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On tail-risk measures for non integrable heavy-tailed random variables

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Abstract

The assessment of risk is a crucial question in various fields of application. An important family of risk measures is provided by the class of distortion risk measures. However, the use of these risk measures is limited by an integrability condition on the quantile function. When heavy-tailed distributions are considered, this condition is violated when the tail index becomes large. In this paper, we first propose a new family of risk measures obtained by minimizing a set of distortion risk measures over a class of probability measures. When a heavy-tailed distribution is considered, we pick in this class a new risk measure whose main feature is to be defined whatever the value of the tail index. The asymptotic behavior of its tail version is investigated and a consistent estimator is proposed. The finite sample performance is discussed on a fire claims dataset.

Keywords: Risk measure; heavy-tailed distribution; extreme value theory; finite sample.

1 Introduction

Nowadays, a crucial question for many companies is to assess the risk of a given phenomenon. For a financial analyst, the phenomenon of interest can be the potential loss of a portfolio. An accurate control of the risk associated to extreme weather events such as hurricane, heat wave, flash flood among many others is of course essential for insurance companies. In a mathematical point of view, the

phenomenon of interest is represented by a real-valued random variable. For a set \mathcal{X} of random variables defined on an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the risk is evaluated by a mapping from \mathcal{X} to \mathbb{R} . To be a proper risk measure, Artzner *et al.* [2] propose a set of properties that should be satisfied by this mapping.

Definition 1 *Let \mathcal{X} be a set of random variables closed under linear combinations i.e., such that for all $m \in \mathbb{R}$, $\lambda > 0$ and $X \in \mathcal{X}$, $\lambda X + m \in \mathcal{X}$. A (monetary) risk measure on \mathcal{X} is a mapping $\rho : \mathcal{X} \rightarrow \mathbb{R}$ which is monotone ($X_1 \geq X_2 \Rightarrow \rho(X_1) \geq \rho(X_2)$), translation-invariant (for all $m \in \mathbb{R}$, $\rho(X + m) = \rho(X) + m$) and homogeneous (for all $\lambda > 0$, $\rho(\lambda X) = \lambda \rho(X)$).*

Another desirable property for a risk measure ρ is to be law-invariant i.e., such that $\rho(X_1) = \rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ sharing the same distribution (in short, $X_1 \stackrel{d}{=} X_2$). Most of the risk measures considered in the literature are law-invariant. One notable counterexample is the Maximum Loss defined for all bounded random variables by $\text{ML}(X) = \sup(-X)$ which is not a law invariant risk measure. Since \mathcal{X} is a set of random variables defined on an atomless probability space, for all law invariant risk measures $\rho : \mathcal{X} \rightarrow \mathbb{R}$, there exists a function $\varrho : \mathcal{Q} := \{Q_X \mid X \in \mathcal{X}\} \rightarrow \mathbb{R}$ such that $\rho(X) = \varrho(Q_X)$, see Lemma 1. Of course, the function ϱ is monotone, translation-invariant and homogeneous. In this paper, only law-invariant risk measures are considered and we make the choice to work with the mapping ϱ instead of ρ .

For each field of application, the use of risk measures has been considered by many authors. In actuarial science, one can cite Brazauskas *et al.* [5], in environmental science, Read and Vogel [23] and in finance Rockafellar and Uryasev [25].

Among the large number of risk measures proposed in the literature, a notable one is the Value-at-Risk of level $\beta \in (0, 1)$ given for $X \in \mathcal{X}$ by

$$\text{VaR}_\beta(X) := \inf \{x \in \mathcal{S}_X \mid F_X(x) \geq \beta\} =: Q_X(\beta),$$

with the convention $\inf(\emptyset) = +\infty$, where $F_X(\cdot) := \mathbb{P}(X \leq \cdot)$, is the cumulative distribution function and $\mathcal{S}_X := \{x \in \mathbb{R} \mid \forall \delta > 0, F_X(x - \delta) < F_X(x + \delta)\}$ the support of the random variable X . The Value-at-Risk correspond to the quantile function Q_X of X evaluated in β .

As mentioned by many authors such as Artzner *et al.* [2], the Value-at-Risk is not a coherent risk measure. A risk measure ρ is said to be coherent if, in addition to the

properties listed in Definition 1, ρ is sub-additive that is to say for all $X_1, X_2 \in \mathcal{X}$, $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$. A classical coherent risk measure is the Tail-Value-at-Risk of level $\beta \in (0, 1)$ defined for instance in Kaas *et al.* [19, Definition 5.6.6] by

$$\text{TVaR}_\beta(X) := \frac{1}{1-\beta} \int_\beta^1 Q_X(u) du.$$

The Tail-Value-at-Risk takes into account the shape of the right-tail distribution unlike the Value-at-Risk. When the quantile function Q_X is continuous and strictly increasing, $\text{TVaR}_\beta(X) = \mathbb{E}(X \mid X > Q_X(\beta))$, which is then referred to as the Conditional Tail Expectation of level β , see for instance Kaas *et al.* [19, Definition 5.6.8]. To simplify, we assume throughout this paper that all the considered quantile functions are continuous and strictly increasing. Since risky situations are associated to large values of X , we also assume from now on and without loss of generality that $Q_X(0) = c_X \in \mathbb{R}$ for all $X \in \mathcal{X}$ i.e., that the left endpoint is finite.

The Value-at-Risk and the Tail-Value-at-Risk of level β belong to the set of distortion measures introduced by Wang [26], see also El Methni and Stupfler [11]. For a given probability measure $\mu : \mathcal{B}([0, 1]) \rightarrow [0, 1]$ where $\mathcal{B}([0, 1])$ is the Borel set on $[0, 1]$, the distortion risk measure with distortion function $g(x) = \nu([0, x])$ is the mapping $\varrho_\mu^{(D)} : \mathcal{Q} \rightarrow \mathbb{R}$ defined for all quantile function $Q_X \in \mathcal{Q}$ by

$$\varrho_\mu^{(D)}(Q_X) = \int_{[0,1]} Q_X d\mu, \tag{1}$$

if the integral exists. The Value-at-risk is obtained with $\mu = \delta_\beta$ where δ_a is the Dirac measure centered on $a \in \mathbb{R}$. When we take in (1) the probability measure ν_β defined for all $A \in \mathcal{B}([0, 1])$ by $\nu_\beta(A) = \nu((A - \beta)/(1 - \beta))$, where ν is the uniform probability on $[0, 1]$, we obtain in return the Tail-Value-at-Risk. A distortion risk measure is coherent if and only if its distortion function g is convex, see Acerbi [1] and Yaari [28].

To be defined, the distortion risk measure (1) requires the quantile function Q_X to be integrable with respect to (w.r.t.) μ . This condition is violated for instance for very heavy-tailed random variables. Recall that a random variable X is said to be

heavy-tailed with tail index $\gamma_X > 0$ if for all $t > 0$,

$$\lim_{u \rightarrow 1} \frac{Q_X(1 - t(1 - u))}{Q_X(u)} = t^{-\gamma_X}. \quad (2)$$

The set of all continuous and strictly increasing quantile functions satisfying (2) with a finite left endpoint is denoted $\mathcal{Q}^{(\text{HT})}$. A standard example of quantile function $Q_X \in \mathcal{Q}^{(\text{HT})}$ is the strict Pareto distribution with, for all $u \in (0, 1)$, $Q_X(u) = \lambda_X(1 - u)^{-\gamma_X} + m_X$ for some scale parameter $\lambda_X > 0$, position parameter $m_X \in \mathbb{R}$ and shape parameter $\gamma_X > 0$ which is nothing else but the tail index. This tail index controls the heaviness of the tail distribution of X . A large value of γ_X corresponds to a very heavy-tailed distribution and one can expect that the condition of integrability w.r.t. μ will be violated in this case. More specifically, introducing the constant

$$\gamma_\mu^* := \sup \left\{ \gamma > 0 \mid \int_{[0,1]} (1 - u)^{-\gamma} d\mu(u) < \infty \right\}, \quad (3)$$

with the convention $\sup \emptyset = 0$, we show in Lemma 3 that Q_X is not integrable as soon as $\gamma_X > \gamma_\mu^*$. For instance if $\mu = \nu$ is the uniform probability measure, $\gamma_\nu^* = 1$, see de Haan and Ferreira [17, Exercice 1.16]. As a consequence, the Tail-Value-at-Risk is not defined for $Q_X \in \mathcal{Q}^{(\text{HT})}$ when $\gamma_X > 1$.

Since heavy-tailed distributions are often considered in application, see for instance Glasserman *et al.* [16] and since large tail index can be observed in financial series, see Moscadelli [21] and Neslehova *et al.* [22], there is a real need to propose risk measures taking finite values even for non-integrable random variables. This is the main objective of this work. As mentioned by Delbaen [8], ‘there is not immediate solution for this problem’ if we are searching for a coherent risk measure. A first attempt to propose a risk measure defined for large values of the tail index can be found in Gardes *et al.* [15]. This risk measure is obtained as the solution of a L^p optimization problem with $p \geq 1$ and it exists as soon as $\gamma_X < 1/(p - 1)$. When $p = 1$, this measure coincides with the Value-at-Risk and with the Tail-Value-at-Risk when $p = 2$.

In this paper, a different approach is investigated. In a first step, we propose a new risk measure obtained by minimizing a set of distortion risk measures over a

collection \mathcal{P} of probability measures. Next, a particular set \mathcal{P} is proposed in order to deal with the particular case of heavy-tailed distributions. The main features of the obtained measure is i) to be well defined and finite whatever the value of the tail index and ii) to coincide with a chosen distortion risk measure for small values of the tail index. In view of point ii), our new risk measure for heavy-tailed distributions can be understood as an extended version of a given distortion measure.

The paper is organized as follows. The new risk measure is defined and commented in Section 2. A tail version of this risk measure is also introduced in order to focus on the tail behavior of X . Section 3 is the central part of this work where the focus is made on heavy-tailed distributions. First, an adaptation of the risk measure defined in Section 2 is given in Section 3.1. Next, some asymptotic properties of its tail version are obtained in Section 3.2. Finally, Section 3.3 is devoted to the estimation of the new risk measure. An application to an insurance dataset (fire claims of a Norwegian insurance company) is also discussed. All the proofs are gathered in Section 4.

2 Minimization of distortion risk measures over a set of probability measures

As mentioned in the introduction, the first part of this paper is devoted to the proposal of a new risk measure defined on a set $\mathcal{Q} = \{Q_X \mid X \in \mathcal{X}\}$ of quantile functions. Recall that throughout this paper, we assume that all quantile functions in \mathcal{Q} are continuous, strictly increasing and with a finite left endpoint ($Q_X(0) = c_X$ for all $X \in \mathcal{X}$).

