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Numerical control of a semilinear wave equation on an interval

M.M. Cavalcanti, V.N. Domingos Cavalcanti, C. Rosier, and L. Rosier

Abstract We are concerned with the numerical exact controllability of the semilinear wave equation on the interval $(0,1)$. We introduce a Picard iterative scheme yielding a sequence of approximated solutions which converges towards a solution of the null controllability problem, provided that the initial data are small enough. The boundary control, which is applied at the endpoint $x = 1$, is taken in the space $H_0^1(0,T)$ for $T = 2$. For the linear part, the control input is obtained by imposing a transparent boundary condition at $x = 1$. Next, we provide several simulations to show the efficiency of the algorithm, using collocation pseudospectral methods on Chebychev grids to discretize the second order derivative in space in the wave equation.

1 Introduction

The issue of the exact controllability of the wave equation is well understood for a long time [24, 4]. Unfortunately, the numerical control of the wave equation turns out to be delicate, as the discrete control may not converge towards

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the continuous control of the wave equation. This is due to the emergence of spurious high-frequencies oscillations for the discrete control problem. Several strategies have been proposed to face this problem: multigrid strategy, filtering of the high frequency modes, introduction of viscous terms, etc. We refer to [14, 3, 26, 21, 8, 9, 20, 22] for some works about the numerical control of the wave equation, and to [13, 27] for some surveys. See also [11, 6] for the numerics corresponding to the wave equation with damping. We refer also to [5] for the stabilization of hyperbolic systems, and to [1, 10] for the reduction of initial-boundary value problems for hyperbolic PDEs to initial-value problems for integro-differential-difference equations.

In most of the papers concerned with the theoretical or numerical control of the wave equation, the control strategy is based upon the Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions. However, in dimension one there exists another strategy based upon the use of transparent boundary conditions leading to the finite-time stability of the system. This was first noticed by Majda in [19], and next used in [17, 22] to give explicit control inputs for the wave equation in dimension one. The transparent boundary conditions can be as well used for the finite-time stabilization of systems of conservation laws on networks [18, 23, 2, 15]. The idea is to let waves leave the domain in a natural way without bounce (or damping) at the boundary.

We shall see that the trajectory corresponding to the transparent boundary conditions is the same as those obtained by applying homogeneous Dirichlet conditions on a larger domain, thanks to the finite propagation speed of the wave equation. Thus, to design numerically the control, we can replace HUM (unstable) by a numerical method to solve a classical Cauchy problem for the wave equation (stable).

In this paper, we are concerned with the exact boundary controllability of a semi-linear wave equation

$$y_{tt} - y_{xx} = f(y)$$

on the interval $(0,1)$ from a numerical perspective. By the time reversibility of the wave equation, it is sufficient to restrict ourselves to the null controllability problem. The aim of the paper is to propose an algorithm at the continuous level which gives a sequence of controls $h^n \to h$ and a sequence of "trajectories" $y^n \to y$ such that, at the limit, y satisfies both

$$\begin{aligned} &y_{tt} - y_{xx} = f(y), \qquad (t,x) \in (0,T) \times (0,1) \\ &y(t,0) = 0,\ y(t,1) = h(t), \qquad t \in (0,T) \\ &(y(0,.), y_t(0,.)) = (\xi^0, \xi^1) \end{aligned}$$

and

$$(y(T,.), y_t(T,.)) = (0,0).$$

Here, (ξ^0, ξ^0) denotes any small initial data in some appropriate space, and the function f is locally Lipschitz continuous and it vanishes at 0. The state (y, y_t) and the control h will be taken more regular (see [12]) than in the linear case ($f \equiv 0$) in order to give a sense to the nonlinear term $f(y)$.

The control h^n is computed numerically by solving several linear control problems, and the state y^n, obtained by solving numerically an initial boundary-value problem, provides a numerical approximation of the trajectory y.

The paper is organized as follows. In Section 2, we describe our iterative scheme and prove the convergence of both the trajectories and the control inputs in Theorem 1, the main result in the paper. Section 3 is devoted to the description of the numerical approach used here and Section 4 provides several numerical simulations. Some concluding remarks are displayed in Section 5. Some technical lemmas needed to prove Theorem 1 are gathered in an appendix.

2 Picard iteration scheme

We first propose, at the continuous level, an exponentially convergent iterative scheme providing at the limit both the controlled trajectory and the control input for the semilinear wave equation. Due to obvious storage limitations, it is preferable to keep in the memory of the computer the control input $h(t)$ computed at the previous step than the corresponding trajectory $y(t,x)$. That choice, needed for a practical implementation of the algorithm, results in a scheme for which the convergence turns to be more complicated to establish than for a scheme based on the storage of the trajectory.

To describe the scheme, we need to introduce a few notations. For the sake of simplicity, we write

$$H_x^k = H^k(0,1)\ (k \in \mathbb{Z}), \quad L_x^2 = L^2(0,1), \quad H_t^1 = H^1(0,T)$$

and introduce the spaces

$$\mathcal{H}_0 := H_0^1(0,1) \times L^2(0,1) \subset \mathcal{H} := \{(y,z) \in H_x^1 \times L_x^2;\ y(0) = 0\}$$

endowed with the norm

$$||(y,z)||_{\mathcal{H}} = \left(||y_x||_{L_x^2}^2 + ||z||_{L_x^2}^2 \right)^{\frac{1}{2}}.$$

Let $(S(t))_{t\in\mathbb{R}}$ denote the unitary group on $\mathcal{H}_0$ generated by the operator $A(y,z) = (z, y_{xx})$ with domain

$$D(A) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \subset \mathcal{H}_0.$$

Thus, for given $(\xi^0,\xi^1) \in \mathcal{H}_0$, if $(y(t),z(t)) = S(t)(\xi^0,\xi^1)$, then $z(t) = y_t(t)$ and y solves the system

$$y_{tt} - y_{xx} = 0, \quad (t,x) \in \mathbb{R} \times (0,1) \tag{1}$$
$$y(t,0) = y(t,1) = 0, \quad t \in \mathbb{R} \tag{2}$$
$$(y(0,.), y_t(0,.)) = (\xi^0, \xi^1). \tag{3}$$

Pick $\delta_0 > 0$ and $f \in W^{1,\infty}(-\delta_0, \delta_0)$ such that

$$f(0) = 0, \text{ and } ||f'||_{L^\infty(-\delta_0,\delta_0)} \leq \eta < \eta_0 < \frac{1}{2}. \tag{4}$$

Let finally $\Lambda : \mathcal{H}_0 \to H_0^1(0,T)$ denote a continuous map such that for any given $(\xi^{0,T}, \xi^{1,T}) \in \mathcal{H}_0$, the solution y of the controlled system

$$y_{tt} - y_{xx} = 0, \quad (t,x) \in (0,T) \times (0,1) \tag{5}$$
$$y(t,0) = 0, \ y(t,1) = h(t) := \Lambda(\xi^{0,T}, \xi^{1,T})(t), \quad t \in (0,T) \tag{6}$$
$$(y(0,.), y_t(0,.)) = (0,0) \tag{7}$$

satisfies

$$(y(T,.), y_t(T,.)) = (\xi^{0,T}, \xi^{1,T}).$$

From well-known results, Λ is well-defined and continuous from $L^2_x \times H^{-1}_x$ into $L^2(0,T)$ if $T \geq 2$, but Λ is also well-defined and continuous from $\mathcal{H}_0$ into $H_0^1(0,T)$ (see e.g. [12]). (Note that $h(0) = 0$ by (7) and $h(T) = \xi^{0,T}(1) = 0$, for $\xi^{0,T} \in H_0^1(0,1)$.) Thus, there is some constant $C_\Lambda > 0$ such that

$$||h||_{H_t^1} \leq C_\Lambda ||(\xi^{0,T}, \xi^{1,T})||_{\mathcal{H}} \tag{8}$$

Actually Λ may be constructed explicitly by using d'Alembert's formula, and this is precisely the numerical strategy used here.

