^{i\theta} \quad (35)$$

rotates clockwise as generated by the algebra $\mathbf{i}\mathbf{e}_1 = -\mathbf{e}_2$ and $\mathbf{i}\mathbf{e}_2 = \mathbf{e}_1$, while coordinate positions $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$ define spinors $z = \mathbf{e}_1 \mathbf{x} = x + \mathbf{i}y$ providing a geometric interpretation of imaginary numbers [31].

The concept of parallelism $\mathbf{a} \wedge \mathbf{b} = 0$ and transversality $\mathbf{a} \cdot \mathbf{b} = 0$ is independent of second-order metric structure allowing a nonmetric dual basis (13). Without a measure of length and angle, transverse 2-blades have indefinite shape insensitive to Galilean shear deformations of the form (22), but well-defined area, since the product $\mathbf{a}\mathbf{b}$ remains well-defined. Given a tangent direction, the identity

$$\mathbf{x} = \mathbf{x}\mathbf{v}\mathbf{v}^{-1} = (\mathbf{x} \cdot \mathbf{v})\mathbf{v}^{-1} + (\mathbf{x} \wedge \mathbf{v})\mathbf{v}^{-1}, \quad (36)$$

defines projection and rejection $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$ according to symmetric expressions

$$2\mathbf{x}_{\parallel} = \mathbf{x} + \mathbf{v}\mathbf{x}\mathbf{v}^{-1}, \quad 2\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{v}\mathbf{x}\mathbf{v}^{-1}, \quad (37)$$

where $\mathbf{x}_{\parallel} \cdot \mathbf{x}_{\perp} = 0$ is readily verified and $\mathbf{v}\mathbf{x}\mathbf{v}^{-1} = \mathbf{x}_{\parallel} - \mathbf{x}_{\perp}$ is the second diagonal. Parallel transport may be defined by premetric constructions such as Schild's ladder [32].

Symmetric and anti-symmetric vector products generalizes to arbitrary grade. Scalar multiplication is even $a \cdot b = ab$ without extension $a \wedge b = 0$, while

$$2\mathbf{a} \cdot \mathbf{b} = \mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a} = 0, \quad \mathbf{a} \wedge \mathbf{b} = \mathbf{a}\mathbf{b}, \quad (38)$$

shows that there is no scalar-vector contraction. Bivector multiplication is even $a \cdot \mathbf{S} = a\mathbf{S}$ and $a \wedge \mathbf{S} = 0$, whereas

$$2\mathbf{a} \cdot \mathbf{S} = \mathbf{a}\mathbf{S} - \mathbf{S}\mathbf{a}, \quad 2\mathbf{a} \wedge \mathbf{S} = \mathbf{a}\mathbf{S} + \mathbf{S}\mathbf{a}, \quad (39)$$

define contracted vectors and extended volumes proportional to the pseudoscalar unit $\iota^2 = -1$ as shown below. As a result, the object

$$\mathbf{a}\mathbf{S} = \mathbf{a} \cdot \mathbf{S} + \mathbf{a} \wedge \mathbf{S}, \quad (40)$$

is odd under parity supplementing (32).

A. Geometric duality

The distinction between polar and axial vectors is superseded by Hodge duality relating elements of grade $n - k$ and k in n spatial dimensions. Volumes $k = 3$ are dual to geometric points $k = 0$ in three dimensions, where the unit cube $\iota = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ reverses handedness under parity $\iota^P = -\iota$. Scalar rescaling is possible $a \wedge \iota = a\iota$ and $a \cdot \iota = 0$, there is no vector extension in three dimensions $\mathbf{a} \wedge \iota = 0$, while contraction

$$\mathbf{s} \cdot \iota = \iota\mathbf{s} = \mathbf{S} \quad (41)$$

introduce bivector surface elements superseding normal vectors advocated by Gibbs and Heaviside. Extension provides volume

$$\mathbf{a} \wedge \mathbf{S} = \iota(\mathbf{a} \cdot \mathbf{s}), \quad (42)$$

unless the vector lies within the surface $\mathbf{a} \wedge \mathbf{S} = \mathbf{a} \cdot \mathbf{s} = 0$ and transverse to \mathbf{s} . Contracted vectors are dual 2-blades

$$\mathbf{a} \cdot \mathbf{S} = \iota(\mathbf{a} \wedge \mathbf{s}), \quad (43)$$

replacing cross products $-\mathbf{a} \times \mathbf{s}$. Transverse surfaces are specified by the dual condition $\mathbf{a} \cdot \mathbf{S} = \mathbf{a} \wedge \mathbf{s} = 0$.

B. Multivectors

Following Eqs. (32) and (40), multigrade primitives in three spatial dimensions can be written [33–35],

$$M = a + \mathbf{v} + \iota\mathbf{w} + \iota b, \quad (44)$$

with pre-metric contraction of even grade

$$M \cdot M = a^2 + \mathbf{v}^2 - \mathbf{w}^2 - b^2 + 2\iota(\mathbf{a}\mathbf{w} + \mathbf{b}\mathbf{v}). \quad (45)$$

The set of quaterions $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ introduced by Hamilton [36] must be viewed as even bivectors providing surface basis elements.

Geometric primitives required by particle motion are point events and tangent vectors

$$X = ct + \mathbf{x}, \quad V = c + \mathbf{v}, \quad (46)$$

where comoving derivation $dX/d\tau = \gamma V$ introduces time dilation and individual components are extracted by conjugation $(X \pm X^P)/2$. We notice that light-time conversion (20) inevitably links spatial inversion $\mathbf{x}^P = -\mathbf{x}$ to time reversal. Dual fronts of equal time ιX transverse to tangent directions [37] are intersected by rays $X_p = (c + \mathbf{v}_p)t$ defined by $\mathbf{v} \wedge \mathbf{v}_p = 0$, while contraction $V \cdot X^P = 0$ supersedes Lorentz orthogonality. Isotropic foliations (16) can be written $\mp XX^P$ and causal motion is restricted by the norm $\gamma^2 VV^P = c^2$.