2.1 Definition of the \mathcal{P} -inf risk measure

The new risk measure is obtained by minimizing a set of distortion risk measures over a family \mathcal{P} of probability distributions. We call such risk measure \mathcal{P} -inf risk measure. Its precise definition is given hereafter.

Definition 2 *Let \mathcal{P} be a non-empty set of probability measures on $\mathcal{B}([0, 1])$ such that for all quantile functions $Q_X \in \mathcal{Q}$, there exists at least one measure ν^* for*

which $\int_{[0,1]} |Q_X| d\nu^* < \infty$. The \mathcal{P} -inf risk measure is the mapping $\bar{\varrho}(\cdot | \mathcal{P}) : \mathcal{Q} \rightarrow \mathbb{R}$ defined for all $Q_X \in \mathcal{Q}$ by

$$\bar{\varrho}(Q_X | \mathcal{P}) := \inf \left\{ \int_{[0,1]} Q_X d\nu \mid \nu \in \mathcal{P} \right\}.$$

This definition deserves some discussion. First, for any $Q_X \in \mathcal{Q}$ and $\nu \in \mathcal{P}$, since $Q_X(0) = c_X \in \mathbb{R}$, the integral $\int_{[0,1]} Q_X d\nu$ exists and takes its value in $(c_X, \infty]$.

In Definition 2 it is claimed that the mapping $\bar{\varrho}(\cdot | \mathcal{P})$ is a proper risk measure.

Indeed, for all $Q_X \in \mathcal{Q}$,

$$\bar{\varrho}(Q_X | \mathcal{P}) \in \left[c_X, \int_{[0,1]} Q_X d\nu^* \right] \subset \mathbb{R},$$

and thus $\bar{\varrho}(\cdot | \mathcal{P})$ is a real valued mapping. The monotonicity is guaranteed by the monotonicity of the integral. Finally, it is readily seen that for all $\lambda > 0$ and $m \in \mathbb{R}$, $\bar{\varrho}(\lambda \cdot + m | \mathcal{P}) = \lambda \bar{\varrho}(\cdot | \mathcal{P}) + m$.

It is worth noting that when \mathcal{P} is not reduced to one element, the \mathcal{P} -inf risk measure is not a distortion risk measure. Obviously, when $\mathcal{P} = \{\mu\}$, the \mathcal{P} -inf risk measure coincides with the distortion risk measure $\varrho_\mu^{(D)}$ as defined in (1).

2.2 Tail version of the \mathcal{P} -inf risk measure

In risk management or environmental sciences, we are mostly interested in large values of the random variable X of interest (important losses, high temperatures, etc.). To assess the risk in the right tail of the distribution, it seems reasonable to impose that for any $X \in \mathcal{X}$, the \mathcal{P} -inf risk measure $\bar{\varrho}(Q_X | \mathcal{P})$ does not depend on the values of $Q_X(u)$ for $u \in [0, \beta)$ with a level β close to 1. To do so, a natural idea is to replace all the probability measures in \mathcal{P} by measures supported on the interval $[\beta, 1]$. This can be achieved by the transformation defined hereafter.

Definition 3 For $\beta \in (0, 1)$, the tail-transformation of level β is the mapping \mathcal{T}_β defined for all probability measure ν by $\mathcal{T}_\beta(\nu) = \nu_\beta$ where for all $A \in \mathcal{B}([\beta, 1])$ we have $\nu_\beta(A) = \nu((A - \beta)/(1 - \beta))$. For a given set \mathcal{P} of probability measures, we denote by \mathcal{P}_β its image by the mapping \mathcal{T}_β .

It is readily seen that for all $Q_X \in \mathcal{Q}$,

$$\int_{[0,1]} Q_X d\nu_\beta = \int_{[0,1]} Q_{X,\beta} d\nu,$$

where $Q_{X,\beta}(\cdot) = Q_X((1-\beta)\cdot + \beta)$. It follows from the above that the tail version of level β of the \mathcal{P} -inf risk measure is the mapping $\bar{q}_\beta(\cdot | \mathcal{P})$ defined for all $Q_X \in \mathcal{Q}$ by

$$\bar{q}_\beta(Q_X | \mathcal{P}) := \bar{q}(Q_X | \mathcal{P}_\beta) = \bar{q}(Q_{X,\beta} | \mathcal{P}).$$

Let us emphasize that \mathcal{T}_β is not the only transformation that can be used to define the tail version of the \mathcal{P} -inf risk measure. Another possibility is to consider the transformation which associates to every probability measure ν its conditional version $\nu(\cdot | [\beta, 1]) := \nu(\cdot \cap [\beta, 1]) / \nu([\beta, 1])$. When ν is the standard uniform distribution, the two transformations coincide.

We conclude this section by remarking that the Tail-Value-at-Risk of level β is the tail version of the distortion risk measure defined for all $Q_X \in \mathcal{Q}$ by $\varrho(Q_X | \{v\}) = \varrho_v^{(D)}(Q_X) = \mathbb{E}(X)$, where v is the standard uniform distribution.

3 A risk measure for heavy-tailed distributions

We assume that the variable of interest X is heavy-tailed distributed. We thus consider quantile functions in the set $\mathcal{Q}^{(\text{HT})}$ of all continuous and strictly increasing quantile functions satisfying (2) with a finite left endpoint.

In this section, we are searching for a set \mathcal{P} of probability measures in order to obtain a \mathcal{P} -inf risk measure (see Definition 2) satisfying $\bar{q}(Q_X | \mathcal{P}) < \infty$ for all $Q_X \in \mathcal{Q}^{(\text{HT})}$ and, for a reference probability measure μ , $\bar{q}(Q_X | \mathcal{P}) = \varrho_\mu^{(D)}(Q_X)$ for all $Q_X \in \mathcal{Q}_{0,\mu} \subset \mathcal{Q}^{(\text{HT})}$. For such a set \mathcal{P} , the \mathcal{P} -inf risk measure can be seen as an extension from $\mathcal{Q}_{0,\mu}$ to the set $\mathcal{Q}^{(\text{HT})}$ of the distortion risk measure $\varrho_\mu^{(D)}$. A natural choice for the reference probability measure is the standard uniform distribution. More generally, we can take a probability measure μ such that the function $x \mapsto \mu([0, x])$ is convex. For such a probability measure, the distortion risk measure $\varrho_\mu^{(D)}$ is coherent, see the introduction for more details.

3.1 Definition

A proposition for a set of probability measures as requested in the introduction of Section 3 is given hereafter.

Definition 4 *Let μ be a reference probability measure. For $\kappa \in [1, \infty]$ and $\gamma_0 \in (0, \gamma_\mu^*)$, the set $\mathcal{P}_\mu(\kappa, \gamma_0)$ is the set of probability measures containing the reference probability measure μ and all probability measures ν such that*

$$\int_{[0,1]} (1-u)^{-\gamma} d\nu(u) \geq \int_{[0,1]} (1-u)^{-\gamma} d\mu(u) \text{ for all } \gamma \in (0, \gamma_0], \quad (4)$$

and

$$\left(\int_{[0,1]} (1-u)^{-\gamma} d\nu(u) \right)^{\gamma_0/\gamma} \leq \kappa \int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) \text{ for all } \gamma > 0. \quad (5)$$

Let us say a few words about this set of probability measures. First, according to the definition of the constant γ_μ^* given in (3), conditions (4) and (5) make sense since for all $\gamma \in (0, \gamma_0]$,

$$\int_{[0,1]} (1-u)^{-\gamma} d\mu(u) < \infty.$$

The tuning parameter κ in (5) controls the size of the set $\mathcal{P}_\mu(\kappa, \gamma_0)$. The larger set is obtained when $\kappa = +\infty$, i.e., when (5) is always satisfied. Note that if $\kappa < 1$, it is easy to check that $\mathcal{P}_\mu(\kappa, \gamma_0) = \{\mu\}$.

With this set of probability measures, we obtained the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure as defined in Definition 2. Its properties are given in the next result.

Proposition 1 *Let μ be a reference probability measure such that $\gamma_\mu^* > 0$. For all $\kappa \in [1, \infty]$ and $\gamma_0 \in (0, \gamma_\mu^*)$ such that $\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) > 0$, the risk measure $\bar{\varrho}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))$ is finite for all $Q_X \in \mathcal{Q}^{(\text{HT})}$. Moreover,*

$$\bar{\varrho}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) = \int_{[0,1]} Q_X d\mu = \varrho_\mu^{(\text{D})}(Q_X),$$

for all $Q_X \in \mathcal{Q}_{0,\mu}$ where $\mathcal{Q}_{0,\mu}$ is the set of strict Pareto distributions with a tail index $\gamma_X \in (0, \gamma_0]$.

The $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure is thus an extension of the distortion risk measure (1) from $\mathcal{Q}_{0,\mu}$ to the set $\mathcal{Q}^{(\text{HT})}$. The size of the set $\mathcal{Q}_{0,\mu}$ is controlled by the

parameter γ_0 . One may be tempted to choose a value close to γ_μ^* in order to have a large set $\mathcal{Q}_{0,\mu}$ for which the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure coincides with the distortion risk measure. But, as pointed out by El Methni and Stupfler [10], the finite-sample performance of an estimator of the distortion risk measure $\varrho_\mu^{(D)}(Q_X)$ is badly affected by large values of γ_X . This is due to the fact that a large variability in the tail is observed when the tail index goes to γ_μ^* . As suggested by Gardes *et al.* [15], one can take γ_0 in order to ensure the existence of the second moment w.r.t. μ of a strict Pareto distribution with tail index γ_0 that is to say

$$\gamma_0 = \sup \left\{ \gamma > 0 \left| \int_{[0,1]} (1-u)^{-2\gamma} d\mu(u) < \infty \right. \right\}, \quad (6)$$

This choice will be adopted in the real data study.

In general, the explicit expression of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure cannot be obtained. However, for a strict Pareto distribution Q_X with tail index $\gamma_X > 0$, we can show (see Lemma 4) that under the conditions of Proposition 1,

$$\bar{\varrho}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) = \left(\int_{[0,1]} (1-u)^{-(\gamma_X \wedge \gamma_0)} d\mu(u) \right)^{(\gamma_X \vee \gamma_0)/\gamma_0},$$

where $a \vee b$ (resp. $a \wedge b$) is the maximum (resp. minimum) value of the set $\{a, b\}$. Let us emphasize that the value of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure does not depend on κ .