We are in a position to describe our Picard iterative scheme. Pick $T = 2$ and let $(\xi^0, \xi^1) \in \mathcal{H}_0$.

1. First iteration.
We set $h^1(t) = 0$ for $t \in [0,T]$. Let y_1^1 solve the nonlinear system

$$y^1_{1,tt} - y^1_{1,xx} = f(y^1_1), \quad (t,x) \in (0,T) \times (0,1) \tag{9}$$
$$y^1_1(t,0) = 0, \ y^1_1(t,1) = h^1(t) = 0, \quad t \in (0,T) \tag{10}$$
$$(y^1_1(0,.), y^1_{1t}(0,.)) = (\xi^0, \xi^1). \tag{11}$$

Next, we set

$$(\delta h)^1 := -\Lambda(y^1_1(T,.), y^1_{1t}(T,.)), \tag{12}$$

and introduce the solution of the linear system

$$y^1_{2,tt} - y^1_{2,xx} = 0, \quad (t,x) \in (0,T) \times (0,1) \tag{13}$$
$$y^1_2(t,0) = 0,\ y^1_2(t,1) = (\delta h)^1(t), \quad t \in (0,T) \tag{14}$$
$$(y^1_2(0,.), y^1_{2t}(0,.)) = (0,0). \tag{15}$$

Set $y^1 := y^1_1 + y^1_2$. Then, by construction, y^1 solves

$$y^1_{tt} - y^1_{xx} = f(y^1_1), \quad (t,x) \in (0,T) \times (0,1) \tag{16}$$
$$y^1(t,0) = 0,\ y^1(t,1) = h^1(t) + (\delta h)^1(t), \quad t \in (0,T) \tag{17}$$
$$(y^1(0,.), y^1_t(0,.)) = (\xi^0, \xi^1), \tag{18}$$

and it satisfies

$$(y^1(T,.), y^1_t(T,.)) = (0,0). \tag{19}$$

Thus, $(\delta h)^1$ is a correction term added to h^1 in order that (19) be satisfied.

2. *n*TH ITERATION ($n \geq 2$).
Assume that $h^{n-1} \in H^1_0(0,T)$ and $(\delta h)^{n-1} \in H^1_0(0,T)$ have been constructed. Set

$$h^n(t) := h^{n-1}(t) + (\delta h)^{n-1}(t), \tag{20}$$

and let y^n_1 denote the solution of the nonlinear system

$$y^n_{1,tt} - y^n_{1,xx} = f(y^n_1), \quad (t,x) \in (0,T) \times (0,1) \tag{21}$$
$$y^n_1(t,0) = 0,\ y^n_1(t,1) = h^n(t), \quad t \in (0,T) \tag{22}$$
$$(y^n_1(0,.), y^n_{1t}(0,.)) = (\xi^0, \xi^1). \tag{23}$$

Next, set

$$(\delta h)^n = -\Lambda(y^n_1(T,.), y^n_{1t}(T,.)), \tag{24}$$

and let y^n_2 denote the solution of the linear system

$$y^n_{2,tt} - y^n_{2,xx} = 0, \quad (t,x) \in (0,T) \times (0,1) \tag{25}$$
$$y^n_2(t,0) = 0,\ y^n_2(t,1) = (\delta h)^n(t), \quad t \in (0,T) \tag{26}$$
$$(y^n_2(0,.), y^n_{2t}(0,.)) = (0,0). \tag{27}$$

Let

$$y^n := y^n_1 + y^n_2. \tag{28}$$

Then y^n satisfies

$$y^n_{tt} - y^n_{xx} = f(y^n_1), \quad (t,x) \in (0,T) \times (0,1) \tag{29}$$
$$y^n(t,0) = 0,\ y^n(t,1) = h^n(t) + (\delta h)^n(t), \quad t \in (0,T) \tag{30}$$
$$(y^n(0,.), y^n_t(0,.)) = (\xi^0, \xi^1), \tag{31}$$
$$(y^n(T,.), y^n_t(T,.)) = (0,0). \tag{32}$$

We are in a position to state the convergence result for the Picard iterative scheme at the continuous level.

Theorem 1 *Let $T = 2$. Then there exist some constants $\delta \in (0,\delta_0)$, $\varepsilon \in (0,1)$ and $C > 0$ such that for any $(\xi^0,\xi^1) \in \mathcal{H}_0$ with*

$$||(\xi^0,\xi^1)||_{\mathcal{H}} < \delta \tag{33}$$

the sequence $(y_1^n, y_2^n, h^n, (\delta h)^n)_{n\geq 1}$ is well defined in $C([0,T],H_x^1)^2 \times (H_t^1)^2$ and we have

$$||(y_1^n - y, y_{1,t}^n - y_t)||_{C([0,T],\mathcal{H})} + ||(y_2^n, y_{2,t}^n)||_{C([0,T],\mathcal{H})} + ||h^n - h||_{H_t^1} + ||(\delta h)^n||_{H_t^1} \leq C\varepsilon^n, \tag{34}$$

where $(y, y_t, h) \in C([0,T],\mathcal{H}) \times H_0^1(0,T)$ satisfies

$$y_{tt} - y_{xx} = f(y), \qquad (t,x) \in (0,T) \times (0,1) \tag{35}$$

$$y(t,0) = 0,\ y(t,1) = h(t), \qquad t \in (0,T) \tag{36}$$

$$(y(0,.), y_t(0,.)) = (\xi^0,\xi^1), \tag{37}$$

and

$$(y(T,.), y_t(T,.)) = (0,0). \tag{38}$$

Proof We need two lemmas whose proofs are postponed in appendix.