IV. LINEAR WAVES

The geometric optics approximation [12] to complex fields $\psi(\mathbf{x}, t) \sim e^{i\varphi}$ is based on scalar phase

$$\varphi = \mathbf{k} \cdot \mathbf{x} - \omega t, \quad (47)$$

where the absence of a reference value allow local gauge transformations $\varphi \rightarrow \varphi + \varphi_0(\mathbf{x}, t)$. Instant wave vector assumed constant over the domain of interest and proper frequency

$$\mathbf{k} = \partial\varphi/\partial\mathbf{x}, \quad \omega = -\partial\varphi/\partial t, \quad (48)$$

serve as conjugate variables. Wave length $\lambda = 2\pi/\mathbf{k}$ and oscillation period $T = 2\pi/\omega$ introduce phase velocities unrelated to transport

$$\mathbf{v}_p = \lambda/T = \omega/\mathbf{k}, \quad (49)$$

ranging from the stationary limit $\omega \rightarrow 0$ to uniform waves $\mathbf{k} \rightarrow 0$ propagating instantly. The ratio \mathbf{v}_p/c is reversed under conversion $\omega \leftrightarrow c\mathbf{k}$ and parallelism $\mathbf{k} \wedge \mathbf{v}_p = 0$ implies algebraic dispersion

$$\mathbf{k}\mathbf{v}_p = \mathbf{k} \cdot \mathbf{v}_p = \omega, \quad (50)$$

with constant phase along rays. Dual volumes ιt and wave-fronts $\iota\mathbf{x}$ are suggested by complex phasors invoking extension (42).

The space-time derivative $\partial_X = \partial_{ct} + \partial_{\mathbf{x}}$ define conjugates

$$K = \partial_X\varphi = -\omega/c + \mathbf{k}, \quad (51)$$

with scalar contraction $\varphi = K \cdot X$ independent of the speed of light observing gauge freedom $K \rightarrow K + \partial_X\varphi_0$.

The expression $K = (1 - \mathbf{v}_p/c)\mathbf{k}$ represent linear rays, while second-order wave propagation is ensured by the premetric Lorenz condition

$$-\partial_X^P \cdot \partial_X \varphi = (-\partial_t^2/c^2 + \partial_{\mathbf{x}}^2)\varphi = 0, \quad (52)$$

rather than Lorentz orthogonality in Minkowski space.

A. Symplectic structure

Displacements may represent dual surface elements according to the exact 1-form

$$d\varphi = \mathbf{k} \cdot d\mathbf{x} - \omega dt + \mathbf{x} \cdot d\mathbf{k} - t d\omega, \quad (53)$$

where the projection $K \cdot dX$ vanishes along rays and the second part $X \cdot dK$ vanish at wave packets $\mathbf{x} = \mathbf{v}_g t$ constructed by Fourier superposition according to group dispersion

$$d\mathbf{k} \cdot \mathbf{v}_g = d\omega, \quad (54)$$

that follows from Poincaré's lemma and the closed 2-form

$$d(d\varphi) = 2d\mathbf{k} \cdot d\mathbf{x} - 2d\omega dt = 0, \quad (55)$$

introducing symplectic structure $dK \cdot dX = 0$. Tangent velocities $\mathbf{v}_g = \partial\omega/\partial\mathbf{k}$ are defined by frequency dispersion $\omega(\mathbf{k})$ and reciprocal slowness $d\mathbf{k}/d\omega$ are given by the inverse function $\mathbf{k}(\omega)$, where dispersion relations $\omega = \Omega(\mathbf{x}, \mathbf{k}, t)$ may be viewed as Hamiltonian functions providing a link between optics and mechanics with frequency and wave vector as canonical variables [38].

B. Causal structure

While symplectic structure is fundamentally a geometric constraint, causal structure is tied to the propagation of physical signals. Second-order vacuum dispersion

$$\mp \kappa^2 = \mathbf{k}^2 - \omega^2/c^2 = KK^P, \quad (56)$$

introduces light shells $\kappa^2 = 0$ separating disjunct domains, where linear foliations

$$\mp \kappa d\kappa = \mathbf{k} \cdot d\mathbf{k} - \omega d\omega/c^2, \quad (57)$$

are transverse $K \cdot dK^P = 0$ at constant parameter $d\kappa = 0$. Equivalently,

$$(c^2/\mathbf{v}_p) \cdot d\mathbf{k}/d\omega = 1, \quad (58)$$

shows that group dispersion (54) is consistent with dual structure (15) identifying $\mathbf{v}_g = \mathbf{v}$ as well as the relation $\mathbf{v}_g = c^2/\mathbf{v}_p$.

C. Canonical dynamics

Conjugate or proper frequency defined by (48) introduces a Hamilton-Jacobi type equation in phase space

$$\partial\varphi/\partial t + \Omega(\mathbf{x}, \mathbf{k}, t) = 0, \quad (59)$$

showing that global phase plays the role of action in geometric optics [12]. Analytical mechanics is anticipated by the exact 1-form

$$d\Omega = -d\mathbf{x} \cdot \mathbf{f} + d\mathbf{k} \cdot \mathbf{v} + dt\partial\Omega/\partial t, \quad (60)$$

equivalent to a Legendre transformation in thermodynamics, where the first term is equivalent of work based on local gradients $\mathbf{f} = -\partial\Omega/\partial\mathbf{x}$ and the second term defines group motion at fixed event setting $\mathbf{v}_g = \mathbf{v}$. Group dispersion (54) reveals that only the last term is nonzero in conservative systems allowing geometric points to act as sources or sinks of energy.

Second-order group dynamics is introduced by the principle of maximum aging valid also in general relativity [39]. A twin embarking on a non-inertial round trip will return younger than its sibling that stays at home as a consequence time dilation $dt = \gamma d\tau$. However surprising this result may appear, there is nothing paradoxically about the situation, since the two journeys are non-equivalent. Hamiltonian optics introduces a set of first-order equations

$$d\mathbf{k}/dt = -\partial\Omega/\partial\mathbf{x}, \quad d\mathbf{x}/dt = \partial\Omega/\partial\mathbf{k}, \quad (61)$$

compatible with Newton's second law $d\mathbf{k}/dt = \mathbf{f}$ and the comoving derivative

$$d\varphi/dt = \mathbf{k} \cdot \mathbf{v}_p - \omega + (\mathbf{x} - \mathbf{v}t) \cdot \mathbf{f}, \quad (62)$$

plays the role of Lagrangian in configuration space. Wave propagation in least time according to Fermat's Principle [40] complements group dynamics, since reciprocal group and phase velocities also define inverse refractive indices $1/n_g = n_p$, where $n = c/v$.