Tail version – We now use the definition given in Section 2.2 to introduce the tail version of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure. It is given by

$$\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) = \bar{\varrho}(Q_{X,\beta} | \mathcal{P}_\mu(\kappa, \gamma_0)).$$

Since for all $\beta \in (0, 1)$, $Q_{X,\beta} \in \mathcal{Q}^{(HT)}$ if and only if $Q_X \in \mathcal{Q}^{(HT)}$ and $Q_{X,\beta} \in \mathcal{Q}_{0,\mu}$ if and only if $Q_X \in \mathcal{Q}_{0,\mu}$, we have, under the conditions of Proposition 1, that $\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))$ is finite for all $Q_X \in \mathcal{Q}^{(HT)}$ and for all $Q_X \in \mathcal{Q}_{0,\mu}$,

$$\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) = \int_{[0,1]} Q_X d\mu_\beta = \varrho_{\mu_\beta}^{(D)}(Q_X).$$

The tail version of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure is thus an extension of the tail version of the tail distortion risk measure (1). Some interesting asymptotic properties as the level β goes to 1 are given in the next Section.

3.2 Asymptotic properties of the tail version

In a first step, we provide an asymptotic equivalent of the tail version of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure when the tail level $\beta \rightarrow 1$. This result will be very useful to propose an estimator of this tail version when the level $\beta = \beta_n$ goes to 1 as the sample size n increases, see Section 3.3. The following condition on the reference probability measure and on the parameter γ_0 are required.

(H.1) The reference probability measure μ is such that $\gamma_\mu^* > 0$ and the function

$$\gamma \in (0, \gamma_\mu^*) \mapsto \int_{[0,1]} (1-u)^{-\gamma} d\mu(u)$$

is continuous. Moreover, the parameter $\gamma_0 \in (0, \gamma_\mu^*)$ is such that

$$\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) > 0.$$

For the reference probability measure $\mu = \nu$, the uniform distribution on $[0, 1]$, condition (H.1) holds since $\gamma_\nu^* = 1$ and $\int_{[0,1]} (1-u)^{-\gamma} d\nu(u) = 1/(1-\gamma)$.

Proposition 2 *Let $Q_X \in \mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_X > 0$. If μ and γ_0 satisfy condition (H.1) then for all $\kappa \in [1, \infty)$,*

$$\lim_{\beta \rightarrow 1} \frac{\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{Q_X(\beta)} = \left[\int_{[0,1]} (1-u)^{-(\gamma_X \wedge \gamma_0)} d\mu(u) \right]^{(\gamma_X \vee \gamma_0)/\gamma_0}.$$

Roughly speaking, the above equivalence is due to the fact that for a quantile function $Q_X \in \mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_X > 0$, one has for all $u \in (0, 1)$ that $Q_{X,\beta}(u)/Q_X(\beta) \rightarrow (1-u)^{-\gamma_X}$ as $\beta \rightarrow 1$. The infimum is then obtained by using Lemma 4.

It is worth noting that the asymptotic equivalent of the tail version does not depend on κ . In fact, only the rate of convergence is influenced by the choice of κ ; the larger the value of κ is, the slower the convergence is. The rate of convergence is also badly influenced by a large value of γ_X . More specifically, we can show that for all $\varepsilon > 0$,

there exists $\beta(\varepsilon) \in (0, 1)$ such that for all $\beta \geq \beta(\varepsilon)$ and $Q_X \in \mathcal{Q}^{(\text{HT})}$,

$$\left| \frac{\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{Q_X(\beta)} - \left[\int_{[0,1]} (1-u)^{-(\gamma_X \wedge \gamma_0)} d\mu(u) \right]^{(\gamma_X \vee \gamma_0)/\gamma_0} \right| \leq 2\varepsilon g(\kappa, \gamma_X),$$

where $g(\kappa, \gamma_X)$ increases as κ and/or γ_X increases. The expression of $g(\kappa, \gamma_X)$ is given in the proof of Proposition 2.

Some asymptotic properties of the tail version of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure are given hereafter.

Proposition 3 *Assume that condition (H.1) holds for μ and γ_0 .*

i) *For all quantile functions $Q_X \in \mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_X \in (0, \gamma_0]$, one has for all $\kappa \in [1, \infty)$,*

$$\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) \sim \int_{[0,1]} Q_X d\mu_\beta \text{ as } \beta \rightarrow 1.$$

ii) *Let X_1 and X_2 be two random variables such that Q_{X_1} and Q_{X_2} belong to the set $\mathcal{Q}^{(\text{HT})}$ with respective tail index γ_1 and γ_2 . If $0 < \gamma_1 \leq \gamma_2 \leq \gamma_0$ and if*

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_1 > x, X_2 > x)}{\mathbb{P}(X_2 > x)} = 0, \quad (7)$$

then for all $\kappa \in [1, \infty)$ and $\varepsilon > 0$, there exists $\tau_0 \in (0, 1)$ such that for all $\beta > 1 - \tau_0$,

$$\begin{aligned} & \bar{\varrho}_\beta(Q_{X_1+X_2} | \mathcal{P}_\mu(\kappa, \gamma_0)) \\ & \leq [\bar{\varrho}_\beta(Q_{X_1} | \mathcal{P}_\mu(\kappa, \gamma_0)) + \bar{\varrho}_\beta(Q_{X_2} | \mathcal{P}_\mu(\kappa, \gamma_0))] (1 + \varepsilon). \end{aligned}$$

Point i) of Proposition 3 entails that, for β large enough, the tail risk measure $\bar{\varrho}_\beta(\cdot | \mathcal{P}_\mu(\kappa, \gamma_0))$ can be seen as an extended version of the tail distortion risk measure $\varrho_{\mu_\beta}^{(\text{D})}$ from the set of heavy-tailed quantile functions with tail index lower than γ_0 to the whole set $\mathcal{Q}^{(\text{HT})}$ of heavy-tailed distributions.

The result in part ii) of Proposition 3 can be interpreted as an asymptotic sub-additivity property of the tail version of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure. Condition (7) ensures that when X_1 and X_2 are heavy-tailed, the sum $X_1 + X_2$ is also heavy-tailed with tail index $\gamma_1 \vee \gamma_2$.

3.3 Statistical inference

In practice, the distribution of the random variable of interest X is unknown. The available information is contained in a realization of a sample X_1, \dots, X_n of n independent replications of X . In this section, it is assumed that Q_X , the quantile function of X , belongs to the set $\mathcal{Q}^{(\text{HT})}$ of continuous and strictly increasing heavy-tailed quantile functions with a finite left-endpoint. The tail index of X is denoted γ_X .

Let μ be a reference probability measure. The aim of this section is to propose an estimator of $\bar{q}_{\beta_n}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))$ that is to say of the tail version of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure of level β_n where β_n is a sequence converging to 1 as the sample size increases. Two situations for the rate of convergence of β_n to 1 are considered hereafter.

Intermediate case – We first assume that β_n is an intermediate sequence i.e., $\beta_n \rightarrow 1$ and $n(1 - \beta_n) \rightarrow 0$. A direct approach would be to replace in $\bar{q}_{\beta_n}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))$ the quantile function Q_X by its empirical counterpart $\hat{Q}_{n,X}(\cdot) = X_{\lceil n \cdot \cdot \rceil:n}$ where $\lceil \cdot \rceil$ is the ceiling function and $X_{1:n} \leq \dots \leq X_{n:n}$ are the observations arranged in ascending order. This leads to the estimator

$$\bar{q}_{\beta_n}(\hat{Q}_{n,X} | \mathcal{P}_\mu(\kappa, \gamma_0)) = \inf \left\{ \sum_{i=\lceil n\beta_n \rceil}^n X_{i:n} \nu_{\beta_n} \left(\left[\frac{i-1}{n}, \frac{i}{n} \right] \right) \middle| \nu \in \mathcal{P}_\mu(\kappa, \gamma_0) \right\}.$$

Unfortunately, the explicit expression of this estimator is unknown. To overcome this drawback, we take advantage of the asymptotic equivalence obtained in Proposition 2 to propose the estimator

$$\hat{\bar{q}}_{n,\beta_n}(Q_X | \mu, \gamma_0) := X_{\lceil n\beta_n \rceil:n} \left(\int_{[0,1]} (1-u)^{-(\hat{\gamma}_{n,X} \wedge \gamma_0)} d\mu(u) \right)^{(\hat{\gamma}_{n,X} \vee \gamma_0)/\gamma_0}, \quad (8)$$

where $\hat{\gamma}_{n,X}$ is any estimator of γ_X that converges in probability to γ_X . Convergence in probability ($\xrightarrow{\mathbb{P}}$) of (8) is established in the next result.

Proposition 4 *Assume that μ and γ_0 satisfy condition (H.1). If $\beta_n \rightarrow 1$ and $n(1 - \beta_n) \rightarrow \infty$ then for all $\kappa \in [1, \infty)$ and for an estimator $\hat{\gamma}_{n,X}$ that converges in probability to γ_X ,*

$$\frac{\hat{\bar{q}}_{n,\beta_n}(Q_X | \mu, \gamma_0)}{\bar{q}_{\beta_n}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))} \xrightarrow{\mathbb{P}} 1.$$

Let us mention that the consistency holds for all $\gamma_0 \in (0, \gamma_\mu^*)$, whatever the value of the tail index γ_X . When $\gamma_0 = \infty$, estimator (8) is written

$$\widehat{\varrho}_{n,\beta_n}(Q_X | \mu, \infty) = X_{\lceil n\beta_n \rceil:n} \int_{[0,1]} (1-u)^{-\widehat{\gamma}_{n,X}} d\mu(u).$$

The ratio of this estimator with the tail distortion tail-risk measure $\varrho_{\mu\beta_n}^{(D)}$ converges in probability to 1 when $\gamma_X < \gamma_\mu^*$. In the particular case where $\mu = \nu$ is the uniform probability measure, this statistic has been studied as an estimator of the Tail-Value-at-Risk of level β_n in Gardes *et al.* [15, Section 4.2]. Its asymptotically normality has been established under the condition $\gamma_X < 1/2$.

A natural estimator of the tail index γ_X is the Hill estimator, see Hill [18], defined for a sequence $k_n \in \{1, \dots, n\}$ by

$$\widehat{\gamma}_{n,X}^{(H)}(k_n) := \frac{1}{k_n} \sum_{i=1}^{k_n} \log \frac{X_{n-i+1:n}}{X_{n:n}}.$$

The convergence in probability of the Hill estimator is established for instance in de Haan and Ferreira [17, Theorem 3.2.2] under the condition that $k_n \rightarrow \infty$ and $n/k_n \rightarrow \infty$.