Lemma 1 *(Estimates for the linear system) There exists a constant $C_1 > 0$ such that for any $(z^0,z^1) \in \mathcal{H}_0$, any $h \in H_t^1$ with $h(0) = 0$, and any $g \in L^1(0,T,L_x^2)$, the solution of the system*

$$z_{tt} - z_{xx} = g, \qquad (t,x) \in (0,T) \times (0,1) \tag{39}$$

$$z(t,0) = 0,\ z(t,1) = h(t), \qquad t \in (0,T) \tag{40}$$

$$(z(0,.), z_t(0,.)) = (z^0,z^1) \tag{41}$$

□

fulfills

$$||(z,z_t)||_{C([0,T],\mathcal{H})} \leq ||(z^0,z^1)||_{\mathcal{H}} + ||g||_{L^1(0,T,L_x^2)} + C_1||h||_{H_t^1}. \tag{42}$$

Let $(z,z_t) = W(h)$ denote the solution of (39)-(41) for $h \in H^1(0,T)$ with $h(0) = 0$ and $(z^0,z^1,g) = (0,0,0)$.

Lemma 2 *(Estimate for the nonlinear problem) Let C_1 be as in Lemma 1. Then there exist some constants $\delta_1 \in (0,\delta_0)$, $R > 0$ and $C_2 > 0$ such that for $(z^0,z^1,h) \in \mathcal{H}_0 \times H_t^1$ with $h(0) = 0$ and*

$$||(z^0,z^1)||_{\mathcal{H}} + C_1||h||_{H_t^1} \leq \delta_1, \tag{43}$$

the map

$$\Gamma:\ (z,\tilde{z}) \to S(t)(z^0,z^1) + W(h) + \int_0^t S(t-s)(0,f(z(s)))ds \tag{44}$$

has a fixed-point $(z,\tilde{z})$ *in the ball* $B(0,R) \subset C([0,T],\mathcal{H})$. *Furthermore,* $\tilde{z} = z_t$, *and* z *solves the nonlinear system*

$$z_{tt} - z_{xx} = f(z), \qquad (t,x) \in (0,T)\times(0,1) \tag{45}$$
$$z(t,0) = 0,\ z(t,1) = h(t), \qquad t \in (0,T) \tag{46}$$
$$(z(0,.), z_t(0,.)) = (z^0,z^1), \tag{47}$$

□

and we have

$$||(z,z_t)||_{C([0,T],\mathcal{H})} \le C_2\left(||(z^0,z^1)||_{\mathcal{H}} + C_1||h||_{H_t^1}\right). \tag{48}$$

The proof of Theorem 1 is done in two steps. In the first step, we show by induction on n that $||h^n||_{H_t^1}$ is uniformly bounded, and that for some $\varepsilon \in (0,1)$

$$||(\delta h)^{n+1}||_{H_t^1} \le \varepsilon||(\delta h)^n||_{H_t^1}.$$

In the second step, we combine the above estimates to prove the exponential convergence of all the sequences. Pick any $\delta \in (0,\delta_1)$ and set (for notational convenience) $(\delta h)^0(t) := \sin(\pi t/T)$. Let

$$\varepsilon := \frac{2C_\Lambda C_1}{\pi - 2\eta}\eta, \qquad C_3 = \frac{C_\Lambda C_2}{1-\varepsilon}. \tag{49}$$

where C_Λ is as in (8). We have $\varepsilon < 1$ and $C_3 > 0$ if $0 < \eta < \eta_0 < \frac{1}{2}$ with η_0 *small enough*.

Step 1. Estimates of the control inputs.

Proposition 1 *Let* ε *and* C_3 *be as in* (49). *Then we have*

$$||h^n||_{H_t^1} \le C_3\delta, \qquad \forall n \ge 1, \tag{50}$$
$$||(\delta h)^n||_{H_t^1} \le \varepsilon||(\delta h)^{n-1}||_{H_t^1}, \qquad \forall n \ge 1. \tag{51}$$

□

Proof of Proposition 1. We proceed by induction on $n \ge 1$. Assume first that $n = 1$. Then (50) is obvious, for $h^1 = 0$. To prove (51), we estimate $(\delta h)^1$ in H_t^1 as follows:

$$\begin{aligned} ||(\delta h)^1||_{H_t^1} &= ||\Lambda(y_1^1(T,.), y_{1t}^1(T,.))||_{H_t^1} \\ &\leq C_\Lambda ||(y_1^1(T,.), y_{1t}^1(T,.))||_{\mathcal{H}} \\ &\leq C_\Lambda C_2 ||(\xi^0, \xi^1)||_{\mathcal{H}} \\ &\leq C_\Lambda C_2 \delta \\ &\leq \varepsilon ||(\delta h)^0||_{H_t^1} \end{aligned} \tag{52}$$

if δ is small enough.

Assume that (50) and (51) are valid for any integer $n' < n$, where $n \geq 2$. Let us check that (50) and (51) are also true for n. Since $h^n = \sum_{k=1}^{n-1} (\delta h)^k$, we have that

$$\begin{aligned} ||h^n||_{H_t^1} &\leq \sum_{k=1}^{n-1} \varepsilon^{k-1} ||(\delta h)^1||_{H_t^1} \\ &\leq \frac{C_\Lambda C_2}{1-\varepsilon} \delta = C_3 \delta, \end{aligned}$$

where we used (51) for $k < n$ and (52). Let us proceed to the proof of (51). Let $w^n := y^{n-1} - y_1^n = (y_1^{n-1} + y_2^{n-1}) - y_1^n$. Then w^n satisfies

$$\begin{aligned} &w_{tt}^n - w_{xx}^n = f(y_1^{n-1}) - f(y_1^n), \qquad (t,x) \in (0,T) \times (0,1) \\ &w^n(t,0) = 0,\ w^n(t,1) = 0, \qquad t \in (0,T) \\ &w^n(0,.) = w_t^n(0,.) = 0, \\ &(w^n(T,.), w_t^n(T,.)) = -(y_1^n(T), y_{1t}^n(T)). \end{aligned} \tag{53}$$

Thus, by (24), (42) and (53), we have

$$\begin{aligned} ||(\delta h)^n||_{H_t^1} &\leq C_\Lambda ||(y_1^n(T,.), y_{1t}^n(T,.))||_{\mathcal{H}} \\ &\leq C_\Lambda ||(w^n, w_t^n)||_{C([0,T],\mathcal{H})} \\ &\leq C_\Lambda ||f(y_1^{n-1}) - f(y_1^n)||_{L^1(0,T,L_x^2)} \\ &\leq C_\Lambda \eta ||y_1^{n-1} - y_1^n||_{L^1(0,T,L_x^2)}. \end{aligned} \tag{54}$$

For the last line, we used (4) and the estimate $\sup(||y_1^{n-1}||_{L_{t,x}^\infty}, ||y_1^n||_{L_{t,x}^\infty}) \leq \delta_0$ valid if δ is small enough. Indeed, using (48) and (50) for $n-1$ and n,

$$\begin{aligned} ||y_1^n||_{L^\infty((0,T)\times(0,1))} &\leq ||(y_1^n)_x||_{L^\infty(0,T,L^2(0,1))} \\ &\leq C_2 (||(\xi^0, \xi^1)||_{\mathcal{H}} + C_1 ||h^n||_{H_t^1}) \\ &\leq C_2 (\delta + C_1 C_3 \delta) \\ &\leq \delta_0 \end{aligned}$$

and similarly $||y_1^{n-1}||_{L^\infty((0,T)\times(0,1))} \leq \delta_0$, for δ small enough. It remains to estimate $||y_1^{n-1} - y_1^n||_{L^1(0,T,L_x^2)}$ in terms of $||(\delta h)^{n-1}||_{H_t^1}$. Let $v^n = y_1^{n-1} - y_1^n$. Then v^n solves

$$\begin{aligned} &v^n_{tt} - v^n_{xx} = f(y^{n-1}_1) - f(y^n_1), \qquad (t,x) \in (0,T)\times(0,1) \\ &v^n(t,0) = 0,\ v^n(t,1) = -(\delta h)^{n-1}(t), \qquad t \in (0,T) \\ &v^n(0,.) = v^n_t(0,.) = 0. \end{aligned} \tag{55}$$