D. Doppler effect versus aberration

Partial derivatives transforming as

$$\partial_{\mathbf{x}'} = \partial_{\mathbf{x}} + \mathbf{v}_G^{-1} \partial_t, \quad \partial_{t'} = \partial_t + \mathbf{v}_G \cdot \partial_{\mathbf{x}}, \quad (63)$$

under temporal resetting and relative motion, respectively, are responsible for conjugate kinematics

$$\mathbf{k}' = \mathbf{k} - \omega/\mathbf{v}_G, \quad \omega' = \omega - \mathbf{v}_G \cdot \mathbf{k}, \quad (64)$$

introducing wave aberration and Doppler effect [41] compatible with first-order special relativity $\gamma_G \sim 1$. Factorization

$$\mathbf{k}' = \mathbf{k}(1 - \mathbf{v}_p/\mathbf{v}_G), \quad \omega' = \omega(1 - \mathbf{v}_G \cdot \mathbf{v}_p^{-1}), \quad (65)$$

shows that uniform waves $\mathbf{v}_C = \mathbf{v}_p$ are accompanied by stationary groups $\mathbf{v}_G = \mathbf{v}_g$ of infinite slowness $d\omega' = 0$ according to group dispersion (54). Stationary waves at relative velocity $\mathbf{v}_G = \mathbf{v}_p$ are accompanied by elsewhere groups $\mathbf{v}_C = \mathbf{v}_g$ of vanishing slowness $d\mathbf{k}' = 0$ using the differential $d\mathbf{k}'/d\omega' = d\mathbf{k}/d\omega - 1/\mathbf{v}_C$.

The number wave crests passing a given event is proportional to phase which must be invariant. It is easily seen that Galilean point aberration is compensated by Doppler effect, while Carrollian wave aberration is compensated by relative time.

V. CLASSICAL MECHANICS

Newton's first law ensures that local inertial frames exist. The second law and the definition of work

$$d\mathbf{p} = \mathbf{F}dt, \quad dE = \mathbf{F} \cdot d\mathbf{x} \quad (66)$$

relates differential energy-momentum of a single particle to space-time displacements. Forces may be eliminated in favor of group dispersion

$$dE = \mathbf{v} \cdot d\mathbf{p}, \quad (67)$$

relaxing the assumption of singular mass points, where $\mathbf{v} = \partial E / \partial \mathbf{p}$ is a generalized tangent velocity along canonical momentum. Inverse phase velocities directed along the force are projected using (37),

$$d\mathbf{p}/dE = \mathbf{F}/\mathbf{F} \cdot \mathbf{v} = 1/\mathbf{v}_p, \quad (68)$$

allowing local field propagation rather than action at a distance. Rejection $\mathbf{v} \wedge d\mathbf{p} = d\mathbf{x} \wedge \mathbf{F}$ introduces axial torsion and exterior derivatives.

A. Canonical dynamics

Newtonian dispersion (67) can just as well be written as a closed 2-form

$$d(dS) = d\mathbf{p} \cdot d\mathbf{x} - dE dt = 0 \quad (69)$$

introducing symplectic structure [42], where the 1-form $dS = \mathcal{L}dt$ defines a Lagrangian and the action itself

$$S = \mathbf{p} \cdot \mathbf{x} - Et \quad (70)$$

plays the role of wave phase subject to gauge freedom adding a total derivative to the Lagrangian. Premetric dispersion (50) is algebraic,

$$E = \mathbf{v}_p \cdot \mathbf{p} = \mathbf{v}_p \mathbf{p}, \quad (71)$$

with rays along canonical momentum

$$\mathbf{v} \wedge \mathbf{v}_p = 0 \quad (72)$$

introducing wave-like mechanics with reciprocal parameters (19). The term ‘‘momenergy’’ coined by Taylor and Wheeler is adequately represented by (51),

$$P = \partial_X S = -E/c + \mathbf{p}, \quad (73)$$

with scalar contraction $S = P \cdot X$ and symplectic structure $dP \cdot dX = 0$. Factorization $P = (1 - \mathbf{v}_p/c)\mathbf{p}$ applies also to differentials using (68).

Conjugate energy is responsible for the Hamilton-Jacobi equation

$$\partial S / \partial t + \mathcal{H}(\mathbf{x}, \mathbf{p}, t) = 0, \quad (74)$$

introducing wave mechanics [43–45], while Lagrangian physics is set in configuration space

$$\mathcal{L} = \mathbf{p} \cdot \mathbf{v} - E + \mathbf{F} \cdot (\mathbf{x} - \mathbf{v}t), \quad (75)$$

by Legendre transformation $\mathbf{p} = \partial \mathcal{L} / \partial \mathbf{v}$, where the force $\mathbf{F} = \partial \mathcal{L} / \partial \mathbf{x}'$ evaluated at the particle position introduces the Euler-Lagrange equation. In terms of multivectors, contraction $P \cdot V$ vanishes at the phase velocity and work $dP/dt \cdot X$ is defined relative to wave packets [46]. Hamiltonian dynamics

$$d\mathbf{p}/dt = -\partial \mathcal{H} / \partial \mathbf{x}, \quad d\mathbf{x}/dt = \partial \mathcal{H} / \partial \mathbf{p}, \quad (76)$$

is based on the 1-form

$$d\mathcal{H}(\mathbf{x}, \mathbf{p}, t) = -d\mathbf{x} \cdot \mathbf{F} + d\mathbf{p} \cdot \mathbf{v} + dt \partial \mathcal{H} / \partial t, \quad (77)$$

consistent with the principle of maximal aging valid also in general relativity.