If we denote by $\widehat{\varrho}_{n,\beta_n}^{(H)}(Q_X | \mu, \gamma_0)$ the estimator (8) obtained by using the Hill estimator $\widehat{\gamma}_{n,X}^{(H)}(k_n)$ with an intermediate sequence k_n , a direct consequence of Proposition 4 is that

$$\frac{\widehat{\varrho}_{n,\beta_n}^{(H)}(Q_X | \mu, \gamma_0)}{\varrho_{\beta_n}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))} \xrightarrow{\mathbb{P}} 1.$$

To determinate the rate of convergence in Proposition 4 or to establish the asymptotic normality of (8), we need in particular to establish the rate of convergence in Proposition 2 which is a difficult task. This question is not addressed in this paper.

Extreme case – The consistency in Proposition 4 has been obtained for the situation where the level β_n of the tail-risk measure is an intermediate sequence. For an extreme level β_n satisfying $n(1 - \beta_n) \rightarrow b \geq 0$, the quantile $Q(\beta_n)$ cannot be estimated by the order statistic $X_{\lceil n\beta_n \rceil:n}$ since $X_{\lceil n\beta_n \rceil:n}/Q_X(\beta_n)$ does not converge in probability to 1 in this case. As a consequence, estimator (8) is not more consistent. We thus need to extrapolate the estimate at an intermediate level α_n to

the extreme level β_n . The extrapolation procedure is based on the approximation

$$\frac{\bar{\varrho}_{\beta_n}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{\bar{\varrho}_{\alpha_n}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))} \approx \left(\frac{1 - \alpha_n}{1 - \beta_n} \right)^{\gamma_X}.$$

This approximation is a direct consequence of the regularly varying property of the quantile function Q_X . A similar idea was used by Weissman [27] for the estimation of extreme quantiles. Consequently, for an extreme level β_n , we propose the extrapolated estimator

$$\widehat{\varrho}_{n, \beta_n}^{(W)}(Q_X | \mu, \gamma_0, \alpha_n) := \widehat{\varrho}_{n, \alpha_n}(Q_X | \mu, \gamma_0) \left(\frac{1 - \alpha_n}{1 - \beta_n} \right)^{\widehat{\gamma}_{n, X}}, \quad (9)$$

where $\widehat{\gamma}_{n, X}$ is a consistent estimator of γ_X and α_n is an intermediate sequence. Before establishing the consistency of (9), let us mention that the regularly varying property (2) means that $Q_X(u) = (1 - u)^{-\gamma_X} \ell_X((1 - u)^{-1})$ where ℓ_X is a slowly varying function at infinity i.e., for all $t > 0$,

$$\lim_{x \rightarrow \infty} \frac{\ell_X(tx)}{\ell_X(x)} = 1. \quad (10)$$

It is well known, see for instance Bingham *et al.* [4, Theorem 1.3.1], that a slowly varying function can be represented for all $x > 1$ as,

$$\ell_X(x) = c_X(x) \exp\left(\int_1^x \frac{\Delta_X(t)}{t} dt\right), \quad (11)$$

where $c_X(x) \rightarrow c \geq 0$ as $x \rightarrow \infty$ and $\Delta_X(t) \rightarrow 0$ as $t \rightarrow \infty$. The function Δ_X controls the rate of convergence in (10). The consistency of (9) is established in the next result.

Proposition 5 *Let $Q_X \in \mathcal{Q}^{(HT)}$. Assume that μ and γ_0 satisfy condition (H.1) and that $|\Delta_X|$ is asymptotically decreasing. Let α_n and β_n be sequences converging to 1 and such that $n(1 - \alpha_n) \rightarrow \infty$, $n(1 - \beta_n) \rightarrow b \geq 0$ and*

$$\lim_{n \rightarrow \infty} \ln \left(\frac{1 - \alpha_n}{1 - \beta_n} \right) \left| \Delta_X \left(\frac{1}{1 - \alpha_n} \right) \right| = 0.$$

If the estimator $\widehat{\gamma}_{n, X}$ satisfies the condition

$$\ln \left(\frac{1 - \alpha_n}{1 - \beta_n} \right) (\widehat{\gamma}_{n, X} - \gamma_X) \xrightarrow{\mathbb{P}} 0, \quad (12)$$

then for all $\kappa \in [1, \infty)$,

$$\frac{\widehat{\varrho}_{n,\beta_n}^{(W)}(Q_X | \mu, \gamma_0, \alpha_n)}{\overline{\varrho}_{\beta_n}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))} \xrightarrow{\mathbb{P}} 1.$$

This result holds for all $\gamma_0 \in (0, \gamma_\mu^*)$ and $\gamma_X > 0$. If we take $\gamma_0 = \infty$ in (9), we obtain the estimator,

$$\widehat{\varrho}_{n,\beta_n}^{(W)}(Q_X | \mu, \infty, \alpha_n) = \left(\frac{1 - \alpha_n}{1 - \beta_n} \right)^{\widehat{\gamma}_{n,X}} X_{[n\alpha_n]:n} \int_{[0,1]} (1-u)^{-\widehat{\gamma}_{n,X}} d\mu(u),$$

which is a consistent estimator of the tail distortion risk measure $\varrho_{\mu\beta_n}^{(D)}(Q_X)$ as soon as $\gamma_X < \gamma_\mu^*$. For $\mu = v$, this estimator is called the indirect extrapolated estimator of the Tail-Value-at-Risk in Gardes *et al.* [15]. It is equivalent in probability to the Tail-Value-at-Risk when $\gamma_X < 1$ and asymptotically Gaussian when $\gamma_X < 1/2$.

Let us now take a look at the particular case where the Hill estimator is used for the estimation of the tail index. We denote by $\widehat{\varrho}_{n,\beta_n}^{(W,H)}(Q_X | \mu, \gamma_0, \alpha_n)$ the extrapolated estimator (9) computed with the Hill estimator $\widehat{\gamma}_{n,X}^{(H)}(k_n)$ with $k_n = [n(1 - \alpha_n)]$. The following result is a consequence of Proposition 5. It is obtained under a restriction on the representation (11) of the slowly varying function.

- (H.2) In representation (11), the sign of the function Δ_X is asymptotically constant and $|\Delta_X|$ is asymptotically decreasing and regularly varying with index $\rho \leq 0$. Moreover, the function c_X is constant.

Note that when (11) holds with a constant function c_X , the slowly varying function is said to be normalised. As mentioned by Bingham *et al.* [4, Page 15], “we lose nothing by restricting attention to the case of constant c -function in (11)”. The parameter ρ , called the second-order parameter, tunes the rate of convergence in (2). We are now in position to state our last result.

Corollary 1 *Let $Q_X \in \mathcal{Q}^{(HT)}$. Assume that μ and γ_0 satisfy conditions (H.1) and that Δ_X satisfies condition (H.2). If α_n and β_n are sequences converging to 1 and such that $n(1 - \alpha_n) \rightarrow \infty$, $n(1 - \beta_n) \rightarrow b \geq 0$,*

$$[n(1 - \alpha_n)]^{-1/2} \ln \left(\frac{1 - \alpha_n}{1 - \beta_n} \right) \rightarrow 0 \text{ and } [n(1 - \alpha_n)]^{1/2} \Delta_X \left(\frac{1}{1 - \alpha_n} \right) \rightarrow \lambda \in \mathbb{R},$$

as $n \rightarrow \infty$, then for all $\kappa \in [1, \infty)$,

$$\frac{\widehat{\bar{Q}}_{n, \beta_n}^{(W, H)}(Q_X \mid \mu, \gamma_0, \alpha_n)}{\bar{Q}_{\beta_n}(Q_X \mid \mathcal{P}_\mu(\kappa, \gamma_0))} \xrightarrow{\mathbb{P}} 1.$$

The proof of Proposition 1 mainly consists to check that under condition (H.2), the classical second-order condition holds. This condition, given for instance in de Haan and Ferreira [17, Definition 2.3.1] can be written

$$\lim_{u \rightarrow 1} \frac{1}{\Delta_X((1-u)^{-1})} \left(\frac{Q_X(1-t(1-u))}{Q_X(u)} - t^{-\gamma_X} \right) = t^{-\gamma_X} \int_1^{t^{-1}} v^{\rho-1} dv. \quad (13)$$

This condition specifies the rate of converge in (2). Under (13), the Hill estimator is asymptotically gaussian with rate of convergence $k_n^{-1/2}$, see de Haan and Ferreira [17, Theorem 3.2.5].

The rate of convergence of β_n to 1 is slightly restricted by the condition $[n(1-\alpha_n)]^{-1/2} \ln((1-\alpha_n)/(1-\beta_n)) \rightarrow 0$. Note that when $n(1-\beta_n) \rightarrow c > 0$, this condition is always satisfied.

Real data study – We close this section, by a short illustration of the estimation procedure on a fire insurance dataset studied by Beirlant *et al.* [3]. This dataset provides the sizes of 9181 fire insurance claims for a Norwegian insurance company for the period 1972 to 1992. These claim sizes are corrected for inflation using the Norwegian CPI and are expressed in thousands Norwegian Kroner (NKR). The dataset is available for instance in the R package `CASdatasets` that can be downloaded at the address <http://dutangc.perso.math.cnrs.fr/RRepository/pub/>. For the period 1985 to 1992, the annual numbers of claim sizes are similar. We thus focus on this period in the application. For each year $j \in \{1985, \dots, 1992\}$, we denote by $x_1^{(j)}, \dots, x_{n_j}^{(j)}$ the observed n_j fire losses. As in Gardes and Girard [14], it is assumed that these observations are realizations of a sample $X_1^{(j)}, \dots, X_{n_j}^{(j)}$ of n_j independent copies of an heavy-tailed random variable $X^{(j)}$.