An application of Lemma 1 yields

$$\begin{aligned} ||(y^{n-1}_1)_x - (y^n_1)_x||_{C([0,T],L^2_x)} &\le ||(y^{n-1}_1 - y^n_1, y^{n-1}_{1,t} - y^n_{1,t})||_{C([0,T],\mathcal{H})} \\ &\le ||f(y^{n-1}_1) - f(y^n_1)||_{L^1(0,T,L^2_x)} + C_1||(\delta h)^{n-1}||_{H^1_t} \\ &\le \eta||y^{n-1}_1 - y^n_1||_{L^1(0,T,L^2_x)} + C_1||(\delta h)^{n-1}||_{H^1_t}. \end{aligned} \tag{56}$$

It follows from Poincaré's inequality $\|f\|_{L^2_x} \le \pi^{-1}\|f_x\|_{L^2_x}$ (valid for $f \in H^1_x$ with $f(0) = 0$) that

$$||y^{n-1}_1 - y^n_1||_{L^1(0,T,L^2_x)} \le \frac{T}{\pi}||(y^{n-1}_1)_x - (y^n_1)_x||_{C([0,T],L^2_x)} \tag{57}$$

which, combined with (56), gives

$$||y^{n-1}_1 - y^n_1||_{L^1(0,T,L^2_x)} \le (1 - \frac{T\eta}{\pi})^{-1}\frac{C_1 T}{\pi}||(\delta h)^{n-1}||_{H^1_t}. \tag{58}$$

Gathering together (54), (58) and the fact that $T = 2$, we conclude that

$$||(\delta h)^n||_{H^1_t} \le \varepsilon||(\delta h)^{n-1}||_{H^t_1}$$

with ε as in (49). The proof of Proposition 1 is complete.

Step 2. Exponential decay

Let us start with the convergence of the sequence of controls $(h^n)_{n\ge1}$. From (51) and (52), we infer that

$$||h^{n+1} - h^n||_{H^1_t} = ||(\delta h)^n||_{H^1_t} \le C_\Lambda C_2 \delta \varepsilon^{n-1}, \qquad \forall n \ge 1, \tag{59}$$

so that for $n, p \ge 1$

$$\begin{aligned} ||h^{n+p} - h^n||_{H^1_t} &\le C_\Lambda C_2 \delta(\varepsilon^{n-1} + \cdots + \varepsilon^{n+p-2}) \\ &\le \frac{C_\Lambda C_2 \delta}{1-\varepsilon}\varepsilon^{n-1}. \end{aligned} \tag{60}$$

Thus $(h^n)_{n\ge1}$ is a Cauchy sequence in H^1_t, and its limit $h \in H^1_t$ satisfies

$$||h - h^n||_{H^1_t} \le \frac{C_\Lambda C_2 \delta}{1-\varepsilon}\varepsilon^{n-1} \tag{61}$$

by letting $p \to \infty$ in (60).

It follows from (25)-(27), (42), (51) and (52) that

$$||(y_2^n, y_{2,t}^n)||_{C([0,T],\mathcal{H})} \le C_1||(\delta h)^n||_{H_t^1} \le C_1\varepsilon^{n-1}||(\delta h)^1||_{H_t^1}$$
$$\le C_\Lambda C_1 C_2 \delta\varepsilon^{n-1} =: C_4\delta\varepsilon^{n-1},$$

and hence $(y_2^n, y_{2,t}^n) \to (0,0)$ in $C([0,T],\mathcal{H})$ as $n \to \infty$. From (55), (58) and (59), we infer that

$$||(y_1^{n+1}-y_1^n, y_{1,t}^{n+1}-y_{1,t}^n)||_{C([0,T],\mathcal{H})} \le (C_1+\eta(1-\frac{T\eta}{\pi})^{-1}\frac{C_1T}{\pi})||(\delta h)^n||_{H_t^1} \le C_5\delta\varepsilon^{n-1}$$

with $C_5 = (C_1 + \eta(1 - \frac{T\eta}{\pi})^{-1}\frac{C_1T}{\pi})C_\Lambda C_2$. This yields

$$||(y_1^{n+p} - y_1^n, y_{1,t}^{n+p} - y_{1,t}^n)||_{C([0,T],\mathcal{H})} \le \frac{C_5\delta}{1-\varepsilon}\varepsilon^{n-1},$$

so that the sequence $(y_1^n, y_{1,t}^n)$ converges towards some function (y, y_t) in $C([0,T],\mathcal{H})$ with $y(t,0) = 0$ and

$$||(y - y_1^n, y_t - y_{1,t}^n)||_{C([0,T],\mathcal{H})} \le \frac{C_5\delta}{1-\varepsilon}\varepsilon^{n-1}.$$

Thus (34) is proved for some constant $C > 0$.

It remains to check that the pair (y,h) satisfies (35)-(38). (35)-(37) follow from (21)-(23) and (34), while (38) follows from (28), (32) and (34).

3 Numerical schemes

Let us describe the numerical schemes used to compute y_1^n, y_2^n and $(\delta h)^n$. To simplify the notations, we remove the index n (corresponding to the n^{th} Picard iteration) in y_1^n, y_2^n and $(\delta h)^n$. We focus on the computation of y_1 and h, as y_2 is not needed to approximate y.

3.1 Chebychev collocation spectral method

We have to solve numerically several times the linear (or semilinear) wave equation on a bounded domain. After an obvious change of spatial variables, we can assume that $x \in (-1,1)$, incorporating a coefficient c in the wave operator. Consider the Cauchy problem for the semilinear wave equation:

$$u_{tt} - c^2u_{xx} = g(t,x,u), \ \ (t,x) \in (0,T)\times(-1,1), \tag{62}$$
$$u(t,-1) = h_{-1}(t), \ \ u(t,1) = h_1(t), \quad t \in (0,T), \tag{63}$$
$$u(0,x) = u^0(x), \ \ u_t(0,x) = u^1(x), \quad x \in (-1,1), \tag{64}$$