B. Constitutive relations

As pointed out by van Dantzig [47], constitutive relations are required to link energy-momentum to matter. The familiar expressions

$$\mathbf{p} = m\mathbf{v}, \quad E = E_0 + \frac{1}{2}m\mathbf{v}^2, \quad (78)$$

are compatible with group dispersion (67) including an arbitrary constant, while canonical dispersion (71),

$$\mathbf{p} = E/\mathbf{v}_p = E\mathbf{v}/c^2, \quad (79)$$

allows mass to be identified as zero-order energy

$$m = E_0/c^2, \quad (80)$$

observing that wave packets carry internal energy. While avoiding relativistic dynamics, this simple argument does rely on wave/particle duality. Energy dilation $E \sim \gamma E_0$ defines kinetic energy, whereas expressions ignoring internal energy $\mathcal{H} \sim \mathbf{p}^2/2m$ and $\mathcal{L} \sim \frac{1}{2}m\mathbf{v}^2$ based on the symplectic constant $\gamma \mathcal{L} = -E_0$ are only apparently non-relativistic, because factors of c^2 cancel.

Energy exchange in geometric points is allowed by the 1-form (77) based on classical antiparticle production and annihilation implying that conversion will take place unless prevented by infinite barrier $E_0 \rightarrow \infty^2$. In this sense, Galilean physics enforces mass conservation by concealing infinite amounts of energy within matter never to be released.

Photons may be viewed as massless particles with finite product $m\gamma$ in the limit $m \rightarrow 0$ and $\gamma \rightarrow \infty$ subject to light-like dispersion $E_0 = cp_0$ with Compton momentum $p_0 = mc$. Massless dispersion also define Galilean photons with finite momentum and infinite, but conserved energy.

C. Aberration versus Doppler effect

Particle aberration and Doppler effect must be complemented by Carrollian time conventionality $E_0 \rightarrow 0^2$ in order for kinetic and canonical descriptions to coexist in disjunct domains. A Galilean boost applied to the Newtonian expressions (78) implies

$$\mathbf{p}' = \mathbf{p} - m\mathbf{v}_G, \quad E' = E - \mathbf{v}_G \cdot \mathbf{p} + \frac{1}{2}mv_G^2, \quad (81)$$

and replacing mass by energy

$$\mathbf{p}' \sim \mathbf{p} - E\mathbf{v}_G/c^2, \quad E' \sim \gamma_G E - \mathbf{v}_G \cdot \mathbf{p}, \quad (82)$$

allows canonical transformations of the form (64),

$$\mathbf{p}' = \mathbf{p} - E/\mathbf{v}_C, \quad E' = E - \mathbf{v}_G \cdot \mathbf{p}, \quad (83)$$

valid to first order $\gamma_G \sim 1$. Dual kinematics allows point aberration $v \ll c$ at constant energy $cp \ll E$ to coexist with elsewhere wave aberration $v_p \gg c$, while Doppler shifted waves $v_p \ll c$ at constant momentum $cp \gg E$ are consistent with elsewhere rays $v \gg c$. Factorization

$$\mathbf{p}' = \mathbf{p}(1 - \mathbf{v}_p/\mathbf{v}_C), \quad E' = E(1 - \mathbf{v}_G \cdot \mathbf{v}_p^{-1}), \quad (84)$$

shows that momentum vanish in the time frame $\mathbf{v}_C = \mathbf{v}_p$ and propagating waves become stationary in the inertial frame $\mathbf{v}_G = \mathbf{v}_p$ carrying no energy.

D. Charged particles

Electric charges in units of coulomb interact with external fields. Additional momenergy is provided by magnetic vector and scalar potentials

$$\mathbf{p} = \gamma m\mathbf{v} + q_e\mathbf{A}_m, \quad E = \gamma mc^2 + q_e\phi_m, \quad (85)$$

observing that voltage $V=J/C$ induced by non-stationary magnetic flux in weber $V=Wb/s$ imply products of action $CWb=Js$ in SI-units [48]. Canonical transport (79) follows by eliminating scaling factors

$$\mathbf{p} - q_e\mathbf{A}_m = (E - q_e\phi_m)\mathbf{v}/c^2 \quad (86)$$

where momentum behaves like velocity given that energy in minimal coupling is scalar. The action must be a global function

$$S = \gamma E_0(\mathbf{v} \cdot \mathbf{x}/c^2 - t) + q_e\Phi_m, \quad (87)$$

including scalar flux $\Phi = \mathbf{A} \cdot \mathbf{x} - \phi t$ subject to gauge freedom $\Phi \rightarrow \Phi + \Phi_0$, where the contribution of a moving particle

$$S(\mathbf{x} = \mathbf{v}t) = -mc^2t/\gamma - q_e\phi'_m t. \quad (88)$$

can be written in terms of the Doppler shifted scalar potential as given by (83),

$$\mathbf{A}' = \mathbf{A} - \phi/\mathbf{v}_p, \quad \phi' = \phi - \mathbf{v} \cdot \mathbf{A}, \quad (89)$$

identifying $\mathbf{v}_G = \mathbf{v}$ and $\mathbf{v}_C = \mathbf{v}_p$.

Symmetric contraction $\Phi_s = A_s \cdot X$ may be defined for charges of both species $s = e, m$ introducing conjugate variables

$$A_s = \partial_X \Phi_s = -\phi_s/c + \mathbf{A}_s, \quad (90)$$

in units of Wb/m and C/m, respectively. As a result, total momenergy can be written

$$P = m\gamma V \mp q_s A_s^P, \quad (91)$$

where symmetric potentials are denoted by tilded indices, A change of sign is induced by pseudoscalar magnetic charge ιq_m and electric potential ιA_e , Sec. VII.

Newton's equation of motion based on gradient forces in a comoving frame

$$d(\gamma m\mathbf{v} + q_s\mathbf{A}_{\tilde{s}})/dt = -q_s\nabla(\phi_{\tilde{s}} - \mathbf{v} \cdot \mathbf{A}_{\tilde{s}}), \quad (92)$$

includes convection in potential fields with corresponding derivative $(\mathbf{v} \cdot \nabla)\mathbf{A} = \nabla(\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \times \nabla \times \mathbf{A}$. Doppler shifts cancel

$$d(\gamma m\mathbf{v})/dt = q_s(-\nabla\phi_{\tilde{s}} - \partial_t\mathbf{A}_{\tilde{s}} + \mathbf{v} \times \nabla \times \mathbf{A}_{\tilde{s}}), \quad (93)$$

introducing $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ as the electric Lorentz force per unit charge with axial flux density playing the role of vorticity

$$\mathbf{E} = -\nabla\phi_m - \partial_t\mathbf{A}_m, \quad \mathbf{B} = \nabla \times \mathbf{A}_m. \quad (94)$$

The magnetic case involves a change of sign

$$\mathbf{H} = \nabla\phi_e + \partial_t\mathbf{A}_e, \quad \mathbf{D} = -\nabla \times \mathbf{A}_e, \quad (95)$$

when reduced to axial vector calculus. The corresponding Lorentz field $\mathbf{H} - \mathbf{v} \times \mathbf{D}$ acting on magnetic charges is compatible with reverse electric and magnetic handedness. In both cases, point charges are simply flux singularities [49].