Our goal is to compare the annual risks of the fire losses over the considered period. More specifically, for each year j , we estimate the tail version of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure $\bar{Q}_\beta(Q_{X^{(j)}} \mid \mathcal{P}_v(\kappa, \gamma_0))$ where v is the uniform probability measure. Recall that when $\gamma_X \leq \gamma_0$, this tail-risk measure coincides with the Tail-Value-at-Risk. The level β is taken as $1 - 1/\tilde{n}$ where $\tilde{n} = 637.5$ thousands NKR is the median of $\{n_{1985}, \dots, n_{1992}\}$. When the sample size is lower than \tilde{n} , the level β can be consid-

ered as extreme. This is the case for the years 1985, 1990, 1991 and 1992. For each year j , we thus chose to use the extrapolated estimator $\widehat{\varrho}_{n,\beta}^{(W,H)}(Q_{X^{(j)}} \mid v, \gamma_0, \alpha_n)$. Its expression is given by

$$\left(\frac{1-\alpha_n}{1-\beta_n}\right)^{\widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n)} X_{[n\alpha_n]:n}^{(j)} \left(\frac{1}{1-\widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n)\text{wedge}\gamma_0}\right)^{(\widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n)\vee\gamma_0)/\gamma_0}. \quad (14)$$

The Hill estimator is computed with $k_n = \lceil n(1-\alpha_n) \rceil$ order statistics where $\alpha_n = 1 - (\ln(n))^2/(2n)$. The parameter γ_0 is taken as in (6) i.e., $\gamma_0 = 1/2$. In Figure 1, we represent for each year j the value of the Hill estimator $\widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n)$ together with its asymptotic confidence interval of level 0.95 given by

$$\left[\widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n) \left(1 - k_n^{-1/2} u_{0.975}\right) ; \widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n) \left(1 - k_n^{-1/2} u_{0.025}\right)\right], \quad (15)$$

where u_α is the quantile of order α of a standard normal distribution. This asymptotic confidence interval is constructed on the base of the asymptotic normality of the Hill estimator, see de Haan and Ferreira [17, Theorem 3.2.5]. It appears that for the years 1985, 1986 and 1988, the critical value $\gamma_v^* = 1$ is contained in the asymptotic confidence interval (15). Consequently, we can reasonably have some doubt about the existence of the Tail-Value-at-Risk for these years. In the left panel of Figure 2, the values of the extrapolated tail risk measure estimator (14) are depicted for each year j . The interval of variation of the estimator when the tail index estimator varies in the confidence interval (15) is also represented. The lower (resp. upper) bound of this interval of variation is thus the estimator (14) where the Hill estimator is replaced by the lower (resp. upper) bound of the confidence interval (15). Same thing is done in the right panel with the indirect extrapolated estimator of the Tail-Value-at-Risk

$$\left(\frac{1-\alpha_n}{1-\beta_n}\right)^{\widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n)} \frac{X_{[n\alpha_n]:n}^{(j)}}{1-\widehat{\gamma}_{n,X^{(j)}}^{(H)}(k_n)}. \quad (16)$$

Recall that this estimator is valid when the tail index is smaller than 1. As a consequence, for the years 1985, 1986 and 1988, the upper bound of the interval of variation is infinite since for these years, the right bound of the confidence interval (15) is larger than 1.

As expected, the estimator (14) is less sensitive to the variations of the tail index

estimator. For the period 1989 to 1992, the estimators (14) and (16) are similar since for these years, the tail index is not too large. At the opposite, for the years 1985, 1986 and 1988, the indirect extrapolated estimator (16) is very large which is not surprising since the estimation of the tail index is close to 1.

Conclusion

The main contribution of this work is the proposition of a new tail risk measure of level β , denoted $\bar{\varrho}_\beta(\cdot | \mathcal{P}_\mu(\kappa, \gamma_0))$, which is dedicated to heavy-tailed distributions. The main advantage of this new proposal is to be defined whatever the value of the tail index. When the tail index is sufficiently small, our new measure is equivalent to a classical distortion risk measure. In a finite sample point of view, the proposed estimator has a better behavior when the tail index is large than an estimator of the distortion risk measure (when it exists). Our new tail risk measure seems to be a good alternative to the well known distortion risk measure for very heavy-tailed distributions.

4 Proofs

4.1 Preliminary results

The aim of our first lemma is to show that a law invariant risk measure can be also be defined as a mapping from a set of quantile measures.

Lemma 1 *Let \mathcal{X} be a set of random variables. A risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is law invariant if and only if there exists a function $\varrho : \mathcal{Q} \rightarrow \mathbb{R}$ with $\mathcal{Q} := \{Q_X | X \in \mathcal{X}\}$ such that $\rho(X) = \varrho(Q_X)$ for all $X \in \mathcal{X}$.*

Proof – Since the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, one can find a random variable $U : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow ([0, 1], \mathcal{B}([0, 1]))$ uniformly distributed. For all random variables X defined on $(\Omega, \mathcal{F}, \mathbb{P})$, it is easy to check that $X \stackrel{d}{=} Q_X(U)$. Since ρ is law invariant, $\rho(X) = \rho(Q_X(U))$ for all $X \in \mathcal{X}$ and thus the functional ϱ is defined for all $Q_X \in \mathcal{Q}$ by $\varrho(Q_X) = \rho(Q_X(U))$. The converse is straightforward. ■

In the next result, an asymptotic equivalent, as $\beta \rightarrow 1$, of the tail distortion risk measure of level β is given.

Lemma 2 For all probability measure μ such that $\gamma_\mu^* > 0$ and all quantile functions $Q_X \in \mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_X \in (0, \gamma_\mu^*)$

$$\lim_{\beta \rightarrow 1} \int_{[0,1]} \frac{Q_X(u)}{Q_X(\beta)} d\mu_\beta(u) = \int_{[0,1]} (1-u)^{-\gamma_X} d\mu(u).$$

Proof – Let us introduce the function U_X defined for all $x \geq 1$ by $U_X(x) = Q_X(1 - 1/x)$. The function U_X is regularly varying with index γ_X . We have

$$\begin{aligned} \int_{[0,1]} \frac{Q_X(u)}{Q_X(\beta)} d\mu_\beta(u) &= \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\mu(u) \\ &= \int_{[0,1]} \frac{U_X((1-\beta)^{-1}(1-u)^{-1})}{U_X((1-\beta)^{-1})} d\mu(u) \\ &= \int_{[0,1]} (1-u)^{-\gamma_X} d\mu(u) + \int_{[0,1]} |R_X(u, \beta)| d\mu(u), \end{aligned}$$

where

$$R_X(u, \beta) := \frac{U_X((1-\beta)^{-1}(1-u)^{-1})}{U_X((1-\beta)^{-1})} - (1-u)^{-\gamma_X}. \quad (17)$$

Using the result given in de Haan and Ferreira [17, Proposition B.1.10], we know that for all $\varepsilon > 0$ and $\delta > 0$, there exists $\tau_0 = \tau_0(\varepsilon, \delta) \in (0, 1)$ such that for all $Q_X \in \mathcal{Q}^{(\text{HT})}$ and $\beta > 1 - \tau_0$,

$$|R_X(u, \beta)| \leq \varepsilon (1-u)^{-(\gamma_X + \delta)}. \quad (18)$$

Note that τ_0 does not depend on $u \in [0, 1]$. Hence, taking $\delta > 0$ such that $\gamma_X + \delta < \gamma_\mu^*$, one has $\int_{[0,1]} (1-u)^{-(\gamma_X + \delta)} d\mu(u) < \infty$ and for all $\varepsilon > 0$,

$$\int_{[0,1]} |R_X(u, \beta)| d\mu(u) \leq \varepsilon \int_{[0,1]} (1-u)^{-(\gamma_X + \delta)} d\mu(u),$$

leading to the desired result. ■

Next, we show that $Q_X \in \mathcal{Q}^{(\text{HT})}$ is integrable w.r.t. μ if its tail index γ_X is lower than the constant γ_μ^* defined in (3).

Lemma 3 For all probability measure μ such that $\gamma_\mu^* > 0$ and all quantile functions $Q_X \in \mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_X > 0$ one has

$$\begin{cases} \int_{[0,1]} |Q_X| d\mu < \infty & \text{if } \gamma_X \in (0, \gamma_\mu^*), \\ \int_{[0,1]} |Q_X| d\mu = \infty & \text{if } \gamma_X > \gamma_\mu^*. \end{cases}$$

Proof – We start with the decomposition

$$\int_{[0,1]} |Q_X| d\mu = \int_{[0,\beta)} |Q_X| d\mu + \int_{[\beta,1]} |Q_X| d\mu.$$

Since $Q_X(0) \geq c_X \in \mathbb{R}$, it is clear that

$$\int_{[0,\beta)} |Q_X| d\mu < \infty.$$

Moreover, since $Q_X(u) \rightarrow \infty$ as $u \rightarrow 1$, for β large enough, $Q_X(u) > 0$ for all $u \geq \beta$. It thus suffices to work with the integral

$$\int_{[\beta,1]} Q_X d\mu \in (0, \infty].$$

First assume that $\gamma_X \in (0, \gamma_\mu^*)$. We must show that

$$\int_{[\beta,1]} Q_X d\mu < \infty.$$

For all $x \geq 1$, let $U_X(x) = Q_X(1 - 1/x)$. We have

$$\begin{aligned} \int_{[\beta,1]} Q_X d\mu &= Q_X(\beta) \int_{[\beta,1]} \frac{U_X((1-u)^{-1})}{U_X((1-\beta)^{-1})} d\mu(u) \\ &= Q_X(\beta) \int_{[0,1]} \frac{U_X((1-\beta)^{-1}(1-u)^{-1})}{U_X((1-\beta)^{-1})} d\mu_\beta^*(u), \end{aligned}$$

where μ_β^* is the probability measure given for all $A \in \mathcal{B}([0,1])$ by $\mu_\beta^*(A) = \mu((1-\beta)A + \beta)$. Hence,

$$\int_{[\beta,1]} Q_X d\mu \leq Q_X(\beta) \left[\int_{[0,1]} (1-u)^{-\gamma_X} d\mu_\beta^*(u) + \int_{[0,1]} |R_X(u, \beta)| d\mu_\beta^*(u) \right],$$

where $R_X(u, \beta)$ is defined in (17). Using the result given in de Haan and Ferreira [17, Proposition B.1.10], we know that for all $\varepsilon > 0$ and $\delta > 0$, there exists $\tau_0 = \tau_0(\varepsilon, \delta) \in (0, 1)$ such that for all $Q_X \in \mathcal{Q}^{(\text{HT})}$ and $\beta > 1 - \tau_0$, inequality (18) holds. Taking $\delta = (\gamma_\mu^* - \gamma)/2 > 0$, we then have for $\beta > 1 - \tau_0$,

$$\int_{[\beta,1]} Q_X d\mu < Q_X(\beta) \left[\int_{[0,1]} (1-u)^{-\gamma_X} d\mu_\beta^*(u) + \varepsilon \int_{[0,1]} (1-u)^{-(\gamma_X + \delta)} d\mu_\beta^*(u) \right].$$

We conclude the proof by remarking that

$$\int_{[0,1]} (1-u)^{-(\gamma_X+\delta)} d\mu_\beta^*(u) = (1-\beta)^{-(\gamma_X+\delta)} \int_{[\beta,1]} (1-u)^{-(\gamma_X+\delta)} d\mu < \infty$$

since $\gamma_X + \delta < \gamma_\mu^*$.