where g, h_{-1}, h_1, u^0 and u^1 are given functions. In this paper, we shall use the Chebychev collocation spectral method to solve (62)-(64), as described e.g. in [25]. Let $N \in \mathbb{N}^*$ be given. We introduce the Chebychev points $x_j = \cos(\theta_j)$, where $\theta_j = (j-1)\pi/N$ for $1 \leq j \leq N+1$. Note that $x_1 = 1$ and $x_{N+1} = -1$. Any smooth function $v = v(x)$ defined on $[-1,1]$ is conveniently approximated by the Lagrange polynomial $p \in \mathbb{R}[x]$ of degree at most N defined by $p(x_i) = v(x_i)$ for $1 \leq j \leq N+1$. It turns that a good approximation of du/dx can be obtained by using the Chebychev differentiation matrix $D = (D_{ij})_{1 \leq i,j \leq N+1}$ defined as

$$D_{jj} = \sum_{\substack{k=1 \\ k \neq j}}^{N+1} (x_j - x_k)^{-1}, \quad 1 \leq j \leq N+1,$$

$$D_{ij} = \frac{1}{a_j} \sum_{\substack{k=1 \\ k \neq i,\, j}}^{N+1} (x_i - x_k), \quad 1 \leq i \neq j \leq N+1,$$

where $a_j = \sum_{\substack{k=1 \\ k \neq j}}^{N+1} (x_j - x_k)$ for $1 \leq j \leq N+1$. Indeed, introducing the vector $U = (U_j)_{1 \leq j \leq N+1}$ with $U_j = u(x_j)$ $(1 \leq j \leq N+1)$ and setting $\tilde{U} = DU$, $\tilde{U} = (\tilde{U}_j)_{1 \leq j \leq N+1}$, then we notice that the Lagrange polynomial q of degree $\leq N$ defined by $q(x_j) = \tilde{U}_j$ for $1 \leq j \leq N+1$ provides a good approximation of du/dx; namely, we have for some constants $C > 0$ and $K > 1$

$$|\frac{du}{dx}(x_j) - \tilde{U}_j| \leq CK^{-N}, \quad \forall N \geq 1,\ \forall j \in \{1, ..., N+1\}.$$

(See e.g. [7, 25].)

Let $M \in \mathbb{N}^*$ and $\Delta t = T/M$, where $T = 2$. For $1 \leq j \leq N+1$, $0 \leq m \leq M$, let U_j^m stand for the numerical approximation of $u(m\Delta t, x_j)$. To solve (62)-(64) numerically, we use a leap-frog formula in t and Chebychev spectral differentiation in x to get

$$\begin{aligned}
&(\Delta t)^{-2}(U_j^{m+1} - 2U_j^m + U_j^{m-1}) - c^2(D^2\, U)_j = g(m\Delta t, x_j, U_j^m), \\
&\qquad\qquad 2 \leq j \leq N,\ 0 \leq m \leq M-1, \\
&U_1^{m+1} = h_1((m+1)\Delta t),\ U_{N+1}^{m+1} = h_{-1}((m+1)\Delta t), \quad 0 \leq m \leq M-1, \\
&U_j^0 = u^0(x_j),\ U_j^{-1} = u^0(x_j) - (\Delta t)u^1(x_j), \quad 2 \leq j \leq N.
\end{aligned}$$

3.2 Numerical scheme for the linear control problem

To compute $(\delta h)^n$ as in (24), we have to solve numerically the following linear control problem: for given $(y_1(T,.), y_{1,t}(T,.)) \in \mathcal{H}_0$, find $h \in H_0^1(0,T)$ such that the solution y_2 of the system

$$y_{2,tt} - y_{2,xx} = 0, \quad (t,x) \in (0,T) \times (0,1) \tag{65}$$
$$y_2(t,0) = 0,\ y_2(t,1) = h(t), \quad t \in (0,T) \tag{66}$$
$$y_2(0,x) = y_{2,t}(0,x) = 0, \quad x \in (0,1) \tag{67}$$

satisfies

$$y_2(T,x) = -y_1(T,x),\ y_{2,t}(T,x) = -y_{1,t}(T,x), \quad x \in (0,1). \tag{68}$$

The change of unknown function $z(t,x) = y_2(T-t,x)$ transforms the exact controllability problem (65)-(68) into the following null controllability problem

$$z_{tt} - z_{xx} = 0, \quad (t,x) \in (0,T) \times (0,1) \tag{69}$$
$$z(t,0) = 0,\ z(t,1) = h(T-t), \quad t \in (0,T) \tag{70}$$
$$z(0,x) = z^0(x) = -y_1(T,x),\ z_t(0,x) = z^1(x) = y_{1,t}(T,x), \quad x \in (0,1) \tag{71}$$
$$z(T,x) = z_t(T,x) = 0, \quad x \in (0,1). \tag{72}$$

Instead of considering a discretization of the control problem (69)-(72) and solving a finite-dimensional control problem, we design a control for the problem (69)-(72) based upon the finite-time stabilization method widely used for systems of conservation laws (see e.g. [18, 23, 2, 15]). It is well known that the transparent boundary condition $z_x = -z_t$ taken at $x = 1$ (replacing the condition $z(t,1) = h(T-t)$), which is a control in feedback form, leads to the finite-time stability of (69)-(72). The idea is to let the waves leave the domain $(0,1)$ in a free way, as they do naturally when the spatial domain is $\mathbb{R}$ instead of $(0,1)$. To achieve such a natural control, we proceed as follows. We extend z^0 and z^1 as odd functions with compact support. Namely, we set

$$z^0(x) = \begin{cases} -y_1(T,x) & \text{if } 0 < x < 1, \\ y_1(T,-x) & \text{if } -1 < x < 0, \\ 0 & \text{if } x \in (-\infty,1) \cup (1,+\infty), \end{cases} \tag{73}$$

and

$$z^1(x) = \begin{cases} y_{1,t}(T,x) & \text{if } 0 < x < 1, \\ -y_{1,t}(T,-x) & \text{if } -1 < x < 0, \\ 0 & \text{if } x \in (-\infty,1) \cup (1,+\infty). \end{cases} \tag{74}$$

Let $z = z(t,x)$ solve now the system

$$z_{tt} - z_{xx} = 0, \quad 0 < t < T,\ x \in \mathbb{R} \tag{75}$$
$$z(0,x) = z^0(x),\ z_t(0,x) = z^1(x), \quad x \in \mathbb{R}. \tag{76}$$

Since $y_1(T,.) \in H_0^1(0,1)$ and $y_{1,t}(T,.) \in L^2(0,1)$, $(z^0,z^1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(z,z_t) \in C([0,T],H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. Furthermore, z is odd in x, for z^0 and z^1 are odd in x. Thus $z(0,t) = 0$ for all $t \in (0,T)$. Finally, it follows from D'Alembert's formula

$$z(t,x) = \frac{1}{2}[z^0(x+t) + z^0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} z^1(s)ds \tag{77}$$

and from the choice of z^0, z^1, and $T = 2$ that

$$z(T,x) = z_t(T,x) = 0, \qquad \text{for} \quad -1 < x < 1. \tag{78}$$

We infer from (77) that

$$\text{supp } z(t,.) \subset [-1-t, 1+t] \subset [-3,3] \quad \text{ for } 0 \le t \le T = 2.$$

Thus, we can actually limit the computations to the spatial domain $[-3,3]$ by imposing homogenous Dirichlet boundary conditions at $x = \pm 3$, and use e.g. the scheme described in Subsection 3.1. The control $h(t) = z(T-t,1)$ is numerically obtained by computing $z(m\Delta t, 1)$ for $0 \le m \le M$. Note that 1 may not be a Chebychev point $3x_j$ for some $j \in [1, N+1]$. Assuming that $1 \in [3x_{j+1}, 3x_j]$, then a linear interpolation between $z(m\Delta t, 3x_j)$, $z(m\Delta t, 3x_{j+1})$ gives an approximation of $z(m\Delta t, 1)$.