E. Antiparticles

Leaving symmetry breaking aside [50] the idea of classical pair production and annihilation is not new. Following Dirac [51], work by Stückelberg [52] and Feynman [53] within special relativity [54–56] were based on time reversal $t^T = -t$ introducing anti-symmetry $X^{PT} = -X$ represented by the Klein four-group $\{1, -1, \iota, -\iota\}$. The sign of the positron first observed in cloud chambers must be opposite to that of the electron in order to satisfy charge conservation and the full CPT -theorem of quantum field theory includes charge conjugation $q^C = -q$ stating that the distinction between ordinary and anti-matter must be conventional [57]. Neutral particles are symmetric under space-time inversion, while separate P and T violations are believed to be rare.

While magnetic dipoles are abundant, it is less appreciated that permanent electric dipoles along spin directions have remained unobserved [58]. Scalar dipoles $\mathbf{d} = q\mathbf{r}$ violate P and T symmetry, while the possibility of pseudoscalar charge restores inversion symmetry at the same time classifying magnetic dipoles as classical mesons composed of antiparticle pairs.

VI. MATTER WAVES

Wave-particle duality follows from the identification of mechanical action with wave phase

$$S = \hbar\varphi, \quad (96)$$

where $h = 2\pi\hbar \equiv 6.62607015 \times 10^{-34}$ Js is Planck's constant. Term by term comparison $P = \hbar K$ implies

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}. \quad (97)$$

implying that symplectic structure, algebraic dispersion, and geometric duality is all inherited as classical fragments existing within quantum theory [59]. It may be argued that the dichotomy is dissolved rather than resolved by being placed in a proper context [60]. Wave functions $\psi(\mathbf{x}, t) = \sqrt{\rho}e^{iS/\hbar}$ introduce conserved probability distributions

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0, \quad (98)$$

where Fourier transformation

$$-\omega\rho + \mathbf{k} \cdot \mathbf{j} = 0 \quad (99)$$

introduces the inverse phase velocity $\mathbf{v}_p^{-1} = \mathbf{k}/\omega$. Dual projection then follows by defining a characteristic velocity $\mathbf{j} = \rho\mathbf{v}$ implying (15). Using $J = \rho V$, continuity may also be written $\partial_X \cdot J = 0$. The single-particle Schrödinger equation $i\hbar\partial_t\psi = \mathcal{H}\psi$ follows from the Hamilton-Jacobi equation [44] with

$$\mathcal{H} = (\mathbf{p} - q\mathbf{A})^2/2m + q\phi, \quad (100)$$

while the Klein-Gordon equation based on relativistic or quadratic dispersion $E^2 = \mathbf{p}^2c^2 + m^2c^2$ including internal energy introduces negative energies and uncontrolled pair production leading to the concept of a quantum field.

A. de Broglie on relativity

The arguments presented in the 1924 thesis relied heavily on special relativity [61, 62]. Particles were modelled as localized wave packets moving in uniform time with Fourier components interpreted as phase waves. Guided by a principle of “phase harmony” [63] de Broglie extended the proper oscillations of a point particle defined by the Planck-Einstein relation

$$mc^2 = \hbar\omega', \quad (101)$$

to uniform phase oscillations accompanying uniform time in the rest frame. Particle motion introduces relativistic Doppler effect

$$\omega' = \gamma(\omega - \mathbf{v} \cdot \mathbf{k}), \quad (102)$$

showing that the frequency scales as $\omega' = \omega/\gamma$ to second order compatible with energy dilation, when the phase velocity $\mathbf{v}_p^{-1} = \mathbf{k}/\omega$ and light-time reciprocity is invoked. Conjugate momentum is proportional to spatial variations in the dual frame

$$\mathbf{p} = h/\lambda, \quad (103)$$

where apparent superluminal motion that worried de Broglie disappears is replaced by elsewhere time zones. Slow groups $\lambda/\lambda_0 \gg 1$ with $\lambda_0 = h/mc$ the Compton wave length reproduce the geometric optics approximation in classical physics [12]. Inverse velocities imply inverse refractive indices.

B. Landé paradox

Neglect of Carrollian symmetry leads to the so-called Landé paradox [64], when particle kinematics is confronted by wave kinematics

$$m\mathbf{v} = \hbar\mathbf{k}, \quad (104)$$

since invariant wave vector is incompatible with Galilean velocity shifts. To avoid a logical paradox both sides must transform the same way. As we have seen, the left-hand-side represents a wave packet with individual Fourier components propagating at the phase velocity. Galilean particle momentum $v \ll c$ is accompanied by Carrollian phase waves $v_p \gg c$ and the right-hand side must therefore be reset according to Carrollian kinematics allowing wave aberration. The apparent paradox is thus resolved by dual kinematics.

A complementary pseudo-paradox would ask, how both sides of (104) can possibly be invariant for slowly

propagating wave components. In this case, Galilean wave kinematics must be accompanied by Carrollian point kinematics leaving the relativistic velocity $\gamma\mathbf{v}$ invariant even for $\gamma \sim 1$. Once again, kinetic and canonical momentum coexists peacefully. Non-relativistic wave propagation simply require dual group kinematics.