Now assume that $\gamma_X > \gamma_\mu^*$. It suffices to prove that there exists $\beta \in (0, 1)$ such that

$$\int_{[\beta,1]} Q_X d\mu = Q_X(\beta) \int_{[\beta,1]} \frac{U_X((1-u)^{-1})}{U_X((1-\beta)^{-1})} d\mu(u) = \infty.$$

One can take β such that $Q_X(\beta) > 0$. Using the notation introduced in the first part of the proof,

$$\int_{[\beta,1]} \frac{U_X((1-u)^{-1})}{U_X((1-\beta)^{-1})} d\mu(u) = \int_{[0,1]} \frac{U_X((1-\beta)^{-1}(1-u)^{-1})}{U_X((1-\beta)^{-1})} d\mu_\beta^*(u).$$

Using Potter's bounds (see for instance de Haan and Ferreira [17, eq. (B.1.19)]), for $\delta < \gamma_X - \gamma_\mu^*$, there exists β_0 such that for all $\beta > \beta_0$ and $u \in (0, 1)$,

$$\frac{1}{2}(1-u)^{-(\gamma_X-\delta)} \leq \frac{U_X((1-\beta)^{-1}(1-u)^{-1})}{U_X((1-\beta)^{-1})}.$$

Hence,

$$\begin{aligned} \int_{[\beta,1]} \frac{U_X((1-u)^{-1})}{U_X((1-\beta)^{-1})} d\mu(u) &\geq \frac{1}{2} \int_{[0,1]} (1-u)^{-(\gamma_X-\delta)} d\mu_\beta^*(u) \\ &= (1-\beta)^{-(\gamma_X-\delta)} \int_{[0,1]} (1-u)^{-(\gamma_X-\delta)} d\mu(u), \end{aligned}$$

which is infinite since $\gamma_X - \delta > \gamma_\mu^*$. ■

Below, we give the explicit expression of the $\mathcal{P}_\mu(\kappa, \gamma_0)$ -inf risk measure for strict Pareto distributions.

Lemma 4 *Let μ be a reference probability measure such that $\gamma_\mu^* > 0$. For all $\kappa \in [1, \infty]$ and $\gamma_0 \in (0, \gamma_\mu^*)$ such that $\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) > 0$, one has for all strict Pareto distributions with tail index $\gamma_X > 0$,*

$$\bar{q}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) = \int_{[0,1]} (1-u)^{-\gamma_X} d\nu_X^*(u),$$

with $\nu_X^* = \mu$ if $\gamma_X \leq \gamma_0$ and $\nu_X^* = \delta_{b(\mu, \gamma_0)}$ if $\gamma_X > \gamma_0$ where

$$b(\mu, \gamma_0) := 1 - \left(\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) \right)^{-1/\gamma_0}. \quad (19)$$

Proof of Lemma 4 – First, let us show that for all $\kappa \in [1, \infty]$, the Dirac measure $\delta_{b(\mu, \gamma_0)}$ belongs to the set $\mathcal{P}_\mu(\kappa, \gamma_0)$ ensuring that $\nu_X^* \in \mathcal{P}_\mu(\kappa, \gamma_0)$ for all $\gamma_X > 0$. Since by assumption, $\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) \in (0, \infty)$, the constant $b(\mu, \gamma_0)$ belongs to $(0, 1)$. When $\gamma_X \leq \gamma_0$, an application of the Jensen's inequality leads to

$$1 - \left(\int_{[0,1]} (1-u)^{-\gamma_X} d\mu(u) \right)^{-1/\gamma_X} \leq b(\mu, \gamma_0).$$

As a consequence, for all $\gamma_X \in (0, \gamma_0]$,

$$\int_{[0,1]} (1-u)^{-\gamma_X} d\mu(u) \leq (1 - b(\mu, \gamma_0))^{-\gamma_X} = \int_{[0,1]} (1-u)^{-\gamma_X} d\delta_{b(\mu, \gamma_0)}(u),$$

and condition (4) holds. Moreover, for all $\gamma_X > 0$,

$$\int_{[0,1]} (1-u)^{-\gamma_X} d\delta_{b(\mu, \gamma_0)}(u) = \left(\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) \right)^{\gamma_X/\gamma_0}.$$

Condition (5) is thus satisfied with $\kappa = 1$ and thus for all $\kappa \in [1, \infty]$ ensuring that $\delta_{b(\mu, \gamma_0)} \in \mathcal{P}_\mu(\kappa, \gamma_0)$.

Let us now show that the infimum is reached with the measure ν_X^* . When $\gamma_X \leq \gamma_0$, it is easy to check that the infimum is reached with the probability measure μ . It remains to prove that for all $\gamma_X > \gamma_0$,

$$\int_{[0,1]} (1-u)^{-\gamma_X} d\delta_{b(\mu, \gamma_0)}(u) \leq \int_{[0,1]} (1-u)^{-\gamma_X} d\nu(u), \quad (20)$$

for all $\nu \in \mathcal{P}_\mu(\kappa, \gamma_0)$. Let us use an indirect proof to prove (20). Assume that there exists a probability measure $\nu \in \mathcal{P}_\mu(\kappa, \gamma_0)$ such that

$$\left(\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) \right)^{\gamma_X/\gamma_0} > \int_{[0,1]} (1-u)^{-\gamma_X} d\nu(u) \quad (21)$$

Then

$$\frac{\int_{[0,1]} (1-u)^{-\gamma_0} d\nu(u)}{\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u)} < \frac{1}{\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u)} \left(\int_{[0,1]} (1-u)^{-\gamma_X} d\nu(u) \right)^{\gamma_0/\gamma_X},$$

by Jensen's inequality. Then, if (21) holds,

$$\int_{[0,1]} (1-u)^{-\gamma_0} d\nu(u) < \int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u),$$

which is impossible for a measure $\nu \in \mathcal{P}_\mu(\kappa, \gamma_0)$. Hence

$$\begin{aligned} \int_{[0,1]} (1-u)^{-\gamma_X} d\delta_{b(\mu, \gamma_0)}(u) &= \left(\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) \right)^{\gamma_X/\gamma_0} \\ &\leq \int_{[0,1]} (1-u)^{-\gamma_X} d\nu(u), \end{aligned}$$

and the proof is complete. ■

4.2 Proofs of main results

Proof of Proposition 1 – For all strict Pareto quantile function $Q_X \in \mathcal{Q}_{0,\mu}$,

$$\bar{\varrho}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) = \lambda_X \inf \left\{ \int_{[0,1]} (1-u)^{-\gamma_X} d\nu(u) \mid \nu \in \mathcal{P}_\mu(\kappa, \gamma_0) \right\} + m_X.$$

Since $\gamma_X \in (0, \gamma_0]$, from (4) we know that for all $\nu \in \mathcal{P}_\mu(\kappa, \gamma_0)$,

$$\int_{[0,1]} (1-u)^{-\gamma_X} d\nu(u) \geq \int_{[0,1]} (1-u)^{-\gamma_X} d\mu(u).$$

Hence, since $\mu \in \mathcal{P}_\mu(\kappa, \gamma_0)$, one has for all $Q \in \mathcal{Q}_{0,\mu}$ that

$$\bar{\varrho}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) = \lambda_X \int_{[0,1]} (1-u)^{-\gamma_X} d\mu(u) + m_X = \int_{[0,1]} Q_X d\mu.$$

It remains to show that $\bar{\varrho}(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))$ is finite for all $Q_X \in \mathcal{Q}^{(HT)}$. To prove this, it suffices to find a probability measure $\nu^* \in \mathcal{P}_\mu(\kappa, \gamma_0)$ such that

$$\int_{[0,1]} Q_X d\nu^* < \infty,$$

for all $Q_X \in \mathcal{Q}^{(\text{HT})}$. According to Lemma 4, one can take $\nu^* = \delta_{b(\mu, \gamma_0)}$ and the proof is complete. \blacksquare

Proof of Proposition 2 – Let Q_X be a quantile function in $\mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_X > 0$. For all $\varepsilon > 0$ and $\beta \in (0, 1)$, there exists a measure $\nu_{X, \varepsilon}^{(\beta)} \in \mathcal{P}_\mu(\kappa, \gamma_0)$ such that

$$0 \leq \int_{[0,1]} Q_{X, \beta} d\nu_{X, \varepsilon}^{(\beta)} - \bar{q}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) \leq \varepsilon. \quad (22)$$

Let us now introduce the set

$$B_\varepsilon = \left\{ \beta \in (0, 1) \left| \int_{[0,1]} Q_\beta d\nu_{X, \varepsilon}^{(\beta)} < \int_{[0,1]} Q_\beta d\nu_X^* \right. \right\},$$

where the probability measure ν_X^* was introduced in Lemma 4.

If $\beta_1 := \sup B_\varepsilon < 1$, then for all $\beta > \beta_1$ or for all $\beta \notin B_\varepsilon$,

$$\int_{[0,1]} Q_{X, \beta} d\nu_{X, \varepsilon}^{(\beta)} \geq \int_{[0,1]} Q_{X, \beta} d\nu_X^*,$$

and thus, for all $\beta > \beta_1$,

$$0 \leq \int_{[0,1]} Q_{X, \beta} d\nu_X^* - \bar{q}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) \leq \varepsilon. \quad (23)$$

Now, if $\sup B_\varepsilon = 1$, using the result given in de Haan and Ferreira [17, Proposition B.1.10], for all $\delta > 0$ (with $0 < \delta < \gamma_\mu^* - \gamma_Q$ if $\gamma_Q < \gamma_\mu^*$), there exists $\beta_2 \in (0, 1)$ such that for all $\beta \in B_\varepsilon$ with $\beta > \beta_2$,

$$\begin{aligned} 0 &\leq \int_{[0,1]} \frac{Q_{X, \beta}(u)}{Q_X(\beta)} d\nu_X^*(u) - \int_{[0,1]} \frac{Q_{X, \beta}(u)}{Q_X(\beta)} d\nu_{X, \varepsilon}^{(\beta)}(u) \\ &\leq \int_{[0,1]} (1-u)^{-\gamma_X} d\nu_X^*(u) - \int_{[0,1]} (1-u)^{-\gamma_X} d\nu_{X, \varepsilon}^{(\beta)}(u) \\ &\quad + \varepsilon \left[\int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\nu_X^*(u) + \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\nu_{X, \varepsilon}^{(\beta)}(u) \right]. \end{aligned}$$

From Lemma 4,

$$\int_{[0,1]} (1-u)^{-\gamma_X} d\nu_X^*(u) - \int_{[0,1]} (1-u)^{-\gamma_X} d\nu_{X, \varepsilon}^{(\beta)}(u) \leq 0,$$

and thus

$$\begin{aligned} 0 &\leq \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\nu_X^*(u) - \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\nu_{X,\varepsilon}^{(\beta)}(u) \\ &\leq \varepsilon \left[\int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\nu_X^*(u) + \int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\nu_{X,\varepsilon}^{(\beta)}(u) \right]. \end{aligned} \quad (24)$$

Let us now show that there exists $\beta_3 \in (0, 1)$ such that

$$\int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\nu_X^*(u) + \sup_{\substack{\beta \geq \beta_3 \\ \beta \in B_\varepsilon}} \int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\nu_{X,\varepsilon}^{(\beta)}(u) \leq c(\kappa, \gamma_X), \quad (25)$$

where $c(\kappa, \gamma_X)$ is a positive constant independent of ε .