Another way to compute numerically $h(t) = z(T-t,1)$ is to use (77) for $x = 1$, which gives for $t > 0$

$$z(t,1) = \frac{1}{2}z^0(1-t) + \frac{1}{2}\int_{1-t}^{1} z^1(s)ds. \tag{79}$$

(This expression was already derived in [22].) The integral term in (79) can be estimated by using the trapezoid rule.

A third way to compute numerically $h(t) = z(T-t,1)$ is to go back to the transparent boundary condition $z_x(t,1) = -z_t(t,1)$ and to introduce the Riemann invariants $s := z_t + z_x$ and $d := z_t - z_x$. Set $d^0 := z^1 - z^0_x$. Then the transport equation $d_t + d_x = 0$ is integrated as $d(t,x) = d^0(x-t)$. On the other hand $z_t = (s+d)/2$ and $s(t,1) = 0$, for $z_x(t,1) = -z_t(t,1)$. It follows that

$$z(t,1) = \int_0^t z_t(s,1)ds = \frac{1}{2}\int_0^t d(s,1)ds = \frac{1}{2}\int_0^t d^0(1-s)ds \tag{80}$$

which is nothing but (79). Numerically, $h(T - m\Delta t) = z(m\Delta t, 1)$ is approximated by $\frac{1}{4}\sum_{n=0}^{m-1}[d^0(1-n\Delta t) + d^0(1-(n+1)\Delta t)]$. The main avantage of this way to compute the control in the linear problem is that there is no need to solve the wave equation for $(t,x) \in (0,T)\times(-3,3)$. What is needed is the computation of d^0, first on the Chebychev points $(x_j)_{1\le j\le N+1}$ (using the Chebychev differentiation matrix to evaluate z^0_x), and next on the fine grid $(1-n\Delta t)_{0\le n\le M}$ by using the associated Lagrange polynomial.

4 Numerical simulations

4.1 Linear problem

We first test the boundary control of the linear wave equation by using the numerical integration of the linear wave equation (75)-(76) supplemented with the boundary conditions $z(t,\pm 3) = 0$. We take as initial data $z^0(x) = \sin(\pi x)$ and $z^1(x) = 0$ for $x \in (0,1)$ and we extend z^0 and z^1 to $\mathbb{R}$ as in (73)-(74). We use the numerical scheme in Subsection 3.1 with $N = 200$ and $\Delta t = 0.1/N^2$. The simulation is displayed in Figure 1. As expected, the solution z vanishes at $T = 2$ for $x \in (-1,1)$. If we let $e(z^0)$ denote the sup-norm of the vector of the values of z^0 at the Chebychev points inside $(-1,1)$, which gives an estimate of $\|z^0\|_{L^\infty(-1,1)}$, then we obtain as relative error

$$\frac{e(z(T,.))}{e(z^0)} \approx 2.5\,10^{-3}.$$

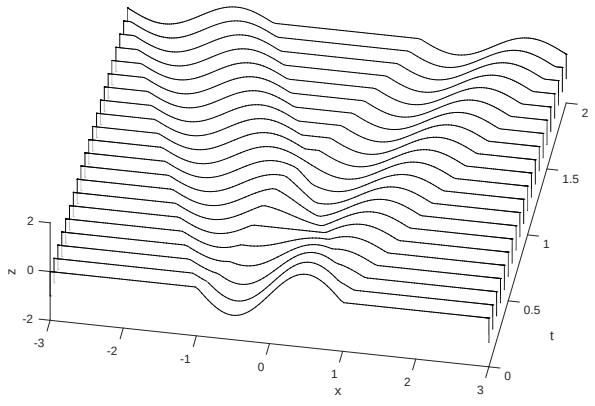

Fig. 1 Free evolution of the linear wave equation.

4.2 Nonlinear control problem

From now on, we compute numerically the control h in the linear control problem (69)-(72) by using a discretization of (80).

4.2.1 The general case

We are concerned with the numerical simulation of the control problem (35)-(11), where f is as in (4). We shall take $f(u) = Cu^2$, where $C \in \mathbb{R}$ is some parameter. We shall see that the parameter C has to take its values in a bounded interval for the control problem to be numerically solved. We take as initial data

$$\xi^0(x) = 1.5\sin(3\pi x), \quad \xi^1(x) = x^2 \quad \text{for } x \in (0,1).$$

For $C = 4$, we apply our Picard iterative scheme with $n = 5$ iterations, $N = 200$, $\Delta t = 1/N^2$. The simulation is displayed in Figure 2 (bottom). The relative error is found to be $\frac{e(z(T,.))}{e(z^0)} \approx 6.9\,10^{-3}$. Another simulation is performed with $C = 0$ (linear case) and is also displayed in Figure 2 (top). The presence of the nonlinear term $4u^2$ in the PDE leads to a visible change of the control input.

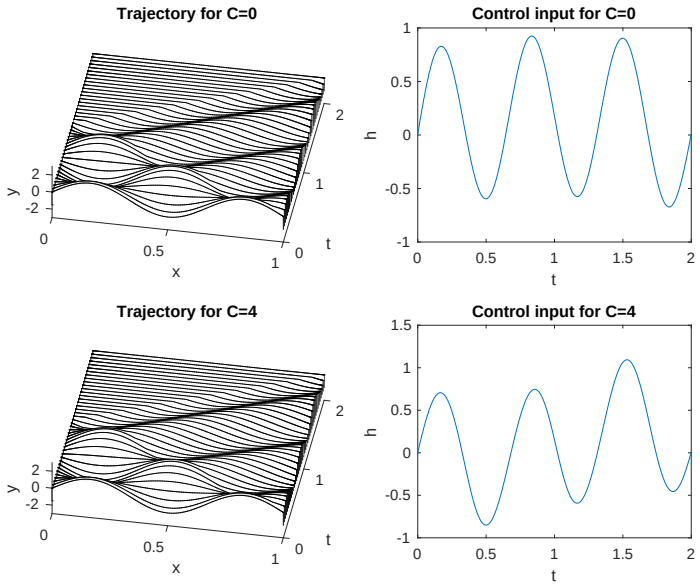

Fig. 2 Control of the nonlinear wave equation: $f(u) = Cu^2$.

4.2.2 The case of an odd function f

When f is odd, then the solution of the system

$$\begin{aligned} y_{tt} - y_{xx} &= f(y), \quad t \in (0,T),\ x \in (-1,1), \\ y(t,\pm 1) &= \pm h(t), \quad t \in (0,T), \\ y(0,x) = \mathrm{sign}(x)\xi^0(|x|),\ y_t(t,0) &= \mathrm{sign}(x)\xi^1(|x|),\ x \in (-1,1). \end{aligned}$$

is *odd*, and hence it vanishes for $x = 0$. This allows to do the computations on the interval $(-1,1)$.