C. EPR correlations

Relative or frame-dependent simultaneity is a signature of special relativity. Whether non-local quantum correlations could be influenced by time reversal has been investigated in space-like separated, EPR-type experiments with one detector moving. Very accurate timing allowed nearly simultaneous measurements in a frame moving at $v_G \sim 100$ m/s relative to the laboratory [65, 66]. Since, $\beta \ll 1$ and $\gamma_G \sim 1$, first-order dual kinematics applies, where the dual velocity parameter is of the order $v_C = c^2/v_G \sim 10^{15}$ m/s. No difference compared to an identical experiment having both detectors at rest in the laboratory frame could be detected. Although Carrollian kinematics reverses time ordering, a null result is to be anticipated, since physical processes including wave function collapse can not depend on a particular choice of time coordinate.

D. Phase shifts

The simple-minded transformation (81) introduces a Galilean shift $S' = S - S_G$ of unbalanced terms

$$S_G = m\mathbf{v}_G \cdot \mathbf{x} - \frac{1}{2}mv_G^2 t, \quad (105)$$

where the additional phase $\psi' = e^{-iS_G/\hbar}\psi$ [67–69] generated in the Schrödinger equation should be detectable by interferometer experiments. Bargmann argued that a back and forth trip suddenly reversing velocities combined with spatial translations lead to similar effects [70] prompting a spurious selection rule forbidding superposition of different mass states [71, 72]. Active transformations are restricted by physical laws and non-causal translations must involve elsewhere kinematics. Single-particle actions are indeed invariant when first-order generators (64) are properly identified.

VII. ELECTROMAGNETISM

Intuitively, the physics of conducting wires, coils, and parallel plates must involve aspects of dual geometry. Point charges define isotropic flux densities

$$\mathbf{D} = q_e \hat{\mathbf{r}}/4\pi r^2, \quad \mathbf{B} = q_m \hat{\mathbf{r}}/4\pi r^2, \quad (106)$$

in units of C/m² and Wb/m² or tesla, respectively, where $\hat{\mathbf{r}}$ is a radial unit vector and $4\pi r^2$ is the area of a sphere.

Although isolated magnetic monopoles have never been observed [73], Maxwell's equations allow sources of both specimen, where linear combinations introduce continuous U(1) symmetry responsible for helicity conservation [74, 75]. Galilean transport define orthogonal field strengths in A/m and V/m, respectively,

$$\mathbf{H} = \mathbf{v} \times \mathbf{D}, \quad \mathbf{E} = -\mathbf{v} \times \mathbf{B}, \quad (107)$$

where the minus sign conforms with Lenz' rule and Maxwell's equations introduced below.

As we have seen, the product of electric and magnetic charge provides action and must be quantized as shown by Dirac considering a semi-infinite string of magnetic dipoles [76],

$$q_e q_m = n\hbar, \quad (108)$$

where n is integer. Given the elementary electric unit $e \equiv 1.602176634 \times 10^{-19}$ C as measured by the Josephson effect with $K_J = 2e/h$ the relevant constant, the magnetic unit or flux quantum obtained for $n = 1$ takes the value $\Phi_0 = \hbar/e \simeq 2.07 \times 10^{-15}$ Wb. The ratio known as von Klitzing's constant

$$R_K = \Phi_0/e = \hbar/e^2 \simeq 25.8 \text{ k}\Omega, \quad (109)$$

defines a quantum of resistance unrelated to particle transport, but compatible with internal de Broglie oscillations.

Empty space is assigned linear response by generalizing the phenomenology of displaced charge $q_e = CU$ with capacitance in farad $F=C/V$ and induced flux $\Phi_m = LI$ with inductance in henry $H=Wb/A$. Closed circuits introduce time constants $\tau = \sqrt{CL}$ and the impedance $Z = \sqrt{L/C}$. By analogy, flux densities are related to field strengths by vector constitutive relations

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad (110)$$

based on electric permittivity $\epsilon_0 \sim 8.85$ pF/m and magnetic permeability $\mu_0 \sim 4\pi \cdot 10^{-7}$ H/m formerly a defined constant. Virtual charge and current fluctuations allowed by classical pair production have been suggested as a possible mechanism [77], where the product

$$\epsilon_0 \mu_0 = 1/c^2 \quad (111)$$

defines a characteristic velocity, while the impedance

$$Z_0 = \frac{1}{\epsilon_0 c} = \mu_0 c = \sqrt{\mu_0/\epsilon_0} \sim 377 \Omega, \quad (112)$$

measures the relative strength of electric and magnetic interactions. The fine-structure constant in SI-units

$$\alpha = \frac{e^2}{(4\pi\epsilon_0)\hbar c} = \frac{Z_0}{2R_K} \sim 1/137 \quad (113)$$

is the ratio of medium to matter impedance in a vacuum populated by virtual particles,

A. Electric-magnetic duality

Dirac quantization is charge symmetric $q_e \leftrightarrow q_m$ reversing the constant (109) and the role of electric and magnetic quantities. Symmetric response $\epsilon_0 \leftrightarrow \mu_0$ reverses the impedance (112) introducing a dual vacuum $\alpha \leftrightarrow 1/4\alpha$ [78] with symmetric Coulomb potentials

$$4\pi\phi_e/c = Z_0q_e/r \quad \leftrightarrow \quad 4\pi\phi_m/c = Z_0^{-1}q_m/r, \quad (114)$$

and vector potentials $\mathbf{A}_s = \phi_s/\mathbf{v}_p$. Ordinary wave propagation is ensured by the Lorenz constraint

$$\chi = -\partial_X \cdot A^P = \partial_t\phi/c^2 + \partial_{\mathbf{x}} \cdot \mathbf{A} = 0 \quad (115)$$

with Fourier transform $K \cdot A^P = 0$ which may be interpreted as “grounding” the particle at $\phi' = 0$ using (19). The capacitive limit $\epsilon_0 \rightarrow \infty$ and $\mu_0 \rightarrow 0$ implies perfect conductivity $Z_0 \rightarrow 0$, while the inductive limit $\epsilon_0 \rightarrow 0$ and $\mu_0 \rightarrow \infty$ defines a perfect insulator $Z_0 \rightarrow \infty$.