↪ We start with the first term.

- If $\gamma_X \leq \gamma_0$, one can pick δ such that $\gamma_X + \delta < \gamma_\mu^*$ and thus

$$\int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\nu_X^*(u) = \int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\mu(u) < \infty. \quad (26)$$

- If $\gamma_X > \gamma_0$, $\nu_X^*(u) = \delta_{b(\mu, \gamma_0)}$ where $b(\mu, \gamma_0)$ was defined in Lemma 4. Hence,

$$\int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\nu_X^*(u) = \left(\int_{[0,1]} (1-u)^{-\gamma_0} d\mu(u) \right)^{(\gamma_X+\delta)/\gamma_0} < \infty. \quad (27)$$

↪ Now, for the second term,

- if $\gamma_X > \gamma_\mu^*$, we know from Lemma 3 that

$$\int_{[0,1]} Q_{X,\beta} d\mu = \infty.$$

Hence, (22) cannot be true if $\nu_{X,\varepsilon}^{(\beta)} = \mu$ and thus $\nu_{X,\varepsilon}^{(\beta)} \in \tilde{\mathcal{P}}_\mu(\kappa, \gamma_0)$ where $\tilde{\mathcal{P}}_\mu(\kappa, \gamma_0)$ is the set of probability measures satisfying (4) and (5). As a consequence, if $\gamma_X > \gamma_\mu^*$, one has

$$\int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\nu_{X,\varepsilon}^{(\beta)}(u) \leq \left(\kappa \int_{[0,1]} (1-u)^{-\gamma_X-\delta} d\mu(u) \right)^{(\gamma_X+\delta)/\gamma_0}, \quad (28)$$

for all $\beta \in B_\varepsilon$.

- If $\gamma_X \leq \gamma_\mu^*$, we first need to prove the following assertion.

(A) There exist $\tilde{\gamma} \in (0, \gamma_\mu^*)$ and $\beta_3 \in (0, 1)$ such that, if $\gamma_X > \tilde{\gamma}$, for all $\beta \in B_\varepsilon$

with $\beta > \beta_3$, the probability measure $\nu_{X,\varepsilon}^{(\beta)}$ is not equal to μ .

As a direct consequence of this assertion, if $\gamma_X \leq \gamma_\mu^*$,

$$\sup_{\substack{\beta \geq \beta_3 \\ \beta \in B_\varepsilon}} \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\nu_{X,\varepsilon}^{(\beta)}(u) < \infty. \quad (29)$$

Indeed, if $0 < \gamma_X \leq \tilde{\gamma}$, since $\gamma_X + \delta < \gamma_\mu^*$,

$$\begin{aligned} & \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\nu_{X,\varepsilon}^{(\beta)}(u) \\ & \leq \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\mu(u) \wedge \left(\kappa \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\mu(u) \right)^{(\gamma_X + \delta)/\gamma_0} < \infty, \end{aligned}$$

and if $\gamma_X > \tilde{\gamma}$, assertion (A) entails that for all $\beta \in B_\varepsilon$ with $\beta \geq \beta_3$,

$$\int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\nu_{X,\varepsilon}^{(\beta)}(u) \leq \left(\kappa \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\mu(u) \right)^{(\gamma_X + \delta)/\gamma_0} < \infty.$$

Let us prove assertion (A). Since the function

$$\gamma \in (0, \gamma_\mu^*) \mapsto \int_{[0,1]} (1-u)^{-\gamma} d\mu(u)$$

is increasing, continuous and converges to ∞ as $\gamma \rightarrow \gamma_\mu^*$, for all $\chi > 0$, there exists $\tilde{\gamma}(\chi) \in (0, \gamma_\mu^*)$ such that for all $\gamma > \tilde{\gamma}(\chi)$,

$$\int_{[0,1]} (1-u)^{-\gamma} d\mu(u) \geq \chi.$$

Now, for some $\delta_1 > 0$, let $\chi(\varepsilon) = \varepsilon + \tilde{\chi}$ where

$$\tilde{\chi} = 3 \max_{\gamma \in (0, \gamma_\mu^* + \delta_1]} \left(\kappa \int_{[0,1]} (1-u)^{-\gamma} d\mu(u) \right)^{\gamma/\gamma_0}.$$

Using Potter's bounds (see for instance de Haan and Ferreira [17, eq. (B.1.19)]), there exists $\beta_4 \in (0, 1)$ such that for all $\beta \in B_\varepsilon$, $\beta > \beta_4$,

$$\int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\mu(u) \geq \frac{1}{2} \int_{[0,1]} (1-u)^{-\bar{\gamma}_X(\chi(\varepsilon), \mu)} d\mu(u),$$

where $\bar{\gamma}_X(\chi(\varepsilon), \mu) = \gamma_X - [\gamma_\mu^* - \tilde{\gamma}(\chi(\varepsilon))]/2$. Hence, if $\gamma_X \geq [\tilde{\gamma}(\chi(\varepsilon)) + \gamma_\mu^*]/2$ then

$\bar{\gamma}_X(\chi(\varepsilon), \mu) > \tilde{\gamma}(\chi(\varepsilon))$ and thus,

$$\int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\mu(u) \geq \frac{\chi(\varepsilon)}{2}.$$

Moreover, for $\delta_1 > 0$, there exists $\beta_5 \in (0, 1)$ such that for all $\beta \in B_\varepsilon$ with $\beta > \beta_5$ and for all probability measure $\nu \in \tilde{\mathcal{P}}_\mu(\kappa, \gamma_0)$,

$$\begin{aligned} \frac{\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{Q_X(\beta)} &\leq \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\nu(u) \\ &\leq \frac{3}{2} \int_{[0,1]} (1-u)^{-\gamma_X - \delta_1} d\nu(u) \leq \frac{\tilde{\chi}}{2}. \end{aligned}$$

Since $Q_X(\beta) \rightarrow \infty$ as $\beta \rightarrow 1$, there exists $\beta_6 \in (0, 1)$ such that $Q_X(\beta) > 2$. Hence, for all $\beta \in B_\varepsilon$ with $\beta > \beta_3 := \beta_4 \vee \beta_5 \vee \beta_6$, if $\gamma_X > [\tilde{\gamma}(\chi(\varepsilon)) + \gamma_\mu^*]/2 =: \tilde{\gamma}$,

$$\int_{[0,1]} Q_{X,\beta} d\mu - \bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0)) \geq Q_X(\beta) \left[\frac{\chi(\varepsilon)}{2} - \frac{\tilde{\chi}}{2} \right] = \frac{Q_X(\beta)\varepsilon}{2} \geq \varepsilon,$$

and thus necessarily, $\nu_{X,\varepsilon}^{(\beta)} \neq \mu$. Assertion (A) is then proved.

Gathering (26) to (29) entails that (25) holds with

$$c(\kappa, \gamma_X) = \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\mu(u) \wedge \left(\kappa \int_{[0,1]} (1-u)^{-\gamma_X - \delta} d\mu(u) \right)^{(\gamma_X + \delta)/\gamma_0},$$

if $\gamma_X \leq \tilde{\gamma}$ and if $\gamma_X > \tilde{\gamma}$,

$$c(\kappa, \gamma_X) = \left(\int_{[0,1]} (1-u)^{-[(\gamma_X + \delta) \wedge \gamma_0]} d\mu(u) \right)^{[(\gamma_X + \delta) \vee \gamma_0]/\gamma_0}.$$

Note that as expected, $c(\kappa, \gamma_X)$ does not depend on ε . Moreover when κ and/or γ_X increase, so does $c(\kappa, \gamma_X)$.

For all $\beta \in B_\varepsilon$ with $\beta > \beta_3$, inequalities (22) and (24) and the fact that $Q_X(\beta) \geq 2$ lead to

$$\begin{aligned} 0 &\leq \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\mu_X^*(u) - \frac{\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{Q_X(\beta)} \\ &\leq \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\mu_X^*(u) - \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\nu_{X,\varepsilon}^{(\beta)}(u) + \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\nu_{X,\varepsilon}^{(\beta)}(u) \\ &\quad - \frac{\bar{\varrho}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{Q_X(\beta)} \leq \varepsilon \left(c(\kappa, \gamma_X) + \frac{1}{2} \right). \end{aligned} \tag{30}$$

From (23) and (30), we finally get that for all $\varepsilon > 0$ and for $\beta > \beta_3$

$$0 \leq \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\mu_X^*(u) - \frac{\bar{Q}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{Q_X(\beta)} \leq \varepsilon g(\kappa, \gamma_X), \quad (31)$$

with

$$g(\kappa, \gamma_X) := \left[1 \vee \left(c(\kappa, \gamma_X) + \frac{1}{2} \right) \right].$$

As shown in the proof of Lemma 2 entails that for all $Q_X \in \mathcal{Q}^{(\text{HT})}$,

$$\left| \int_{[0,1]} \frac{Q_{X,\beta}(u)}{Q_X(\beta)} d\mu_X^*(u) - \int_{[0,1]} (1-u)^{-\gamma_X} d\mu_X^*(u) \right| < \varepsilon g(\kappa, \gamma_X),$$

and thus, from (31),

$$\left| \frac{\bar{Q}_\beta(Q_X | \mathcal{P}_\mu(\kappa, \gamma_0))}{Q_X(\beta)} - \int_{[0,1]} (1-u)^{-\gamma_X} d\mu_X^*(u) \right| \leq 2\varepsilon g(\kappa, \gamma_X),$$

which is the desired result. \blacksquare

Proof of Proposition 3 – From Lemma 2, we have for all quantile functions $Q_X \in \mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_X \in (0, \gamma_0]$,

$$\lim_{\beta \rightarrow 1} \int_{[0,1]} \frac{Q_X(u)}{Q_X(\beta)} d\mu_\beta(u) = \int_{[0,1]} (1-u)^{-\gamma_X} d\mu(u),$$

The point i) is then a direct consequence of Proposition 2.