We first test our numerical scheme with $f(u) = Cu^3$. It is numerically proved to work well for $C \in [-2\,,\,1.8]$. For $C = 2$, we have a blow-up, while for $C \leq -3$ we never reach the null state at $T = 2$. This suggests that the Lipschitz condition (4) cannot be replaced by the passivity-like sector condition $f(y)y > 0$. Note that, even for small values of C, the best result is achieved for a "small" number of iterations: the relative error reaches its minimal value and next increases as the number of iterations tends to infinity. We take $\xi^0(x) = 2\sin(2\pi x)$ and $\xi^1(x) = 0$ as initial data, and $N = 200$ and $\Delta t = 4/N^2$ as parameters. For $C = 1.5$ we obtain as relative error $5.7\ 10^{-3}$ with 5 iterations (see Figure 3). For $C = -4$, the "best" relative

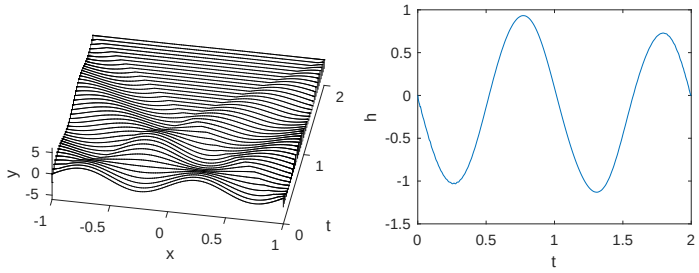

Fig. 3 Control of the semilinear wave equation: $f(u) = 1.5\,u^3$.

error is 0.54 and it is obtained with 4 iterations (see Figure 4). Finally, with the choice $f(u) = C \sin u$ (which is both Lipschitz continuous and bounded), we obtain with $C = 6$ as relative error 0.26 with 3 iterations (see Figure 5). This suggests that our algorithm based on a perturbative argument does not work for large (Lipschitz continuous and bounded) perturbations.

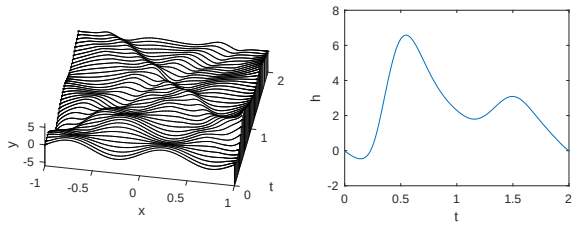

Fig. 4 Control of the semilinear wave equation: $f(u) = -4u^3$.

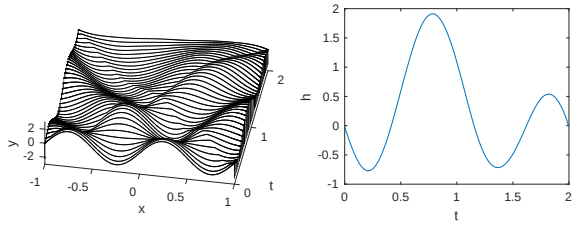

Fig. 5 Control of the semilinear wave equation: $f(u) = 6 \sin u$.

5 Conclusion

An iterative algorithm was proposed to derive a boundary control for the null controllability of a semilinear wave equation in dimension one. Designing the control of the linear part by using a transparent boundary condition and using a collocation pseudospectral method on a Chebychev grid to compute the second order derivative in x in the wave equation, we provided several numerical simulations showing the efficiency of the method, at least for *small* nonlinearities and *small* initial data. Of course, further investigations are required to provide a numerical analysis of the algorithm. It would be also interesting to see whether the method can be extended to the wave equation in dimension two or three. This will be done elsewhere.

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Appendix

Proof of Lemma 1

The estimate for $h \equiv 0$ follows from classical semigroup theory, $(S(t))_{t\in\mathbb{R}}$ being a unitary group on $\mathcal{H}_0$. Assume now that $h \neq 0$ and that $(z^0, z^1, g) = (0,0,0)$. Since $||(z,\tilde{z})||_{\mathcal{H}}$ and $||z||_{H^1_x} + ||\tilde{z}||_{L^2_x}$ are equivalent norms on $\mathcal{H}$, it is sufficient to prove

$$||z||_{C([0,T],H^1_x)} + ||z_t||_{C([0,T],L^2_x)} \leq C||h||_{H^1_t}.$$

(Here, C denotes some constant that may vary from line to line.) It is well-known that

$$||z||_{C([0,T],L^2_x)} + ||z_t||_{C([0,T],H^{-1}_x)} \leq C\left(||z^0||_{L^2_x} + ||z^1||_{H^{-1}_x} + ||h||_{L^2_t}\right) \tag{81}$$

where z denotes the solution by transposition of (39)-(41) with $(z^0, z^1, g) = (0,0,0)$. Introduce $w := z_t$. Then w solves the system

$$\begin{aligned} &w_{tt} - w_{xx} = 0, \qquad (t,x) \in (0,T)\times(0,1)\\ &w(t,0) = 0,\ w(t,1) = h'(t), \qquad t \in (0,T)\\ &(w(0,.), w_t(0,.)) = (z^1, z^0_{xx}) = (0,0). \end{aligned}$$

Since $h \in H^1_t$, we infer from (81) that

$$||w||_{C([0,T],L^2_x)} + ||w_t||_{C([0,T],H^{-1}_x)} \leq C||h||_{H^1_t}, \tag{82}$$

so that the estimate for $||z_t||_{C([0,T],L^2_x)}$ is established. For $||z||_{C([0,T],H^1_x)}$, it is sufficient to notice that $z_{xx} = z_{tt} = w_t$ and to combine

$$||z_{xx}||_{C([0,T],H^{-1}_x)} \leq C||h||_{H^1_t}$$

with the elliptic estimate

$$||z(t,.)||_{H^1_x} \leq C\left(||z_{xx}(t,.)||_{H^{-1}_x} + |z(t,0)| + |z(t,1)|\right) \leq C(||z_{xx}(t,.)||_{H^{-1}_x} + |h(t)|).$$

Proof of Lemma 2

For $(z,\tilde{z}) \in C([0,T],\mathcal{H})$, let

$$|||(z,\tilde{z})||| = \max_{t\in[0,T]} ||(z(t),\tilde{z}(t))||_{\mathcal{H}} = \max_{t\in[0,T]} \left(||z_x(t)||^2_{L^2_x} + ||\tilde{z}(t)||^2_{L^2_x}\right)^{\frac{1}{2}}.$$