At the field level, cross products (107) are replaced by 2-blades and vector contraction using (43)

$$\mathbf{H} = \mathbf{v} \wedge \mathbf{D}, \quad \mathbf{E} = \iota(\mathbf{v} \wedge \mathbf{B}) = \mathbf{v} \cdot \mathbf{B}, \quad (116)$$

where flux lines intersect transverse surface elements $\mathbf{B} \cdot \mathbf{B} = 0$. A choice of spatial orientation need not be specified. Geometric derivation $\partial_{\mathbf{x}}A = \partial_{\mathbf{x}} \cdot A + \partial_{\mathbf{x}} \wedge A$ introduces multivectors reminiscent of complex Riemann-Silberstein fields

$$F = -\partial_X A_m^P = \chi_m + \mathbf{E}/c + \mathbf{B}, \quad (117)$$

superseding Faraday tensors as well as the combination due to Minkowski

$$G = -\partial_X A_e^P = \chi_e + \mathbf{H}/c + \mathbf{D}, \quad (118)$$

where scalar terms vanish in the Lorenz gauge (115).

Scalar electric charge [79] is compatible with polar electrics, while dual magnetics implies pseudoscalar magnetic charge [80]. The terms $q_m \mathbf{A}_e$ and $q_m \phi_e$ then pick up minus signs ... The combination of complementary geometry and charge symmetry is known as electric-magnetic duality

$$\mathbf{E} \leftrightarrow \mathbf{H}, \quad \mathbf{D} \leftrightarrow \mathbf{B}, \quad q_e \leftrightarrow \iota q_m, \quad (119)$$

implying $F \leftrightarrow G$ exchange symmetry.

Symmetric Lorentz forces per unit charge acting on monopoles of both species [81–83] are defined by Galilean relativity [84, 85] allowing zero forces

$$\mathbf{E}' = \mathbf{E} - \mathbf{v}_G \cdot \mathbf{B}, \quad \mathbf{H}' = \mathbf{H} - \mathbf{v}_G \wedge \mathbf{D}, \quad (120)$$

assuming quasi-stationary fields $E \ll cB$ and $H \ll cD$. Quasi-uniform flux densities $E \gg cB$ and $H \gg cD$ introduce Carrollian kinematics

$$\mathbf{D}' = \mathbf{D} - \mathbf{v}_C^{-1} \cdot \mathbf{H}, \quad \mathbf{B}' = \mathbf{B} - \mathbf{v}_C^{-1} \wedge \mathbf{E}, \quad (121)$$

as follows by invoking vacuum relations (110) and (111) as well as reciprocal parameters (28).

B. Maxwell equations

The vector formulation of Gibbs and Heaviside [86] in Euclidean geometry [87] is easily modified to accommodate magnetic monopoles [48]. The laws of Gauss are independent of time

$$\nabla \cdot \mathbf{D} = \rho_e, \quad \nabla \times \mathbf{H} = \partial_t \mathbf{D} + \mathbf{j}_e, \quad (122)$$

$$\nabla \cdot \mathbf{B} = \rho_m, \quad -\nabla \times \mathbf{E} = \partial_t \mathbf{B} + \mathbf{j}_m, \quad (123)$$

while non-stationary flux densities and macroscopic currents $\mathbf{j}_s = \rho_s \mathbf{w}_s$ constituting individual transport velocities induce circulating Ampere and Faraday fields. The equations are linear, uncoupled, and symmetric except for a sign responsible for Lenz’ rule. Nowhere does the speed of light appear and properties of the vacuum must be specified by additional constitutive relations such as (110) and (111).

In the absence of source terms, Fourier transformation reveals that flux densities must be orthogonal to phase velocities and field strengths

$$\mathbf{v}_p^{-1} \times \mathbf{H} = -\mathbf{D}, \quad \mathbf{v}_p^{-1} \times \mathbf{E} = \mathbf{B}, \quad (124)$$

introducing linear wave structure. Eliminating cross products is consistent with Carrollian kinematics (121), where flux densities vanish in uniform time $\mathbf{k} = 0$ at $\mathbf{v}_G = \mathbf{v}$ and $\mathbf{v}_C = \mathbf{v}_p$. Eliminating either fluxes or fields using the particle solutions (107) leads to vector relations of the form

$$\mathbf{E} = -\mathbf{v} \times (\mathbf{v}_p^{-1} \times \mathbf{E}), \quad (125)$$

where the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ and orthogonality $\mathbf{v} \cdot \mathbf{E} = 0$ implies

$$\mathbf{E} = (\mathbf{v} \cdot \mathbf{v}_p^{-1})\mathbf{E} \quad (126)$$

showing that inverse projection (15) define linear dispersion with phase velocity identified as slowness.

1. Charge and energy conservation

Macroscopic sources must satisfy charge conservation (98) independent of field dynamics and metric, since only counting operations are involved. Kinematical symmetries separate into Galilean relativity $w_s \ll c$ leaving volumes and densities invariant in absolute time

$$\rho'_s = \rho_s, \quad \mathbf{j}'_s = \mathbf{j}_s - \rho_s \mathbf{v}_G, \quad (127)$$

and Carrollian kinematics $w_s \gg c$ in absolute space

$$\rho'_s = \rho_s - \mathbf{v}_C^{-1} \cdot \mathbf{j}_s, \quad \mathbf{j}'_s = \mathbf{j}_s, \quad (128)$$

allowing absolute currents, where the product $\rho_s \mathbf{w}_s$ is first-order invariant, while densities are shifted in relative time and vanish at $\mathbf{v}_C = \mathbf{v}_p$ regardless of optional

charge cancellations. Charge-current exchange symmetry applies along tangent directions in accordance with the multisource $J = c\rho_s + \mathbf{j}_s$.

Field energy $S = cu + \mathbf{S}$ symmetric under light-time exchange is conserved $\partial_X \cdot S = 0$ according to Poynting's theorem

$$\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}_e - \mathbf{H} \cdot \mathbf{j}_m, \quad (129)$$

where the right-hand side is the power density supplied by source currents,

$$\partial_t u = \mathbf{E} \cdot \partial_t \mathbf{D} + \mathbf{H} \cdot \partial_t \mathbf{B}, \quad (130)$$

is the energy density rate of change, and the power flux through a surface

$$\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot (\nabla \times \mathbf{H}) + \mathbf{H} \cdot (\nabla \times \mathbf{E}), \quad (131)$$

defines the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H} = -\mathbf{E} \cdot \mathbf{H}$. In a linear medium such as (110), the energy density is dipole-like and symmetric

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}), \quad (132)$$

where $\mathbf{H} \cdot \mathbf{B} = -\mathbf{H} \cdot \mathbf{B}$ and electric-magnetic symmetry suggests a corresponding momentum density due to Minkowski that agrees with that of Abraham $\mathbf{D} \times \mathbf{B} = \mathbf{S}/c^2$ in vacuum [48].