Let us now prove the second point of the Proposition. Under (7), the sum $X_1 + X_2$ is such that $Q_{X_1+X_2} \in \mathcal{Q}^{(\text{HT})}$ with tail index $\gamma_2 \leq \gamma_0$, see for instance Davis and Resnick [7, Lemma 2.1]. Hence, as $\beta \rightarrow 1$, we have from Lemma 2

$$\begin{aligned} \bar{Q}_\beta(Q_{X_1+X_2} | \mathcal{P}_\mu(\kappa, \gamma_0)) &\sim Q_{X_1+X_2}(\beta) \int_{[0,1]} (1-u)^{-\gamma_2} d\mu(u) \\ &\sim \int_{[0,1]} Q_{X_1+X_2} d\mu_\beta \leq \int_{[0,1]} Q_{X_1} d\mu_\beta + \int_{[0,1]} Q_{X_2} d\mu_\beta. \end{aligned}$$

The conclusion follows with another use of Lemma 2. \blacksquare

Proof of Proposition 4 – Let us first introduce the function $\Phi_{\mu, \gamma_0} : (0, \infty) \rightarrow (0, \infty)$, defined for all $\gamma > 0$ by

$$\Psi_{\mu, \gamma_0}(\gamma) = \left[\int_{[0,1]} (1-u)^{-(\gamma \wedge \gamma_0)} d\mu(u) \right]^{(\gamma \vee \gamma_0) / \gamma_0}.$$

We thus have

$$\widehat{\varrho}_{n,\beta_n}(Q_X \mid \mu, \gamma_0) = X_{[n\beta_n]:n} \Psi_{\mu,\gamma_0}(\widehat{\gamma}_{n,X}).$$

Using the asymptotic equivalence obtained in Proposition 2, it suffices to prove that

$$\frac{X_{[n\beta_n]:n}}{Q_X(\beta_n)} \xrightarrow{\mathbb{P}} 1 \text{ and } \Psi_{\mu,\gamma_0}(\widehat{\gamma}_{n,X}) \xrightarrow{\mathbb{P}} \Psi_{\mu,\gamma_0}(\gamma_X).$$

Let's start with the first convergence in probability. If V_1, \dots, V_n is a sample of n independent copies of a standard uniform random variables, it is well known that

$$X_{[n\beta_n]:n} \stackrel{d}{=} Q_X(1 - V_{n-[n\beta_n]+1:n}) = U_X(V_{n-[n\beta_n]+1:n}^{-1}),$$

where $V_{1:n} \leq \dots \leq V_{n:n}$ is the sample V_1, \dots, V_n arranged in ascending order and $U_X(\cdot) = Q_X(1 - 1/\cdot)$. From the representation of uniform order statistics, see David [6] and Reiss [24],

$$V_{n-[n\beta_n]+1:n} \stackrel{d}{=} \frac{\sum_{i=1}^{n-[n\beta_n]+1} E_i}{\sum_{i=1}^n E_i},$$

where E_1, \dots, E_n are independent copies of a standard exponential random variable. Since $n(1 - \beta_n) \rightarrow \infty$, it is readily seen that $(n - [n\beta_n] + 1)/[n(1 - \beta_n)] \rightarrow 1$. An application of the law of large numbers leads to

$$\frac{V_{n-[n\beta_n]+1:n}}{1 - \beta_n} \xrightarrow{\mathbb{P}} 1.$$

Since U_X is a regularly varying function,

$$U_X(V_{n-[n\beta_n]+1:n}^{-1}) / U_X((1 - \beta_n)^{-1}) \xrightarrow{\mathbb{P}} 1$$

showing the first convergence. The proof of the second convergence in probability is straightforward since $\widehat{\gamma}_{n,X} \xrightarrow{\mathbb{P}} \gamma_X$ and since under (H.1) the function Ψ_{μ,γ_0} is continuous. ■

Proof of Proposition 5 – Using the function Ψ_{μ,γ_0} introduced in the proof of Proposition 4, we have

$$\widehat{\varrho}_{n,\beta_n}^{(W)}(Q_X \mid \mu, \gamma_0, \alpha_n) := X_{[n\alpha_n]:n} \left(\frac{1 - \alpha_n}{1 - \beta_n} \right)^{\widehat{\gamma}_{n,X}} \Psi_{\mu,\gamma_0}(\widehat{\gamma}_{n,X}).$$

According to the asymptotic equivalence obtained in Proposition 2, it thus suffices to show that

$$\frac{X_{[n\alpha_n]:n}}{Q_X(\beta_n)} \left(\frac{1-\alpha_n}{1-\beta_n} \right)^{\hat{\gamma}_{n,X}} \xrightarrow{\mathbb{P}} 1.$$

Since $X_{[n\alpha_n]:n}/Q(\alpha_n) \xrightarrow{\mathbb{P}} 1$, see the proof of Proposition 4, it is equivalent to prove that

$$\frac{Q_X(\alpha_n)}{Q_X(\beta_n)} \left(\frac{1-\alpha_n}{1-\beta_n} \right)^{\hat{\gamma}_{n,X}} = \frac{\ell_X((1-\alpha_n)^{-1})}{\ell_X((1-\beta_n)^{-1})} \exp \left((\hat{\gamma}_{n,X} - \gamma_X) \ln \left(\frac{1-\alpha_n}{1-\beta_n} \right) \right) \xrightarrow{\mathbb{P}} 1.$$

Under (12),

$$\exp \left((\hat{\gamma}_{n,X} - \gamma_X) \ln \left(\frac{1-\alpha_n}{1-\beta_n} \right) \right) \xrightarrow{\mathbb{P}} 1.$$

Now, using the representation (11) of a slowly varying function,

$$\frac{\ell_X((1-\alpha_n)^{-1})}{\ell_X((1-\beta_n)^{-1})} = \frac{c_X((1-\alpha_n)^{-1})}{c_X((1-\beta_n)^{-1})} \exp \left(- \int_{(1-\alpha_n)^{-1}}^{(1-\beta_n)^{-1}} \frac{\Delta_X(t)}{t} dt \right).$$

Since α_n and β_n converge to 1 as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{c_X((1-\alpha_n)^{-1})}{c_X((1-\beta_n)^{-1})} = 1.$$

Moreover, since $|\Delta_X|$ is asymptotically decreasing,

$$\left| \int_{(1-\alpha_n)^{-1}}^{(1-\beta_n)^{-1}} \frac{\Delta_X(t)}{t} dt \right| \leq \left| \Delta_X \left(\frac{1}{1-\alpha_n} \right) \right| \ln \left(\frac{1-\alpha_n}{1-\beta_n} \right) \rightarrow 0,$$

and the proof is complete. ■

Proof of Corollary 1 – Our goal is to prove that

$$(n(1-\alpha_n))^{1/2} \left(\hat{\gamma}_{n,X}^{(H)} - \gamma_X \right) \xrightarrow{d} \mathcal{N}(0, \gamma_X^2). \quad (32)$$

Corollary 1 is then a direct consequence of Proposition 5. To prove (32), we use de Haan and Ferreira [17, Theorem 3.2.5] in which the asymptotic normality of the Hill estimator is established. We thus have to check that under the assumptions of Corollary 1, the conditions of de Haan and Ferreira [17, Theorem 3.2.5] are satisfied. After a comparison between the set of conditions in Corollary 1 and the one in de Haan and Ferreira [17, Theorem 3.2.5], we notice that we only have to check that

the second order condition (13) holds or equivalently that for all $t > 1$,

$$\lim_{x \rightarrow \infty} \frac{\ell_X(tx)/\ell_X(x) - 1}{\Delta_X(x)} = \int_1^t u^{\rho-1} du = \frac{t^\rho - 1}{\rho}.$$

Under the conditions of Corollary 1,

$$\frac{\ell_X(tx)}{\ell_X(x)} = \exp\left(\int_x^{tx} \frac{\Delta_X(u)}{u} du\right) = \exp\left(\Delta_X(x) \int_1^t \frac{\Delta_X(ux)}{\Delta_X(x)} \frac{1}{u} du\right).$$

Since the sign of Δ_X is asymptotically constant and $|\Delta_X|$ is regularly varying of index $\rho \leq 0$, we have from Bingham *et al.* [4, Theorem 1.2.1] that

$$\lim_{x \rightarrow \infty} \frac{\Delta_X(ux)}{\Delta_X(x)} = u^\rho,$$

uniformly on u in a compact set. Hence,

$$\lim_{x \rightarrow \infty} \int_1^t \frac{\Delta_X(ux)}{\Delta_X(x)} \frac{1}{u} du = \int_1^t u^{\rho-1} du,$$

and the conclusion follows since $\Delta_X(x) \rightarrow 0$ as $x \rightarrow \infty$ and $(\exp(u) - 1)/u \rightarrow 1$ as $u \rightarrow 0$.

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Figure 1: For each year $j \in \{1985, \dots, 1992\}$, the point \circ represents the value of the Hill estimator and the vertical full line its confidence interval of level 0.95. The horizontal line is the critical value $\gamma_v^* = 1$.

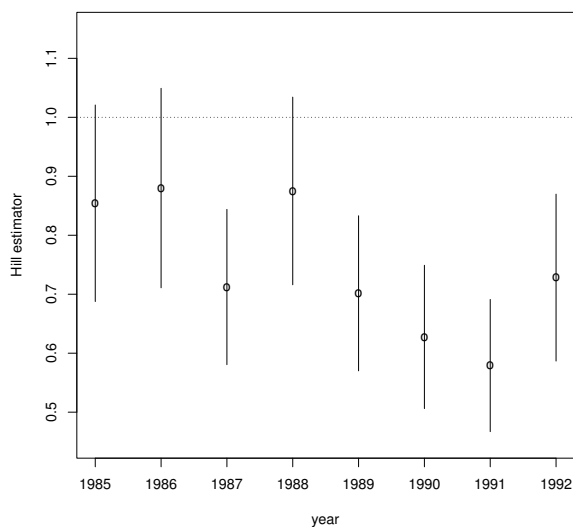


Figure 2: Left panel: for each year $j \in \{1985, \dots, 1992\}$, value of the estimator (14) together with its interval of variation when the Hill estimator varies in the confidence interval (15). Right panel: idem with the Tail-Value-at-Risk estimator (16).