We shall apply the contraction mapping principle in the ball $B(0,R) \subset C([0,T],\mathcal{H})$, where $R > 0$ will be chosen later on. Note that

$$||z||_{L^\infty_x(0,1)} \le ||z_x||_{L^2_x}$$

if $z \in H^1_x$ with $z(0) = 0$. Assume that $z^0 \in H^1_0(0,1)$, and that

$$||(z^0,z^1)||_{\mathcal{H}} + C_1||h||_{H^1_t} =: \delta \le \delta_0. \tag{83}$$

$$R < \delta_0. \tag{84}$$

Note that, for $(z,\tilde{z}) \in B(0,R)$,

$$||z(t)||_{L^\infty_x(0,1)} \le ||z_x(t)||_{L^2_x} \le |||(z,\tilde{z})||| < \delta_0 \qquad \forall t \in [0,T].$$

Pick two pairs $(z_1,\tilde{z}_1)$ and $(z_2,\tilde{z}_2)$ in $B(0,R)$. Then by Lemma 1

$$\begin{aligned} |||\Gamma(z_1,\tilde{z}_1) - \Gamma(z_2,\tilde{z}_2)||| &= ||| \int_0^t S(t-s)(0, f(z_1(s)) - f(z_2(s)))ds||| \\ &\le ||f(z_1) - f(z_2)||_{L^1(0,T,L^2_x)} \\ &\le \eta||z_1 - z_2||_{L^1(0,T,L^2_x)} \\ &\le T\eta|||(z_1 - z_2, \tilde{z}_1 - \tilde{z}_2)|||. \end{aligned} \tag{85}$$

Note that Γ contracts in $B(0,R)$, for $T = 2$ and η as in (4). On the other hand, for any $(z,\tilde{z}) \in B(0,R)$, we have by Lemma 1 and (85)

$$\begin{aligned} |||\Gamma(z,\tilde{z})||| &\le |||S(t)(z^0,z^1) + W(h)||| + ||| \int_0^t S(t-s)(0, f(z(s)))ds||| \\ &\le \delta + 2\eta|||(z,\tilde{z}))|||. \end{aligned} \tag{86}$$

Thus $\Gamma(B(0,R)) \subset B(0,R)$ if

$$\delta + 2\eta R \le R.$$

Pick

$$R = \frac{\delta}{1-2\eta}\cdot$$

with

$$0 < \delta < \delta_1 = (1-2\eta)\delta_0 < \delta_0.$$

Then Γ has a unique fixed point $(z,\tilde{z})$ in $B(0,R)$ by the contraction mapping principle. Clearly, $\tilde{z} = z_t$ and (48) holds with $C_2 = (1-2\eta)^{-1}$

References

1. Abolinya, V. È., Myshkis, A. D.: Mixed problems for quasi-linear hyperbolic systems in the plane. (Russian) Mat. Sb. (N.S.) **50 (92)** (1960), 423–442.
2. Alabau-Boussouira, F., Perrollaz, V., Rosier, L.: Finite-time stabilization of a network of strings, Math. Control Relat. Fields **5**, no. 4, 721–742 (2015)
3. Asch, M., Lebeau, G.: Geometrical aspects of exact boundary controllability for the wave equation- a numerical study, ESAIM: COCV **3**, 163–212 (May 1998)
4. Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control Optim. **30** (5), 1024–1065 (1992)
5. Bastin, G., Coron, J.-M.: Stability and Boundary Stabilization of 1-D Hyperbolic Systems. Progress in Nonlinear Differential Equations and their Applications, 88. Subseries in Control, Birkhaüser/Springer, 2016.
6. Cavalcanti, M. M., Corrêa, W. J., Rosier, C., Dias Silva, F. R.: General decay rate estimates and numerical analysis for a transmission problem with locally distributed nonlinear damping, Comput. Math. Appl. **73** (10), 2293–2318 (2017)
7. Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A.: Spectral methods in Fluid Dynamics, Springer Series in Computational Physics, Springer-Verlag Berlin Heidelberg (1988)
8. Castro, C., Micu, S.: Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method, Numer. Math. **102** (3), 413–462 (2006)
9. Castro, C., Micu, S., Münch, A.: Numerical approximation of the boundary control for the wave equation with mixed finite elements in a square, IMA J. Numer. Anal. **28** (1), 186–214 (2008)
10. Cooke, K. L., Krumme, D. W.: Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations, J. Math. Anal. Appl. **24** (1968), 372–387.
11. Domingos Cavalcanti, V. N., Rodrigues, J. H., Rosier, C.: Numerical analysis for the wave equation with locally nonlinear distributed damping, J. Comput. Appl. Math. **301**, 144–160 (2016)
12. Ervedoza, S., Zuazua, E.: A systematic method for building smooth controls for smooth data, Discrete Contin. Dyn. Syst., Ser. B **14** (4), 1375–1401 (2010)
13. Ervedoza, S., Zuazua, E., Numerical approximation of exact controls for waves. Springer Briefs in Mathematics. Springer, New York (2013)
14. Glowinski, R., Li, C., Lions, J.L.: A numerical approach to the exact boundary controllability of the wave equation, Japan J. Appl. Math. **3** 1–76 (1990)
15. Gugat, M., Perrollaz, V., Rosier, L.: Boundary stabilization of quasilinear hyperbolic systems of balance laws: exponential decay for small source terms, J. Evol. Equ. **18**, no. 3, 1471–1500 (2018)
16. Ignat, L., Zuazua, E.: Convergence of a multi-grid method for the control of waves, J. Eur. Math. Soc. **11**, 351–391 (2009)
17. Komornik, V.: Exact Controllability and Stabilization. The Multiplier Method. RAM: Research in Applied Mathematics, Masson, Paris (1994)
18. Leugering, G., Schmidt, E. J. P. G.: On the modelling and stabilization of flows in networks of open canals, SIAM J. Control Optim. **41** 164–180 (2002)
19. Majda, A.: Disappearing solutions for the dissipative wave equation, Indiana Univ. Math. J. **24**, 1119–1133 (1975)
20. Micu, S.: Uniform boundary controllability of a semidiscrete 1-D wave equation with vanishing viscosity, SIAM J. Control Optim. **47** no. 6, 2857–2885 (2008)
21. Münch, A.: A uniformly controllable and implicit scheme for the 1-D wave equation, M2AN Math. Model. Numer. Anl. **39** (2) 377–418 (2005)
22. Pedregal, P., Periago, F., Villena, J.: A numerical method of local energy decay for the boundary controllability of time-reversible distributed parameter systems, Stud. Appl. Math. **121**, no. 1, 27–47 (2008)

23. Perrollaz, V., Rosier, L.: Finite-time stabilization of 2×2 hyperbolic systems on tree-shaped networks, SIAM J. Control Optim. **52**, no. 1, 143–163 (2014)
24. Russell, D. L.: Controllability and stabilization theory for linear partial differential equations: recent progress and open questions, SIAM Rev. **20** (4), 639–737 (1978)
25. Trefethen, L.N.: Spectral Methods in Matlab, SIAM Philadelphia (2000)
26. Zuazua, E.: Boundary observability for the finite difference space semi-discretizations of the 2-D wave equation in the square, J. Math. Pures Appl. **78**, 523—563 (1999)
27. Zuazua, E.: Propagation, observation, control and numerical approximation of waves, SIAM Rev. **47**, No. 2, 197–243 (2005)