2. Dual equations

Algebraic expressions replace coordinate based vector calculus $\nabla \mathbf{A} = \nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A}$ introducing dual relations equivalent to Eqs. (42) and (43) interchanging contraction and extension

$$\nabla \cdot \mathbf{A} = -\iota \nabla \wedge \mathbf{A}, \quad \nabla \times \mathbf{A} = -\iota \nabla \wedge \mathbf{A} = -\nabla \cdot \mathbf{A}, \quad (133)$$

where $\mathbf{A} = \iota \mathbf{A}$ is a dual vector. Applied to Maxwell's equations, Ampere contraction separates from Faraday extension [88–90],

$$\nabla \cdot \mathbf{D} = \rho_e, \quad -\nabla \cdot \mathbf{H} = \partial_t \mathbf{D} + \mathbf{j}_e, \quad (134)$$

$$\nabla \wedge \mathbf{B} = \iota \rho_m, \quad -\nabla \wedge \mathbf{E} = \partial_t \mathbf{B} + \iota \mathbf{j}_m, \quad (135)$$

driven by pseudoscalar magnetic sources. Invoking multivectors (117) and (118) in the Lorenz gauge, only two equations are needed

$$\nabla \cdot G = \partial_{ct} \mathbf{D} + J_e/c, \quad \nabla \wedge F = \partial_{ct} \mathbf{B} + \iota J_m/c, \quad (136)$$

where electric-magnetic duality is the equivalence between Ampere and Faraday parts $F \leftrightarrow G$.

C. Macroscopic constitutive relations

Separate electric and magnetic relations are unaffected by dual magnetics. Macroscopic polarization and magnetization is specified by homogeneous and isotropic linear response at rest [91], where permittivity $\epsilon(\omega, \mathbf{k})$ and permeability $\mu(\omega, \mathbf{k})$ relative to the vacuum are local functions in Fourier space and $n^2 = \epsilon\mu$ is the refractive index [92]. Vector relations in a medium moving at velocity \mathbf{v}_G were obtained by Minkowski

$$\mathbf{D} + \mathbf{v}_G^{-1} \times \mathbf{H} = \epsilon \epsilon_0 (\mathbf{E} + \mathbf{v}_G \times \mathbf{B}), \quad (137)$$

$$\mathbf{B} - \mathbf{v}_G^{-1} \times \mathbf{E} = \mu \mu_0 (\mathbf{H} - \mathbf{v}_G \times \mathbf{D}), \quad (138)$$

where scaling factors cancel [93, 94]. Dual kinematics (120) and (121) thus adequately describe linear electrodynamics. Galilean symmetry eliminates cross products on the left-hand side, where perfect vacuum conductivity is possible at finite permittivity without induction $\mathbf{B} \rightarrow 0^2$ known as the electric limit, while finite permeability defines a perfect insulator without displacement in the magnetic limit $\mathbf{D} \rightarrow 0^2$. Likewise, Carrollian symmetry eliminates cross products on the right-hand side, where finite permeability defines perfect conductivity in the magnetic limit $\mathbf{E} \rightarrow 0^2$, while finite permittivity defines a perfect insulator in the electric limit $\mathbf{H} \rightarrow 0^2$ [95]. Singular contraction in reciprocal limits is ultimately a consequence of the speed of light (111) being factorized.

VIII. CONCLUSIONS

Perceived by our senses everyday phenomena seem to be connected by instantly propagating light signals as argued by Descartes. Römer's observation that the speed of light is finite eventually led to the idea of causal structure, where only two-way velocities measured by a single clock are physically meaningful. One-way propagation is linked to synchrony by a vicious circle introducing kinematical redundancy comparable to gauge freedom. We have argued that uniform synchrony set by light-time conversion must coexist with equal-time zones transverse to tangent directions. The question raised by Dyson [96] as to how special relativity seemingly arises out of thin air starting from Galilean symmetry $c \rightarrow \infty$ has thus been answered: Relative time defined by non-causal extension at finite c may ultimately be traced to kinematical gauge freedom introducing Carrollian time conventionality in the global limit $c \rightarrow 0$.

Fundamental insights are highlighted by the following simple facts: Geometry and physical laws transcend coordinate bookkeeping, kinematics and dynamics can not be sharply distinguished, and metric structure is second order. Massive particles carry internal de Broglie clocks and dynamics is provided by the principle of maximum aging circumventing action at a distance. As demonstrated, dual geometric structure is implied by macro-

scopic continuity involving only counting operations. Relying on dual vector spaces, we have shown that light-time permutation introduces first-order relativity and wave-particle duality in classical mechanics, where the celebrated contribution $E_0 = mc^2$ provides internal energy to localized wave packets. As a result canonical and kinetic particle momenergy coexists resolving the Landé paradox by wave aberration. Remarkably, the concept of de Broglie matter waves reduces to elsewhere time zones in ordinary space whether Euclidean or premetric with no extra time dimension required.

The awkward distinction between polar electrics and axial magnetics required by vector calculus is superseded by electric-magnetic duality based on charge symmetry and geometric duality. Galilean kinematics provides symmetric Lorentz forces acting on point charges, while Carrollian kinematics applies to linear flux propagation. Macroscopic Maxwell equations augmented by the simplest possible constitutive relations define dual vacuum

limits that are either perfectly conducting with no electric field or magnetic induction or perfectly insulating lacking either magnetic field or electric displacement.

Pair production and annihilation is predicted at finite c with antiparticles simply defined as classical *CPT*-inversions linked by light-time conversion. Save rare exceptions, separate inversion symmetries are expected to hold separately [97] as suggested by the absence of permanent electric dipoles. Most interestingly, pseudoscalar magnetic dipoles then qualify as classical antiparticle pairs or mesons explaining the observed elusiveness of free monopoles.

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